7. Suppose $f : \Omega \to \mathbb{C}$ is holomorphic, where Ω is connected. Prove that if any of $\mathrm{Re}(f)$, $\mathrm{Im}(f)$, or $|f|$ are constant, then f is constant.

Proof. By Corollary I.3.6, since Ω is connected and f is holomorphic on Ω, f is constant if $f' = 0$. Recall that a differentiable function $g : U \to \mathbb{R}^n$ (where $U \subseteq \mathbb{R}^m$ is open) is constant on its domain if its derivative is 0. Clearly, any constant function $U \to \mathbb{R}^n$ is differentiable. So if any one of $\mathrm{Re}(f)$, $\mathrm{Im}(f)$, or $|f|$ is constant, then its derivative is the 0 matrix.

First, assume $\mathrm{Re}(f)(z) = u(z)$ is constant. Then its derivative is the 0 matrix, so its partial derivatives $u_x(z)$ and $v_x(z)$ are 0. By the Cauchy-Riemann equations, we have $v_x(z) = -u_y(z) = 0$. Also, in the proof of the Cauchy-Riemann equations, we showed that if f is holomorphic then its derivative must satisfy

$$f'(z) = u_x(z) + v_x(z)i$$

which is 0, given our observations. Therefore, f is constant by the corollary.

The same argument applies if $\mathrm{Im}(f)(z) = v(z)$ is constant. Its partial derivatives $v_x(z)$ and $v_y(z)$ are 0, and the Cauchy-Riemann equations gives $u_y = -v_x = 0$. Also, we derived that the derivative of f must be

$$f'(z) = v_y(z) - u_y(z)i$$

which is again 0, so f is constant.

Finally, assume $|f(z)| = u^2 + v^2$ is constant. The partial derivatives of $|f|$ are then $\frac{\partial}{\partial x}|f(z)| = 2uu_x + 2vv_x = 0$ and $\frac{\partial}{\partial y}|f(z)| = 2uu_y + 2vv_y = 0$. Dividing by 2 and substituting with the Cauchy-Riemann equations gives the system

$$uu_x - vu_y = 0$$

$$uu_y + vu_x = 0$$

Multiplying the first equation by u and the second by v and summing them gives

$$u^2u_x - uvu_y + uvu_y + v^2u_x = (u^2 + v^2)u_x = |f(z)|^2 u_x = 0.$$

If $|f(z_0)|^2 = 0$ for some $z_0 \in \Omega$, then the function must be 0 there, since, for any $z \in \mathbb{C}$, $|z|^2 = 0$ if and only if $z = 0$. Since $|f|$ is constant, then, $f = 0$ and hence f is constant. So we may assume that $|f| \neq 0$, which then implies, by the above equation, that $u_x = 0$.

Substituting $u_x = 0$ into the system we derived, we find that $vu_y = 0$ and $uu_y = 0$. Multiplying the first equation by v and the second by u then summing gives

$$v^2u_y + u^2u_y = |f(z)|^2 u_y = 0.$$

We have assumed $|f(z)|^2 \neq 0$, which gives that $u_y = 0$.

By applying the Cauchy-Riemann equations again, we now have $u_x = u_y = v_x = v_y = 0$. So, by the formula for the derivative we stated earlier, $f' = 0$. Again, by the corollary, f is constant. □

8. Say that a function $f : \Omega \to \mathbb{C}$ is $\overline{z}$-*differentiable* at $z_0 \in \Omega$ if the limit

$$\frac{\partial}{\partial \overline{z}} f(z_0) = \lim_{z \to 0} \frac{f(z_0 + z) - f(z_0)}{\overline{z}}$$

exists. Show that, if the limit exists, it must equal $\frac{1}{2}[(u_x - v_y) + i(v_x + u_y)]$ Explain why, if f is holomorphic, we must have $\frac{1}{2}[(u_x - v_y) + i(v_x + u_y)] = 0$. Thus, in a reasonable sense, holomorphic functions are truly functions of one variable z and are independent of $\overline{z}$.

