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First order logic—syntax

First order languages extend propositional ones by adding the apparatus to refer to elements of a structure and to assert their properties.

Our language consists of (certain) finite sequences of symbols, as described below.

- The *logical symbols* are the following.

$$(\quad) \quad \neg \quad \to \quad \forall$$

- $\hat{=}$ is the *equality symbol*.
- The *variable symbols* are x_i, for $i \in \mathbb{N}$.
- The *constant symbols* are c_i, for $i \in \mathbb{N}$.
- The *function symbols* are F_i, for $i \in \mathbb{N}$.
- The *predicate symbols* are P_i, for $i \in \mathbb{N}$.

We fix a function π mapping the set of function and predicate symbols to $\mathbb{N}$ so that for each $k \geq 1$, each of the sets

$$\{i \in \mathbb{N} \mid \pi(F_i) = k\}$$

and

$$\{i \in \mathbb{N} \mid \pi(P_i) = k\}$$

is infinite. For example, we could define $\pi : F_i \mapsto k$, where the kth prime is the least prime which divides i. (So $e(j) = 2$ if j is even etc.) The purpose of the function π is to specify the number of arguments or *arity* of each function and predicate symbol.

2.1 Terms

Recall our notation; if $s = \langle s_1, \ldots, s_n \rangle$ and $t = \langle t_1, \ldots, t_m \rangle$ are finite sequences of symbols, then $s + t$ denotes the finite sequence $\langle s_1, \ldots, s_n, t_1, \ldots, t_m \rangle$.

Definition 2.1 The set of *terms*, T, is defined as the smallest set of finite sequences T satisfying the following properties.

(1) For each $i \in \mathbb{N}$, the sequences of length one,

$$\langle x_i \rangle$$

and

$$\langle c_i \rangle$$

belong to T.

(2) If F_i is a function symbol, $n = \pi(F_i)$, and $\tau_1, \ldots, \tau_n$ belong to T, then

$$\langle F_i \rangle + \langle (\rangle + \tau_1 + \cdots + \tau_n + \langle) \rangle$$

belongs to T. More briefly, the concatenation $F_i(\tau_1 \ldots \tau_n)$ belongs to T.

We will assume familiarity with the methods of the previous chapter and omit the proof that T is well defined.

Remark 2.2 We shall adopt several notational conventions.

(1) Often we shall say that x_i is a term. Of course we are referring to the sequence of length 1, $\langle x_i \rangle$.

(2) More generally we shall indicate terms informally and use

$$F_i(\tau_1, \ldots, \tau_n)$$

to indicate the term

$$\langle F_i \rangle + \langle (\rangle + \tau_1 + \cdots + \tau_n + \langle) \rangle$$

The elements of T are uniquely readable, as is pointed out in the next sequence of lemmas.

Lemma 2.3 (Readability) *For each term τ in T, exactly one of the following conditions applies.*

(1) There is an $i \in \mathcal{N}$ greater than or equal to 1 such that τ is x_i or τ is c_i.

(2) There is an $i \in \mathcal{N}$ greater than or equal to 1 such that $\pi(F_i) = n$ and there are terms $\tau_1, \ldots, \tau_n$ in T such that τ is $F_i(\tau_1, \ldots, \tau_n)$.

Proof. As in the proof of Lemma 1.11, we let T be the subset of T whose elements satisfy one of the above clauses. We observe that T satisfies the closure properties of Definition 2.1. Consequently, $\mathrm{T} \subseteq T$, as required.

The two conditions are mutually exclusive, as the first symbol in τ determines which condition applies. $\square$

Lemma 2.4 *If $\tau \in \mathrm{T}$, then no proper initial segment of τ is an element of T.*

Proof. We proceed by induction on the length of $\tau \in \mathrm{T}$.

If τ is a term of length 1, then the only proper initial segment is the null sequence, which by Lemma 2.3 is not an element of T.

Suppose that τ has length greater than 1 and assume the lemma for all terms of length less than that of τ. By Lemma 2.3, τ is of the form $F_i(\tau_1, \ldots, \tau_n)$. Suppose that σ is a proper initial segment of τ such that $\sigma \in \mathrm{T}$. As above, σ is not the null sequence, so the first symbol in σ is F_i. By Lemma 2.3, σ must have the form $F_i(\sigma_1, \ldots, \sigma_n)$, where each σ_i belongs to T. But then, σ_1 and τ_1 must be identical, since neither can be a proper initial segment of the other. It follows by an induction up to n, that for each i, σ_i is equal to τ_i. But then $\sigma = \tau$, contradicting the choice of σ. Thus, τ has no proper initial segment in T, as required. □

Theorem 2.5 (Unique Readability) *For each term τ in* T*, exactly one of the following conditions applies.*

(1) There is an $i \in \mathcal{N}$ greater than or equal to 1 such that τ is x_i or τ is c_i.
(2) There is an $i \in \mathcal{N}$ greater than or equal to 1 such that $\pi(F_i) = n$ and there are terms $\tau_1, \ldots, \tau_n$ in T *such that τ is $F_i(\tau_1, \ldots, \tau_n)$.*

Further, in (2), the function symbol F_i and the terms $\tau_1, \ldots, \tau_n$ are unique.

