Homework 2

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Exercise 3. Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

Proof. Suppose that $\mathbb{R}$ is homeomorphic to $\mathbb{R}^2$. Then there exists a homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}$, and thus $f(x, y) = 0$ for some $(x, y) \in \mathbb{R}^2$. Now, let B_ϵ be an open ball around (x, y). Any open ball is connected and continuous maps preserve connectedness. So, since f^{-1} is continuous, we know that $f(B_\epsilon)$ is some open, connected set in $\mathbb{R}$. The only connected sets in $\mathbb{R}$ are intervals, so $f(B_\epsilon) = (a, b) \subseteq \mathbb{R}$ for some $a, b \in \mathbb{R}$.

Now, since $(x, y) \in B_\epsilon$, we know that $0 = f(x, y) \in (a, b)$. So, if we remove (x, y) from B_ϵ, the image of the resulting set will be (a, b) with 0 removed, i.e. $f(B_\epsilon \setminus \{(x, y)\}) = (a, b) \setminus \{0\}$. However, this is a contradiction, because f should preserve connectedness, yet $B_\epsilon \setminus \{(x, y)\}$ is still connected (since it is obviously still path connected) while $(a, b) \setminus \{0\}$ can be separated into a union of the nonempty disjoint sets $(a, 0)$ and $(0, b)$, and is thus disconnected.

Therefore, no such map f exists, so $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

$\square$

Exercise 4. Let $(X, \mathcal{T}_X)$ be a topological space and $(Y, \mathcal{T}_Y)$ be a topological space that is Hausdorff. For a function $f : X \to Y$, let Γ_f denote the graph of f. Show that if f is continuous, then Γ_f is closed in the product topology on $X \times Y$.

Proof. Assume that f is continuous. We will show that Γ_f^C is open by proving that every point is interior.

Let $(x, y) \in \Gamma_f^C$. We know that $y \neq f(x)$ because (x, y) is not on the graph of f. Thus, since Y is Hausdorff, we can find disjoint open sets $\mathcal{U}_y, \mathcal{U}_{f(x)} \subset Y$ that contain y and $f(x)$, respectively. Since f is continuous, we know that $f^{-1}(\mathcal{U}_{f(x)})$ is open in X. So $f^{-1}(\mathcal{U}_{f(x)}) \times \mathcal{U}_y$ is open in the product topology on $X \times Y$. Now, $\mathcal{U}_{f(x)}$ contains $f(x)$, so $f^{-1}(\mathcal{U}_{f(x)})$ contains x. Also, $\mathcal{U}_y$ contains y. So $f^{-1}(\mathcal{U}_{f(x)}) \times \mathcal{U}_y$ contains (x, y).

Now we will show that $f^{-1}(\mathcal{U}_{f(x)}) \times \mathcal{U}_y$ is contained within Γ_f^C. Let $(x_0, y_0) \in f^{-1}(\mathcal{U}_{f(x)}) \times \mathcal{U}_y$, and so $f(x_0) \in \mathcal{U}_{f(x)}$ and $y_0 \in \mathcal{U}_y$. Therefore, if $y_0 = f(x_0)$ then $y_0 \in \mathcal{U}_y \cap \mathcal{U}_{f(x)}$, contradicting that $\mathcal{U}_y$ and $\mathcal{U}_{f(x)}$ are disjoint. Thus $y_0 \neq f(x_0)$, and so $(x_0, y_0) \notin \Gamma_f$. So $(x_0, y_0) \in \Gamma_f^C$. Since (x_0, y_0) was arbitrary, we know that $f^{-1}(\mathcal{U}_{f(x)}) \times \mathcal{U}_y$ is contained within Γ_f^C.

Since every point $(x, y) \in \Gamma_f^C$ has an open neighborhood $f^{-1}(\mathcal{U}_{f(x)}) \times \mathcal{U}_y$ which is completely contained within Γ_f^C, every point of Γ_f^C is interior. So Γ_f^C is open in the product topology on $X \times Y$. Therefore, Γ_f is closed. $\square$

Exercise 5. Show that if $(X, \mathcal{T})$ is second-countable and $S \subset X$, then every limit point of S is a limit of a sequence in S.

Proof. Suppose that $(X, \mathcal{T})$ is second-countable and $S \subset X$, and assume that x is a limit point of S. A countable basis exists for $(X, \mathcal{T})$, so we can arrange all the basis elements that contains x into a sequence $\{B_n\}$. Next, construct another sequence $\{S_n\}$ defined by

$$S_n = \left(\bigcap_{k=1}^{n} B_k \right) \cap S \text{ for each } n \in \mathbb{N}.$$

Note that S_n is nonempty for all $n \in \mathbb{N}$ because every B_k contains x, so this finite iterated intersection is an open set containing x, and thus it has a nonempty intersection with S (since S has x as a limit point). Also, note that for all $i > j$ we have $S_i \subseteq B_j$. Finally, we can construct a sequence $\{x_n\}$ in S that converges to x by choosing x_n, for each $n \in \mathbb{N}$, to be any element of S_n.

Now, let $\mathcal{U}$ be a subset of X that contains x. We know that there must be a basis element contained within $\mathcal{U}$ that contains x. Thus this basis element is in our sequence $\{B_n\}$, so let it be B_N. Now, for all $n > N$, we know that $x_n \in S_n \subseteq B_N \subseteq \mathcal{U}$. Therefore, since $\mathcal{U}$ was arbitrary, $\{x_n\}$ converges to x.

Since x was an arbitrary limit point of S, we have shown that every limit point of S is a limit of a sequence in S.

$\square$