# Mathematical Logic
(The Berkeley undergraduate course)

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# Contents

# 1
# Propositional logic

Propositional logic governs the way by which propositions are combined in compound sentences.

Informally, a proposition is a declarative sentence, such as any of the following.

- Life is nothing but a competition to be the criminal rather than the victim. (B. Russell)
- Life is as tedious as a twice-told tale. (W. Shakespeare)
- Life is a dead-end street. (H. L. Menken)
- Life is too short to learn German. (R. Porson)

Propositions may be combined by logical connectives to form more complicated statements, such as “If life is a dead-end street, then life is too short to learn German” or “If it is not the case that life is too short to learn German, then life is as tedious as a twice told tale”. The truth or falsity of a compound statement depends solely on that of its parts. Understanding propositional logic is just understanding that dependence.

In the next section, we will introduce symbols $A_n$ for propositions, $\neg$ for negation, and $\rightarrow$ for implication. Using $A_1, \ldots, A_4$ to denote the above propositions, our two compound sentences would be denoted by $(A_3 \rightarrow A_4)$ and $((\neg A_4) \rightarrow A_2)$.

## 1.1 The language

Our language for propositional logic consists of (certain) finite sequences of symbols.

**Definition 1.1** • The *logical symbols* are the following symbols.

$$( \quad ) \quad \neg \quad \rightarrow$$

- The *propositional symbols* are $A_n$, for $n$ in $\mathbb{N}$. ($\mathbb{N}$ is the set of non-negative integers; i.e. the set of *natural numbers*)

**Definition 1.2** If $s = \langle s_1, \ldots, s_n \rangle$ and $t = \langle t_1, \ldots, t_m \rangle$ are finite sequences, we let $s + t$ denote the finite sequence

$$s + t = \langle s_1, \ldots, s_n, t_1, \ldots, t_m \rangle.$$

**Definition 1.3** The *propositional language* $\mathcal{L}_0$ is the smallest set $L$ of finite sequences of the above symbols satisfying the following properties.

(1) For each propositional symbol $A_n$ with $n \in \mathbb{N}$,

$$A_n \in L.$$

(2) For each pair of finite sequences $s$ and $t$, if $s$ and $t$ belong to $L$, then

$$(\neg s) \in L$$

and

$$(s \to t) \in L.$$

For the duration of Chapter 1, we will use *propositional formula* or just *formula* to refer to an element of $\mathcal{L}_0$.

In Definition 1.3, we defined the propositional language $\mathcal{L}_0$ as the smallest set which is closed under Conditions 1 and 2. In the following, we show that $\mathcal{L}_0$ is well defined.

**Theorem 1.4** *$\mathcal{L}_0$ is the intersection of all of the sets which satisfy the two conditions of Definition 1.3.*

*Proof.* Let $L_0$ be the intersection of all of the sets which satisfy the two conditions of Definition 1.3. There is at least one such set, since the set of all finite sequences of symbols does satisfy the two conditions. We claim that $L_0$ is a set which satisfies those two conditions.

For each $n \in \mathbb{N}$, $A_n$ is an element of every set which satisfies Condition 1. Consequently, $A_n$ is an element of the intersection of all such sets, and thus it is an element of $L_0$.

Now, suppose that $s$ and $t$ belong to $L_0$. Then they belong to every set which satisfies Conditions 1 and 2. But then, for every such set, we can apply Condition 2 to conclude that $(\neg s)$ and $(s \to t)$ also belong to that set. Therefore, $(\neg s)$ and $(s \to t)$ belong to the intersection of all such sets, and thus belong to $L_0$.

Hence, $L_0$ satisfies Conditions 1 and 2. Since it is contained in every set which also satisfies those conditions, it must the smallest such set. Consequently, $\mathcal{L}_0$ is equal to $L_0$. □

### 1.1.1 Subformulas

**Definition 1.5** Suppose that $s = \langle s_1, \ldots, s_n \rangle$ is a finite sequence. A finite sequence $t$ is a *block-subsequence* of $s$ if there exist non-negative integers $i$ and $j$ such that

(1) $i + j \leq n$,
(2) $t = \langle s_i, s_{i+1}, \ldots, s_{i+j} \rangle$.

**Example 1.6** (1) $\langle 3 \rangle$ is a block-subsequence of $\langle 1, 2, 3, 4, 5, 6 \rangle$.

(2) $\langle 3,4,5\rangle$ is a block-subsequence of $\langle 1,2,3,4,5,6\rangle$.
(3) $\langle 1,6\rangle$ is not a block-subsequence of $\langle 1,2,3,4,5,6\rangle$.
(4) If $s$ is a finite sequence and $s$ has length $n$, then there are at most

$$\sum_{i=1}^{n}(n-i)+1 = n^2+n-(1/2)n(n+1) = 1/2(n+1)n$$

block-subsequences of $s$.

**Definition 1.7** (1) A sequence $\langle a_1,\ldots,a_k\rangle$ is an *initial segment* of another sequence $\langle b_1,\ldots,b_m\rangle$ if and only if $k$ is less than or equal to $m$ and for all $i\leq k$, $a_i=b_i$. In other words, $\langle b_1,\ldots,b_m\rangle$ is equal to $\langle a_1,\ldots,a_k\rangle+\langle b_{k+1},\ldots,b_m\rangle$, where $\langle b_{k+1},\ldots,b_m\rangle$ could be the empty sequence.
(2) When $m$ is greater than $k$, we say that $\langle a_1,\ldots,a_k\rangle$ is a *proper initial segment* of $\langle a_1,\ldots,a_k,b_{k+1},\ldots,b_m\rangle$.

**Definition 1.8** Suppose that $\varphi$ is a formula. A formula $\psi$ is a *subformula* of $\varphi$ if $\psi$ is a block-subsequence of $\varphi$.

The subformulas of $\varphi$ are precisely the formulas which arise in the definition of $\varphi$.

**Definition 1.9** Suppose that

$$\varphi=\langle a_1,\ldots,a_n\rangle$$

is a formula.

Suppose $s$ is a finite sequence. An *occurrence* of $s$ in $\varphi$ is an interval

$$[j_1,j_2]$$

such that $s=\langle a_{j_1},\ldots,a_{j_2}\rangle$.

**Remark 1.10** Suppose that $\varphi$ is a formula. A formula $\psi$ is a subformula of $\varphi$ if and only if $\psi$ has an occurrence in $\varphi$.

**Lemma 1.11 (Readability)** *Suppose that $\varphi$ is a formula in $\mathcal{L}_0$. Then exactly one of the following conditions applies.*
*(1) There is an $n$ such that $\varphi=\langle A_n\rangle$.*
*(2) There is a $\psi\in\mathcal{L}_0$ such that $\varphi=(\neg\psi)$.*
*(3) There are $\psi_1$ and $\psi_2$ in $\mathcal{L}_0$ such that $\varphi=(\psi_1\rightarrow\psi_2)$.*

*Proof.* Consider the subset $L$ of $\mathcal{L}_0$ which consists of those formulas which satisfy the above three clauses. By the first clause, if $n\in\mathbb{N}$, then $\langle A_n\rangle\in L$. Consequently, $L$ satisfies Condition 1 of Definition 1.3. Secondly, if $\psi$ is in $L$, then $\psi\in\mathcal{L}_0$ and so $(\neg\psi)\in\mathcal{L}_0$. But then $(\neg\psi)$ is an element of $\mathcal{L}_0$ which satisfies the second of the above clauses, and hence $(\neg\psi) \in L$. Similarly, if $\psi_1$ and $\psi_2$ belong to $L$, then so does $(\psi_1 \to \psi_2)$. Thus, $L$ satisfies Condition 2 of Definition 1.3. It follows that $\mathcal{L}_0 \subseteq L$, and so $\mathcal{L}_0 = L$.