Proof. By Lemma 2.3, it will be sufficient to verify the uniqueness of $\tau_1, \ldots, \tau_n$. This follows as in the proof of Lemma 2.4. Suppose that τ could be written as $F_i(\tau_1, \ldots, \tau_n)$ and as $F_j(\sigma_1, \ldots, \sigma_m)$. Then F_i and F_j both occur as the first symbol in τ, and hence are equal. Consequently, $n = m = \pi(F_i)$. Then, τ_1 and σ_1 must also be equal, as neither can be a proper initial segment of the other. By induction on i less than or equal to n, for each i, τ_i is equal to σ_i, as required. □

2.2 Formulas

Definition 2.6 The set of formulas, $\mathcal{L}$, is the smallest set L of finite sequences of symbols as above satisfying the following properties.

(1) If P_i is a predicate symbol, $n = \pi(P_i)$ is the arity of P_i and $\tau_1, \ldots, \tau_n$ are terms, then

$$P_i(\tau_1 \ldots \tau_n)$$

is an element of L

(2) If τ_1 and τ_2 are terms, then

$$(\tau_1 \hat{=} \tau_2)$$

is an element of L.

(3) If $\varphi \in L$, then

$$(\neg\varphi)$$

is an element of L

(4) If φ_1 and φ_2 are elements of L, then

$$(\varphi_1 \to \varphi_2)$$

is an element of L

(5) If $\varphi \in L$ and x_i is a variable symbol, then

$$(\forall x_i \varphi)$$

is an element of L.

As in the case of T, we will not repeat the argument to show that $\mathcal{L}$ is well defined.

Definition 2.7 The *atomic formulas* are the ones indicated in (1) and (2) of Definition 2.6.

2.3 Subformulas

We define the relation ψ is a subformula of φ for formulas in $\mathcal{L}$.

Definition 2.8 Suppose that φ is a formula. A formula ψ is a *subformula* of φ if ψ is a block-subsequence of φ. (See Definition 1.5.)

Definition 2.9 Suppose that $\varphi = \langle a_1, \ldots, a_n \rangle$ is a formula and s is a finite sequence. An *occurrence* of s in φ is an interval $[j_1, j_2]$ such that $s = \langle a_{j_1}, \ldots, a_{j_2} \rangle$.

Remark 2.10 Suppose that φ is a formula. A formula ψ is a subformula of φ if and only if ψ has an occurrence in φ.

We will give an abbreviated proof that every formula in $\mathcal{L}$ is uniquely readable, as stated in Theorem 2.13. As above, we proceed by proving a readability lemma, a proper initial segment lemma, and then a uniqueness lemma.

Lemma 2.11 (Readability) *Suppose that φ is a formula. Then exactly one of the following conditions applies.*

(1) There is an i and terms $\tau_1, \ldots, \tau_n$ are terms, where $n = \pi(P_i)$, such that $\varphi = P_i(\tau_1 \ldots \tau_n)$.
(2) There are terms τ_1 and τ_2 such that $\varphi = (\tau_1 \hat{=} \tau_2)$.
(3) There is a formula ψ such that $\varphi = (\neg\psi)$.
(4) There are formulas ψ_1 and ψ_2 such that $\varphi = (\psi_1 \to \psi_2)$.
(5) There is a formula ψ and a variable x_i such that $\varphi = (\forall x_i \psi)$.

The proof Lemma 2.11 is analogous to that of Lemma 1.11.

Lemma 2.12 *If $\varphi \in \mathcal{L}$, then no proper initial segment of φ is an element of $\mathcal{L}$.*

Proof. We consider the cases of Lemma 2.11.

Suppose that φ is of the form $P_i(\tau_1 \dots \tau_n)$ and ψ is a proper initial segment of φ which also belongs to $\mathcal{L}$. Then the first symbol in ψ is P_i and so ψ must also be of the form $P_i(\sigma_1 \dots \sigma_n)$. But then τ_1 must equal σ_1, or they would be a pair of distinct terms for which one is a proper initial segment of the other, contradicting Lemma 2.4. It follows by induction on i less than or equal to n that each σ_i is equal to τ_i, and hence that φ is equal to ψ.

The case when φ is an equality between terms can be analyzed similarly, using Lemma 2.4.

The cases when φ is $(\neg\psi)$ or $(\psi_1 \to \psi_2)$ are analogous to the same cases in the propositional case. See Lemma 1.12.

Finally, consider the case when φ is $(\forall x_i \varphi_1)$. If ψ is an initial segment of φ, then ψ must be of the form $(\forall x_i \psi_1)$, as φ and ψ must have the same first three symbols. But then induction applies to φ_1, and ψ_1 must equal φ_1. It follows that φ is equal to ψ. □

Theorem 2.13 (Unique Readability) *Suppose that φ is a formula. Then exactly one of the following conditions applies.*

(1) There is an i and terms $\tau_1, \dots, \tau_n$ are terms, where $n = \pi(P_i)$, such that $\varphi = P_i(\tau_1 \dots \tau_n)$.
(2) There are terms τ_1 and τ_2 such that $\varphi = (\tau_1 \hat{=} \tau_2)$.
(3) There is a formula ψ such that $\varphi = (\neg\psi)$.
(4) There are formulas ψ_1 and ψ_2 such that $\varphi = (\psi_1 \to \psi_2)$.
(5) There is a formula ψ and a variable x_i such that $\varphi = (\forall x_i \psi)$.

