In all exercises, you may assume R is a commutative ring with identity where $1 \neq 0$.

1. (Exercise 14 in DF §10.4.) Let I be an arbitrary nonempty index set, and for each $i \in I$ let N_i be an R-module. Let M be an R-module. Prove that there is an R-module isomorphism

$$M \otimes_R \left(\bigoplus_{i \in I} N_i \right) \cong \bigoplus_{i \in I} (M \otimes_R N_i).$$

Proof. Define a map $f : M \times \left(\bigoplus_{i \in I} N_i\right) \to \bigoplus_{i \in I}(M \otimes_R N_i)$ by $(m, \prod n_i) \mapsto \prod m \otimes n_i$. Observe that f is bilinear: for any $r_1, r_2 \in R, m_1, m_2 \in M$, and $\prod x_i, \prod y_i \in \oplus_{i \in I} N_i$, we have

$$\begin{aligned}
f((r_1 m_1 + r_2 m_2, \prod x_i)) &= \prod (r_1 m_1 + r_2 m_2) \otimes x_i \\
&= r_1 \prod (m_1 \otimes x_i) + r_2 \prod (m_2 \otimes x_i) \\
&= r_1 f(m_1, x_i) + r_2 f(m_2, x_i) \\
f((m_1, r_1 \prod x_i + r_2 \prod y_i) &= m_1 \otimes (r_1 \prod x_i + r_2 \prod y_i) \\
&= \prod m_1 \otimes (r_1 x_i + r_2 y_i) \\
&= r_1 \prod m_1 \otimes x_i + r_2 \prod m_1 \otimes y_i \\
&= r_1 f(m_1, \prod x_i) + r_2 f(m_1, \prod y_i)
\end{aligned}$$

therefore, by the universal property of tensor product (the version from Corollary 12), f induces a unique R-module homomorphism $\varphi : M \otimes_R \left(\bigoplus_{i \in I} N_i\right) \to \bigoplus_{i \in I}(M \otimes_R N_i)$ defined by $(m \otimes \prod n_i) \mapsto \prod (m \otimes n_i)$.

Next, for each $i \in I$ define a map $g_i : M \times N_i \to M \otimes_R (\oplus_{i \in I} N_i)$ by $(m, n) \mapsto m \otimes \iota_i(n)$ where ι_i is the natural inclusion of N_i into $\oplus_{i \in I} N_i$. Recall that ι_i is an R-module homomorphism. Observe that g_i is bilinear: for each $r_1, r_2 \in R, m_1, m_2 \in M$, and $x, y \in N_i$, we have

$$\begin{aligned}
g_i(r_1 m_1 + r_2 m_2, x) &= (r_1 m_1 + r_2 m_2) \otimes \iota_i(x) \\
&= r_1 (m_1 \otimes \iota_i(x)) + r_2 (m_2 \otimes \iota_i(x)) \\
&= r_1 g_i(m_1, x) + r_2 g_i(m_2, x) \\
g_i(m_1, r_1 x + r_2 y) &= m_1 \otimes \iota_i(r_1 x + r_2 y) \\
&= r_1 (m_1 \otimes \iota_i(x)) + r_2 (m_1 \iota_i(y)) \\
&= r_1 g(m_1, x) + r_2 g(m_1, y)
\end{aligned}$$

therefore each g_i induces a unique R-module homomorphism $\psi_i : M \otimes_R N_i \to M \otimes_R (\oplus_{i \in I} N_i)$ satisfying $\psi_i(m \otimes n) = m \otimes \iota_i(n)$.

By the universal property of direct sum, there is a unique R-module homomorphism $\psi : \oplus_{i \in I}(M \otimes N_i) \to M \otimes (\oplus_{i \in I} N_i)$ such that $\psi \circ \hat{\iota}_i = \psi_i$ for each $i \in I$, where $\hat{\iota}_i$ is the inclusion of $M \otimes_R N_i$ into $\oplus_{j \in I}(M \otimes_R N_j)$.

