17. Compute, for all values of $n \in \mathbb{Z}$, $(1+i)^n + (1-i)^n$.

Proof. Observe that

$$(1+i)^n + (1-i)^n = 2^{\frac{n}{2}}(e^{i\frac{n\pi}{4}} + e^{-i\frac{n\pi}{4}}) = 2^{\frac{n}{2}} \cdot 2\,\mathrm{Re}(e^{i\frac{n\pi}{4}}) = 2^{\frac{n}{2}+1}\cos\left(\frac{n\pi}{4}\right).$$

Since $\cos\left(\frac{n\pi}{4}\right)$ has a period of 8, we can evaluate the given expression based on the value of $n \pmod 8$.

$$(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1}\cos\left(\frac{n\pi}{4}\right) = \begin{cases} 2^{\frac{n}{2}+1}\cdot 1 & n \equiv 0 \pmod 8 \\ 2^{\frac{n}{2}+1}\cdot(-1) & n \equiv 4 \pmod 8 \\ 2^{\frac{n}{2}+1}\cdot 2^{-\frac{1}{2}} & n \equiv 1 \text{ or } 7 \pmod 8 \\ 2^{\frac{n}{2}+1}\cdot(-2^{-\frac{1}{2}}) & n \equiv 3 \text{ or } 5 \pmod 8 \end{cases}$$

$$= \begin{cases} 2^{\frac{n}{2}+1} & n \equiv 0 \pmod 8 \\ -2^{\frac{n}{2}+1} & n \equiv 4 \pmod 8 \\ 2^{\frac{n+1}{2}} & n \equiv 1 \text{ or } 7 \pmod 8 \\ -2^{\frac{n+1}{2}} & n \equiv 3 \text{ or } 5 \pmod 8 \end{cases}$$

□

18. Prove that, for all $z \in \mathbb{C}$, $\sin^2(z) + \cos^2(z) = 1$.

Proof.

$$\begin{aligned} \sin^2(z) + \cos^2(z) &= \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}+e^{-iz}}{2i}\right)^2 \\ &= \frac{e^{2zi}+e^{-2zi}+2}{4} + \frac{e^{2zi}+e^{-2zi}-2}{-4} \\ &= \frac{e^{2zi}+e^{-2zi}+2-e^{2zi}-e^{-2zi}+2}{4} \\ &= 1 \end{aligned}$$

□

19. Find the real and imaginary parts of $\sin(z)$, $\cos(z)$, and e^{e^z}.

Proof. For any $x \in \mathbb{R}$, $\cos(ix) = \dfrac{e^{i(ix)}+e^{-i(ix)}}{2} = \dfrac{e^{-x}+e^{x}}{2} = \cosh(x)$, where cosh is the hyperbolic cosine function. Similarly, $\sin(ix) = \dfrac{e^{i(ix)}-e^{-i(ix)}}{2i} = \dfrac{e^{-x}-e^{x}}{2i} = i\dfrac{e^{x}-e^{-x}}{2} = i\sinh(x)$, where sinh is the hyperbolic sine function.

Letting $z = x+iy$, we have

$$\begin{aligned} \cos(z) + i\sin(z) &= e^{iz} \\ &= e^{ix}e^{i(iy)} \\ &= (\cos(x)+i\sin(x))(\cos(iy)+i\sin(iy)) \\ &= [\cos(x)\cos(iy) - \sin(x)\sin(iy)] + i[\sin(x)\cos(iy)+\cos(x)\sin(iy)]. \end{aligned}$$

Thus, $\cos(z) = \cos(x)\cos(iy) - \sin(x)\sin(iy)$ and $\sin(z) = \sin(x)\cos(iy)+\cos(x)\sin(iy)$. From the first paragraph, this implies

$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$

Therefore,

$$\begin{aligned}\text{Re}(\cos(z)) &= \cos(x)\cosh(y)\\ \text{Im}(\cos(z)) &= \sin(x)\sinh(y)\\ \text{Re}(\sin(z)) &= \sin(x)\cosh(y)\\ \text{Im}(\sin(z)) &= \cos(x)\sinh(y)\end{aligned}$$

□

20. Prove that $e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$.

Proof. In the proof of Theorem I.5.1, we proved the following:

- $e^{i\frac{\pi}{2}} = i$.
- $\cos(x)$ and $\sin(x)$ are strictly positive for $x \in (0, \frac{\pi}{2})$.

