1/27/15

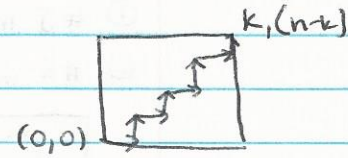
Binomial Coefficient

$\binom{n}{k}: \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$

picking k (order doesn't matter) out of n

$(x_1+x_2)^n = \sum_{k=0}^{n} \binom{n}{k} x_1^k x_2^{n-k}$

ways (non-dumb) to go from $(0,0)$ to $k,(n-k)$



Recurrence

$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

k-subsets of [n] $\quad n \notin$ set $\quad n \in$ set

$\binom{0}{0}$

$\binom{1}{0}\binom{1}{1}$

$\binom{2}{0}\binom{2}{1}\binom{2}{2}$

Goal: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ ... extends Induction

1) prove for boundary cases: for $\binom{n}{n}, \binom{n}{0}$, we get $\frac{n!}{0!\,n!} = 1$
2) prove satisfies recurrence relation

$$\frac{n!}{k!(n-k)!} = \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!}$$

Page 16 VLW

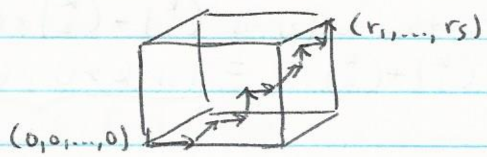
Multinomial Coefficient

$\binom{n}{r_1 \ldots r_s}: \frac{n!}{r_1! r_2! \ldots r_s!} \longleftrightarrow \binom{n}{k} = \binom{n}{k, n-k}$

$r_1 + \ldots + r_s = n$

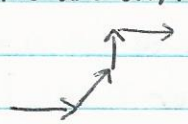
$$(x_1+x_2+\ldots+x_s)^n = \sum_{r_1+\ldots+r_s=n} \binom{n}{r_1,\ldots,r_s} x_1^{r_1} \ldots x_s^{r_s}$$

ways to go from $(0,0,\ldots,0)$ to $(r_1,\ldots,r_s)$



$\longleftrightarrow$ n distinguishable balls into boxes of size $r_1,\ldots,r_s$

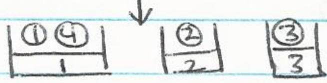
paths are alphabet $x_1,\ldots,x_s \rightarrow$ bins



1 2 3 4 $\rightarrow$ balls

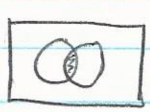
$= x_1 x_2 x_3 x_1$

$\updownarrow$



Chapter 10 (VLW) Inclusion-Exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$



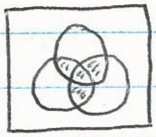
$\underline{10.1}$ | $N_j - \sum_{|M|=j} \underbrace{N(M)}_{\bigcap_{i \in M} E_i}$

set S

$E_1, \dots, E_n \subseteq S$

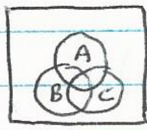
① $\# S$ in no $E_i = N - N_1 + N_2 - N_3 \dots$

② $\# S$ in at least one $E_i = N_1 - N_2 + N_3 \dots$



$N_j = \sum j$ - intersections of E_i

roughly "$\binom{n}{j}$"



②

$$\overset{E_1}{\overset{\shortparallel}{|A}} \cup \overset{E_2}{\overset{\shortparallel}{B}} \cup \overset{E_3}{\overset{\shortparallel}{C|}} = |A| + |B| + |C|$$
$$- |A \cap B| - |A \cap C| - |B \cap C|$$
$$+ |A \cap B \cap C|$$

in general $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n} = 2^n$

1) $(1+1)^n = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$

2) partition all subsets by size of set, sum principle

<u>Proof</u> |

<u>Idea</u>: for each $s \in S$, ask for its contribution to N_i's

If s is in <u>exactly</u> k sets $E_{s_1}, \dots, E_{s_k}$

s contributes k to N_1

$\binom{k}{2}$ to N_2

s's contribution in total to (2) is $\binom{k}{1} - \binom{k}{2} + \binom{k}{3} - \dots = 1 - (1-1)^k$

$(1-1)^k = 1 - \binom{k}{1} + \binom{k}{2} \dots$ $\underbrace{1 \text{ if } k > 0, \ 0 \text{ if } k = 0}$

5 guards — a team needs: 2 guards

5 forwards — 2 forwards

4 centers — 1 center

1 michael Jordan (wildcard)

1) MJ case

2) no MJ case $\binom{5}{2}\binom{5}{2}\binom{4}{1}$

- don't need extras because guards / forwards are indistinguishable

MJ case

MJ is a center: $\binom{5}{2}\binom{5}{2}\binom{4}{0}$

" " forward: $\binom{5}{2}\binom{5}{1}\binom{4}{1}$

" " guard: $\binom{5}{1}\binom{5}{2}\binom{4}{1}$

Trigger| (seeing these → use Inclusion-Exclusion)

① Count blah where at least 1 "good" thing happens

② Count blah where no "bad" thing happens

Step

① identify "good", "bad" sets

② I-E

How many numbers in $[n]$ are relatively prime to n?

no common prime factors

1) $n = p_1^{r_1} \cdots p_k^{r_k}$

2) "bad sets" are E_i, $1 \le i \le k$

$E_i = \{\text{numbers divisible by } p_i\}$

$|E_i| = \frac{n}{p_i}$

$|E_i \cap E_j \cap E_k| = \frac{n}{p_i p_j p_k}$

Answer| $n - \left(\frac{n}{p_1} + \frac{n}{p_2} + \frac{n}{p_3} \cdots - \frac{n}{p_i p_j} - \frac{n}{p_i p_k} \cdots + \frac{n}{p_i p_j p_k} \cdots\right)$

$= n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\left(1-\frac{1}{p_3}\right)\cdots = \phi(n)$ $\quad a^{\phi(n)} \equiv 1 \pmod{n}$

$1-a-b+ab = (1-a)(1-b)$

$1-a-b-c+ab+bc+ac-abc = (1-a)(1-b)(1-c)$

Derangements|

A derangement $f: [n] \to [n]$ is a permutation where for all $i \in [n]$, $f(i) \neq i$

what is the # of derangements (D_n) for $[n] \to [n]$

Idea:| use I-E

"bad permutation": $E_1: 1 \to 1$
$E_2: 2 \to 2$
$E_3: 3 \to 3$

- recurrence (induction +)
- multinomial coefs
- inclusion - exclusion
  - derangements, $\phi(n)$