We will now show that ψ is the inverse of φ, thus giving that φ is an isomorphism. Let $m \otimes \prod n_i \in M \otimes_R (\oplus_{i \in I} N_i)$, where $n_i \in N_i$ for each $i \in I$. Note that there are only finitely many $i \in I$ for which $n_i \neq 0$, so $\prod_{i \in I} m \otimes n_i = \sum_{\{i : n_i \neq 0\}} \hat{\iota}_i(m \otimes n_i)$ is a sum of finitely many terms. We have

$$\begin{aligned}
\psi \circ \varphi \left(m \otimes \prod n_i\right) &= \psi \left(\prod m \otimes n_i\right) \\
&= \psi \left(\sum_{\{i : n_i \neq 0\}} \hat{\iota}_i (m \otimes n_i) \right) \\
&= \sum_{\{i : n_i \neq 0\}} \psi \circ \hat{\iota}_i (m \otimes n_i) \\
&= \sum_{\{i : n_i \neq 0\}} \psi_i (m \otimes n_i) \\
&= \sum_{\{i : n_i \neq 0\}} m \otimes \iota_i (n_i) \\
&= m \otimes \sum_{\{i : n_i \neq 0\}} \iota_i (n_i) \\
&= m \otimes \prod n_i
\end{aligned}$$

so ψ is a left inverse of φ.

Next, let $\prod m_i \otimes n_i \in \oplus_{i \in I} (M \otimes_R N_i)$, where $m_i \in M, n_i \in N_i$ for each $i \in I$. We have

$$\begin{aligned}
\varphi \circ \psi \left(\prod m_i \otimes n_i\right) &= \varphi \circ \psi \left(\sum_{\{i : m_i \otimes n_i \neq 0\}} \hat{\iota}_i (m_i \otimes n_i) \right) \\
&= \varphi \left(\sum_{\{i : m_i \otimes n_i \neq 0\}} \psi_i (m_i \otimes n_i) \right) \\
&= \varphi \left(\sum_{\{i : m_i \otimes n_i \neq 0\}} m_i \otimes \iota_i (n_i) \right) \\
&= \sum_{\{i : m_i \otimes n_i \neq 0\}} \varphi \left(m_i \otimes \iota_i (n_i)\right) \\
&= \sum_{\{i : m_i \otimes n_i \neq 0\}} \hat{\iota}_i \left(\prod m_i \otimes n_i\right) \\
&= \prod m_i \otimes n_i
\end{aligned}$$

therefore ψ is also a right inverse of φ, so φ is an isomorphism. $\square$

2. (Exercise 16 in DF §10.4.) Let I and J be ideals of R, so that R/I and R/J are naturally R-modules.

(a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $\overline{1_R} \otimes \overline{r}$, where $r \in R$ and the bar in the left (resp. right) factor denotes the equivalence class modulo I (resp. modulo J).

Proof. Every element of $R/I \otimes_R R/J$ can be written as $\sum_{i=1}^n \overline{x_i} \otimes \overline{y_i}$ for some n, where $x_i, y_i \in R$ for each $i \in \{1, \dots, n\}$. Using the natural action of R on $R/I \otimes_R R/J$, we have

$$\sum_{i=1}^n \overline{x_i} \otimes \overline{y_i} = \sum_{i=1}^n \overline{1_R} \cdot x_i \otimes y_i \cdot \overline{1_R} = \sum_{i=1}^n \overline{1_R} \otimes (x_i y_i) \cdot \overline{1_R} = \sum_{i=1}^n \overline{1_R} \otimes \overline{x_i y_i} = \overline{1_R} \otimes \sum_{i=1}^n \overline{x_i y_i} = \overline{1_R} \otimes \overline{\sum_{i=1}^n x_i y_i}$$

hence the result follows. $\square$

(b) Prove that there is an R-module isomorphism $R/I \otimes_R R/J \cong R/(I+J)$ mapping $\overline{r} \otimes \overline{r'}$ to $\overline{rr'}$ (where $r, r' \in R$ and the bars denote the equivalence class modulo I, J, and $I+J$ respectively). (Recall that $I+J$ denotes the ideal generated by the set $I \cup J$, or equivalently the set of all elements $a+b$ where $a \in I$ and $b \in J$.)