We also showed, immediately afterwards, that every complex number has exactly two square roots (which are negatives of each other, as we saw on the previous homework). Since $e^{i\frac{\pi}{4}} = (e^{i\frac{\pi}{2}})^{\frac{1}{2}}$, this number is a square root of i, and therefore must be either $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ or $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$. However, the second option implies that $\cos(\frac{\pi}{2})$ and $\sin(\frac{\pi}{2})$ are negative, contradicting that these functions are positive for $x \in (0, \frac{\pi}{2})$. Thus we must have $e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$. □

21. Find all $z \in \mathbb{C}$ such that $\cos(z) = 2$.

Proof. We want to find all solutions to $\cos(z) = \frac{e^{iz}+e^{-iz}}{2} = 2$. Since e^{iz} is never 0 for $z \in \mathbb{C}$, we can subtract 2 and multiply by e^{iz} to obtain an equation with the same solution set. This equation is

$$(e^{iz})^2 - 4(e^{iz})^2 + 1 = 0.$$

The quadratic formula tells us that this equation is equivalent to

$$e^{iz} = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3},$$

which is equivalent to

$$iz = \log(2 \pm \sqrt{3})$$

where, here, log is the "multi-valued" function of complex numbers. All solutions to this equation are given by

$$iz = \log|2 \pm \sqrt{3}| + 2\pi n i$$

for $n \in \mathbb{Z}$, where log here is the real-valued function of real numbers. Multiplying by $-i$ gives $z = 2\pi n - i\log(2 \pm \sqrt{3})$, since $2 \pm \sqrt{3} > 0$. So the solution set is

$$\{2\pi n - i\log(2 \pm \sqrt{3}) : n \in \mathbb{Z}\}.$$

□

22. Let Log denote the principal branch of the logarithm. Find $z_1, z_2 \in \mathbb{C} \setminus (-\infty, 0]$ such that $\text{Log}(z_1 z_2) \neq \text{Log}\, z_1 + \text{Log}\, z_2$.

Proof. Let $z_1 = z_2 = e^{i\frac{2\pi}{3}}$. Then

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log}(e^{i\frac{4\pi}{3}}) = -\frac{2\pi}{3}i.$$

However,

$$\operatorname{Log}(z_1) + \operatorname{Log}(z_2) = 2\operatorname{Log}(e^{i\frac{2\pi}{3}}) = 2\left(\frac{2\pi}{3}\right)i = \frac{4\pi}{3}i.$$

$\square$

Note: I abuse notation in the following problem. When n appears in an expression, that expression actually represents "the set of all numbers of that form" for $n \in \mathbb{Z}$. Sometimes, two expressions I claim are equal are actually only equal up to a relabeling of n; but this is okay, since the sets they represent (under this notation) are actually equal. I only do this to turn as many negatives into positives as possible without having to create new variables.

23. Compute all values of

(a) $\sin i$, $\cos i$, $\tan(1+i)$.

Proof. In exercise 19, we showed that for any $x \in \mathbb{R}$, $\cos(ix) = \cosh(x)$ and $\sin(ix) = i\sinh(x)$. So

$$\cos(i) = \cosh(i) = \frac{e + e^{-1}}{2}.$$

$$\sin(i) = i\sinh(i) = i\frac{e - e^{-1}}{2}.$$

$$\tan(1+i) = \frac{\sin(1+i)}{\cos(1+i)} = \frac{\frac{e^{(1+i)i} - e^{-(1+i)i}}{2i}}{\frac{e^{(1+i)i} + e^{-(1+i)i}}{2}} = \frac{e^{1-i} - e^{-(1-i)}}{e^{1-i} + e^{-(1-i)}}i$$

$\square$

(b) $\log(-1)$, $\log(i)$, $\log(1+i)$, $\log(\log(i))$.

