4

The logic of first order structures

4.1 Isomorphisms between structures

We begin with the fundamental notion of isomorphism for $\mathcal{L}_{\mathcal{A}}$-structures. Notice that if $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure, then $\mathcal{L}_{\mathcal{A}}$ is uniquely specified by the domain of I.

Definition 4.1 Suppose that $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$ are $\mathcal{L}_{\mathcal{A}}$-structures. A bijection $e : M \to N$ defines an isomorphism between $\mathcal{M}$ and $\mathcal{N}$ if and only if the following conditions hold.

(1) For each constant symbol c_i in the domain of I,

$$e(I(c_i)) = J(c_i).$$

(2) For each function symbol F_i in the domain of I, if $n = \pi(F_i)$ then for each $\langle a_1, \ldots, a_{n+1} \rangle \in M^{n+1}$,

$$I(F_i)(a_1, \ldots, a_n) = a_{n+1} \;\leftrightarrow\; J(F_i)(e(a_1), \ldots, e(a_n)) = e(a_{n+1}).$$

(3) For each predicate symbol P_i in the domain of I, if $n = \pi(P_i)$ then for each $\langle a_1, \ldots, a_n \rangle \in M^n$,

$$\langle a_1, \ldots, a_n \rangle \in I(P_i) \;\leftrightarrow\; \langle e(a_1), \ldots, e(a_n) \rangle \in J(P_i).$$

When $\mathcal{M}$ is equal to $\mathcal{N}$, we say that e is an *automorphism*.

For any structure $\mathcal{M}$, the identity function $e : x \mapsto x$ is an example, though a trivial one, of an automorphism of $\mathcal{M}$. It follows directly from Definition 4.1 that the inverse of an isomorphism is also an isomorphism and that the composition of two isomorphisms is also an isomorphism.

Theorem 4.2 *Suppose that $e : M \to N$ is an isomorphism of $\mathcal{L}_{\mathcal{A}}$-structures $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$. Suppose that ν is an $\mathcal{M}$-assignment. Then, the composition of e and ν, $e \circ \nu$, is an $\mathcal{N}$-assignment, and for each $\mathcal{L}_{\mathcal{A}}$-formula φ,*

$$(\mathcal{M}, \nu) \vDash \varphi \;\leftrightarrow\; (\mathcal{N}, e \circ \nu) \vDash \varphi.$$

Proof. Clearly, $e \circ \nu$ is an $\mathcal{N}$-assignment. It maps each variable symbol x_i to an element of M via ν and then maps that element of M to an element of N via e. Consequently, $e \circ \nu$ maps the variable symbols to N, as required.

Next, we note that the isomorphism e preserves the interpretation of terms. In other words, if τ is an $\mathcal{L}_{\mathcal{A}}$-term, then $e(\overline{\nu}(\tau)) = \overline{e \circ \nu}(\tau)$. The proof is by induction on the length of terms. In the atomic cases,

$$\begin{aligned} e(\overline{\nu}(\langle c_i \rangle)) &= e(I(c_i)) \\ &= J(c_i) \qquad \text{(since } e \text{ is an isomorphism)} \\ &= \overline{e \circ \nu}(\langle c_i \rangle). \end{aligned}$$

and

$$\begin{aligned} e(\overline{\nu}(\langle x_i \rangle)) &= e(\nu(x_i)) \\ &= e \circ \nu(x_i) \\ &= \overline{e \circ \nu}(\langle x_i \rangle). \end{aligned}$$

Assuming the claim for all shorter terms, consider $F_i(\tau_1, \ldots, \tau_n)$.

$$\begin{aligned} e(\overline{\nu}(F_i(\tau_1, \ldots, \tau_n))) &= e(I(F_i)(\overline{\nu}(\tau_1), \ldots, \overline{\nu}(\tau_n))) \\ &= J(F_i)(e(\overline{\nu}(\tau_1)), \ldots, e(\overline{\nu}(\tau_n))) \\ &\qquad \text{(since } e \text{ is an isomorphism)} \\ &= J(F_i)(\overline{e \circ \nu}(\tau_1), \ldots, \overline{e \circ \nu}(\tau_n)) \\ &\qquad \text{(by induction)} \\ &= \overline{e \circ \nu}(F_i(\tau_1, \ldots, \tau_n) \end{aligned}$$

Finally, we verify the statement of the theorem, by induction on the length of formulas.

The atomic cases follow from the above observation about terms.

$$\begin{aligned} &(\mathcal{M}, \nu) \models P_i(\tau_1 \ldots \tau_n) \ \leftrightarrow \\ &\quad \leftrightarrow \ \langle \overline{\nu}(\tau_1), \ldots, \overline{\nu}(\tau_n) \rangle \in I(P_i) \\ &\qquad\qquad \text{(by definition)} \\ &\quad \leftrightarrow \ \langle e(\overline{\nu}(\tau_1)), \ldots, e(\overline{\nu}(\tau_n)) \rangle \in J(P_i) \\ &\qquad\qquad \text{(since } e \text{ is an isomorphism)} \\ &\quad \leftrightarrow \ \langle \overline{e \circ \nu}(\tau_1), \ldots, \overline{e \circ \nu}(\tau_n)) \rangle \in J(P_i) \\ &\qquad\qquad \text{(by the observation on terms)} \\ &\quad \leftrightarrow \ (\mathcal{N}, e \circ \nu) \models P_i(\tau_1 \ldots \tau_n) \end{aligned}$$

The case of an equality uses the fact that an isomorphism is injective.

$$\begin{aligned} (\mathcal{M}, \nu) \models \tau_1 \mathrel{\hat{=}} \tau_n \ &\leftrightarrow \ \overline{\nu}(\tau_1) = \overline{\nu}(\tau_2) && \text{(by definition)} \\ &\leftrightarrow \ e(\overline{\nu}(\tau_1)) = e(\overline{\nu}(\tau_2)) && \text{(as } e \text{ is injective)} \\ &\leftrightarrow \ \overline{e \circ \nu}(\tau_1) = \overline{e \circ \nu}(\tau_2) && \text{(as above)} \\ &\leftrightarrow \ (\mathcal{N}, e \circ \nu) \models (\tau_1 \mathrel{\hat{=}} \tau_2) && \text{(by definition)} \end{aligned}$$

Now, we consider the propositional connectives.

$$
\begin{aligned}
(\mathcal{M},\nu) \models (\neg\psi) &\leftrightarrow (\mathcal{M},\nu) \not\models \psi && \text{(by definition)} \\
&\leftrightarrow (\mathcal{N}, e\circ\nu) \not\models \psi && \text{(by induction)} \\
&\leftrightarrow (\mathcal{N}, e\circ\nu) \models (\neg\psi) && \text{(by definition)}
\end{aligned}
$$

The analysis of implication is similar.

