In all exercises, you may assume $R$ is a commutative ring with identity where $1 \neq 0$. For some of the questions, you may find Theorem 17 and its corollaries (pp. 373-374) helpful; we have not covered these yet in class but will next week.

1. (Exercise 2 in DF §10.4.) Show that the element $2 \otimes \overline{1}$ is zero in the $\mathbf{Z}$-module $\mathbf{Z} \otimes_{\mathbf{Z}} (\mathbf{Z}/2\mathbf{Z})$, but is nonzero in $(2\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{Z}/2\mathbf{Z})$.

*Proof.* By the relations given at the beginning of section 10.4, in $\mathbf{Z} \otimes_{\mathbf{Z}} (\mathbf{Z}/2\mathbf{Z})$ we have

$$2 \otimes \overline{1} = (1+1) \otimes \overline{1} = 1 \otimes \overline{1} + 1 \otimes \overline{1} = 1 \otimes (\overline{1} + \overline{1}) = 1 \otimes 0 = 0$$

because $1 \otimes 0 = 1 \otimes (0+0) = 1 \otimes 0 + 1 \otimes 0$.

Now, assume that $2 \otimes \overline{1} = 0$ in $(2\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{Z}/2\mathbf{Z})$. We will show that this impies that $(2\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{Z}/2\mathbf{Z}) = \{0\}$. Any element of $(2\mathbf{Z}) \times (\mathbf{Z}/2\mathbf{Z})$ is of the form $(2x, \overline{y})$ where $x, y \in \mathbf{Z}$. By example 1 on pg. 368, $m \otimes \overline{0} = 0$ for all $m$. So we may assume that $\overline{y} = 1$. This gives

$$(2x) \otimes \overline{1} = (x2) \otimes \overline{1} = x(2 \otimes \overline{1}) = 0$$

therefore $(2\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{Z}/2\mathbf{Z}) = \{0\}$. By the universal property of the tensor product, there are no nonzero $\mathbf{Z}$-bilinear maps from $(2\mathbf{Z}) \times (\mathbf{Z}/2\mathbf{Z})$ to any abelian group. However, the map $\varphi : (2\mathbf{Z}) \times (\mathbf{Z}/2\mathbf{Z}) \to \mathbf{Z}$ defined by $\varphi(x, y) = xy$ is a nonzero $\mathbf{Z}$-bilinear to an abelian group:

$$\begin{aligned}
\varphi(r_1 a + r_2 b, x) &= (r_1 a + r_2 b)x = r_1 ax + r_2 bx = r_1 \varphi(a, x) + r_2 \varphi(b, x) \\
\varphi(x, r_1 a + r_2 b) &= x(r_1 a + r_2 b) = r_1 ax + r_2 bx = r_1 \varphi(x, a) + r_2 \varphi(x, b).
\end{aligned}$$

This is a contradiction, thus $2 \otimes \overline{1} \neq 0$. $\square$

2. (Exercise 8 in DF §10.4.) Let $R$ be an integral domain with quotient field (a.k.a. field of fractions; to review this, see §7.5) $Q$, and let $N$ be any $R$-module. Let $U = R \backslash \{0\}$ denote the set of nonzero elements in $R$, and define $U^{-1}N$ to be the set of equivalence classes of ordered pairs $(u, n)$ with $u \in U$ and $n \in N$, under the equivalence relation $(u, n) \sim (u', n')$ if and only if there exists $v \in U$ such that $v(u'n - un') = 0$. (That is, $U^{-1}N$ is the quotient of the set $U \times N$ by the given equivalence relation.) Given an ordered pair $(u, n) \in U \times N$, let $\overline{(u, n)} \in U^{-1}N$ denote the equivalence class of $(u, n)$.

(a) Prove that $U^{-1}N$ is an abelian group under the addition defined by

$$\overline{(u_1, n_1)} + \overline{(u_2, n_2)} = \overline{(u_1 u_2, u_2 n_1 + u_1 n_2)}.$$

Prove that the operation $r \cdot \overline{(u, n)} = \overline{(u, rn)}$ defines an action of $R$ on $U^{-1}N$ making it into an $R$-module. [This is an example of *localization*, considered in general in §15.4.]

