3

First order logic—semantics

3.1 Formulas and structures

Suppose that $\mathcal{A}$ contains some of the constant, predicate, and function symbols of our language. A finite sequence φ is an $\mathcal{L}_{\mathcal{A}}$-formula if and only if

(1) φ is a formula,
(2) and the constant, predicate, and function symbols occurring in φ are all in $\mathcal{A}$.

Definition 3.1 An $\mathcal{L}_{\mathcal{A}}$-structure is a pair (M, I) as follows.

(1) $M \neq \emptyset$.
(2) I is a function with domain $\mathcal{A}$ such that for each $i \in \mathbb{N}$ the following conditions hold:
   a) If $c_i \in \mathcal{A}$ then $I(c_i) \in M$;
   b) if $F_i \in \mathcal{A}$ then $I(F_i)$ is a function

$$I(F_i) : M^n \to M$$

   where $n = \pi(F_i)$;
   c) if $P_i \in \mathcal{A}$ then

$$I(P_i) \subseteq M^n$$

   where $n = \pi(P_i)$.

3.2 The satisfaction relation

3.2.1 Interpreting terms

Definition 3.2 Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure. A function ν is an $\mathcal{M}$-*assignment* if ν is a map from $\{x_i : i \in \mathbb{N}\}$ to M.

Thus, an $\mathcal{M}$-assignment is simply a function which associates to each variable symbol an element of the universe of the structure $\mathcal{M}$.

Definition 3.3 Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure and that ν is an $\mathcal{M}$-assignment. We define a function

$$\overline{\nu} : \{\tau : \tau \text{ is an } \mathcal{L}_{\mathcal{A}}\text{-term}\} \to M$$

by induction on the length of $\mathcal{L}_{\mathcal{A}}$-terms as follows.

Base step. If τ has length 1, then we define $\overline{\nu}(\tau)$ by whichever equation applies.

$$\overline{\nu}(\langle x_i \rangle) = \nu(x_i)$$
$$\overline{\nu}(\langle c_i \rangle) = I(c_i)$$

Recursion step. If $\tau = F_i(\tau_1, \ldots, \tau_n)$, where $n = \pi(F_i)$, then

$$\overline{\nu}(\tau) = I(F_i)(\overline{\nu}(\tau_1), \ldots, \overline{\nu}(\tau_n)).$$

By Theorem 2.5, unique readability for terms, $\overline{\nu}$ is well defined.

Definition 3.4 Suppose that $\mathcal{M}$ is an $\mathcal{L}_{\mathcal{A}}$-structure, τ is an $\mathcal{L}_{\mathcal{A}}$-term, and ν and μ are $\mathcal{M}$-assignments.

- Then, ν and μ *agree on the free variables of* τ if and only if for all variables x_i, if x_i appears in τ, then $\nu(x_i) = \mu(x_i)$.
- Similarly, ν and μ *agree on the free variables of* φ if and only if for all variables x_i, if x_i appears freely in φ, then $\nu(x_i) = \mu(x_i)$.

Lemma 3.5 *Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure and τ is an $\mathcal{L}_{\mathcal{A}}$-term. Suppose that ν and μ are $\mathcal{M}$-assignments which agree on the free variables of τ. Then $\overline{\mu}(\tau) = \overline{\nu}(\tau)$*

Proof. We prove Lemma 3.5 by induction on the length of τ. If τ has length 1, then by Theorem 2.5, unique readability for terms, there is an i such that either τ is $\langle x_i \rangle$ and $\overline{\mu}(\tau) = \mu(x_i) = \nu(x_i) = \overline{\nu}(\tau)$ or τ is $\langle c_i \rangle$ and $\overline{\mu}(\tau) = I(c_i) = \overline{\nu}(\tau)$. In either case, the lemma is verified. Now, suppose that τ has length greater than 1 and assume the lemma for all terms which are shorter than τ. Again by Theorem 2.5, τ has the form $F_i(\tau_1, \ldots, \tau_n)$, and we use the inductive hypothesis as follows.

