Problem Set 9, Math 172 Spring '15

This problem set is due Tuesday, April 14th, 2015 at **the beginning of class**. All class guide rules apply. "VLW" refers to Van Lint and Wilson, our textbook. **All questions, when graded, are worth an equal number of points unless stated otherwise, though not all questions would end up being graded.**

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1. Recall that the extended Hamming code takes the Hamming code and adds a "parity check" bit; it has dimension 4 and uses 8 bits. The code generator matrix that I like is (which is slightly different from the order I may have given in class):

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

   Show how it could be used to **correct** 1 error and **detect** 2 errors.

2. A "High low mid" pattern in a permutation $(a_1, \ldots, a_n)$ of length n is 3 terms a_i, a_j, a_k where $i < j < k$ and $a_i > a_k > a_j$. (example: in the permuation 165243 the $(6, 2, 3)$ terms form a "high low mid"). Count the number of permutations of length n which avoid any "high low mid" patterns.

3. Find all subsets $F \subset \mathbf{Z}$ such that the equation $a + 2b = n$ with $a, b \in F$ has exactly one solution for every positive integer n. (Hint: you may want to skip the following text to avoid overthinking, but it may also help you.)

   You: "WTF?"

   Problem: "Well... I'm designed to be kind of difficult. You probably need to think outside the box a bit."

   You: "I guess this means dual graphs, generating functions, or Hall's Marriage Theorem."

   Problem: "Do you really think I'm Hall's Marriage Theorem? How is that remotely possible?"

   You: "There was a problem with chessboards that looked like it had nothing to do with Hall's, but Hall's was totally what you were supposed to do there."

   Problem: "You have a point."

   You: "So are you Hall's Marriage Theorem?"

   Problem: "... no?"

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4. There are $2n$ people at a party. Each person has an even number of friends. Prove (**using 2 different methods**) that there are 2 people with an even number of common friends.

5. A couple of problems to practice with the *absorption identity*

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1},$$

which is really helpful for moving things in and out of binomial coefficients (such as cancelling out terms you don't want.) **You must use the identity somewhere for both of these.**

   (a) Simplify $\sum_{k=0}^{n} k\binom{n}{k}$. (Have you seen this before?)

   (b) An (actual!!) paper has its final answer "simplified" to

$$\sum_{k=0}^{n} k\frac{\binom{m-k-1}{m-n-1}}{\binom{m}{n}}.$$

   Simplify it further.

6. Let l be a product of n primes. Count the number of ways to write l as a product of m positive integers, none of which are 1.

7. How much time did you spend on this problem set? What comments do you have of the problems? (difficulty, type, enjoyment, fairness, etc.)