1. Prove that for any pair z, w of complex numbers, the following properties hold:
   - $\overline{z+w} = \overline{z} + \overline{w}$
   - $\overline{zw} = \overline{z}\overline{w}$
   - $|zw| = |z||w|$

   Proof. Let $z = a + bi$ and $w = c + di$.
   - $\overline{z+w} = \overline{(a+bi)+(c+di)} = \overline{(a+c)+(b+d)i} = (a+c)-(b+d)i = (a-bi)+(c-di) = \overline{z}+\overline{w}$
   - $\overline{zw} = \overline{(a+bi)(c+di)} = \overline{(ac-bd)+(ad+bc)i} = (ac-bd)-(ad+bc)i = (a-bi)(c-di) = \overline{z}\overline{w}$
   - $|zw| = |(a+bi)(c+di)| = |(ac-bd)+(ad+bc)i| = (ac-bd)^2+(ad+bc)^2 = a^2c^2+b^2c^2+a^2d^2+b^2d^2 = (a^2+b^2)(c^2+d^2) = |a+bi||c+di| = |z||w|$

   ◻

2. Prove that the modulus function $\mathbb{C} \to [0, \infty)$ is continuous.

   Lemma. *A function $f : X \to Y$ is continuous if the preimage of every basic open set, for a fixed basis of Y, is open.*

   Proof. Let $\mathcal{B}$ be a basis for Y, and suppose $f^{-1}(B)$ is open for every $B \in \mathcal{B}$. Let $U \subseteq Y$ be open. Then there exists some collection $\mathcal{C} \subseteq \mathcal{B}$ such that $U = \bigcup_{B\in\mathcal{C}} B$. Therefore,

   $$g^{-1}(U) = g^{-1}(\bigcup_{B\in\mathcal{C}} B) = \bigcup_{B\in\mathcal{C}} g^{-1}(B)$$

   is a union of open sets, and hence open. So the preimage of every open set is open, thus f is continuous. ◻

   Main proof. The set $\mathcal{B} = \{[0,b) : b > 0\} \cup \{(a,b) : a, b \geq 0\}$ of open balls in $[0,\infty)$ forms a basis for $[0,\infty)$. The preimage of any open ball of the form $[0,b)$ is the open ball $\{x \in \mathbb{C} : d(0,x) < b\}$ in $\mathbb{C}$, and the preimage of any ball of the form (a,b) is the open annulus $\{x \in \mathbb{C} : a < d(0,x) < b\}$. (Clearly, this annulus is open - it is the intersection of two open sets: the open ball of radius b centered at 0, and the complement of the closed ball of radius a centered at 0). Hence, by the lemma, the modulus function is continuous. ◻

3. Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial with real coefficients. Prove that the roots of p occur in conjugate pairs.

   Proof. Let $p(x) = \sum_{k=0}^{n} a_k x^k$ for some $a_0, \dots, a_n \in \mathbb{R}$, and suppose that $p(z) = 0$. Recall that conjugation distributes over addition and multiplication, and also that conjugation is the identity on the real numbers. Observe that

   $$p(\overline{z}) = \sum_{k=0}^{n} a_k \overline{z}^k = \sum_{k=0}^{n} \overline{a_k}\overline{z^k} = \overline{\sum_{k=0}^{n} a_k z^k} = \overline{p(z)} = \overline{0} = 0$$

   so $\overline{z}$ is a root of p as well.

   Now, suppose that $z \in \mathbb{C} \setminus \mathbb{R}$ has multiplicity r as a root. We will show by induction on r that $\overline{z}$ has multiplicity r as well. This is trivial if $r = 0$, since then neither z nor $\overline{z}$ is a root. Assuming this holds for some r, suppose z has multiplicity $r + 1$. We know that $\overline{z}$ is a root, so we may consider the polynomial

   $$q(x) = \frac{p(x)}{(x-z)(x-\overline{z})}.$$

   We know that z is a root of $q(x)$ with multiplicity r. By the inductive hypothesis, $\overline{z}$ is a root with multiplicity r as well. Clearly, then, $p(x) = (x-z)(x-\overline{z})q(x)$ has $\overline{z}$ as a root with multiplicity $r+1$. ◻

4. Describe the following sets geometrically and draw a picture of each:

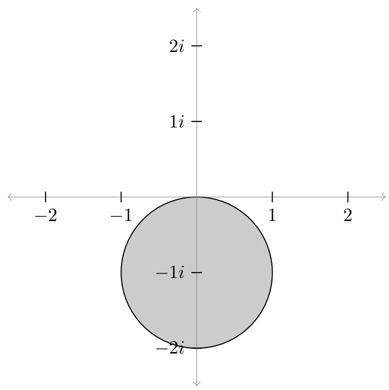
   (a) $\{z \in \mathbb{C} : |z + i| \leq 1\}$
   This is the set of all points of distance at most 1 from the point $-i$.

   (b) $\{z \in \mathbb{C} : |z - 1| = |z - i|\}$
   This is the set of points equidistant from 1 and i, which we know from geometry is the perpendicular bisector of the line segment between these two points.