Finally, we consider applications of quantification, say $\varphi = (\forall x_i \psi)$. By definition, $(\mathcal{M},\nu) \models (\forall x_i \psi)$ if and only if for every $\mathcal{M}$-assignment μ, if ν and μ agree on the free variables of φ, then $(\mathcal{M},\mu) \models \psi$. Since e is surjective, for every $\mathcal{N}$-assignment μ^* which agrees with $e\circ\nu$ on the free variables of φ, there is an $\mathcal{M}$-assignment μ which agrees with ν on the free variables of φ such that $e\circ\mu = \mu^*$. By induction, for each such μ and μ^*, $(\mathcal{M},\mu) \models \psi \ \leftrightarrow\ (\mathcal{N},\mu^*) \models \psi$. Thus, if $(\mathcal{M},\nu) \models (\forall x_i\psi)$, then $(\mathcal{N}, e\circ\nu) \models (\forall x_i\psi)$. Similarly, if $(\mathcal{M},\nu) \not\models (\forall x_i\psi)$, then let μ be an $\mathcal{M}$-assignment such that ν and μ agree on the free variables of φ and $(\mathcal{M},\mu) \not\models \psi$. Then, $e\circ\mu$ is such an $\mathcal{N}$-assignment for $e\circ\nu$, and so $(\mathcal{N}, e\circ\nu) \not\models (\forall x_i\psi)$. □

4.1.1 Exercises

(1) Let $A = \{F_1\}$ be the alphabet with one unary function symbol. Give an examples of different infinite $\mathcal{L}_A$-structures $\mathcal{M} = (M, I)$ with the following properties.
   a) $\mathcal{M}$ has no nontrivial automorphisms.
   b) $\mathcal{M}$ has a countably infinite set of automorphisms.
   c) For each element a of M there are only finitely many b's in M such that there is an automorphism f of $\mathcal{M}$ with $f(a) = b$. However, there are uncountably many automorphisms of $\mathcal{M}$.

Theorem 4.2 suggests the following definition.

Definition 4.3 Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}_{\mathcal{A}}$-structures. Then $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent* if and only if for each $\mathcal{L}_{\mathcal{A}}$-sentence φ

$$\mathcal{M} \models \varphi \ \leftrightarrow\ \mathcal{N} \models \varphi.$$

We write $\mathcal{M} \equiv \mathcal{N}$ that these structures are elementarily equivalent.

4.1.2 Exercises

(1) Characterize the collection of automorphisms of the integers $\mathbb{Z}$ with the binary relation $<$.
(2) Suppose that $\mathcal{A}$ is finite and that $\mathcal{M}$ is a finite $\mathcal{L}_{\mathcal{A}}$-structure. Prove that there is an $\mathcal{L}_{\mathcal{A}}$-sentence φ such that for every $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{N}$, if $\mathcal{N} \models \varphi$ then $\mathcal{N} \cong \mathcal{M}$.

(3) Suppose that $\mathcal{M}$ and $\mathcal{N}$ are finite $\mathcal{L}_{\mathcal{A}}$-structures. Prove that the following are equivalent.
 a) $\mathcal{M} \cong \mathcal{N}$
 b) $\mathcal{M} \equiv \mathcal{N}$

4.2 Substructures and elementary substructures

Definition 4.4 Suppose that $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$ are $\mathcal{L}_{\mathcal{A}}$-structures. $\mathcal{M}$ is a *substructure* of $\mathcal{N}$ if and only if $M \subseteq N$ and the following conditions hold.

(1) If c_i is a constant symbol of $\mathcal{L}_{\mathcal{A}}$ then $I(c_i) = J(c_i)$.
(2) If F_i is a function symbol of $\mathcal{L}_{\mathcal{A}}$ with $n = \pi(F_i)$, then $I(F_i)$ is the restriction of $J(F_i)$ to M^n.
(3) If P_i is a predicate symbol of $\mathcal{L}_{\mathcal{A}}$ with $n = \pi(P_i)$, then $I(P_i)$ is equal to $J(P_i) \cap M^n$.

We will write $\mathcal{M} \subseteq \mathcal{N}$ to indicate that $\mathcal{M}$ is a substructure of $\mathcal{N}$.

Theorem 4.5 *Suppose that $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$ are $\mathcal{L}_{\mathcal{A}}$-structures with $M \subseteq N$. Then the following are equivalent.*

(1) $\mathcal{M}$ is a substructure of $\mathcal{N}$.
(2) For all atomic $\mathcal{L}_{\mathcal{A}}$-formulas, φ and for all $\mathcal{M}$-assignments ν,

$$(\mathcal{M}, \nu) \models \varphi \;\leftrightarrow\; (\mathcal{N}, \nu) \models \varphi.$$

Proof. The theorem follows directly from the definitions. $\square$

The equivalence given in Theorem 4.5 suggests the following definition.

Definition 4.6 Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}_{\mathcal{A}}$-structures with $\mathcal{M} \subseteq \mathcal{N}$. $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$, if and only if for all $\mathcal{L}_{\mathcal{A}}$-formulas φ and for all $\mathcal{M}$-assignments ν,

$$(\mathcal{M}, \nu) \models \varphi \;\leftrightarrow\; (\mathcal{N}, \nu) \models \varphi.$$

We write $\mathcal{M} \preceq \mathcal{N}$ to indicate that $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$.