It remains to show that the three possibilities are mutually exclusive.

Clearly, the first case excludes the other two since both of the formulas in the latter two cases begin with the symbol (. Now, if $\varphi = (\neg\psi)$, then the second symbol in $\varphi$ is $\neg$. However, if $\varphi = (\psi_1 \to \psi_2)$, then the second symbol in $\varphi$ is the first symbol in $\psi_1$, which by the above is either an $A_n$ or (. Consequently, these two cases are mutually exclusive. $\square$

We now prove a technical lemma which we will apply to show that the subformulas of $\varphi$ mentioned above are uniquely determined.

**Lemma 1.12** *If $\varphi \in \mathcal{L}_0$, then no proper initial segment of $\varphi$ is an element of $\mathcal{L}_0$.*

*Proof.* We prove Lemma 1.12 by induction on $n$.

If $\varphi$ has length 1 then the only subsequence to be considered is the empty sequence, which by Lemma 1.11 is not an element of $\mathcal{L}_0$.

Now suppose that $\varphi \in \mathcal{L}_0$ has length $n$, $n > 1$, and Lemma 1.12 holds for all $m$ less than $n$. By Lemma 1.11, since $\varphi$ has length greater than 1, $\varphi$ has one of two forms: $(\neg\psi)$ or $(\psi_1 \to \psi_2)$.

Suppose that $\varphi$ is $(\neg\psi)$. For a contradiction, suppose that $\theta \in \mathcal{L}_0$ is a proper initial segment of $(\neg\psi)$. Then the first symbol in $\theta$ is (, so $\theta$ is not of the form $\langle A_i \rangle$, and by Lemma 1.11 the length of $\theta$ is greater than one. Thus, the second symbol in $\theta$ is $\neg$, which by Lemma 1.11 is not the first symbol of any element of $\mathcal{L}_0$, and so $\theta$ cannot be of the form $(\theta_1 \to \theta_2)$. Consequently, there is a $\theta_1 \in \mathcal{L}_0$ such that $\theta$ is equal to $(\neg\theta_1)$. But then $(\neg\psi)$ has $(\neg\theta_1)$ as a proper initial segment, and so $\psi$ has $\theta_1$ as a proper initial segment, contradiction to the induction hypothesis.

Finally, suppose that $\varphi$ is $(\psi_1 \to \psi_2)$ and that $\theta \in \mathcal{L}_0$ is a proper initial segment of $\varphi$. We can apply Lemma 1.11 and argue as in the previous paragraph that there are $\theta_1$ and $\theta_2$ in $\mathcal{L}_0$ such that $\theta = (\theta_1 \to \theta_2)$. But then either $\theta_1$ is a proper initial segment of $\psi_1$ (a contradiction), $\psi_1$ is a proper initial segment of $\theta_1$ (a contradiction), or $\psi_1 = \theta_1$ and $\theta_2$ is a proper initial segment of $\psi_2$ (a contradiction).

In either case, $\varphi$ has no proper initial segment in $\mathcal{L}_0$. $\square$

**Theorem 1.13 (Unique Readability)** *Suppose that $\varphi$ is a formula in $\mathcal{L}_0$. Then exactly one of the following conditions applies.*

*(1) There is an $n$ such that $\varphi = \langle A_n \rangle$.*

*(2) There is a $\psi \in \mathcal{L}_0$ such that $\varphi = (\neg\psi)$.*

*(3) There are $\psi_1$ and $\psi_2$ in $\mathcal{L}_0$ such that $\varphi = (\psi_1 \to \psi_2)$.*

*Further, in cases (2) and (3), the formulas $\psi$, and $\psi_1$ and $\psi_2$ are unique, respectively.*

*Proof.* By Lemma 1.11, it is enough to check the claim of uniqueness.

First, suppose that $\varphi = (\neg\psi)$ and $\varphi = (\neg\theta)$. Thus, the sequence of symbols $\varphi$ can be read as $\langle (, \neg \rangle + \psi + \langle ) \rangle$ and as $\langle (, \neg \rangle + \theta + \langle ) \rangle$. The occurrences of $\psi$ and $\theta$ within $\varphi$ have the same length and the same elements, and therefore are equal.

Finally, suppose that $\varphi = (\psi_1 \to \psi_2)$ and $\varphi = (\theta_1 \to \theta_2)$. Since both $\psi_1$ and $\theta_1$ belong to $\mathcal{L}_0$, by Lemma 1.12 neither can be a proper initial segment of the other. Since they are initial segments of each other, they must be equal. As in the case of negation, it follows that $\psi_2$ and $\theta_2$ are also equal. □

### 1.1.2 Exercises

(1) Give three examples of elements of $\mathcal{L}_0$ with at least 15 symbols each. The examples should have an interesting structure. For one of these examples, give a meaningful sentence which has the same structure.

(2) For which natural numbers $n$ are there elements of $\mathcal{L}_0$ of length $n$?

(3) Show that a sequence $\varphi$ is an element of $\mathcal{L}_0$ if and only if there is a finite sequence of sequences $\langle \varphi_1, \dots, \varphi_n \rangle$ such that $\varphi_n = \varphi$, and for each if $i$ less than or equal to $n$ either there is an $m$ such that $\varphi_i = \langle A_m \rangle$, or there is a $j$ less than $i$ such that $\varphi_i$ is equal to $(\neg\varphi_j)$, or there are $j_1$ and $j_2$ less than $i$ such that $\varphi_i$ is equal to $(\varphi_{j_1} \to \varphi_{j_2})$.

(4) Give an algorithm (suitable to be programmed for a computer) to determine whether a given finite sequence belongs to $\mathcal{L}_0$.

(5) Consider the set of symbols $*$ and $\#$. Let $\mathcal{L}^*$ be the smallest set $L$ of sequences of these symbols with the following properties.

a) The length one sequences $\langle * \rangle$ and $\langle \# \rangle$ belong to $L$.

b) If $\sigma$ and $\tau$ belong to $L$, then so do $\langle * \rangle + \sigma + \langle \# \rangle$ and $\langle * \rangle + \sigma + \tau + \langle \# \rangle$.

State Readability and Unique Readability for $\mathcal{L}^*$ and determine for each whether it holds.

(6) (Polish Notation) Let $\mathcal{P}_0$ be the smallest set of sequences $P$ such that the following conditions hold.

a) For each $n$, $\langle A_n \rangle \in P$.

b) If $\psi_1$ and $\psi_2$ belong to $P$, then so do $\neg\psi_1 = \langle \neg \rangle + \psi_1$ and $\to\psi_1\psi_2 = \langle \to \rangle + \psi_1 + \psi_2$.

State and prove the unique readability theorem for $\mathcal{P}_0$. Note, the Polish system of notation does away with parentheses.

## 1.2 Truth assignments

We can now describe the semantics for propositional logic.

