Sample Problems from Past 125 Midterms

(some were assigned as homework)

In the following, you must prove that your answers are correct.

1. Give an example of an element of $\mathcal{L}_0$ which has at least ten symbols. Prove that your example belongs to $\mathcal{L}_0$.

The sequence $(\neg(\neg(\neg A_1)))$ is such an example. To show that it is is an element of $\mathcal{L}_0$, consider the sequence of formulas

$$A_1, (\neg A_1), (\neg(\neg A_1)), (\neg(\neg(\neg A_1))).$$

Since it is a propositional symbol, A_1 is an element of $\mathcal{L}_0$. Each of the other elements the sequence is of the form $(\neg\psi)$, where ψ is its immediate predecessor. Since, $(\neg\psi)$ belongs to $\mathcal{L}_0$ whenever ψ does, by induction on the sequence, all of its elements belong to $\mathcal{L}_0$.

2. Consider the set of symbols $*$ and $\#$. Let $\mathcal{L}^*$ be the smallest set L of sequences of these symbols with the following properties.

1. *The length one sequences $\langle * \rangle$ and $\langle \# \rangle$ belong to L.*
2. *If σ and τ belong to L, then so do $\langle * \rangle + \sigma + \langle \# \rangle$ and $\langle * \rangle + \sigma + \tau + \langle \# \rangle$.*

State Readability and Unique Readability for $\mathcal{L}^*$ and determine for each whether it holds.

Readability of $\mathcal{L}^*$ is the following assertion.

> Suppose that φ is a formula in $\mathcal{L}^*$. Then exactly one of the following conditions applies.
>
> 1. $\varphi = \langle * \rangle$ or $\varphi = \langle \# \rangle$ or
> 2. There is a $\sigma \in \mathcal{L}^*$ such that $\varphi = \langle * \rangle + \sigma + \langle \# \rangle$.
> 3. There are σ and τ in $\mathcal{L}^*$ such that $\varphi = \langle * \rangle + \sigma + \tau + \langle \# \rangle$.

Unique Readability extends Readability by further asserting that in cases (1) and (2), the formulas σ, and σ and τ are unique, respectively.

Both assertions are false for $\mathcal{L}^*$. Consider the following formula.

$$* * *\#\#\#$$

Since $\sigma = * * \#\#$ is a formula by clauses (1) and (3), there is a $\sigma \in \mathcal{L}^*$ such that $* * *\#\#\# = \langle * \rangle + \sigma + \langle \# \rangle$. Since, $* * \#$ is a formula by clauses (1) and (2) and $\#$ is a formula by clause (1), there are σ and τ in $\mathcal{L}^*$ such that $* * *\#\#\# = \langle * \rangle + \sigma + \tau + \langle \# \rangle$. Thus, $* * *\#\#\#$ satisfies more than one of the three clauses in the statement of Readability, which contradicts the asserted condition that elements of $\mathcal{L}^*$ should satisfy exactly one clause. Since Unique Readability includes Readability, it also fails for $\mathcal{L}^*$.

3. (Prove the Inference Lemma.) Suppose that φ and ψ are in $\mathcal{L}_0$ and $\Gamma \subseteq \mathcal{L}_0$. Use the logical axioms to show the following:

$$\Gamma \cup \{\varphi\} \vdash \psi \textit{ if and only if } \Gamma \vdash (\varphi \to \psi)$$

The proof of the Inference Lemma is available in the lecture notes.

4. Show that the set of logical consequences of

$$\{A_i : i \neq 1 \textit{ and } i \in \mathbb{N}\}$$

is consistent but not maximally consistent. Show that the set of logical consequences of

$$\{A_i : i \in \mathbb{N}\}$$

is maximally consistent.

Let Γ denote the set of logical consequences of $\{A_i : i \neq 1 \text{ and } i \in \mathbb{N}\}$. In other words, Γ is the set of formulas φ in $\mathcal{L}_0$ such that for any truth assignment ν, if for all $i \neq 1$, $\nu(A_i) = T$ then $\overline{\nu}(\varphi) = T$. Then, *Gamma* is satisfied by the truth assignment that assigns every propositional symbol the value T. By the soundness theorem, since Γ is satisfiable, so is every one of its logical consequences. Thus, Γ is consistent.

