33. Let E be a nonempty set, and let $f_n, g_n : E \to \mathbb{C}$ be sequences of functions converging uniformly to f, g, respectively.

(a) Prove that $f_n + g_n \to f + g$ uniformly.

Proof. For all $\epsilon > 0$, there exist N_1 and N_2 such that, for all $x \in E$ and all $n \geq \max(N_1, N_2)$, we have $|f_n(x) - f(x)| \leq \frac{\epsilon}{2}$ and $|g_n(x) - g(x)| \leq \frac{\epsilon}{2}$. Therefore, for all $x \in E$ and all $n \geq \max(N_1, N_2)$, we have

$$\begin{aligned}|(f_n(x) + g_n(x)) - (f(x) - g(x))| &= |(f_n(x) - f(x)) - (g_n(x) - g(x))| \\ &\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

□

(b) Prove that if f and g are bounded functions, then $f_n g_n \to fg$ uniformly.

Proof. Observe that

$$\begin{aligned}|f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)g_n(x) - f(x)g_n(x) + f(x)g_n(x) - f(x)g(x)| \\ &= |g_n(x)(f_n(x) - f(x)) + f(x)(g_n(x) - g(x))| \\ &\leq |g_n(x)||f_n(x) - f(x)| + |f(x)||g_n(x) - g(x)|.\end{aligned}$$

Let $\epsilon > 0$. Since f and g are bounded, there exists some $M \in \mathbb{R}$ such that $f(x), g(x) < M$ for all $x \in E$. Since both sequences are uniformly convergent, there exists some $N \in \mathbb{N}$ such that if $n > N$, then $|f_n(x) - f(x)|, |g_n(x) - g(x)| < \sqrt{\epsilon + M^2} - M$.

Now, we see that, for $n > N$, g_n must be bounded as well, since

$$|g_n(x)| - |g(x)| \leq |g_n(x) - g(x)| < \sqrt{\epsilon + M^2} - M$$

$$\implies |g_n(x)| < \sqrt{\epsilon + M^2} - M + |g(x)| < \sqrt{\epsilon + M^2}.$$

Substituting these bounds into the inequality from the first paragraph gives

$$|f_n(x)g_n(x) - f(x)g(x)| \leq \sqrt{\epsilon + M^2}(\sqrt{\epsilon + M^2} - M) + M(\sqrt{\epsilon + M^2} - M) = \epsilon.$$

□

(c) Prove that if f is bounded away from 0 then $\frac{1}{f_n} \to \frac{1}{f}$ uniformly.

Proof. First, note that there exists some $N_1 \in \mathbb{N}$ such that, for all $n > N_1$, $|f_n|$ is bounded away from zero. Simply choose N_1 so that $|f_n(x) - f(x)| < \frac{a}{2}$ for all $x \in E$. Then, whenever $n \geq N_1$, we have $|f_n(x)| > a - \frac{a}{2} = \frac{a}{2} > 0$.

Now, let $\epsilon > 0$, and let N_2 be such that whenever $n > N_2$, we have $|f_n(x) - f(x)| < \frac{a^2}{2}\epsilon$ for all $x \in E$. Then, whenever $n > \max(N_1, N_2)$, we have

$$\left|\frac{1}{f_n(x)} - \frac{1}{f(x)}\right| = \frac{|f(x) - f_n(x)|}{|f(x)||f_n(x)|} < \frac{\frac{a^2}{2}\epsilon}{\frac{a}{2}a} = \epsilon.$$

□

(d) Suppose E is the image of a piecewise-C^1 path γ, and suppose each f_n is continuous. Prove that $\displaystyle\int_\gamma f_n(z)dz \to \int_\gamma f(z)dz$.

Proof. Let $\epsilon > 0$. There exists some N such that, whenever $n > N$ we have $|f_n(z) - f(z)| < \frac{\epsilon}{L(\gamma)}$ for all $z \in E$. This gives

$$\left|\int_\gamma f_n(z)dz - \int_\gamma f(z)dz\right| = \left|\int_\gamma f_n(z) - f(z)dz\right| \leq \int_\gamma |f_n(z) - f(z)|dz \leq \int_\gamma \frac{\epsilon}{L(\gamma)}dz = \epsilon.$$

Without uniform convergence, the second inequality would not hold. □

34. Let $\Omega \subseteq \mathbb{C}$ be open, and let $f_n : \Omega \to \mathbb{C}$ be a sequence of functions. We say that (f_n) *converges locally uniformly* to f if for each $z \in \Omega$, there exists an open neighborhood $\mathcal{U}$ of z such that f_n converges uniformly to f on $\mathcal{U}$. Prove that $f_n \to f$ locally uniformly if and only if $f_n \to f$ uniformly on compact sets of Ω.