4.2.1 Exercise

(1) Let $\mathcal{A} = \emptyset$ and let $\mathcal{N}$ be the $\mathcal{L}_{\mathcal{A}}$ structure whose universe is $\mathbb{N}$, the natural numbers. Show that for every infinite subset S of $\mathbb{N}$, the $\mathcal{L}_{\mathcal{A}}$-structure with universe S is an elementary substructure of $\mathcal{N}$.

4.3 Definable sets and Tarski's Criterion

Suppose that $\mathcal{N}$ is an $\mathcal{L}_{\mathcal{A}}$-structure. The problem of constructing elementary substructures of $\mathcal{N}$ looks difficult because the criterion for success involves truth in the substructure to be constructed and, in particular, anticipating quantification over the whole substructure while still in the process of its construction.

Tarski's Theorem below gives an elegant characterization of when a substructure of $\mathcal{N}$ is an elementary substructure. Tarski's criterion is given in terms of definable sets.

Definition 4.7 Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure.

(1) Suppose that $X \subseteq M$. A set $Y \subseteq M^n$ is *definable in* $\mathcal{M}$ *with parameters from* X if and only if there are elements $b_1, \ldots, b_m$ of X and an $\mathcal{L}_{\mathcal{A}}$-formula φ such that the following conditions hold.
  a) φ has $n + m$ free variables $x_{j_1}, \ldots, x_{j_n}$ and $x_{k_1}, \ldots, x_{k_m}$.
  b) For each $\langle a_1, \ldots, a_n \rangle \in M^n$, $\langle a_1, \ldots, a_n \rangle \in Y$ if and only if there is an $\mathcal{M}$-assignment ν such that
    i. for each $i \leq n$, $\nu(x_{j_i}) = a_i$,
    ii. for each $i \leq m$, $\nu(x_{k_i}) = b_i$,
    iii. and $(\mathcal{M}, \nu) \models \varphi$.

(2) A set $Y \subseteq M^n$ is *definable* in $\mathcal{M}$ without parameters if and only if it is definable with parameters from $\emptyset$.

When n is equal to 1, we identify M^1 with M and speak of definable subsets of M.

Lemma 4.8 *Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure and that $Y \subseteq M^n$ is definable in $\mathcal{M}$ with parameters from X. Then, there are $b_1, \ldots, b_m$ in X and there is a formula $\varphi = \varphi(x_1, \ldots, x_{n+m})$ such that*

$$Y = \{\langle a_1, \ldots, a_n \rangle : \mathcal{M} \models \varphi[a_1, \ldots, a_n, b_1, \ldots, b_m]\}$$

Proof. We will leave the proof of Lemma 4.8 to the Exercises. □

Theorem 4.9 *Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure and that $X \subseteq M$. Suppose that $Y \subset M^n$ is definable in $\mathcal{M}$ with parameters from X and that $e : M \to M$ is an automorphism of $\mathcal{M}$.*

If for each $b \in X$, $e(b) = b$, then

$$Y = \{\langle e(a_1), \ldots, e(a_n) \rangle : \langle a_1, \ldots, a_n \rangle \in Y\}.$$

Proof. Let $b_1, \ldots, b_m$ be elements of X and let $\varphi = \varphi(x_1, \ldots, x_{n+m})$ be an $\mathcal{L}_{\mathcal{A}}$-formula such that for all $a_1, \ldots, a_n$ in M,

$$\langle a_1, \ldots, a_n \rangle \in Y \;\leftrightarrow\; \mathcal{M} \models \varphi[a_1, \ldots, a_n, b_1, \ldots, b_m].$$

Suppose that $\langle a_1, \ldots, a_n \rangle \in Y$, then we can apply Theorem 4.2 to conclude that $\mathcal{M} \models \varphi[e(a_1), \ldots, e(a_n), e(b_1), \ldots, e(b_m)]$. Since e fixes all of the elements of X, $\mathcal{M} \models \varphi[e(a_1), \ldots, e(a_n), b_1, \ldots, b_m]$ and so $\langle e(a_1), \ldots, e(a_n) \rangle \in Y$. To verify the reverse inclusion, suppose that $\langle c_1, \ldots, c_n \rangle \in Y$, that is $\mathcal{M} \models \varphi[c_1, \ldots, c_n, b_1, \ldots, b_m]$. Since e is an automorphism, e is surjective. Let $a_1, \ldots, a_n$ be elements of

M such that for each i less than or equal to n, $e(a_i) = c_i$. Consequently, $\mathcal{M} \models \varphi[e(a_1), \ldots, e(a_n), b_1, \ldots, b_m]$. Applying Theorem 4.2 in the other direction, $\mathcal{M} \models \varphi[a_1, \ldots, a_n, b_1, \ldots, b_m]$ and so $\langle a_1, \ldots, a_n \rangle \in Y$. Thus, there is a sequence $\langle a_1, \ldots, a_n \rangle \in Y$ such that $\langle c_1, \ldots, c_n \rangle$ is equal to $\langle e(a_1), \ldots, e(a_n) \rangle$, as required.

Remark 4.10 Definability within a structure is one of the central concepts in Mathematical Logic. In the next section, we shall consider the problem of classifying the definable sets of various specific structures. In many cases, the analysis requires that careful attention be paid to parameters.

Example 4.11 Suppose that $\mathcal{A} = \emptyset$, so that $\mathcal{L}_{\mathcal{A}}$ is the trivial language. Suppose that M is a nonempty set. Then, $\mathcal{M} = (M, \emptyset)$ is an $\mathcal{L}_{\mathcal{A}}$-structure. Further *any* bijection $e : M \to M$ defines an isomorphism of $\mathcal{M}$ to $\mathcal{M}$. We can use Theorem 4.2 to prove the following.

(1) Suppose that $A \subset M$. Then A is definable in $\mathcal{M}$ without parameters if and only if $A = \emptyset$ or $A = M$.
(2) Suppose that $A \subset M$. Then A is definable in $\mathcal{M}$ from parameters if and only if A is finite or $M \setminus A$ is finite.