Proof.

$$\begin{aligned}
\frac{\partial}{\partial \overline{z}} f(z_0) &= \lim_{z\to 0} \frac{f(z_0+z)-f(z_0)}{\overline{z}} \\
&= \lim_{x\to 0} \frac{f(x_0+x,y_0)-f(x_0,y_0)}{\overline{x}} \\
&= \lim_{x\to 0} \frac{u(x_0+x,y_0)+v(x_0+x,y_0)i-u(x_0,y_0)-v(x_0,y_0)i}{x} \\
&= \lim_{x\to 0} \frac{[u(x_0+x,y_0)-u(x_0,y_0)]+[v(x_0+x,y_0)-v(x_0,y_0)]i}{x} \\
&= \lim_{x\to 0} \frac{u(x_0+x,y_0)-u(x_0,y_0)}{x} + i\lim_{x\to 0} \frac{v(x_0+x,y_0)-v(x_0,y_0)}{x} \\
&= u_x(z_0)+v_x(z_0)i
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial \overline{z}} f(z_0) &= \lim_{z\to 0} \frac{f(z_0+z)-f(z_0)}{\overline{z}} \\
&= \lim_{y\to 0} \frac{f(x_0,y_0+y)-f(x_0,y_0)}{\overline{yi}} \\
&= \lim_{y\to 0} \frac{u(x_0,y_0+y)+v(x_0,y_0+y)i-u(x_0,y_0)-v(x_0,y_0)i}{-yi} \\
&= -\lim_{y\to 0} \frac{v(x_0,y_0+y)-v(x_0,y_0)}{y} + i\lim_{y\to 0} \frac{u(x_0,y_0+y)-u(x_0,y_0)}{y} \\
&= -v_y(z_0)+u_y(z_0)i
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial}{\partial \overline{z}} f(z_0) &= \frac{1}{2}\frac{\partial}{\partial \overline{z}} f(z_0) + \frac{1}{2}\frac{\partial}{\partial \overline{z}} f(z_0) \\
&= \frac{1}{2}[u_x(z_0)+v_x(z_0)i] + \frac{1}{2}[-v_y(z_0)+u_y(z_0)i] \\
&= \frac{1}{2}[(u_x(z_0)-v_y(z_0))+(v_x(z_0)+u_y(z_0))i].
\end{aligned}$$

Applying the Cauchy-Riemann equations reduces this equation to

$$\frac{\partial}{\partial \overline{z}} f(z_0) = \frac{1}{2}[(u_x(z_0)-v_y(z_0))+(v_x(z_0)+u_y(z_0))i] = \frac{1}{2}[(u_x(z_0)-u_x(z_0))+(v_x(z_0)-v_x(z_0))i] = 0.$$

□

9. Find the radii of convergence of the following power series:

(a) $\sum_{n=0}^{\infty} n! z^n$ $\qquad R = 0$

Proof. Let $r > 0$. There exists some $N \in \mathbb{N}$ such that $N > \frac{2}{r}$, which gives $N(\frac{r}{2}) > 1$. So for $z = \frac{r}{2}$ and $n > N$, the summand satisfies

$$\begin{aligned}
n!\left(\frac{r}{2}\right)^n &= N!\left(\frac{r}{2}\right)^N \frac{n!}{N!}\left(\frac{r}{2}\right)^{n-N} \\
&> N!\left(\frac{r}{2}\right)^N N^{n-N}\left(\frac{r}{2}\right)^{n-N} \\
&= N!\left(\frac{r}{2}\right)^N \left(N\frac{r}{2}\right)^{n-N} \\
&> N!\left(\frac{r}{2}\right)^N 1^{n-N} \\
&= N!\left(\frac{r}{2}\right)^N > 0.
\end{aligned}$$

Therefore, for the choice $z = \frac{r}{2}$, which is in the disc of radius r, the series diverges, since its summand does not converge to 0. So for every $r > 0$, we know $R \neq r$. Thus $R = 0$. □