Proof. Suppose $f_n \to f$ locally uniformly. Let $C \subseteq \Omega$ be a compact set, and for each $z \in C$ choose an open neighborhood $\mathcal{U}_z$ on which $f_n \to f$ uniformly, and let $\mathscr{C} = \{\mathcal{U}_z : z \in C\}$. Since $\mathscr{C}$ forms an open cover of C, it has a finite subcover $\mathscr{C}' = \{\mathcal{U}_1, \dots, \mathcal{U}_m\}$.

Let $\epsilon > 0$. For each $\mathcal{U}_k \in \mathscr{C}'$, there is some N_k such that whenever $n > N_k$ we have $|f_n(x) - f(x)| < \epsilon$ for all $x \in \mathcal{U}_k$. Letting $N = \max(N_1, \dots, N_m)$, we have $|f_n(x) - f(x)| < \epsilon$ for all $x \in C$ whenever $n > N$, thus $f_n \to f$ uniformly on C.

For the reverse direction, suppose $f_n \to f$ uniformly on compact sets. Let $z \in \Omega$. Since Ω is open, there is some closed ball $B_r(z)$ of radius r, centered at z, that is contained within Ω. Closed balls are compact, so $f_n \to f$ uniformly on $B_r(z)$. In particular, this occurs on the open ball of radius $\frac{r}{2}$ around z, which is an open neighborhood of z. So $f_n \to f$ locally uniformly. □

35. Let f be an entire function, and suppose that for all sufficiently large $|z|$, $|f(z)| \leq C \cdot |z|^n$. Prove that f is a polynomial of degree at most n.

Proof. By hypothesis, there exists some $R \in \mathbb{R}^+$ such that, whenever $|z| \geq R$, we have $|f(z)| \leq C \cdot |z|^n$. So, for all w on the circle C_r of radius $r > R$ centered at 0, we have $|w| = r > R$. Therefore,

$$|f(w)| \leq C \cdot |w|^n = C \cdot r^n.$$

Since f is holomorphic, Cauchy's estimate gives us

$$f^{(k)}(0) \leq \frac{k!}{r^k} \sup_{z \in C_r} |f(z)| \leq Ck! \frac{r^n}{r^k}.$$

When $k > n$, the righthand side approaches 0 as $r \to \infty$. Since this inequality holds for arbitrarily large r, we must have $f^{(k)}(0) = 0$ whenever $k > n$.

Since f is entire, we know it is analytic and that the coefficient of the kth term in its power series expansion about 0 is $\dfrac{f^{(k)}(0)}{k!}$. Thus, all terms for which $k > n$ are 0, so f must be a polynomial of degree at most n. □

36. Let $f : \Omega \to \mathbb{C}$ be holomorphic. Prove that for any $z_0 \in \Omega$, the sequence of derivatives cannot satisfy $|f^{(n)}(z_0)| > n! \cdot n^n$.

Proof. Suppose, for a contradiction, that this inequality does hold for all n. Let $z_0 \in \Omega$ (since Ω is nonempty), and consider any nonempty open ball $B_r(z_0)$ whose closure is contained in Ω (we know one exists because Ω is nonempty). Since f is holomorphic on Ω, it has a power series expansion $\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$ that converges to f on all of $B_r(z_0)$. However, Hadamard's formula tells us that its radius of convergence satisfies

$$\frac{1}{R} = \limsup \left| \frac{f^{(n)}(z_0)}{n!} \right|^{\frac{1}{n}} \leq \limsup \left| \frac{n! \cdot n^n}{n!} \right|^{\frac{1}{n}} = \limsup n = \infty$$

thus $R = 0$. This is a contradiction, since it implies that $r \leq 0$, but we assumed that $B_r(z_0)$ is open and nonemtpy. □

