6

The Compactness Theorem

Theorem 6.1 *Suppose that* Γ *is a set of formulas and that for every finite subset* Γ_0 *of* Γ*,* Γ_0 *is satisfiable. Then,* Γ *is satisfiable.*

Proof. For the sake of a contradiction, suppose that Γ is not satisfiable. Then, by the Completeness Theorem, Γ is not consistent. Let $\langle\psi_1, \dots, \psi_n\rangle$ be a deduction from Γ such that ψ_n is equal to' $(\neg(x_1 \doteq x_1))$. Let Γ_0 be $\Gamma \cap \{\psi_1, \dots, \psi_n\}$. Note that Γ_0 is a finite set. Then $\langle\psi_1, \dots, \psi_n\rangle$ is a deduction from Γ_0 of $(\neg(x_1 \doteq x_1))$. By the Soundness Theorem, Γ_0 is not satisfiable. This contradicts our assumptions on Γ, and the theorem follows. $\square$

6.1 Applications of the Compactness Theorem

6.1.1 Satisfiability in finite structures

Suppose that Γ is a set of sentences. An immediate corollary of the Lowenheim-Skolem Theorem is that if Γ is satisfiable, then there is an $\mathcal{M}$ such that the universe of $\mathcal{M}$ is countable and $\mathcal{M} \vDash \Gamma$. Here is another proof of this corollary, at least for the case that there exist infinitely many constant symbols which do not occur in any formula of Γ:

Suppose that Γ is satisfiable. Then Γ is consistent, and by the construction of Chapter 5, there is a model of Γ whose universe consists of equivalence classes of constant symbols. Since there are only countably many constant symbols, there are only countably many such equivalence classes. Thus, there is an $\mathcal{M}$ such that the universe of $\mathcal{M}$ is countable and $\mathcal{M} \vDash \Gamma$.

But now, consider the problem of determining whether Γ has an infinite model or even whether Γ has only infinite models.

Theorem 6.2 *Suppose that* Γ *is a set of sentences such that for every* $n \in \mathcal{N}$*, there is an* $\mathcal{M}$ *such that* $(\mathcal{M}, \nu) \vDash \Gamma$ *and the universe of* $\mathcal{M}$ *has at least* n *elements. Then there is an* $\mathcal{M}$ *such that the universe of* $\mathcal{M}$ *is infinite and* $\mathcal{M} \vDash \Gamma$.

Proof. Consider the set of formulas Δ defined as follows.

$$\Delta = \Gamma \cup \{(\neg(x_i \doteq x_j)) : i \text{ and } j \text{ are distinct elements of } \mathbb{N}\}$$

Suppose that Δ_0 is a finite subset of Δ, and let n be the size of Δ_0. By assumption, choose $\mathcal{M}$ so that $\mathcal{M} \models \Gamma$ and so that the universe of $\mathcal{M}$ has at least $2n$ elements.

Γ is a set of sentences, so no variable occurs freely in any element of Γ. The elements of $\Delta_0 \setminus \Gamma$ are formulas of the form $(\neg(x_i \mathrel{\hat{=}} x_j))$. Each of these has at most two freely occurring variables. So, there are at most $2n$ variables which occur freely in Δ_0. Choose an $\mathcal{M}$-assignment ν such that for any pair of distinct variables which occur freely in Δ_0, ν assigns these variables to distinct element of the universe of $\mathcal{M}$. Since $\mathcal{M} \models \Gamma$, $(\mathcal{M}, \nu) \models \Delta_0 \cap \Gamma$. By the choice of ν, $(\mathcal{M}, \nu) \models \Delta_0 \setminus \Gamma$. Thus, $(\mathcal{M}, \nu) \models \Delta_0$.

