7.6 Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable, $a \in \mathbb{R}$, and we define

$$F(x) = \int_a^x f(y)dy.$$

Show that F is a continuous function.

Proof. In the previous homework, we proved exercise 6.4, which can be applied to the measure space $\mathbb{R}$ because it is σ-finite. It states that for each $\epsilon > 0$ there is a $\delta > 0$ such that if $\mu(A) < \delta$ then $\int_A |f| < \epsilon$. Now, let $x \in \mathbb{R}$ and $\epsilon > 0$. Choose a δ so that the bound just mentioned holds. Then if $|x - x_0| < \frac{\delta}{3}$, we have $x_0 \in A = (x - \frac{\delta}{3}, x + \frac{\delta}{3})$ where $\mu(A) < \delta$. So we have

$$\begin{aligned}
|F(x) - F(x_0)| &= \left| \int_a^x f - \int_a^{x_0} f \right| \\
&= \left| \int_a^x f - \int_a^x f - \int_x^{x_0} f \right| \\
&= \left| \int_x^{x_0} f \right| \\
&\le \int_x^{x_0} |f| \\
&\le \int_A |f| < \epsilon.
\end{aligned}$$

We have applied here the fact that, for any $r, s, t \in \mathbb{R}$, $\int_r^t f = \int_r^s f + \int_s^t$. According to the definition given in Bass, $\int_a^b f = \int_{[a,b]} f$. This is nonsense if $a > b$, since then $[a, b] = \emptyset$, which would create a truly stupid inconsistency in notation between the Lebesgue the Riemann integrals. I will assume that $\int_a^b f = -\int_b^a f$ if $a > b$, since this makes more sense. Then, for $r \le s \le t$, we have

$$\int_r^t f = \int (f \cdot \mathbb{1}_{(-\infty,s)} + f \cdot \mathbb{1}_{[s,\infty)})\mathbb{1}_{[r,t]} = \int f \cdot \mathbb{1}_{[r,s)} + \int f \cdot \mathbb{1}_{[s,t]} = \int_r^s f + \int_s^t f.$$

If $r \le t \le s$, we have from the previous result $\int_r^s f = \int_r^t f + \int_t^s f$, so subtracting $\int_t^s f$ gives

$$\int_r^t f = \int_r^s f - \int_t^s f = \int_r^s f + \int_s^t f$$

as desired. All other cases follow from rearranging and/or negating both sides of one of these equalities. □

7.10 Prove that the limit exists and find its value:

$$\lim_{n\to\infty} \int_0^1 \frac{1 + nx^2}{(1 + x^2)^n} \log(2 + \cos(x/n))dx$$

Proof. Assume $x \in [0, 1]$. Note that $0 < 1 + nx^2 \le \sum_{k=0}^n \binom{n}{k} x^{2n} = (1 + x^2)^n$, therefore $0 < \left| \frac{1+nx^2}{(1+x^2)^n} \right| \le 1$. Also, $0 < \cos(x/n) \le 1$, so $0 < |\log(2 + \cos(x/n))| \le \log(3)$. So if f_n is the integrand, then $|f_n| \le \log(3)$, hence we may apply the dominated convergence theorem.

Since $\{0\}$ is a null set, the expression equals $\lim_{n\to\infty} \int_{(0,1]} f_n$. Applying L'Hospital's rule to the first factor by differentiating with respect to n gives

$$\lim_{n\to\infty} \frac{1+nx^2}{(1+x^2)^n} = \lim_{n\to\infty} \frac{x^2}{\log(1+x^2)(1+x^2)^n} = \frac{x^2}{\log(1+x^2)} \lim_{n\to\infty} \frac{1}{(1+x^2)^n} = 0.$$

The limit of the second factor is clearly $\log 3$, which is finite. So the limit of the integrand is the product of the limits of these factors, which is 0. Thus, by the dominated convergence theorem, the expression equals $\int_{(0,1]} 0 = 0$. $\square$

7.16 Let $(X, \mathcal{A}, \mu)$ be a measure space. A family of measurable functions $\{f_n\}$ is *uniformly integrable* if given ϵ there exists M such that

$$\int_{\{x:|f_n(x)|>M\}} |f_n(x)| d\mu < \epsilon$$

for each n. The sequence is *uniformly absolutely continuous* if, given ϵ, there exists δ such that

$$\left| \int_A f_n d\mu \right| < \epsilon$$

for each n if $\mu(A) < \delta$.

Suppose μ is a finite measure. Prove that $\{f_n\}$ is uniformly integrable if and only if $\sup_n \int |f_n| d\mu < \infty$ and $\{f_n\}$ is uniformly absolutely continuous.

