Homework 3

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Note: $\mathbb{N}$ denotes the set $\{0, 1, 2, \dots\}$.

1.4.1 Exercises

Exercise 2. For $\Gamma \subseteq \mathcal{L}_0$ and $\psi \in \mathcal{L}_0$, show that $\Gamma \cup \{\varphi\}$ logically implies ψ if and only if Γ logically implies $(\varphi \to \psi)$.

Proof. This proof relies on the fact that $\mathcal{L}_0$ is sound and complete: because $\mathcal{L}_0$ is complete (in the sense of Completeness; Version II), any subset that logically implies some formula proves that formula as well; because $\mathcal{L}_0$ is sound, any subset that proves some formula logically implies that formula as well.

First, suppose $\Gamma \cup \{\varphi\}$ logically implies ψ. By Version II of Completeness of $\mathcal{L}_0$, $\Gamma \cup \{\varphi\} \vdash \psi$. By the deduction lemma, $\Gamma \vdash (\varphi \to \psi)$. Thus, by the soundness lemma, Γ logically implies $(\varphi \to \psi)$.

Now, suppose Γ logically implies $(\varphi \to \psi)$. By Version II of Completeness of $\mathcal{L}_0$, $\Gamma \vdash (\varphi \to \psi)$. Since $\Gamma \subseteq \Gamma \cup \{\varphi\}$, we know that any Γ-proof is also a $\Gamma \cup \{\varphi\}$-proof. So, $\Gamma \cup \{\varphi\} \vdash (\varphi \to \psi)$. So, by the inference lemma, $\Gamma \cup \{\varphi\} \vdash \psi$. Thus, by the soundness lemma, $\Gamma \cup \{\varphi\}$ logically implies ψ.

Therefore, $\Gamma \cup \{\varphi\}$ logically implies ψ if and only if Γ logically implies $(\varphi \to \psi)$.

$\square$

Exercise 4. For Γ_1 and Γ_2 subsets of $\mathcal{L}_0$, Γ_1 is *logically equivalent* to Γ_2 if and only if, for all $\varphi \in \mathcal{L}_0$, Γ_1 logically implies φ if and only if Γ_2 logically implies φ. For $\Gamma \subseteq \mathcal{L}_0$, Γ is *independent* if it is not logically equivalent to any of its proper subsets. Prove the following.

a) If Γ is finite, then there is a Γ_0 such that $\Gamma_0 \subseteq \Gamma$, Γ and Γ_0 are logically equivalent, and Γ_0 is independent.

Proof. Suppose $\Gamma \subseteq \mathcal{L}_0$ is finite. So we may induct on the number of formulas in Γ. If Γ contains 0 formulas, then Γ is empty, and thus has no proper subsets. So Γ is independent, vacuously. Thus $\Gamma_0 = \Gamma$ satisfies the conditions of the proposition.

For the inductive step, we will need to first show that logical equivalence is transitive (this is basically obvious). Suppose that Γ_1 is logically equivalent to Γ_2 and Γ_2 is logically equivalent to Γ_3. Then, for all $\varphi \in \mathcal{L}_0$, if Γ_1 logically implies φ then so does Γ_2. Since Γ_2

logically implies φ, so does Γ_3. Switching the roles of Γ_1 and Γ_3 gives that Γ_3 logically implies φ only if Γ_1 does, too. So Γ_1 and Γ_3 are logically equivalent, and thus logical equivalence is transitive.

Now, assume for some $k \in \mathbb{Z}^+ \cup \{0\}$ that the proposition holds for all subsets of $\mathcal{L}_0$ containing less than or equal to k formulas. Suppose Γ contains $k+1$ formulas. If Γ is independent, then $\Gamma_0 = \Gamma$ is a subset of Γ that satisfies the conditions of the proposition.

So assume that Γ is not independent. Then Γ has some proper subset Γ_1 that is logically equivalent to Γ. Since Γ_1 is a proper subset, it contains less formulas than Γ does. So the number of formulas in Γ_1 is less than or equal to k. By the inductive hypothesis, then, there is a subset Γ_0 of Γ_1 such that Γ_0 is independent and logically equivalent to Γ_1 (which is, in turn, logically equivalent to Γ).

