In these exercises, μ denotes Lebesgue measure. A subset of $\mathbb{R}$ is called Borel if it is an element of the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$.

1. Find an example of Lebesgue measurable subsets $\{A_i : i \in \mathbb{N}\}$ of $[0,1]$ such that $\mu(A_n) > 0$ for each n, $\mu(A_n \Delta A_m) > 0$ if $n \neq m$, and $\mu(A_n \cap A_m) = \mu(A_n)\mu(A_m)$ if $n \neq m$.

$$A_n = \bigcup_{i=0}^{2^{n-1}-1} \left(\frac{2i}{2^n}, \frac{2i+1}{2^n} \right), \qquad n \geq 1$$

Proof. First, note that for each n, A_n is a union of 2^{n-1} intervals, each of measure 2^{-n}. So A_n has measure $\frac{1}{2}$. Now, let $m < n$. We will show that a given interval in the union defining A_m intersects exactly 2^{n-m-1} of the intervals in the union defining A_n, and that each of these intersections is actually a containment.

Fix two such intervals, $\left(\frac{2i}{2^n}, \frac{2i+1}{2^n}\right) \subset A_n$ and $\left(\frac{2j}{2^n}, \frac{2j+1}{2^n}\right) \subset A_m$ for some i and j such that $0 \leq i < 2^n$ and $0 \leq j < 2^m$. Assume, for a contradiction, that $\left(\frac{2i}{2^n}, \frac{2i+1}{2^n}\right) \cap \left(\frac{2j}{2^n}, \frac{2j+1}{2^n}\right) \neq \emptyset$ but $\left(\frac{2i}{2^n}, \frac{2i+1}{2^n}\right) \not\subset \left(\frac{2j}{2^n}, \frac{2j+1}{2^n}\right)$. This means we must have either

$$\frac{2i}{2^n} < \frac{2j}{2^m} < \frac{2i+1}{2^n} \qquad \text{or} \qquad \frac{2i}{2^n} < \frac{2j+1}{2^m} < \frac{2i+1}{2^n}.$$

However, these inequalities reduce to

$$2^{n-m}j - \frac{1}{2} < i < 2^{n-m}j \qquad \text{or} \qquad 2^{n-m-1}(2j+1) - \frac{1}{2} < i < 2^{n-m-1}(2j+1),$$

both of which contradict that i is an integer. So we must have $\left(\frac{2i}{2^n}, \frac{2i+1}{2^n}\right) \subset \left(\frac{2j}{2^n}, \frac{2j+1}{2^n}\right)$.

The inclusion $\left(\frac{2i}{2^n}, \frac{2i+1}{2^n}\right) \subset \left(\frac{2j}{2^n}, \frac{2j+1}{2^n}\right)$ holds if and only if we have

$$\frac{2j}{2^m} \leq \frac{2i}{2^n} < \frac{2i+1}{2^n} \leq \frac{2j+1}{2^m}.$$

Solving shows that these inequalities are satisfies if and only if $2^{n-m}j \leq i < 2^{n-m-1}(2j+1)$. Therefore, there are exactly $2^{n-m-1}(2j+1) - 2^{n-m}j = 2^{n-m-1}$ choices of i for which these inequalities hold, each corresponding to an interval $\left(\frac{2i}{2^n}, \frac{2i+1}{2^n}\right)$ that is contained in $\left(\frac{2j}{2^n}, \frac{2j+1}{2^n}\right)$.

A_m is a disjoint union of 2^{m-1} intervals, and we have just shown that each of these contains 2^{n-m-1} intervals from A_n. Each interval in A_n has measure 2^{-n}. Therefore

$$\mu(A_n \cap A_m) = 2^{m-1} \cdot 2^{n-m-1} \cdot 2^{-n} = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \mu(A_n)\mu(A_m).$$

Finally, we have $\mu(A_n \Delta A_m) = \mu(A_n) + \mu(A_m) - 2\mu(A_n \cap A_m) = \frac{1}{2} > 0$. ∎

2. (F 1.5, exercise 30) If $E \in \mathcal{L}$ and $m(E) > 0$, for any $\alpha < 1$ there is an open interval I such that $m(E \cap I) > \alpha m(I)$.

Proof. First, we will prove Proposition 1.20 as a lemma, which claims that for any $E \in \mathcal{M}_\mu$ with $\mu(E) < \infty$, there is a set A that is a finite union of disjoint open intervals such that $\mu(E \Delta A) < \epsilon$ (it does not actually state "disjoint", but this is just as easy to prove). By Theorem 1.18, there is an open set $U \supset E$ such that $\mu(U) \leq \mu(E) + \frac{\epsilon}{2}$. U can be written as a union of disjoint open intervals I_k. Since $\mu(U) < \infty$, we know that there is some n such that $\sum_{n+1}^{\infty} I_k < \frac{\epsilon}{2}$, so let $A = \bigcup_{1}^{n} I_k$. We have $\mu(E \setminus A) \leq \mu(U \setminus A) = \sum_{n+1}^{\infty} I_k < \frac{\epsilon}{2}$ and $\mu(A \setminus E) \leq \mu(U \setminus E) = \mu(U) - \mu(E) \leq \frac{\epsilon}{2}$, thus $\mu(E \Delta A) = \mu(E \setminus A) + \mu(A \setminus E) < \epsilon$.