$$\begin{aligned}
\overline{\mu}(\tau) &= \overline{\mu}(F_i(\tau_1, \ldots, \tau_n)) \\
&= I(F_i)(\overline{\mu}(\tau_1), \ldots, \overline{\mu}(\tau_n)) \\
&= I(F_i)(\overline{\nu}(\tau_1), \ldots, \overline{\nu}(\tau_n)) && \text{(by induction)} \\
&= \overline{\nu}(F_i(\tau_1, \ldots, \tau_n)) \\
&= \overline{\nu}(\tau)
\end{aligned}$$

Thus, $\overline{\mu}(\tau) = \overline{\nu}(\tau)$ as required. $\square$

Definition 3.6 Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure and ν is an $\mathcal{M}$-assignment. We define the *satisfaction relation*,

$$(\mathcal{M}, \nu) \models \varphi$$

by recursion on the length of φ as follows.

Atomic cases. Suppose that φ is an atomic formula.

(1) Suppose that $\varphi = P_i(\tau_1 \dots \tau_n)$ where $n = \pi(P_i)$ and where $\tau_1, \dots, \tau_n$ are terms. Then

$$(\mathcal{M}, \nu) \models \varphi \text{ if and only if } \langle \overline{\nu}(\tau_1), \dots, \overline{\nu}(\tau_n) \rangle \in I(P_i).$$

(2) Suppose that $\varphi = (\sigma \mathrel{\hat{=}} \tau)$ where σ and τ are terms. Then

$$(\mathcal{M}, \nu) \models \varphi \text{ if and only if } \overline{\nu}(\sigma) = \overline{\nu}(\tau).$$

Inductive cases. Suppose that φ is not an atomic formula.

(3) Suppose that $\varphi = (\neg \psi)$. Then

$$(\mathcal{M}, \nu) \models \varphi \text{ if and only if } (\mathcal{M}, \nu) \not\models \psi.$$

Here, we use $(\mathcal{M}, \nu) \not\models \psi$ to indicate that it is not the case that $(\mathcal{M}, \nu) \models \psi$.

(4) Suppose that $\varphi = (\psi_1 \rightarrow \psi_2)$. Then

$$(\mathcal{M}, \nu) \models \varphi \text{ if and only if}$$
$$\text{either} (\mathcal{M}, \nu) \not\models \psi_1 \text{ or } (\mathcal{M}, \nu) \models \psi_2.$$

(5) Suppose that $\varphi = (\forall x_i \psi)$. Then $(\mathcal{M}, \nu) \models \varphi$ if and only if for all $\mathcal{M}$-assignments μ, if ν and μ agree on the free variables of φ, then $(\mathcal{M}, \mu) \models \psi$. Since x_i is bound in φ, the values of these μ's on x_i range over all of M.

By Theorem 2.13, unique readability for formulas, $(\mathcal{M}, \nu) \models \varphi$ is well defined for all $\mathcal{M}$-assignments ν and $\mathcal{L}_{\mathcal{A}}$-formulas φ. We will sometimes say that $(\mathcal{M}, \nu)$ *satisfies* φ to indicate $(\mathcal{M}, \nu) \models \varphi$.

Theorem 3.7 *Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure, φ is an $\mathcal{L}_{\mathcal{A}}$-formula, and ν and μ are $\mathcal{M}$-assignments which agree on the free variables of φ. Then,*

$$(\mathcal{M}, \nu) \models \varphi \ \leftrightarrow \ (\mathcal{M}, \mu) \models \varphi.$$

Proof. We proceed by induction on the length of φ. We now suppose that the length of φ is n and that the theorem holds for all $\mathcal{L}_{\mathcal{A}}$-formulas of length less than n. To be precise, we can assume the following.

Induction hypothesis. *For all ψ, ν_1, and μ_1, if ψ is an $\mathcal{L}_{\mathcal{A}}$-formula of length less than n, ν_1 is an $\mathcal{M}$-assignment, μ_1 is an $\mathcal{M}$-assignment, and ν_1 and μ_1 agree on the free variables of ψ, then $(\mathcal{M}, \nu_1) \models \psi$ if and only if $(\mathcal{M}, \mu_1) \models \psi$.*

To ground the induction, note that there are no formulas of length 0, so the induction hypothesis holds when n is equal to 1.

