5

The Gödel Completeness Theorem

5.1 The notion of proof

Definition 5.1 A *validity* is an $\mathcal{L}$ formula φ which is satisfied in any and every interpretation. That is, for all $\mathcal{M}$ and all ν, $(\mathcal{M}, \nu) \models \varphi$.

Validities are the most uninteresting $\mathcal{L}$ formulas, since their being satisfied by a particular $(\mathcal{M}, \nu)$ provides no information whatsoever about $(\mathcal{M}, \nu)$. But the set of validities, which can be regarded as the set of first order absolute truths, is a fascinating set. In this chapter, we will give a syntactic characterization of this set, describing it in terms of pure logic. We will show that a first order $\mathcal{L}$ formula is valid if and only if it can be proven.

The formal notion of proof involves specifying the *logical axioms*. Every logical axiom is valid, and there will be a straightforward algorithm to determine whether any given $\mathcal{L}$ formula is a logical axiom.

Some of the logical axioms involve the deduction of instances of a formula φ from the hypothesis $\forall x_i \varphi$. Others involve deducing that τ_1 has the property asserted by φ from the hypothesis that $\tau_1 \hat{=} \tau_2$ and τ_2 has that property. To state these axioms, we have to extend our notation from Definitions 3.9 and 3.12.

Definition 5.2 Suppose that φ is an $\mathcal{L}$ formula and that τ is a term.

(1) Suppose that x_i is a free variable of φ.
   a) The term τ is *substitutable* for x_i if and only if every variable x_j of τ is free for x_i in φ.
   b) If τ is substitutable for x_i in φ, then $\varphi(x_i; \tau)$ denotes the $\mathcal{L}$ formula obtained by substituting τ for each free occurrence of x_i in φ. Similarly, $\varphi(x_{i_1}, \dots, x_{i_n}; \tau_{i_1}, \dots, \tau_{i_n})$ denotes the $\mathcal{L}$ formula obtained by simultaneously substituting each τ_{i_j} for the free occurrences of x_{i_j} in φ.

(2) Suppose that c_i is a constant symbol.
   a) The term τ is *substitutable* for c_i if and only if for every variable x_j of τ, no occurrence of c_i in φ is within the scope of an occurrence of $\forall x_j$.
   b) If τ is substitutable for c_i in φ, then $\varphi(c_i; \tau)$ denotes the $\mathcal{L}$ formula obtained by substituting τ for each occurrence of c_i in φ. Similarly, $\varphi(c_{i_1}, \dots, c_{i_n}; \tau_{i_1}, \dots, \tau_{i_n})$ denotes the $\mathcal{L}$ formula obtained by simultaneously substituting each τ_{i_j} for the occurrences of c_{i_j} in φ.

Definition 5.3 The set of logical axioms, denoted Δ, is the smallest set of $\mathcal{L}$ formulas which satisfies the following closure properties.

(1) (Instances of Propositional Tautologies) Suppose that φ_1, φ_2 and φ_3 are $\mathcal{L}$ formulas. Then each of the following $\mathcal{L}$ formulas is a logical axiom:
(Group I axioms)
a) $((\varphi_1 \to (\varphi_2 \to \varphi_3)) \to ((\varphi_1 \to \varphi_2) \to (\varphi_1 \to \varphi_3)))$
b) $(\varphi_1 \to \varphi_1)$
c) $(\varphi_1 \to (\varphi_2 \to \varphi_1))$
(Group II axioms)
a) $(\varphi_1 \to ((\neg\varphi_1) \to \varphi_2))$
(Group III axioms)
a) $(((\neg\varphi_1) \to \varphi_1) \to \varphi_1)$
(Group IV axioms)
a) $((\neg\varphi_1) \to (\varphi_1 \to \varphi_2))$
b) $(\varphi_1 \to ((\neg\varphi_2) \to (\neg(\varphi_1 \to \varphi_2))))$

(2) Suppose that φ is an $\mathcal{L}$ formula, τ is a term, and that τ is substitutable for x_i in φ. Then

$$((\forall x_i \varphi) \to \varphi(x_i; \tau)) \in \Delta.$$

(3) Suppose that φ_1 and φ_2 are $\mathcal{L}$ formulas. Then

$$((\forall x_i(\varphi_1 \to \varphi_2)) \to ((\forall x_i \varphi_1) \to (\forall x_i \varphi_2))) \in \Delta.$$

(4) Suppose that φ is an $\mathcal{L}$ formula and that x_i is not a free variable of φ. Then

$$(\varphi \to (\forall x_i \varphi)) \in \Delta.$$

(5) For every variable x_i, $(x_i \mathrel{\hat{=}} x_i) \in \Delta$.

(6) Suppose that φ_1 and φ_2 are $\mathcal{L}$ formulas and that x_j is substitutable for x_i in φ_1 and in φ_2.

$$\text{If } \varphi_2(x_i; x_j) = \varphi_1(x_i; x_j),$$
$$\text{then } ((x_i \mathrel{\hat{=}} x_j) \to (\varphi_1 \to \varphi_2)) \in \Delta.$$

(7) Suppose that $\varphi \in \Delta$. Then $(\forall x_i \varphi) \in \Delta$.

Definition 5.4 Suppose that Γ is a set of $\mathcal{L}$ formulas and that φ is an $\mathcal{L}$ formula. Then

$$\Gamma \vdash \varphi$$

if and only if there exists a finite sequence $\langle \varphi_1, \ldots, \varphi_n \rangle$ of $\mathcal{L}$ formulas such that

(1) $\varphi_1 \in \Gamma \cup \Delta$,
(2) $\varphi_n = \varphi$,

(3) for each $i \leq n$, either
   a) $\varphi_i \in \Gamma \cup \Delta$, or
   b) there exist $i_0 < i$ and $i_1 < i$ such that φ_{i_1} is equal to $(\varphi_{i_0} \to \varphi_i)$. This rule of inference is called *modus ponens*.

$\langle \varphi_1, \ldots, \varphi_n \rangle$ is called a *deduction* of φ_n from Γ.

When $\Gamma \vdash \varphi$, we say that Γ *proves* φ or that *there is a proof of* φ *from* Γ.

5.2 Soundness

Definition 5.5 Suppose that Γ is a set of $\mathcal{L}$ formulas.

(1) Γ is *consistent* if and only if for every φ, if $\Gamma \vdash \varphi$, then $\Gamma \nvdash (\neg\varphi)$.
(2) Γ is *satisfiable* if and only if there exists a structure $\mathcal{M}$ and an $\mathcal{M}$-assignment ν such that $(\mathcal{M}, \nu) \models \Gamma$.

We can now state the Gödel Completeness Theorem.

Theorem 5.6 (Gödel Completeness) *For any set of* $\mathcal{L}$ *formulas* Γ*, the following conditions are equivalent.*

(1) Γ *is consistent.*
(2) Γ *is satisfiable.*

The implication from satisfiability to consistency can be expressed heuristically—if $(\mathcal{M}, \nu)$ satisfies Γ, then $(\mathcal{M}, \nu)$ satisfies all of the deductive consequences of Γ. We check this implication in the following theorem.

Theorem 5.7 (Soundness) *Suppose that* Γ *is a set of* $\mathcal{L}$ *formulas, that* φ *is an* $\mathcal{L}$ *formula, and that* $\Gamma \vdash \varphi$*. Suppose that* $\mathcal{M}$ *is a structure and that* ν *is an* $\mathcal{M}$*-assignment such that* $(\mathcal{M}, \nu) \models \Gamma$*. Then,* $(\mathcal{M}, \nu) \models \varphi$*.*

Proof. We show by induction on n that if $\langle \varphi_1, \ldots, \varphi_n \rangle$ is a deduction from Γ, then for each i less than or equal to n, $\mathcal{M} \models \varphi_i$. We assume that the claim holds for every i less than n, and we check that it holds for n.

If $\varphi_n \in \Gamma$, then since $(\mathcal{M}, \nu) \models \Gamma$, $(\mathcal{M}, \nu) \models \varphi_n$.

If there are i and j less than n such that φ_j is equal to $(\varphi_i \to \varphi_n)$, then by induction $(\mathcal{M}, \nu) \models \varphi_i$ and $(\mathcal{M}, \nu) \models (\varphi_i \to \varphi_n)$. By the definition of satisfaction $(\mathcal{M}, \nu) \models \varphi_n$.

It remains to consider the case in which $\varphi_n \in \Delta$. For this, we must consider each of the clauses in Definition 5.3.

Clause 1. If φ_n is one of the propositional tautologies of Clause 1, then $(\mathcal{M}, \nu) \models \varphi_n$ by the definition of satisfaction for the logical connectives. (See Exercise 3 on page 37.)

Clause 2. If φ_n is obtained by Clause 2, then it has the form

$$((\forall x_i \varphi) \to \varphi(x_i; \tau)),$$

where τ is substitutable for x_i in φ. If $(\mathcal{M}, \nu) \not\models (\forall x_i \varphi)$, then trivially $(\mathcal{M}, \nu) \models \varphi_n$, so we may assume that $(\mathcal{M}, \nu) \models (\forall x_i \varphi)$. Then for every $\mathcal{M}$-assignment μ which agrees with ν on the free variables of $(\forall x_i \varphi)$, $(\mathcal{M}, \mu) \models \varphi$. In particular, if μ agrees with ν on all of the variables except for x_j and $\mu(x_j) = \overline{\nu}(\tau)$, then $(\mathcal{M}, \mu) \models \varphi$. By the Substitution Theorem 3.13, $(\mathcal{M}, \nu) \models \varphi(x_i; \tau)$, and so $(\mathcal{M}, \nu) \models \varphi_n$.