Now, for the main proof, begin by assuming $m(E) < \infty$. Then there is a finite union A of open intervals such that $m(E\Delta A) < (1-\alpha)m(E)$. We have

$$\begin{aligned} m(E) &= m(E \cap A) + m(E \setminus A) \\ &\leq m(E \cap A) + m(E\Delta A) \\ &< m(E \cap A) + (1-\alpha)m(E) \end{aligned}$$

which gives $\alpha m(E) < m(E \cap A)$. Using this, we find

$$\begin{aligned} m(A) &= m(E \cap A) + m(A \setminus E) \\ &\leq m(E \cap A) + m(E\Delta A) \\ &< m(E \cap A) + (1-\alpha)m(E) \\ &< m(E \cap A) + (1-\alpha)\frac{m(E \cap A)}{\alpha} \\ &= \frac{m(E \cap A)}{\alpha} \end{aligned}$$

so $\alpha m(A) < m(E \cap A)$. We know there are disjoint open intervals $I_1, \ldots, I_n$ such that $A = I_1 \cup \cdots \cup I_n$. This means

$$\sum_1^n m(E \cap I_k) = m(E \cap A) > \alpha m(A) = \sum_1^n \alpha m(I_k)$$

so there must be some k for which $m(E \cap I_k) > \alpha m(I_k)$.

Finally, consider the case where $m(E) = \infty$. There must be some open interval $J = (j, j+2)$ for $j \in \mathbb{Z}$ such that $E \cap J$ has finite, positive measure. Otherwise, $m(E) = m(\bigcup_1^\infty E \cap (i, i+2)) \leq \sum_1^\infty m(E \cap (i, i+2)) = \sum_1^\infty 0 = 0$, a contradiction. We can apply the finite case of the proposition, which we have just proven, to the set $E \cap J$ to obtain an open interval I such that

$$m(E \cap (I \cap J)) = m((E \cap J) \cap I) > \alpha m(I) \geq \alpha m(I \cap J).$$

Therefore, $I \cap J$ is the interval we seek (it is an intersection of open intervals, thus is an open interval).

$\square$

3. Suppose that $A \subset \mathbb{R}$ is a Borel set of $\mathbb{R}$ with $\mu(A) > 0$. Prove that the set of differences

$$\{x - y : x, y \in A\}$$

contains a nonempty open interval that includes the origin.

Proof. For any set S, denote the set of differences by $S - S = \{x - y : x, y \in S\}$. Apply the previous exercise to obtain an open interval I such that $\frac{3}{4}\mu(I) < \mu(A \cap I)$. Clearly, I has finite measure, since it is strictly less than something; it must also be nonempty because otherwise we would have

$$0 = \frac{3}{4}\mu(\emptyset) = \frac{3}{4}\mu(I) < \mu(A \cap I) = \mu(\emptyset) = 0.$$

So $J = (-\frac{1}{2}\mu(I), \frac{1}{2}\mu(I))$ is nonempty and includes the origin. Let $E = A \cap I$.

Suppose, for a contradiction, that $J \not\subset E - E$. Then there is some $x \in J$ such that, for all $y \in E$, $x + y \notin E$; otherwise, there would be some $a \in E$ such that $x = y - a \in E - E$, a contradiction. Thus, $(x + E) \cap E = \emptyset$, and so

$$\mu((x+E) \cup E) = \mu(x+E) + \mu(E) - \mu((x+E) \cap E) = \mu(E) + \mu(E) - 0 = 2\mu(E).$$

Also, $E \subset I$, so $(x+E) \cup E \subset (x+I) \cup I$. But $|x| < \frac{1}{2}\mu(I)$, therefore $\mu((x+I) \cap I) \geq \frac{1}{2}\mu(I)$. This means

$$2\mu(E) = \mu((E+x) \cup E) \leq \mu((I+x) \cup I) \leq 2\mu(I) - \frac{1}{2}\mu(I) = \frac{3}{2}\mu(I)$$

thus $\mu(E) \leq \frac{3}{4}\mu(I)$, a contradiction. So we must have $J \subset E - E \subset A - A$. $\square$

4. Construct a Borel set $A \subset \mathbb{R}$ such that $0 < \mu(A \cap I) < \mu(I)$ for every open interval I. We may wish to consider variants of Cantor-like sets, and F 1.5 exercise 32 may assist you in the construction of a Cantor set of positive measure (something that was also an exercise in a preceding assignment).

