F 2.4.35 $f_n \to f$ in measure iff for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) < \epsilon$ for $n \geq N$.

Proof. The definition of $f_n \to f$ in measure is as follows: for every $\epsilon > 0$, for every $\delta > 0$, there exists some N_δ such that whenever $n > N_\delta$ we have

$$\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) < \delta.$$

If $f_n \to f$ in measure, then simply taking $\delta = \epsilon$ transforms this statement into exactly what we are trying to prove, so the forward direction is trivial.

For convenience, denote $\{x : |f_n(x) - f(x)| \geq t\}$ by S_t. Suppose now that for every $t > 0$ there exists $N_t \in \mathbb{N}$ such that $\mu(S_t) < t$ for $n \geq N_t$, and let $\delta, \epsilon > 0$. If $\epsilon \leq \delta$, then choosing $N = N_\epsilon$ gives $\mu(S_\epsilon) < \epsilon \leq \delta$ when $n > N$, as desired. If $\delta < \epsilon$, then $S_\epsilon \subseteq S_\delta$, thus taking $N = N_\delta$ gives $\mu(S_\epsilon) \leq \mu(S_\delta) < \delta$ when $n > N$. □

F 2.4.38 Suppose $f_n \to f$ in measure and $g_n \to g$ in measure.

(a) $f_n + g_n \to f + g$.

Proof. Let $\epsilon > 0$ and let N_f, N_g be such that $\mu(\{x : |f_n(x) - f(x)| \geq \frac{\epsilon}{2}\}) < \frac{\epsilon}{2}$ if $n > N_f$ and $\mu(\{x : |g_n(x) - g(x)| \geq \frac{\epsilon}{2}\}) < \frac{\epsilon}{2}$ if $n > N_g$, then take $N = \max(N_f, N_g)$. Suppose that that $|(f_n + g_n) - (f + g)| \geq \epsilon$. Then $|f_n - f| + |g_n - g| \geq \epsilon$, and so either $|f_n - f| \geq \frac{\epsilon}{2}$ or $|f_n - f| \geq \frac{\epsilon}{2}$. This means

$$\{x : |(f_n(x) - g_n(x)) - (f(x) - g(x))| \geq \epsilon\} \subseteq \{x : |f_n(x) - f(x)| \geq \frac{\epsilon}{2}\} \cup \{x : |g_n(x) - g(x)| \geq \frac{\epsilon}{2}\}$$

and thus

$$\mu(\{x : |(f_n(x) - g_n(x)) - (f(x) - g(x))| \geq \epsilon\})$$
$$\leq \mu(\{x : |f_n(x) - f(x)| \geq \frac{\epsilon}{2}\}) + \mu(\{x : |g_n(x) - g(x)| \geq \frac{\epsilon}{2}\}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

whenever $n > N$, satisfying the definition from the previous exercise. □

(b) $f_n g_n \to fg$ in measure if $\mu(X) < \infty$, but not necessarily if $\mu(X) = \infty$.

Proof. Suppose $\mu(X) < \infty$. For any function h and any $\epsilon > 0$, there is some $M \in \mathbb{N}$ such that $\mu(\{x : |h(x)| > M\}) < \epsilon$. If this were false, we would have $\mu(\{x : |h(x)| > M\}) \geq \epsilon$ for all M, and so

$$\mu(\{x : |h(x)| \leq M\}) \leq \mu(X) - \epsilon$$

for all M. By continuity from below, we can take limits to produce $\mu(X) = \mu(\{x : |h(x)| \leq \infty\}) \leq \mu(X) - \epsilon$, a contradiction.

