Problem Set 3, Math 172 Spring '15

This problem set is due Tuesday, February 10th, 2015 at **the beginning of class**. All class guide rules apply. "VLW" refers to Van Lint and Wilson, our textbook. **All questions, when graded, are worth an equal number of points unless stated otherwise, though not all questions would end up being graded.**

1. (VLW, 3B) Let the edges of K_7 be colored with red and blue colors. Show that there are at least 4 subgraphs isomorphic to K_3 with all three edges having the same color. Also show that equality is possible.

2. (VLW 1A) Find all automorphisms of the Petersen Graph (`http://en.wikipedia.org/wiki/Petersen_graph`). (**Note a question like this implicitly requires you to prove that you've actually found all of them.**) (Possibly useful hints: the wikipedia page tells you there are 120, and the automorphisms form a group isomorphic to S_5, which may be familiar to you if you took 113. No algebra required to do this problem though, and you can't use this information as part of your proof)

3. Fix positive integers n and k, and let $S = [n]$. Find the number of k-tuples $(T_1, T_2, \ldots, T_k)$ of subsets T_i of S such that (a) the T_i are pairwise disjoint; (b) $T_1 \cup T_2 \cup \cdots \cup T_k = S$. **These are two separate problems.** (Hint: for both of these, try to **think from the perspective of the individual element**.)

4. Give a good definition of "isomorphism" for simple *directed* graphs (in terms of $V(G)$ and $E(G)$). Count the number of isomorphic labeled **and** unlabeled directed trees (strictly speaking we need to define it: basically, between any two vertices v and w there is either no edge, and edge (v, w), or an edge (w, v) (but not both!), such that if we "forget the direction" of the edges the underlying undirected graph is a tree) on 5 vertices.

5. Prove that a graph is *bipartite* (this is a graph where the vertices can be written as a disjoint union $A \cup B$ such that the edges only go **between** A and B, and never, say, between two vertices in A or two vertices in B) if and only if there is no subgraph of form C_n, where n is odd. (**This is a very fundamental result that will come up again, so remember it.**).

6. (optional) Prove the theorem mentioned in class about Eulerian paths: show that you have an Eulerian path in a graph G if and only if the number of vertices with odd degree is 0 or 2 (you may use without proof the well-known fact that the number of vertices with odd degree must be even in **any** graph; prove it to yourself if you haven't seen this, or find it in the textbook/any book on graph theory). This problem is optional not because it is hard (it isn't), but because many of you have already seen it. Make sure you know how to do it, though.

7. How much time did you spend on this problem set? What comments do you have of the problems? (difficulty, type, enjoyment, fairness, etc.)