**Definition 1.14** A *truth assignment for* $\mathcal{L}_0$ is a function $\nu$ from the set of propositional symbols $\{A_n : n \in \mathbb{N}\}$ into the set $\{T, F\}$.

Now, $(\neg\psi)$ should have the opposite truth value from that of $\psi$ and the truth value of $(\psi_1 \to \psi_2)$ should reflect whether, if $\psi_1$ has truth value $T$, then $\psi_2$ has truth value $T$.

**Theorem 1.15** *Suppose that $\nu$ is a truth assignment for $\mathcal{L}_0$. Then there is a unique function $\overline{\nu}$ defined on $\mathcal{L}_0$ with the following properties.*

*(1) For all $n$, $\overline{\nu}(\langle A_n\rangle) = \nu(A_n)$.*

*(2) For all $\psi \in \mathcal{L}_0$,*

$$\overline{\nu}((\neg\psi)) = \begin{cases} T, & \textit{if } \overline{\nu}(\psi) = F; \\ F, & \textit{otherwise.} \end{cases}$$

*(3) For all $\psi_1$ and $\psi_2$ in $\mathcal{L}_0$,*

$$\overline{\nu}((\psi_1 \to \psi_2)) = \begin{cases} F, & \textit{if } \overline{\nu}(\psi_1) = T \textit{ and } \overline{\nu}(\psi_2) = F; \\ T, & \textit{otherwise.} \end{cases}$$

*Proof.* We can define $\overline{\nu}$ by recursion on the natural numbers greater than or equal to 1. During the $s+1$st step of the recursion, we may assume that $\overline{\nu}$ is already defined on all elements of $\mathcal{L}_0$ which have length less than or equal to $s$.

**Base step.** For each $n \in \mathbb{N}$, define $\overline{\nu}(\langle A_n\rangle) = \nu(A_n)$.

**Recursion step.** Suppose that $s \geq 1$, that $\overline{\nu}$ is defined on all sequences from $\mathcal{L}_0$ of length less than or equal to $s$, and that $\varphi$ is an element of $\mathcal{L}_0$ of length $s+1$.

If $\varphi = (\neg\psi)$, we define $\overline{\nu}(\varphi)$ as in (2); if $\varphi = (\psi_1 \to \psi_2)$, we define $\overline{\nu}(\varphi)$ as in (3).

It remains to show that $\overline{\nu}$ is well defined on $\mathcal{L}_0$, it satisfies (1), (2), and (3), and that it is the unique such function—existence and uniqueness.

We will prove the first two claims (existence) by induction on the natural numbers greater than or equal to 1.

Clearly, $\overline{\nu}$ is well defined on the elements of $\mathcal{L}_0$ of length 1 and satisfies (1).

Suppose that $s \geq 1$ and, by induction, that $\overline{\nu}$ is well defined on the set of elements of $\mathcal{L}_0$ of length less than or equal to $s$. Suppose that $\varphi \in \mathcal{L}_0$ and $\varphi$ has length $s+1$. By the Unique Readability Theorem 1.13, either there is a $\psi \in \mathcal{L}_0$ such that $\varphi = (\neg\psi)$, or there are $\psi_1$ and $\psi_2$ such that $\varphi = (\psi_1 \to \psi_2)$, the two cases are mutually exclusive, and, in either case, the subformulas of $\varphi$ which appear in the way described are unique. Thus, the condition to define $\overline{\nu}(\varphi)$ is unambiguous, showing that $\overline{\nu}$ is well defined at $\varphi$. Further, $\overline{\nu}$ is defined at $\varphi$ so as to satisfy whichever of (2) or (3) is relevant.

By induction, $\overline{\nu}$ is well defined on $\mathcal{L}_0$ and satisfies (1), (2), and (3).

Now, we verify uniqueness. Suppose that $\overline{\nu}^* : \mathcal{L}_0 \to \{T, F\}$ and satisfies (1), (2), and (3). For the sake of a contradiction, suppose that $\overline{\nu}^*$ is not equal to $\overline{\nu}$. Fix $\varphi$ so that $\overline{\nu}^*(\varphi) \neq \overline{\nu}(\varphi)$ and so that there is no $\psi \in \mathcal{L}_0$ such that $\psi$ is strictly shorter than $\varphi$ and $\overline{\nu}^*(\psi) \neq \overline{\nu}(\psi)$.

Since $\overline{\nu}^*$ satisfies (1), for every $n$, $\overline{\nu}^*(\langle A_n\rangle) = \nu(A_n)$. By definition, $\overline{\nu}^(\langle A_n\rangle) = \nu(A_n)$. Hence, for every $n$, $\overline{\nu}^*(\langle A_n\rangle) = \overline{\nu}(\langle A_n\rangle)$.

Consequently, the length of $\varphi$ must be greater than 1. By the Unique Readability Theorem 1.13, $\varphi$ is either a negation $(\neg\psi)$ or an implication $(\psi_1 \to \psi_2)$

and uniquely so. In the first case, since $\overline{\nu}^*$ satisfies (2), $\overline{\nu}^*((\neg\psi))$ has the opposite value from $\overline{\nu}^*(\psi)$. Since $\psi$ is shorter than $\varphi$, $\overline{\nu}^*(\psi) = \overline{\nu}(\psi)$. By definition, $\overline{\nu}((\neg\psi))$ has the opposite value from $\overline{\nu}(\psi)$. It follows that $\overline{\nu}^*((\neg\psi))$ and $\overline{\nu}((\neg\psi))$ are equal. Thus, $\varphi$ cannot be a negation. An analogous argument shows that $\varphi$ cannot be an implication. This is a contradiction to the Readability Lemma 1.11. Consequently, $\overline{\nu}^*$ is equal to $\overline{\nu}$, which completes the proof of the theorem. □

**Theorem 1.16** *Suppose that $\varphi \in \mathcal{L}_0$ and that $\nu$ and $\mu$ are truth assignments which agree on the propositional symbols which occur in $\varphi$. Then $\overline{\nu}(\varphi) = \overline{\mu}(\varphi)$.*

*Proof.* Proceed just as in the uniqueness part of the proof of Theorem 1.13. Show that there cannot be a shortest subformula of $\varphi$ where $\overline{\nu}$ and $\overline{\mu}$ disagree. □

**Definition 1.17** (1) A truth assignment $\nu$ *satisfies* a formula $\varphi$ if and only if $\overline{\nu}(\varphi) = T$. Similarly, $\nu$ *satisfies* a set of formulas $\Gamma$ if and only if it satisfies all of the elements of $\Gamma$.

(2) $\varphi$ is a *tautology* if and only if every truth assignment satisfies $\varphi$.

(3) $\varphi \in \mathcal{L}_0$ or $\Gamma \subset \mathcal{L}_0$ are *satisfiable* if and only if there is a truth assignment which satisfies $\varphi$ or $\Gamma$, respectively.

(4) $\varphi$ is a *contradiction* if and only if there is no truth assignment which satisfies $\varphi$.