*Proof.* Note: $R$ is an integral domain, so we will commute under multiplaction as necessary.

First, suppose that $(a, x) \sim (c, z)$ and $(b, y) \sim (d, w)$ for some $a, b, c, d \in U, x, y, z, w \in N$. Then $u(az - cx) = 0$ and $v(bw - dy) = 0$ in $N$ for some $u, v \in U$. We have

$$\begin{aligned}
\overline{(a, x)} + \overline{(b, y)} &= \overline{(ab, bx + ay)} \\
\overline{(c, z)} + \overline{(d, w)} &= \overline{(cd, dz + cw)}.
\end{aligned}$$

But also

$$\begin{aligned}
uv(ab(dz + cw) - cd(bx + ay)) &= uvabdz + uvabcw - uvbcdx - uvacdy \\
&= vbd(uaz) + uac(vbw) - vbd(ucx) - uac(vdy) \\
&= vbd(u(az - cx)) + uac(v(bw - dy)) \\
&= vbd(0) + uac(0) \\
&= 0
\end{aligned}$$

thus the righthand sides are equivalent under $\sim$. So the operation is well-defined.

It is clear from the definition of + that the righthand side is of the form $\overline{(u,n)}$ for $u \in U$ and $n \in N$: $u_1u_2 \in U$ because $R$ is an integral domain (so $u_1u_2 \neq 0$) and $u_2n_1 + u_1n_2 \in N$ because $N$ is a $R$-module. Therefore the set is closed under this operation.

We will check associativity. Let $a, b, c \in U$ and $x, y, z \in N$.

$$\begin{aligned}(\overline{(a,x)} + \overline{(b,y)}) + \overline{(c,z)} &= \overline{(ab, bx+ay)} + \overline{(c,z)} \\ &= \overline{(abc, c(bx+ay)+abz)} \\ &= \overline{(abc, bcx+acy+abz)}\end{aligned}$$

where in the last step we have taken advantage of the commutativity of multiplication in $R$, since it is an integral domain. Also,

$$\begin{aligned}\overline{(a,x)} + (\overline{(b,y)} + \overline{(c,z)}) &= \overline{(a,x)} + \overline{(bc, cy+bz)} \\ &= \overline{(abc, a(cy+bz)+bcx)} \\ &= \overline{(abc, acy+abz+bcx)} \\ &= \overline{(abc, bcx+acy+abz)}\end{aligned}$$

where in the last step we have taken advantage of the commutativity of addition in $R$, simply because it is a ring. So the operation + is associative.

Denote the identity element of $N$ by $0_N$. The element $(1_R, 0_N)$ serves as an additive identity in $U^{-1}N$ since, for any $a \in U, x \in N$, we have

$$\overline{(a,x)} + \overline{(1_R, 0_N)} = \overline{(a(1_R), 1_Rx + a(0_N))} = \overline{(a,x)} = \overline{(1_R, 0_N)} + \overline{(a,x)}.$$

We will use the following fact several times, so I will state it here:

$$\text{for any } u, v \in U, n \in N \text{ we have } (u,n) \sim (uv, vn) \tag{1}$$

This is clear because $1_R(u(vn) - (uv)n) = 0$.

For any $a \in U, x \in N$, the element $(a - x)$ is an additive inverse (where $-x$ is the inverse of $x$ in $N$):

$$\overline{(a,x)} + \overline{(a,-x)} = \overline{(a^2, ax-ax)} = \overline{(a^2, 0_N)} = \overline{(a^2(1_R), a^2(0_N))} = \overline{(1_R, 0_N)}$$

where the last equality is given by (1). It is clear that the operation is commutative, since it is symmetric in $u_1, u_2$ and in $n_1, n_2$. Thus, $U^{-1}N$ is an abelian group.

