1. Let μ^* denote the outer measure on the power-set of the real numbers arising from the length function defined on the set of open intervals. Give an example to show that it is not the case that, for all $E \subset \mathbb{R}$,

$$\mu^*(E) = \sup_{U \subset E,\ U \text{ open}} \mu^*(U)$$

Proof. Let I be the set of irrationals. We know $\mathbb{Q}$ has outer measure 0, so

$$\mu^*(E) = \mu^*(E \cap \mathbb{Q}) + \mu^*(E \cap \mathbb{Q}^c) = \mu^*(E \cap \mathbb{Q}^c) \le \mu^*(E)$$

for any set $E \subset \mathbb{R}$. Thus the inequality is an equality, so $\mathbb{Q}$ is μ^*-measurable. Since μ^* gives a measure on the set of all μ^*-measurable sets, we must have $\mu^*(I) = \mu^*(\mathbb{R}) - \mu^*(\mathbb{Q}) = \infty$. However, the rationals are dense in $\mathbb{R}$, thus $\{U \subset I : U \text{ open}\} = \{\emptyset\}$. So $\mu^*(I) = \infty \ne 0 = \mu^*(\emptyset) = \sup_{U \subset E,\ U \text{ open}} \mu^*(U)$. □

2. (F 1.4, exercise 17) If μ^* is an outer measure on X and $\{A_j\}$ is a sequence of disjoint μ^*-measurable sets, then
$\mu^*(E \cap (\bigcup_1^\infty A_j)) = \sum_1^\infty \mu^*(E \cap A_j)$ for any $E \subset X$.

Proof. The inequality $\mu^*(E \cap (\bigcup_1^\infty A_j)) \le \sum_1^\infty \mu^*(E \cap A_j)$ follows immediately from the definition of an outer measure; thus, if $\mu^*(E \cap (\bigcup_1^\infty A_j)) = \infty$, then we obviously have equality. So we may assume $\mu^*(E \cap (\bigcup_1^\infty A_j))$ is finite.

First, we will show by induction on n that

$$\mu^*(E \cap \bigcup_1^\infty A_i) = \sum_1^n \mu^*(E \cap A_i) + \mu^*(E \cap \bigcup_{n+1}^\infty A_i).$$

This is an identity for $n = 0$. Now, suppose it holds for some n. Since A_{n+1} is measurable,

$$\begin{aligned}
\mu^*(E \cap \bigcup_1^\infty A_i) &= \sum_1^n \mu^*(E \cap A_i) + \mu^*(E \cap \bigcup_{n+1}^\infty A_i) \\
&= \sum_1^n \mu^*(E \cap A_i) + \mu^*((E \cap \bigcup_{n+1}^\infty A_i) \cap A_{n+1}) + \mu^*((E \cap \bigcup_{n+1}^\infty A_i) \cap A_{n+1}^c) \\
&= \sum_1^n \mu^*(E \cap A_i) + \mu^*(E \cap A_{n+1}) + \mu^*(E \cap \bigcup_{n+2}^\infty A_i) \\
&= \sum_1^{n+1} \mu^*(E \cap A_i) + \mu^*(E \cap \bigcup_{n+1}^\infty A_i)
\end{aligned}$$

as desired.

Now, $0 \le \mu^*(E \cap \bigcup_{n+1}^\infty A_i) \le \sum_{n+1}^\infty \mu^*(E \cap A_i)$, and $\lim_{n\to\infty} \sum_{n+1}^\infty \mu^*(E \cap A_i) = 0$, so we must have $\lim_{n\to\infty} \mu^*(E \cap \bigcup_{n+1}^\infty A_i) = 0$. Also, $\sum_1^{n+1} \mu^*(E \cap A_i)$ is an increasing sequence, hence its limit must exist. Therefore,

$$\mu^*(E \cap \bigcup_1^\infty A_i) = \lim_{n\to\infty} \mu^*(E \cap \bigcup_1^\infty A_i) = \lim_{n\to\infty} \left[\sum_1^{n+1} \mu^*(E \cap A_i) + \mu^*(E \cap \bigcup_{n+1}^\infty A_i)\right] = \sum_1^\infty \mu^*(E \cap A_i).$$

□

3. (F 1.4, exercise 18) Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and μ^* the induced outer measure.

(a) For any $E \subset X$ and $\epsilon > 0$ there exists $A \in \mathcal{A}_\sigma$ with $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$.

