Homework 1
Stat 150

Michael Knopf

February 6, 2015

Q 1. Let S and T be stopping times with respect to some sequences X_n and Y_n. Which of the following are stopping times?

- $T - 1$
  No
- $\min\{S, T\}$
  Yes
- $\max\{S, T\}$
  Yes

Proof. A random variable T is a stopping time with respect to a sequence X_n if and only if, for all n, $\mathbb{P}(T = n \mid \mathcal{F}_n)$ is either 0 or 1. Equivalent definitions for stopping time, when the index set is $\mathbb{N}$, are that $\mathbb{P}(T \leq n \mid \mathcal{F}_n)$ $\mathbb{P}(T < n \mid \mathcal{F}_{n-1})$, $\mathbb{P}(T > n \mid \mathcal{F}_n)$, or $\mathbb{P}(T \geq n \mid \mathcal{F}_{n-1})$ are either 1 or 0 for all n. The first case follows from the fact that

$$\mathbb{P}(T \leq n|\mathcal{F}_n) = \sum_{i=1}^{n} \mathbb{P}(T = i|\mathcal{F}_n).$$

If T is a stopping time, then exactly one term on the righthand side is 1 and the rest are 0, thus the lefthand side is either 1 or 0 – and conversely. The other cases are proven in a similar way, by a combination of lowering the upper index to $n - 1$ and subtracting the righthand side from 1.

$T - 1$ may not be a stopping time because the event $T - 1 = n \iff T = n + 1$ could depend on the value of X_{n+1}. For a counterexample, let $X_n \sim \text{Bern}\left(\frac{1}{2}\right)$ and let $T = \min\{n : X_n = 1\}$. T is a stopping time, but $\mathbb{P}(T - 1 = 1|X_1)$ is 0 if $X_1 = 1$ and $\frac{1}{2}$ if $X_1 = 0$. Thus $T - 1$ is not a stopping time.

$\min\{S, T\}$ is a stopping time because it is simply the rule that says to stop when either of T or S have occured. Let $\mathcal{F}_n = \{S_1 = x_1, \ldots S_n = x_n\}$ and $\mathcal{G}_n = \{T_1 = y_1, \ldots, T_n = y_n\}$. More formally, $\min\{S, T\}$ is a stopping time relative to the sequence (X_n, Y_n) because

$$\begin{aligned}
&\mathbb{P}(\min\{S, T\} > n \mid (S_1, T_1) = (x_1, y_1), \ldots, (S_n, T_n) = (x_n, y_n)) \\
&= 1 - \mathbb{P}(S \leq n, T \leq n \mid \mathcal{F}_n, \mathcal{G}_n) \\
&= 1 - \mathbb{P}(S \leq n \mid \mathcal{F}_n, \mathcal{G}_n)\, \mathbb{P}(T \leq n \mid \mathcal{F}_n, \mathcal{G}_n) \\
&= 1 - \mathbb{P}(S \leq n \mid \mathcal{F}_n)\, \mathbb{P}(T \leq n|\mathcal{G}_n) = 0 \text{ or } 1
\end{aligned}$$

where we were able to factor the joint probability on the second line because S and T are conditionally independent given $\mathcal{F}_n \cap \mathcal{G}_n$.

Similarly, $\max\{S,T\}$ is a stopping time relative to the sequence (X_n, Y_n) because

$$\begin{aligned}
&\mathbb{P}\left(\max\{S,T\} \leq n \mid \ (S_1,T_1) = (x_1,y_1), \ldots, (S_n,T_n) = (x_n,y_n)\right) \\
&= \ \mathbb{P}\left(S \leq n, T \leq n \mid \mathcal{F}_n, \mathcal{G}_n\right) \\
&= \ \mathbb{P}\left(S \leq n \mid \mathcal{F}_n, \mathcal{G}_n\right) \mathbb{P}\left(T \leq n \mid \mathcal{F}_n, \mathcal{G}_n\right) \\
&= \ \mathbb{P}\left(S \leq n \mid \mathcal{F}_n\right) \mathbb{P}\left(T \leq n | \mathcal{G}_n\right) = 0 \text{ or } 1.
\end{aligned}$$

Again, $\max\{S,T\}$ is the rule that says to stop when both of S and T have occured. □

Q 2. Let $S_n = \sum_{i=1}^n W_i$ be a simple random walk with W_i iid and $\mathbb{P}[W_i = 1] = \mathbb{P}[W_i = -1] = \frac{1}{2}$. Find $\mathbb{E}\left[S_m \mid S_n\right]$ when (a) $m > n$ and (b) $m < n$.