Clause 3. If φ_n is obtained by Clause 3, then it has the form

$$((\forall x_i (\psi_1 \to \psi_2)) \to ((\forall x_i \psi_1) \to (\forall x_i \psi_2))).$$

If $(\mathcal{M}, \nu) \not\models (\forall x_i (\psi_1 \to \psi_2))$ or if $(\mathcal{M}, \nu) \not\models (\forall x_i \psi_1)$, then $(\mathcal{M}, \nu) \models \varphi_n$. Otherwise, for every $\mathcal{M}$-assignment μ which agrees with ν on the free variables of $(\forall x_i (\psi_1 \to \psi_2))$, $(\mathcal{M}, \mu) \models (\psi_1 \to \psi_2)$ and $(\mathcal{M}, \mu) \models \psi_1$. Consequently, for every such μ, $(\mathcal{M}, \mu) \models \psi_2$. Since every free variable of $(\forall x_i \psi_2)$ is also free in $(\forall x_i (\psi_1 \to \psi_2))$, for every $\mathcal{M}$-assignment μ which agrees with ν on the free variables of $(\forall x_i \psi_2)$, $(\mathcal{M}, \mu) \models \psi_2$. It follows that $(\mathcal{M}, \nu) \models (\forall x_i \psi_2)$, and hence that $(\mathcal{M}, \nu) \models \varphi_n$.

Clause 4. If φ_n is obtained by Clause 4, then it has the form

$$(\psi \to (\forall x_i \psi))$$

where x_i is not free in ψ. If $(\mathcal{M}, \nu) \not\models \psi$, then $(\mathcal{M}, \nu) \models \varphi_n$. Assume that $(\mathcal{M}, \nu) \models \psi$. Then, by Theorem 3.7, for every $\mathcal{M}$-assignment μ, if ν and μ agree on the free variables of ψ, then $(\mathcal{M}, \mu) \models \psi$. Since x_i is not free in ψ, the variables which occur freely ψ also occur freely in $(\forall x_i \psi)$. Thus, if ν and μ agree on the free variables of $(\forall x_i \psi)$, then $(\mathcal{M}, \mu) \models \psi$. It follows that $(\mathcal{M}, \nu) \models (\forall x_i \psi)$.

Clause 5. If φ_n is obtained by Clause 5, then it has the form $(x_i \mathrel{\hat{=}} x_i)$, which is satisfied in every $(\mathcal{M}, \nu)$.

Clause 6. If φ_n is obtained by Clause 6, then it has the form

$$((x_i \mathrel{\hat{=}} x_j) \to (\psi_1 \to \psi_2)),$$

where ψ_1 and ψ_2 are $\mathcal{L}$ formulas such that x_j is substitutable for x_i in ψ_1 and in ψ_2 and such that $\psi_2(x_i; x_j) = \psi_1(x_i; x_j)$. If $(\mathcal{M}, \nu) \not\models (x_i \mathrel{\hat{=}} x_j)$ or $(\mathcal{M}, \nu) \not\models \psi_1$, then $(\mathcal{M}, \nu) \models \varphi_n$. Thus, we may assume that $(\mathcal{M}, \nu) \models (x_i \mathrel{\hat{=}} x_j)$ and $(\mathcal{M}, \nu) \models \psi_1$. Since $(\mathcal{M}, \nu) \models (x_i \mathrel{\hat{=}} x_j)$, $\nu(x_i) = \overline{\nu}(\langle x_j \rangle)$ and we can apply the Substitution Theorem 3.13 to the $\mathcal{L}$ formula obtained by substituting the term $\langle x_j \rangle$ for the variable x_i in ψ_1. Thus, $(\mathcal{M}, \nu) \models \psi_1(x_i; x_j)$. Since $\psi_2(x_i; x_j) = \psi_1(x_i; x_j)$, $(\mathcal{M}, \nu) \models \psi_2(x_i; x_j)$. Again noting that $\nu(x_i) = \overline{\nu}(\langle x_j \rangle)$, we may apply Theorem 3.13 and conclude from $(\mathcal{M}, \nu) \models \psi_2(x_i; x_j)$ that $(\mathcal{M}, \nu) \models \psi_2$. Consequently, $(\mathcal{M}, \nu) \models \varphi_n$.

Clause 7. If φ_n is obtained by Clause 7, then it has the form $(\forall x_i \psi)$, where $\psi \in \Delta$. By induction, for every $\mathcal{M}$-assignment μ, $(\mathcal{M}, \mu) \models \psi$. Consequently, $(\mathcal{M}, \nu) \models (\forall x_i \psi)$, as required. □

Corollary 5.8 *Suppose that* Γ *is a set of* $\mathcal{L}$ *formulas. If* Γ *is satisfiable, then* Γ *is consistent.*

5.2.1 Exercises

(1) Show that for every pair of $\mathcal{L}$ formulas φ and ψ, $\{\varphi, (\neg\varphi)\} \vdash \psi$.

5.3 Deduction and generalization theorems

Theorem 5.9 (Deduction) *Suppose that* Γ *is a set of* $\mathcal{L}$ *formulas and that* φ_1 *and* φ_2 *are* $\mathcal{L}$ *formulas. Then*

$$\Gamma \cup \{\varphi_1\} \vdash \varphi_2 \textit{ if and only if } \Gamma \vdash (\varphi_1 \rightarrow \varphi_2).$$

Proof. We first verify the implication from right to left. Suppose that $\langle \theta_1, \ldots, \theta_n \rangle$ is a deduction from Γ of $(\varphi_1 \rightarrow \varphi_2)$. In particular, θ_n is equal to $(\varphi_1 \rightarrow \varphi_2)$. Then,

$$\langle \theta_1, \ldots, \theta_{n-1}, (\varphi_1 \rightarrow \varphi_2), \varphi_1, \varphi_2 \rangle$$

is a deduction of φ_2 from Γ, as required.

For the implication from left to right, we proceed by induction on the length of deductions to show that if $\Gamma \cup \{\varphi_1\} \vdash \varphi_2$, then $\Gamma \vdash (\varphi_1 \rightarrow \varphi_2)$. So, let $\langle \theta_1, \ldots, \theta_n \rangle$ be a deduction from $\Gamma \cup \{\varphi_1\}$. Assume that j is less than or equal to n and that for each i less than j, $\Gamma \vdash (\varphi_1 \rightarrow \theta_i)$. By Definition 5.4, either θ_j is equal to φ_1, $\theta_j \in \Gamma \cup \Delta$, or there are i_1 and i_2 less than j such that θ_{i_2} is equal $(\theta_{i_1} \rightarrow \theta_j)$.

If θ_j is equal to φ_1, then we apply Clause 6 of Definition 5.3. On trivial grounds, x_1 is free for x_1 in φ_1 and $\varphi_1(x_1; x_1) = \varphi_1(x_1; x_1)$. Consequently, $((x_1 \hat{=} x_1) \rightarrow (\varphi_1 \rightarrow \varphi_1))$ is an element of Δ. By Clause 5, $(x_1 \hat{=} x_1)$ is also an element of Δ. Thus,

$$\begin{aligned}&\langle((x_1 \hat{=} x_1) \rightarrow (\varphi_1 \rightarrow \varphi_1),\\ &\ (x_1 \hat{=} x_1),\\ &\ (\varphi_1 \rightarrow \varphi_1)\rangle\end{aligned}$$

is a deduction from Γ of $(\varphi_1 \rightarrow \varphi_1)$.

In fact, $(\varphi_1 \rightarrow \varphi_1)$ is an instance of a propositional tautology, and any such can be deduced using only Clause 1 and modus ponens. We give a second deduction of $(\varphi_1 \rightarrow \varphi_1)$ of this sort.

$\langle(\varphi_1 \to ((\varphi_1 \to \varphi_1) \to \varphi_1)),$	Clause 1a
$((\varphi_1 \to ((\varphi_1 \to \varphi_1) \to \varphi_1)) \to$ $((\varphi_1 \to (\varphi_1 \to \varphi_1)) \to (\varphi_1 \to \varphi_1))),$	Clause 1b
$((\varphi_1 \to (\varphi_1 \to \varphi_1)) \to (\varphi_1 \to \varphi_1))),$	Modus ponens
$(\varphi_1 \to (\varphi_1 \to \varphi_1)),$	Clause 1a
$(\varphi_1 \to \varphi_1)\rangle$	Modus ponens

If θ_j is an element of $\Gamma \cup \Delta$, then we apply Clause 1a of Definition 5.3. Namely, $(\theta_j \to (\varphi_1 \to \theta_j))$ is an element of Δ. But then $\langle \theta_j, (\theta_j \to (\varphi_1 \to \theta_j)), (\varphi_1 \to \theta_j)\rangle$ is a deduction from Γ. The first two elements of the sequence belong to $\Gamma \cup \Delta$ and the third element of the sequence is obtained from its predecessors by an application of modus ponens.

If there are i_1 and i_2 less than j such that θ_{i_2} is equal $(\theta_{i_1} \to \theta_j)$, then we apply Clause 1b of Definition 5.3 as follows. We may assume that Γ proves $(\varphi_1 \to \theta_{i_1})$ by means of the deduction $\langle \alpha_1, \dots, \alpha_n\rangle$ and that Γ proves $(\varphi_1 \to (\theta_{i_1} \to \theta_j))$ by means of the deduction $\langle \beta_1, \dots, \beta_m\rangle$. By Clause 1b, $((\varphi_1 \to (\theta_{i_1} \to \theta_j)) \to ((\varphi_1 \to \theta_{i_1}) \to (\varphi_1 \to \theta_j)))$ is an element of Δ. Consequently,

$$\begin{aligned}
&\langle \alpha_1, \dots, \alpha_{n-1}, (\varphi_1 \to \theta_{i_1}),\\
&\quad \beta_1, \dots, \beta_{m-1}, (\varphi_1 \to (\theta_{i_1} \to \theta_j)),\\
&\quad ((\varphi_1 \to (\theta_{i_1} \to \theta_j)) \to ((\varphi_1 \to \theta_{i_1}) \to (\varphi_1 \to \theta_j))),\\
&\quad ((\varphi_1 \to \theta_{i_1}) \to (\varphi_1 \to \theta_j)), (\varphi_1 \to \theta_j)\rangle
\end{aligned}$$

is a deduction of $(\varphi_1 \to \theta_j)$ from Γ, as required. □

Theorem 5.10 (Generalization) *Suppose that Γ is a set of $\mathcal{L}$ formulas, that φ is an $\mathcal{L}$ formula, and that $\Gamma \vdash \varphi$. Suppose that x_i is a variable and that x_i is not a free variable of any formula in Γ. Then, $\Gamma \vdash (\forall x_i \varphi)$.*

Proof. Once again we go by induction on the lengths of deductions. Let $\langle \theta_1, \dots, \theta_n\rangle$ be a deduction from Γ of φ, and assume that for each j less than n, $\Gamma \vdash (\forall x_i \theta_j)$.