So $\Gamma_0 \subseteq \Gamma_1 \subset \Gamma$ and, by the transitivity of logical equivalence, Γ_0 is logically equivalent to Γ. Thus Γ_0 satisfies the conditions of the proposition, since it is an independent subset of Γ that is logically equivalent to Γ. Therefore, we have shown inductively that the proposition holds for all finite $\Gamma \subseteq \mathcal{L}_0$.

$\square$

b) There is an infinite set Γ such that Γ has no independent and logically equivalent subset.

Proof. Define a sequence of formulas $\{\varphi_n\}$ by $\varphi_n = A_0 \wedge A_1 \wedge A_2 \wedge \cdots \wedge A_n$ for each $n \in \mathbb{N}$. Notice that, for any $i < j$, $\{\varphi_j\}$ logically implies φ_i. Take Γ to be the set $\{\varphi_n : n \in \mathbb{N}\}$.

Assume, for a contradiction, that Γ_0 is an independent and logically equivalent subset of Γ. Clearly, Γ is not empty because the empty set only implies tautologies, yet Γ implies A_0, for instance, which is not a tautology. Thus, the set $M = \{n : \varphi_n \in \Gamma_0\}$ is nonempty.

If this set M has no maximum, then for any $\varphi_i \in \Gamma_0$ there is some $j > i$ such that $\varphi_j \in \Gamma_0$ as well. But by our statement in the first paragraph, φ_j logically implies φ_i. So Γ_0 is logically equivalent to its own proper subset $\Gamma_0 \setminus \{\varphi_i\}$. Therefore Γ_0 is not independent, a contradiction.

So we may assume that M has a maximum m. Thus for all $k > m$, $\varphi_k \notin \Gamma_0$, and it is clear that φ_k is not logically implied by Γ_0. Thus Γ_0 is not logically equivalent to Γ, a contradiction.

Therefore, Γ is an infinite subset of $\mathcal{L}_0$ that has no independent and logically equivalent subset.

$\square$

c) For every $\Gamma \subseteq \mathcal{L}_0$, there is a $\Delta \subseteq \mathcal{L}_0$ such that Δ is independent and logically equivalent to Γ.

Proof. For this proof, we will need to use the fact that $\mathcal{L}_0$ is countable. So we will begin by proving this. We have already shown that $\mathcal{L}_0$ can be constructed recursively as

$$\mathcal{L}_0 = \bigcup_{i=0}^{\infty} F_m$$

where

$$F_0 = \{A_n : n \in \mathbb{N}\} \qquad \text{and} \qquad F_{m+1} = F_m \cup \{(\neg\varphi) : \varphi \in F_m\} \cup \{(\varphi \to \psi) : \varphi, \psi \in F_m\}.$$

We can induct on m to show that each F_m is countable. F_0 is countable because it the set of elements of a sequence. Now, assume that F_m is countable for some $m \in \mathbb{N}$. Then $\{(\neg\varphi) : \varphi \in F_M\}$ is countable because it is clearly in bijection with F_m. Also, $\{(\varphi \to \psi) : \varphi, \psi \in F_m\}$ is countable because it is in bijection with $F_m \times F_m$. So F_{m+1} is a union of three countable sets, and is thus countable. So each F_m is countable, so $\mathcal{L}_0$ is a countable union of countable sets, and is therefore countable.

Now, to prove the main claim, let $\Gamma \subseteq \mathcal{L}_0$ and let Φ be the set of all consequences of Γ, i.e. $\Phi = \{\varphi \in \mathcal{L}_0 : \Gamma \models \varphi\}$. Since $\Phi \subseteq \mathcal{L}_0$, we know that Φ is countable. So there is a sequence $\{\varphi_n\}$, for $n \in \mathbb{N}$, such that $\Phi = \{\varphi_n : n \in \mathbb{N}\}$.