Γ is not maximally consistent since $\Gamma \cup \{A_1\}$ and $\Gamma \cup \{(negA_1)\}$ are satisfied by the constantly T truth assignment and be the F for A_1 and T elsewhere truth assignment, respectively. Thus both sets are consistent, and $\Gamma \cup \{A_1\}$ is a consistent proper extension of Γ, as required.

5. Let $A = \{F_1\}$ be the alphabet with one unary function symbol. Give an examples of different infinite $\mathcal{L}_A$-structures $\mathcal{M} = (M, I)$ with the following properties.

1. *$\mathcal{M}$ has no nontrivial automorphisms.*
2. *$\mathcal{M}$ has a countably infinite set of automorphisms.*
3. *For each element a of M there are only finitely many b's in M such that there is an automorphism f of $\mathcal{M}$ with $f(a) = b$. However, there are uncountably many automorphisms of $\mathcal{M}$.*

For (1), consider the natural numbers $\{0, 1, 2, \dots\}$ with the unary function F $n \mapsto n + 1$. Any automorphism π of this structure must map 0 to itself, since 0 is the unique point which is not in the range of F. If $\pi(n) = n$, then, since π commutes with F, $\pi(n + 1) = \pi(F(n)) = F(\pi(n)) = n + 1$. By induction π is the identity.

For (2), consider the integers$\{\dots, -2, -1, 0, 1, 2, \dots\}$ with the unary function F. Suppose π is an automorphism, then by the same argument π must be the function $n \mapsto n+\pi(0)$. Since there are only countably many possibilities for $\pi(0)$, there are only countably many automorphisms.

For (3), define an F-cycle to be a finite set $\{a_1, a_2, \dots, a_n\}$ with the unary function $F(a_j) = a_{j+1}$, when $j < n$, and $F(a_n) = a_1$. Consider the structure $\mathcal{M}$ consisting of the disjoint union of F-cycles, with exactly one cycle of each length. For every subset A of we can define an automorphism of π_A of $\mathcal{M}$ by letting $\pi_A(a) = a$, if a belongs to an F-cycle of length n and $n \in A$, and letting $\pi_A(a) = F(a)$ otherwise. If $A \neq B$, then $pi_A \neq \pi_B$. Since the set of subsets of $\{1, 2, \dots\}$ is not countable, neither is the set of automorphisms of $\mathcal{M}$.

6. Let $A = \{P_1\}$ be the alphabet with one unary predicate symbol. For each of i equal to 1 or 2, suppose that $\mathcal{M}_i = (M_i, I_i)$ is an A structure such that M_i, $I_i(P_1)$, and $M_i \setminus I_i(P_1)$ are all infinite. Here $M_i \setminus I_i(P_1)$ consists of those elements of M_i which are not in $I_i(P_1)$.

Show that $\mathcal{M}_1 \equiv \mathcal{M}_2$.

By the Lowenheim-Skolem Theorem, there are countable substructures $\mathcal{M}_i^* \preceq \mathcal{M}_i$. Each $\mathcal{M}_i^*$ consists of a countable set M_i^* and a predicate which divides that set into two infinite pieces. For any two infinite countable sets, there is a bijection between them. Thus, there is a bijection between the set of elements of M_1^* that satisfy P_1 in $\mathcal{M}_1^*$ and the the set of elements of M_2^* that satisfy P_1 in $\mathcal{M}_2^*$. Similarly, there is a bijection between the set of elements of M_1^* that do not satisfy P_1 in $\mathcal{M}_1^*$ and the the set of elements of M_2^* that do not satisfy P_1 in $\mathcal{M}_2^*$. The union of these two functions is a bijection between M_1^* and M_2^* that respects P_1, hence is an isomorphism. Since isomorphic structures are elementarily equivalent, $\mathcal{M}_1^* \equiv \mathcal{M}_2^*$. In summary,$\mathcal{M}_1 \equiv \mathcal{M}_1^* \equiv \mathcal{M}_2^* \equiv \mathcal{M}_2$. So $\mathcal{M}_1 \equiv \mathcal{M}_2$.