Furthermore, suppose that $h_n \to h$ in measure. Then, using the M from the previous paragraph which satisfies $\frac{\epsilon}{2}$, there is also some N such that $n > N$ implies $\mu(\{x : |h(x) - h_n(x)| > M\}) < \frac{\epsilon}{2}$. If $|h_n(x)| > 2M$, then $|h(x)| + |h(x) - h_n(x)| > 2M$ and so either $|h(x)| > M$ or $|h(x) - h_n(x)| > M$. Thus, we have

$$\mu(\{x : |h_n(x)| > 2M\}) \leq \mu(\{x : |h(x)| > M\}) + \mu(\{x : |h(x) - h_n(x)| > M\}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So, for all $\epsilon > 0$, there exists some N and some M' (namely the value $2M$ which we found) for which $\mu(\{x : |h_n(x)| > 2M\}) < \epsilon$ whenever $n > N$.

Therefore, we can find some c_g such that $\mu(\{x : |g(x)| > c_g\}) < \frac{\epsilon}{4}$, and we can also find some c_f and some N_0 such that $\mu(\{x : |f_n(x)| > c_f\}) < \frac{\epsilon}{4}$ whenever $n > N_0$. We can then find some N_1 so that $\{x : |g_n(x) - g(x)| > \frac{\epsilon}{2c_f}\}) < \frac{\epsilon}{4}$ whenever $n > N_1$ and some N_2 so that

$\{x : |f_n(x) - f(x)| > \frac{\epsilon}{2c_g}\}) < \frac{\epsilon}{4}$ whenever $n \geq N_2$. Let $N = \max(N_0, N_1, N_2)$. Then for $n > N$ we have

$$\begin{aligned}
&\mu(\{x : |f_n(x)||g_n(x) - g(x)| > \frac{\epsilon}{2}\}) \\
&\leq \mu(\{x : |f_n(x)| \leq c_f\} \cap \{x : |g_n(x) - g(x)| > \frac{\epsilon}{2c_f}\}) + \mu(\{x : |f_n(x)| \geq c_f\}) \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.
\end{aligned}$$

and

$$\begin{aligned}
&\mu(\{x : |g(x)||f_n(x) - f(x)| > \frac{\epsilon}{2}\}) \\
&\leq \mu(\{x : |f_g(x)| \leq c_g\} \cap \{x : |f_n(x) - f(x)| > \frac{\epsilon}{2c_g}\}) + \mu(\{x : |g(x)| \geq c_f\}) \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.
\end{aligned}$$

Since $|f_n g_n - fg| = |f_n g_n + f_n g - f_n g - fg| \leq |f_n||g_n - g| + |g||f_n - f|$, we know $|f_n g_n - fg| > \epsilon$ implies that $|f_n(x)||g_n(x) - g(x)| > \frac{\epsilon}{2}$ or $|g(x)||f_n(x) - f(x)| > \frac{\epsilon}{2}$. Thus,

$$\begin{aligned}
&\mu(\{x : |f_n(x)g_n(x) - f(x)g(x)| > \epsilon) \\
&\leq \mu(\{x : |f_n(x)||g_n(x) - g(x)| > \frac{\epsilon}{2}\}) + \mu(\{x : |g(x)||f_n(x) - f(x)| > \frac{\epsilon}{2}\}) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

So our chosen N satisfies the definition.

To see that the requirement $\mu(X) < \infty$ is necessary, consider the sequences $f_n(x) = \frac{1}{n}\chi_{[0,n)}$ and $g_n(x) = \sum\limits_{k=1}^{n} k \cdot \chi_{[k-1,k)}$. Then $f_n \to 0 = f$ in measure and $g_n \to \sum\limits_{k=1}^{\infty} k \cdot \chi_{[k-1,k)} = g$, which is finite for all x. So $fg = 0$. However, $f_n g_n = \sum\limits_{k=1}^{n} \frac{k}{n} \cdot \chi_{[k-1,k)}$ does not converge to 0 in measure. For all n, the set $\{x : |f_n(x)g_n(x) - 0| > \frac{1}{2}\}$ contains $[n-1, n)$, which has measure 1. $\square$

F 2.4.43 Suppose that $\mu(X) < \infty$ and $f : X \times [0,1] \to \mathbb{C}$ is a function such that $f(\cdot, y)$ is measurable for each $y \in [0,1]$ and $f(x, \cdot)$ is continuous for each $x \in X$.