37. Prove that the range of a nonconstant entire function is dense in $\mathbb{C}$.

Proof. Suppose the range of f is not dense in $\mathbb{C}$. Then there exist $r \in \mathbb{R}$ and $w \in \mathbb{C}$ such that $|f(z) - w| > r$ for all $z \in \mathbb{C}$. But then $\frac{1}{f(z)-w}$ is an entire, bounded function (the denominator is always nonzero since $w \notin f(\mathbb{C})$, and its modulus is bounded by r). Therefore, $\frac{1}{f(z)-w} = c$ for some nonzero $c \in \mathbb{C}$. Thus, $f(z) = w + \frac{1}{c}$ is constant. $\square$

38. Compute the following integrals:

(a) $\displaystyle\int_{|z|=1} \frac{\sin z}{z^{38}} dz$

Proof.

$$\int_{|z|=1} \frac{\sin z}{z^{38}} dz = \frac{2\pi i}{37!} \left[\frac{37!}{2\pi i} \int_C \frac{\sin z}{(z-0)^{37+1}} dz \right] = \frac{2\pi i}{37!} \frac{d^{37}}{dz^{37}} [\sin z]_{z=0} = \frac{2\pi i}{37!} \cos 0 = \frac{2\pi i}{37!}$$

$\square$

(b) $\displaystyle\int_{|z|=1} \left[\frac{z-2}{2z-1} \right]^3 dz$

Proof.

$$\int_{|z|=1} \left[\frac{z-2}{2z-1} \right]^3 dz = \frac{1}{8} \int_{|z|=1} \frac{(z-2)^3}{(z-\frac{1}{2})^3} dz = \frac{1}{8} \frac{2\pi i}{2!} \frac{d^2}{dz^2} [(z-2)^3]_{z=\frac{1}{2}} = \frac{\pi i}{8} (3)(2)(\frac{1}{2} - 2) = -\frac{9\pi i}{8}$$

$\square$

(c) $\displaystyle\frac{1}{2\pi i} \int_{|z|=1} \frac{(z-b)^m}{(z-a)^n} dz$, $|a| < 1 < |b|$; $m, n \in \mathbb{Z}$

Proof. Since $|a| < 1 < |b|$, we know that a is within the circle of integration. Also, $z - b$ is nonzero on this circle. So whenever $n \geq 1$, we may apply Theorem II.3.1.

However, first observe that if $n \leq 0$, then the integrand is holomorphic on an open ball containing the circle $|z| = 1$. So, in this case, the integral is 0. This holds regardless of the value of m, since $z - b$ is always nonzero within the open ball.

Next, assume $n \geq 1$:

$$\begin{aligned} \frac{1}{2\pi i} \int_{|z|=1} \frac{(z-b)^m}{(z-a)^n} dz &= \frac{1}{(n-1)!} \left(\frac{(n-1)!}{2\pi i} \int_{|z|=1} \frac{(z-b)^m}{(z-a)^{(n-1)+1}} dz \right) \\ &= \frac{1}{(n-1)!} \left((n-1)! \binom{m}{n-1} (z-a)^{m-n+1} \right) \\ &= \binom{m}{n-1} (z-a)^{m-n+1} \end{aligned}$$

$\square$

39. Let f be an entire function.

(a) Prove that if f has uncountably many zeros, then $f = 0$.

Proof. Let $S = \{z \in \mathbb{C} : f(z) = 0\}$. We will show that there exists some $N \in \mathbb{N}$ such that the open ball B_N of radius N, centered at 0, is such that $B_N \cap S$ is uncountable.

Suppose this is false. Then, for every N, $B_N \cap S$ is countable. But then $\bigcup_{N=1}^{\infty} B_N \cap S = S$ is a countable union of countable sets, and is thus countable - a contradiction.

So let N be such that $B_N \cap S$ is uncountable. Take any sequence $\{z_n\}$ from this set. Since $\{z_n\} \subseteq B_N \cap S \subset \overline{B_N}$, and $\overline{B_N}$ is compact, we know that z_n has a convergent subsequence z_n'. Since $f(z_n') = 0$ for all n, and z_n' converges to some point in the domain of f (since f is entire), the analytic continuation theorem states that $f = 0$.