First, consider the case when φ is an element of Γ. We can apply Clause 4 of Definition 5.3 and conclude that $(\varphi \to (\forall x_i \varphi))$ is an element of Δ. Then, $\langle \varphi, (\varphi \to (\forall x_i \varphi)), (\forall x_i \varphi)\rangle$ is a deduction of $(\forall x_i \varphi)$ from Γ, as required.

Next, consider the case when there are i_1 and i_2 less than n such that θ_{i_2} is equal to $(\theta_{i_1} \to \varphi)$. By induction, Γ proves $(\forall x_i \theta_{i_1})$ and $(\forall x_i(\theta_{i_1} \to \varphi))$. By Clause 3 of Definition 5.3,

$$((\forall x_i(\theta_{i_1} \to \varphi)) \to ((\forall x_i \theta_{i_1}) \to (\forall x_i \varphi)))$$

is an element of Δ. By concatenating deductions as in the proof of the previous theorem, it follows that $\Gamma \vdash (\forall x_i \varphi)$, as required. □

The next theorem requires two lemmas.

Lemma 5.11 *Suppose that φ is an $\mathcal{L}$ formula, x_i is free for x_j in φ, and x_i does not occur freely in $(\forall x_j \varphi)$. Then*

$$\emptyset \vdash ((\forall x_j \varphi) \to (\forall x_i \varphi(x_j; x_i)))$$

Proof. Since x_i is free for x_j in φ, we may apply Clause 2 of Definition 5.3 to conclude that $((\forall x_j \varphi) \to \varphi(x_j; x_i))$ is an element of Δ. By the Deduction Theorem 5.9, $\{(\forall x_j \varphi)\} \vdash \varphi(x_j; x_i)$. Since x_i does not occur freely in $(\forall x_j \varphi)$, we can apply the Generalization Theorem 5.10 to conclude that $\{(\forall x_j \varphi)\} \vdash (\forall x_i \varphi(x_j; x_i))$. By the Deduction Theorem again, $\emptyset \vdash ((\forall x_j \varphi) \to (\forall x_i \varphi(x_j; x_i)))$, as required. □

Lemma 5.12 *Suppose that Γ is a set of $\mathcal{L}$ formulas and that the constant symbol c_i does not occur in any formula of Γ. Suppose that $\langle \theta_1, \dots, \theta_m \rangle$ is a deduction from Γ and that the variable x_j does not occur in any of the formulas θ_n for which $n \leq m$. Then $\langle \theta_1(c_i; x_j), \dots, \theta_m(c_i; x_j) \rangle$ is a deduction from Γ.*

Proof. Note, if c_i does not occur in φ, then $\varphi(c_i; x_j) = \varphi$. By assumption c_i does not occur in any formula of Γ, so for each $\varphi \in \Gamma$, $\varphi(c_i; x_j) = \varphi$.

It can be verified by inspection of Definition 5.3 that if φ is a logical axiom and x_j does not occur in φ, then $\varphi(c_i; x_j)$ is a logical axiom.

Finally, if φ_1 and φ_2 are $\mathcal{L}$ formulas then

$$(\varphi_1 \to \varphi_2)(c_i; x_j) = (\varphi_1(c_i; x_j) \to \varphi_2(c_i; x_j)).$$

It follows by induction on $n \leq m$, that $\langle \theta_1(c_i; x_j), \dots, \theta_n(c_i; x_j) \rangle$ is a proof from Γ. □

Theorem 5.13 (Constants) *Suppose that Γ is a set of $\mathcal{L}$ formulas, that φ is an $\mathcal{L}$ formula, and that $\Gamma \vdash \varphi$. Suppose that c_i is a constant and that c_i does not occur in any formula of Γ. Let x_j be a variable which is substitutable for c_i in φ and which does not occur freely in φ. Then the following conditions hold.*

(1) $\Gamma \vdash (\forall x_j \varphi(c_i; x_j))$.

(2) There is a deduction $\langle \varphi_1, \dots, \varphi_n \rangle$ of $(\forall x_j \varphi(c_i; x_j))$ from Γ such that
  - *a) for each $m \leq n$, c_i does not occur in φ_m,*
  - *b) and for each $m < n$ and each c_k, if c_k occurs in φ_m, then either c_k occurs in $(\forall x_j \varphi(c_i; x_j))$ or c_k occurs in some formula of Γ.*

Proof. Let $\langle \theta_1, \dots, \theta_n \rangle$ be a deduction of φ from Γ. Let x_{j_1} be a variable such that for each m less than or equal to n, x_{j_1} does not appear in θ_m. Let Γ_0 be $\{\theta_m : m \leq n\} \cap \Gamma$. By Lemma 5.12, $\langle \theta_1(c_i; x_{j_1}), \dots, \theta_n(c_i; x_{j_1}) \rangle$ is a deduction of $\varphi(c_i; x_{j_1})$ from Γ_0. By the Generalization Theorem, $\Gamma_0 \vdash (\forall x_{j_1} \varphi(c_i; x_{j_1}))$. Since

x_j is substitutable for c_i in φ and does not occur freely in φ, x_j is substitutable for x_{j_1} and does not occur freely in $(\forall x_{j_1}\varphi(c_i;x_{j_1}))$. By Lemma 5.11,

$$\emptyset \vdash ((\forall x_{j_1}\varphi(c_i;x_{j_1})) \to (\forall x_j\varphi(c_i;x_{j_1})(x_{j_1};x_j))).$$

Of course, $\varphi(c_i;x_{j_1})(x_{j_1};x_j)$ is equal to $\varphi(c_i;x_j)$ and so

$$\emptyset \vdash ((\forall x_{j_1}\varphi(c_i;x_{j_1})) \to (\forall x_j\varphi(c_i;x_j))).$$

Thus,

$$\Gamma_0 \vdash (\forall x_{j_1}\varphi(c_i;x_{j_1})) \text{ and}$$
$$\Gamma_0 \vdash ((\forall x_{j_1}\varphi(c_i;x_{j_1})) \to (\forall x_j\varphi(c_i;x_j)))$$

and so $\Gamma_0 \vdash (\forall x_j\varphi(c_i;x_j))$.

Now, we verify the second claim. Let $\langle\theta_1,\dots,\theta_n\rangle$ be a deduction of $(\forall x_j\varphi(c_i;x_j))$ from Γ. By an application of Lemma 5.12, if m is less than n, c_k occurs in θ_m, and c_k does not occur in φ or in any element of Γ_0, then taking x_{k_1} so that x_{k_1} does not appear in any of the θ_i's, $\langle\theta_1(c_m;x_{k_1}),\dots,\theta_n(c_m;x_{k_1})\rangle$ is a deduction of φ from Γ_0. By sequential application of this observation, we may assume that for each $m<n$ and each c_k, if c_k occurs in φ_m, then either c_k occurs in φ or c_k occurs in some formula of Γ. □

5.3.1 Exercises

(1) Suppose that $\Gamma \cup \{(\neg\varphi)\}$ is not consistent. Show that $\Gamma \vdash \varphi$. (This is a technical formulation of the legitimacy of proofs by contradiction.)

(2) Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and ν is an $\mathcal{M}$-assignment. Show that $\{\varphi : (\mathcal{M},\nu) \models \varphi\}$ is maximally consistent.

5.4 The Henkin property

Definition 5.14 We use the notation $(\exists x_i\varphi)$ to represent the $\mathcal{L}$ formula $(\neg(\forall x_i(\neg\varphi)))$.

By inspection of Definition 3.6, $(\mathcal{M},\nu) \models (\exists x_i\varphi)$ if and only if there is an $\mathcal{M}$-assignment μ which agrees with ν on the free variables of $(\exists x_i\varphi)$ such that $(\mathcal{M},\mu) \models \varphi$.

To complete the proof the Gödel Completeness Theorem, we must develop the machinery to show that if Γ is consistent then Γ is satisfiable.

Definition 5.15 Suppose that Γ is a consistent set of $\mathcal{L}$ formulas. Γ is *maximally consistent* if and only if for any $\mathcal{L}$ formula φ, either $\varphi \in \Gamma$ or $\Gamma \cup \{\varphi\}$ is not consistent.

Lemma 5.16 *Suppose that Γ is a maximally consistent set of $\mathcal{L}$ formulas. Then for each $\mathcal{L}$ formula φ, either $\varphi \in \Gamma$ or $(\neg\varphi) \in \Gamma$.*

Proof. Suppose that $(\neg\varphi)$ does not belong to Γ. By the maximality of Γ, $\Gamma \cup \{(\neg\varphi)\}$ is inconsistent. By the exercise at the end of the previous section, for any $\mathcal{L}$ formula θ, $\Gamma \cup \{(\neg\varphi)\} \vdash \theta$. Consequently, letting θ be $(\neg(x_1 \doteq x_1))$, $\Gamma \cup \{(\neg\varphi)\} \vdash (\neg(x_1 \doteq x_1))$. By the Deduction Theorem 5.9, $\Gamma \vdash ((\neg\varphi) \to (\neg(x_1 \doteq x_1)))$. Applying Clause 1 in the definition of Δ, $\Gamma \vdash (((\neg\varphi) \to (\neg(x_1 \doteq x_1))) \to ((x_1 \doteq x_1) \to \varphi))$. Two applications of modus ponens yield $\Gamma \vdash \varphi$. Now, since $\Gamma \vdash \varphi$, any deduction from $\Gamma \cup \{\varphi\}$ can be converted into a deduction from Γ by replacing each instance of φ with a deduction of φ from Γ. Thus, for each θ, if $\Gamma \cup \{\varphi\} \vdash \theta$ then $\Gamma \vdash \theta$. Since Γ is consistent, $\Gamma \cup \{\varphi\}$ is also consistent. Since no proper superset of Γ is consistent, $\varphi \in \Gamma$, as required. $\square$

Definition 5.17 A set of $\mathcal{L}$ formulas Γ has the *Henkin Property* if and only if for each $\mathcal{L}$ formula φ and for each variable x_i, if $(\exists x_i \varphi) \in \Gamma$ then there exists a constant c_j such that $\varphi(x_i; c_j) \in \Gamma$.