Proof. First, we need to show that the given map φ is well-defined. It is not enough to check that $\varphi(\overline{1}_R \otimes \overline{x}) = \varphi(\overline{1}_R \otimes \overline{y})$ when $x - y \in J$, since this is not equivalent to $\overline{1}_R \otimes \overline{x} = \overline{1}_R \otimes \overline{y}$. For instance, in $\mathbf{Z}/2\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/3\mathbf{Z} = 0$ we have $\overline{1} \otimes \overline{1} = \overline{0} = \overline{1} \otimes \overline{2}$, even though $1 - 2 \notin 3\mathbf{Z}$. So we will again invoke the universal property.

Let $f : R/I \times R/J \to R/(I+J)$ be defined by $f(\overline{x}, \overline{y}) = \overline{xy}$. First, we will show that f is well-defined. Suppose $x - s \in I$ and $y - r \in J$. Then $xy \in I$ and $sr \in J$ simply because $x, s \in R$, and $I+J$ is an R-module. Thus $xy - rs \in I + J$, so $f(\overline{x}, \overline{y}) = \overline{xy} = \overline{rs} = f(\overline{r}, \overline{s})$.

Now, we will show that f is bilinear. For any $r, s, x, y, z \in R$, we have

$$f(\overline{rx+sy}, \overline{z}) = \overline{(rx+sy)z} = r \cdot \overline{xz} + s \cdot \overline{yz} = rf(\overline{x}, \overline{z}) + sf(\overline{y}, \overline{z})$$

Since R is commutative and f is symmetric in its two arguments, linearity holds in the second component as well. Thus f induces an R-module homomorphism $R/I \otimes R/J \to R/(I+J)$, which is precisely φ. Thus φ is a well-defined homomorphism.

To see that φ is injective, let $\overline{1}_R \otimes \overline{x} \in \ker(\varphi)$. Then $1_R \cdot x = x \in I + J$, so $x = i + j$ for some $i \in I, j \in J$. Thus

$$\overline{1}_R \otimes \overline{x} = \overline{1}_R \otimes \overline{i+j} = \overline{1}_R \otimes \overline{i} + \overline{1}_R \otimes \overline{j} = \overline{i} \otimes \overline{1}_R + \overline{1}_R \otimes \overline{j} = \overline{0_R} \otimes \overline{1}_R + \overline{1}_R \otimes \overline{0_R} = 0.$$

Since φ is a group homomorphism, and its kernel is 0, it must be injective.

φ is clearly surjective, since for any $x + (I+J) \in R/(I+J)$ we have $\varphi((1_R + I) \otimes (x + J)) = (1_R \cdot x) + (I+J) = x + (I+J)$. Thus φ is an isomorphism. □

3. (Exercise 18 in DF §10.4.) Suppose R is an integral domain and I is a principal ideal in R. Prove that the R-module $I \otimes_R I$ has no nonzero torsion elements; i.e., if $r \in R \backslash \{0\}$ and $m \in I \otimes_R I$ satisfy $rm = 0$, then $m = 0$.

Proof. Let $I = (a)$, so that every element of I is of the form ra for some $r \in R$. First, note that every tensor is of the form $\sum_i r_i a \otimes s_i a = (\sum_i r_i s_i) \cdot (a \otimes a) = r \cdot (a \otimes a)$. It is natural to guess that $I \otimes_R I \cong R$.

Define $f : I \times I \to R$ by $(ra, sa) \mapsto rs$. Check that f is bilinear: for any $c, d, r, s, t \in R$, we have

$$\begin{aligned} f(c(ra) + d(sa), ta) &= f((cr+ds)a, ta) \\ &= (cr+ds)t \\ &= c(rt) + d(st) \\ &= cf(r,t) + df(s,t). \end{aligned}$$

Again, since f is symmetric in its arguments we know it is linear in its second component as well. So f is bilinear, and thus induces a unique R-module homomorphism $\varphi : I \otimes_R I \to R$ such that $\varphi(ra \otimes sa) = rs$. Since every tensor is of the form $r \cdot (a \otimes a)$, φ is equivalently defined by $\varphi(r \cdot (a \otimes a)) = r$.