Since Δ_0 was an arbitrary finite subset of Δ, every finite subset of Δ is satisfiable. By the Compactness Theorem 6.1, Δ is satisfiable. Suppose that $(\mathcal{M}^*, \nu^*) \models \Delta$, and hence $\mathcal{M}^* \models \Gamma$. Then the map $i \mapsto \nu(x_i)$ is injective, and so the universe of $\mathcal{M}^*$ is infinite. $\square$

Theorem 6.3 *Suppose that φ is an $\mathcal{L}$-sentence such that for every $\mathcal{L}$-structure $\mathcal{M}$, if the universe of $\mathcal{M}$ is infinite, then $\mathcal{M} \models \varphi$. Then, there is an n such that for every $\mathcal{L}$-structure $\mathcal{M}$, if the universe of $\mathcal{M}$ has at least n elements, then $\mathcal{M} \models \varphi$.*

Proof. For the sake of a contradiction, suppose that for every n there is an $\mathcal{M}$ such that the universe of $\mathcal{M}$ has at least n elements and $\mathcal{M} \models (\neg\varphi)$. By Theorem 6.2, there is an $\mathcal{L}$-structure $\mathcal{M}^*$ such that $\mathcal{M}^*$ has an infinite universe and $\mathcal{M}^* \models (\neg\varphi)$. The existence of $\mathcal{M}^*$ contradicts our assumption on φ, proving the theorem. $\square$

6.1.2 Wellordered sets

Definition 6.4 Suppose that $\succ$ is a total ordering of a set X. Then, $\succ$ is a *wellorder* of X if and only if there is no infinite sequence $\langle a_n : n \in \mathbb{N}\rangle$ such that for each n, $a_n \succ a_{n+1}$.

Theorem 6.5 *Suppose that Γ is a set of sentences in the language $\{\succ\}$, that is in the language with one binary relation symbol. Suppose that Γ is satisfied in at least one infinite total order. Then there is a total ordering $\mathcal{M}$ such that $\mathcal{M}$ is not a wellorder and $\mathcal{M} \models \Gamma$.*

Proof. Let $TOTAL$ be the finite set of first order formulas which axiomatize the properties of a total order. Let Δ be the following set of formulas.

$$\Delta = \Gamma \cup \{x_i \succ x_j : i < j \text{ in } \mathbb{N}\} \cup TOTAL$$

Any finite subset of Δ can be satisfied in the infinite total order which satisfies Γ. Consequently, Δ is satisfiable. Suppose that $(\mathcal{M}, \nu) \models \Delta$. Then $\mathcal{M}$ is a total order which satisfies Γ and which is not a well order. $\square$

6.1.3 Exercises

(1) Suppose that T_1 and T_2 are sets of sentences such that $T_1 \cup T_2$ has no models. Show that there is a sentence φ such that $T_1 \vdash \varphi$ and $T_2 \vdash (\neg\varphi)$.

(2) Say that a set of sentences T is *finitely axiomitizable* if and only if there is a finite set of sentences Γ such that for all $\mathcal{M}$, $\mathcal{M} \models \Gamma$ if and only if $\mathcal{M} \models T$. Suppose that T_1 and T_2 are sets of sentences such that for every structure $\mathcal{M}$, $\mathcal{M} \models T_1$ if and only if $\mathcal{M} \not\models T_2$. Show that T_1 and T_2 are finitely axiomitizable.

(3) Suppose that $\mathcal{M}$ is an infinite structure. Show that there is an $\mathcal{M}_1$ such that $\mathcal{M}$ and $\mathcal{M}_1$ are elementarily equivalent and $\mathcal{M}_1$ has an element which is not the interpretation of any constant symbol.

(4) Let $\mathbb{R}$ be the structure of the real numbers with 0, 1, $+$, $\times$, and $<$. Show that there are two countable structures $\mathcal{M}_1$ and $\mathcal{M}_2$ with the following properties.
   - $\mathcal{M}_1 \equiv \mathbb{R}$ and $\mathcal{M}_2 \equiv \mathbb{R}$
   - $\mathcal{M}_1$ and $\mathcal{M}_2$ are not isomorphic

6.2 Types

Definition 6.6 An *n-type* is a maximally consistent set of formulas Γ such that every variable which occurs freely in an element of Γ is one of $x_1, \ldots, x_n$.

We write $\Gamma(x_1, \ldots, x_n)$ to indicate that Γ is an n-type.

Example 6.7 If $\mathcal{M} = (M, I)$ is a structure and $m_1, \ldots, m_n$ are elements of M, then

$$\Gamma(x_1, \ldots, x_n) = \{\varphi(x_1, \ldots, x_n) : \mathcal{M} \models \varphi[m_1, \ldots, m_n]\}$$

is an n-type, called the type of $(m_1, \ldots, m_n)$ in $\mathcal{M}$.

