Estimating $n!$

$n! < n^n$ $\qquad\qquad$ $n! \approx \frac{n^n}{e^n}\sqrt{2\pi n}$

$\log n! < n \log n$

$n! > \left(\frac{n}{2}\right)^{\frac{n}{2}}$

$\frac{n}{2}\log\frac{n}{2} < \log n!$

$\approx \frac{n}{2}(\log n - \log 2)$ for big n $\qquad$ $\frac{n}{2}\log n < \log n!$

$\Rightarrow \log n! \sim C\, n \log n$

2/3/15

VLW 1

graph $G = \{V(G), E(G)\}$ $\qquad$ $\exists\ (x,y) \in E(G) \Rightarrow (y,x) \in E(G)$

set vertices: $V(G)$; edges: $E(G)$, $(x,y) \in V(G) \times V(G)$

K_n

Complete Graph

$\binom{n}{2}$ edges $= |E(K_n)|$

loops (x,x)

double edges $\begin{cases} E(G) \text{ multiset} \\ (x,y) \text{ appears } >1 \text{ time} \end{cases}$

Simple: a graph w/ no loops or double edges

C_n

cycle graph $\quad$ C_4  $\quad$ C_5  $\qquad$ $V(G) = \{0, 1, \ldots, n-1\}$

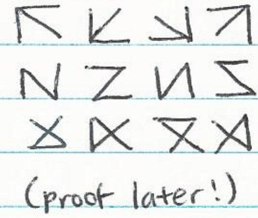
polygon $\qquad$ $E(G) = \{(x, x+1 \pmod n) \mid x \in V(G)\}$

Tree a graph w/ no cycles, connected $:= \forall\, x, y \in V(G)$ can find path $x \to y$

edges $(x, x_1); (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x)$

$V(G) = [3]$ 

$V(G) = [4]$ 



n^{n-2} trees (proof later!)

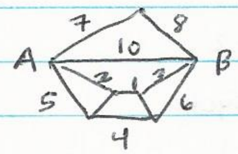
<u>directed graph</u> allow: $(x,y) \in E(G)$
$(y,x) \notin E(G)$

  Sometimes allow 

undirected C_4 directed C_4

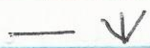
<u>Weighted graph</u> function $W: E(G) \to S$ ← set usually $\mathbb{R}$

<u>Ex: traffic</u>



G_1 and G_2 are <u>isomorphic</u> if $\exists\ f: V(G) \to V(G)$
↳ $f': E(G) \to E(G)$
$(x,y) \to (f(x), f(y))$
Which sends G_1 to G_2.

an isomorphism from G to itself is an automorphism

$V(G) = 4$ # of trees up to isomorphism:  unlabeled
the others were labeled distinguishable indistinguishable

<u>7 bridges problem</u> 18th century

Euler: a path where all edges are used exactly once (Eulerian)

Hamiltonian: a path where all vertices are used exactly once
↑ optimizing is "hard"

Eulerian condition: all vertices (except possibly two) have even degree
- degree of $v \in V$ = # edges $e \in E(G)$ s.t. edge is (v,u) for some v.

 degree 3  degree 2

<u>Ex 2.2</u>

trees with 5 vertices <u>unlabeled</u>

<u>labeled</u>

$\frac{5!}{4!}$ $\frac{5!}{2}$ $\frac{5!}{2}$

$5 + 60 + 60 = 125 = 5^3$

2.1 | There are n^{n-2} labeled trees on n vertices.

Proof 3 | Let $t(d_1, \dots, d_n)$ be # of trees with vertex degrees multi set $\{d_1, \dots, d_n\}$

 degrees 3,1,1,1 $\qquad t(5,2) = t(2,5)$

Fact a tree on n vertices has $n-1$ edges $\rightarrow$ n edges $\exists$ a cycle somewhere

WLOG $d_1 \geq d_2 \dots \geq d_n$

$d_n = 1$ because $\sum d_i = 2n-2$

must be cycle

$t(d_1, \dots, d_n) = t(d_1, \dots, d_{n-1}, 1)$

$= \sum_{i=1}^{n-1} t(d_1, \dots, d_{i-1}, d_i - 1, d_{i+1}, \dots, d_{n-1})$

take away this edge and you have a tree with one less vertex and one less edge

Guess $t(d_1, \dots, d_n) = \binom{n-2}{d_1 - 1, d_2 - 1, \dots d_n - 1}$ works

check boundary cases

then check recurrence.

$(x_1 + x_2 + \dots + x_k)^{n'} = \sum \binom{n'}{r_1 \dots r_k} x_1^{r_1} \dots x_k^{r_k}$

pick $x_1 = \dots = x_k = 1$

$k = n$ $\qquad n^{n-2} = \sum t(d_1, \dots, d_n)$

$n' = n-2$

- Graphs
  - check C_n, K_n $\qquad$ - Eulerian / Hamiltonian
  - labeled v/ unlabeled
- Trees
  - split into equivalence classes & count automorphisms
  - labeled n^{n-2}