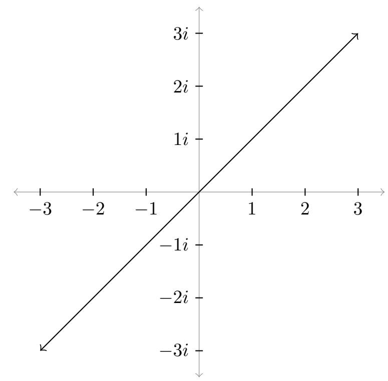
   (c) $\{z \in \mathbb{C} : |z - 4i| + |z + 4i| = 10\}$
   An ellipse is the set of points for which the sum of the distances to some two foci is constant. The major axis must have length 10. Since $|3 - 4i| + |3 + 4i| = 10$, the minor axis has length 6.

   (d) $\{z \in \mathbb{C} : \frac{1}{z} = \overline{z}\}$
   Multiplying both sides by z gives the equation $|z| = 1$. So this is the unit circle centered at the origin.

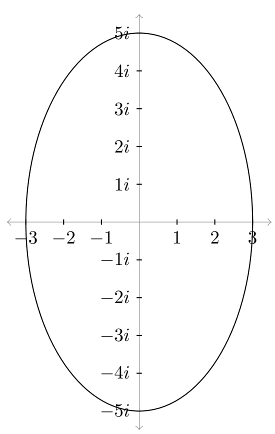
a)



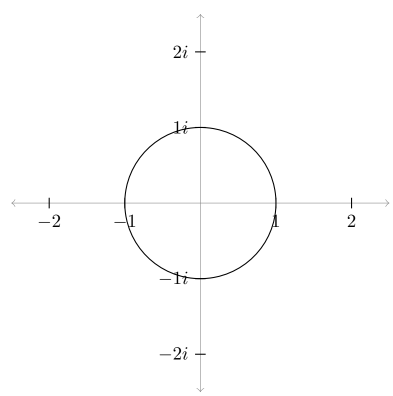
b)



c)



d)



5. Prove that $\mathbb{C}$ cannot be made into an ordered field.

   Proof. Suppose $(\mathbb{C}, <)$ is an ordered field. First, assume $i > 0$. This means that $-1 = i \cdot i > 0$, and by the same logic $1 = (-1)(-1) > 0$. But then we must have $0 = 1 + (-1) > 0$, a contradiction because no element is strictly less than itself in a total ordering.

   Next, assume $i < 0$. Subtracting i from both sides gives $-i > 0$. Again, we have $(-i)^2 = -1 > 0$. We have just shown this to be a contradiction. The only remaining possibility now is $i = 0$, which is absurd. Therefore, no such ordered field $(\mathbb{C}, <)$ exists. $\square$

6. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Fix $w \in \mathbb{D}$ and define $f : \overline{\mathbb{D}} \to \mathbb{C}$ by

$$f(z) = \frac{w - z}{1 - \overline{w}z}.$$

Show the following:

- f maps $\mathbb{D}$ to $\mathbb{D}$ and $\partial\mathbb{D}$ to $\partial\mathbb{D}$.
- f is a bijection on $\mathbb{D}$.
- f is holomorphic on $\mathbb{D}$.

Proof. We are given that $|w| < 1$. To prove that f maps $\mathbb{D}$ to $\mathbb{D}$ and $\partial\mathbb{D}$ to $\partial\mathbb{D}$, we need to show that if $|z| < 1$ then $|f(z)| < 1$ and if $|z| = 1$ then $|f(z)| = 1$. Note that

$$|f(z)| = \left(\frac{w-z}{1-\overline{w}z}\right)\overline{\left(\frac{w-z}{1-\overline{w}z}\right)} = \frac{(w-z)(\overline{w}-\overline{z})}{(1-\overline{w}z)(1-w\overline{z})} = \frac{|w|+|z|-(w\overline{z}+\overline{w}z)}{1+|w||z|-(w\overline{z}+\overline{w}z)}.$$

If $|z| = 1$ then $f(z) = 1$, since the numerator equals the denominator. So f maps $\partial\mathbb{D}$ into itself.

To see that f maps $\mathbb{D}$ into itself as well, it suffices to show that if $|z| < 1$ then the numerator is less than the denominator, which is equivalent to showing that $|w| + |z| < 1 + |w||z|$. This is equivalent to showing

$$0 < 1 + |w||z| - |w| - |z| = (|w|-1)(|z|-1).$$

Given the restriction $|w|, |z| < 1$, this inequality holds since both factors on the right-hand side are negative.

f is its own inverse, so it is a bijection:

$$f(f(z)) = \frac{w - \dfrac{w-z}{1-\overline{w}z}}{1-\overline{w}\dfrac{w-z}{1-\overline{w}z}} = \frac{(1-\overline{w}z)w-(w-z)}{(1-\overline{w}z)-\overline{w}(w-z)} = \frac{w-|w|z-w+z}{1-\overline{w}z-|w|+\overline{w}z} = z\left(\frac{1-\overline{w}}{1-\overline{w}}\right) = z.$$

We will now show that f is holomorphic on $\mathbb{D}$. The numerator and denominator of f are both linear combinations of holomorphic functions, since z is holomorphic and any constant function is holomorphic. Also, the denominator is nonzero on $\mathbb{D}$: we know that $|1-\overline{w}z| \geq ||1| - |\overline{w}z|| > 0$ because $|\overline{w}z| = |w||z| < 1$ for $z, w \in \mathbb{D}$. Therefore, by the quotient rule, f must be holomorphic on $\mathbb{D}$. $\square$