Proof. By Theorem 1.14, μ^* is given by

$$\mu^*(E) = \inf \left\{ \sum_1^\infty \mu_0(A_j) : A_j \in \mathcal{A}, E \subset \bigcup_1^\infty A_j \right\}$$

Therefore, for every $\epsilon > 0$ there is a collection $A_j \in \mathcal{A}$ for which $E \subset \bigcup_1^\infty A_j$ and $\sum_1^\infty \mu_0(A_j) \leq \mu^*(E) + \epsilon$.

Let $A = \bigcup_1^\infty A_j$. Each A_j is μ^*-measurable, and so μ^* yields a measure on the σ-algebra they generate. Thus, $\mu^*(A) = \sum_1^\infty \mu^*(A_j) = \sum_1^\infty \mu_0(A_j) \leq \mu^*(E) + \epsilon$, as desired. □

(b) If $\mu^*(E) < \infty$, then E is μ^*-measurable iff there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$.

Proof. Assume $\mu^*(E) < \infty$. By part (a), we know that for each $i \in \mathbb{N}$ there is some $A_i \in \mathcal{A}_\sigma$ such that $E \subset A_i$ and $\mu^*(A_i) \leq \mu^*(E) + \frac{1}{i}$. Let $B = \bigcap_1^\infty A_i$. Then $E \subset B$ and, for each i, $\mu^*(B) \leq \mu^*(A_i) \leq \mu^*(E) + \frac{1}{i}$. Therefore, we must have $\mu^*(B) \leq \mu^*(E)$. Since $E \subset B$, this gives $\mu^*(B) = \mu^*(E)$. Also, B is μ^*-measurable because it is in the σ-algebra generated by $\mathcal{A}$.

Now, suppose E is μ^*-measurable. Then $\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c) = \mu^*(E) + \mu^*(B \setminus E)$. Since $\mu^*(E) < \infty$, this gives $\mu^*(B \setminus E) = \mu^*(B) - \mu^*(E) = 0$.

For the reverse direction, suppose there is some $B \in \mathcal{A}_{\sigma\delta}$ such that $E \subset B$ and $\mu^*(B \setminus E) = 0$. Then for any $F \subset X$,

$$\mu^*(F) \leq \mu^*((B \setminus E) \cap F) + \mu^*((B \setminus E) \cap F^c) \leq \mu^*(B \setminus E) + \mu^*(B \setminus E) = 0$$

so all inequalities are equalities, hence $B \setminus E$ is μ^*-measurable. We already know that B is μ^*-measurable, since it is in the σ-algebra generated by $\mathcal{A}$. Therefore, $E = B \setminus (B \setminus E)$ is μ^*-measurable. □

(c) If μ_0 is σ-finite, the restriction $\mu^*(E) < \infty$ in (b) is superfluous.

Proof. Suppose μ_0 is σ-finite. Then there is a sequence X_i such that $\mu_0(X_i) < \infty$ for each i and $X = \bigcup_i^\infty X_i$. Define $\mathcal{A}_i = \{F \in \mathcal{A} : F \subset X_i\}$. This set is clearly closed under finite unions and contains the empty set, and the complement of F in X_i is $F^c \cap X_i$, which is in $\mathcal{A}$; thus, $\mathcal{A}_i$ is an algebra on X_i.

Let $E \subset X$, and let $E_i = E \cap X_i$ for each i. Since $E_i \subset X_i$, we have $\mu^*(E_i) \leq \mu^*(X_i) < \infty$. By the same construction as in (b), we can construct a set $B_i \in (\mathcal{A}_i)_{\sigma\delta}$ such that $E_i \subset B_i$ and $\mu^*(E_i) = \mu^*(B_i)$. Again, B_i is μ^*-measurable because it is generated by $\mathcal{A}_i$, which consists of measurable sets.

Now, suppose E is μ^*-measurable. Then by closure, $E_i = E \cap X_i$ is μ^*-measurable. Since $\mu^*(E_i) < \infty$, we have $\mu(B_i \setminus E_i) = 0$, just as in part (b). Also, $B_i \setminus E_i$ is generated by measurable sets and is hence measurable. Next, define $B = \bigcup_1^\infty B_i$. Then $E \subset B$, and

$$\begin{aligned}
\mu^*(B \setminus E) &= \mu^*(\bigcup_{i=1}^\infty B_i \cap (\bigcup_{j=1}^\infty E_i)^c) \\
&= \mu^*(\bigcup_{i=1}^\infty B_i \cap \bigcap_{j=1}^\infty E_i^c) \\
&= \mu^*(\bigcup_{i=1}^\infty \bigcap_{j=1}^\infty B_j \cap E_i^c) \\
&= \mu^*(\bigcup_{i=1}^\infty \bigcap_{j=1}^\infty B_j \setminus E_i) \\
&= \mu^*(\bigcup_{i=1}^\infty B_i \setminus E_i) \text{ since } E_i \subset B_j \text{ iff } i = j \\
&= \sum_1^\infty \mu^*(B_i \setminus E_i) \\
&= 0.
\end{aligned}$$