The following application of Tarski's Theorem motivates the definition of the Henkin property.

Theorem 5.18 *Suppose that $\mathcal{M} = (M, I)$ is a structure and ν is an $\mathcal{M}$-assignment such that*

$$\{\nu(x_i) : i \in \mathbb{N}\} \subseteq \{I(c_i) : i \in \mathbb{N}\}.$$

Let $\Gamma = \{\varphi : (\mathcal{M}, \nu) \models \varphi\}$. Then the following conditions hold.

(1) Γ is maximally consistent.

(2) Γ has the Henkin property if and only if there exists an elementary substructure $(M_0, I_0) \preceq \mathcal{M}$ such that

$$M_0 = \{I(c_i) : i \in \mathbb{N}\}.$$

Proof. We begin with the first claim. For the consistency of Γ, note that every formula in Γ is satisfied by $(\mathcal{M}, \nu)$. By the Soundness Theorem 5.7, if $\Gamma \vdash \varphi$, then $(\mathcal{M}, \nu) \models \varphi$. By the definition of satisfaction, if $(\mathcal{M}, \nu)$ satisfies φ, then $(\mathcal{M}, \nu)$ does not satisfy $(\neg\varphi)$. Consequently, if $\Gamma \vdash \varphi$, then $\Gamma \nvdash (\neg\varphi)$, and so Γ is consistent.

For the maximality of Γ, suppose that $\varphi \notin \Gamma$. Then $(\mathcal{M}, \nu) \not\models \varphi$, hence $(\mathcal{M}, \nu) \models (\neg\varphi)$ and so $(\neg\varphi) \in \Gamma$. Thus, φ and $(\neg\varphi)$ can be deduced from $\Gamma \cup \{\varphi\}$, showing that it is not consistent. Since Γ is consistent and no proper extension of Γ is consistent, Γ is maximally consistent.

Now, we consider the second claim.

For the implication from left to right, suppose that Γ has the Henkin property. We will show that M_0 satisfies Tarski's Criterion.

Let $m_1, \ldots, m_n$ be elements of M_0 and suppose that $A \subseteq M$ is definable in $\mathcal{M}$ from these elements as follows.

$$a \in A \leftrightarrow \mathcal{M} \models \varphi[a, m_1, \ldots, m_n]$$

We must show that $A \cap M_0$ is not empty.

Since each element of M_0 is in the range of I applied to the set of constant symbols, we fix $c_{i_1}, \dots, c_{i_n}$ so that for each $j \leq n$, $I(c_{i_j}) = m_j$. By the Substitution Theorem,

$$a \in A \;\leftrightarrow\; \mathcal{M} \models \varphi(x_1, \dots, x_n; c_{i_1}, \dots, c_{i_n})[a].$$

Since A is not empty,

$$(\mathcal{M}, \nu) \models (\exists x_0 \varphi(x_1, \dots, x_n; c_{i_1}, \dots, c_{i_n})).$$

Then, $(\exists x_0 \varphi(x_1, \dots, x_n; c_{i_1}, \dots, c_{i_n}))$ is an element of Γ, and by the Henkin property, there is a c_{i_0} such that

$$\varphi(x_1, \dots, x_n; c_{i_1}, \dots, c_{i_n})(x_0; c_{i_0}) \in \Gamma.$$

Note that

$$\varphi(x_1, \dots, x_n; c_{i_1}, \dots, c_{i_n})(x_0; c_{i_0}) = \varphi(x_0, \dots, x_n; c_{i_0}, \dots, c_{i_n}).$$

Consequently,

$$\mathcal{M} \models \varphi(x_0, \dots, x_n; c_{i_0}, c_{i_1}, \dots, c_{i_n}),$$

and so

$$\mathcal{M} \models \varphi[I(c_{i_0}), I(c_{i_1}), \dots, I(c_{i_n})].$$

By the above, each $I(c_{i_j})$ is equal to m_j, so

$$\mathcal{M} \models \varphi[I(c_{i_0}), m_1, \dots, m_n].$$

$I(c_{i_0})$ is the desired element of $M_0 \cap A$.

For the implication from right to left, suppose that $\mathcal{M}_0$ is an elementary substructure of $\mathcal{M}$. To check that Γ has the Henkin property, suppose that $(\exists x_i \varphi) \in \Gamma$ and that the variables of φ are included in $x_1, \dots, x_n$. Let $c_{k_1}, \dots, c_{k_{i-1}}, c_{k_{i+1}}, \dots, c_{k_n}$ be constant symbols of $\mathcal{L}_{\mathcal{A}}$ such that for each j less then or equal to n other than i, $\nu(x_j) = I(c_{k_j})$. Since $(\exists x_i \varphi) \in \Gamma$, by the definition of Γ,

$$(\mathcal{M}, \nu) \models (\exists x_i \varphi),$$

the Substitution Theorem 3.13 implies that

$$\mathcal{M} \models (\exists x_i \varphi(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n; c_{k_1}, \dots, c_{k_{i-1}}, c_{k_{i+1}}, \dots, c_{k_n})).$$

Now, $\mathcal{M}_0 \preceq \mathcal{M}$, so

$$\mathcal{M}_0 \models (\exists x_i \varphi(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n; c_{k_1}, \dots, c_{k_{i-1}}, c_{k_{i+1}}, \dots, c_{k_n})).$$

Then, there is an $\mathcal{M}_0$ assignment μ such that

$$(\mathcal{M}_0, \mu) \models \varphi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n; c_{k_1}, \ldots, c_{k_{i-1}}, c_{k_{i+1}}, \ldots, c_{k_n}).$$

By the fact that every element of M_0 is the denotation of a constant symbol, there is a c_{k_i} such that $\mu(x_i) = I_0(c_{k_i})$. But then, by applying the Substitution Theorem 3.13 again,

$$\mathcal{M}_0 \models \varphi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n; c_{k_1}, \ldots, c_{k_{i-1}}, c_{k_{i+1}}, \ldots, c_{k_n})(x_i; c_{k_i}).$$

The order of substitution does not matter, and so

$$\mathcal{M}_0 \models \varphi(x_i; c_{k_i})(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n; c_{k_1}, \ldots, c_{k_{i-1}}, c_{k_{i+1}}, \ldots, c_{k_n}).$$

But then, by application of Theorem 3.13, $(\mathcal{M}, \nu) \models \varphi(x_i; c_{k_i})$ and so $\varphi(x_i; c_{k_i}) \in \Gamma$, as required. $\square$

5.4.1 Exercises

(1) Suppose that Γ is a maximally consistent set of $\mathcal{L}$ formulas. Define the relation $\sim_\Gamma$ between terms by $\tau_1 \sim_\Gamma \tau_2$ if and only if $(\tau_1 \mathrel{\hat{=}} \tau_2) \in \Gamma$. Show that $\sim_\Gamma$ is an equivalence relation. That is to say that it is reflexive, symmetric, and transitive.

Lemma 5.19 *Suppose that φ is an $\mathcal{L}$ formula with no quantifiers and that $\tau_1, \ldots, \tau_n$ and $\sigma_1, \ldots, \sigma_n$ are terms. Then, for any sequence of variables $x_{m_1}, \ldots, x_{m_n}$,*

$$\{(\tau_i \mathrel{\hat{=}} \sigma_i) : i \leq n\} \cup \{\varphi(x_{m_1}, \ldots, x_{m_n}; \tau_1, \ldots, \tau_n)\} \vdash \\ \varphi(x_{m_1}, \ldots, x_{m_n}; \sigma_1, \ldots, \sigma_n).$$

Proof. We prove Lemma 5.19 by induction on n. Suppose that it holds for all $\mathcal{L}$ formulas with no quantifiers and for all m less than n. Since we are substituting terms for all of the x_{m_j}'s, we may assume that none of these variables occur in any of the τ_i's or σ_i's. By induction

$$\{(\tau_i \mathrel{\hat{=}} \sigma_i) : i \leq n\} \cup \{\varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \tau_1, \ldots, \tau_{n-1})\} \vdash \\ \varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \sigma_1, \ldots, \sigma_{n-1})$$

By the deduction theorem,

$$\{(\tau_i \mathrel{\hat{=}} \sigma_i) : i \leq n\} \vdash \\ \begin{pmatrix} \varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \tau_1, \ldots, \tau_{n-1}) \rightarrow \\ \varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \sigma_1, \ldots, \sigma_{n-1}) \end{pmatrix}$$

Since x_{m_n} does not appear in any of the hypotheses,

$$\{(\tau_i \hat{=} \sigma_i) : i \leq n\} \vdash \left(\forall x_{m_n} \begin{pmatrix} \varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \tau_1, \ldots, \tau_{n-1}) \rightarrow \\ \varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \sigma_1, \ldots, \sigma_{n-1}) \end{pmatrix}\right)$$

Then Clause 2 applies and so

$$\{(\tau_i \hat{=} \sigma_i) : i \leq n\} \vdash \begin{pmatrix} \varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \tau_1, \ldots, \tau_{n-1})(x_{m_n}; \sigma_n) \rightarrow \\ \varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \sigma_1, \ldots, \sigma_{n-1})(x_{m_n}; \sigma_n) \end{pmatrix}.$$

Using the fact that the variables x_{m_j} do not appear in any of the τ_i's or σ_i's, we can rewrite this formula as

$$\{(\tau_i \hat{=} \sigma_i) : i \leq n\} \vdash \begin{pmatrix} \varphi(x_{m_1}, \ldots, x_{m_n}; \tau_1, \ldots, \tau_{n-1}, \sigma_n) \rightarrow \\ \varphi(x_{m_1}, \ldots, x_{m_n}; \sigma_1, \ldots, \sigma_n) \end{pmatrix}$$