To give an example, consider the formula $(\neg((\neg A_1) \to A_2))$ and a truth assignment $\nu$ such that $\nu(A_1) = \nu(A_2) = F$. By Theorem 1.16, the values of $\nu$ on $A_1$ and $A_2$ determine the value of $\overline{\nu}$ on $(\neg((\neg A_1) \to A_2))$. In Figure 1.1, we show the values of $\overline{\nu}$ on $(\neg((\neg A_1) \to A_2))$ and its subformulas.

| $A_1$ | $A_2$ | $(\neg A_1)$ | $((\neg A_1) \to A_2)$ | $(\neg((\neg A_1) \to A_2))$ |
|---|---|---|---|---|
| $F$ | $F$ | $T$ | $F$ | $T$ |

**Fig. 1.1** Extending a truth assignment

We can expand the table to systematically examine all possible truth assignments on $(\neg((\neg A_1) \to A_2))$, as in Figure 1.2.

| $A_1$ | $A_2$ | $(\neg A_1)$ | $((\neg A_1) \to A_2)$ | $(\neg((\neg A_1) \to A_2))$ |
|---|---|---|---|---|
| $T$ | $T$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $T$ |

**Fig. 1.2** The truth table for $(\neg((\neg A_1) \to A_2))$

Truth tables, such as the one in Figure 1.2, provide a systematic method to examine all the possible truth assignments for a given formula. Given a formula $\varphi$, we generate a truth table for $\varphi$ as follows.

(1) The top row of the table consists of a list $\psi_1, \psi_2, \ldots, \psi_n = \varphi$ consisting of the subformulas of $\varphi$, ordered from left to right as follows.
   a) The subformulas of $\varphi$ of the form $\langle A_m \rangle$ appear in the list without repetition before any of the other subformulas of $\varphi$.
   b) For each $i \leq n$ all of the proper subformulas of $\psi_i$ appear in the list $\psi_1, \psi_2, \ldots, \psi_{i-1}$.
   c) The last element of the list is $\varphi$.
(2) Letting $k$ be the number of subformulas of $\varphi$ of the form $\langle A_m \rangle$, we consider all of the $2^k$ possible truth assignments for their propositional symbols. We use a row in the table for each such truth assignment $\nu$, and we fill in the cell below $\langle A_m \rangle$ in that row with the value of $\nu$ at $A_m$.
(3) Finally, we work our way across each row and fill in the values of $\overline{\nu}$ at $\psi_i$ as determined by the values already filled in for its subformulas.

We give another example in Figure 1.3. This time we have chosen the tautology expressing the principle that if $A_1$ implies $A_2$, then the contrapositive implication from $(\neg A_2)$ to $(\neg A_1)$ also holds.

| $A_1$ | $A_2$ | $(A_1 \to A_2)$ | $(\neg A_1)$ | $(\neg A_2)$ | $((\neg A_2) \to (\neg A_1))$ | $((A_1 \to A_2) \to ((\neg A_2) \to (\neg A_1)))$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |

**Fig. 1.3** The truth table for $((A_1 \to A_2) \to ((\neg A_2) \to (\neg A_1)))$

**Theorem 1.18** *There are algorithms to determine whether a propositional formula $\varphi$ is a tautology, satisfiable, or a contradiction.*

*Proof.* Starting with a formula $\varphi$, we can systematically generate its truth table. Then $\varphi$ is a tautology if and only if every entry in the last column of its truth table is equal to $T$. It is satisfiable if and only if there is an entry in the last column of its truth table which is equal to $T$. It is a contradiction if and only if every entry in the last column of its truth table is equal to $F$.

**Remark 1.19** Roughly speaking, if $\varphi$ has $n$ many symbols, then the analysis of $\varphi$ by the method of truth tables involves $2^n$ many steps. A question which has received a considerable amount of attention is whether there is a more efficient method which when given $\varphi$ determines whether $\varphi$ is satisfiable. For more information on this problem, known as the $P = NP$ problem, and even a cash prize, see the following web site.

*http://www.claymath.org/millennium/P_vs_NP/*

### 1.2.1 Truth functions

**Definition 1.20** An $n$-place *truth function* is a function whose domain is the set of sequences of $T$'s and $F$'s of length $n$, written $\{T, F\}^n$ and whose range is contained in $\{T, F\}$.

If $\varphi$ is a formula in $\mathcal{L}_0$ and the propositional symbols which occur in $\varphi$ are contained in the set $\{A_0, \dots, A_{n-1}\}$, then we can define the truth function $f_\varphi$ derived from $\varphi$. Given $\sigma \in \{T, F\}^n$, we let $\nu$ be the truth assignment on $\{A_0, \dots, A_{n-1}\}$ such that $\nu(A_{i-1})$ is equal to the $i$th element of $\sigma$, and we define $f_\varphi(\sigma)$ to be $\overline{\nu}(\varphi)$.

In the next theorem, we show that $\mathcal{L}_0$ is as expressive as is possible. By this, we mean that every truth function is represented by a formula in $\mathcal{L}_0$.

**Theorem 1.21** *Suppose that $f : \{T, F\}^n \to \{T, F\}$ is a truth function. Then there is a formula $\varphi$ such that $f_\varphi = f$.*

*Proof.* We build up to the formula $\varphi$ by a sequence of smaller steps.

For $\sigma \in \{T, F\}^n$, define $\theta_{\sigma,i}$ so that

$$\theta_{\sigma,i} = \begin{cases} A_{i-1}, & \text{if } \sigma(i) = T; \\ (\neg A_{i-1}), & \text{if } \sigma(i) = F. \end{cases}$$

Given two formulas $\psi_1$ and $\psi_2$, we define the conjunction of $\psi_1$ and $\psi_2$ to be the formula $(\neg(\psi_1 \to (\neg\psi_2)))$. As is seen in Figure 1.2.1, a truth assignment satisfies the conjunction of $\psi_1$ and $\psi_2$ if and only if it satisfies both $\psi_1$ and $\psi_2$.

| $\psi_1$ | $\psi_2$ | $(\neg\psi_2)$ | $(\psi_1 \to (\neg\psi_2))$ | $(\neg(\psi_1 \to (\neg\psi_2)))$ |
|---|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $F$ |
| $F$ | $T$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $F$ |

**Fig. 1.4** The conjunction of $\psi_1$ and $\psi_2$.

Given more than two formulas $\psi_1, \dots, \psi_n$, we use recursion and define their conjunction to be the conjunction of $\psi_1$ with the conjunction of $\psi_2, \dots, \psi_n$. For example, the conjunction of $\psi_1$, $\psi_2$, and $\psi_3$ is the formula

$$(\neg(\psi_1 \to (\neg(\neg(\psi_2 \to (\neg\psi_3)))))).$$

By induction, if $\nu$ is a truth assignment, then $\overline{\nu}$ maps the conjunction of $\psi_1, \dots, \psi_n$ to $T$ if and only if $\overline{\nu}$ maps each of $\psi_1, \dots, \psi_n$ to $T$.

For $\sigma \in \{T, F\}^n$, we let $\psi_\sigma$ be the conjunction of the formulas $\theta_{\sigma,i}$ for $i$ less than or equal to $n$. The only truth assignments that satisfy $\psi_\sigma$ are those which assign $\sigma(i)$ to $A_{i-1}$.