To verify the first claim, suppose that φ is an $\mathcal{L}_{\mathcal{A}}$-formula and x_1 is the only free variable in φ. If there is no m in $\mathcal{M}$ such that $\mathcal{M} \models \varphi[m]$, then φ defines $\emptyset$ in $\mathcal{M}$. Otherwise, suppose that $m \in M$ and $\mathcal{M} \models \varphi[m]$. If n is another element of M, then the function e from M to M obtained by transposing m and n is a bijection from M to M, and therefore an isomorphism from $\mathcal{M}$ to $\mathcal{M}$. By Theorem 4.2, since $\mathcal{M} \models \varphi[m]$ we also have $\mathcal{M} \models \varphi[e(m)]$, that is $\mathcal{M} \models \varphi[n]$. Consequently, if φ defines a nonempty set, then that set is all of M.

We leave the proof of the second claim to the Exercises.

Theorem 4.12 (Tarski) *Suppose that $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$ are $\mathcal{L}_{\mathcal{A}}$-structures, and $\mathcal{M}$ is a substructure of $\mathcal{N}$. The following are equivalent.*

(1) $\mathcal{M} \preceq \mathcal{N}$
(2) $\mathcal{M} \subseteq \mathcal{N}$ *and for each nonempty set $A \subseteq N$, if A is definable in $\mathcal{N}$ with parameters from M, then $A \cap M \neq \emptyset$.*

Proof. The easier direction is the implication from (1) to (2). Suppose that $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, and suppose that A is a nonempty set which is definable in $\mathcal{N}$ with parameters from M. Let $\varphi(x_1, \ldots, x_{1+n})$ be a $\mathcal{L}_{\mathcal{A}}$-formula and let $m_1, \ldots, m_n$ be elements of M such that for all $a \in N$, $a \in A \leftrightarrow \mathcal{N} \models \varphi[a, m_1, \ldots, m_n]$. Since A is not empty, $\mathcal{N} \not\models (\forall x_1(\neg\varphi))[m_1, \ldots, m_n]$. Since $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, $\mathcal{M} \not\models (\forall x_1(\neg\varphi))[m_1, \ldots, m_n]$. Fix m in M so that $\mathcal{M} \models \varphi[m, m_1, \ldots, m_n]$. Since $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, $\mathcal{N} \models \varphi[m, m_1, \ldots, m_n]$, and so m is an element of A. Thus, $A \cap M$ is not empty, as required.

Now, we prove the implication from (2) to (1). Suppose that $\mathcal{M} \subseteq \mathcal{N}$ and for each nonempty set $A \subseteq N$, if A is definable in $\mathcal{N}$ with parameters from M, then $A \cap M \neq \emptyset$. We prove by induction on the length of formulas φ that for all $\mathcal{M}$-assignments ν, $(\mathcal{M}, \nu) \models \varphi$ if and only if $(\mathcal{N}, \nu) \models \varphi$. Note that since $M \subseteq N$, every $\mathcal{M}$-assignment is also an $\mathcal{N}$-assignment.

The atomic cases follow from Theorem 4.5, and the propositional cases follow directly from the inductive hypothesis. We consider only the case when φ is $(\forall x_i \psi)$.

Suppose that ν is an $\mathcal{M}$-assignment and $(\mathcal{N}, \nu) \models \varphi$. Then, for every $\mathcal{N}$-assignment μ which agrees with ν on the free variables of φ, $(\mathcal{N}, \mu) \models \psi$. In particular, for every $\mathcal{M}$-assignment μ which agrees with ν on the free variables of φ, $(\mathcal{N}, \mu) \models \psi$. By induction, for these $\mathcal{M}$-assignments, $(\mathcal{N}, \mu) \models \psi$ if and only if $(\mathcal{M}, \mu) \models \psi$. Consequently, for every $\mathcal{M}$-assignment μ which agrees with ν on the free variables of φ, $(\mathcal{M}, \mu) \models \psi$, and $(\mathcal{M}, \nu) \models \varphi$ as required.

Now, suppose that ν is an $\mathcal{M}$-assignment and $(\mathcal{N}, \nu) \not\models (\forall x_i \psi)$. Then there is an $\mathcal{N}$-assignment μ which agrees with ν on the free variables of $(\forall x_i \psi)$ such that $(\mathcal{N}, \mu) \not\models \psi$. Let $x_{k_1}, \ldots, x_{k_m}$ be the free variables of φ and let Y be the subset of N defined in $\mathcal{N}$ by the formula $\neg\psi$ using x_i for the elements of Y and $x_{k_1}, \ldots, x_{k_m}$ for the parameters $\nu(x_{k_1}), \ldots, \nu(x_{k_m})$. Since $(\mathcal{N}, \mu) \not\models \psi$, $\mu(x_i)$ is an element of Y and so Y is not empty.

Now, we use the assumption (2). By (2), $Y \cap M$ is not empty, and we let b be an element of $Y \cap M$. Consequently, if ρ is an $\mathcal{M}$-assignment such that ρ agrees with ν on the free variables of $(\forall x_i \psi)$ and $\rho(x_i) = b$, then $(\mathcal{N}, \rho) \models \neg\psi$. By induction, $(\mathcal{M}, \rho) \models \neg\psi$ and so $(\mathcal{M}, \nu) \not\models (\forall x_1 \psi)$. Taking the contrapositive of the above, if $(\mathcal{M}, \nu) \models (\forall x_1 \psi)$ then $(\mathcal{N}, \nu) \models (\forall x_1 \psi)$, as required. $\square$

4.3.1 Exercises

(1) Let $\mathcal{N}$ be the structure with universe $\mathbb{N}$, interpreting constants for 0 and 1, and functions for $+$ and $\times$. Show that if $\mathcal{M} \subseteq \mathcal{N}$, then $\mathcal{M} = \mathcal{N}$.

(2) Let $\mathcal{L}_A$ be the language with one unary predicate symbol P. Let $\mathcal{M}$ be the finite structure (M, I) such that $M = \{a, b, c, d, e\}$ and $I(P) = \{a, b\}$. In other words, $\mathcal{M}$ interprets P as holding of a and b and as not holding of c, d, or e.

   a) Which subsets of M which are definable in $\mathcal{M}$ without parameters?
   b) Which subsets of M are are definable in $\mathcal{M}$ with parameters?

