Worked with Sydney Wong and Taylor Hines

Q 1. Let Z_n be a branching process with offspring distribution X. Suppose that X has mean $\mu < 1$. Calculate the total expected number of offspring in all the generations.

Proof. $\mathbb{E}[Z_n] = \mu^n$, so

$$\mathbb{E}\left[\sum_{n=0}^{\infty} Z_n\right] = \sum_{n=0}^{\infty} \mathbb{E}[Z_n] = \sum_{n=0}^{\infty} \mu^n = \frac{1}{1-\mu}.$$

We know that this sum converges to the given value because $\mu < 1$. □

Q 2. Let Z_n be a branching process with offspring distribution $X \sim \text{Geom}(\frac{1}{2})$. We showed in class that the branching process becomes extinct with probability 1. Calculate the expected number of steps to extinction.

Proof. Let T be the time of extinction. We know that $\mathbb{P}(T < \infty) = 1$, and we can express T as $T = \sum_{n=0}^{\infty} I_n$, where I_n indicates the event $\{Z_n > 0\}$. Note that $\mathbb{E}[I_n] = \mathbb{P}(Z_n > 0) = \frac{1}{n+1}$, since we have shown in lecture that this is true when $X \sim \text{Geom}(\frac{1}{2})$. Therefore,

$$\mathbb{E}[T] = \mathbb{E}\left[\sum_{n=0}^{\infty} I_n\right] = \sum_{n=0}^{\infty} \mathbb{E}[I_n] = \sum_{n=0}^{\infty} \frac{1}{n+1} = -1 + \sum_{n=0}^{\infty} \frac{1}{n} = \infty$$

by the divergence of the harmonic series. □

Q 3. Let Z_n be a branching process with offspring distribution X. Suppose that X has mean μ and variance σ^2. Find

$$\text{Cov}(Z_t, Z_s).$$

Proof. We have shown in class that

$$\text{Var}(Z_n) = \begin{cases} n\sigma^2 & \mu = 1 \\ \sigma^2 \mu^{n-1} \frac{\mu^n - 1}{\mu - 1} & \mu \neq 1 \end{cases}.$$

We may assume WLOG that $s \leq t$. Note that

$$\begin{aligned}
\mathbb{E}[Z_t Z_s] &= \mathbb{E}[\mathbb{E}[Z_t Z_s \mid Z_s]] = \mathbb{E}[Z_s \mathbb{E}[Z_t \mid Z_s]] = \mathbb{E}[Z_s \mathbb{E}[Z_{t-s} \mid Z_0 = Z_s]] \\
&= \mathbb{E}[Z_s^2 \mu^{t-s}] = \mu^{t-s}(\text{Var}(Z_s) + \mathbb{E}[Z_s]^2) \\
&= \begin{cases} \mu^{t-s}(s\sigma^2 + \mu^{2s}) & \mu = 1 \\ \mu^{t-s}(\sigma^2 \mu^{s-1} \frac{\mu^s - 1}{\mu - 1} + \mu^{2s}) & \mu \neq 1 \end{cases} \\
&= \begin{cases} s\sigma^2 + 1 & \mu = 1 \\ \sigma^2 \mu^{t-1} \frac{\mu^s - 1}{\mu - 1} + \mu^{t+s} & \mu \neq 1. \end{cases}
\end{aligned}$$

So

$$\begin{aligned}
\text{Cov}(Z_s, Z_t) &= \mathbb{E}[Z_s Z_t] - \mathbb{E}[Z_s]\mathbb{E}[Z_t] \\
&= \begin{cases} s\sigma^2 + 1 - \mu^{t+s} & \mu = 1 \\ \sigma^2 \mu^{t-1} \frac{\mu^s - 1}{\mu - 1} + \mu^{t+s} - \mu^{t+s} & \mu \neq 1 \end{cases} \\
&= \begin{cases} s\sigma^2 & \mu = 1 \\ \sigma^2 \mu^{t-1} \frac{\mu^s - 1}{\mu - 1} & \mu \neq 1. \end{cases}
\end{aligned}$$

Obviously, if $t < s$ then all we need to do is to switch t and s in this expression. □

Q 4. Let Z_n be a branching process with offspring distribution $X \sim \text{Geom}(p)$. Find the extinction probability as a function of p.

Proof. We have shown in lecture that the extinction probability, in this case, is the smaller fixed point of the generating function $G_X(s)$. Thus s^* solves

$$\begin{aligned} s = G_X(s) &= \frac{1}{1-(1-p)s} \\ \implies 0 &= (1-p)s^2 - s + p \\ &= \frac{1}{1-p}(s-1)\left(s - \frac{p}{1-p}\right). \end{aligned}$$

If $p \le \frac{1}{2}$, then $\frac{p}{1-p}$ is the smaller root, and thus the extinction probability. Otherwise, it is 1. □

Q 5. Students come to office hours according to a rate 5 per hour Poisson process. They stay for 10 minutes and then leave. Conditional that 8 came during the hour, what is the distribution of the number still there at the end?

Proof. A person is still in office hours at the end of the hour if and only if they arrived during the last 10 minutes. The arrivals in a Poisson process fall uniformly throughout the interval, so it is clear that the distribution is $\text{Bin}(8, \frac{1}{6})$, since this 10 minute interval is $\frac{1}{6}$ of the entire interval. To see this proven, suppose that N is the number of arrivals in some length 1 time interval of a rate λ Poisson process, and that M is the number of arrivals in some length p subinterval. Then M is independent from $N-M$, since $N-M$ is the number of arrivals in the complement of the length p interval, and these two intervals are disjoint. So

$$\begin{aligned} \mathbb{P}(M = k \mid N = n) &= \frac{\mathbb{P}(M = k, N - M = n - k)}{\mathbb{P}(N = n)} \\ &= \frac{\mathbb{P}(M = k)\mathbb{P}(N - M = n - k)}{\mathbb{P}(N = n)} \\ &= \frac{e^{-p\lambda}\frac{(p\lambda)^k}{k!}e^{-(1-p)\lambda}\frac{((1-p)\lambda)^{n-k}}{(n-k)!}}{e^{-\lambda}\frac{(\lambda)^n}{n!}} \\ &= \frac{n!}{k!(n-k)!}p^k(1-p)^{n-k} \\ &= \binom{n}{k}p^k(1-p)^{n-k}. \end{aligned}$$

Thus the number of arrivals in the length p subinterval, conditional on the number of arrivals n in the entire interval, is distributed $\text{Binomial}(n, p)$. □