Given two formulas $\psi_1$ and $\psi_2$, we define the disjunction of $\psi_1$ and $\psi_2$ to be the formula $((\neg\psi_1) \to \psi_2)$. As is seen in Figure 1.2.1, a truth assignment satisfies

| $\psi_1$ | $\psi_2$ | $(\neg\psi_1)$ | $((\neg\psi_1) \to \psi_2)$ |
|---|---|---|---|
| $T$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $F$ |

**Fig. 1.5** The disjunction of $\psi_1$ and $\psi_2$.

the conjunction of $\psi_1$ and $\psi_2$ if and only if it satisfies at least one of $\psi_1$ or $\psi_2$. As above, when $n$ is greater than two, we define the disjunction of $\psi_1, \ldots, \psi_n$ to be the disjunction of $\psi_1$ with the disjunction of $\psi_2, \ldots, \psi_n$. By another induction, if $\nu$ is a truth assignment, then $\overline{\nu}$ maps the disjunction of $\psi_1, \ldots, \psi_n$ to $T$ if and only if it maps at least one of $\psi_1, \ldots, \psi_n$ to $T$.

Now we let $\varphi_f$ be the disjunction of the set of formulas $\psi_\sigma$ for which $f(\sigma) = T$. By construction, if $\nu$ is a truth assignment that satisfies $\varphi_f$, then there is a $\sigma$ such that $f(\sigma) = T$ and for all $i$ less than or equal to $n$, $\nu(A_{i-1})$ is equal to the $i$th element of $\sigma$. Consequently, $f$ is equal to $f_{\varphi_f}$, as required. $\square$

**Remark 1.22** It is not unusual to include symbols $\wedge$ for conjunction, $\vee$ for disjunction, and $\leftrightarrow$ for "if and only if". By Theorem 1.21, these and all other logical connectives can be expressed in the language with only $\neg$ and $\to$.

Of course, the fewer symbols there are in the language, the fewer the number of cases there are in proofs by induction, so we decided in favor a small number of logical symbols. Occasionally, we pay a price for that decision: for example, with the lengths of the formulas that appeared in the proof of Theorem 1.21.

**Remark 1.23** In some applications, it important to the best possible representative of a truth function $f$. Best possible could mean having the shortest length or having the fewest logical connectives of a certain type. When $n$ is large, it is computationally prohibitive to generate the truth tables for all of the possible formulas with the desired functionality. Finding the optimal $\varphi$ for a specified $f$ remains an interesting problem.

## 1.3 A proof system for $\mathcal{L}_0$

Suppose that $\Gamma$ is a subset of $\mathcal{L}_0$ so that $\Gamma$ is a set of propositional formulas. We shall define a formal notion of proof. Intuitively a proof from $\Gamma$ will be a finite sequence,

$$\langle \varphi_1, \ldots, \varphi_n \rangle$$

of propositional formulas which satisfies certain conditions. In order to make the definition precise we need to first define the set of *Logical Axioms.*

**Definition 1.24** Suppose that $\varphi_1$, $\varphi_2$ and $\varphi_3$ are propositional formulas. Then each of the following propositional formulas is a logical axiom:
(Group I axioms)

(1) $((\varphi_1 \to (\varphi_2 \to \varphi_3)) \to ((\varphi_1 \to \varphi_2) \to (\varphi_1 \to \varphi_3)))$
(2) $(\varphi_1 \to \varphi_1)$
(3) $(\varphi_1 \to (\varphi_2 \to \varphi_1))$

(Group II axioms)

(1) $(\varphi_1 \to ((\neg\varphi_1) \to \varphi_2))$

(Group III axioms)

(1) $(((\neg\varphi_1) \to \varphi_1) \to \varphi_1)$

(Group IV axioms)

(1) $((\neg\varphi_1) \to (\varphi_1 \to \varphi_2))$
(2) $(\varphi_1 \to ((\neg\varphi_2) \to (\neg(\varphi_1 \to \varphi_2))))$ □

It is easily verified that each logical axiom is a tautology.

**Definition 1.25** Suppose that $\Gamma \subseteq \mathcal{L}_0$.

(1) Suppose that

$$s = \langle \varphi_1, \dots, \varphi_n \rangle$$

is a finite sequence of propositional formulas. The finite sequence $s$ is a $\Gamma$-proof if for each $i \leq n$ at least one of

a) $\varphi_i \in \Gamma$; or
b) $\varphi_i$ is a logical axiom; or
c) there exist $j_1 < i$ and $j_2 < i$ such that

$$\varphi_{j_2} = (\varphi_{j_1} \to \varphi_i).$$

(2) $\Gamma \vdash \varphi$ if and only if there exists a finite sequence

$$s = \langle \varphi_1, \dots, \varphi_n \rangle$$

such that $s$ is a $\Gamma$-proof and such that $\varphi_n = \varphi$. □

Notice that if $s = \langle \varphi_1, \dots, \varphi_n \rangle$ is a $\Gamma$-proof and if $t = \langle \psi_1, \dots, \psi_m \rangle$ is a $\Gamma$-proof then so is $s + t = \langle \varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_m \rangle$.

We shall prove a sequence of simple lemmas about the proof system. For each lemma we shall note which logical axioms are actually used.

The first lemma, which concerns inference, requires no logical axioms whatsoever.

**Lemma 1.26 (Inference)** *Suppose that $\Gamma \subseteq \mathcal{L}_0$, $\varphi$ is a formula and that $\psi$ is a formula. Suppose that $\Gamma \vdash \psi$ and that $\Gamma \vdash (\psi \to \varphi)$.*

*Then $\Gamma \vdash \varphi$.*

*Proof.* Let $\langle \varphi_1, \dots, \varphi_n \rangle$ be a $\Gamma$-proof of $\psi$, thus $\varphi_n = \psi$. Let $\langle \psi_1, \dots, \psi_m \rangle$ be a $\Gamma$-proof of $(\psi \to \varphi)$. Then, $\langle \varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_m, \varphi \rangle$ is a $\Gamma$-proof and $\Gamma \vdash \varphi$. □

The second lemma of our series is the Soundness Lemma, this also is independent of the choice of logical axioms, provided that every logical axiom is a tautology.

**Lemma 1.27 (Soundness)** *Suppose that* $\Gamma \subseteq \mathcal{L}_0$*,* $\varphi$ *is a formula and that* $\Gamma \vdash \varphi$*. Suppose that*

$$\nu : \{A_1, \ldots, A_n, \ldots\} \to \{T, F\}$$

*is a truth assignment such that* $\overline{\nu}(\psi) = T$ *for all* $\psi \in \Gamma$*.*

*Then* $\overline{\nu}(\varphi) = T$*.*

*Proof.* We leave the proof to the reader, but one argues by induction on the length of the $\Gamma$-proof.

The next lemma is the Deduction Lemma. This lemma requires the logical axioms from Group I.

**Lemma 1.28 (Deduction)** *Suppose that* $\Gamma \subseteq \mathcal{L}_0$*,* $\varphi$ *is a formula,* $\psi$ *is a formula and*

$$\Gamma \cup \{\varphi\} \vdash \psi.$$

*Then* $\Gamma \vdash (\varphi \to \psi)$*.*

*Proof.* Let

$$\langle \psi_1, \ldots, \psi_n \rangle$$

be a $(\Gamma \cup \{\varphi\})$-proof of $\psi$. We prove by induction on $i \leq n$ that

$$\Gamma \vdash (\varphi \to \psi_i).$$

First we consider the case $i = 1$. Either $\psi_1 \in \Gamma \cup \{\varphi\}$ or $\psi_1$ is a logical axiom (possibly both). So there are three subcases of this case.