(3) Suppose that T is a set of sentences and that there is an $\mathcal{N} = (N, J)$ such that $\mathcal{N} \models T$ and N is infinite. Show that there is an $\mathcal{M} = (M, I)$ and an element a of M such that $\mathcal{M} \models T$ and a is not definable in $\mathcal{M}$ without parameters.

(4) Prove Lemma 4.8.

(5) Prove the second claim of Example 4.11.

4.4 Dense orders

We consider the following example and show how, by constructing automorphisms one can in this case easily determine which sets are definable.

Suppose $\mathcal{L}_{\mathcal{A}}$ has only one 2-place predicate symbol. Thus $\mathcal{L}_{\mathcal{A}}$-structures are naturally of the form (M, P), where $M \neq \emptyset$ and $P \subseteq M \times M$. We consider the $\mathcal{L}_{\mathcal{A}}$-structure $(\mathbb{R}, <)$, given by the set of real numbers with the usual order. Suppose that $X \subset \mathbb{R}$ is finite. Define for reals a and b, $a \sim_X b$ if and only if there exists a bijection $e : \mathbb{R} \to \mathbb{R}$ such that e is an automorphism of the $\mathcal{L}_{\mathcal{A}}$-structure $(\mathbb{R}, <)$, such that $e(a) = b$ and such that for all $t \in X$, $e(t) = t$.

The relation $\sim_X$ is an equivalence relation on $\mathbb{R}$. That is to say that for all t_1, t_2, t_3 in $\mathbb{R}$ the following conditions hold.

(1) $t_1 \sim_X t_1$; since the identity map $x \mapsto x$ is an automorphism.
(2) If $t_1 \sim_X t_2$ then $t_2 \sim_X t_1$; since the inverse of an automorphism is an automorphism.
(3) if $t_1 \sim_X t_2$ and $t_2 \sim_X t_3$ then $t_1 \sim_X t_3$; since the composition of automorphisms is an automorphism.

For each $t \in \mathbb{R}$ let

$$[t]_X = \{w \in \mathbb{R} : w \sim_X t\}$$

be the equivalence class of t.

Definition 4.13 (1) A set $I \subseteq \mathbb{R}$ is an *interval* if for all a, b, c in $\mathbb{R}$, if $a \leq b \leq c$ and $\{a, c\} \subseteq I$ then $b \in I$.
(2) If $I \subseteq \mathbb{R}$ is an interval, then a real number a is an *endpoint* of I if and only if either a is the greatest lower bound of I or a is the least upper bound of I.
(3) If $c < d$ are real numbers, then we use the following notation to represent intervals with endpoints c and d.

$$(c, d) = \{x : c < x < d\} \tag{1}$$
$$[c, d] = \{x : c \leq x \leq d\} \tag{2}$$

Lemma 4.14 *Suppose that $X \subset \mathbb{R}$ is finite. Then for each $a \in \mathbb{R}$, $[a]_X$ is an interval. Further*

(1) if $a \in X$ then $[a]_X = \{a\}$,
(2) and if $a \notin X$ then $[a]_X$ is the maximum interval $I \subseteq \mathbb{R}$ such that $a \in I$ and $I \cap X = \emptyset$.

Proof. If a is an element of X, then any automorphism of $(\mathbb{R}, \leq)$ which fixes all of the elements of X must fix a. But then for all b, if $a \sim_X b$ then $a = b$. In other words, $[a]_X = \{a\}$.

Otherwise, let I be the maximum interval such that $a \in I$ and $I \cap X = \emptyset$. To show that I is equal to $[a]_X$, let b be an element of I. Let (c, d) be a subinterval of I such that a and b belong to (c, d) and $[c, d]$ is a subset of I. We may assume that a is less than b, since otherwise we can consider the inverse of the map constructed as below to send b to a. First, we define an order preserving bijection e_0 from $[c, d]$ to itself so that e_0 maps a to b.

$$e_0(x) = \begin{cases} c + \frac{b-c}{a-c}(x-c), & \text{if } x \in [c, a]; \\ b + \frac{d-b}{d-a}(x-a), & \text{if } x \in [a, d]. \end{cases}$$

The function e_0 consists of stretching the interval $[c, a]$ to match $[c, b]$ and compressing $[a, d]$ to match $[b, d]$. Then, we extend e_0 to an order preserving bijection of $\mathbb{R}$ by mapping every real number not in $[c, d]$ to itself. The resulting function, e is an automorphism of $\mathbb{R}$ which shows that $a \sim_X b$. □

Theorem 4.15 *Suppose that $X \subseteq \mathbb{R}$ and that $A \subseteq \mathbb{R}$. Then the following are equivalent.*

(1) A is definable in $(\mathbb{R}, <)$ with parameters from X.
(2) A is a finite union of intervals I such that the endpoints of I belong to X.

Proof. We will take the implication from (2) to (1) as being self-evident, and we will prove the implication from (1) to (2).

Suppose that A is definable in $(\mathbb{R}, <)$ with parameters from X. Let φ be a formula in the first order language with $\leq$, let $a_1, \dots, a_n$ be elements of X, and suppose that for all real numbers b,

$$b \in A \leftrightarrow (\mathbb{R}, <) \models \varphi[b, a_1, \dots, a_n].$$

We first show that for each b, if $b \in A$ then $[b]_X \subseteq A$. So, suppose that b and c are real numbers, $b \in A$, and $b \sim_X c$. By Lemma 4.14, there is an automorphism e of $(\mathbb{R}, \leq)$ such that e maps b to c and e fixes the elements of $a_1, \dots, a_n$. By Theorem 4.9, $b \in A$ if and only if $e(b) \in A$. Consequently, $b \in A$ implies $c \in A$, as required.

But then, A is a union of $\sim_{\{a_1,\dots,a_n\}}$ equivalence classes. Each of these classes is an interval, and since $\{a_1, \dots, a_n\}$ is finite, there are only finitely many of them. Theorem 4.15 follows immediately. □

By applying Tarski's Theorem 4.12, we can characterize the elementary substructures of $(\mathbb{R}, <)$.