(a) If $0 < \epsilon, \delta < 1$ then $E_{\epsilon,\delta} = \{x : |f(x,y) - f(x,0)| \leq \epsilon \text{ for all } y < \delta\}$ is measurable.

Proof. For each $y \in [0,1]$, define $g_y(x) = |f(x,y) - f(x,0)|$. g_y is the absolute value of a difference of measurable functions, thus g_y is measurable. So we can express $E_{\epsilon,\delta}$ as

$$\bigcap_{y \in [0,\delta)} g_y^{-1}([0,\epsilon]).$$

We will show that this set equals

$$\bigcap_{y \in [0,\delta) \cap \mathbb{Q}} g_y^{-1}([0,\epsilon]),$$

which is a countable intersection of measurable sets (since $[0,\epsilon] \in \mathcal{B}_{\mathbb{R}}$ and g_y is measurable for all y), hence measurable.

The forward inclusion is obvious, because $[0,\delta) \cap \mathbb{Q} \subset [0,\delta)$. For the reverse inclusion, suppose that $x \in g_y^{-1}([0,\epsilon])$ for all $y \in [0,\delta) \cap \mathbb{Q}$. For a given $y \in [0,\delta)$, take a sequence y_n in $[0,\delta) \cap \mathbb{Q}$ converging to y. By assumption, we have $g_{y_n}(x) \leq \epsilon$ for all y_n. Since $g_y(x)$ is the absolute value of a difference of functions that are continuous in y, it is continuous in y as well. So taking limits gives $g_y(x) = \lim\limits_{n \to \infty} g_{y_n}(x) \leq \epsilon$, hence $x \in g_y^{-1}([0,\epsilon])$. Because y was arbitrary, the reverse inclusion is established. $\square$

(b) For any $\epsilon > 0$ there is a set $E \subset X$ such that $\mu(E) < \epsilon$ and $f(\cdot, y) \to f(\cdot, 0)$ uniformly on E^c as $y \to 0$.

Proof. Since $f(x,y)$ is continuous in y for all $x \in X$, we know $\lim_{y\to 0} f(x,y) = f(x,0)$. Any sequence $y_n \to 0$ gives a sequence $f_n(x) = f(x, y_n)$ which then converges to $f(x,0)$. So by Egoroff's theorem, there is such a set E for which $f_n \to f$ uniformly on E^c. This holds for all sequences y_n converging to 0, thus $f(\cdot, y) \to f(\cdot, 0)$ uniformly on E^c as $y \to 0$. □

B 12.1 Suppose μ is a signed measure. Prove that A is a null set with respect to μ if and only if $|\mu|(A) = 0$.

Proof. Decompose X into a disjoint union $E \cup F$, where $\mu = \mu^+ - \mu^-$ for some positive measures with $\mu^+(F) = 0 = \mu^-(E)$. First, suppose that A is a null set. We know $0 = \mu(A\cap E) = \mu^+(A\cap E) - \mu^-(A\cap E) = \mu^+(A\cap E)$ and similarly $0 = \mu(A \cap F) = -\mu^-(A\cap F)$, since these are subsets of A. Therefore,

$$\begin{aligned}|\mu|(A) &= \mu^+(A) + \mu^-(A)\\ &= \mu^+(A\cap E) + \mu^-(A\cap E) + \mu^+(A\cap F) + \mu^-(A\cap F)\\ &= 0.\end{aligned}$$

Now, assume that $|\mu|(A) = 0$ and let $B \subset A$. B decomposes as $(B\cap E)\cup(B\cap F)$ with $B\cap E \subset A\cap E$ and $B\cap F \subset A\cap F$. Since $|\mu|(A) = 0$ we know that $\mu^+(A\cap E) + \mu^-(A\cap F) = 0$. Since both of these are positive measures, we must have $\mu^+(A\cap E) = 0 = \mu^-(A\cap F)$. Again, because they are positive measures, we then have $\mu^+(B\cap E) \le \mu^-(A\cap E) = 0$ and $\mu^-(B\cap F) \le \mu^-(A\cap F) = 0$, so $\mu(B) = \mu^+(B\cap E) - \mu^-(B\cap F) = 0 - 0 = 0$. □