(b) $\sum_{n=1}^{\infty} \dfrac{z^n}{n^3}$ $\qquad R = 1$

Proof. $\dfrac{1}{R} = \lim_{n\to\infty} \left| \dfrac{(n+1)^3}{n^3} \right| = \lim_{n\to\infty} \dfrac{n^3 + 3n^2 + 3n + 1}{n^3} = 1$ thus $R = 1$. □

(c) $\sum_{n=1}^{\infty} z^{n!}$ $\qquad R = 1$

Proof. Note that $\sum_{n=1}^{\infty} z^{n!} = \sum_{n=1}^{\infty} a_n z^n$ where $a_n = 1$ if $n = k!$ for some $k > 0$ and $a_n = 0$ otherwise. This allows us to apply Hadamard's formula. Since $a_n = 1$ for infinitely many n, we know that $\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n} = 1$, thus $R = 1$. □

(d) $\sum_{n=1}^{\infty} n! z^{n!}$ $\qquad R = 0$

Proof. Using the same method as in (d), express the series as $\sum_{n=1}^{\infty} z^{n!} = \sum_{n=1}^{\infty} n a_n z^n$, where a_n is defined as before. For every $M \in \mathbb{R}$, there exists some N such that $k! > M$ for all $k > N$. Therefore, $\limsup_{n\to\infty} |n a_n|^{1/n} = \infty$, so $R = 0$. □

(e) $\sum_{n=1}^{\infty} n^n z^{n^2}$ $\qquad R = 0$

Proof. Using the same method, define $a_n = 1$ if n is a perfect square and $a_n = 0$ otherwise. Thus, $\sum_{n=1}^{\infty} n^n z^{n^2} = \sum_{n=1}^{\infty} \sqrt{n}^{\sqrt{n}} a_n z^n$. Again, $\sqrt{n}^{\sqrt{n}}$ is unbounded and $a_n = 1$ for infinitely many n. Thus $\limsup_{n\to\infty} |\sqrt{n}^{\sqrt{n}} a_n|^{1/n} = \infty$, so $R = 0$. □

(f) $\sum_{n=0}^{\infty} q^{n^2} z^n$, where $q \in \mathbb{C}$ is such that $|q| < 1$. $\qquad R = \infty$

Proof. This is a straightforward application of Hadamard's formula and the fact that the modulus function is multiplicative:

$$\limsup_{n\to\infty} |q^{n^2}|^{1/n} = \limsup_{n\to\infty} |q|^n = 0$$

so $R = \infty$. □

10. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of complex numbers. For all $n \in \mathbb{N}$, define $B_n = \sum_{k=1}^{n} b_k$ and define $B_0 = 0$. Prove that, for any integers $N > M \geq 1$,

$$\sum_{n=M}^{N} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n.$$

Proof. This actually holds for $N = M \geq 1$ as well, so I will prove this stronger statement by induction because the base case is simpler.

Fix $M > 0$, and define $f(n)$ to be the right-hand side of the given equation. First, let $N = M$. Then $\sum_{n=M}^{N} a_n b_n = a_M b_M$ and

$$\begin{aligned} f(N) &= a_M B_M - a_M B_{M-1} - \sum_{n=M}^{M-1} (a_{n+1} - a_n) B_n \\ &= a_M b_M - a_M \cdot 0 - 0 \\ &= \sum_{n=M}^{N} a_n b_n \end{aligned}$$

since the third term and B_{M-1} are both the empty sum.

Now, assume the proposition holds for some $N \geq M$. Since $\sum_{n=M}^{N+1} a_n b_n = \sum_{n=M}^{N} a_n b_n + a_{N+1} b_{N+1}$, we need to show that $f(N+1) = f(N) + a_{N+1} b_{N+1}$.

$$\begin{aligned}
f(N+1) &= a_{N+1} B_{N+1} - a_M B_{M-1} - \sum_{n=M}^{N} (a_{n+1} - a_n) B_n \\
&= a_{N+1}(B_N + b_{N+1}) - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n - (a_{N+1} - a_N) B_N \\
&= a_{N+1} b_{N+1} - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n + a_N B_N \\
&= f(N) + a_{N+1} b_{N+1}
\end{aligned}$$

as desired. □

11. The following power series all have radius of convergence equal to 1. Prove that

(a) $\sum_{n=1}^{\infty} n z^n$ converges at no point on $|z| = 1$. We begin a lemma:

Lemma. *Let M be a nonnegative integer, and define $s_N = \sum_{n=M}^{N} a_n$ for some sequence a_n of complex numbers. If s_N converges to a complex number (not to infinity), then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} |a_n| = 0$.*

Proof. Suppose s_N converges to $s \in \mathbb{C}$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_{n+1} - s_n) = \lim_{n\to\infty} s_{n+1} - \lim_{n\to\infty} s_n = s - s = 0.$$

Since the modulus function is continuous, we know that

$$\lim_{n\to\infty} |a_n| = |\lim_{n\to\infty} a_n| = 0$$

as well. □

Proof. This series diverges by the lemma, since

$$\lim_{n\to\infty} |nz^n| = \lim_{n\to\infty} |n||1| = \lim_{n\to\infty} n = \infty.$$

□

(b) $\sum_{n=1}^{\infty} \dfrac{z^n}{n^2}$ converges at every point on $|z| = 1$.

Proof. We know from real analysis that the series converges when $z = 1$. Therefore, it is Cauchy. So for every $\epsilon > 0$ there exists an N such that $m \geq n > N$ implies

$$\left| \sum_{k=m}^{n} \frac{z^k}{k^2} \right| \leq \sum_{k=m}^{n} \left| \frac{z^k}{k^2} \right| = \sum_{k=m}^{n} \frac{1}{k^2} = \left| \sum_{k=m}^{n} \frac{1}{k^2} \right| < \epsilon$$

So, $\sum_{n=1}^{\infty} \dfrac{z^n}{n^2}$ is Cauchy as well, and thus convergent because $\mathbb{C}$ is complete. □

(c) $\sum_{n=1}^{\infty} \dfrac{z^n}{n}$ converges at every point on $|z| = 1$ except $z = 1$.

Lemma. *If a_n is a decreasing sequence of non-negative real numbers converging to 0 and b_n is a sequence of complex numbers such that, for some $C \in \mathbb{R}$,*

$$\left| \sum_{n=1}^{N} b_n \right| \leq C$$

for every $N \in \mathbb{N}$, then $\sum\limits_{n=1}^{\infty} a_n b_n$ converges.

Proof. Define B_n as $B_n = \sum_{k=1}^{N} b_k$ for $n \geq 1$ and $B_0 = 0$. We can now apply Exercise 10 with $M = 1$ to obtain

$$\sum_{n=1}^{N} a_n b_n = a_N B_N - a_1 B_0 - \sum_{n=1}^{N-1} (a_{n+1} - a_n) B_n = a_N B_N - \sum_{n=1}^{N-1} (a_{n+1} - a_n) B_n.$$

We will now show that the right-hand side converges as $N \to \infty$.

Since $\lim\limits_{N\to\infty} a_N = 0$, we know that $\lim\limits_{N\to\infty} C a_N = 0$. But $0 \leq |a_N B_N| \leq C a_N$, thus $a_N B_N$ is absolutely convergent by the squeeze theorem, and thus convergent. Now, $|a_{n+1} - a_n| = a_n - a_{n+1}$ because a_n is a decreasing sequence. So

$$|B_n(a_n - a_{n+1})| = |B_n|(a_n - a_{n+1}) \leq C(a_n - a_{n+1})$$

for all n. Note that

$$\sum_{n=1}^{N-1} C(a_n - a_{n+1}) = C(a_1 - a_N)$$

by telescoping, and

$$\lim_{N\to\infty} (a_1 - a_N) = \lim_{N\to\infty} a_1 - \lim_{N\to\infty} a_N = a_1 - 0 = a_1.$$

Therefore, $\lim\limits_{N\to\infty} \sum\limits_{n=1}^{N-1} C(a_n - a_{n+1}) = \lim\limits_{N\to\infty} C(a_1 - a_N) = C a_1$. Since

$$0 \leq \sum_{n=1}^{N-1} |(a_{n+1} - a_n) B_n| \leq \sum_{n=1}^{N-1} C(a_n - a_{n+1})$$

we know that $\sum\limits_{n=1}^{N-1} (a_{n+1} - a_n) B_n$ converges absolutely (and thus converges) by comparison. Consequentially, $\sum\limits_{n=1}^{N} a_n b_n$ converges. □

