1/29/15

Derangements

a <u>derangement</u> $f:[n] \to [n]$ is a permutation with <u>no</u>
$f(i)=i$ for any i

<u>Strategy</u>: IE , <u>events</u>: $E_1: 1 \to 1$

$\vdots$

$E_n: n \to n$

$n! - |E_1| - |E_2| - \dots - |E_n| + |E_1 \cap E_2| + |E_1 \cap E_3| \dots - |E_1 \cap E_2 \cap E_3| \dots$

$n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)!$

$\binom{n}{3} = \frac{n(n-1)(n-2)}{3!}$

$= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots \pm \frac{1}{n!}\right) \approx$ almost $\frac{1}{e}$

Probability

Ω: <u>sample space</u> (set)

E <u>events</u> $E \subseteq \Omega$ $\quad E \in \{0,1\}^{\Omega}$ power set.

$Pr: \underbrace{\{0,1\}^{\Omega}}_{\text{events}} \to [0,1]$

1) $P(\Omega) = 1$

2) $P(E_1 \cup E_2 \dots \cup E_n) = P(E_1) + P(E_2) + \dots + P(E_n)$

E_i's disjoint

<u>Roll a die</u>

$\Omega = \{1,2,3,4,5,6\}$

$P(\{x\}) = \frac{1}{6}$

$P(\{2,4,6\}) = \frac{1}{2}$

uniform distribution $\to P(\{x\}) = P(\{y\}) \ \forall x, y$

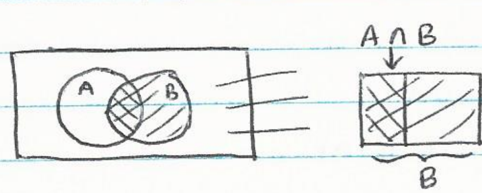
In uniform distribution, Pr(permutation is derangement)

$= \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \cdots\right)$

<u>Figure out for yourself:</u>

events A, B are independent iff $P(A)P(B) = P(A \cap B)$

prob. of "A given B" $\quad P(A|B) = \frac{P(A \cap B)}{P(B)}$



$p(\{1 \text{ or } 3\} \mid \text{the die is odd}) = \frac{2}{3}$

random variable : $X : \Omega \to S$ set

Ex| Ω : {52 cards of a deck}

"suit" $X : \Omega \to$ {spades, clubs, ..

"rank" $Y : \Omega \to [13]$

$P(X = \text{hearts}) := Pr(\{x \in \Omega \mid X(x) = \text{"hearts"}\})$

allows this shorthand

Answers to the little quiz

① ranks suits

$13 \times 12 \times \binom{4}{3}\binom{4}{2}$

Incorrect: $\binom{13}{2}\binom{4}{3}\binom{4}{2}$

② case/sum principle

0 elements going to 3 $= 2^{10}$

1 element going to 3 $= \binom{10}{1} 2^9$

2 elements going to 3 $= \binom{10}{2} 2^8$

3 " " " $= \binom{10}{3} 2^7$

Asymptotics

$f(x) = O(g(x))$ as $x \to x_0$ means $\frac{f(x)}{g(x)} < C$ as $x \to x_0$

"f(x) grows at most as fast as g(x)"

$n^2 = O(n^2)$

$n^2 = O(n^3)$

Example:

want to say "usually get 3 different people"

total ways to pick 3 from n (with replacement) $= n^3$ * temporarily a bad example

3 different people $= n(n-1)(n-2)$

$n^3 - n(n-1)(n-2) = O(n^2)$

$an^2 + bn + c = O(n^2)$

$f(x) = o(g(x))$ as $x \to x_0$ if $\frac{f(x)}{g(x)} \to 0$ as $x \to x_0$

$f(x) \sim g(x)$ if $\frac{f(x)}{g(x)} \to 1$ as $x \to x_0$ $o(n^3)$

$O \approx \leq$

$o \approx <$

$\sim \approx =$

Estimating $n!$

$n! < n^n$ $\qquad\qquad$ $n! \approx \frac{n^n}{e^n}\sqrt{2\pi n}$

$\log n! < n \log n$

$n! > \left(\frac{n}{2}\right)^{\frac{n}{2}}$

$\frac{n}{2}\log\frac{n}{2} < \log n!$

$\approx \frac{n}{2}(\log n - \log 2)$ for big n $\qquad$ $\frac{n}{2}\log n < \log n!$

$\Rightarrow \log n! \sim C\, n \log n$