Define a sequence $\{\Delta_n\}$ as follows:

If $\{\varphi_0\}$ is independent, let $\Delta_0 = \{\varphi_0\}$. Otherwise, let $\Delta_0 = \emptyset$. Next, for $n \geq 0$, follow these instructions:

1. If $\Delta_n \cup \{\varphi_{n+1}\}$ is independent, let $\Delta_{n+1} = \Delta_n \cup \{\varphi_{n+1}\}$.
2. Otherwise, if Δ_n is logically equivalent to $\Delta_n \cup \{\varphi_{n+1}\}$, then let $\Delta_{n+1} = \Delta_n$.
3. Otherwise, let
$$\Delta_{n+1} = \Delta_n \cup \left\{ \left(\left(\bigwedge_{\psi \in \Delta_n} \psi \right) \to \varphi_{n+1} \right) \right\}.$$

Finally, let

$$\Delta = \bigcup_{i=0}^{\infty} \Delta_i.$$

We will now show that Δ and Γ are logically equivalent. Suppose that Γ logically implies some $\varphi \in \mathcal{L}_0$. Then $\varphi \in \Phi$, so $\varphi = \varphi_n$ for some $n \in \mathbb{N}$. We will induct on n to show that Δ logically implies φ_n. If $n = 0$, then either $\varphi_0 \in \Delta_0 \subseteq \Delta$ or $\{\varphi_0\}$ is not independent, meaning that φ_0 is a tautology. In either case, Δ logically implies φ.

Now, assume for some $k \in \mathbb{N}$ that Δ logically implies φ_n for all $n \leq k$. Then Δ_{n+1} is constructed using the set of instructions listed above. In (1) we know $\varphi_{n+1} \in \Delta_{n+1}$, and in (2) we know $\Delta_{n+1} = \Delta_n$ is logically equivalent to $\Delta_n \cup \{\varphi_{n+1}\}$. So clearly Δ_{n+1} logically implies φ_{n+1} in if either (1) or (2) are followed.

Otherwise (3) is followed, so $\left(\left(\bigwedge_{\psi \in \Delta_n} \psi\right) \to \varphi_{n+1}\right) \in \Delta_{n+1}$. By the inductive hypothesis, Δ_n implies each of $\varphi_1, \varphi_2, \ldots,$ and φ_n (thus so does Δ_{n+1}). So by the completeness of $\mathcal{L}_0$, Δ_{n+1} proves $\varphi_1, \varphi_2, \ldots,$ and φ_n. By concatenating these Δ_{n+1}-proofs with $\left(\bigwedge_{\psi \in \Delta_n} \psi\right)$, then with $\left(\left(\bigwedge_{\psi \in \Delta_n} \psi\right) \to \varphi_{n+1}\right)$ (we also need to include some other formulas which are needed to express conjunction in terms of $\neg$ and $\to$, but all of these are present in Δ_{n+1}), we can form a Δ_{n+1}-proof for φ_{n+1}. Therefore, by the soundness of $\mathcal{L}_0$, Δ_{n+1} logically implies φ_{n+1}. So Δ logically implies φ_{n+1}.

It is not hard to see that Γ logically implies every formula in Δ. We can show this simply by showing that Δ is a subset of Φ, which is the set of all formulas that Γ implies. If $\varphi \in \Delta$, then $\varphi = \varphi_0 \in \Phi$, or $\varphi = \varphi_n \in \Phi$, or $\varphi = \left(\left(\bigwedge_{\psi\in\Delta_n} \psi\right) \to \varphi_{n+1}\right)$ for some $n \in \mathbb{N}$. We know that $\varphi_1, \ldots, \varphi_{n+1} \in \Phi$ and that these formulas together logically imply $\left(\left(\bigwedge_{\psi\in\Delta_n} \psi\right) \to \varphi_{n+1}\right)$. Thus, by the transitivity of logical implication, $\varphi = \left(\left(\bigwedge_{\psi\in\Delta_n} \psi\right) \to \varphi_{n+1}\right)$ is implied by Γ, so $\varphi \in \Phi$. Thus, in all cases, $\varphi \in \Phi$. So $\Delta \subseteq \Phi$, and therefore Γ logically implies every formula in Δ. So Γ and Δ are logically equivalent.

Finally, we wish to show that Δ is independent. We will now explain that this amounts to showing that each Δ_n is independent. Suppose there was some φ which was implied by $\Delta \setminus \{\varphi\}$. Then there would be a $(\Delta \setminus \{\varphi\})$-proof of φ. A proof is a finite sequence of formulas, and each formula of Δ must be in some Δ_n, so let N be such that $\varphi \in \Delta_N$ and all symbols used in the $(\Delta \setminus \{\varphi\})$-proof of φ are in Δ_N. Then Δ_N is not independent. Thus Δ is independent if each Δ_n is independent.

