Stat 150 Practice Midterm Spring 2015

Instructor: Allan Sly

Name:

SID:

There are 4 questions worth a total of 61 points plus 4 bonus points. Attempt all questions and show your working - solutions without explanation will **not** receive full credit. Answer the questions in the space provided. Additional space is available on the final page. One double sided sheet of notes is permitted. Answers can be left in numerical form without simplification except where specified.

	Score
Q 1	
Q 2	
Q 3	
Q 4	
Total	

Question 1

Let X_n be a Markov chain with states $\{1, 2, 3, 4\}$, with X_0 the uniform distribution and transition matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(a) [2 points] What are the classes of communicating states of the Markov chain?

(b) [4 points] Find $\mathbb{P}[X_1 = 3]$.

(c) [4 points] Find $\mathbb{P}[X_2 = 3]$.

(d) [4 points] Find $\lim_n \mathbb{P}[X_n = 3 \mid X_0 = 3]$.

(e) [4 points] Find $\lim_n \mathbb{P}[X_n = 3 \mid X_0 = 2]$.

Question 2

Three jars contain n balls numbered $1, \ldots, n$. At each step a uniformly chosen ball is removed from its jar and then placed in a randomly chosen jar. Let X_n, Y_n and Z_n be the number of balls in the first, second and third jars respectively.

(a) [6 points] What are transition probabilities of X_n? Is it a martingale?

(b) [6 points] What is the stationary distribution of X_n?

(c) [4 Bonus points] The vector $W_n = (X_n, Y_n, Z_n)$ is also a Markov chain, find its stationary distribution.

Question 3

In a population of kangaroos, each kangaroo has 10 offspring each of which survive to maturity with probability p. Thus if Z_n is the population size in generation.

(a) [6 points] Calculate $\mathbb{E}[Z_{n+1} \mid Z_0, \ldots, Z_n]$. For which p is Z_n a martingale?

(b) [6 points] If $Z_0 = 10$ calculate $\mathbb{E}[Z_2]$.

(c) [6 points] If $Z_0 = 10$ calculate the variance of Z_2.

Question 4

A birth and death chain has an absorbing state at 0, so $P_{00} = 1$ and transition probabilities for $i \geq 1$ given by,

$$P_{ij} = \begin{cases} \frac{1}{i+1} & j = i-1 \\ 1 - \frac{1}{i+1} - \frac{1}{i+3} & j = i \\ \frac{1}{i+3} & j = i+1 \end{cases}$$

(a) [6 points] Find a non-constant P-harmonic function for the Markov chain.

(b) [4 points] Starting from position 3 find the probability of reaching state n before state 0.

(c) [3 points] Find the probability of ever reaching state 0?

Additional Space