Proof. For each n, M, let $T_M^n = \{x : |f_n(x)| > M\}$. Suppose first that $\{f_n\}$ is uniformly integrable, and choose an M so that $\int_{T_M^n} |f_n(x)| d\mu < \frac{\epsilon}{2}$. Let $\delta = \frac{\epsilon}{2M}$. Then for any A with $\mu(A) < \delta$, we have

$$\begin{aligned}
\left| \int_A f_n \right| &\leq \int_A |f_n| \\
&= \int_{T_M^n \cap A} |f_n| + \int_{(T_M^n)^c \cap A} |f_n| \\
&\leq \int_{T_M^n} |f_n| + \int_A M \\
&< \frac{\epsilon}{2} + M\mu(A) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Now, fix any positive ϵ. There is some M such that, for all n,

$$\int |f_n| = \int_{T_M^n} |f_n| + \int_{(T_M^n)^c} |f_n| < \epsilon + M\mu((T_M^n)^c) \leq \epsilon + M\mu(X) < \infty.$$

Thus, since the sequence $\int |f_n|$ is bounded, its supremum is finite.

Next, suppose that $\sup_n \int |f_n| d\mu < \infty$ and $\{f_n\}$ is uniformly absolutely continuous. There exists some δ such that

$$\left| \int_A f_n d\mu \right| < \frac{\epsilon}{2}$$

for each n if $\mu(A) < \delta$. Thus, if $\mu(A) < \delta$ we have

$$\int_A |f_n| = \int_{A\cap\{x:f_n(x)\geq 0\}} f_n - \int_{A\cap\{x:f_n(x)<0\}} f_n < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Suppose we were to have, for all M, that $\mu(\{x : |f_n(x)| > M\}) \geq \delta$. Then, for all M, we would have

$$M\delta \leq M\mu(\{x : |f_n(x)| > M\}) \leq \int_{\{x:|f_n(x)|>M\}} M \leq \int_{\{x:|f_n(x)|>M\}} |f_n| \leq \int |f_n| \leq \sup \int |f_n|$$

contradicting that the supremum is finite. Thus $\mu(\{x : |f_n(x)| > M\}) \leq \delta$ for some M. Therefore, $\displaystyle\int_{\{x:|f_n(x)|>M\}} |f_n(x)| d\mu < \epsilon$, so $\{f_n\}$ is uniformly integrable. □

8.3 Suppose A is a Borel measurable subset of $[0, 1]$, μ is the Lebesgue measure, and $\epsilon \in (0, 1)$. Prove that there exists a continuous function $f : [0, 1] \to \mathbb{R}$ such that $0 \leq f \leq 1$ and

$$\mu(\{x : f(x) \neq \mathbb{1}_A(x)\}) < \epsilon.$$

Proof. If A is a null set, then $g = 0$ suffices. So we may assume A has positive measure. Since A is Borel measurable, there exists a compact set F and an open set G such that $F \subseteq A \subseteq G$ and $\mu(G \setminus F) < \epsilon$. Since $\mu(A) > 0$, we may assume $F \neq \emptyset$, since we could take ϵ to be $\min(\epsilon, \mu(A))$ instead, forcing $\mu(F) > 0$ (otherwise $\epsilon > \mu(G \setminus F) = \mu(G) > \mu(A)$, a contradiction). So there exists some minimum distance δ from F to G^c. Define

$$g(x) = \left(1 - \frac{\text{dist}(x, F)}{\delta}\right)^+.$$

g is a composition of continuous functions, hence continuous. g is 0 on G^c and 1 on F, and $0 \leq g \leq 1$. So we have $g = \mathbb{1}_A$ on both F and G^c. Therefore, $\{x : f(x) \neq \mathbb{1}_A(x)\} \subseteq G \setminus F$, hence $\mu(\{x : f(x) \neq \mathbb{1}_A(x)\}) \leq \mu(G \setminus F) < \epsilon$. □

8.7 Let μ be a measure, not necessarily σ-finite, and suppose f is real-valued and integrable with respect to μ. Prove that $A = \{x : f(x) \neq 0\}$ has σ-finite measure, that is, there exists $F_n \uparrow A$ such that $\mu(F_n) < \infty$ for each n.

Proof. For each $n \in \mathbb{N}$ (including zero) define $S_n = f^{-1}((n, n + 1])$ and $T_n = f^{-1}([-(n + 1), -n))$. Then $A = \bigcup_{n=0}^{\infty} S_n \cup \bigcup_{n=0}^{\infty} T_n$. Suppose that, for some n, $\mu(S_n) = \infty$. Then

$$\int_{S_n} |f| = \int_{S_n} f \geq \int_{S_n} n = n\mu(S_n) = \infty,$$

a contraction. Suppose that, for some n, $\mu(T_n) = \infty$. Then

$$\int_{T_n} |f| = \int_{T_n} (-f) \geq \int_{T_n} n = n\mu(T_n) = \infty,$$

a contradiction. So $\mu(S_n), \mu(T_n) < \infty$ for all n, thus A is σ-finite. □