We can induct on n to show that each Δ_n is independent. We have clearly defined Δ_0 to be independent. Now, assume Δ_n is independent. If (1) or (2) is used, then clearly Δ_{n+1} is independent. So assume (3) is used. We know that no subset of Δ_n implies $\left(\bigwedge_{\psi\in\Delta_n} \psi\right)$ because Δ_n is independent. Also, Δ_n does not imply φ_{n+1}, or else (2) would have been used. Therefore, if any element of Δ_n is removed from Δ_{n+1}, then φ_{n+1} is no longer a consequence. So the only possibility would be to remove $\left(\left(\bigwedge_{\psi\in\Delta_n} \psi\right) \to \varphi_{n+1}\right)$. But without this formula, Δ_{n+1} again cannot imply φ_{n+1}, since it would be reduced to the subset Δ_n, which does not imply φ_{n+1} because (2) was not used. Thus we cannot remove any element from Δ_{n+1} and still have that Δ_{n+1} logically implies φ_{n+1}. So Δ_{n+1} is independent.

Therefore, since each Δ_n is independent, Δ is independent. So Δ is an independent set which is logically equivalent to Γ.

□

Extra Problems

Exercise 1. Does there exist a $\varphi \in \mathcal{L}_0$ such that the following conditions hold?

a) φ is neither a contradiction nor a tautology.

b) For every $\psi \in \mathcal{L}_0$ using only the propositional letters that appear in φ, if φ does not logically imply ψ then ψ logically implies φ.

No such ψ exists.

Proof. We will show that, if φ satisfies the first condition, then $\psi = (\neg\varphi)$, which uses only the propositional symbols found in φ, necessarily fails the second condition.

Assume that φ is neither a tautology nor a contradiction. Then there exist truth assignments ν_T and ν_F such that $\overline{\nu}_T(\varphi) = T$ and $\overline{\nu}_F(\varphi) = F$. By the recursive definitions of $\overline{\nu}_T$ and $\overline{\nu}_F$, we know then that

$$\overline{\nu}_T(\psi) = \overline{\nu}_T((\neg\varphi)) = F \text{ and } \overline{\nu}_F(\psi) = \overline{\nu}_F((\neg\varphi)) = T.$$

Therefore, φ does not logically imply ψ because $\overline{\nu}_T$ satisfies φ but does not satisfy ψ. Likewise, ψ does not logically imply φ because $\overline{\nu}_F$ satisfies ψ but does not satisfy φ.

□

Exercise 2. Given two truth assignments ν_1 and ν_2, show that there is an infinite set Γ such that Γ is satisfied by ν_1 and ν_2 and by no other truth assignments.

Note: If $\nu_1(A_i) = \nu_2(A_i)$, say that ν_1 and ν_2 *agree* on A_i. Otherwise, say ν_1 and ν_2 *disagree* on A_i.

Proof. We construct Γ as follows:

For every $i \in \mathbb{N}$, if $\nu_1(A_i) = \nu_2(A_i) = T$, include A_i in Γ. If $\nu_1(A_i) = \nu_2(A_i) = F$, include $(\neg A_i)$ in Γ.

Next, if ν_1 and ν_2 disagree on some A_i and A_j for $i < j$, then include $\varphi_{i,j}$ in Γ, which we define in the table below. It is obvious that Γ is infinite (if it is not obvious, then simply add infinitely many distinct tautologies to Γ and it will still satisfy the claim).