□

(b) Suppose that, for each $z_0 \in \Omega$, at least one coefficient in the power series expansion of f at z_0 is 0. Prove that f is a polynomial.

Proof. Let $S_n = \{z \in \mathbb{C} : f^{(n)}(z) = 0\}$. The nth coefficient in the power series expansion of f about z_0 can be 0 only if $f^{(n)}(z_0) = 0$. Therefore, every $z \in \mathbb{C}$ falls into at least one S_n. Thus, $\bigcup_{n=1}^{\infty} S_n = \mathbb{C}$. Since $\mathbb{C}$ is uncountable, there must be some n such that S_n is uncountable (otherwise this would be a countable union of countable sets, hence countable). By part (a), this implies that $f^{(n)} = 0$. But then $f^{(k)} = 0$ for all $k \geq n$. This implies that only finitely many coefficients in the power series expansion of f are nonzero, therefore f is a polynomial.

□

40. Prove that, for all $\xi \in \mathbb{C}$, $\displaystyle\int_{-\infty}^{\infty} e^{-\pi x^2} \cdot e^{-2\pi i x\xi} dx = e^{-\pi\xi^2}$.

Proof. In lecture, this equality was proven for all $\xi \in \mathbb{R}$. Therefore, if we can show that $f(\xi) = \displaystyle\int_{-\infty}^{\infty} e^{-\pi x^2} \cdot e^{-2\pi i x\xi} dx$ is entire, then the analytic continuation theorem states that we must have $f(\xi) = e^{-\pi\xi^2}$ for all $\xi \in \mathbb{C}$, since these two functions agree on an entire line.

By Morera's theorem, if we can show that $\displaystyle\int_T f(\xi)d\xi = 0$ for every triangle $T \subseteq \mathbb{C}$, then f is entire. In order to do this, however, it will be useful to show that

$$\int_T \int_{-\infty}^{\infty} e^{-\pi x^2} \cdot e^{-2\pi i x\xi} dx d\xi = \int_{-\infty}^{\infty} \int_T e^{-\pi x^2} \cdot e^{-2\pi i x\xi} d\xi dx.$$

Fubini's theorem tells us that this equality holds if

$$\int_T \int_{-\infty}^{\infty} |e^{-\pi x^2} \cdot e^{-2\pi i x\xi}| dx d\xi < \infty.$$

First, observe that

$$\begin{aligned} |e^{-\pi x^2} \cdot e^{-2\pi i x\xi}| &= |e^{-\pi x^2} \cdot e^{-2\pi i x(\operatorname{Re}(\xi) + i\operatorname{Im}(\xi))}| \\ &= |e^{-\pi(x^2 - 2\operatorname{Im}(\xi)x)}||e^{-2\pi x\operatorname{Re}(\xi)i}| \\ &= e^{-\pi(x^2 - 2\operatorname{Im}(\xi)x + \operatorname{Im}(\xi)^2 - \operatorname{Im}(\xi)^2)} \\ &= e^{-\pi(x - \operatorname{Im}(\xi))^2 - \pi\operatorname{Im}(\xi)^2)}. \end{aligned}$$

So we have

$$\int_T \int_{-\infty}^{\infty} |e^{-\pi x^2} \cdot e^{-2\pi i x\xi}| dx d\xi = \int_T \int_{-\infty}^{\infty} e^{-\pi(x - \operatorname{Im}(\xi))^2 - \pi\operatorname{Im}(\xi)^2)} dx d\xi = \int_T e^{-\pi\operatorname{Im}(\xi)^2)} d\xi \ < \infty$$

for any triangle T.

By Fubini, we have

$$\int_T \int_{-\infty}^{\infty} e^{-\pi x^2} \cdot e^{-2\pi i x\xi} dx d\xi = \int_{-\infty}^{\infty} \int_T e^{-\pi x^2} \cdot e^{-2\pi i x\xi} d\xi dx = \int_{-\infty}^{\infty} 0 d\xi = 0$$

because $e^{-\pi x^2} \cdot e^{-2\pi i x\xi}$ is holomorphic. Therefore, f is holomorphic, and so by analytic continuation we must have $f(\xi) = e^{-\pi\xi^2}$.

□