Definition 6.8 If $\Gamma = \Gamma(x_1, \ldots, x_n)$ is the type of some $(m_1, \ldots, m_n)$ in $\mathcal{M}$, then we say that $\mathcal{M}$ *realizes* Γ. Otherwise, $\mathcal{M}$ *omits* Γ.

Similarly, we can speak of $\mathcal{M}$'s realizing or omitting a set of formulas even when that set is not a type, that is when that set is not maximally consistent.

Theorem 6.9 *Let T be a set of sentences and suppose that Γ is a type extending T. The following are equivalent.*

(1) There is an $\mathcal{M}$ such that $\mathcal{M} \models T$ and $\mathcal{M}$ realizes Γ.

(2) For every finite subset Γ_0 of Γ, there is an $\mathcal{M}$ such that $\mathcal{M} \models T$ and $\mathcal{M}$ realizes Γ_0.

(3) The set

$$T \cup \left\{ (\exists x_1 \exists x_2 \dots \exists x_n (\varphi_1 \wedge \dots \wedge \varphi_k)) : \begin{array}{c} k \in \mathbb{N} \text{ and} \\ \varphi_1, \dots, \varphi_k \in \Gamma \end{array} \right\}$$

is consistent.

Proof.

(1) $\Longrightarrow$ (2):

Clearly, (1) implies (2). Any model of T which realizes Γ also realizes every finite subset of Γ.

(2) $\Longrightarrow$ (3):

Suppose that $\varphi_1, \dots, \varphi_k$ are elements of Γ. By (2), let $\mathcal{M} = (M, I)$ be a model which realizes $\{\varphi_1, \dots, \varphi_k\}$. Thus, we may fix $m_1, \dots, m_n$ in M so that for all $i \leq k$, $\mathcal{M} \vDash \varphi_i[m_1, \dots, m_n]$. So, if ν is an assignment such that for all $i \leq n$, $\nu(x_i) = m_i$, then

$$(\mathcal{M}, \nu) \vDash (\varphi_1 \wedge \dots \wedge \varphi_k).$$

Consequently, $\mathcal{M} \vDash (\exists x_1 \exists x_2 \dots \exists x_n (\varphi_1 \wedge \dots \wedge \varphi_k))$. If $k > k_1$, then $(\exists x_1 \exists x_2 \dots \exists x_n (\varphi_1 \wedge \dots \wedge \varphi_k))$ implies $(\exists x_1 \exists x_2 \dots \exists x_n (\varphi_1 \wedge \dots \wedge \varphi_{k_1})$. Claim (3) follows.

(3) $\Longrightarrow$ (1):

By the Completeness Theorem 5.6, there are $\mathcal{M}$ and ν such that $(\mathcal{M}, \nu) \vDash T \cup \Gamma$. Letting m_i denote $\nu(x_i)$, we can rewrite this condition by saying that for each $\varphi \in \Gamma$, $\mathcal{M} \vDash \varphi[m_1, \dots, m_n]$. Thus, $\mathcal{M}$ realizes Γ, as required. □

Definition 6.10 A set of sentences T is *complete* if and only if for every sentence φ in the language of T, either $\varphi \in T$ or $(\neg \varphi) \in T$.

Definition 6.11 Suppose that T is a complete set of sentences and that Γ is an n-type extending T. Γ is a *principal* type if and only if there is a formula φ in Γ such that for all $\psi(x_1, \dots, x_n)$,

$$\psi \in \Gamma \Leftrightarrow T \cup \{\varphi\} \vdash \psi.$$

Lemma 6.12 *Suppose that T is a consistent and complete set of sentences, and suppose that $\Gamma = \Gamma(x_1, \dots, x_n)$ is an n-type which contains T. If T has a model which omits Γ, then Γ is not a principal type.*