First consider the case in which φ is an atomic formula. There are two subcases.

First, φ could be instance of a predicate, say $\varphi = P_i(\tau_1 \dots \tau_n)$, where $\tau_i, \dots, \tau_n$ are $\mathcal{L}_{\mathcal{A}}$-terms and $n = \pi(P_i)$. Then, every variable occurring in φ is necessarily a free variable of φ. By Lemma 3.5, for each $i \leq n$, $\overline{\nu}(\tau_i) = \overline{\mu}(\tau_i)$. By definition,

$$(\mathcal{M}, \nu) \models P_i(\tau_1 \dots \tau_n) \ \leftrightarrow \ \langle \overline{\nu}(\tau_1), \dots, \overline{\nu}(\tau_n) \rangle \in I(P_i);$$

and similarly

$$(\mathcal{M}, \mu) \models P_i(\tau_1 \dots \tau_n) \ \leftrightarrow \ \langle \overline{\mu}(\tau_1), \dots, \overline{\mu}(\tau_n) \rangle \in I(P_i);$$

Thus since,

$$\langle \overline{\nu}(\tau_1), \dots, \overline{\nu}(\tau_n) \rangle = \langle \overline{\mu}(\tau_1), \dots, \overline{\mu}(\tau_n) \rangle,$$

it follows that

$$(\mathcal{M}, \nu) \models \varphi \ \leftrightarrow \ (\mathcal{M}, \mu) \models \varphi.$$

Second, φ could assert an equality, say $\varphi = (\tau \mathrel{\hat{=}} \sigma)$, where σ and τ are terms. Again, every variable occurring in φ is necessarily a free variable of φ. Then, by Lemma 3.5, $\overline{\nu}(\tau) = \overline{\mu}(\tau)$ and $\overline{\nu}(\sigma) = \overline{\mu}(\sigma)$. It follows from the definition of satisfaction as above that $(\mathcal{M}, \nu) \models (\tau \mathrel{\hat{=}} \sigma)$ if and only if $(\mathcal{M}, \mu) \models (\tau \mathrel{\hat{=}} \sigma)$

This finishes the case that φ is an atomic formula, and now we consider the three cases in which φ is not atomic.

First, φ could be a negation, say $\varphi = (\neg\psi)$. Then, φ and ψ have the same free variables, and so we may apply the induction hypotheses as follows.

$$\begin{aligned}
(\mathcal{M}, \nu) \models \varphi \ &\leftrightarrow \ (\mathcal{M}, \nu) \not\models \psi && \text{(by definition)} \\
&\leftrightarrow \ (\mathcal{M}, \mu) \not\models \psi && \text{(by induction)} \\
&\leftrightarrow \ (\mathcal{M}, \mu) \models \varphi. && \text{(by definition)}
\end{aligned}$$

Second, ψ could be an implication, say $\varphi = (\psi_1 \to \psi_2)$. In this case,

$$\begin{aligned}
&\{x_j : x_j \text{ is a free variable of } \varphi\} = \\
&\qquad \{x_j : x_j \text{ is a free variable of } \psi_1\} \cup \{x_j : x_j \text{ is a free variable of } \psi_2\}.
\end{aligned}$$

We apply the induction hypothesis to conclude that $(\mathcal{M}, \nu) \models \psi_1$ if and only if $(\mathcal{M}, \mu) \models \psi_1$ and $(\mathcal{M}, \nu) \models \psi_2$ if and only if $(\mathcal{M}, \mu) \models \psi_2$. By definition,

$(\mathcal{M}, \nu) \models \varphi$ if and only if, either $(\mathcal{M}, \nu) \not\models \psi_1$ or $(\mathcal{M}, \nu) \models \psi_2$, and $(\mathcal{M}, \mu) \models \varphi$ if and only if, either $(\mathcal{M}, \mu) \not\models \psi_1$ or $(\mathcal{M}, \mu) \models \psi_2$. Thus (by the induction hypothesis), $(\mathcal{M}, \nu) \models \varphi$ if and only if $(\mathcal{M}, \mu) \models \varphi$.