Clearly the map $\psi : R \to I \otimes_R I$ given by $r \mapsto r \cdot (a \otimes a)$ is a homomorphism and both a left and right inverse of φ:

$$\begin{aligned} \psi(x + ry) &= (x+ry) \cdot (a \otimes a) = x \cdot (a \otimes a) + r \cdot (y \cdot (a \otimes a)) = \psi(x) + r\psi(y) \\ \varphi \circ \psi(r) &= \varphi(r \cdot (a \otimes a)) = r \\ \psi \circ \varphi(r \cdot (a \otimes a)) &= \psi(r) = r \cdot (a \otimes a) \end{aligned}$$

Thus φ is an isomorphism.

The action of R on itself is simply multiplication in R, thus R being torsion free is equivalent to it being an integral domain. Therefore, R is torsion free and so $I \otimes_R I$ must also be torsion free. □

4. (Exercise 19 in DF §10.4.) Let $I = (2, X)$ be the ideal generated by 2 and X in the ring $R = \mathbf{Z}[X]$, as in Exercise 17 (assigned on HW10). Show that the nonzero element $2 \otimes X - X \otimes 2$ in $I \otimes_R I$ is a torsion element. Show in fact that $2 \otimes X - X \otimes 2$ is annihilated by both 2 and X, and that the submodule of $I \otimes_R I$ generated by $2 \otimes X - X \otimes 2$ is isomorphic to R/I.

Proof.

$$2(2 \otimes x - x \otimes 2) = (2 \otimes 2x - 2x \otimes 2) = (2x \otimes 2 - 2x \otimes 2) = 0$$
$$x(2 \otimes x - x \otimes 2) = (2x \otimes x - x \otimes 2x) = (2x \otimes x - 2x \otimes x) = 0$$

thus $(2 \otimes x - x \otimes 2)$ is annihilated by both 2 and x. The only reason we could not do this before (in hw 10) was that $(2 \otimes x)$ cannot be written as $2(1 \otimes x)$, since $1 \notin I$.

This implies that both 2 and x annihilate the submodule A of $I \otimes_R I$ generated by $2 \otimes x - x \otimes 2$: in general, let S be a commutative ring, let $\{v\}$ a basis for an S-module V, and suppose $a \in S$ annihilates v. Any element of V is of the form sv for some $s \in S$, so we have

$$a(sv) = (as)v = (sa)v = s(av) = s(0) = 0$$

thus a is contained in the annihilator of V.

Furthermore, this implies that I is contained in the annihilator of A: In homework 7, problem 5, we were allowed to take for granted the result of exercise 9 from section 10.1, which states that the annihilator of A is an ideal. This would be easy to show anyway, since the annihilator of a module is the kernel of the homomorphism $r \mapsto rx$, thus is an ideal. Since 2 and x annihilate A, the annihilator of A must contain the ideal generated by 2 and x. So I annihilates A.

Define a map $\varphi : R \to A$ by $r(x) \mapsto r(x) \cdot (2 \otimes x - x \otimes 2)$. φ is clearly an R-module homomorphism:

$$\begin{aligned}
\varphi(p(x) + r(x)q(x)) &= p(x) + r(x)q(x)(2 \otimes x - x \otimes 2) \\
&= p(x)(2 \otimes x - x \otimes 2) + r(x)(q(x)(2 \otimes x - x \otimes 2)) \\
&= \varphi(p(x)) + r(x) \cdot \varphi(q(x))
\end{aligned}$$

Since I annihilates A, it is contained in the kernel of φ. We will now show that I is *precisely* this kernel. Suppose $p(x) \in \mathbf{Z}[x]$, but $p(x) \notin I$. Then $p(x)$ has an odd constant term, thus can be written as $p(x) = 1 + q(x)$ for some $q(x) \in I$. So we have

$$\begin{aligned}
\varphi(p(x)) &= p(x) \cdot (2 \otimes x - x \otimes 2) \\
&= (1 + q(x)) \cdot (2 \otimes x - x \otimes 2) \\
&= 2 \otimes x - x \otimes 2 + q(x) \cdot (2 \otimes x - x \otimes 2) \\
&= 2 \otimes x - x \otimes 2 + 0 \\
&= 2 \otimes x - x \otimes 2.
\end{aligned}$$