Proof. For the sake of a contradiction, suppose that T does not locally omit Γ. Then, there is a $\psi(x_1, \dots, x_n)$ such that for all $\varphi \in \Gamma$, $T \vdash (\psi \rightarrow \varphi)$ and such that $T \cup \psi$ is consistent. Since T is consistent, it must be the case that $(\forall x_1 \dots \forall x_n (\neg \psi)) \notin T$. Since T is complete, $(\neg(\forall x_1 \dots \forall x_n (\neg \psi))) \in T$. In other words, $(\exists x_1 \ldots \exists x_n \psi) \in T$. Now suppose that $\mathcal{M} = (M, I)$ is given so that $\mathcal{M} \vDash T$. Since $\mathcal{M}$ satisfies $(\exists x_1 \ldots \exists x_n \psi)$, we may fix $m_1, \ldots, m_n$ in M so that $\mathcal{M}$ satisfies $\psi[m_1, \ldots, m_n]$. As every element of Γ can be deduced from ψ, it follows that for every $\varphi \in \Gamma$, $\mathcal{M} \vDash \varphi[m_1, \ldots, m_n]$. So, $\mathcal{M}$ realizes Γ. Since $\mathcal{M}$ was arbitrary, there is no model of T which omits Γ. This contradicts our assumption on T, and the lemma follows. □

Theorem 6.13 (Omitting Types) *Suppose that T is a complete and consistent set of sentences. Suppose that $\Gamma = \Gamma(x_1, \ldots, x_n)$ is an n-type which contains T and which is not a principal type.*

Then T has a model which omits Γ.

Proof. To keep our notation simple, assume that $\Gamma = \Gamma(x_1)$ is a 1-type.

As in the proof of the Completeness Theorem 5.6, we can assume that there is an infinite set of constant symbols which do not appear in T. Let $\{c_{i_n} : n \in \mathbb{N}\}$ be this set.

We proceed as in the proof of the Completeness theorem to construct a set of formulas T_∞ so that the model built from constants using T_∞ has the required properties. Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be an enumeration of all first order formulas, presented so that for each n, c_{i_n} does not appear in $\varphi_0, \varphi_1, \ldots, \varphi_n$. We construct a sequence of sets

$$T = T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots$$

so that the following properties hold.

(1.1) For each n, T_{n+1} is a finite consistent extension of T.

(1.2) For each n, $\varphi_n \in T_{n+1}$ or $(\neg\varphi_n) \in T_{n+1}$.

(1.3) For each n, if $\varphi_n \in T_n$ and φ_n is of the form $(\exists x_j \psi)$, then $\psi(x_j; c_{i_n}) \in T_{n+1}$.

Given T_n, construct T_{n+1} by extending T_n in three steps as follows.

A. First, if $T_n \cup \{\varphi_n\}$ is consistent, then add φ_n to T_n. Otherwise, add $(\neg\varphi_n)$ to T_n. Call the result T^a_{n+1}.

B. If φ_n is of the form $(\exists x_j \psi)$ and $\varphi_n \in T^a_{n+1}$, then add $\psi(x_j; c_{i_n})$ to T^a_{n+1}. Call the result T^b_{n+1}.

Steps A and B are identical with the ones that we took in the proof of Theorem 5.23. By the argument given there, if T_n is consistent then so is T^b_{n+1}. Further, in each of these steps we add at most one formula to T_n. Consequently, if T_n is a finite extension of T, then so is T^b_{n+1}.

C. Let T^b_{n+1} be $T \cup \{\psi_1, \ldots, \psi_r\}$. By substituting variables of large index, we may assume that $x_1, \ldots, x_n$ do not appear in $\{\psi_1, \ldots, \psi_r\}$ and that the variables which do occur there are included in $x_k, \ldots, x_{k+m}$. Consider the formula $\theta(x_1)$ equal to

$$\exists x_2 \ldots \exists x_n \exists x_k \ldots \exists x_{k+m} (\psi_1 \wedge \cdots \wedge \psi_r)[c_{i_n}, \ldots, c_{i_1}; x_1, \ldots, x_n]).$$

Note that x_1 is the only variable which might occur freely in θ. Since Γ is not a principal type (and since Γ contains T), there is a formula $\sigma \in \Gamma$ such that

$$T \cup \{\theta\} \nvdash \sigma.$$

Thus, $T \cup \{\theta\} \cup \{(\neg\sigma)\}$ is consistent. We add $(\neg\sigma(x_1; c_{i_n}))$ to T^b_{n+1} and let T_{n+1} be the result, finishing the definition of T_{n+1}. The consistency of T_{n+1} follows from Theorem 5.13, the theorem on constants. See Exercise 1, on page 85.

