6.2. Suppose f is non-negative and measurable and μ is σ-finite. Show there exist simple functions s_n increasing to f at each point such that $\mu(\{x : s_n(x) \neq 0\}) < \infty$ for each n.

Proof. Let $X = \bigcup_1^\infty X_i$ where $\mu(X_i) < \infty$ for each i, and $X_i \cap X_j = \emptyset$ for $i \neq j$. Define $f_i = f \cdot \mathbb{1}(X_i)$. Then f_i is non-negative and measurable, so for each i there is a sequence $\{s_i^n\}_{n=0}^\infty$ of simple functions increasing to f_i. Since $f_i = 0$ on X_i^c and this sequence is increasing, we must have $s_i^n = 0$ on X_i^c.

Define a sequence t_n of functions by $t_n = \sum_{i=1}^{n} s_i^n$. Each t_n is a sum of simple functions, hence simple. For $x \in X_i$ we have $t_n(x) = 0$ if $i > n$ and $t_n = s_i^n(x)$ if $i \leq n$. So $t_n(x) \to f(x)$ for all x. Also, for all n we have

$$\mu(\{x : t_n(x) \neq 0\}) \leq \mu\left(\bigcup_1^n X_i\right) \leq \sum_1^n \mu(X_i) < \infty.$$

□

6.3. Let f be a non-negative measurable function. Prove that

$$\lim_{n\to\infty} \int (f \wedge n) \to \int f.$$

Proof. This is a straightforward application of the monotone convergence theorem. Let $f_n = f \wedge n$. Let $x \in X$. Then one of $f(x) \leq n$, or $n+1 \leq f(x)$, or $n < f(x) < n+1$ holds. In the first case, $f(x) \wedge n = f(x) = f(x) \wedge (n+1)$. In the second, $f(x) \wedge n = n \leq n+1 = f(x) \wedge (n+1)$. In the last, $f(x) \wedge n = n \leq f(x) = f(x) \wedge (n+1)$. Therefore, $f_n \leq f_{n+1}$.

Let $N = f(x)$. Whenever $n > N$, we have $|(f(x) \wedge n) - f(x)| = |f(x) - f(x)| = 0 < \epsilon$ for a given ϵ. Thus $\lim_{n\to\infty} f \wedge n = f$. Clearly, the minimum of two non-negative functions (f and the constant function n) is non-negative. Thus f_n satisfies the hypotheses of the monotone convergence theorem, of which the conclusion is the desired result. □

6.4. Let $(X, \mathcal{A}, \mu)$ be a measure space and suppose μ is σ-finite. Suppose f is integrable. Prove that given ϵ there exists δ such that

$$\int_A |f(x)|\mu(dx) < \epsilon$$

whenever $\mu(A) < \delta$.

Proof. Since f is integrable, f is measurable. Therefore, so is $|f|$. So, by the previous exercise, there exists an n such that $\left|\int |f| - \int (|f| \wedge n)\right| < \frac{\epsilon}{2}$. However, $|f| \wedge n \leq |f|$, so we may drop the outside absolute values. We can also apply linearity to obtain

$$\int |f| - (|f| \wedge n) < \frac{\epsilon}{2}.$$

For any measurable function g and measurable set A, we have $g \cdot \mathbb{1}(A) \leq g$, so $\int_A g \leq \int g$. This yields

$$\int_A |f| - (|f| \wedge n) \leq \int |f| - (|f| \wedge n) < \frac{\epsilon}{2}.$$

Let $\delta = \frac{\epsilon}{2n}$, and assume $\mu(A) < \delta$. Since n is measurable and $|f| \wedge n \leq n$, we have

$$\int_A |f| \wedge n \leq \int_A n = n\mu(A) \leq \frac{\epsilon}{2}.$$

Note that we have applied here the fact that $\displaystyle\int_A n = n\mu(A)$, which we acquire from Proposition 6.3 (1) by taking $a = b = n$.

Combining these inequalities, we find

$$\int_A |f| = \int_A |f| - (|f| \wedge n) + (|f| \wedge n) = \int_A |f| - (|f| \wedge n) + \int_A |f| \wedge n < \epsilon.$$

The fact that μ is σ-finite seems irrelevant. $\square$

6.5. Suppose $\mu(X) < \infty$ and f_n is a sequence of bounded real-valued measurable functions that converge to f uniformly. Prove that

$$\int f_n d\mu \to \int f d\mu.$$

This is called the *bounded convergence theorem.*

Proof. Given $\epsilon > 0$, let $\epsilon' = \frac{\epsilon}{\mu(X)}$. Since each f_n is bounded by some M_n, $\displaystyle\int f_n \leq M_n\mu(X)$. So $\displaystyle\int f_n$ is finite. Since $f_n \to f$ uniformly, there is some N such that $|f(x) - f_n(x)| < \epsilon'$ for all x whenever $n > N$. So f is bounded as well, since $|f| = |f - f_{n+1}| + |f_{n+1}| \leq \epsilon' + M_{n+1}$. Thus $\displaystyle\int f$ is also finite (by the same reasoning). So

$$\left|\int f - \int f_n\right| = \left|\int f - f_n\right| \leq \int |f - f_n| \leq \int \epsilon' = \frac{\epsilon}{\mu(X)}\mu(X) = \epsilon$$

whenever $n > N$. Therefore, $\displaystyle\int f_n \to \int f$. $\square$