We will now check that the operation $r \cdot \overline{(u,n)} = \overline{(u, rn)}$ defines an action of $R$ onto $U^{-1}N$, making it into an $R$-module. Let $r, s \in R$, $a, b \in U$, and $x, y \in N$.

$$\begin{aligned}
r \cdot (\overline{(a,x)} + \overline{(b,y)}) &= r \cdot (\overline{(ab, bx + ay)}) \\
&= \overline{(ab, r(bx+ay))} \\
&= \overline{(ab, b(rx) + a(ry)))} \\
&= \overline{(a, rx)} + \overline{(b, ry)} \\
&= r \cdot \overline{(a,x)} + r \cdot \overline{(b,y)} \\
(r+s) \cdot \overline{(a,x)} &= \overline{(a, (r+s)x)} \\
&= \overline{(a^2, a(r+s)x)} \qquad \text{(by (1))} \\
&= \overline{(a^2, arx + asx)} \\
&= \overline{(a, rx)} + \overline{(a, sx)} \\
&= r \cdot \overline{(a,x)} + s \cdot \overline{(a,x)} \\
(rs)\overline{(a,x)} &= \overline{(a, rsx)} \\
&= r \cdot \overline{(a, sx)} \\
&= r \cdot (s \cdot \overline{(a,x)}) \\
1_R \cdot \overline{(a,x)} &= \overline{(a, 1_R x)} \\
&= \overline{(a,x)}
\end{aligned}$$

So this operation makes $U^{-1}N$ into an $R$-module. □

(b) Show that the map from $Q \times N$ to $U^{-1}N$ defined by sending $(a/b, n)$ to $\overline{(b, an)}$ for $a \in R$, $b \in U$, $n \in N$, is an $R$-bilinear map, so induces a homomorphism $f$ from $Q \otimes_R N$ to $U^{-1}N$. Show that the map $g : U^{-1}N \to Q \otimes_R N$ defined by

$$g(\overline{(u,n)}) = (1/u) \otimes n$$

is well defined and is an inverse homomorphism to $f$. Conclude that $Q \otimes_R N \cong U^{-1}N$ as $R$-modules.

*Proof.* Let $\varphi : Q \times N \to U^{-1}N$ be defined by $(a/b, n) \mapsto \overline{(b, an)}$. Let $a, b, c, d \in U$, $r, s \in$, and $m, n \in N$.

$$\begin{aligned}
\varphi\left(\left(r\frac{a}{b} + s\frac{c}{d}, n\right)\right) &= \varphi\left(\left(\frac{rad + scb}{bd}, n\right)\right) \\
&= \overline{(bd, (rad + scb)\, n)} \\
&= \overline{(bd, (rad)\, n + (scb)\, n)} \\
&= \overline{(b, ran)} + \overline{(d, scn)} \\
&= r \cdot \overline{(b, an)} + s \cdot \overline{(d, cn)} \\
&= r\varphi\left(\frac{a}{b}, n\right) + s\varphi\left(\frac{c}{d}, n\right) \\
\varphi\left(\left(\frac{a}{b}, r \cdot m + s \cdot n\right)\right) &= \overline{(b, a\,(rm + sn))} \\
&= \overline{(b^2, ab\,(rm + sn))} \qquad \text{(by (1))} \\
&= \overline{(b, arm)} + \overline{(b, asn)} \\
&= r \cdot \overline{(b, am)} + s \cdot \overline{(b, an)} \\
&= r\varphi\left(\frac{a}{b}, m\right) + s\varphi\left(\frac{a}{b}, n\right)
\end{aligned}$$

Therefore, $\varphi$ is an $R$-bilinear map. By the universal property of the tensor product, $\varphi$ induces a homomorphism $f : Q \otimes_R N \to U^{-1}N$ such that $\varphi = f \circ \iota$, where $\iota : Q \times N \to Q \otimes_R N$ is given by $\iota(\frac{a}{b}, n) = \frac{a}{b} \otimes n$. This means that $f$ must be defined by $f(\frac{a}{b} \otimes n) = \varphi(\frac{a}{b}, n) = \overline{(b, an)}$.