The other direction of the proof did not require that $\mu^*(E) < \infty$, therefore (b) holds without this restriction as long as μ_0 is σ-finite. □

4. (F 1.4, exercise 19) Let μ^* be an outer measure on X induced from a finite premeasure μ_0. If $E \subset X$, define the inner measure of E to be $\mu_*(E) = \mu_0(X) - \mu^*(E^c)$. Then E is μ^*-measurable iff $\mu^*(E) = \mu_*(E)$.

Proof. If E is μ^* measurable, then

$$\mu_0(X) = \mu^*(X) = \mu^*(X \cap E) + \mu^*(X \cap E^c) = \mu^*(E) + \mu^*(E^c)$$

so clearly $\mu_*(E) = \mu^*(E)$.

Now, suppose $\mu_*(E) = \mu_0(X) - \mu^*(E^c) = \mu^*(E)$. We may construct, as we did in part (b) of the previous exercise, a set $B \in \mathcal{A}_{\sigma\delta}$ such that $E \subset B$ and $\mu^*(B) = \mu^*(E)$. Since B is μ^*-measurable, we have

$$\mu^*(E^c) = \mu^*(E^c \cap B) + \mu^*(E^c \cap B^c) = \mu^*(B \setminus E) + \mu^*(B^c)$$

Since the inner measure equals the outer measure, this gives

$$\mu_0(X) - \mu^*(E) = \mu^*(E^c) = \mu^*(B \setminus E) + \mu^*(B^c).$$

Now, using the fact that $\mu^*(B) = \mu^*(E)$ and that μ_0 is finite, we finally have

$$\mu^*(B \setminus E) = \mu_0(X) - \mu^*(E) - \mu^*(B^c) = \mu_0(X) - \mu^*(B) - \mu^*(B^c) = \mu_0(X) - \mu_0(X) = 0.$$

Therefore, $\mu^*(B \setminus E) = 0$, so E is μ^*-measurable. □

5. (F 1.4, exercise 23) Let $\mathcal{A}$ be the collection of finite unions of sets of the form $(a, b] \cap \mathbb{Q}$ where $-\infty \le a < b \le \infty$.

   (a) $\mathcal{A}$ is an algebra on $\mathbb{Q}$.

   Proof. Let $\mathcal{B} = \{(a, b] \cap \mathbb{Q} : -\infty \le a < b \le \infty\}$, and let $\mathcal{E} = \mathcal{B} \cup \{\emptyset\}$. First, we will show that $\mathcal{E}$ is an elementary family.

   By construction, $\emptyset \in \mathcal{E}$. Now, let $E, F \in \mathcal{E}$. If either of E or F is empty, then $E \cap F \in \mathcal{E}$. So assume $E = (a, b] \cap \mathbb{Q}$ and $F = (c, d] \cap \mathbb{Q}$ where $-\infty \le a < b \le \infty, -\infty \le c < d \le \infty$. We may assume, without loss of generality, that $a \le c$, since otherwise we could switch the labels of E and F. If $a < b \le c < d$, then $E \cap F = \emptyset \in \mathcal{E}$. If $a < c \le b < d$, then $E \cap F = (c, b] \cap \mathbb{Q} \in \mathcal{E}$. The only remaining possibility is $a \le c < d \le b$, in which case $E \cap F = (c, d] \cap \mathbb{Q} \in \mathcal{E}$. Finally, $E^c = ((-\infty, a] \cup (b, \infty)) \cap \mathbb{Q} = ((-\infty, a] \cap \mathbb{Q}) \cup ((b, \infty] \cap \mathbb{Q})$ is a finite disjoint union of sets in $\mathcal{E}$. Therefore, $\mathcal{E}$ is an elementary family.

Now, $\mathcal{E}$ generates an algebra by Proposition 1.7. All we need to show is that this algebra equals $\mathcal{A}$. Firstly, the $\emptyset$ is the empty union of sets from $\mathcal{B}$, so $\emptyset \in \mathcal{A}$. A finite disjoint union is a union, so the collection of finite disjoint unions from $\mathcal{E}$ is contained in $\mathcal{A}$. Now, let $A = A_1 \cup \cdots \cup A_n$ for $A_i \in \mathcal{B}$. We will induct on on n to show that A is a disjoint union of sets from $\mathcal{E}$.