**Subcase 1.1:** $\psi_1 \in \Gamma$. So we must show that $\Gamma \vdash (\varphi \to \psi_1)$. However

$$\Gamma \vdash (\psi_1 \to (\varphi \to \psi_1))$$

since $(\psi_1 \to (\varphi \to \psi_1))$ is a logical axiom. Further

$$\Gamma \vdash \psi_1$$

since $\psi_1 \in \Gamma$. Therefore by the Inference Lemma 1.26, $\Gamma \vdash (\varphi \to \psi_1)$.

**Subcase 1.2:** $\psi_1 = \varphi$. Note that $(\varphi \to \varphi)$ is a logical axiom and so

$$\Gamma \vdash (\varphi \to \varphi).$$

**Subcase 1.3:** $\psi_1$ is a logical axiom. This is just like subcase 1.1; $(\psi_1 \to (\varphi \to \psi_1))$ is a logical axiom and so

$$\Gamma \vdash (\psi_1 \to (\varphi \to \psi_1)).$$

Since $\psi_1$ is a logical axiom, $\Gamma \vdash \psi_1$. Therefore by the inference Lemma, $\Gamma \vdash (\varphi \to \psi_1)$.

We now suppose that $k \leq n$ and assume as an induction hypothesis that for all $i < k$,

$$\Gamma \vdash (\varphi \to \psi_i).$$

There are two subcases.

**Subcase 2.1:** $\psi_k \in \Gamma \cup \{\varphi\}$ or $\psi_k$ is a logical axiom. But then exactly as in the case of $\psi_1$, $\Gamma \vdash (\varphi \to \varphi_k)$.

**Subcase 2.2:** There exist $j_1 < k$ and $j_2 < k$ such that $\psi_{j_2} = (\psi_{j_1} \to \psi_k)$.

By the induction hypothesis; $\Gamma \vdash (\varphi \to \psi_{j_1})$ and $\Gamma \vdash (\varphi \to \psi_{j_2})$. Now we use the logical axiom

$$((\varphi \to (\psi_{j_1} \to \psi_k)) \to ((\varphi \to \psi_{j_1}) \to (\varphi \to \psi_k))).$$

By the induction hypothesis,

$$\Gamma \vdash (\varphi \to (\psi_{j_1} \to \psi_k)),$$

and so by the Inference Lemma,

$$\Gamma \vdash ((\varphi \to \psi_{j_1}) \to (\varphi \to \psi_k)).$$

Again by the induction hypothesis,

$$\Gamma \vdash (\varphi \to \psi_{j_1}),$$

and so by the Inference Lemma one last time,

$$\Gamma \vdash (\varphi \to \psi_k).$$

This completes the induction and so $\Gamma \vdash (\varphi \to \psi)$. Finally we note that only Group I logical axioms were used. □

**Definition 1.29** Suppose that $\Gamma \subseteq \mathcal{L}_0$.

(1) $\Gamma$ is *inconsistent* if for some formula $\varphi$, $\Gamma \vdash \varphi$ and $\Gamma \vdash (\neg\varphi)$.

(2) $\Gamma$ is *consistent* if $\Gamma$ is not inconsistent. □

If $\Gamma$ is an inconsistent set of formulas then $\Gamma \vdash \psi$ for *every* formula $\psi$. This is the content of the next lemma the proof of which appeals to the Deduction Lemma and logical axioms in Group II. Therefore only logical axioms from Groups I and II are needed.

**Lemma 1.30** *Suppose that $\Gamma \subseteq \mathcal{L}_0$ and that $\Gamma$ is inconsistent. Suppose that $\psi$ is a formula.*

*Then $\Gamma \vdash \psi$.*

*Proof.* Since $\Gamma$ is inconsistent there exists a formula $\varphi$ such that

$$\Gamma \vdash \varphi$$

and $\Gamma \vdash (\neg\varphi)$.

But

$$\Gamma \vdash (\varphi \rightarrow ((\neg\varphi) \rightarrow \psi))$$

since $(\varphi \rightarrow ((\neg\varphi) \rightarrow \psi))$ is a logical axiom. Therefore by the Inference Lemma,

$$\Gamma \vdash ((\neg\varphi) \rightarrow \psi)$$

and by the Inference Lemma again,

$$\Gamma \vdash \psi.$$

This completes the proof. □

**Definition 1.31** Suppose that $\Gamma \subseteq \mathcal{L}_0$ and that $\Gamma$ is consistent. Then $\Gamma$ is *maximally consistent* if and only if for each formula $\psi$ if $\Gamma \cup \{\psi\}$ is consistent then $\psi \in \Gamma$. □

**Lemma 1.32** *Suppose that $\Gamma \subseteq \mathcal{L}_0$ and that $\Gamma$ is consistent. Suppose that $\varphi$ is a formula.*

*Then, at least one of $\Gamma \cup \{\varphi\}$ or $\Gamma \cup \{(\neg\varphi)\}$ is consistent, possibly both.*

*Proof.* Suppose that $\Gamma \cup \{(\neg\varphi)\}$ is inconsistent. Therefore, for each formula $\psi$,

$$\Gamma \cup \{(\neg\varphi)\} \vdash \psi$$

and in particular, $\Gamma \cup \{(\neg\varphi)\} \vdash \varphi$.

Thus by the Deduction Lemma, $\Gamma \vdash ((\neg\varphi) \rightarrow \varphi)$. But $(((\neg\varphi) \rightarrow \varphi) \rightarrow \varphi)$ is a logical axiom (Group III), and so by the Inference Lemma, $\Gamma \vdash \varphi$.

Now assume toward a contradiction that $\Gamma \cup \{\varphi\}$ is inconsistent. By Lemma 1.30, for each formula $\psi$,

$$\Gamma \cup \{\varphi\} \vdash \psi.$$

By the Deduction Lemma, for each formula $\psi$,

$$\Gamma \vdash (\varphi \rightarrow \psi).$$

But $\Gamma \vdash \varphi$ and so by the Inference Lemma, for each formula $\psi$, $\Gamma \vdash \psi$. Thus $\Gamma$ is inconsistent, which is a contradiction. Therefore $\Gamma \cup \{\varphi\}$ is consistent.

So we have proved, assuming the consistency of $\Gamma$, that if $\Gamma \cup \{(\neg\varphi)\}$ is inconsistent then $\Gamma \cup \{\varphi\}$ is consistent. □

**Corollary 1.33** *Suppose that $\Gamma \subseteq \mathcal{L}_0$ and that $\Gamma$ is maximally consistent. Suppose that $\varphi$ is a formula.*

*Then:*

*(1) Either $\varphi \in \Gamma$ or $(\neg\varphi) \in \Gamma$;*
*(2) If $\Gamma \vdash \varphi$ then $\varphi \in \Gamma$.*

*Proof.* We first prove (1). By Lemma 1.32, either $\Gamma \cup \{\varphi\}$ is consistent or $\Gamma \cup \{(\neg\varphi)\}$ is consistent. Therefore since $\Gamma$ is maximally consistent (1) must hold.

We finish by proving (2). We are given that $\Gamma \vdash \varphi$. By (1), if $\varphi \notin \Gamma$ then $(\neg\varphi) \in \Gamma$ which implies that $\Gamma \vdash (\neg\varphi)$. But $\Gamma \vdash \varphi$ and so this contradicts the consistency of $\Gamma$. $\square$

We now use the logical axioms in Group IV.