Corollary 4.16 *Let $\mathcal{R} = (\mathbb{R}, <)$. Suppose that $M \subseteq \mathbb{R}$ and that $\mathcal{M} = (M, <_M)$ is the induced substructure of $\mathcal{R}$. Then the following are equivalent.*

(1) $\mathcal{M} \preceq \mathcal{R}$.
(2) $(M, <_M)$ is a dense total order without endpoints.

Proof. The implication from (1) to (2) is direct: being a dense total order without endpoints is a first order property of $\mathcal{R}$ which must apply to any of its elementary substructures.

We now prove the implication from (2) to (1). Suppose that $(M, <_M)$ is a dense total order without endpoints. We will apply Tarski's Criterion to show that $\mathcal{M} \preceq \mathcal{R}$. Let $\{m_1, \ldots, m_n\}$ be a finite subset of M, and let A be a nonempty subset of $\mathbb{R}$ which is definable in $\mathcal{R}$ using parameters from $\{m_1, \ldots, m_n\}$. It is sufficient to show that $A \cap M$ is not empty.

By Lemma 4.15, A is a finite union of intervals I in $\mathcal{R}$ whose endpoints belong to $\{m_1, \ldots, m_n\}$. Let I be a nonempty such interval. If I is a singleton $\{m_i\}$, then $m_i \in (A \cap M)$. Secondly, there could be $m_i < m_j$ such every real number between m_i and m_j belongs to I. Since $\mathcal{M}$ is dense and m_i and m_j are elements of M, there is an $m \in M$ such that $m_i < m < m_j$. Then, $m \in (A \cap M)$ as required. Finally, I could be an unbounded interval. Since M has no endpoints, there must be an element of m in I in this case as well. □

Another corollary is the following version of Theorem 4.15 but for the structure $(\mathbb{Q}, <)$.

Theorem 4.17 *Suppose that $X \subseteq \mathbb{Q}$ and that $A \subseteq \mathbb{Q}$. Then the following are equivalent.*

(1) A is definable in $(\mathbb{Q}, <)$ with parameters from X.
(2) A is a finite union of intervals I such that the endpoints of I belong to X.

Proof. Given a formula $\varphi(x_1, \ldots, x_{n+1})$ and parameters $q_1, \ldots, q_n$ from X, we can evaluate φ in $(\mathbb{R}, <)$ relative to $q_1, \ldots, q_n$. The set A so defined is a finite union of intervals I such that the endpoints of I belong to $\{q_1, \ldots, q_n\}$. The equivalence between satisfying φ relative to $q_1, \ldots, q_n$ and belonging to the finite union of intervals is a first order property of $(\mathbb{R}, <)$. Since $(\mathbb{Q}, <)$ is a dense total order without endpoints, $(\mathbb{Q}, <) \prec (\mathbb{R}, <)$. Thus, the equivalence between satisfying φ relative to $q_1, \ldots, q_n$ and belonging to the finite union of intervals is satisfied by $(\mathbb{Q}, <)$. □

4.5 Countable sets

Definition 4.18 A set A is *countable* if either it is empty or there is a surjective map from $\mathbb{N}$ to A.

In particular, every finite set is countable, and every subset of $\mathbb{N}$ is countable. Intuitively, the countable sets are those sets whose size is less than or equal to the size of $\mathbb{N}$.

Theorem 4.19 *Suppose that $\langle A_i : i \in \mathbb{N}\rangle$ is a countable sequence of countable sets. Then $A = \cup\{A_i \mid i \in \mathbb{N}\}$ is a countable set.*

Proof. If all of the A_i's are empty, then their union is empty and hence countable. Otherwise, we may assume that none of the A_i's are empty, since discarding the empty A_i's does not change the value of A. Similarly, we may assume that there are infinitely many A_i's by allowing sets to appear more than once.

Fix a sequence of functions, $\langle f_i : i \in \mathbb{N}\rangle$ so that for each i, f_i is a surjection from $\mathbb{N}$ to A_i.

A side remark. To make sense of the above sentence, we must appeal to the axiom of choice (AC). AC is the assertion that if F is a set of nonempty sets, then there is a function c with domain F such that for each element x in F, $c(x) \in x$. In other words, c chooses an element from each element of F. In our application, we are given that for each i, there is at least one function mapping $\mathbb{N}$ onto A_i. In fact, if A_i has more than one element then there are infinitely many distinct such functions. We use the axiom of choice to choose particular countings of the A_i's.

Returning to our proof, let a be an element of A. Define $f : \mathbb{N} \to A$ as follows.

$$f(n) = \begin{cases} f_i(j), & \text{if } n = 2^i 3^j; \\ a, & \text{otherwise.} \end{cases}$$

f is well defined since every element of $\mathbb{N}$ is uniquely factored as a product of prime numbers. If b is an element of A, then there is an i such that $b \in A_i$ and hence there is a j such that $f_i(j) = b$. But then, $f(2^i 3^j) = b$. Consequently, f is a surjection. □

Even though $\mathcal{N}$ is infinite, there are sets whose size is not less than or equal to the size of $\mathcal{N}$.

Theorem 4.20 (Cantor) *The set of real numbers is not countable.*

Proof. We show first that the set $\mathcal{P}(\mathbb{N})$ of all subsets of $\mathbb{N}$ is not countable.
Suppose that

$$f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$$

Define

$$A = \{k \in \mathbb{N} \mid k \notin f(k)\}.$$

We claim that A is not in the range of f. Suppose toward a contradiction that $f(i) = A$. Then $i \in A$ if and only if $i \notin A$ which is a contradiction. This proves that f is not a surjection.

Thus $\mathcal{P}(\mathbb{N})$ is uncountable. Finally we show that set of real numbers is uncountable by producing a function

$$g; \mathcal{P}(\mathbb{N}) \to \mathbb{R}$$

which is one to one. Define

$$g(A) = \sum_{i=1}^{\infty} \epsilon_i^A 3^{-i}$$

where for each i, $\epsilon_i^A = 1$ if $i \in A$ and 0 otherwise. It follows that for A, B in $\mathcal{P}(\mathbb{N})$, if $A \neq B$ then $g(A) \neq g(B)$ and so g is one to one as required. Thus the range of g is uncountable and so the set of real numbers is uncountable. □

4.5.1 Exercises

(1) Show that $\mathbb{Q}$, the set of rational numbers, is countable.
(2) Show that there is a bijection between $\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$.
(3) Show that if $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure and X is a countable subset of M, then the collection of all sets $A \subseteq M$ such that A is definable in $\mathcal{M}$ with parameters from X is countable. (Hint: Show that there are only countably many formulas and countably many finite sequences from X.)