Clearly, if $n = 1$ this holds. Now, consider the set $A_n \setminus (A_1 \cup \cdots \cup A_{n-1}) = A_1^c \cap \cdots \cap A_{n-1}^c \cap A_n$. For each i, A_i^c is in the algebra of finite disjoint unions from $\mathcal{E}$ (we have already shown this). Also, A_n is clearly in this algebra. So by closure under finite intersections, we must have that $A_n \setminus (A_1 \cup \cdots \cup A_{n-1})$ is in this algebra, and hence is a finite disjoint union of sets from $\mathcal{E}$. Also, it is disjoint from $A_1 \cup \cdots \cup A_{n-1}$, which by induction is a finite disjoint union of sets from $\mathcal{E}$. Therefore, A is a finite disjoint union of sets from $\mathcal{E}$, and $\mathcal{A}$ is contained in this algebra.

We have shown both inclusions, therefore $\mathcal{A}$ equals the algebra of finite disjoint unions from $\mathcal{E}$, and is thus an algebra.

□

(b) The σ-algebra generated by $\mathcal{A}$ is $\mathcal{P}(\mathbb{Q})$.

Proof. Let $x \in \mathbb{Q}$. For each i, define $E_i = (x - \frac{1}{i}, x] \cap \mathbb{Q}$. Then $\cap_1^\infty E_i = \{x\}$, so the σ-algebra generated by $\mathcal{A}$ contains all singletons from $\mathbb{Q}$. $\mathbb{Q}$ is countable, so every subset of $\mathbb{Q}$ is a countable union of singletons, thus $\mathcal{P}(\mathbb{Q}) = \mathcal{A}$ (because the reverse inclusion is trivial).

□

(c) Define μ_0 on $\mathcal{A}$ by $\mu_0(\emptyset) = 0$ and $\mu_0(A) = \infty$ for $A \neq \emptyset$. Then μ_0 is a premeasure on $\mathcal{A}$, and there is more than one measure on $\mathcal{P}(\mathbb{Q})$ whose restriction to $\mathcal{A}$ is μ_0.

Proof. First, we will show that μ_0 is a premeasure on $\mathcal{A}$. By definition, $\mu_0(\emptyset) = 0$. Also, if A_i is a sequence of disjoint sets in $\mathcal{A}$, then $\mu_0(\bigcup_1^\infty A_i)$ is 0 if all the A_is are empty and ∞ otherwise. Similiary, $\sum_1^\infty \mu_0(A_i)$ is 0 if all the A_is are empty and ∞ otherwise. So $\mu_0(\bigcup_1^\infty A_i) \le \sum_1^\infty \mu_0(A_i)$.

The outer measure which μ_0 induces on $\mathbb{Q}$ is simply $\mu^*(A) = 0$ if $A = \emptyset$ and $\mu(A) = \infty$ otherwise. This is because every nonempty subset of $\mathbb{Q}$ is contained in $(-\infty, \infty] \cap \mathbb{Q}$, which has premeasure ∞, and is not contained in the empty set, hence ∞ is the infinum given in (1.12).

Now, define μ to be the counting measure, i.e. $\mu_(A) = \text{card}(A)$ (taken to by ∞ if A is infinite). μ is a premeasure because $\mu(\emptyset) = 0$ and, if A_i is a sequence of disjoint sets in $\mathcal{A}$ such that $\bigcup_1^\infty A_i \in \mathcal{A}$, then $\mu(\bigcup_1^\infty A_i) = \text{card}(\bigcup_1^\infty A_i) = \sum_1^\infty \text{card}(A_i) = \sum_1^\infty \mu(A_i)$.

Now, let $A \in \mathcal{A}$. If $A = \emptyset$, then $\mu(A) = 0 = \mu_0(A)$. So assume A is nonempty. Note that, for any $a < b$, $(a, b] \cap \mathbb{Q}$ is infinite by the denseness of the rationals. Since A is a union of sets of this form, A must be infinite, thus $\mu(A) = \infty$. So the restriction of μ to $\mathcal{A}$ is μ_0, although $\mu \neq \mu_0$ on all of $\mathcal{P}(\mathbb{Q})$ - for instance, $\mu(\{0\}) = 1 \neq \infty = \mu_0(\{0\}$. □