**Lemma 1.34** *Suppose that $\Gamma \subseteq \mathcal{L}_0$ and that $\Gamma$ is maximally consistent. Suppose that $\varphi_1$ and $\varphi_2$ are formulas.*

*Then $(\varphi_1 \to \varphi_2) \in \Gamma$ if and only if at least one of $\varphi_1 \notin \Gamma$ or $\varphi_2 \in \Gamma$.*

*Proof.* We first suppose that $\varphi_1 \notin \Gamma$. We must show that $(\varphi_1 \to \varphi_2) \in \Gamma$.

Since $\varphi_1 \notin \Gamma$, by Corollary 1.33, $(\neg\varphi_1) \in \Gamma$.

Thus $\Gamma \vdash (\neg\varphi_1)$. But

$$\Gamma \vdash ((\neg\varphi_1) \to (\varphi_1 \to \varphi_2))$$

since $((\neg\varphi_1) \to (\varphi_1 \to \varphi_2))$ is a logical axiom, and so by the Inference Lemma,

$$\Gamma \vdash (\varphi_1 \to \varphi_2).$$

Therefore by Corollary 1.33, $(\varphi_1 \to \varphi_2) \in \Gamma$.

Next we suppose that $\varphi_2 \in \Gamma$. Now

$$(\varphi_2 \to (\varphi_1 \to \varphi_2))$$

is a logical axiom and so $\Gamma \vdash (\varphi_2 \to (\varphi_1 \to \varphi_2))$. By the Inference Lemma, since $\varphi_2 \in \Gamma$,

$$\Gamma \vdash (\varphi_1 \to \varphi_2).$$

Therefore, again by Corollary 1.33, $(\varphi_1 \to \varphi_2) \in \Gamma$.

To finish, we suppose that $\varphi_1 \in \Gamma$ and $\varphi_2 \notin \Gamma$. Now we must show that $(\varphi_1 \to \varphi_2) \notin \Gamma$.

Since $\varphi_2 \notin \Gamma$, by Corollary 1.33, $(\neg\varphi_2) \in \Gamma$.

Thus $\Gamma \vdash \varphi_1$ and $\Gamma \vdash (\neg\varphi_2)$. But

$$\Gamma \vdash (\varphi_1 \to ((\neg\varphi_2) \to (\neg(\varphi_1 \to \varphi_2)))),$$

since $(\varphi_1 \to ((\neg\varphi_2) \to (\neg(\varphi_1 \to \varphi_2))))$ is a logical axiom. Therefore by the Inference Lemma,

$$\Gamma \vdash ((\neg\varphi_2) \to (\neg(\varphi_1 \to \varphi_2))),$$

and by the Inference Lemma once again,

$$\Gamma \vdash (\neg(\varphi_1 \to \varphi_2)).$$

Finally by Corollary 1.33, $(\neg(\varphi_1 \to \varphi_2)) \in \Gamma$ and so $(\varphi_1 \to \varphi_2) \notin \Gamma$ as required.

This completes the proof of the lemma. $\square$

Our goal is to show that if $\Gamma$ is consistent then $\Gamma$ is satisfiable. We first consider the special case that $\Gamma$ is maximally consistent. This case will turn out to be an easy case for $\Gamma$ uniquely specifies the truth assignment which witnesses that $\Gamma$ is satisfiable.

**Lemma 1.35** *Suppose that $\Gamma \subseteq \mathcal{L}_0$ and that $\Gamma$ is maximally consistent. Then $\Gamma$ is satisfiable.*

*Proof.* Define a truth assignment $\nu$ as follows. For each $i \in \mathbb{N}$, let

$$\nu(A_i) = \begin{cases} T, & \text{if } \langle A_i \rangle \in \Gamma; \\ F, & \text{if } \langle A_i \rangle \notin \Gamma. \end{cases}$$

We claim that for each formula $\varphi$, $\overline{\nu}(\varphi) = T$ if $\varphi \in \Gamma$ and $\overline{\nu}(\varphi) = F$ if $\varphi \notin \Gamma$. We organize our proof of this claim by induction on the length of $\varphi$.

The case that $\varphi$ has length 1 is immediate.

Suppose that $\varphi$ has length $n > 1$ and that as induction hypothesis, for all formulas $\psi$ if $\psi$ has length less than $n$ then $\overline{\nu}(\psi) = T$ if $\psi \in \Gamma$ and $\overline{\nu}(\psi) = F$ if $\psi \notin \Gamma$.

There are two cases.

**Case 1:** $\varphi = (\neg\psi)$. Since $\Gamma$ is maximally consistent, $\varphi \in \Gamma$ if and only if $\psi \notin \Gamma$. But $\overline{\nu}(\varphi) = T$ if and only if $\overline{\nu}(\psi) = F$. By the induction hypothesis $\overline{\nu}(\psi) = T$ if and only if $\psi \in \Gamma$.

Thus if $\varphi \in \Gamma$ then $\overline{\nu}(\varphi) = T$ and $\overline{\nu}(\varphi) = F$ if $\varphi \notin \Gamma$.

**Case 2:** $\varphi = (\psi_1 \to \psi_2)$. Since $\Gamma$ is maximally consistent, $\varphi \in \Gamma$ if and only if at least one of $\psi_1 \notin \Gamma$ or $\psi_2 \in \Gamma$. This is by Lemma 1.34.

By the definition of $\overline{\nu}$, $\overline{\nu}(\varphi) = T$ if and only if either $\overline{\nu}(\psi_1) = F$ or $\overline{\nu}(\psi_2) = T$. Therefore by the induction hypothesis, $\overline{\nu}(\varphi) = T$ if and only if either $\psi_1 \notin \Gamma$ of $\psi_2 \in \Gamma$.

Thus, $\overline{\nu}(\varphi) = T$ if and only if $\varphi \in \Gamma$.

This completes the induction. $\square$

**Theorem 1.36** *Suppose that* $\Gamma \subseteq \mathcal{L}_0$ *and that* $\Gamma$ *is consistent.*
*Then there exists a set* $\Gamma^* \subset \mathcal{L}_0$ *such that*

$$\Gamma \subseteq \Gamma^*$$

*and such that* $\Gamma^*$ *is maximally consistent.*

*Proof.* Let $(\varphi_i : i \in \mathbb{N})$ be an enumeration of all of the formulas of $\mathcal{L}_0$. For example, we could enumerate the finitely many length 1 formulas which use only the propositional symbol $A_1$; then, we could enumerate the finitely many formulas of length less than or equal to 2 which use no propositional symbols other than $A_1$ and $A_2$; and in subsequent steps, enumerate the finitely many $n$ formulas of length less than or equal to $n$ which use no propositional symbols other than $A_1, \ldots, A_n$.

We construct a sequence of sets $(\Gamma_n : m \in \mathbb{N})$ by recursion on $n$. To begin, let $\Gamma_0$ equal $\Gamma$. Given $\Gamma_n$, let $\Gamma_{n+1}$ be defined as follows.