Main proof. Let $a_n = \frac{1}{n}$ and $b_n = z^n$. Clearly, a_n is a non-negative decreasing sequence of real numbers. By the lemma, it suffices to show find some $C \in \mathbb{R}$ such that, for all $N \in \mathbb{N}$, $\left|\sum\limits_{n=1}^{N} z^n\right| \leq C$. However, this is a geometric series, so

$$\left|\sum_{n=1}^{N} z^n\right| = \left|\frac{1 - z^{N+1}}{1 - z} - 1\right| = \frac{|z^{N+1} + z|}{|1 - z|} = \frac{|z|^N |z + 1|}{|1 - z|} = \frac{|z + 1|}{|1 - z|} \leq \frac{|1| + |z|}{|1 - z|} = \frac{2}{|1 - z|}$$

shows that $C = \dfrac{2}{|1 - z|}$ suffices. □

12. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

Prove that f is infinitely differentiable. What is the Taylor series of f centered at 0? Does it equal f?

Proof. We will show by induction on n that there exists a sequence p_n of polynomials such that, for all $x > 0$, $\frac{d^n}{dx^n} e^{-\frac{1}{x}} = p_n(\frac{1}{x}) e^{-\frac{1}{x}}$. For $n = 0$, the polynomial $p_0(x) = 1$ suffices. Now, suppose the proposition holds for some n. Then, for $x > 0$,

$$\begin{aligned} \frac{d^{n+1}}{dx^{n+1}} e^{-\frac{1}{x}} &= \frac{d}{dx}[p_n(1/x) e^{-\frac{1}{x}}] \\ &= x^{-2} p_n(1/x) e^{-\frac{1}{x}} - x^{-2} p_n'(1/x) e^{-\frac{1}{x}} \\ &= x^{-2}[p_n(1/x) - p_n'(1/x)] e^{-\frac{1}{x}} \end{aligned}$$

Letting $y = \frac{1}{x}$, we see that

$$x^{-2}[p_n(1/x) - p_n'(1/x)] = y^2[p_n(y) - p_n'(y)].$$

The derivative of a polynomial is a polynomial, and sums and products of polynomials are polynomials. Thus this expression is a polynomial in y, or equivalently a polynomial $p_{n+1}(1/x)$ in $1/x$. Thus, the proposition is true by induction.

So $e^{-\frac{1}{x}}$ is infinitely differentiable, and its nth derivative of the form $p(\frac{1}{x})e^{-\frac{1}{x}}$ for some polynomial p. The nth derivative of 0 is 0, so all that remains to show is that the limit as $x \to 0$ of any function of this form is 0, since this will prove that the function is infinitely differentiable at $x = 0$ as well.

Lemma. *Suppose* $g : (0, \infty) \to (0, \infty)$ *and* $h : (0, \infty) \to \mathbb{R}$ *and*

$$\lim_{x \to 0} g(x) = \infty, \qquad \lim_{x \to \infty} h(x) = 0$$

then

$$\lim_{x \to 0} h(g(x)) = 0.$$

Proof. Let $\epsilon > 0$. Since $\lim\limits_{x \to \infty} h(x) = 0$, there exists some M such that, whenever $g(x) > M$, we have $h(g(x)) < \epsilon$. Since $\lim\limits_{x \to 0} g(x) = \infty$, there exists some $\delta > 0$ such that whenever $0 < |x| < \delta$, we have $g(x) > M$. Therefore, whenever $0 < |x| < \delta$, we have $h(g(x)) < \epsilon$. ☐

Letting $g(x) = \frac{1}{x}$ and $h(x) = p(x)e^{-x}$ for $x \in (0, \infty)$, we see that $\lim\limits_{x \to 0} g(x) = \infty$ and $\lim\limits_{x \to \infty} h(x) = 0$. Therefore, by the lemma, $\lim\limits_{x \to 0} p(1/x)e^{-\frac{1}{x}} = \lim\limits_{x \to 0} h(g(x)) = 0$.