Let x_k be a variable which does not appear in any of the τ_i's or σ_i's and does not appear in φ. By Clause 6 in Definition 5.3,

$$\emptyset \vdash \left((x_{m_n} \hat{=} x_k) \rightarrow \begin{pmatrix} \varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \tau_1, \ldots, \tau_{n-1}) \rightarrow \\ \varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \tau_1, \ldots, \tau_{n-1})(x_{m_n}; x_k) \end{pmatrix}\right).$$

By Clauses 7 and then 2,

$$\emptyset \vdash \left((x_{m_n} \hat{=} x_k) \rightarrow \begin{pmatrix} \varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \tau_1, \ldots, \tau_{n-1}) \rightarrow \\ \varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \tau_1, \ldots, \tau_{n-1})(x_{m_n}; x_k) \end{pmatrix}\right)(x_{m_n}, x_k; \tau_n, \sigma_n)$$

Making the substitutions,

$$\emptyset \vdash \left((\tau_n \hat{=} \sigma_n) \rightarrow \begin{pmatrix} \varphi(x_{m_1}, \ldots, x_{m_n}; \tau_1, \ldots, \tau_n) \rightarrow \\ \varphi(x_{m_1}, \ldots, x_{m_n}; \tau_1, \ldots, \tau_{n-1}, \sigma_n) \end{pmatrix}\right)$$

By the deduction theorem,

$$\{(\tau_n \hat{=} \sigma_n)\} \vdash \begin{pmatrix} \varphi(x_{m_1}, \ldots, x_{m_n}; \tau_1, \ldots, \tau_n) \rightarrow \\ \varphi(x_{m_1}, \ldots, x_{m_n}; \tau_1, \ldots, \tau_{n-1}, \sigma_n) \end{pmatrix}.$$

Combining the previous two paragraphs,

$$\{(\tau_n \hat{=} \sigma_n)\} \cup \{\varphi(x_{m_1}, \ldots, x_{m_n}; \tau_1, \ldots, \tau_n)\} \vdash \varphi(x_{m_1}, \ldots, x_{m_n}; \sigma_1, \ldots, \sigma_n)$$

as required. $\square$

We now prove a special case of the Gödel Completeness Theorem.

Theorem 5.20 *Suppose that* Γ *is a maximally consistent set of* $\mathcal{L}$ *formulas with the Henkin property. Then* Γ *is satisfiable.*

Proof. Our proof is naturally divided into parts. First we must define a model $\mathcal{M}$ and an assignment ν for that model. Then we must verify that $(\mathcal{M}, \nu)$ satisfies Γ.

Defining $\mathcal{M}$ and ν. We say that two constants c_i and c_j are *equivalent mod* Γ, written $c_i \sim_\Gamma c_j$, if and only if $(c_i \mathrel{\hat{=}} c_j) \in \Gamma$. As was stated in the exercise at the end of the previous section, $\sim_\Gamma$ is an equivalence relation on the set of all constant symbols. That is to say that for all i, j, and k,

(1.1) $c_i \sim_\Gamma c_i$,
(1.2) if $c_i \sim_\Gamma c_j$ then $c_j \sim_\Gamma c_i$,
(1.3) and if $c_i \sim_\Gamma c_j$ and $c_j \sim_\Gamma c_k$ then $c_i \sim_\Gamma c_k$.

For each $i \in \mathbb{N}$, let

$$[c_i]_\Gamma = \{c_j : j \in \mathbb{N} \text{ and } c_i \sim_\Gamma c_j\}.$$

$[c_i]_\Gamma$ is the equivalence class of c_i under the equivalence relation $\sim_\Gamma$. This set of equivalence classes is the universe of our model. Let

$$M = \{[c_i]_\Gamma : i \in \mathbb{N}\}.$$

We now define our $\mathcal{M}$-assignment ν. For each variable x_i, chose j so that $(x_i \mathrel{\hat{=}} c_j) \in \Gamma$ and let $\nu(x_i) = [c_j]_\Gamma$.

To see that ν is well defined, we must show that for each x_i there is at least one c_j such that $(x_i \mathrel{\hat{=}} c_j) \in \Gamma$. Further, we must show that for any two constant symbols c_{j_1} and c_{j_2}, if $(x_i \mathrel{\hat{=}} c_{j_1}) \in \Gamma$ and $(x_i \mathrel{\hat{=}} c_{j_2}) \in \Gamma$ then $c_{j_1} \sim_\Gamma c_{j_2}$.

For the first of these claims, consider the formula $(\exists x_{i+1}(x_i \mathrel{\hat{=}} x_{i+1}))$. If it is not an element of Γ, then by Lemma 5.16 $(\forall x_{i+1}(\neg(x_i \mathrel{\hat{=}} x_{i+1})))$ is an element of Γ. But then x_i is substitutable for x_{i+1} in $(\neg(x_i \mathrel{\hat{=}} x_{i+1}))$, and so $\Gamma \vdash (\neg(x_i \mathrel{\hat{=}} x_i))$. But $(x_i \mathrel{\hat{=}} x_i) \in \Delta$ and so $\Gamma \vdash (x_i \mathrel{\hat{=}} x_i)$. Thus, Γ is not consistent, contrary to assumption. Thus, $(\exists x_{i+1}(x_i \mathrel{\hat{=}} x_{i+1})) \in \Gamma$. Since Γ has the Henkin property, there exists a constant c_j such that $(x_i \mathrel{\hat{=}} c_j) \in \Gamma$. Thus, for each x_i, there is a c_j as required by the definition of ν.

The second claim follows from Lemma 5.19. Thus, ν is well defined.

We next define the interpretation map I.

(2.1) Suppose c_i is a constant. Then

$$I(c_i) = [c_i]_\Gamma.$$

(2.2) Suppose that P_i is a predicate symbol and that $n = \pi(P_i)$. Then

$$I(P_i) = \{\langle [c_{k_1}]_\Gamma, \ldots, [c_{k_n}]_\Gamma \rangle \in M^n : P_i(c_{k_1} \ldots c_{k_n}) \in \Gamma\}.$$

(2.3) Suppose that F_i is a function symbol and that $n = \pi(F_i)$. Then

$$I(F_i)([c_{k_1}]_\Gamma, \ldots, [c_{k_n}]_\Gamma) = [c_{k_{n+1}}]_\Gamma$$

if and only if

$$(F_i(c_{k_1}, \ldots, c_{k_n}) \mathrel{\hat{=}} c_{k_{n+1}}) \in \Gamma.$$

The proofs that $I(P_i)$ and $I(F_i)$ are well defined are analogous to the proof that ν is well defined.

Claim 5.21 *For any term τ, $\overline{\nu}(\tau) = [c_i]_\Gamma$ if and only if $(\tau \mathrel{\hat{=}} c_i) \in \Gamma$.*

Proof. We prove Claim 5.21 by induction on the length of τ. If τ has length 1 then for some $k \in \mathbb{N}$, $\tau = \langle x_k \rangle$ or $\tau = \langle c_k \rangle$. If $\tau = \langle x_k \rangle$ then $\overline{\nu}(\tau) = \nu(x_k)$ and the claim follows by the fact that ν is well defined. If $\tau = \langle c_k \rangle$ then $\overline{\nu}(\tau) = I(c_k) = [c_k]_\Gamma$ and the claim follows from the definition of $\sim_\Gamma$.

Now suppose that τ has length $n > 1$ and that:

Induction Hypothesis: *If σ is a term of length less than n, then for all constants c_k, $\overline{\nu}(\sigma) = [c_k]_\Gamma$ if and only if $(\sigma \mathrel{\hat{=}} c_k) \in \Gamma$.*

Since τ has length > 1, there are terms $\tau_1, \ldots, \tau_m$ and a function symbol F_i such that $\tau = F_i(\tau_1, \ldots, \tau_m)$, where $m = \pi(F_i)$. By the definition of $\overline{\nu}$,

$$\overline{\nu}(\tau) = I(F_i)(\overline{\nu}(\tau_1), \ldots, \overline{\nu}(\tau_m)).$$

Let $c_{j_1}, \ldots, c_{j_m}$ be constants such that for each $k \leq m$, $\overline{\nu}(\tau_k) = [c_{j_k}]_\Gamma$. Thus,

$$\overline{\nu}(\tau) = I(F_i)([c_{j_1}]_\Gamma, \ldots, [c_{j_m}]_\Gamma).$$

By the definition of $I(F_i)$, for each constant symbol c_s,

$$I(F_i)([c_{j_1}]_\Gamma, \ldots, [c_{j_m}]_\Gamma) = [c_s]_\Gamma$$

if and only if

$$(F_i(c_{j_1}, \ldots, c_{j_m}) \mathrel{\hat{=}} c_s) \in \Gamma.$$

Consequently, $\overline{\nu}(\tau) = [c_s]_\Gamma$ if and only if $(F_i(c_{j_1}, \ldots, c_{j_m}) \mathrel{\hat{=}} c_s) \in \Gamma$.

By the induction hypothesis, for each k less than or equal to m, $(\tau_k \mathrel{\hat{=}} c_{j_k})$ is an element of Γ. Therefore, by Lemma 5.19,

$$(F_i(\tau_1, \ldots, \tau_n) \mathrel{\hat{=}} F_i(c_{j_1}, \ldots, c_{j_m})) \in \Gamma.$$

We can conclude that

$$(F_i(c_{j_1}, \ldots, c_{j_m}) \mathrel{\hat{=}} c_s) \in \Gamma \text{ if and only if } (\tau \mathrel{\hat{=}} c_s) \in \Gamma.$$

Consequently, $\overline{\nu}(\tau) = [c_s]_\Gamma$ if and only if $(\tau \mathrel{\hat{=}} c_s) \in \Gamma$. This completes the inductive step and so proves the claim. $\square$

We now prove that for each formula φ, $\varphi \in \Gamma$ if and only if $(\mathcal{M}, \nu) \models \varphi$.