Proof. Let $\{q_i\}$ be an enumeration of the rationals, and for each k let I_k be an open interval of length 3^{-i}, centered at q_i. Now, we can create a sequence of disjoint sets U_n by

$$U_n = I_n \setminus \bigcup_{n+1}^{\infty} I_i.$$

Take note of three facts:

1. Any open interval $I \subset \mathbb{R}$ contains some U_n:
   Let I' be an interval of $\frac{1}{3}$ the length of I, but with the same center as I. I' contains an infinite number of rationals, but there are only finitely many n for which the width of U_n (meaning $\sup U_n - \inf U_n$) is at least that of I'. So I' must contain some rational q_n for which the width of U_n is less than that of I'. Since U_n is centered at $q_n \in I'$, we must have $U_n \subset I$.
2. The U_n are disjoint: This is clear from their construction.
3. Every U_n contains a subset A_n such that $0 < \mu(A_n) < \mu(U_n)$:
   The U_n have positive measure because
   $$\mu(U_n) \geq \mu(U_n) - \mu(\bigcup_{n+1}^{\infty} I_i) \geq 3^{-n} - \sum_{n+1}^{\infty} 3^{-k} = \frac{3^{-n}}{2} > 0.$$
   For each n, let $a_n = \inf(U_n)$ and $b_n = \sup(U_n)$, and take an increasing sequence $\{s_k^n\}$ such that $s_1^n = a$, $s_{k+1}^n - s_k^n < 3^{-n} = \mu(U_n)$ for each k, and $s_k^n \to b$. Now define sets $A_k^n = U_n \cap (a, s_k)$ for each k. For each k, if $\mu(A_k) = 0$ then $0 \leq \mu(A_{k+1}^n) = \mu(A_k^n) \cap \mu(U_n \cap (s_k, s_{k+1})) < \mu(U_n)$ by construction. But we cannot have $\mu(A_k^n) = 0$ for all k, since then $\mu(U_n) = \mu(\bigcup_1^\infty A_k^n) \leq \sum_1^\infty \mu(A_k^n) = 0$, a contradiction. So we may let K be the smallest index such that $\mu(A_K^n) > 0$. We must have $K > 1$ because $A_1^n = U_n \cap (a, a) = \emptyset$. Therefore, $\mu(A_{K-1}^n) = 0$, and hence by the previous argument we have
   $$0 < \mu(A_K^n) < \mu(U_n).$$
   So $A_n = A_K^n$ satisfies $0 < \mu(A_n) < \mu(U_n)$.

Let $A = \bigcup_1^\infty A_n$, $U = \bigcup_1^\infty U_n$, and let I be any open interval. By (1), I contains some U_k. Thus, $A_k \subset A \cap I$. Thus, $0 < \mu(A_k) \leq \mu(A \cap I)$. Since the U_n are disjoint, so are the A_n, thus

$$\mu(A \cap I) = \sum_1^\infty \mu(A_n \cap I) < \sum_1^\infty \mu(U_n \cap I) = \mu(U \cap I) \leq \mu(I).$$

The middle inequality comes from the fact that at least one of the terms on the left is strictly less than one on the right, specifically the term $\mu(A_k \cap I) < \mu(U_k \cap I)$. For all other n, we have $A_n \cap I \subset U_n \cap I$, so we at least have $\mu(A_n \cap I) \leq \mu(U_n \cap I)$. Therefore, putting these inequalities together gives $0 < \mu(A \cap I) < \mu(I)$.

□

5. Read the proof of Theorem 1.19 from the text. Complete the missing details.

Proof. The theorem is proven for all $E \subset \mathbb{R}$ with finite measure. Since μ is finite on bounded sets, $\mathbb{R}$ is σ-finite. So $\mathbb{R} = \bigcup_1^\infty X_i$ where $\mu(X_i) < \infty$ for each i. Let $E \in \mathcal{M}_\mu$ and define $E_i = E \cap X_i$.

We will show that (a) implies (c). By the finite case of the theorem, there exist $H_i \in F_\sigma$ and N_i such that $E_i = H_i \cup N_i$ and $\mu(N_i) = 0$. Each H_i is a countable union of closed sets, so $H = \bigcup_1^\infty H_i$ is also a countable union of closed sets, so $H \in F_\sigma$. Also, letting $N = \bigcup_1^\infty N_i$ we have $\mu(N) \leq \sum_1^\infty \mu(N_i) = 0$, so $\mu(N) = 0$. Therefore,

$$E = \bigcup_1^\infty H_i \cup N_i = \left(\bigcup_1^\infty H_i\right) \cup \left(\bigcup_1^\infty N_i\right) = H \cup N$$

giving the desired result.

Now, we will show that (a) implies (b). Again, let $E \in \mathcal{M}_\mu$. By theorem 1.9, $\mathcal{M}_\mu$ is a σ-algebra, therefore $E^c \in \mathcal{M}_\mu$. We have just proven that (a) implies (c), therefore we can write $E^c = H \cup N$ for some $H \in F_\sigma$ and $\mu(N) = 0$. H is a countable union of closed sets, therefore H^c is a countable intersection of open sets, i.e. $H^c \in G_\delta$. So

$$E = (E^c)^c = (H \cup N)^c = H^c \setminus N$$

thus E takes the form given in (b). $\square$