$\nu_1(A_i)$	$\nu_1(A_j)$	$\nu_2(A_i)$	$\nu_2(A_j)$	$\varphi_{i,j}$
T	T	F	F	$(\neg A_i) \to (\neg A_j)$
T	F	F	T	$(\neg A_i) \to A_j$
F	T	T	F	$A_i \to (\neg A_j)$
F	F	T	T	$A_i \to A_j$

Next, we show that ν_1 and ν_2 satisfy Γ. Let $\theta \in \Gamma$. There are only six forms which θ can take: $\theta = A_i$ for some $i \in \mathbb{N}$, $\theta = (\neg A_i)$ for some $i \in \mathbb{N}$, or θ is of one of the four forms in the rightmost column of the above table. If $\theta = A_i$, then $\nu_1(A_i) = \nu_2(A_i) = T$, since this is the only situation where A_i would have been included in Γ. So ν_1 and ν_2 both satisfy θ. If $\theta = (\neg A_i)$, then $\nu_1(A_i) = \nu_2(A_i) = F$, since this is the only situation where $(\neg A_i)$ would have been included in Γ. So ν_1 and ν_2 both satisfy θ. If θ is of one of the four forms in the rightmost column of the above table, then inspection of the left four columns reveals that ν_1 and ν_2 both satisfy θ. So ν_1 and ν_2 both satisfy Γ.

Now, suppose ν_3 is a truth assignment that is not equal to ν_1 or ν_2. Then, for some A_i and A_j, ν_3 disagrees with ν_1 on A_i and disagrees with ν_2 on A_j. If $i = j$, then ν_1 and ν_2 must agree on A_i, since otherwise ν_3 could not disagree with both of them. Therefore, either $\nu_1(A_i) = \nu_2(A_i) = T$ and $\nu_3(A_i) = F$, so $A_i \in \Gamma$ but ν_3 does not satisfy A_i, or $\nu_1(A_i) = \nu_2(A_i) = F$ and $\nu_3(A_i) = T$, so $(\neg A_i) \in \Gamma$ but ν_3 does not satisfy $(\neg A_i)$.

So we assume that $i \neq j$. Further, assume without loss of generality that $i < j$. The table below shows that ν_3 does not satisfy $\varphi_{i,j}$, which was included in Γ as a result of the disagreement of ν_1 and ν_2 on A_i and A_j.

$\nu_1(A_i)$	$\nu_1(A_j)$	$\nu_2(A_i)$	$\nu_2(A_j)$	$\nu_3(A_i)$	$\nu_3(A_j)$	$\varphi_{i,j}$	$\overline{\nu}_3(\varphi)$
T	T	F	F	F	T	$(\neg A_i) \to (\neg A_j)$	F
T	F	F	T	F	F	$(\neg A_i) \to A_j$	F
F	T	T	F	T	T	$A_i \to (\neg A_j)$	F
F	F	T	T	T	F	$A_i \to A_j$	F

Since, in all cases, ν_3 does not satisfy some formula in Γ, ν_3 does not satisfy Γ. Because ν_3 was arbitrary, Γ is satisfied by ν_1 and ν_2 but by no other truth assignments.

$\square$

Exercise 3. Show that the axioms in Group IV (2) are tautologies.

Proof. We wish to show that the following logical axioms are tautologies:

1) $((\neg\varphi_1) \to (\varphi_1 \to \varphi_2))$
2) $(\varphi_1 \to ((\neg\varphi_2) \to (\neg(\varphi_1 \to \varphi_2))))$

A formula ψ is a tautology if it is satisfied by every truth assignment. We can show this with a truth table, since there are only finitely many values a truth assignment could take for the formulas specified within ψ.

$\overline{\nu}(\varphi_1)$	$\overline{\nu}(\varphi_2)$	$\overline{\nu}((\neg\varphi_1))$	$\overline{\nu}((\varphi_1 \to \varphi_2))$	$\overline{\nu}(((\neg\varphi_1) \to (\varphi_1 \to \varphi_2)))$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

Let $\psi = (\varphi_1 \to ((\neg\varphi_2) \to (\neg(\varphi_1 \to \varphi_2))))$.

$\overline{\nu}(\varphi_1)$	$\overline{\nu}(\varphi_2)$	$\overline{\nu}((\varphi_1 \to \varphi_2))$	$\overline{\nu}((\neg(\varphi_1 \to \varphi_2)))$	$\overline{\nu}((\neg\varphi_2))$	$\overline{\nu}(((\neg\varphi_2) \to (\neg(\varphi_1 \to \varphi_2))))$	$\overline{\nu}(\psi)$
T	T	T	F	F	T	T
T	F	F	T	T	T	T
F	T	T	F	F	T	T
F	F	T	F	T	F	T

$\square$