1. (F 2.1, exercise 3) If $\{f_n\}$ is a sequence of measurable functions on X, then $\{x : \lim f_n(x) \text{ exists}\}$ is a measurable set.

   Proof. We know that the difference of two measurable functions is measurable, and the maximum of two measurable functions is measurable, therefore

$$\begin{aligned}\{x : \lim f_n(x) \text{ exists}\} &= \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcup_{n=N+1}^{\infty} \bigcup_{m=N+1}^{\infty} \{x : |f_n(x) - f_m(x)| < \frac{1}{k}\} \\ &= \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcup_{n=N+1}^{\infty} \bigcup_{m=N+1}^{\infty} \{x : \max(f_n(x) - f_m(x), f_m(x) - f_n(x)) < \frac{1}{k}\}\end{aligned}$$

   is a measurable set because $\max(f_n(x) - f_m(x), f_m(x) - f_n(x))$ is a measurable function. □

2. (F 2.1, exercise 4) If $f : X \to \overline{\mathbb{R}}$ and $f^{-1}((r, \infty]) \in \mathcal{M}$ for each $r \in \mathbb{Q}$, then f is measurable.

   Proof. It is not entirely clear what the measure space of $\overline{\mathbb{R}}$ is intended to be. I will assume it is $\mathcal{B}_{\overline{\mathbb{R}}}$, which is the σ-algebra generated by the open sets of $\overline{\mathbb{R}}$. I cannot find it anywhere mentioned in Folland what the "usual" topology on $\overline{\mathbb{R}}$ is. The metric on $\mathbb{R}$ does not extend to this set in a natural way, since distances must be finite. One condition it is natural to require is that the topology on $\overline{\mathbb{R}}$ restrict to the usual topology on $\mathbb{R}$. Also, we know that the entire space $[-\infty, \infty]$ must be open. But then $\{-\infty, +\infty\} = [-\infty, +\infty] \cap (-\infty, +\infty)^c$ is open, hence $U \cup \{-\infty, +\infty\}$ is open for any open $U \subseteq \mathbb{R}$. $\{+\infty\}$ is open if and only if $\{-\infty\}$ is open, since $\{-\infty\} = \{-\infty, +\infty\} \cap \{+\infty\}^c$. Also, $\{+\infty\}$ is open if and only if $U \cup \{+\infty\}$ is open for all (or for any) $U \subseteq \mathbb{R}$. Continuing this train of logic leads us to the conclusion that the only choice to make is whether or not $\{+\infty\}$ is open. In every source I can find, this is set is taken to be open. Therefore, the topology on $\overline{\mathbb{R}}$ should be taken to be

$$\{U \cup E : U \subseteq \mathbb{R} \text{ is open and } E \in \{\emptyset, \{+\infty\}, \{-\infty\}, \{-\infty, +\infty\}\}\}.$$

   We will show that $B = \{(x, \infty] : x \in \mathbb{R}\}$ generates the Borel algebra on $\overline{\mathbb{R}}$. Let $a < b$. Then $(a, b] = (a, \infty] \cap (b, \infty]^c$. Also, $\{+\infty\} = \bigcap_1^\infty (n, \infty]$ and $\{-\infty\} = \left(\bigcup_1^\infty (-n, \infty]\right)^c$. Obviously, $\{-\infty, +\infty\} = \{-\infty\} \cup \{+\infty\}$ and $\emptyset = \{-\infty\} \cap \{+\infty\}$. Also, given $a < b$ we have $(a, b) = \bigcap_1^\infty (a, b + \frac{1}{n}]$, so B generates every open interval, hence it generates every open set of $\mathbb{R}$, along with $\emptyset, \{+\infty\}, \{-\infty\}$, and $\{-\infty, +\infty\}$. Thus the topology on $\overline{\mathbb{R}}$ is contained in the σ-algebra generated by B, and hence so is the Borel algebra on $\overline{\mathbb{R}}$. Also, every set in B is open in $\overline{\mathbb{R}}$, and hence B is contained in the Borel algebra on $\overline{\mathbb{R}}$. So B generates the Borel algebra on $\overline{\mathbb{R}}$.