The Taylor series for $f(x)$ centered at 0 is the 0 function, since $f^{(n)}(0) = 0$ for all nonnegative integers n. Although this Taylor series has an infinite radius of convergence, it does not equal f. ☐

13. Let $a, b, c \in \mathbb{C}$ and consider the equation $az^2 + bz + c = 0$. Complete the square to show that the quadratic formula still holds.

Proof. Assume $a \neq 0$.

$$\begin{aligned}
&az^2 + bz + c = 0 \\
\implies &z^2 + \frac{b}{a}z = -\frac{c}{a} \\
\implies &z^2 + \frac{b}{a}z + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} \\
\implies &\left(z + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \\
\implies &\left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}
\end{aligned}$$

From the discussion on nth roots of complex numbers, we know that any nonzero complex number $w = re^{i\phi}$ has exactly two square roots: $\sqrt{r}e^{i\phi/2}$ and $\sqrt{r}e^{i(\phi/2+\pi)} = -\sqrt{r}e^{i\phi/2}$. The former root we call the *principal square root*, and define the image of w under the square root function $\sqrt{\ } : \mathbb{C} \to \mathbb{C}$ to be the principal square root of w. This means that the square roots of w are $\sqrt{w}$ and $-\sqrt{w}$. Therefore, the final equation in the above algebraic argument is equivalent to

$$z + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{or} \qquad z + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}}.$$

The square root function is still multiplicative, since

$$\sqrt{(re^{i\theta})(se^{i\phi})} = \sqrt{rse^{i(\theta+\phi)}} = \sqrt{rs}e^{i(\theta+\phi)/2} = (\sqrt{r}e^{i\frac{\theta}{2}})(\sqrt{s}e^{i\phi/2}) = \sqrt{re^{i\theta}}\sqrt{se^{i\phi}}.$$

So the above equations reduce to

$$z + \frac{b}{2a} = \frac{\sqrt{b^2-4ac}}{2a} \qquad \text{or} \qquad z + \frac{b}{2a} = -\frac{\sqrt{b^2-4ac}}{2a}$$

since two of the four equations that result from all the possible combinations of positive and negative numerators and denominators would be redundant. The statement above is precisely the definition of $z + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$, which gives us

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

□

14. Let $n \in \mathbb{N}, m \in \mathbb{Z}$. Define $\omega = e^{\frac{2\pi i}{n}}$. Compute $\sum_{k=0}^{n-1} \omega^{mk}$ and $\sum_{k=0}^{n-1} (-1)^k \omega^{mk}$.

$$\sum_{k=0}^{n-1} \omega^{mk} = \begin{cases} n & \text{if } n \mid m \\ 0 & \text{if } n \nmid m \end{cases}$$

$$\sum_{k=0}^{n-1} (-1)^k \omega^{mk} = \begin{cases} n & \text{if } \frac{m}{n} \in \mathbb{Z} + \frac{1}{2} \\ 0 & \text{if } \frac{m}{n} \notin \mathbb{Z} + \frac{1}{2} \text{ and } 2 \mid n \\ \dfrac{2}{1+\omega^m} & \text{if } \frac{m}{n} \notin \mathbb{Z} + \frac{1}{2} \text{ and } 2 \nmid n \end{cases}$$

where $\mathbb{Z} + \frac{1}{2} = \{n + \frac{1}{2} : n \in \mathbb{Z}\}$.

Proof. The formula for a finite geometric series can be applied to both of these sums, but only if the ratio is not 1. For the first sum, this means $\omega^m \neq 1$; for the second, $-\omega^m \neq 1$. It is clear that $\omega^m = 1$ if and only if n divides m. For the second situation, observe that $-\omega^m = 1$ if and only if

$$\omega^m = e^{2\pi \frac{m}{n} i} = -1 = e^{\pi i}$$

which is equivalent to $\frac{2\pi m}{n} = \pi + 2\pi k$ for some integer k. This equation can be rewritten as

$$\frac{m}{n} = \frac{2k+1}{2} = k + \frac{1}{2}.$$

In this case, where the ratio equals 1, the first sum is n. In the second case, the sum is 0 if n is even and 1 if n is odd. However, the condition we have derived requires that n be even, thus the second sum is always 0 in this case.