B 12.2 Let μ be a signed measure. Define

$$\int f d\mu = \int f d\mu^+ - \int f d\mu^-.$$

Prove that

$$\left|\int f d\mu\right| \le \int |f| d|\mu|.$$

Proof. First note that, for any two positive measures μ_1 and μ_2, we have the property

$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int d\mu_2$$

which we will now prove. This is obvious for simple functions, since

$$\int s d(\mu_1+\mu_2) = \sum_1^n a_i(\mu_1+\mu_2)(E_i) = \sum_1^n a_i\mu_1(E_i) + \sum_1^n a_i\mu_2(E_i) = \int s d\mu_1 + \int s d\mu_2.$$

For an arbitrary nonnegative function f, one inequality is now straightforward:

$$\begin{aligned}\int f d(\mu_1+\mu_2) &= \sup\left\{\int s d(\mu_1+\mu_2) : 0 \le s \le f, s \text{ simple}\right\}\\ &= \sup\left\{\int s d\mu_1 + \int s d\mu_2 : 0 \le s \le f, s \text{ simple}\right\}\\ &\le \sup\left\{\int s d\mu_1 : 0 \le s \le f, s \text{ simple}\right\} + \sup\left\{\int s d\mu_2 : 0 \le s \le f, s \text{ simple}\right\}\\ &= \int f d\mu_1 + \int f d\mu_2.\end{aligned}$$

For the other inequality, we know that for every ϵ there exists a simple function $s_i \leq f$ such that $\int f d\mu_i - \frac{\epsilon}{2} \leq \int s_i d\mu_i$ (for $i = 1, 2$). Taking $s = \max(s_1, s_2)$, it is clear that s is a simple function (if $\{E_i\}$ and $\{F_i\}$ are the partitions used for s_1 and s_2, respectively, then s uses the partition $\{E_i \cap F_j\}$, and value of s on some $E_i \cap F_j$ is the max of the values of s_1 and s_2 on this set, which is constant), and that $\int f d\mu_i - \frac{\epsilon}{2} \leq \int s d\mu_i$ (for $i = 1, 2$). The inequality follows from $\int s_i d\mu_i \leq \int s d\mu_i$, since $s_i \leq s$. So

$$\int f d\mu_1 + \int f d\mu_2 - \epsilon \leq \int s d\mu_1 + \int s d\mu_2 = \int s d(\mu_1 + \mu_2).$$

Since this holds for an arbitrary ϵ, we have

$$\int f d(\mu_1 + \mu_2) = \sup \left\{ \int s d(\mu_1 + \mu_2) : 0 \leq s \leq f \right\} \geq \int f d\mu_1 + \int f d\mu_2$$

as desired. The proof for an arbitrary f (not necessarily positive) follows immediately, but we will not even be needing this case, since the function we will be applying this fact to is $|f|$.

Finally, we have

$$\begin{aligned}
\left| \int f d\mu \right| &= \left| \int f d\mu^+ - \int f d\mu^- \right| \\
&\leq \left| \sup \left\{ \sum_1^n a_i \mu^+(E_i) : 0 \leq \sum_1^n a_i \chi_{E_i} \leq f \right\} - \sup \left\{ \sum_1^n a_i \mu^-(E_i) : 0 \leq \sum_1^n a_i \chi_{E_i} \leq f \right\} \right| \\
&= \left| \int f d\mu^+ - \int f d\mu^- \right| \\
&\leq \int |f| d\mu^+ + \int |f| d\mu^- \\
&= \int |f| d(\mu^+ + \mu^-) \\
&= \int |f| d|\mu|.
\end{aligned}$$

$\square$