Finally, φ could be obtained by quantification, say $\varphi = (\forall x_i \psi)$. By definition, $(\mathcal{M}, \nu) \models \varphi$ if and only if for all $\mathcal{M}$-assignments ρ, if ρ and ν agree on the free variables of φ, then $(\mathcal{M}, \rho) \models \psi$. Similarly, $(\mathcal{M}, \mu) \models \varphi$ if and only if for all $\mathcal{M}$-assignments ρ, if ρ and μ agree on the free variables of φ, then $(\mathcal{M}, \rho) \models \psi$. By assumption, μ and ν agree on the free variables of φ. Therefore (trivially) for all $\mathcal{M}$-assignments ρ, ρ and μ agree on the free variables of φ if and only if ρ and ν agree on the free variables of φ. But then, $(\mathcal{M}, \nu) \models \varphi$ if and only if $(\mathcal{M}, \mu) \models \varphi$

This completes the proof of the theorem. □

Definition 3.8 (1) If $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure, $\tau = \tau(x_1, \ldots, x_n)$ is an $\mathcal{L}_{\mathcal{A}}$-term, and $a_1, \ldots, a_n$ are elements of M, then

$$\tau[a_1, \ldots, a_n]$$

indicates $\overline{\nu}(\tau)$, where ν is any $\mathcal{M}$-assignment such that for all $i \leq n$, $\nu(x_i) = a_i$.

(2) If $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure, $\varphi = \varphi(x_1, \ldots, x_n)$ is an $\mathcal{L}_{\mathcal{A}}$-formula, and $a_1, \ldots, a_n$ are elements of M, then

$$\mathcal{M} \models \varphi[a_1, \ldots, a_n]$$

indicates that $(\mathcal{M}, \nu) \models \varphi$, where ν is any $\mathcal{M}$-assignment such that for all $i \leq n$, $\nu(x_i) = a_i$.

By Lemma 3.5 and Theorem 3.7, the definitions of $\tau[a_1, \ldots, a_n]$ and of the relation $\mathcal{M} \models \varphi[a_1, \ldots, a_n]$ given above do not depend on the choice of ν.

In particular, if φ is a sentence, with no free variables, then we write $\mathcal{M} \models \varphi$ or $\mathcal{M} \not\models \varphi$.

3.3 Substitution and the satisfaction relation

Definition 3.9 (1) If τ is a term and $\tau_1, \ldots, \tau_n$ are terms, we write $\tau(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$ to indicate the term obtained by simultaneously, for each i substituting the term τ_i for each occurrence of x_i in τ.

(2) If $\varphi(x_1, \ldots, x_n)$ is a formula and $\tau_1, \ldots, \tau_n$ are terms, we write $\varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$ to indicate the formula obtained by simultaneously, for each i substituting τ_i for each free occurrence of x_i in φ.

Lemma 3.10 *(1) For any term $\tau(x_1, \ldots, x_n)$ and for any sequence of terms $\tau_1, \ldots, \tau_n$, $\tau(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$ is a term.*

(2) For any formula $\varphi(x_1, \ldots, x_n)$ and any sequence of terms $\tau_1, \ldots, \tau_n$, $\varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$ is a formula.

The proof of Lemma 3.10 is by induction, first on the length of terms to prove (1) and then on the length of formulas using (1) to prove (2).

Example 3.11 We think of $\varphi(x_1;\tau)$ as saying that φ holds of τ. However, blind substitution can have unintended results. Consider the following formula φ.

$$\varphi = \varphi(x_1) = (\forall x_2(x_1 \hat{=} x_2))$$

Now, we substitute x_2 for x_1.

$$\varphi(x_1;x_2) = (\forall x_2(x_2 \hat{=} x_2))$$

There is a substantial difference between the two formulas. Every structure satisfies $\varphi(x_1;x_2)$, but for every structure $\mathcal{M}$ and every $a \in M$, $\mathcal{M} \models \varphi[a]$ if and only if $M = \{a\}$.