4.6 The Lowenheim-Skolem Theorem

The (Downward) Lowenheim-Skolem Theorem is an important application of Tarski's Theorem.

Definition 4.21 If $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure, then we say $\mathcal{M}$ *is countable* to indicate that M is a countable set.

Theorem 4.22 (Lowenheim-Skolem) *Suppose that $\mathcal{N}$ is an $\mathcal{L}_{\mathcal{A}}$-structure. Then there exists an elementary substructure $(M, I) \preceq \mathcal{N}$ such that M is countable.*

Proof. By Tarski's criterion, it suffices to find a countable set $M \subseteq N$ such that for each nonempty set $A \subseteq N$, if A is definable in the structure $\mathcal{N}$ from parameters in M, then $A \cap M \neq \emptyset$.

We construct a countable sequence $\langle M_k : k \in \mathbb{N}\rangle$ of countable subsets of N such that

(1.1) $M_0 = \emptyset$,
(1.2) for each $k \in \mathbb{N}$, $M_k \subseteq M_{k+1}$,
(1.3) and for each $k \in \mathbb{N}$, if $A \subseteq N$ is definable in $\mathcal{N}$ with parameters from M_k and $A \neq \emptyset$ then $A \cap M_{k+1} \neq \emptyset$.

Given M_k, we can find M_{k+1} as follows. Fix a counting $\langle A_i : i \in \mathbb{N}\rangle$ of the collection of subsets of N which are definable in $\mathcal{N}$ using parameters from M_k. (See the Exercises at the end of the previous section.) For each i such that A_i is not empty, let a_i be an element of A_i. Let M_{k+1} be $M_k \cup \{a_i : i \in \mathbb{N}\}$. M_{k+1} is a union of two countable sets and hence is countable. By construction, it satisfies (2) and (3) relative to M_k. Note that we have used the Axiom of Choice to choose the a_i's from the A_i's.

By another use of the Axiom of Choice, it follows that there is a sequence $\langle M_k : k \in \mathbb{N}\rangle$ which satisfies. (1)–(3). Let $M = \cup\{M_k : k \in \mathbb{N}\}$. By Theorem 4.19, M is countable.

Suppose that $A \subseteq N$ and A is definable in the structure $\mathcal{N}$ from parameters in M. Then since $M = \cup\{M_k : k \in \mathbb{N}\}$, it follows (by (2)) that for sufficiently large $k \in \mathbb{N}$, A is definable in the structure $\mathcal{N}$ from parameters in M_k. Therefore, if $A \neq \emptyset$, then $M \cap A \neq \emptyset$. Finally, for each constant symbol c_i of $\mathcal{L}_{\mathcal{A}}$, $J(c_i) \in M$

and and for each function symbol F_i of $\mathcal{L}_{\mathcal{A}}$, and for each $(a_1, \ldots, a_n) \in M^n$ where $n = \pi(F_i)$, $J(F_i)(a_1, \ldots, a_n) \in M$.

Thus, there exists I such that the structure $\mathcal{M} = (M, I)$ is a substructure of $\mathcal{N}$. By Tarski's Theorem, $\mathcal{M} \preceq \mathcal{N}$ and so $\mathcal{M}$ is a countable elementary substructure of $\mathcal{N}$. □

In the proof of Theorem 4.22, we could have started by setting M_0 to be any given countable set. Thus, our proof establishes the stronger statement given below.

Theorem 4.23 *Suppose that $\mathcal{N} = (N, J)$ is an $\mathcal{L}_{\mathcal{A}}$-structure and X is a countable subset of N. Then there exists an elementary substructure $(M, I) \preceq \mathcal{N}$ such that M is countable and $X \subseteq M$.*

4.7 Arbitrary dense total orders

Suppose that $\mathcal{M} = (M, <)$ is a dense total order without endpoints. Must $\mathcal{M}$ be elementarily equivalent to $(\mathbb{Q}, <)$, and can one characterize the subsets of M which are definable in $\mathcal{M}$?

Our analysis of the definable sets of the structure $\mathcal{R}$ made essential use of the existence of automorphisms. One can construct examples of structures $\mathcal{M} = (M, <)$ which are dense orders without endpoints and with the additional property that if

$$e : M \to M$$

is a bijection which defines an automorphism of the structure $\mathcal{M}$ then e is the identity. In fact one can construct $\mathcal{M}$ as a substructure of $\mathcal{R}$. So one cannot hope to use the method of automorphisms to directly analyze the definable sets of an arbitrary dense order without endpoints.

We begin with a characterization of the countable dense total orders without endpoints.

Theorem 4.24 *Suppose that $\mathcal{M} = (M, \leq_M)$ is a countable dense total order without endpoints. Then $\mathcal{M}$ and $(\mathbb{Q}, \leq)$ are isomorphic.*

Proof. Let $m_1, m_2, \ldots$ be a counting of M, and let $q_1, q_2, \ldots$ be a counting of $\mathbb{Q}$. (See the Exercises in the previous section.)

We build a function $f : M \to \mathbb{Q}$ by recursion, specifying finitely many of the values of f during each stage. Let f_n denote the finite function defined by the end of stage n. We will ensure that f_n preserves order: for all x and y in the domain of f_n, $x <_M y$ if and only if $f_n(x) < f_n(y)$.

To start, let f_0 be the function with empty domain. In other words, we will not have specified any of the values of f at the beginning of step 1.

During step $n+1$, we extend f_n as follows. First, let $a_1, \ldots, a_k$ be the domain of f_n written in increasing order with respect to $<_M$. For each i less than or equal to k, we let $f_{n+1}(a_i)$ be equal to $f_n(a_i)$.