Let T_∞ be the union of the T_n. T_∞ is complete, consistent, and has the Henkin property. Thus, the structure built from constants using T_∞ satisfies T_∞. Every element of this model is the interpretation of some constant symbol. If c_i appears in T, then since Γ is a not a principal type, there is an element σ of Γ such that $(\neg\sigma)[x_1; c_i]$ is an element of T and hence of T_∞. If c_i does not appear in T, then for some n, c_i is equal to c_{i_n} and we ensured the existence of such a σ during stage $n+1$ of the construction. Thus, the structure built from constants using T_∞ omits the type Γ, as required. □

6.2.1 ω-categorical theories

Lemma 6.14 *Suppose that T is a complete, consistent, theory. Suppose that $n \in \mathbb{N}$.*

Then the following conditions are equivalent.

(1) There are only finitely many n-types which contain T.

(2) Every n-type containing T is principal.

Proof. We first show that if there are infinitely many principal n-types which contain T, then there is a nonprincipal n-type which contains T.

Let $\langle \Gamma_k : k \in \mathbb{N}\rangle$ enumerate all principal n-types which contain T. For each $k \in \mathbb{N}$ let $\varphi_k(x_1, \ldots, x_n)$ be a formula in Γ_k such that

$$\Gamma_k = \{\psi(x_1, \ldots, x_n) \mid T \vdash (\varphi_k \rightarrow \psi)\}.$$

Thus for each $k \in \mathbb{N}$, if $i \neq k$ then

$$(\neg\varphi_k) \in \Gamma_i.$$

Let

$$\Sigma = T \cup \{(\neg\varphi_k) \mid k \in \mathbb{N}\}.$$

Suppose $S \subset \Sigma$ is finite. Then $S \subset \Gamma_i$ for all sufficiently large $i \in \mathbb{N}$. Therefore Σ is consistent. Let Γ be an n-type which contains Σ. Then $\Gamma \neq \Gamma_k$ for each $k \in \mathbb{N}$ and so Γ is not a principal type.

Thus if if there are infinitely many principal n-types which contain T, then there is a nonprincipal n-type which contains T.

Finally suppose there are only finitely many n-types which contain T. We prove that every n-type which contains T is principal.

Let $\langle \Gamma_1, \ldots, \Gamma_k \rangle$ enumerate all the n-types which contain T. For each $i, j \leq k$, if $i \neq j$ let $\varphi_i^j(x_1, \ldots, x_n) \in \Gamma_i$ be such that $\varphi_i^j \notin \Gamma_j$; and if $i = j$ let φ_i^i be the formula $(x_1 \hat{=} x_1)$ (i. e. a formula in Γ_i).

For each $i \leq k$ let

$$\varphi_i = (\varphi_i^1 \wedge \cdots \wedge \varphi_i^k)$$

It follows that for each $i \leq k$, $\varphi_i \in \Gamma_i$ and for each $j \neq i$, $(\neg\varphi_i) \in \Gamma_j$.

Fix $i \leq k$. We claim that

$$\Gamma_i = \{\varphi(x_1, \ldots, x_n) \mid T \vdash (\varphi_i \to \varphi)\}.$$

Suppose not. Then there is a formula $\varphi \in \Gamma_i$ such that $T \cup \{(\neg(\varphi_i \to \varphi))\}$ is consistent. Let Γ be an n-type which contains $T \cup \{(\neg(\varphi_i \to \varphi))\}$. Since

$$\vdash (\neg(\varphi_i \to \varphi)) \to (\neg\varphi),$$

necessarily, $(\neg\varphi) \in \Gamma$. Similarly $\varphi_i \in \Gamma$ since,

$$\vdash (\neg(\varphi_i \to \varphi)) \to \varphi_i.$$

This is a contradiction. Since $\varphi_i \in \Gamma$, $\Gamma \neq \Gamma_j$ for all $j \leq k$ with $j \neq i$. So $\Gamma = \Gamma_i$. But $(\neg\varphi) \in \Gamma$ and $\varphi \in \Gamma_i$.

Thus for each $i \leq k$,

$$\Gamma_i = \{\varphi(x_1, \ldots, x_n) \mid T \vdash (\varphi_i \to \varphi)\},$$

and so every n-type which contains T is principal. $\square$

Definition 6.15 Suppose that T is a complete, consistent, theory. T is *ω-categorical* if and only if any two countable models of T are isomorphic.

Lemma 6.16 *Suppose that T is a complete, consistent, theory. Suppose $n \in \mathbb{N}$ and that there is a nonprincipal n-type which contains T.*

Then T is not ω-categorical.