Let $g : U^{-1}N \to Q \otimes_R N$ be defined by

$$g(\overline{(u,n)}) = \frac{1}{u} \otimes n.$$

First, we will show that $g$ is well-defined. Suppose $\overline{(a,x)} \sim \overline{(b,y)}$ for some $a, b \in U$, $x, y \in N$. Then $u(ay - bx)$ for some $u \in U$. We have

$$g(\overline{(a,x)}) = \frac{1}{a} \otimes x = \frac{1}{uab}(ub) \otimes x = \frac{1}{uab} \otimes ubx = \frac{1}{uab} \otimes uay = \frac{1}{uab} ua \otimes y = \frac{1}{b} \otimes y = g(\overline{(b,y)}).$$

Next, we will show that $g$ is an $R$-module homomorphism. Let $r, a, b \in R$ and $x, y \in N$. We have

$$\begin{aligned} g(\overline{(a,x)} + r\overline{(b,y)}) &= g(\overline{(ab, bx + ary)}) \\ &= \frac{1}{ab} \otimes (bx + ary) \\ &= \frac{1}{ab} \otimes bx + \frac{1}{ab} \otimes ray \\ &= \frac{1}{a} \otimes x + r(\frac{1}{b} \otimes y) \\ &= g(\overline{(a,x)}) + rg(\overline{(b,y)}). \end{aligned}$$

Finally, we will show that $g$ is an inverse of $f$:

$$g \circ f\left(\frac{a}{b} \otimes n\right) = g(\overline{(b,an)}) = \frac{1}{b} \otimes an = \frac{1}{b}(a) \otimes n = \frac{a}{b} \otimes n$$

$$f \circ g(\overline{(u,n)}) = f\left(\frac{1}{u} \otimes n\right) = \overline{(u, 1(n))} = \overline{(u,n)}$$

thus $g$ is an inverse of $f$. So $f$ is an isomorphism, therefore $Q \otimes_R N \cong U^{-1}N$ as $R$-modules. □

(c) Conclude from (b) that $(1/d) \otimes n$ is 0 in $Q \otimes_R N$ if and only if $rn = 0$ for some nonzero $r \in R$.

*Proof.* Since $g$ is an isomorphism, $(\frac{1}{d}) \otimes n$ is 0 in $Q \otimes_R N$ if and only if $g(\frac{1}{d} \otimes n) = \overline{(d,n)} = 0_{U^{-1}N} = \overline{(1,0)}$. This means that $\overline{(d,n)} = \overline{(1,0)}$ and thus $u(1(n) - d(0)) = un = 0$ for some $u \in U$. Since $U$ is the nonzero elements of $R$, the result follows. □

(d) If $A$ is an abelian group, show that $\mathbf{Q} \otimes_{\mathbf{Z}} A = 0$ (where 0 denotes the zero module $\{0\}$) if and only if $A$ is a torsion abelian group (i.e., every element of $A$ has finite order). (Caution: this does not mean $A$ itself has finite order!)

*Proof.* $\mathbf{Q}$ is the field of fractions of $\mathbf{Z}$, and $A$ is naturally a $\mathbf{Z}$ module under the action $n \cdot a = a + \cdots + a$ ($n$ times). Therefore, the above result applies to $\mathbf{Q} \otimes_{\mathbf{Z}} A$; so, for any $a \in A$, $\frac{1}{d} \otimes a$ is 0 in $\mathbf{Q} \otimes_{\mathbf{Z}} A$ if and only if $ra = 0$ for some nonzero $r \in \mathbf{Z}$.

First, suppose that $A$ is a torsion abelian group, and let $a \in A$. $a$ has some finite order $n$ in $A$, therefore $na = a + \cdots + a = 0$. Since $a$ was arbitrary, $\mathbf{Q} \otimes_{\mathbf{Z}} A = 0$.

Next, assume $\mathbf{Q} \otimes_{\mathbf{Z}} A = 0$, and let $a \in A$. There is some $n \in \mathbf{Z}$ such that $na = a + \cdots + a = 0$, thus $a$ has finite order and divides $n$. So every element of $A$ has finite order, therefore $A$ is a torsion abelian group. □

3. (Exercise 10 in DF §10.4.) Suppose $N \cong R^n$ is a free $R$-module of rank $n$ with $R$-module basis $\{e_1, \ldots, e_n\}$.

(a) Let $M$ be a nonzero $R$-module. Show that for each element $\alpha \in M \otimes_R N$ there is a unique sequence of elements $m_1, \ldots, m_n \in M$ such that $\alpha = \sum_{i=1}^n m_i \otimes e_i$. Deduce that if $\sum_{i=1}^n m_i \otimes e_i = 0$ in $M \otimes_R N$, then $m_1 = \ldots = m_n = 0$.