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_n\}, & \text{if } \Gamma_n \cup \{\varphi_n\} \text{ is consistent;} \\ \Gamma_n \cup \{\neg\varphi_n\}, & \text{otherwise.} \end{cases}$$

We check by induction that each $\Gamma_n$ is consistent. Clearly, $\Gamma_0$ is consistent, since we are given that $\Gamma$ is consistent. Assuming that $\Gamma_n$ is consistent, we can apply Lemma 1.32 to conclude that at least one of $\Gamma_n \cup \{\varphi_n\}$ or $\Gamma_n \cup \{\neg\varphi_n\}$ is also consistent. But then, $\Gamma_{n+1}$ is also consistent.

Now, define $\Gamma^*$ so that

$$\Gamma^* = \cup_{n \in \mathbb{N}} \Gamma_n.$$

For the sake of a contradiction, suppose that $\Gamma^*$ is not consistent, then there is are $\Gamma^*$-proofs of some formula and of its negation. These $\Gamma^*$-proofs are finite, and so there is finite set of formulas $\Gamma^*_F$ from $\Gamma^*$ which appear in these proofs. But then $\Gamma^*_F$ must be a subset of some $\Gamma_n$, and so one of the $\Gamma_n$'s must be inconsistent. Since we have already checked that all of the $\Gamma_n$'s are consistent, this is impossible. Thus $\Gamma^*$ is consistent.

$\Gamma^*$ is also maximally consistent: For every formula $\varphi$, there is an $n$ such that $\varphi$ is equal to $\varphi_n$. But we chose $\Gamma_n$ so that either $\varphi_n \in \Gamma_n$ or $(\neg\varphi_n) \in \Gamma_n$. Since $\varphi = \varphi_n$ and $\Gamma_n \subseteq \Gamma^*$, either $\varphi \in \Gamma^*$ or $(\neg\varphi) \in \Gamma^*$, as required for maximality. □

**Theorem 1.37 (Completeness; Version I)** *Suppose that* $\Gamma \subseteq \mathcal{L}_0$ *and that* $\Gamma$ *is consistent.*
*Then* $\Gamma$ *is satisfiable.*

*Proof.* By Theorem 1.36 extend $\Gamma$ to a maximally consistent set, and then apply Lemma 1.35. □

## 1.4 Logical implication and compactness

**Definition 1.38** Let $\Gamma$ be a subset of $\mathcal{L}_0$ and let $\varphi$ be an element of $\mathcal{L}_0$. Then, $\Gamma$ *logically implies* $\varphi$ if and only if, for every truth assignment $\nu$, if $\nu$ satisfies $\Gamma$ in the sense of Definition 1.17, then $\nu$ satisfies $\varphi$.

For example, $\{\varphi\}$ logically implies $\varphi$, and $\{A_1, (A_1 \to A_2)\}$ logically implies $A_2$.

If $\Gamma$ is a set a formulas, we write $\Gamma \models \varphi$ to indicate that $\Gamma$ logically implies $\varphi$.

**Definition 1.39** A subset $\Gamma$ of $\mathcal{L}_0$ is *finitely satisfiable* if and only if for every finite subset $\Gamma_0$ of $\Gamma$, there is a truth assignment $\nu$ such that for all $\psi \in \Gamma_0$, $\overline{\nu}(\psi) = T$.

**Theorem 1.40 (Compactness for $\mathcal{L}_0$)** *Suppose that $\Gamma \subseteq \mathcal{L}_0$, $\varphi \in \mathcal{L}_0$, and $\Gamma$ logically implies $\varphi$. Then there is a finite set $\Gamma_0$ such that $\Gamma_0 \subseteq \Gamma$ and $\Gamma_0$ logically implies $\varphi$.*

*Proof.* Since $\Gamma \models \varphi$, $\Gamma \cup \{(\neg\varphi)\}$ is *not satisfiable*. Therefore by the Completeness Theorem, $\Gamma \cup \{(\neg\varphi)\}$ is inconsistent. But this implies that there exists a finite set $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \cup \{(\neg\varphi)\}$ is inconsistent. Therefore by Lemma 1.27, the Soundness Lemma, $\Gamma_0 \cup \{(\neg\varphi)\}$ is not satisfiable and so $\Gamma_0 \models \varphi$. □

We end this chapter by discussing a reformulation of the Completeness Theorem.

**Theorem 1.41 (Completeness; Version II)** *Suppose that $\Gamma \subseteq \mathcal{L}_0$, $\varphi$ is a formula and that $\Gamma \models \varphi$.*

*Then $\Gamma \vdash \varphi$.*

*Proof.* Since $\Gamma \models \varphi$, $\Gamma \cup \{(\neg\varphi)\}$ is not satisfiable. Therefore by the Completeness Theorem, $\Gamma \cup \{(\neg\varphi)\}$ is inconsistent and so by Lemma 1.30,

$$\Gamma \cup \{(\neg\varphi)\} \vdash \varphi.$$

By Lemma 1.28, the Deduction Lemma,

$$\Gamma \vdash ((\neg\varphi) \to \varphi).$$

But $(((\neg\varphi) \to \varphi) \to \varphi)$ is a logical axiom and so by Lemma 1.26, the Inference Lemma, $\Gamma \vdash \varphi$. □

### 1.4.1 Exercises

(1) Which of the formulas

$$(((A_1 \to A_1) \to A_2) \to A_2)$$
$$((((A_1 \to A_2) \to A_2) \to A_2) \to A_2)$$

is a tautology?

(2) For $\Gamma \subseteq \mathcal{L}_0$ and $\varphi$ and $\psi$ in $\mathcal{L}_0$, show that $\Gamma \cup \{\varphi\}$ logically implies $\psi$ if and only if $\Gamma$ logically implies $(\varphi \to \psi)$.

(3) Two physicists, $A$ and $B$, and a logician $C$, are wearing hats, which they know are either black or white but not all white. $A$ can see the hats of $B$ and $C$; $B$ can see the hats of $A$ and $C$; $C$ is blind. Each is asked in turn if they know the color of their own hat. The answers are: $A$: "No." $B$: "No." $C$: "Yes." What color is $C$'s hat and how does $C$ know?

(4) For $\Gamma_1$ and $\Gamma_2$ subsets of $\mathcal{L}_0$, $\Gamma_1$ is *logically equivalent* to $\Gamma_2$ if and only if, for all $\varphi \in \mathcal{L}_0$, $\Gamma_1$ logically implies $\varphi$ if and only if $\Gamma_2$ logically implies $\varphi$. For $\Gamma \subseteq \mathcal{L}_0$, $\Gamma$ is *independent* if it is not logically equivalent to any of its proper subsets. Prove the following.

   a) If $\Gamma$ is finite, then there is a $\Gamma_0$ such that $\Gamma_0 \subseteq \Gamma$, $\Gamma$ and $\Gamma_0$ are logically equivalent, and $\Gamma_0$ is independent.
   b) There is an infinite set $\Gamma$ such that $\Gamma$ has no independent and logically equivalent subset.
   c) For every $\Gamma \subseteq \mathcal{L}_0$, there is a $\Delta \subseteq \mathcal{L}_0$ such that $\Delta$ is independent and logically equivalent to $\Gamma$.

(5) Show that the set of logical consequences of

$$\{A_i : i \neq 1 \text{ and } i \in \mathbb{N}\}$$

is consistent but not maximally consistent. Show that the set of logical consequences of

$$\{A_i : i \in \mathbb{N}\}$$

is maximally consistent.