Note: $\mathbb{1}(P(x))$ is the indicator function that evaluates as 1 if the proposition P holds of x, and 0 otherwise.

1. Prove that $d(f,g) = \int_{[0,1]} |f(x) - g(x)|dx$ is a metric on the space of continuous functions on $[0,1]$.

Proof. Let C be the space of continuous functions on $[0,1]$. For any $f \in C$,

$$d(f,f) = \int_{[0,1]} |f(x) - f(x)|dx = \int_{[0,1]} 0dx = 0.$$

Now, suppose $d(f,g) = \int_{[0,1]} |f(x) - g(x)|dx = 0$. Since the integrand is non-negative on the whole domain, the integral can only be zero if the integrand is zero on the whole domain. So $|f(x) - g(x)| = 0$ for all $x \in [0,1]$, thus $f = g$. So d is positive definite. Clearly, $d(f,g) = \int_{[0,1]} |f(x) - g(x)|dx = \int_{[0,1]} |g(x) - f(x)|dx = d(g,f)$, so d is symmetric.

Finally, let $f, g, h \in C$. Then

$$\begin{aligned}
d(f,h) &= \int_{[0,1]} |f(x) - h(x)|dx \\
&= \int_{[0,1]} |f(x) - g(x) + g(x) - h(x)|dx \\
&= \int_{[0,1]} |f(x) - g(x)| + |g(x) - h(x)|dx \\
&= \int_{[0,1]} |f(x) - g(x)|dx + \int_{[0,1]} |g(x) - h(x)|dx \\
&= d(f,g) + d(g,h).
\end{aligned}$$

So d satisfies the triangle inequality as well, hence it is a metric.

□

2. Prove that $d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$ is also such a metric. Show that the set of piecewise linear functions is a dense subset of this metric space.

Proof. Let C be the space of continuous functions on $[0,1]$. For any $f \in C$, $d(f,f) = \sup_{x\in[0,1]} |f(x) - f(x)| = \sup_{x\in[0,1]} 0 = 0$. Now, let $f, g \in C$. If $d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)| = 0$, then at every point $x \in [0,1]$ we have $|f(x) - g(x)| \leq 0$, thus $f(x) = g(x)$ for all $x \in [0,1]$. Thus, d is positive definite. Clearly, $d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)| = \sup_{x\in[0,1]} |g(x) - f(x)| = d(g,f)$. So d is symmetric.

Let $f, g, h \in C$. Then

$$\begin{aligned}
d(f,h) &= \sup_{x\in[0,1]} |f(x) - h(x)| \\
&= \sup_{x\in[0,1]} |f(x) - g(x) + g(x) - h(x)| \\
&= \sup_{x\in[0,1]} |f(x) - g(x)| + |g(x) - h(x)| \\
&\leq \sup_{x\in[0,1]} |f(x) - g(x)| + \sup_{x\in[0,1]} |g(x) - h(x)| \\
&= d(f,g) + d(g,h)
\end{aligned}$$

so d satisfies the triangle inequality, therefore d is a metric.

We will now show that the subset of piecewise linear functions is a dense subset of C by constructing a piecewise linear function that converges to an arbitrary function in C. Let $f \in C$. Define a sequence of functions (f_n), each from $[0,1]$ to $\mathbb{R}$, by

$$f_n(x) = \sum_{i=0}^{n-1} \left[f(\tfrac{i}{n}) + (x - \tfrac{i}{n}) \frac{f(\frac{i+1}{n}) - f(\frac{i}{n})}{\frac{1}{n}} \right] \cdot \mathbb{1}(x \in [\tfrac{i}{n}, \tfrac{i+1}{n})) + f(1) \cdot \mathbb{1}(x = 1).$$

On each interval $\left[\frac{i}{n}, \frac{i+1}{n}\right]$, f_n is simply the line connecting $f(\frac{i}{n})$ and $f(\frac{i+1}{n})$, so it is clearly continuous and piecewise linear.

We will show that $f_n \to f$ in C. Let $\epsilon < 0$. Since $[0,1]$ is compact under the usual metric, and f is continuous under this metric, f is uniformly continuous under it. So for any $x, y \in [0,1]$, there exists some $\delta > 0$ such that whenever $|x - y| < \delta$ we have $|f(x) - f(y)| < \frac{\epsilon}{2}$. Let N be any natural number such that $\frac{1}{N} < \delta$.

Next, let $x \in [0,1]$ and $n > N$. If $x = 1$, then $|f_n(x) - f(x)| = 0 < \epsilon$ by construction. Otherwise, there is some $i < n$ such that $\frac{i}{n} \leq x < \frac{i+1}{n}$. We have

$$\begin{aligned} |f(x) - f_n(x)| &= \left| f(x) - f(\tfrac{i}{n}) - (x - \tfrac{i}{n}) \frac{f(\frac{i+1}{n}) - f(\frac{i}{n})}{\frac{1}{n}} \right| \\ &\leq \left| f(x) - f(\tfrac{i}{n}) \right| + n \left| (x - \tfrac{i}{n}) \right| \left| f(\tfrac{i+1}{n}) - f(\tfrac{i}{n}) \right| \\ &< \frac{\epsilon}{2} + n \cdot \frac{1}{n} \cdot \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Since $|f(x) - f_n(x)| < \epsilon$ for all $n > N$, clearly $\sup\limits_{[0,1]} |f(x) - f_n(x)| < \epsilon$ for all $n > N$. Therefore, $f_n \to f$ in C, hence the subset of piecewise linear functions is a dense subset of C. $\square$

3. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of functions. For $a > 0$ and $m \in \mathbb{N}$, set

$$E_m^a = \{x \in [0,1] : |f_m(x)| < a\}.$$

Prove that

$$\bigcap_{k=1}^{\infty} \bigcup_{l=1}^{\infty} \bigcap_{m=l}^{\infty} E_m^{1/k} = \{x \in [0,1] : f_n(x) \to 0 \text{ as } n \to \infty\}.$$

If we consider instead the set of points $x \in [0,1]$ for which $f_n(x)$ converges as $n \to \infty$, the new set may also be written in a similar form, where instead we consider

$$E_{m,n}^a = \{x \in [0,1] : |f_m(x) - f_n(x)| < a\}$$

for $m, n \in \mathbb{N}$, and again use several unions or intersections, each of which ranges only over a countable set. Find such an expression for the new set of points.

Proof. We will show that each set in the stated equivalence is a subset of the other. First, suppose that $x \in \bigcap\limits_{k=1}^{\infty} \bigcup\limits_{l=1}^{\infty} \bigcap\limits_{m=l}^{\infty} E_m^{1/k}$. Then for all $k \geq 1$, there exists some $l \geq 1$ such that for all $m \geq l$, $|f_m(x)| < \frac{1}{k}$. Now, let $\epsilon > 0$. Choose k so that $\frac{1}{k} < \epsilon$. Taking our N to be the l given in the statement, we have that $|f_m(x) - 0| < \frac{1}{k} < \epsilon$ whenever $m > N$. Therefore, $f_n(x) \to 0$ by definition. So $x \in \{x \in [0,1] : f_n(x) \to 0 \text{ as } n \to \infty\}$, and thus the inclusion $\subseteq$ has been verified.

Next, suppose $x \in \{x \in [0,1] : f_n(x) \to 0 \text{ as } n \to \infty\}$. Then $f_n(x) \to 0$, so for every $\epsilon > 0$ there exists some $l > 0$ such that $|f_n(x)| < \epsilon$ whenever $m \geq l$. In particular, this holds whenever ϵ is any positive integer k. This proves the inclusion $\supseteq$.

For the next part, we will show that

$$\bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{m=N}^{\infty} \bigcap_{n=N}^{\infty} E_{m,n}^{1/k} = \{x \in [0,1] : f_n(x) \to c \text{ for some } c \in \mathbb{R}\}.$$

A sequence in $\mathbb{R}$ is convergent if and only if it is Cauchy, so the set on the righthand side is equal to the set of all $x \in [0,1]$ such that for all $\epsilon > 0$, there exists some $N \geq 1$ such that for all $m, n \geq N$ we have $|f_m(x) - f_n(x)| \leq \epsilon$. It is clear that this statement implies the statement obtained by replacing ϵ with $\frac{1}{k}$, where k is an integer. The converse is true as well, since given any $\epsilon > 0$ we can find some integer k such that $\frac{1}{k} < \epsilon$. Therefore,

$$\bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{m=N}^{\infty} \bigcap_{n=N}^{\infty} E_{m,n}^{1/k} = \{x \in [0,1] : \forall \epsilon > 0 \ \exists N \geq 1 \ \forall m \geq N \ \forall n \geq N \ |f_m(x) - f_n(x)| \leq \epsilon\}$$

$$= \{x \in [0,1] : f_n(x) \to c \text{ for some } c \in \mathbb{R}\}$$

$\square$

4. Let X denote the subset of $[0,1]$ of real numbers having a decimal expansion without any appearance of the numeral 6 in the expansion. What is the cardinality of the set X?

$$|X| = |\mathbb{R}|$$

Proof. After establishing injections $X \to \mathbb{R}$ and $\mathbb{R} \to X$, we will invoke the Schröder-Bernstein theorem to prove the claim. The inclusion map $X \overset{\iota}{\hookrightarrow} \mathbb{R}$ is obviously an injection, so $|X| \leq |\mathbb{R}|$.

There is a natural injection of $\mathbb{R}$ into $\mathcal{P}(\mathbb{Q})$ taking each real number to the Dedekind cut that defines it, so $|\mathbb{R}| \leq |\mathcal{P}(\mathbb{Q})|$. But $|\mathbb{Q}| = |\mathbb{N}|$, so $|\mathcal{P}(\mathbb{Q})| = |\mathcal{P}(\mathbb{N})|$. Finally, let B be the set of all infinite binary sequences in $\{0,1\}$. $|B| = |\mathcal{P}(\mathbb{N})|$, since each binary sequence corresponds to the subset of $\mathbb{N}$ which contains the $i \in \mathbb{N}$ if and only if the ith component of the sequence is 1.

We also have an injection from B into X defined as follows: for any $(s_n) \in B$, map (s_n) to the number in $[0,1]$ which it represents as a base-10 decimal expansion with s_i being the ith digit after the decimal. Clearly this number has no decimal representation in which a 6 occurs, so the map is well-defined. The map is injective because any number that has multiple base-10 representations must end in an infinite string of nines, however these strings contain only zeros and ones. Therefore, $|B| \leq |X|$, and so $|\mathbb{R}| \leq |\mathcal{P}(\mathbb{Q})| = |\mathcal{P}(\mathbb{N})| = |B| \leq |X|$. $\square$