

Homework 2

Stat 150

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Q 1. Can a reversible chain be periodic (demonstrate with an explanation or example)? *Yes.*

Proof. Consider the Markov chain defined by the transition matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This is a chain with 2 states, where at each step we transition to the other state with probability 1. The chain can be defined by the sequence of distributions

$$\mathbb{P}(X_n = n \pmod{2}) = 1.$$

Then $\mathbb{P}(X_{n+1} = i \mid X_n = j) = \mathbb{P}(X_{n+1} = j \mid X_n = i)$ for any $i, j \in \mathbb{Z}_2$, since both sides of the equation are 0 if $i = j$ and 1 if $i \neq j$. Thus the chain is reversible. Also, $\gcd(\{m : P_{ii}^m > 0\}) = 2$. So the chain is periodic.

□

Q 2. Consider the probability transition matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0.1 & 0.9 \\ 0.2 & 0.2 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0.9 & 0 & 0 & 0 & 0 & 0.1 & 0 \end{pmatrix}.$$

What are the classes of communicating states? Which are recurrent? Which states are aperiodic?

Proof. The classes of communicating states are $\{1, 6, 7\}$, $\{2\}$, and $\{3, 4, 5\}$, since these are the states which the process always has a nonzero probability of transitioning between.

The recurrent states are 1, 6, 7, 3, 4, and 5 since, given that we start in any of these states, there is a probability of 1 that we will return to them. This is because it is impossible to leave either of the classes $\{1, 6, 7\}$ or $\{3, 4, 5\}$.

The only transient state is 2, since the probability of never returning to 2, given that we start there, is 0.8, which is nonzero.

The only aperiodic states are 1, 2, 6, and 7. We can get from state 2 to itself in 1 step. For any $i \in \{1, 6, 7\}$, we can return to i in both 2 or 3 steps, thus its period is $\gcd\{2, 3, \dots\} = 1$.

□

Q 3. If states i and j communicate and i is aperiodic, show that j must also be aperiodic.

Proof. Suppose i and j communicate. Then there exist some $a, b > 0$ such that $P_{ij}^a > 0$ and $P_{ji}^b > 0$. So $P_{ii}^{a+b} > 0$. So the period p of state i must divide $a + b$. Now, let $M = \{m : P_{ii}^m > 0\}$. If $m \in M$, then $P_{ii}^{a+m+b} > 0$, so p divides $a + b + m$. But p divides $a + b$, so p divides m . Thus the period of i divides every $m \in M$, so the period of i divides the period of j . However, this argument is symmetric in i and j , thus the period of j divides that of i , as well. So i and j have the same period. So clearly, if i is aperiodic, then so is j . □

Q 4. A biased coin lands heads with probability $\frac{2}{3}$ and tails with probability $\frac{1}{3}$. We toss the coin repeatedly and let U_n be the indicator of the event that there are an even number of heads in the first n coin tosses.

- What is the transition matrix of U_n ?
- What is its stationary distribution?
- After n coin tosses, what is the probability of an even number of heads?

Proof. The transition matrix is $\begin{pmatrix} 1/3 & 2/3 \\ 2/3 & 1/2 \end{pmatrix}$. The matrix is doubly stochastic, thus the stationary distribution must be uniform. So the stationary distribution is $\pi = (\frac{1}{2} \quad \frac{1}{2})$.

The probability s_n of an even number of heads after n coin tosses is simply the distribution of U_n . We know that

$$\begin{aligned} s_{n+1} &= \mathbb{P}(U_{n+1} = 1) = \mathbb{P}(U_{n+1} = 1 \mid U_n = 0)\mathbb{P}(U_n = 0) + \mathbb{P}(U_{n+1} = 1 \mid U_n = 1)\mathbb{P}(U_n = 1) \\ &= \frac{2}{3}(1 - s_n) + \frac{1}{3}s_{n-1} = \frac{2}{3} - \frac{1}{3}s_{n-1} \end{aligned}$$

and $s_0 = 1$, since $\mathbb{P}(U_0 = 1) = 1$. We can easily solve this recurrence relation by adding the homogenous solution to a particular solution.

The homogeneous solution is $c(-\frac{1}{3})^n$ for some c . A particular solution is $\frac{1}{2}$, since $\frac{2}{3} - (\frac{1}{3})(\frac{1}{2}) = \frac{1}{2}$. Solving for c , we find

$$c \left(-\frac{1}{3}\right)^0 + \frac{1}{2} = c + \frac{1}{2} = 1$$

so $c = \frac{1}{2}$. Thus the general solution is $\mathbb{P}(U_n = 1) = s_n = \frac{1}{2} \left(1 + \left(-\frac{1}{3}\right)^n\right)$. □

Q 5. Two bags each contain n balls. In total there are $2n$ balls, n are green and n are red. After each unit of time, a ball is selected from each bag and the two are swapped over (i.e. the selected balls are put each put in the opposite bag). Let X_n be the number of green balls in the first bag.

