Sample Problems for Final Exam

Questions from Math 125a Final Exam, Spring 2006

In the following, you must prove that your answers are correct.

1. Let $\mathbb{R}$ be the structure of the real numbers with 0, 1, +, $\times$, and $<$. Show that there are two countable structures $\mathcal{M}_1$ and $\mathcal{M}_2$ with the following properties.
   - $\mathcal{M}_1 \equiv \mathbb{R}$ and $\mathcal{M}_2 \equiv \mathbb{R}$
   - $\mathcal{M}_1$ and $\mathcal{M}_2$ are not isomorphic
2. Let $\mathcal{L}_A$ be the language with one unary predicate symbol P. Let $\mathcal{M}$ be the finite structure (M, I) such that $M = \{a, b, c, d, e\}$ and $I(P) = \{a, b\}$. In other words, $\mathcal{M}$ interprets P as holding of a and b and as not holding of c, d, or e.
   (a) Which subsets of M which are definable in $\mathcal{M}$ without parameters?
   (b) Which subsets of M are are definable in $\mathcal{M}$ with parameters?
3. Let A be finite, let k be a natural number, and V_k be the set of sentences φ such that for all $\mathcal{L}_A$-structures $\mathcal{M} = (M, I)$, if M has exactly k elements then $\mathcal{M} \models \varphi$. Give an algorithm to determine when given a sentence ψ in $\mathcal{L}_A$ whether $\psi \in V_k$.
4. Let $A = \{c_i : i \in \mathbb{N}\}$.
   (a) Give an example of an $\mathcal{L}_A$ structure $\mathcal{M}$ such that
   $$T_{\mathcal{M}} = \{\varphi : \ \varphi \text{ is a sentence and } \mathcal{M} \models \varphi\}$$
   does not have the Henkin property.
   (b) Is there an example $\mathcal{M} = (M, I)$ such that $T_{\mathcal{M}}$ does not have the Henkin property and $\{I(c_i) : i \in \mathbb{N}\}$ is infinite?
5. Suppose that $\mathcal{M} \subseteq \mathcal{N}$ are infinite $\mathcal{L}_\emptyset$ structures. Show that $\mathcal{M} \preceq \mathcal{N}$.
6. Give the proof to show that if, for every set of formulas Γ
   $$\Gamma \text{ is consistent iff } \Gamma \text{ is satisfiable}$$
   then, for every set of formulas Γ and every formula φ
   $$\Gamma \vdash \varphi \text{ iff every } (\mathcal{M}, \nu) \text{ which satisfies } \Gamma \text{ also satisfies } \varphi.$$

Extra credit. Suppose that T is a set of sentences and that there is an $\mathcal{N} = (N, J)$ such that $\mathcal{N} \models T$ and N is infinite. Show that there is an $\mathcal{M} = (M, I)$ and an element a of M such that $\mathcal{M} \models T$ and a is not definable in $\mathcal{M}$ without parameters.