Definition 3.12 Suppose φ is a formula, x_i is a free variable of φ, and τ is a term. The term τ is *free for x_i in φ* if for each variable x_j occurring in τ, no free occurrence of x_i in φ is within the scope of an occurrence of $\forall x_j$.

Theorem 3.13 (Substitution) *Let $\mathcal{M} = (M, I)$ be an $\mathcal{L}_{\mathcal{A}}$-structure and ν be an $\mathcal{M}$-assignment.*

(1) If $\tau(x_1,\dots,x_n)$ is an $\mathcal{L}_{\mathcal{A}}$-term and $\tau_1,\dots,\tau_n$ are $\mathcal{L}_{\mathcal{A}}$-terms, then

$$\overline{\nu}(\tau(x_1,\dots,x_n;\tau_1,\dots,\tau_n)) = \tau[b_1,\dots,b_n],$$

where for each $i \le n$, $b_i = \overline{\nu}(\tau_i)$.

(2) If $\varphi(x_1,\dots,x_n)$ is an $\mathcal{L}_{\mathcal{A}}$-formula, $\tau_1,\dots,\tau_n$ are $\mathcal{L}_{\mathcal{A}}$-terms, and for each $i \le n$, τ_i is free for x_i in φ, then

$$(\mathcal{M},\nu) \models \varphi(x_1,\dots,x_n;\tau_1,\dots,\tau_n) \ \leftrightarrow\ \mathcal{M} \models \varphi[b_1,\dots,b_n],$$

where for each $i \le n$, $b_i = \overline{\nu}(\tau_i)$.

Proof. The two parts are proven by induction on the lengths of τ and φ, respectively. We leave the proof of the first to the reader and present the proof of the second.

So, assume that (1) is proven, that φ is an $\mathcal{L}_{\mathcal{A}}$-formula, $\tau_1,\dots,\tau_n$ are $\mathcal{L}_{\mathcal{A}}$-terms such that for each $i \le n$, τ_i is free for x_i in φ, and that (2) holds for every formula of length less than that of φ. For each $i \le n$, let b_i denote $\overline{\nu}(\tau_i)$.

When φ is an atomic $\mathcal{L}_{\mathcal{A}}$-formula, the claim follows directly from the definitions and (1).

Now suppose that φ is not an atomic $\mathcal{L}_{\mathcal{A}}$-formula. We must prove, using the induction hypothesis, that $(\mathcal{M},\nu) \models \varphi(x_1,\dots,x_n;\tau_1,\dots,\tau_n)$ if and only if $\mathcal{M} \models \varphi[b_1,\dots,b_n]$.

There are three cases to consider.

First, φ could be a negation, say $\varphi = (\neg\psi)$. Then, the free variables of φ are exactly the same as those of ψ, and for each $i \leq n$, τ_i is free for x_i in ψ. Thus,

$$\varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n) = (\neg\psi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)).$$

By the induction hypothesis,

$$(\mathcal{M}, \nu) \models \psi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n) \ \leftrightarrow\ \mathcal{M} \models \psi[b_1, \ldots, b_n].$$

But then by definition of the satisfaction of a negation,

$$(\mathcal{M}, \nu) \models \varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n) \ \leftrightarrow\ \mathcal{M} \models \varphi[b_1, \ldots, b_n].$$

Second, φ could be an implication, say $\varphi = (\psi_1 \to \psi_2)$. In this case, the free variables of φ are those variables which are free in at least one of ψ_1 or ψ_2. Further, as in the previous case, $\varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$ is equal to $(\psi_1(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n) \to \psi_2(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n))$. So again we may apply the induction hypothesis to obtain the following equivalences.

