Sample Problems from Past 125 Midterms

(some were assigned as homework)

In the following, you must prove that your answers are correct.

1. Give an example of an element of $\mathcal{L}_0$ which has at least ten symbols. Prove that your example belongs to $\mathcal{L}_0$.

2. Consider the set of symbols $*$ and $\#$. Let $\mathcal{L}^*$ be the smallest set L of sequences of these symbols with the following properties.

   (a) The length one sequences $\langle * \rangle$ and $\langle \# \rangle$ belong to L.

   (b) If σ and τ belong to L, then so do $\langle * \rangle + \sigma + \langle \# \rangle$ and $\langle * \rangle + \sigma + \tau + \langle \# \rangle$.

   State Readability and Unique Readability for $\mathcal{L}^*$ and determine for each whether it holds.

3. (Prove the Inference Lemma.) Suppose that φ and ψ are in $\mathcal{L}_0$ and $\Gamma \subseteq \mathcal{L}_0$. Use the logical axioms to show the following:

$$\Gamma \cup \{\varphi\} \vdash \psi \text{ if and only if } \Gamma \vdash (\varphi \rightarrow \psi)$$

4. Show that the set of logical consequences of

$$\{A_i : i \neq 1 \text{ and } i \in \mathbb{N}\}$$

   is consistent but not maximally consistent. Show that the set of logical consequences of

$$\{A_i : i \in \mathbb{N}\}$$

   is maximally consistent.

5. Let $A = \{F_1\}$ be the alphabet with one unary function symbol. Give an examples of different infinite $\mathcal{L}_A$-structures $\mathcal{M} = (M, I)$ with the following properties.

   (a) $\mathcal{M}$ has no nontrivial automorphisms.

   (b) $\mathcal{M}$ has a countably infinite set of automorphisms.

(c) For each element a of M there are only finitely many b's in M such that there is an automorphism f of $\mathcal{M}$ with $f(a) = b$. However, there are uncountably many automorphisms of $\mathcal{M}$.

6. Let $A = \{P_1\}$ be the alphabet with one unary predicate symbol. For each of i equal to 1 or 2, suppose that $\mathcal{M}_i = (M_i, I_i)$ is an A structure such that M_i, $I_i(P_1)$, and $M_i \setminus I_i(P_1)$ are all infinite. Here $M_i \setminus I_i(P_1)$ consists of those elements of M_i which are not in $I_i(P_1)$.

   Show that $\mathcal{M}_1 \equiv \mathcal{M}_2$.

The Logical Axioms

Suppose that φ_1, φ_2 and φ_3 are propositional formulas. Then each of the following propositional formulas is a logical axiom:

(Group I axioms)

1. $((\varphi_1 \to (\varphi_2 \to \varphi_3)) \to ((\varphi_1 \to \varphi_2) \to (\varphi_1 \to \varphi_3)))$
2. $(\varphi_1 \to \varphi_1)$
3. $(\varphi_1 \to (\varphi_2 \to \varphi_1))$

(Group II axioms)

1. $(\varphi_1 \to ((\neg\varphi_1) \to \varphi_2))$

(Group III axioms)

1. $(((\neg\varphi_1) \to \varphi_1) \to \varphi_1)$

(Group IV axioms)

1. $((\neg\varphi_1) \to (\varphi_1 \to \varphi_2))$
2. $(\varphi_1 \to ((\neg\varphi_2) \to (\neg(\varphi_1 \to \varphi_2))))$