Proof.

$$\log(-1) = \log|-1| + i\arg(-1) = 0 + (\pi + 2\pi n)i = (2n+1)\pi i$$

$$\log(i) = \log|i| + i\arg(i) = 0 + \left(\frac{\pi}{2} + 2\pi n\right)i = \left(\frac{\pi}{2} + 2\pi n\right)i$$

$$\log(1+i) = \log|1+i| + i\arg(1+i) = \log(\sqrt{2}) + \left(\frac{\pi}{4} + 2\pi n\right)i$$

for any $n \in \mathbb{Z}$.

$$\log(\log(i)) = \log\left(\left(\frac{\pi}{2} + 2\pi n\right)i\right) = \log\left|\left(\frac{\pi}{2} + 2\pi n\right)i\right| + i\arg\left(\left(\frac{\pi}{2} + 2\pi n\right)i\right)$$

$$= \log\left|\frac{\pi}{2} + 2\pi n\right| + i\left(\frac{\pi}{2} + 2\pi k\right)i$$

for any $n, k \in \mathbb{Z}$, where log in the final expressions denotes the real-valued function of real numbers, and $\left|\frac{\pi}{2} + 2\pi n\right|$ denotes the absolute value of this number, since it will be negative if $n < 0$. $\square$

(c) 2^i, i^i, $(-1)^{2i}$, $(1+i)^i$, $(-1)^{\frac{1}{\pi}}$.

Proof.

$$2^i = e^{i\log(2)} = e^{i(\log(2) + 2\pi n i)} = e^{2\pi n + i\log(2)}$$

$$i^i = e^{i\log(i)} = e^{i\left(\frac{\pi}{2} + 2\pi n\right)i} = e^{2\pi n - \frac{\pi}{2}}$$

$$(-1)^{2i} = e^{2i\log(-1)} = e^{2i(2n+1)\pi i} = e^{2(2n+1)\pi}$$

$$(1+i)^i = e^{i\log(1+i)} = e^{i\left(\log(\sqrt{2}) + \left(\frac{\pi}{4} + 2\pi n\right)i\right)} = e^{-\frac{\pi}{4} + 2\pi n + i\log(\sqrt{2})}$$

$$(-1)^{\frac{1}{\pi}} = e^{\frac{1}{\pi}\log(-1)} = e^{\frac{1}{\pi}(2n+1)\pi i} = e^{(2n+1)i}$$

for any $n \in \mathbb{Z}$. $\square$

24. Show that $\log(i^{\frac{1}{2}}) = \frac{1}{2}\log(i)$, in the sense that each denotes the same infinite set of complex numbers. However, show that $\log(i^2) \neq 2\log(i)$.

Proof. First, note that the square roots of i are $e^{\frac{\pi}{4}i}$ and $e^{\frac{5\pi}{4}i}$. So, if we consider the values of log and arg to be sets, we have

$$\begin{aligned}
\log(i^{\frac{1}{2}}) &= \log(e^{i\frac{\pi}{4}}) \cup \log(e^{i\frac{5\pi}{4}}) \\
&= (\log|e^{i\frac{\pi}{4}}| + i\arg(e^{i\frac{\pi}{4}})) \cup (\log|e^{i\frac{5\pi}{4}}| + i\arg(e^{i\frac{5\pi}{4}})) \\
&= \{i\left(\frac{\pi}{4} + 2\pi n\right) : n \in \mathbb{Z}\} \cup \{i\left(\frac{5\pi}{4} + 2\pi n\right) : n \in \mathbb{Z}\} \\
&= \{i\left(\frac{\pi}{4} + 2\pi n\right) : n \in \mathbb{Z}\} \cup \{i\left(\frac{\pi}{4} + \pi + 2\pi n\right) : n \in \mathbb{Z}\} \\
&= \{i\left(\frac{\pi}{4} + 2\pi n\right) : n \in \mathbb{Z}\} \cup \{i\left(\frac{\pi}{4} + \pi(2n+1)\right) : n \in \mathbb{Z}\} \\
&= \{i\left(\frac{\pi}{4} + \pi n\right) : n \in \mathbb{Z}\}.
\end{aligned}$$

Also,

$$\frac{1}{2}\log(i) = \{\frac{1}{2}(\frac{\pi}{2} + 2\pi n)i : n \in \mathbb{Z}\} = \{i\left(\frac{\pi}{4} + \pi n\right) : n \in \mathbb{Z}\}.$$

Thus, $\log(i^{\frac{1}{2}}) = \frac{1}{2}\log(i)$.

Next, observe that

$$\log(i^2) = \log(-1) = \{(2n+1)\pi i : n \in \mathbb{Z}\}$$

but

$$2\log(i) = \left\{2(\frac{\pi}{2} + 2\pi n)i : n \in \mathbb{Z}\right\} = \{(4n+1)\pi i : n \in \mathbb{Z}\}.$$

Therefore, $-\pi i \in \log(i^2)$, but $-\pi i \notin 2\log(i)$. So $\log(i^2) \neq 2\log(i)$. $\square$