$$(\mathcal{M}, \nu) \models \psi_1(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n) \ \leftrightarrow\ \mathcal{M} \models \psi_1[b_1, \ldots, b_n]$$
$$(\mathcal{M}, \nu) \models \psi_2(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n) \ \leftrightarrow\ \mathcal{M} \models \psi_2[b_1, \ldots, b_n]$$

By definition, $(\mathcal{M}, \nu)$ satisfies $\varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$ if and only if either $(\mathcal{M}, \nu)$ does not satisfy $\psi_1(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$ or it does satisfy $\psi_2(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$. Similarly, $\mathcal{M}$ satisfies $\varphi[b_1, \ldots, b_n]$ if and only if either $\mathcal{M}$ does not satisfy $\psi_1[b_1, \ldots, b_n]$ or it does satisfy $\psi_2[b_1, \ldots, b_n]$. Thus, we have the required equivalence.

$$(\mathcal{M}, \nu) \models \varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n) \ \leftrightarrow\ \mathcal{M} \models \varphi[b_1, \ldots, b_n]$$

Finally, φ could be obtained by quantification, say $\varphi = (\forall x_i \psi)$.

First, we may assume that n is greater than or equal to i, since our notation $\varphi(x_1, \ldots, x_n)$ merely indicates that the free variables of φ are a subset of $\{x_1, \ldots, x_n\}$. Second, for each j such that x_j does not appear freely in φ, $\varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$ does not depend on the value of τ_j. Consequently, we may assume that each such τ_j is equal to $\langle x_j \rangle$. In particular, we may assume that τ_i is $\langle x_i \rangle$.

By definition, $(\mathcal{M}, \nu)$ satisfies $\varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$ if and only if Condition 1 holds.

Condition 1. For all $\mathcal{M}$-assignments μ, if μ and ν agree on the free variables of $\varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$, then

$$(\mathcal{M}, \mu) \models \psi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n).$$

Now, we can apply the inductive hypothesis in the conclusion of Condition 1 and see that it is equivalent to Condition 2.

Condition 2. For all $\mathcal{M}$-assignments μ, if μ and ν agree on the free variables of $\varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$, then

$$\mathcal{M} \models \psi[\overline{\mu}(\tau_1), \ldots, \overline{\mu}(\tau_n)]$$

We will show Condition 2 is equivalent to the following one, Condition 3.

Condition 3. For all $\mathcal{M}$-assignments ρ, if for each j such that x_j appears freely in φ, $\rho(x_j) = \overline{\nu}(\tau_j)$, then $(\mathcal{M}, \rho) \models \psi$

We give separate proofs for the implications between the two conditions.

First, suppose that Condition 2 holds. To prove Condition 3, suppose that ρ is an $\mathcal{M}$-assignment such that for each x_j such that j appears freely in φ, $\rho(x_j) = \overline{\nu}(\tau_j)$. Let μ_ρ be the $\mathcal{M}$-assignment defined as follows.

$$\mu_\rho(x_j) = \begin{cases} \nu(x_j), & \text{if } x_j \text{ occurs freely in} \\ & \qquad \varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n); \\ \rho(x_j), & \text{otherwise.} \end{cases}$$

By Condition 2, $\mathcal{M} \models \psi[\overline{\mu_\rho}(\tau_1), \ldots, \overline{\mu_\rho}(\tau_n)]$. Because each τ_j was free for x_j in φ, if x_j occurs freely in φ, then all of the variables which occur in τ_j occur freely in $\varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$. We may apply Lemma 3.5 to conclude that for each such j, $\overline{\mu_\rho}(\tau_j) = \overline{\nu}(\tau_j)$ and, since $\rho(x_j) = \overline{\nu}(\tau_j)$, $\overline{\mu_\rho}(\tau_j) = \rho(x_j)$. Since x_i does not occur freely in $\varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$ and $\tau_i = \langle x_i \rangle$, $\overline{\mu_\rho}(\tau_i) = \rho(x_i)$. If a variable occurs freely in ψ, then either it occurs freely in φ or it is equal to x_i. Consequently, for each variable x_j such that x_j appears freely in ψ, $\rho(x_j) = \overline{\mu_\rho}(\tau_j)$. Since $\mathcal{M} \models \psi[\overline{\mu_\rho}(\tau_1), \ldots, \overline{\mu_\rho}(\tau_n)]$, it follows from Theorem 3.7 that $(\mathcal{M}, \rho) \models \psi$, as required.