We first reduce to the case in which φ is a sentence. Suppose that φ is a formula, x_i is a variable and that $\nu(x_i) = [c_j]_\Gamma$. Suppose that x_k is a variable not occurring in φ. Thus, $((x_i \mathrel{\hat{=}} x_k) \to (\varphi \to \varphi(x_i; x_k)))$ is a logical axiom, and by application of Clause 7 and then 2,

$$\emptyset \vdash ((x_i \mathrel{\hat{=}} c_j) \to (\varphi \to \varphi(x_i; c_j))),$$

since $\varphi(x_i; x_k)(x_k; c_j) = \varphi(x_i; c_j)$. It follows that if $\nu(x_i) = [c_j]_\Gamma$, then $\Gamma \vdash (\varphi \to \varphi(x_i; c_j))$. In a similar way, if $\nu(x_i) = [c_j]_\Gamma$, then $\Gamma \vdash (\varphi(x_i; c_j) \to \varphi)$: use the above argument for $(\neg\varphi)$ and then apply Clause 1. Consequently, $\Gamma \vdash (\varphi \leftrightarrow \varphi(x_i; c_j))$ and so $(\varphi \leftrightarrow \varphi(x_i; c_j)) \in \Gamma$.

Let n be large enough so that all the free variables of φ belong to $\{x_1, \ldots, x_n\}$. For each $k \leq n$ let m_k be such that $\nu(x_k) = c_{m_k}$. By the above analysis,

$$(\varphi \leftrightarrow \varphi(x_1, \ldots, x_n; c_{m_1}, \ldots, c_{m_n})) \in \Gamma.$$

Notice that $\varphi(x_1, \ldots, x_n; c_{m_1}, \ldots, c_{m_n})$ is a sentence. Thus, for each formula φ there exists a sentence φ^* such that $(\varphi \leftrightarrow \varphi^*) \in \Gamma$.

Claim 5.22 *For every sentence φ, $\varphi \in \Gamma$ if and only if $\mathcal{M} \models \varphi$.*

Proof. We proceed by induction on the length of φ.

We first suppose that φ is a sentence and that φ is an atomic formula. There are two subcases.

First, φ could be of the form $(\tau_1 \mathrel{\hat{=}} \tau_2)$, where τ_1 and τ_2 are terms. Let c_{i_1} be a constant such that $\overline{\nu}(\tau_1) = [c_{i_1}]_\Gamma$ and let c_{i_2} be a constant such that $\overline{\nu}(\tau_2) = [c_{i_2}]_\Gamma$. By Claim 5.21, $(\tau_1 \mathrel{\hat{=}} c_{i_1})$ and $(\tau_2 \mathrel{\hat{=}} c_{i_2})$ are elements of Γ. Lemma 5.19 applies, and so $(\tau_1 \mathrel{\hat{=}} \tau_2) \in \Gamma$ if and only if $(c_{i_1} \mathrel{\hat{=}} c_{i_2})$ is an element of Γ. Further $(c_{i_1} \mathrel{\hat{=}} c_{i_2})$ is an element of Γ if and only if $[c_{i_1}]_\Gamma = [c_{i_2}]_\Gamma$. Consequently, $(\tau_1 \mathrel{\hat{=}} \tau_2) \in \Gamma$ if and only if $\mathcal{M} \models (\tau_1 \mathrel{\hat{=}} \tau_2)$, as required.

The second subcase is that $\varphi = P_i(\tau_1 \ldots \tau_n)$, where $\tau_1, \ldots, \tau_n$ are terms and $n = \pi(P_i)$. By definition,

$$(\mathcal{M}, \nu) \models \varphi \text{ if and only if } \langle \overline{\nu}(\tau_1), \ldots, \overline{\nu}(\tau_n) \rangle \in I(P_i).$$

For each $k \leq n$, let c_{j_k} be a constant such that $\overline{\nu}(\tau_k) = [c_{j_k}]_\Gamma$. Thus,

$$\langle \overline{\nu}(\tau_1), \ldots, \overline{\nu}(\tau_n) \rangle \in I(P_i) \text{ if and only if } \langle [c_{j_1}]_\Gamma, \ldots, [c_{j_n}]_\Gamma \rangle \in I(P_i).$$

By the definition of $I(P_i)$,

$$\langle [c_{j_1}]_\Gamma, \ldots, [c_{j_n}]_\Gamma \rangle \in I(P_i) \text{ if and only if } P_i(c_{j_1}, \ldots, c_{j_n}) \in \Gamma.$$

By Claim 5.21, for each $k \leq n$, $(\tau_k \mathrel{\hat{=}} c_{j_k}) \in \Gamma$ and so we can apply Lemma 5.19 to conclude that

$$P_i(c_{j_1} \ldots c_{j_n}) \in \Gamma \text{ if and only if } P_i(\tau_1 \ldots \tau_n) \in \Gamma.$$

Thus, $\mathcal{M} \models \varphi$ if and only if $\varphi \in \Gamma$. This finishes the case in which φ is an atomic formula.

We now suppose that the length of φ is N, φ is not an atomic formula and that the following condition holds.

Induction Hypothesis: *Suppose that ψ is a sentence of length less than N. Then $\mathcal{M} \vDash \psi$ if and only if $\psi \in \Gamma$.*

There are three subcases.

Negation. Suppose that $\varphi = (\neg\psi)$. Since φ is a sentence, so is ψ. Consequently, the induction hypothesis applies, and so $\mathcal{M} \vDash \psi$ if and only if $\psi \in \Gamma$. Therefore $\mathcal{M} \vDash (\neg\psi)$ if and only if $\mathcal{M} \nvDash \psi$, if and only if $\psi \notin \Gamma$, if and only if $(\neg\psi) \in \Gamma$ (as Γ is maximally consistent).

Implication. Suppose that $\varphi = (\psi_1 \to \psi_2)$. Again, the induction hypothesis applies and we obtain both: $\mathcal{M} \vDash \psi_1$ if and only if $\psi_1 \in \Gamma$ and $\mathcal{M} \vDash \psi_2$ if and only if $\psi_2 \in \Gamma$. By definition, $\mathcal{M} \vDash \varphi$ if and only if either $\mathcal{M} \nvDash \psi_1$ or $\mathcal{M} \vDash \psi_2$. Since Γ is maximally consistent, $\varphi \in \Gamma$ if and only if either $(\neg\psi_1) \in \Gamma$ or $\psi_2 \in \Gamma$. Thus, $\mathcal{M} \vDash \varphi$ if and only if $\varphi \in \Gamma$.

Quantification. Suppose that $\varphi = (\forall x_i \psi)$. By the Henkin property, $(\exists x_i(\neg\psi)) \in \Gamma$ if and only if for some constant c_j, $(\neg\psi)(x_i; c_j) \in \Gamma$. By a straightforward deduction, $(\forall x_i \psi) \in \Gamma$ if and only if $(\exists x_i(\neg\psi)) \notin \Gamma$, if and only if for every constant c_j, $(\neg\psi)(x_i; c_j) \notin \Gamma$. But $(\neg\psi)(x_i; c_j)$ is equal to $(\neg\psi(x_i; c_j))$, and so $(\forall x_i \psi) \in \Gamma$ if and only if for every constant c_j, $\psi(x_i; c_j) \in \Gamma$.

By definition, $\mathcal{M} \vDash \varphi$ if and only if for all $\mathcal{M}$-assignments μ, $(\mathcal{M}, \mu) \vDash \psi$. (Since φ is a sentence, every $\mathcal{M}$-assignment agrees with every other one on the free variables of φ.)

Suppose that μ is an $\mathcal{M}$-assignment. Let c_j be a constant such that $\mu(x_i) = [c_j]_\Gamma$. Then, since x_i is the only free variable of ψ and since $I(c_j) = \mu(x_i)$, it follows that $(\mathcal{M}, \mu) \vDash \psi$ if and only if $\mathcal{M} \vDash \psi(x_i; c_j)$. Thus, the condition that

for all $\mathcal{M}$-assignments μ, $(\mathcal{M}, \mu) \vDash \psi$

is equivalent to the one that

for all constants c_j, $\mathcal{M} \vDash \psi(x_i; c_j)$.

By the induction hypothesis, for each constant c_j, $\mathcal{M} \vDash \psi(x_i; c_j)$ if and only if $\psi(x_i; c_j) \in \Gamma$.

Thus, $\varphi \in \Gamma$ if and only if for each constant c_j, $\psi(x_i; c_j) \in \Gamma$, if and only if for each constant c_j, $\mathcal{M} \vDash \psi(x_i; c_j)$, if and only if $\mathcal{M} \vDash \varphi$.

This completes the final case, and so we have proved that for each sentence φ, $\mathcal{M} \vDash \varphi$ if and only if $\varphi \in \Gamma$. □

By Claim 5.22, Γ is satisfiable. □

5.5 Extensions of consistent sets of $\mathcal{L}$ formulas

Unfortunately there are consistent sets, Γ, of $\mathcal{L}$ formulas which cannot be extended to maximally consistent sets with the Henkin property. The difficulty is the Henkin property. However with certain restrictions on the sets Γ one can in fact extend Γ to a maximally consistent set with the Henkin property.

Theorem 5.23 *Suppose that Γ is a consistent set of $\mathcal{L}$ formulas and that there are infinitely many constants, c_i, which do not occur in any formula of Γ. Then there is a set of formulas Σ such that*

(1) $\Gamma \subseteq \Sigma$,
(2) Σ *is maximally consistent,*
(3) and Σ *has the Henkin property.*

Proof. Let $\langle c_{n_i} : i \in \mathbb{N}\rangle$ enumerate the constants which do not occur in any formula of Γ. Let $\langle \varphi_i : i \in \mathbb{N}\rangle$ be an enumeration of all $\mathcal{L}$ formulas which satisfies

(1.1) for each formula φ there exist distinct positive integers i_0 and i_1 such that $\varphi = \varphi_{i_0} = \varphi_{i_1}$,
(1.2) no constant in the set, $\{c_{n_k} : k \geq i\}$, occurs in φ_i.

Define by induction on $i \in \mathbb{N}$ an increasing sequence $\langle \Sigma_i : i \in \mathbb{N}\rangle$ of sets of formulas as follows.

(2.1) $\Sigma_0 = \Gamma$.
(2.2) a) If $\varphi_i \notin \Sigma_i$ and if $\Sigma_i \cup \{\varphi_i\}$ is consistent, then $\Sigma_{i+1} = \Sigma_i \cup \{\varphi_i\}$.
b) If $\varphi_i \in \Sigma_i$ and if φ_i is an existential formula $\varphi_i = (\exists x_j \psi)$, then $\Sigma_{i+1} = \Sigma_i \cup \{\psi(x_j; c_{n_i})\}$.
c) Otherwise, $\Sigma_{i+1} = \Sigma_i$.