- What is the transition matrix of X_n ?
- Find the stationary distribution.
- Is the Markov chain reversible?

Proof. The transition matrix at any time is given by

$$P_{ij} = \begin{cases} \left(\frac{i}{n}\right)^2 & j = i - 1 \\ 2\left(\frac{i}{n}\right)\left(\frac{n-i}{n}\right) & j = i \\ \left(\frac{n-i}{n}\right)^2 & j = i + 1 \end{cases}.$$

This is, by definition, a birth-death process. By the results derived in lecture, since the state-space is finite, the chain is reversible and its stationary distribution is given by

$$\pi_j = \pi_0 \prod_{i=1}^j \frac{\left(\frac{n-i+1}{n}\right)^2}{\left(\frac{i}{n}\right)^2} = \pi_0 \left(\prod_{i=1}^j \frac{n-i+1}{i} \right)^2 = \pi_0 \binom{n}{j}^2.$$

and

$$\pi_0 = \sum_{j=0}^n \frac{\pi_j}{\binom{n}{j}^2} = \frac{1}{\sum_{j=0}^n \binom{n}{j}^2} = \frac{1}{\binom{2n}{n}}.$$

Thus the stationary distribution is given, for all $j \in \{0, \dots, n\}$, by

$$\pi_j = \frac{\binom{n}{j}^2}{\binom{2n}{n}} = \frac{\binom{n}{j} \binom{n}{n-j}}{\binom{2n}{n}}.$$

So the stationary distribution is hypergeometric. □

Q 6. For $0 < p < 1$ let X_n be an auto-regressive process,

$$X_{n+1} = pX_n + W_{n+1}$$

where the W_n are IID $N(0, 1)$ random variables. What is the stationary distribution of the Markov chain? What happens if $p > 1$?

Proof. First, note that the sequence X_n is given by

$$X_n = p^n X_0 + Z_n$$

where $Z_n = p^{n-1}W_1 + p^{n-2}W_2 + \dots + pW_{n-1} + W_n$. Z_n is a linear combination of standard normal random variables, thus it is normal with mean 0 and variance

$$\begin{aligned} \text{Var}(Z_n) &= \text{Var}\left(\sum_{i=1}^n p^{n-i}W_i\right) = \sum_{i=1}^n \text{Var}(p^{n-i}W_i) \\ &= \sum_{i=1}^n (p^{n-i})^2 \text{Var}(W_i) = \sum_{i=1}^n p^{2n-2i} \\ &= p^{2n-2} \sum_{i=0}^{n-1} (p^{-2})^i = p^{2n-2} \left(\frac{1 - p^{-2n}}{1 - p^{-2}} \right) \\ &= \frac{1 - p^{2n}}{1 - p^2}. \end{aligned}$$

Thus X_n is normal as well with parameters $\mathbb{E}[X_n] = p^n X_0$ and $\text{Var}(X_n) = \text{Var}(p^n X_0 + Z_n) = 0 + \text{Var}(Z_n) = \frac{1 - p^{2n}}{1 - p^2}$.

The stationary distribution of X_n is its limiting distribution, so we may take the limit of these parameters as $n \rightarrow \infty$ to obtain the stationary distribution $N\left(0, \frac{1}{1 - p^2}\right)$, since $0 < p < 1$.

If $p > 1$, both parameters of the normal distribution of X_n exhibit divergent behavior. Its variance approaches $+\infty$ and its expected value approaches $\pm\infty$, depending on the sign of X_0 . However, since its expectation and variance are both divergent, we would naturally wonder if X_n is overwhelmingly likely to be large if its expectation approaches ∞ and heavily negative if its expectation is $-\infty$, or if its divergent variance “wins out” and causes the value of X_n to be completely unpredictable for large values of n .

Let $c \in \mathbb{R}$. What is $\mathbb{P}(X_n > c)$ for large n ?

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P}(X_n < c) &= \lim_{n \rightarrow \infty} \mathbb{P}\left(\frac{X_n - p^n X_0}{\frac{1 - p^{2n}}{1 - p^2}} < \frac{c - p^n X_0}{\frac{1 - p^{2n}}{1 - p^2}}\right) \\ &= \lim_{n \rightarrow \infty} \mathbb{P}\left(Z < -\frac{c - cp^2 + p^n X_0 - p^{n+2} X_0}{1 - p^{2n}}\right) \\ &= \lim_{n \rightarrow \infty} \mathbb{P}\left(Z < -\frac{\frac{c}{p^{2n}} - \frac{c}{p^{2n-2}} + \frac{X_0}{p^n} - \frac{X_0}{p^{n-2}}}{\frac{1}{p^{2n}} - 1}\right) \\ &= \mathbb{P}(Z < 0) = \frac{1}{2}. \end{aligned}$$

Thus, for any value c and large values of n , it is equally likely that the chain will be on either side of c . No matter how long the process has run, every interval of the form $(-\infty, c)$ or (c, ∞) , for any c , will be visited at some future step. Essentially, the divergent variance “wins out” against the divergent expectation.

□