*Proof.* We aim to show that the map $f : M^n \to M \otimes_R N$ defined by $f(m_1, \ldots, m_n) = \sum_{i=1}^n m_i \otimes e_i$ is a bijection. Define a map $\varphi : M \times N \to M^n$ by $(m, \sum_{i=1}^n r_i e_i) \mapsto (r_1 m, \ldots, r_n m)$ (since every element of $M \otimes_R N$ takes this form in a unique way). Observe that $\varphi$ is $R$-balanced:

$$\begin{aligned}
\varphi(x+y, \sum_{i=1}^n r_i e_i) &= (r_1(x+y), \ldots, r_n(x+y)) \\
&= (r_1 x + r_1 y, \ldots, r_n x + r_n y) \\
&= (r_1 x, \ldots, r_n x) + (r_1 y, \ldots, r_n y) \\
&= \varphi(x, \sum_{i=1}^n r_i e_i) + \varphi(y, \sum_{i=1}^n r_i e_i) \\
\varphi(x, \sum_{i=1}^n r_i e_i + \sum_{i=1}^n s_i e_i) &= \varphi(x, \sum_{i=1}^n (r_i + s_i) e_i) \\
&= ((r_1 + s_1)x, \ldots, (r_n + s_n)x) \\
&= (r_1 x, \ldots, r_n x) + (s_1 x, \ldots, s_n x) \\
&= \varphi(x, \sum_{i=1}^n r_i e_i) + \varphi(x, \sum_{i=1}^n s_i e_i).
\end{aligned}$$

By the universal property of tensor product, $\varphi$ induces a unique group homomorphism $\Phi : M \otimes_R N \to M^n$ such that $\Phi(m \otimes \sum_{i=1}^n r_i e_i) = (r_1 m, \cdots, r_n m)$.

To see that $f$ is a bijection, we may simply show that it is the inverse of $\Phi$:

$$f \circ \Phi(m \otimes \sum_{i=1}^n r_i e_i) = f(r_1 m, \ldots, r_n m) = \sum_{i=1}^n r_i m \otimes e_i = m \otimes \sum_{i=1}^n r_i e_i$$

$$\Phi \circ f(m_1, \ldots, m_n) = \Phi(\sum_{i=1}^n m_i \otimes e_i) = \sum_{i=1}^n \Phi(m_i \otimes e_i) = \sum_{i=1}^n (0, \ldots, m_i, \ldots, 0) = (m_1, \ldots, m_n)$$

where we have distributed $\Phi$ through the summation because it is a group homomorphism with respect to addition in $M \otimes_R N$. So $f$ is invertible and, thus, a bijection.

Since the representation of 0 as $\sum_{i=1}^n m_i \otimes e_i$ is unique with respect to the sequence $m_1, \ldots, m_n$, and we know that $\sum_{i=1}^n 0 \otimes e_i = 0$, we must have $m_1 = \cdots = m_n = 0$ if $\sum_{i=1}^n m_i \otimes e_i = 0$. □

(b) Show that if $f_1, \ldots, f_n \in N$ are merely $R$-linearly independent elements (but do not form a basis for $N$ over $R$), then it is not necessarily true that $\sum_{i=1}^n m_i \otimes f_i = 0$ implies $m_1 = \ldots = m_n = 0$. (Hint: Consider $R = \mathbf{Z}$, $n = 1$, $M = \mathbf{Z}/2\mathbf{Z}$, and the element $\overline{1} \otimes 2$.)

*Proof.* Let $R = \mathbf{Z}$, $n = 1$, and $M = \mathbf{Z}/2\mathbf{Z}$. We have

$$\overline{1} \otimes 2 = \overline{1}(2) \otimes 1 = \overline{2} \otimes 1 = 0 \otimes 1 = 0$$

However, $\overline{1}$ is nonzero in $(\mathbf{Z}/2\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}$. Clearly, 2 is linearly independent in $\mathbf{Z}$ because $z \cdot 2 = 0$ implies $z = 0$. □

4. (Exercise 11 in DF §10.4.) Let $\{e_1, e_2\}$ be a basis of the $\mathbf{R}$-module $V = \mathbf{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_{\mathbf{R}} V$ cannot be written as a simple tensor $v \otimes w$ for any $v, w \in V$.