   Finally, let $x \in \mathbb{R}$. Taking an increasing sequence $r_n \to x$ of rationals, we have

$$f^{-1}((x, \infty]) = f^{-1}\left(\bigcap_1^\infty (r_n, \infty]\right) = \bigcap_1^\infty f^{-1}((r_n, \infty])$$

   is a countable intersection of measurable sets, hence measurable. By Proposition 2.1, f is measurable because $f^{-1}(E)$ is measurable for each $E \in B$, and B generates the Borel algebra on $\overline{\mathbb{R}}$. □

3. Let $F : [0, \infty) \to \mathbb{R}$ be a Lebesgue measurable function such that

$$f(x + y) = f(x) + f(y)$$

   for all $x, y \geq 0$. For $a \in \mathbb{R}$, set $\Lambda_a \subseteq [0, \infty), \Lambda_a = \{x \geq 0 : f(x) \geq ax\}$.

   (1) Show that if $a \in F$ is such that $\mu(\Lambda_a) > 0$, then Λ_a contains an interval of the form $[b, \infty)$ for some $b > 0$.

   (2) Show that, in fact, we may take $b = 0$ in part (1).

   (3) Prove that there exists $\lambda \in \mathbb{R}$ such that $f(x) = \lambda x$ for all $x \geq 0$.

Proof. We begin by proving f is right continuous. It suffices simply to show that f is right continuous at 0, since given a sequence x_n converging to some $x \geq 0$ from above, we know $x_n - x \to 0$. Thus, $f(x_n - x) \to 0$, giving

$$f(x_n) = f(x + (x - x_n)) = f(x) + f(x - x_n) \to f(x) + 0 = f(x)$$

hence f is right continuous at x.

Note first that $f(0) = f(0+0) = f(0) + f(0)$, therefore $f(0) = 0$. Let $\epsilon > 0$; we want to find a δ such that $|f(x)| = |f(x) - f(0)| < \epsilon$ whenever $|x| = |x - 0| < 0$ (note this also implies that $f(0) = 0$). Consider the restriction of f to $[0,1]$. By Lusin's theorem, there exists a closed interval $F \subseteq [0,1]$ such that $\mu([0,1] \setminus F) < \frac{2}{3}$, i.e. $\mu(F) > \frac{2}{3}$, and $f\mid_F$ is uniformly continuous on F (since F is compact). Thus, for any $x, y \in F$ there is a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

Let $0 \leq x < \min(\delta, \frac{1}{3})$. If $F \cap F - x$ were empty, then we would have

$$x + 1 \geq \mu(F \cup F - x) = 2\mu(F) > \frac{4}{3}$$

contradicting that $x < \frac{1}{3}$. Therefore, there is some $y \in F \cap F - x$. So $|(x+y) - y| = |x| < \delta$ by assumption, giving us

$$|f(x)| = |f(x+y) - f(y)| < \epsilon.$$

So f is continuous at 0, and hence continuous on all of $[0, \infty)$.

We have shown already that $f(0 \cdot x) = f(0) = 0 = 0 \cdot f(x)$; if $f(nx) = nf(x)$ for some integer $n \geq 0$ then

$$f((n+1)x) = f(nx + x) = nf(x) + f(x) = (n+1)f(x).$$

Thus, $f(nx) = nf(x)$ for all integers $n \geq 0$. Also, for integers $a, b \geq 0$ we have $af(x) = bf(\frac{a}{b}x)$, so dividing by b gives us $\frac{a}{b}f(x) = f(\frac{a}{b}x)$. Therefore, for nonnegative $q \in \mathbb{Q}$, we have $f(q) = qf(1)$. So let $\lambda = f(1)$. Using the denseness of $\mathbb{Q}$ in $\mathbb{R}$, we can construct a sequence q_n of rationals converging to an arbitrary nonnegative $x \in \mathbb{R}$ from above. By right continuity, we have $f(q_n) \to f(x)$; but we know $f(q_n) = \lambda q_n \to \lambda x$. So $f(x) = \lambda x$ for all nonnegative $x \in \mathbb{R}$.