Second, if m_{n+1} is not in the domain of f_n, then we define $f_{n+1}(m_{n+1})$ so as to preserve order. If m_{n+1} is less than a_1, then using the fact that $\mathbb{Q}$ has no least element, we let j be the least integer such that q_j is less than $f(a_1)$ and let $f_{n+1}(m_{n+1}) = q_j$; if there is an i such that $a_i <_M m_{n+1} <_M a_{i+1}$, then using the fact that $\mathbb{Q}$ is dense, we let j be the least integer such that $f_{n+1}(a_i) < q_j < f_{n+1}(a_{i+1})$ and let $f_{n+1}(m_{n+1}) = q_j$; if m_{n+1} is greater than a_k, then using the fact that $\mathbb{Q}$ has no greatest element, we let j be the least integer such that q_j is greater than $f_{n+1}(a_k)$ and let $f_{n+1}(m_{n+1}) = q_j$

We complete step $n+1$ as follows. If q_{n+1} is not in the range of f_{n+1}, even after we have defined f_{n+1} on m_{n+1}, then we find the least j such that m_j has the same order theoretic properties relative to $a_1, \ldots, a_k$ and m_{n+1} as q_{n+1} has relative to $f_{n+1}(a_1), \ldots, f_{n+1}(a_k)$ and $f_{n+1}(M_{n+1})$. The only properties of $\mathbb{Q}$ that we used of in the above were that it has no least or greatest elements, that has no endpoints, and it is totally ordered. $\mathcal{M}$ shares these properties, and so we can find m_j in M just as we found q_j in $\mathbb{Q}$.

Let f be the function defined by $f(m) = q$ if and only if there is an n such that $f_n(m) = q$. f is well defined since each f_n extends all of the functions defined at steps before n. Every element of M belongs to the domain of f, since for every $m \in M$ there is an n such that $m = m_n$. Then either m was added to the domain of f during an earlier step, or we add m to the domain of f by setting a value for f_n at m_n during step n. Similarly, every $q \in \mathbb{Q}$ is in the range of f. f preserves order by construction, hence f is injective and for all x and y in M, $x <_M y$ if and only if $f(x) < f(y)$. Thus, $f : M \to \mathbb{Q}$ is an isomorphism.□

Theorem 4.25 *Suppose that $\mathcal{M}$ is a dense total order without endpoints. Then, the following conditions hold.*

(1) $\mathcal{M} \equiv (\mathbb{Q}, \leq)$.

(2) If $X \subseteq M$ and $A \subseteq M$ is definable in the structure $\mathcal{M}$ with parameters from X, then A is a finite union of intervals with endpoints in X.

Proof. By the Downward Lowenheim-Skolem Theorem there exists an elementary substructure

$$(M_0, <_0) = \mathcal{M}_0 \preceq \mathcal{M}$$

such that M_0 is countable. But then, $\mathcal{M}_0 \cong (\mathbb{Q}, <)$ and so $\mathcal{M} \equiv (\mathbb{Q}, <)$. This proves (1).

We now prove (2). In fact (2) follows from (1) (why?) but we shall prove (2) more directly. Fix $X \subseteq M$ and $A \subseteq M$ such that A is definable in $\mathcal{M}$ with parameters from X. Let $\varphi(x_1, x_2, \ldots, x_n)$ be a formula and let $a_2, \ldots, a_n$ be elements of X such that

$$A = \{a \in M : \mathcal{M} \vDash \varphi[a, a_2, \ldots, a_n]\}.$$

We prove that A is a union of intervals with endpoints from $\{a_2, \ldots, a_n\}$.

Assume toward a contradiction that this fails. By Theorem 4.23, Choose $\mathcal{M}_0 = (M_0, I_0)$ so that $\{a_2, \ldots, a_n\} \subseteq M_0$ and so that $\mathcal{M}_0$ is a countable elementary substructure $\mathcal{M}$. Thus, since $\mathcal{M}_0 \preceq \mathcal{M}$,

$$A \cap M_0 = \{a \in M_0 : \mathcal{M}_0 \vDash \varphi[a, a_2, \ldots, a_n]\}$$

and $A \cap M_0$ is not a union of intervals of $\mathcal{M}_0$ with endpoints from $\{a_2, \ldots, a_n\}$. But $\mathcal{M}_0 \cong (\mathbb{Q}, <)$ and this contradicts Theorem 4.17. □

Thus we have managed to analyze the definable sets in an arbitrary structure $\mathcal{M} = (M, <)$ which is a dense order without endpoints. The analysis succeeds by using automorphisms of countable elementary substructures.

What about the definable subsets of the structure

$$(\mathbb{N}, <)?$$

We shall eventually show that if $A \subseteq \mathbb{N}$ is definable from parameters in the structure, $(\mathbb{N}, <)$, then A is either finite or the complement of A is finite. There are no automorphisms of the structure $(\mathbb{N}, <)$ except for the trivial automorphism (given by the identity function). However automorphisms *can* be used to analyze the structure $(\mathbb{N}, <)$ by first constructing a structure $(M, <)$ such that

$$(\mathbb{N}, <) \prec (M, <)$$

and then using automorphisms of $(M, <)$ to show that if $A \subset M$ is definable in the structure $(M, <)$ (from parameters) then A is a finite union of intervals.

This suggests the general problem of given an $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M}$ under what circumstances must there exist an $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{N}$ such that

$$\mathcal{M} \prec \mathcal{N}$$

and $\mathcal{M} \neq \mathcal{N}$. The only requirement is that $\mathcal{M}$ be infinite. We shall prove this as a corollary of the the Gödel Completeness Theorem which is the topic of the next chapter.

4.7.1 Exercises

(1) Let $A = \{P_1\}$ be the alphabet with one unary predicate symbol. For each of i equal to 1 or 2, suppose that $\mathcal{M}_i = (M_i, I_i)$ is an A structure such that M_i, $I_i(P_1)$, and $M_i \setminus I_i(P_1)$ are all infinite. Here $M_i \setminus I_i(P_1)$ consists of those elements of M_i which are not in $I_i(P_1)$.
Show that $\mathcal{M}_1 \equiv \mathcal{M}_2$.