Proof. Let Γ be a nonprincipal n-type which contains T. By the Completeness Theorem, there is a countable structure $\mathcal{M}$ such that

$$\mathcal{M} \vDash T,$$

and such that $\mathcal{M}$ realizes Γ.

By the Omitting Types Theorem, Theorem 6.13, there is a countable structure $\mathcal{N}$ such that

$$\mathcal{N} \vDash T$$

and such that $\mathcal{N}$ omits Γ. Clearly $\mathcal{M}$ and $\mathcal{N}$ are not isomorphic. Thus T is not ω-categorical. $\square$

Theorem 6.17 (Ryll-Nardzewski) *Suppose that T is a complete, consistent, theory. The following conditions are equivalent.*

(1) T is ω-categorical.
(2) For all $n \in \mathbb{N}$, there are finitely many n-types.
(3) For every type Γ extending T, Γ is principal.

Proof. By Lemma 6.14 and Lemma 6.16, if T is ω-categorical then both (2) and (3) must hold.

We now suppose that (3) holds and prove that T is ω-categorical.

Let $\mathcal{M}_1 = (M_1, I_1)$ and $\mathcal{M}_2 = (M_2, I_2)$ be countable models of T. We build an isomorphism between $\mathcal{M}_1$ and $\mathcal{M}_2$ by a back-and-forth construction reminiscent of the proof that any two countable dense linear orders without endpoints are isomorphic.

Let $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ be enumerations of M_1 and M_2, respectively. We go by recursion to define $f : M_1 \to M_2$. In the recursion step, we will make use of the following lemma.

Lemma 6.18 *Suppose that $a_{i_1}, \ldots, a_{i_k}$ and $b_{j_1}, \ldots, b_{j_k}$ are given so that $(a_{i_1}, \ldots, a_{i_k})$ realizes the same type in $\mathcal{M}_1$ as $(b_{j_1}, \ldots, b_{j_k})$ does in $\mathcal{M}_2$. For any $a \in M_1$, there is a $b \in M_2$ such that $(a_{i_1}, \ldots, a_{i_k}, a)$ realizes the same type in $\mathcal{M}_1$ as $(b_{j_1}, \ldots, b_{j_k}, b)$ does in $\mathcal{M}_2$.*

Proof. Let Γ be the type of $(a_{i_1}, \ldots, a_{i_k})$ in $\mathcal{M}_1$ and Let Γ^+ be the type of $(a_{i_1}, \ldots, a_{i_k}, a)$ in $\mathcal{M}_1$. By assumption, every type extending T is principal. Fix $\psi(x_1, \ldots, x_{k+1}) \in \Gamma^+$ so that every element of Γ^+ is a consequence of $T \cup \{\psi\}$. Then, $\mathcal{M}_1 \vDash (\exists x_{k+1}\psi)[a_{i_1}, \ldots, a_{i_k}]$ and so $(\exists x_{k+1}\psi) \in \Gamma$. Since Γ is also the type of $(b_{j_1}, \ldots, b_{j_k})$ in $\mathcal{M}_2$, $\mathcal{M}_2 \vDash (\exists x_{k+1}\psi)[b_{j_1}, \ldots, b_{j_k}]$. Let b be an element of $\mathcal{M}_2$ such that $\mathcal{M}_2 \vDash \psi[b_{j_1}, \ldots, b_{j_k}, b]$. Since Γ^+ is the set of consequence of $T \cup \{\psi\}$, $(b_{j_1}, \ldots, b_{j_k}, b)$ realizes Γ^+ in $\mathcal{M}_2$, as required to prove the lemma. □

By symmetry, it is also the case that for each b in M_2 there is an a in M_1 as above.

Now we can define the isomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$. At step $n+1$, assume that there is a finite subset $\{a_{i_1}, \ldots, a_{i_k}\}$ of M_1 on which we have defined f so that $(a_{i_1}, \ldots, a_{i_k})$ realizes the same type in $\mathcal{M}_1$ that $(f(a_{i_1}), \ldots, f(a_{i_k}))$ realizes in $\mathcal{M}_2$. Using the previous lemma, we extend the definition of f to include a_n in its domain, and then to include b_n in its range, in each case preserving the property that the resulting sequences realize the same type in their respective models.