Parts 1 and 2 are now obvious. If $\mu(\Lambda_a)$ has positive measure, then it is nonempty. So there is some $x \geq 0$ such that $\lambda x = f(x) \geq ax$, meaning $\lambda \geq a$. So $f(x) = \lambda x \geq ax$ for all $x \geq 0$, thus $[0, \infty) \subseteq \Lambda_a$. ☐

4. Let $A \subset [0,1]$ be a Lebesgue measurable set with $\mu(A) > 0$. Show that, for any $\epsilon > 0$, there exists an interval $O \subset [0,1]$ such that the relative density of A in O - i.e., the ratio $\mu(A \cap O)/\mu(O)$ - exceeds $1 - \epsilon$.

5. Let $r \in [0,1)$. Consider the map $\tau = \tau_r : [0,1) \to [0,1)$ that sends $x \in [0,1)$ to $(x + r) \pmod 1$, the fractional part of $x + r$. For any $A \subset [0,1)$, consider the union $A^* = \bigcup_{n=0}^{\infty} \tau^n(A)$, where $\tau^0(A) = A$ and $\tau^n(A)$ is the image of A under the nth iterate of the function τ, for $n \geq 1$.

   (1) If $r \in \mathbb{Q}$, find an example of a Lebesgue measurable set $A \subset [0,1)$ of positive Lebesgue measure for which A^* has Lebesgue measure strictly between zero and one.

   Proof. There are integers a and b such that $r = \frac{a}{b}$. Let

   $$A = \bigcup_{n=0}^{b-1} \left[\frac{2n}{2b}, \frac{2n+1}{2b}\right).$$

   If $x \in A$, then $\frac{2n}{2b} \leq x < \frac{2n+1}{2b}$ for some n. Thus, $\tau(x) = x + \frac{a}{b} \pmod 1$ satisfies $\frac{2(n+a)}{2b} \pmod 1 \leq \tau(x) < \frac{2(n+a)+1}{2b} \pmod 1$. If $n < b - 1$, then the $\pmod 1$s can simply be ignored, so $x \in A$. Otherwise, $b = n - 1$ and we have $0 \leq \tau(x) < \frac{1}{2b}$, so again $\tau(x) \in A$. Thus, $\tau(A) \subseteq A$. By the same argument in reverse, if $x \in \tau(A)$ then $x \in A$. So we must have $\tau(A) = A$.

This means that $A^* = A$, since the union is simply the union of A with itself an infinite number of times. We know that A has measure strictly between 0 and 1, since $A \subseteq [0, 1)$, but $\left[\frac{1}{2b}, \frac{2}{2b}\right) \cap A = \emptyset$ and this interval has positive measure. Hence, the same applies to A^*.

□

(2) If $r \notin \mathbb{Q}$, prove that, for any such set A, the set A^* has Lebesgue measure one. You may wish to consider the interval O discussed in Q4 and its iterates under τ_r.

Proof. First, note that τ is measure-preserving. If $X \subseteq [0, 1)$, then we can decompose X as a disjoint union of Y and Z where τ is a left translation on Y and a right translation on Z (Y is the part that is "pushed over" the point 1). The images of Y and Z are still disjoint, since τ is a bijection, thus $\mu(X) = \mu(Y) + \mu(Z) = \mu(\tau(Y)) + \mu(\tau(Z)) = \mu(\tau(X))$.