*Proof.* Suppose, for a contradiction, that there do exist $v, w \in \mathbf{R}^2$ such that $v \otimes w = e_1 \otimes e_2 + e_2 \otimes e_1$. Since $\{e_1, e_2\}$ forms a basis for $\mathbf{R}^2$ over $\mathbf{R}$, there exist unique $a, b, c, d \in \mathbf{R}$ such that $v = ae_1 + be_2$ and $w = ce_1 + de_2$. So

$$v \otimes w = ac(e_1 \otimes e_1) + ad(e_1 \otimes e_2) + bc(e_2 \otimes e_1) + bd(e_2 \otimes e_2) = e_1 \otimes e_2 + e_2 \otimes e_2.$$

By the result of part (a) in the previous exercise, we must have $ac = bd = 0$ and $ad = bc = 1$. So either $a = 0$ or $c = 0$. However, if $a = 0$ then $ad \neq 0$ and if $c = 0$ then $bc \neq 0$, a contradiction. So the result follows. $\square$

5. (Exercise 15 in DF §10.4.) Show that tensor products do not in general commute with direct products; that is, there exist $R$, $M$, $N_i$ such that $M \otimes_R (\prod N_i) \not\cong \prod(M \otimes_R N_i)$. (Hint: consider the direct product of the $\mathbf{Z}$-modules $\mathbf{Z}/2^i\mathbf{Z}$ where $i = 1, 2, \ldots$; tensor this with $\mathbf{Q}$ over $\mathbf{Z}$.)

*Proof.* We want to show that $\prod(\mathbf{Z}/2^i\mathbf{Z}) \otimes_{/Z} \mathbf{Q} \not\cong \prod(\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Z}/2^i\mathbf{Z})$. We have shown in exercise 2 that, if $A$ is an abelian group, then $\mathbf{Q} \otimes_{\mathbf{Z}} A = 0$ if and only if $A$ is a torsion abelian group. Note that $\prod(\mathbf{Z}/2^i\mathbf{Z})$ is an abelian group, and also that $\mathbf{Z}/2^i\mathbf{Z}$ is an abelian group for every $i$.

For every $i$, $\mathbf{Z}/2^i\mathbf{Z}$ is torsion because every element has order at most $2^i$. Therefore, $\prod(\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Z}/2^i\mathbf{Z}) = \prod 0 = 0$. However, $\prod(\mathbf{Z}/2^i\mathbf{Z})$ is not torsion because the element $\prod 1_{\mathbf{Z}/2^i\mathbf{Z}}$ has infinite order. Thus, $\prod(\mathbf{Z}/2^i\mathbf{Z}) \otimes_{/Z} \mathbf{Q} \neq 0$. So clearly these tensor products cannot be isomorphic. $\square$

6. (Exercise 17 in DF §10.4.) Let $I = (2, X)$ be the ideal generated by 2 and $X$ in the ring $R = \mathbf{Z}[X]$. The ring $\mathbf{Z}/2\mathbf{Z} = R/I$ is naturally an $R$-module (the $R$-action being given by multiplication in $R$ followed by reduction mod $I$) that is annihilated by both 2 and $X$.

(a) Show that the map $\phi : I \times I \to \mathbf{Z}/2\mathbf{Z}$ defined by

$$\phi(a_0 + a_1X + \ldots + a_nX^n, b_0 + b_1X + \ldots + b_mX^m) = \overline{\frac{a_0}{2}b_1}$$

(where the bar over the integer denotes its equivalence class in $\mathbf{Z}/2\mathbf{Z}$) is $R$-bilinear.

*Proof.* Note that $I$ is the set of all polynomials in $\mathbf{Z}[x]$ with even constant term. This is stated in example 3 on pg. 252, although it should be clear. Therefore, the action of $\mathbf{Z}[x]$ on $\mathbf{Z}/2\mathbf{Z}$ is simply $p(x) \cdot a = (p(x))(a) + I = \overline{a(p(0))}$, i.e. it is the constant term of $p(x)$, times $a$, taken modulo 2. Further, this map can be written as $\phi(p(x), q(x)) = \overline{\frac{p(0)}{2}r'(0)}$, where $r'(0)$ is the derivative of $r(x)$ evaluated at 0.