Claim 5.24 *For each i, the following properties hold.*

(3.1) $\Gamma \subseteq \Sigma_i$.
(3.2) $\Sigma_i \subseteq \Sigma_{i+1}$.
(3.3) Σ_i *is consistent.*
(3.4) No constant in the set, $\{c_{n_k} : k \geq i\}$ *occurs in any formula of* Σ_i.
(3.5) $\varphi_i \in \Sigma_{i+1}$ *or* $\Sigma_i \cup \{\varphi_i\}$ *is not consistent.*

Proof. Proceed by induction on i. The only subtle point is to show that Σ_{i+1} is consistent when it is defined by means of case 2(b). We give the argument for this case below.

Suppose that Claim 5.24 holds for i and that $\Sigma_{i+1} = \Sigma_i \cup \{\psi(x_j; c_{n_j})\}$ as specified in case 2(b). Suppose for a contradiction that Σ_{i+1} is not consistent. Then, $\Sigma_i \cup \{\psi(x_j; c_{n_i})\} \vdash (\neg(x_1 \hat{=} x_1))$, as every formula can be derived from an inconsistent set. By the Deduction Theorem 5.9, $\Sigma_i \vdash (\psi(x_j; c_{n_i}) \to (\neg(x_1 \hat{=} x_1)))$ and so $\Sigma_i \vdash (\neg\psi(x_j; c_{n_i}))$. By the Theorem on Constants 5.13, $\Sigma_i \vdash (\forall x_j(\neg\psi(x_; c_{n_i})))(c_{n_i}; x_j)$. Making the substitution, $\Sigma_i \vdash (\forall x_j(\neg\psi))$. But since case 2(b) applied, $(\exists x_j \psi) \in \Sigma_i$ and so $(\neg(\forall x_j(\neg\psi))) \in \Sigma_i$. Thus, Σ_i is not consistent, contrary to assumption. Thus, Σ_{i+1} is consistent, as required. □

Let $\Sigma = \cup\{\Sigma_i : i \in \mathbb{N}\}$. By the first four items in Claim 5.24, $\Gamma \subseteq \Sigma$ and Σ is consistent. Further, since every $\mathcal{L}$ formula appears in the list $\langle \varphi_i : i \in \mathbb{N}\rangle$, the fifth item in Claim 5.24 implies that Σ is maximally consistent.

Finally, we verify that Σ has the Henkin property. Suppose that $(\exists x_j\psi) \in \Sigma$. By our choice of the enumeration $\langle \varphi_i : i \in \mathbb{N}\rangle$, there exist $i_0 < i_1$ such that $\varphi_{i_0} = \varphi_{i_1} = (\exists x_j\psi)$. Since $(\exists x_j\psi) \in \Sigma$, $\Sigma_{i_0} \cup \{(\exists x_j\psi)\}$ is consistent. By case 2(a) in the definition of Σ_{i+1} from Σ_i, $(\exists x_j\psi) \in \Sigma_{i_0+1} \subseteq \Sigma_{i_1}$. Thus, case 2(b) applies in the definition of Σ_{i_1+1}, and so $\psi(x_j; c_{n_{i_1}}) \in \Sigma_{i_1+1}$, as required to verify the Henkin property. □

As an immediate corollary of Theorem 5.23 we obtain the following special case of the Gödel Completeness Theorem.

Theorem 5.25 *Suppose that Γ is a consistent set of $\mathcal{L}$ formulas and that there are infinitely many constants, c_i, which do not occur in any formula of Γ. Then Γ is satisfiable.*

Proof. By Theorem 5.23 there exists a set of $\mathcal{L}$ formulas Σ containing Γ such that Σ is maximally consistent and such that Σ has the Henkin property. By Theorem 5.20, Σ is satisfiable. Therefore Γ is satisfiable. □

5.5.1 Exercises

(1) Show that the Compactness Theorem for $\mathcal{L}_0$ follows from Theorem 5.25.

5.6 The Gödel Completeness Theorem

We fix some notation. Suppose that φ is an $\mathcal{L}$ formula. Let φ^* be the formula obtained from φ by substituting c_{2i} for c_i for each constant c_i occurring in φ. Similarly, if Γ is a set of $\mathcal{L}$ formulas, let Γ^* denote the set $\{\varphi^* : \varphi \in \Gamma\}$.

The following is a corollary of the theorem on constants.

Lemma 5.26 *Suppose that Γ is a consistent set of $\mathcal{L}$ formulas. Then, Γ^* is also consistent.*

Proof. Note that if φ is a formula and every constant which occurs in φ is in the set $\{c_{2i} : i \in \mathbb{N}\}$, then $\varphi = \psi^*$ for some formula ψ. Of course ψ is unique and obtained from φ by substituting, for each i, c_i for c_{2i} in φ.

We assume toward a contradiction that Γ^* is not consistent. Let $\langle \varphi_1, \dots, \varphi_n\rangle$ be a deduction from Γ^* such that $\varphi_n = (\neg(x_1 \hat{=} x_1))$. By the Theorem on Constants, Theorem 5.13, there is a deduction $\langle \psi_1^*, \dots, \psi_m^*\rangle$ from Γ^* such that if c_i is a constant symbols which appears in one of the ψ_k^*'s then i is even. Thus for each $k \leq n$ there exists a formula ψ_k such that $\psi_k^* = (\psi_k)^*$.

By a straightforward induction on n, $\langle \psi_1, \dots, \psi_n\rangle$ is a proof from Γ. But $\psi_n = \varphi_n = (\neg(x_1 \hat{=} x_1))$, and so Γ is not consistent. This contradicts our assumption that Γ is consistent, and therefore Γ^* is consistent.

Lemma 5.27 *Suppose that Γ is a set of $\mathcal{L}$ formulas and Γ^* is satisfiable. Then Γ is satisfiable.*

Proof. Let $A_{\mathcal{L}^*}$ be the alphabet of Γ^*. Let $\mathcal{M}^* = (M, I^*)$ be a structure and let ν be an $\mathcal{M}^*$-assignment such that

$$(\mathcal{M}^*, \nu) \vDash \Gamma^*.$$

Let $\mathcal{M} = (M, I)$ be the structure obtained from $\mathcal{M}^*$ by changing I^* to produce I such that

(1.1) $I(F_k) = I^*(F_k)$ for all function symbols, F_k,
(1.2) $I(P_k) = I^*(P_k)$ for all function symbols, P_k,
(1.3) $I(c_k) = I^*(c_{2k})$ for all $k \in \mathbb{N}$.

Since $\mathcal{M}$ and $\mathcal{M}^*$ have the same universe, for each function μ, μ is an $\mathcal{M}$-assignment if and only if μ is an $\mathcal{M}^*$-assignment.

For each term τ^* in $A_{\mathcal{L}^*}$, let τ be the term obtained from τ^* by substituting, for each $i \in \mathbb{N}$, c_{2i} for c_i in τ^*.

Now consider an arbitrary $\mathcal{M}$-assignment μ. Thus μ is both an $\mathcal{M}$-assignment and an $\mathcal{M}^*$-assignment. Let $(\overline{\mu})^{\mathcal{M}}$ denote the extension of μ to terms as calculated relative to the structure $\mathcal{M}$, and let $(\overline{\mu})^{\mathcal{M}^*}$ denote the extension of μ to terms as calculated relative to the structure $\mathcal{M}^*$.

It is easily verified by induction on the length of τ that for all terms τ,

$$(\overline{\mu})^{\mathcal{M}}(\tau) = (\overline{\mu})^{\mathcal{M}^*}(\tau^*).$$

We claim that for all formulas φ and for all $\mathcal{M}$-assignments μ,

$$(\mathcal{M}, \mu) \vDash \varphi \text{ if and only if } (\mathcal{M}^*, \mu) \vDash \varphi^*.$$

This is proved by induction on the length of φ. We leave the details as an exercise.

Since $(\mathcal{M}, \nu) \vDash \Gamma^*$, it follows that $(\mathcal{M}^*, \nu) \vDash \Gamma$, and so Γ is satisfiable. □

With these two lemmas and the results of the previous sections, of the Gödel Completeness Theorem is immediate.

Theorem 5.28 (Gödel Completeness) *A set of $\mathcal{L}$ formulas is consistent if and only if it is satisfiable.*

Proof. By soundness, if Γ is satisfiable then Γ is consistent. Therefore it suffices to prove that if Γ is consistent then Γ is satisfiable.

By Lemma 5.26, since Γ is consistent, Γ^* is consistent. By Theorem 5.25, Γ^* is satisfiable. By Lemma 5.27, Γ is satisfiable. □

The Gödel Completeness Theorem is often succinctly reformulated as follows.

Theorem 5.29 *For any set of $\mathcal{L}$ formulas Γ and any $\mathcal{L}$ formula φ,*

$$\Gamma \vDash \varphi \textit{ if and only if } \Gamma \vdash \varphi.$$

Here $\Gamma \vDash \varphi$ is used to express the condition that φ is satisfied whenever Γ is satisfied.

5.7 The Craig Interpolation Theorem

The Completeness Theorem states that if φ is satisfied whenever Γ is satisfied then there is a proof of φ from Γ. In this section, we generalize the Completeness Theorem to the restricted languages $\mathcal{L}_{\mathcal{A}}$. Of course one can fairly easily convince oneself that the proof of the Completeness Theorem we have given works for the languages $\mathcal{L}_{\mathcal{A}}$. We shall take a slightly different approach for it will reveal some additional interesting features of our formal notion of proof, this approach culminates with the statement of the Craig Interpretation Theorem.

Suppose that

$$\mathcal{A} \subseteq (6i+2 : i \in \mathbb{N}\} \cup \{6i+3 : i \in \mathbb{N}\} \cup \{6i+4 : i \in \mathbb{N}\}) .$$

is a first order alphabet.