Denote $x \pmod 1$ by $\overline{x}$. Let $x \in [0, 1)$. If $\tau^n(x) = x$ for some n, then $\overline{x + nr} = x$, hence $nr \in \mathbb{Z}$, contradicting that r is irrational. Similarly, if there are some $m < n$ such that $\tau^n(x) = \tau^m(x)$, then $\tau^n(x) = \tau^{m+(n-m)}(x) = \tau^{n-m}(\tau^m(x)) = \tau^m(x)$, contradicting what we have just proven. Thus, the images of x under the iterates of τ form an infinite set within $[0, 1)$.

Partition $[0, 1)$ into intervals of size $\frac{1}{k}$. Consider the iterates of 0, i.e. $\{nr : n \geq 0\}$. Two iterates must fall into the same interval, by the pigeonhole principle. So there are some $m \neq n$ such that $0 < \overline{nr} - \overline{mr} < \frac{1}{k}$. So we can write $nr = t + s$ and $mr = u + v$ where $t, u \in \mathbb{Z}$, $s, v < 1$, and $s - v < \frac{1}{k}$. Thus, $(n - m)r = (t - u) + (s - v)$ and $t - u \in \mathbb{Z}$, so $\overline{n - m} < \frac{1}{k}$. If $n - m > 0$, then we have produced a positive iterate of 0 that is less than $\frac{1}{k}$.

Suppose this is not the case. Since $0 < s - v < \frac{1}{k}$, there is some positive integer N such that $\frac{k-1}{k} < N(s - v) < 1$. Therefore,

$$N(m - n) = [N(u - t) - 1] + [N(v - s) + 1]$$

and $0 < N(v - s) + 1 < \frac{1}{k}$. Therefore, $N(m - n)$ is a positive iterate of 0 that is less than $\frac{1}{k}$.

For any $x \in [0, 1)$ and any positive $y < \frac{1}{k}$, there is some N such that $|x - Ny| < \frac{1}{k}$. But if y is an iterate of 0, then so is Ny. Hence, the iterates of 0 are dense in $[0, 1)$. If we shift a dense set and take the image $\pmod 1$, we still have a dense set. The iterates of a given $x \in [0, 1)$ are simply the iterates of 0 shifted by x, thus they are also dense.

Now, let O be a nonempty interval in $[0, 1)$. Let x be the center of O. The iterates of O cover the iterates of x, and the iterates of x are dense in $[0, 1)$. If $2l$ is the length of O, then any point within l of an iterate of x is covered by an iterate of O. But *every* point of $[0, 1)$ is within l of an iterate of x, thus the iterates of O cover $[0, 1)$.

Let $B = (A^*)^c$, and suppose for a contradiction that $\mu(B) > 0$. For any ϵ such that $0 < \epsilon < 1$, we can find an interval O such that $\frac{\mu(A \cap O)}{\mu(O)} > 1 - \epsilon$. Let A_i and O_i be the ith iterates of A and O, respectively. Then $\frac{\mu(A_i \cap O_i)}{\mu(O_i)} > 1 - \epsilon$ as well, since τ is measure-preserving. Since $B \cap A_i = \emptyset$, for all i we have

$$\begin{aligned}
\mu(O_i) &= \mu((A^* \cup B) \cap O_i) \\
&= \mu((A^* \cap O_i) \cup (B \cap O_i)) \\
&= \mu(A^* \cap O_i) + \mu(B \cap O_i) \\
&\geq \mu(A_i \cap O_i) + \mu(B \cap O_i) \\
&> (1 - \epsilon)\mu(O_i) + \mu(B \cap O_i)
\end{aligned}$$

thus $\mu(B \cap O_i) < \epsilon\mu(O_i)$. Since ϵ is arbitrary, this means $\mu(B \cap O_i) = 0$ for all i. But the O_i cover $[0, 1)$, thus $\bigcup_0^\infty B \cap O_i = B$. So

$$\mu(B) = \mu(\bigcup_0^\infty B \cap O_i) \leq \sum_0^\infty \mu(B \cap O_i) = 0.$$

Since $B = (A^*)^c$ has measure 0, A^* must have measure 1. □