Next, we may assume the ratio is not 1 and apply the formula for a finite geometric series:

$$\sum_{k=0}^{n-1} \omega^{mk} = \frac{1-\omega^{mn}}{1-\omega^m} = \frac{1-1^m}{1-\omega^m} = 0.$$

$$\sum_{k=0}^{n-1} (-1)^k \omega^{mk} = \sum_{k=0}^{n-1} (-\omega^m)^k = \frac{1-(-\omega^m)^n}{1-(-\omega^m)} = \frac{1-(-\omega^m)^n}{1+\omega^m} = \begin{cases} \dfrac{1-\omega^{mn}}{1+\omega^m} & \text{if } 2 \mid n \\ \dfrac{1+\omega^{mn}}{1+\omega^m} & \text{if } 2 \nmid n \end{cases}$$

$$= \begin{cases} 0 & \text{if } 2 \mid n \\ \dfrac{2}{1+\omega^m} & \text{if } 2 \nmid n \end{cases}.$$

□

15. Prove that, for any θ that is not an integer multiple of 2π and for any $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} \cos k\theta = \frac{1}{2} + \frac{\sin((n+\frac{1}{2})\theta)}{2\sin\frac{\theta}{2}}.$$

Proof.

$$\begin{aligned}
\sum_{k=0}^{n} \cos k\theta &= \sum_{k=0}^{n} \mathrm{Re}[e^{ik\theta}] \\
&= \mathrm{Re}\left[\sum_{k=0}^{n} e^{ik\theta}\right] \\
&= \mathrm{Re}\left[\frac{e^{i(n+1)\theta}-1}{e^{i\theta}-1}\right] \\
&= \mathrm{Re}\left[\frac{e^{-i\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}}} \cdot \frac{e^{i(n+1)\theta}-1}{e^{i\theta}-1}\right] \\
&= \mathrm{Re}\left[\frac{e^{i(n+\frac{1}{2})\theta}-e^{-i\frac{\theta}{2}}}{e^{i\frac{\theta}{2}}-e^{-i\frac{\theta}{2}}}\right] \\
&= \mathrm{Re}\left[\frac{e^{i(n+\frac{1}{2})\theta}-e^{-i\frac{\theta}{2}}}{2i\,\mathrm{Im}[e^{i\frac{\theta}{2}}]}\right] \\
&= \mathrm{Re}\left[\frac{-i}{-i} \cdot \frac{e^{i(n+\frac{1}{2})\theta}-e^{-i\frac{\theta}{2}}}{2i\sin(\frac{\theta}{2})}\right] \\
&= \frac{\mathrm{Re}[-i(\cos((n+\frac{1}{2})\theta)+i\sin((n+\frac{1}{2})\theta)-(\cos(-\frac{\theta}{2})+i\sin(-\frac{\theta}{2})))]}{2\sin(\frac{\theta}{2})} \\
&= \frac{\sin(\frac{\theta}{2})+\sin((n+\frac{1}{2})\theta)}{2\sin(\frac{\theta}{2})} \\
&= \frac{1}{2} + \frac{\sin((n+\frac{1}{2})\theta)}{2\sin(\frac{\theta}{2})}
\end{aligned}$$

□

16. Prove that if f is an entire function such that $f' = f$, then for some $c \in \mathbb{C}$, $f(z) = ce^z$.

Proof. Observe that

$$\frac{d}{dz}f(z)e^{-z} = f'(z)e^{-z} - f(z)e^{-z} = f'(z)e^{-z} - f'(z)e^{-z} = 0$$

therefore $f(z)e^{-z}$ is constant. So $f(z)e^{-z} = c$ for some $c \in \mathbb{C}$, thus $f(z) = ce^z$. □