Further, in each of the above cases, the terms and/or subformulas which are mentioned in that case are unique.

We leave the proof of Theorem 2.13 to the Exercises.

2.3.1 Exercises

(1) Prove Theorem 2.13.
(2) Consider the set of sequences defined as in Definition 2.6 except that the last clause is changed to read, “If $\varphi \in L$ and x_i is a variable symbol, then $\forall x_i \varphi$ is an element of L” in which the parentheses are omitted. Is this set uniquely readable?
(3) Consider the set of sequences defined as in Definition 2.6 except that the fourth clause is changed to read, “If φ_1 and φ_2 are elements of L, then $\varphi_1 \to \varphi_2$ is an element of L” in which the parentheses are omitted. Is this set uniquely readable?

2.4 Free variables, bound variables

Suppose that φ is a formula and that x_i is a variable. Then each occurrence of $\forall x_i$ in φ defines a unique subformula of φ. This is the content of the next lemma.

Lemma 2.14 *Suppose that φ is a formula, x_i is a variable, s and t are finite sequences, and that*

$$\varphi = s + \langle \forall, x_i \rangle + t.$$

Then there is a finite sequence $\hat{s}$, there is a formula ψ, and there is a finite sequence $\hat{t}$ such that $s = \hat{s} + \langle (\rangle$ and

$$\varphi = \hat{s} + \psi + \hat{t}.$$

Further, ψ is unique.

Proof. Note that the uniqueness of ψ follows by observing that if there were two such formulas, then one would be a proper initial segment of the other and contradict Lemma 2.12.

We prove the existence claims of Lemma 2.14 by induction on the length of φ. There are no formulas of length 1, and so the lemma is true of all length 1 formulas on trivial grounds. Now assume the lemma is true of every formula which is shorter than φ. By Lemma 2.11, we can analyze φ by considering the various cases of the Lemma. If φ is atomic, then φ does not contain an occurrence of $\langle \forall, x_i \rangle$, and again the claim is true on trivial grounds. If φ is $(\neg\theta)$, then any occurrence of $\langle \forall, x_i \rangle$ in φ is also one in θ and by induction there is a ψ contained in θ as required. Similarly, if φ is $(\psi_1 \to \psi_2)$ and there is an occurrence of $\langle \forall, x_i \rangle$ in φ, then it must be contained completely in ψ_1 or in ψ_2 (there is no $\to$ in $\langle \forall, x_i \rangle$) and the induction hypothesis applies. If φ is $(\forall x_j \varphi_1)$, then either the occurrence of $\langle \forall, x_i \rangle$ is the block of the second and third symbols in φ, $\hat{s}$ and $\hat{t}$ are both equal to the empty sequence, and the formula φ is the desired ψ, or the occurrence of $\langle \forall, x_i \rangle$ is entirely contained in φ_1 and the induction hypothesis applies. $\square$

This suggests the following definition.

Definition 2.15 Suppose that $\varphi = \langle a_1, \ldots, a_n \rangle$ is a formula and x_i is a variable.

(1) An occurrence of $\forall x_i$ in φ is an occurrence of $\langle \forall, x_i \rangle$ in φ (as a block-subsequence).
(2) The *scope* of a particular occurrence of $\forall x_i$ in φ is the unique interval $[j_1, j_2]$ with the following properties.
   a) $[j_1 + 1, j_1 + 2]$ is the given occurrence of $\forall x_i$.
   b) $\langle a_{j_1}, \ldots, a_{j_2} \rangle$ is a formula (which of course is a subformula of φ).

Note that Lemma 2.12 implies that the sequence $\langle a_{j_1}, \ldots, a_{j_2} \rangle$ is unique.

Definition 2.16 Suppose that φ is a formula and that x_i is a variable which occurs in φ.

(1) An occurrence of x_i in φ is *free* if and only if it is not within the scope of any occurrence of $\forall x_i$ in φ. Otherwise, the occurrence is *bound*.

(2) x_i is a *free variable* if and only if there is a free occurrence of x_i in φ.
(3) x_i is a *bound variable* of φ if and only if x_i occurs in φ and is not a free variable of φ.

Definition 2.17 (1) If τ is a term, we write $\tau(x_1, \ldots, x_n)$ to indicate that the variables of τ are included in the set $\{x_1, \ldots, x_n\}$.
(2) If φ is a formula, we write $\varphi(x_1, \ldots, x_n)$ to indicate that the free variables of φ are included in the set $\{x_1, \ldots, x_n\}$.

Definition 2.18 A formula φ is a *sentence* if and only if it has no free variables.