Definition 5.30 Suppose that Γ is a set of $\mathcal{L}_{\mathcal{A}}$-formulas and that φ is an $\mathcal{L}_{\mathcal{A}}$-formula. Then $\Gamma \vdash_{\mathcal{L}_{\mathcal{A}}} \varphi$ if and only if there exists a proof $\langle \varphi_1, \ldots, \varphi_n \rangle$ of φ from Γ such that for each $i \leq n$, φ_i is an $\mathcal{L}_{\mathcal{A}}$-formula.

Eliminating predicate symbols. Suppose that φ is a formula and that P_i is a predicate symbol. Let $[\varphi]_{P_i}$ denote the formula defined as follows by induction on the length of φ:

(1) If φ is an atomic formula, then

$$[\varphi]_{P_i} = \begin{cases} (\tau_1 \hat{=} \tau_1), & \text{if } \varphi = P_i(\tau_1 \ldots \tau_n), \text{ where } n = \pi(P_i); \\ \varphi, & \text{otherwise.} \end{cases}$$

(2) $[(\neg\psi)]_{P_i} = (\neg[\psi]_{P_i})$.
(3) $[(\psi_1 \to \psi_2)]_{P_i} = ([\psi_1]_{P_i} \to [\psi_2]_{P_i})$.
(4) $[(\forall x_i \psi)]_{P_i} = (\forall x_i [\psi]_{P_i})$.

Thus $[\varphi]_{P_i}$ is obtained from φ by replacing every instance of P_i by a trivial formula.

Lemma 5.31 (Predicates) *Suppose that Γ is a set of formulas and that P_i is a predicate symbol which does not occur in any formula of Γ. Suppose that $\langle \psi_1, \ldots, \psi_m \rangle$ is a proof from Γ. Then $\langle [\psi_1]_{P_i}, \ldots, [\psi_m]_{P_i} \rangle$ is a proof from Γ.*

Proof. Note that if P_i does not occur in φ, then $[\varphi]_{P_i} = \varphi$. Thus, for each $\varphi \in \Gamma$, $[\varphi]_{P_i} = \varphi$.

By inspection of Definition 5.3, if φ is a logical axiom then $[\varphi]_{P_i}$ is a logical axiom.

Finally, if φ_1 and φ_2 are formulas then

$$[(\varphi_1 \to \varphi_2)]_{P_i} = ([\varphi_1]_{P_i} \to [\varphi_2]_{P_i}).$$

It follows by induction on $n \leq m$, that $\langle [\psi_1]_{P_i}, \ldots, [\psi_n]_{P_i} \rangle$ is a proof from Γ. $\square$

Eliminating function symbols. Suppose that τ is a term and that F_i is a function symbol. Let $[\tau]_{F_i}$ denote the term defined by induction on the length of τ as follows.

(1) $[x_j]_{F_i} = x_j$, $[c_j]_{F_i} = c_j$;
(2) Suppose $\tau = F_j(\tau_1 \dots \tau_m)$, where $m = \pi(F_j)$. Then

$$[\tau]_{F_i} = \begin{cases} F_j([\tau_1]_{F_i}, \dots, [\tau_m]_{F_i}), & \text{if } F_i \neq F_j; \\ [\tau_1]_{F_i}, & \text{otherwise.} \end{cases}$$

Let $[\varphi]_{F_i}$ denote the formula defined as follows by induction on the length of φ.

(1) If φ is an atomic formula and $\varphi = (\tau_1 \mathrel{\hat{=}} \tau_2)$, then $[\varphi]_{F_i} = ([\tau_1]_{F_i} \mathrel{\hat{=}} [\tau_2]_{F_i})$.
(2) If φ is an atomic formula and $\varphi = P_j(\tau_1 \dots \tau_n)$, where $n = \pi(P_j)$, then $[\varphi]_{F_i} = P_j([\tau_1]_{F_i}, \dots, [\tau_n]_{F_i})$.
(3) $[(\neg\psi)]_{F_i} = (\neg[\psi]_{F_i})$.
(4) $[\psi_1 \to \psi_2]_{F_i} = ([\psi_1]_{F_i} \to [\psi_2]_{F_i})$.
(5) $[(\forall x_i \psi)]_{F_i} = (\forall x_i [\psi]_{F_i})$.

Thus $[\varphi]_{F_i}$ is obtained from φ by replacing those terms which express application of F_i by simpler terms which do not refer to F_I.

Lemma 5.32 (Functions) *Suppose that Γ is a set of formulas and that F_i is a function symbol which does not occur in any formula of Γ. Suppose that $\langle \psi_1, \dots, \psi_m \rangle$ is a proof from Γ. Then $\langle [\psi_1]_{F_i}, \dots, [\psi_m]_{F_i} \rangle$ is a proof from Γ.*

Proof. The proof is quite similar to that of the Lemma on Predicates 5.31.

Note that if F_i does not occur in φ then $[\varphi]_{F_i} = \varphi$. Thus for each $\varphi \in \Gamma$, $[\varphi]_{F_i} = \varphi$. By inspection of Definition 5.3, if φ is a logical axiom then $[\varphi]_{F_i}$ is a logical axiom. Finally, if φ_1 and φ_2 are formulas, then $[(\varphi_1 \to \varphi_2)]_{F_i} = ([\varphi_1]_{F_i} \to [\varphi_2]_{F_i})$.

It follows by induction on $n \leq m$, that $\langle [\psi_1]_{F_i}, \dots, [\psi_n]_{F_i} \rangle$ is a proof from Γ. $\square$

As a corollary we obtain:

Theorem 5.33 *Suppose that Γ is a set of $\mathcal{L}_\mathcal{A}$-formulas and that φ is an $\mathcal{L}_\mathcal{A}$-formula. Then*

$\Gamma \vdash \varphi$ *if and only if* $\Gamma \vdash_{\mathcal{L}_\mathcal{A}} \varphi$.

Proof. This is an immediate corollary of Lemma 5.12 the Lemmas on Constants 5.12, on Predicates 5.31, and on Functions 5.32.

The implication from right to left is immediate, so we have only to prove that if $\Gamma \vdash \varphi$ then $\Gamma \vdash_{\mathcal{L}_\mathcal{A}} \varphi$. Let $\langle \psi_1, \dots, \psi_m \rangle$ be a proof from Γ of φ.

Let $\langle \mathcal{A}_k : k \leq n \rangle$ be an increasing sequences of sets such that for all $k < n$,

(1.1) $\mathcal{A}_k \subseteq (\{6i+2 : i \in \mathbb{N}\} \cup \{6i+3 : i \in \mathbb{N}\} \cup \{6i+4 : i \in \mathbb{N}\})$,
(1.2) $\mathcal{A}_{k+1} \backslash \mathcal{A}_k$ contains at most one element,
(1.3) $\mathcal{A}_0 = \mathcal{A}$,
(1.4) for each $j \le m$, ψ_j is an $\mathcal{L}_{\mathcal{A}_k}$-formula.

Thus by condition (4), $\Gamma \vdash_{\mathcal{L}_n} \varphi$. Observe that for each $k < n$ if $\Gamma \vdash_{\mathcal{L}_{k+1}} \varphi$, then $\Gamma \vdash_{\mathcal{L}_k} \varphi$. This follows from Lemma 5.12 if $A_{\mathcal{L}_{k+1}} \setminus A_{\mathcal{L}_k}$ contains only a constant symbol; it follows from Lemma 5.31 if $A_{\mathcal{L}_{k+1}} \setminus A_{\mathcal{L}_k}$ contains only a predicate symbol; and it follows from Lemma 5.32 if $A_{\mathcal{L}_{k+1}} \setminus A_{\mathcal{L}_k}$ contains only a predicate symbol.

Thus (by reverse induction), $\Gamma \vdash_{\mathcal{L}} \varphi$. □

Theorem 5.34 (Gödel Completeness Theorem for $\mathcal{L}_{\mathcal{A}}$) *Suppose that* Γ *is a set of* $\mathcal{L}_{\mathcal{A}}$*-formulas and that* φ *is an* $\mathcal{L}_{\mathcal{A}}$*-formula. Then*

$$\Gamma \vdash_{\mathcal{L}_{\mathcal{A}}} \varphi \text{ if and only if } \Gamma \vDash \varphi.$$

Proof. By Theorem 5.29, $\Gamma \vDash \varphi$ if and only if $\Gamma \vdash \varphi$. By Theorem 5.34, $\Gamma \vdash \varphi$ if and only if $\Gamma \vdash_{\mathcal{L}_{\mathcal{A}}} \varphi$. Thus, $\Gamma \vDash \varphi$ if and only if $\Gamma \vdash_{\mathcal{L}} \varphi$, as required. □

The three lemmas on Constants, on Functions, and on Predicates can be generalized and combined into a single theorem, the Craig Interpolation Theorem, which we cite below without proof.

Suppose that Γ is a set of formulas. Let $\mathcal{A}_\Gamma$ be the set of constant symbols, predicate symbols and function symbols which occur in some formula of Γ. Thus $\mathcal{A}_\Gamma$ is the minimum set, $\mathcal{A}$, such that each formula of Γ is an $\mathcal{L}_{\mathcal{A}}$-formula.

Theorem 5.35 (Craig Interpolation Theorem) *Suppose that* φ_1 *and* φ_2 *are formulas,* Γ *is a set of formulas and that*

$$\Gamma \vdash (\varphi_1 \to \varphi_2).$$

Suppose that

$$\mathcal{A}_{\{\varphi_1\}} \cap \mathcal{A}_{\{\varphi_2\}} \subseteq \mathcal{A}_\Gamma.$$

Then there is a formula ψ *such that:*
(1) ψ *is an* $\mathcal{L}_{\mathcal{A}_\Gamma}$*-formula,*
(2) $\Gamma \vdash (\varphi_1 \to \psi)$*,*
(3) $\Gamma \vdash (\psi \to \varphi_2)$*.*

5.7.1 Exercise

(1) Suppose that φ and ψ are propositional formulas and that φ logically implies ψ. Show that there is a propositional formula θ such that every sentence symbol in θ appears both in φ and in ψ and such that φ logically implies θ and θ logically implies ψ. (This is the propositional form of the Interpolation Theorem.)