Now, suppose that Condition 3 holds. To prove Condition 2, suppose that μ is an $\mathcal{M}$-assignment which agrees with ν on the free variables of $\varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$. Since each τ_j is free for x_j in φ, μ and ν agree on all of the variables that appear in any τ_j such that x_j occurs freely in φ. Applying Lemma 3.5, if x_j appears freely in φ, then $\overline{\mu}(\tau_j) = \overline{\nu}(\tau_j)$. Now, define ρ_μ as follows.

$$\rho_\mu(x_j) = \begin{cases} \overline{\mu}(\tau_j), & \text{if } x_j \text{ occurs freely in } \varphi; \\ \mu(x_j), & \text{otherwise.} \end{cases}$$

By Condition 3, $(\mathcal{M}, \rho_\mu) \models \psi$. Since all of the free variables of ψ are included in $\{x_1, \ldots, x_n\}$, $\mathcal{M} \models \psi[\rho_\mu(x_1), \ldots, \rho_\mu(x_n)]$. By the earlier remarks, if x_j does not appear freely in φ then $\tau_j = \langle x_j \rangle$. Consequently, $\langle \rho_\mu(x_1), \ldots, \rho_\mu(x_n) \rangle$ is equal to $\langle \overline{\mu}(\tau_1), \ldots, \overline{\mu}(\tau_n) \rangle$. Thus, $\mathcal{M} \models \psi[\overline{\mu}(\tau_1), \ldots, \overline{\mu}(\tau_n)]$ as required.

By Theorem 3.7, Condition 3 is equivalent to $\mathcal{M} \models \varphi[b_1, \ldots, b_n]$, where for each i, b_i is equal to $\overline{\nu}(\tau_i)$. Therefore, we have the desired equivalence

$$(\mathcal{M}, \nu) \models \varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n) \;\leftrightarrow\; \mathcal{M} \models \varphi[b_1, \ldots, b_n]$$

where for each i, b_i is equal to $\overline{\nu}(\tau_i)$.

This completes the final case (and hence the proof). □

3.3.1 Exercises

(1) Let $\mathcal{A}$ be a language with one binary relation symbol. Give an example of a sentence φ in this language and $\mathcal{L}_{\mathcal{A}}$-structures $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $\mathcal{M}_1 \models \varphi$ and $\mathcal{M}_2 \not\models \varphi$.

(2) Do there exist an $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M}$, an $\mathcal{M}$-assignment ν, and an $\mathcal{L}_{\mathcal{A}}$-formula φ such that $(\mathcal{M}, \nu) \models \varphi$ and $(\mathcal{M}, \nu) \models (\neg\varphi)$? Do there exist such $\mathcal{M}$ and ν such that $(\mathcal{M}, \nu) \not\models \varphi$ and $(\mathcal{M}, \nu) \not\models (\neg\varphi)$?

(3) Suppose that $A_1, \ldots, A_n$ are propositional symbols, that θ is a propositional tautology, and that $\varphi_1, \ldots, \varphi_n$ are $\mathcal{L}_{\mathcal{A}}$ formulas. Let ψ be the result of substituting for each i, the formula φ_i for each occurrence of the propositional symbol A_i in θ. Prove that for every $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M}$ and every $\mathcal{M}$-assignment ν, $(\mathcal{M}, \nu) \models \psi$.

(4) Give an example of an $\mathcal{A}$ and an $\mathcal{L}_{\mathcal{A}}$-formula φ such that
  a) φ is a sentence,
  b) there is at least one $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M}$ such that $\mathcal{M} \models \varphi$,
  c) and for all $\mathcal{L}_{\mathcal{A}}$-structures $\mathcal{M}$, if $\mathcal{M} \models \varphi$, then the universe of $\mathcal{M}$ is infinite.