However, $2 \otimes x - x \otimes 2 \neq 0$, thus $p(x) \notin \ker \varphi$. Thus $\ker \varphi = I$. By the first isomorphism theorem, φ induces an isomorphism $\overline{\varphi} : R/I \to A$ onto the image of φ, where $\overline{\varphi} \circ \pi = \varphi$ for the natural projection π of R onto R/I. φ is, by construction, surjective, since

$$A = \{r(x) \cdot (2 \otimes x - x \otimes 2) : r(x) \in R\} = \{\varphi(r(x)) : r(x) \in R\} = \mathrm{Im}(\varphi).$$

So $\overline{\varphi}$ is an isomorphism from R/I to A, thus $A \cong R/I$. $\square$

5. (Exercise 2 in DF §10.5.) Suppose that

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[d, "\delta"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D'
\end{tikzcd}$$

is a commutative diagram of groups (you may assume the groups are abelian, if you find this psychologically helpful—it won't make a difference), and that the rows are exact (that is, the diagrams $A \to B \to C$, $B \to C \to D$, $A' \to B' \to C'$, $B' \to C' \to D'$ are exact at B, C, B', C' respectively). Prove that

(a) if α is surjective, and β and δ are injective, then γ is injective;

Proof. Let the maps be $\psi : A \to B$, $\varphi : B \to C$, $\theta : C \to D$, $\psi' : A' \to B'$, $\varphi' : B' \to C'$, and $\theta' : C' \to D'$. Suppose $c \in \ker(\gamma)$, so the $\gamma(c) = 1$.

By commutativity, we have $\delta \circ \theta(c) = \theta' \circ \gamma(c) = 1$, thus $\theta(c) = 1$ because δ is injective. So $c \in \ker(\theta) = \mathrm{Im}(\varphi)$ by exactness. So there exists $b \in B$ such that $\varphi(b) = c$.

By commutativity, we have $\varphi' \circ \beta(b) = \gamma \circ \varphi(b) = \gamma(c) = 1$. Thus $\beta(b) \in \ker(\varphi') = \mathrm{Im}(\psi')$, by exactness. So there exists $a' \in A'$ such that $\psi'(a') = \beta(b)$.

α is surjective, so there exists $a \in A$ such that $\alpha(a) = a'$. So $\psi' \circ \alpha(a) = \psi'(a') = \beta(b)$. By commutativity, we have $\beta \circ \psi(a) = \psi' \circ \alpha(a) = \beta(b)$. Since β is injective, this means $\psi(a) = b$.

By this last fact, $b \in \mathrm{Im}(\psi) = \ker(\phi)$, so $\phi(b) = 1$. b was defined so that $\phi(b) = c$, thus $c = 1$. So γ is injective. □

(b) if δ is injective, and α and γ are surjective, then β is surjective.

Proof. Let $b' \in B'$. Since γ is surjective, there exists $c \in C$ such that $\gamma(c) = \varphi'(b')$. So $\gamma(c) \in \mathrm{Im}(\varphi') = \ker(\theta')$, thus $\theta'(\gamma(c)) = 1$. By commutativity, $\delta(\theta(c)) = \theta'(\gamma(c)) = 1$.

δ is injective, so $\theta(c) = 1$. So $c \in \ker(\theta) = \mathrm{Im}(\varphi)$, by exactness, thus there exists $b \in B$ such that $\varphi(b) = c$.

By commutativity, $\varphi'(\beta(b)) = \gamma(\varphi(b)) = \gamma(c) = \varphi'(b')$. So $\varphi'(\beta(b)) \cdot (\varphi'(b'))^{-1} = \varphi'(\beta(b) \cdot (b')^{-1}) = 1$. Hence, $\beta(b) \cdot (b')^{-1} \in \ker \varphi' = \mathrm{Im}(\psi')$, so there exists $a' \in A'$ such that $\psi'(a') = \beta(b) \cdot (b')^{-1}$.