Let $a(x), b(x) \in \mathbf{Z}[x]$ and $p(x), q(x), r(x) \in I$. Since multiplication in $\mathbf{Z}[x]$ is equivalent to pointwise of polynomials as functions, we have

$$\begin{aligned}
\phi(a(x)p(x) + b(x)q(x), r(x)) &= \phi((a \cdot p + b \cdot q)(x), r(x)) \\
&= \overline{\frac{(a \cdot p + b \cdot q)(0)}{2}r'(0)} \\
&= \overline{\frac{a(0)p(0) + b(0)q(0)}{2}r'(0)} \\
&= a(x) \cdot \overline{\frac{p(0)}{2}r'(0)} + b(x) \cdot \overline{\frac{q(0)}{2}r'(0)} \\
&= a(x) \cdot \phi(p(x), r(x)) + b(x) \cdot \phi(q(x), r(x))
\end{aligned}$$

Note now that, for any $s(x) \in I$, $s(0) = s(x) + I = \overline{0}$. So we have

$$\begin{aligned}
\phi(r(x), a(x)p(x) + b(x)q(x)) &= \phi(r(x), (a \cdot p + b \cdot q)(x)) \\
&= \overline{\frac{r(0)}{2}(a \cdot p + b \cdot q)'(0)} \\
&= \overline{\frac{r(0)}{2}(a'p + ap' + b'q + bq')(0)} \\
&= \overline{\frac{r(0)}{2}}(\overline{a'(0)} \cdot \overline{p(0)} + \overline{a(0)} \cdot \overline{p'(0)} + \overline{b'(0)} \cdot \overline{q(0)} + \overline{b(0)} \cdot \overline{q'(0)}) \\
&= \overline{\frac{r(0)}{2}}(\overline{a'(0)} \cdot \overline{0} + \overline{a(0)} \cdot \overline{p'(0)} + \overline{b'(0)} \cdot \overline{0} + \overline{b(0)} \cdot \overline{q'(0)}) \\
&= \overline{\frac{r(0)}{2}a(0)p'(0)} + \overline{\frac{r(0)}{2}b(0)q'(0)} \\
&= a(x) \cdot \overline{\frac{r(0)}{2}p'(0)} + b(x) \cdot \overline{\frac{r(0)}{2}q'(0)} \\
&= a(x) \cdot \phi(r(x), p(x)) + b(x) \cdot \phi(r(x), q(x))
\end{aligned}$$

thus $\phi$ is $Z[x]$-bilinear. $\square$

(b) Show that there is an $R$-module homomorphism from $I \otimes_R I \to \mathbf{Z}/2\mathbf{Z}$ mappyng $p(X) \otimes q(X)$ to $\overline{\frac{p(0)}{2}q'(0)}$, where $q'$ denotes the usual polynomial derivative of $q$.

*Proof.* This follows immediately from Corrolary 12, which says that there is an $R$-module homomorphism $\Phi : I \otimes_R I \to \mathbf{Z}/2\mathbf{Z}$ such that $\phi = \Phi \circ \iota$. Thus, $\Phi(p(x) \otimes q(x)) = \phi((p(x), q(x)) = \phi(p(x), q(x)) = \overline{\frac{p(0)}{2}q'(0)}$. $\square$

(c) Show that $2 \otimes X \neq X \otimes 2$ in $I \otimes_R I$.

*Proof.* Suppose that $2 \otimes x = x \otimes 2$ in $I \otimes_R I$. Then we must have $\Phi(2 \otimes x) = \Phi(x \otimes 2)$, simply because $\Phi$ is a well-defined function. However, $\Phi(x \otimes 2) = \overline{\frac{0}{2}0} = \overline{0}$ but $\Phi(2 \otimes x) = \overline{\frac{2}{2}1} = \overline{1}$, a contradiction because $\overline{0} \neq \overline{1}$ in $\mathbf{Z}/2\mathbf{Z}$. So $2 \otimes x \neq x \otimes 2$. $\square$