Let f be the function defined in the limit. M_1 is the domain of f and M_2 is its range. Additionally, f is an elementary function, since for each formula φ and each sequence from M_1, $\mathcal{M}_1 \vDash \varphi[a_{i_1}, \ldots, a_{i_k}]$, if and only if φ is in the type of $(a_{i_1}, \ldots, a_{i_k})$, if and only if φ is in the type of $(f(a_{i_1}), \ldots, f(a_{i_k}))$, if and only if $\mathcal{M}_2 \vDash \varphi[f(a_{i_1}), \ldots, f(a_{i_k})]$. Of course, any elementary surjection is an isomorphism, as required to conclude (1). □

6.2.2 Exercises

(1) In the proof of Theorem 6.13, complete the argument that T_{n+1} is consistent from the assumption that T^b_{n+1} is consistent.
(2) Suppose that $\mathcal{A}$ is finite and that $\mathcal{M}$ is a finite $\mathcal{L}$-structure. Show that every type realized in $\mathcal{M}$ is principal.
(3) Suppose that T is a consistent set of sentences such that (for some n) every n-type consistent with T is principal. Show that T has only finitely many consistent completions.
(4) Suppose that T is ω-categorical and M is a countable infinite model of T. Show that M has a nontrivial automorphism.

6.2.3 A theory with uncountably many models

In contrast with ω-categorical theories, there are natural theories which have not just infinitely many nonisomorphic countable models, but uncountably many. In fact, the theory of elementary arithmetic is one such.

Definition 6.19 Let $Th(\mathbb{N})$ denote the set of first order sentences satisfied by the natural numbers with constants for 0 and 1 and with binary function symbols for addition and multiplication.

Lemma 6.20 *There are uncountably many 1-types extending* $Th(\mathbb{N})$.

Proof. Let p_i denote the ith prime number greater than or equal to 2, and let τ_i denote the term

$$\tau = \underbrace{1 + 1 + \cdots + 1}_{p_i \text{ times}}$$

For each $X \subseteq \mathbb{N}$, let G_X be the set of formulas

$$\begin{aligned} &Th(\mathbb{N}) \cup \{\tau_i \text{ is a factor of } x_1 : i \in X\} \\ &\quad \cup \{\tau_i \text{ is not a factor of } x_1 : i \notin X\} \end{aligned}$$

For each X, let Γ_X be a type extending G_X. Since each of these types are consistent, no two of them are equal. Since there are uncountably many subsets of $\mathbb{N}$, the lemma follows. ◻

Corollary 6.21 *There are uncountably many distinct isomorphism types of countable models of* $Th(\mathbb{N})$.

Proof. Suppose that $\{\mathcal{M}_i : i \in \mathbb{N}\}$ is a countable set of countable models of $Th(\mathbb{N})$. We must show that there is a countable model of $Th(\mathbb{N})$ which is not isomorphic to any of the $\mathcal{M}_i$'s.

Each of the $\mathcal{M}_i$'s realizes only countably many 1-types. Consequently, the set of 1-types Γ such that Γ is realized in at least one of the $\mathcal{M}_i$'s is a countable set.

Since there are uncountably many 1-types extending $Th(\mathbb{N})$, there is a 1-type Δ which is not realized in any of the $\mathcal{M}_i$'s. By Theorem 6.9, let $\mathcal{M}$ be a countable model of $Th(\mathbb{N})$ such that $\mathcal{M}$ realizes Δ. $\mathcal{M}$ is the required model not isomorphic to any of the $\mathcal{M}_i$'s. □

6.3 Vaught's conjecture

After 86 pages of defintions and theorems, it might seem that first order logic is well understood. However, there are fundamental questions which remain unsolved. For example, the following is still open.

Conjecture 6.22 (Vaught) *Suppose that T is a complete set of sentences. Show that one of the following two conditions holds.*

(1) There is a countable set of models $\{\mathcal{M}_i : i \in \mathbb{N}\}$ such that for every countable $\mathcal{M}$, if $\mathcal{M} \models T$, then there is an i such that $\mathcal{M}$ is isomorphic to $\mathcal{M}_i$.

(2) There is a set of models $\{\mathcal{M}_X : X \subseteq \mathbb{N}\}$ such that each $\mathcal{M}_X$ satisfies T, and if $X \neq Y$, then $\mathcal{M}_X$ is not isomorphic to $\mathcal{M}_Y$.