Since α is surjective, there exists $a \in A$ such that $\alpha(a) = a'$. By exactness, $\beta(\psi(a)) = \psi'(\alpha(a)) = \psi'(a') = \beta(b) \cdot (b')^{-1}$. So $b' = \beta(\psi(a))^{-1} \cdot \beta(b) = \beta(\psi(a^{-1}) \cdot b)$. Therefore, β is surjective. □

6. (Exercise 12 in DF §10.5.) Let A be an R-module, let I be any nonempty index set, and for each $i \in I$ let B_i be an R-module. Prove that we have the following R-module isomorphisms:

(a) $\mathrm{Hom}_R(\bigoplus_{i\in I} B_i, A) \cong \prod_{i\in I} \mathrm{Hom}_R(B_i, A)$;

Proof. Define a map $\varphi : \mathrm{Hom}_R(\bigoplus_{i\in I} B_i, A) \to \prod_{i\in I} \mathrm{Hom}_R(B_i, A)$ as follows:

Let $f \in \mathrm{Hom}_R(\bigoplus_{i\in I} B_i, A)$. For each $i \in I$, the map $f \circ \iota_i$ gives a homomorphism from B_i to A, where ι_i is the natural inclusion of B_i into $\bigoplus_{i\in I} B_i$. So let $\varphi(f)$ be $\prod f \circ \iota_i \in \prod_{i\in I} \mathrm{Hom}_R(B_i, A)$.

We will show that φ is injective. Suppose $\varphi(f) = 0$. Then $\prod f \circ \iota_i = 0$, thus $f \circ \iota_i = 0$ for each $i \in I$. Let $x \in \bigoplus_{i\in I} B_i$. Since this is a direct sum, x can be expressed as $x = \sum_{j\in J} \iota_j(b_j)$, where J is some finite subset of I and $b_j \in B_j$ for each $j \in J$. Thus,

$$\begin{aligned} f(x) &= f(\sum_{j\in J} \iota_j(b_j)) \\ &= \sum_{j\in J} f \circ \iota_j(b_j) \\ &= \sum_{j\in J} 0 \\ &= 0 \end{aligned}$$

so f is the zero map, thus φ is injective.

Now, let $\prod g_i \in \prod_{i\in I} \operatorname{Hom}_R(B_i, A)$, where $g_i : B_i \to A$ for each $i \in I$. We will show that g is the image of f under φ, where $f : \bigoplus_{i\in I} B_i \to A$ is the map given by $f(\sum_{j\in J} \iota_j(b_j)) = \sum_{j\in J} g_j(b_j)$ for any finite subset J of I where $b_j \in B_j$ for each $j \in J$ (again, every element of the direct sum is of this form).

To show that $\varphi(f) = \prod f \circ \iota_i$ equals $\prod g_i$, all we need to show is that $f \circ \iota_i = g_i$ for each $i \in I$. But this is clear, since for any $x \in B_i$ we have defined f so that $f \circ \iota_i(x) = g_i(x)$. Thus φ is surjective, therefore an isomorphism. □

(b) $\operatorname{Hom}_R(A, \prod_{i\in I} B_i) \cong \prod_{i\in I} \operatorname{Hom}_R(A, B_i)$.

Proof. Define $\varphi : \operatorname{Hom}_R(A, \prod_{i\in I} B_i) \to \prod_{i\in I} \operatorname{Hom}_R(A, B_i)$ by $\varphi(f) = \prod \pi_i \circ f$.

φ is clearly injective. If $f(x) \neq 0$ for some $x \in A$, then $\pi \circ f(x) \neq 0$ for some $i \in I$, thus $\varphi(f) = \prod \pi_i \circ f \neq 0$, so $f \notin \ker(\varphi)$.

Now let $\prod g_i \in \prod_{i\in i} \operatorname{Hom}_R(A, B_i)$, where $g_i : A \to B_i$ for each $i \in I$. Let $f : A \to \prod_{i\in I} B_i$ be defined by $f(x) = \prod g_i(x)$. Then for any $x \in A$, we have

$$\begin{aligned}\varphi(f) &= \prod \pi_i \circ f \\ &= \prod \pi_i \circ \prod g_j \\ &= \prod g_i\end{aligned}$$

therefore $\varphi(f) = g$. So φ is surjective, thus an isomorphism. □