Proof. (a) Assume $m > n$. Then

$$\begin{aligned}
\mathbb{E}\left[S_m \mid S_n\right] &= \mathbb{E}\left[S_n + \sum_{i=n+1}^{m} W_i \mid S_n\right] \\
&= \mathbb{E}\left[S_n \mid S_n\right] + \mathbb{E}\left[\sum_{i=n+1}^{m} W_i \mid S_n\right] \\
&= S_n + \sum_{i=n+1}^{m} \mathbb{E}\left[W_i \mid S_n\right] \\
&= S_n.
\end{aligned}$$

(b) Assume $m < n$. Note that, since the W_i are *iid*, $\mathbb{E}\left[W_i \mid S_n\right] = \mathbb{E}\left[W_j \mid S_n\right] = \dfrac{S_n}{n}$ for $i, j \leq n$. So

$$\mathbb{E}\left[S_m \mid S_n\right] = \mathbb{E}\left[\sum_{i=1}^{m} W_i \mid S_n\right] = \sum_{i=1}^{m} \mathbb{E}\left[W_i \mid S_n\right] = \sum_{i=1}^{m} \frac{S_n}{n} = \frac{m}{n} S_n.$$

□

Q 3. A bag contains red and blue balls. At each step a ball is chosen uniformly from the bag, and it is returned to the bag together with another of the same color. After n rounds, let R_n and B_n be the number of red and blue balls respectively. Show that

$$\frac{R_n}{R_n + B_n}$$

is a martingale. If the bag starts with 5 red and 3 blue balls, what is the expected number of blue balls after n rounds?

Proof. First, note that $R_{n+1} + B_{n+1} = R_n + B_n + 1$ with probability 1 for every n, since we always add one ball to the bag at each step. Also, $\mathbb{P}\left(R_{n+1} = R_n + 1\right) = \dfrac{R_n}{R_n + B_n}$ and $\mathbb{P}\left(R_{n+1} = R_n\right) = \dfrac{B_n}{R_n + B_n}$. So

$$\mathbb{E}\left[\frac{R_{n+1}}{R_{n+1} + B_{n+1}} \mid \mathcal{F}_n\right] = \frac{\mathbb{E}\left[R_n \mid \mathcal{F}_n\right]}{R_n + B_n + 1} = \frac{R_n \frac{R_n}{R_n + B_n} + \left(R_n + 1\right) \frac{B_n}{R_n + B_n}}{R_n + B_n + 1}$$

$$= \frac{R_n (R_n + B_n + 1)}{(R_n + B_n)(R_n + B_n + 1)} = \frac{R_n}{R_n + B_n},$$

thus $\dfrac{R_n}{R_n + B_n}$ is a martingale.

Now, assume $R_0 = 5$ and $B_0 = 3$. Then

$$\mathbb{E}[B_n] = \mathbb{E}[(B_n + R_n) - R_n] = \mathbb{E}\left[8 + n - (8+n)\frac{R_n}{R_n + B_n}\right]$$

$$= (8+n)\left(1 - \mathbb{E}\left[\frac{R_n}{R_n + B_n}\right]\right) = (8+n)\left(1 - \frac{R_0}{R_0 + B_0}\right) = (8+n)\left(1 - \frac{5}{8}\right) = 3 + \frac{3}{8}n.$$

□

Q 4. A die is rolled repeatedly. Which of the following are Markov chains?

- The number N_n of sixes rolled in n rolls. *Yes*
- At time n the number of rolls C_n since the most recent six. *Yes*
- At time n the number of rolls B_n until the next six. *Yes*

For each one that is a Markov chain, find the transition matrix.

Proof. N_n is a Markov Chain because, given N_n, $N_{n+1} = N_n$ with probability $\frac{5}{6}$ and $N_{n+1} = N_n + 1$ with probability $\frac{1}{6}$, regardless of the values of $N_1, \dots, N_{n-1}$. C_n is a Markov Chain because, given C_n, $C_{n+1} = C_n + 1$ with probability $\frac{5}{6}$ and $C_{n+1} = 0$ with probability $\frac{1}{6}$, regardless of the values of $C_1, \dots, C_{n-1}$. B_n is a Markov Chain because, given B_n, $B_{n+1} = B_n - 1$ with probability 1 if $B_n > 1$, and $B_{n+1} \sim \text{Geom}\left(\frac{1}{6}\right)$ (with support $\{1, 2, \dots\}$) if $B_n = 1$, regardless of the values of $B_1, \dots, B_{n-1}$.

From the definition of a transition matrix given in class, it is difficult to decide what the transition matrices for N_n and C_n should be. The definition given was that the i,jth entry in the matrix is $\mathbb{P}(X_{n+1} = j \mid X_n = i)$. However, there are different conventions on how to define $\mathbb{P}(A \mid B)$ when $\mathbb{P}(B) = 0$. I am going to choose to consider a conditional probability undefined in this case. So the nth transition matrix will only contain an ith row if $\mathbb{P}(X_n = i) \neq 0$. Also, it might make sense to not have a jth column in the matrix if $\mathbb{P}(X_{n+1} = k) = 0$ for all $k \geq j$. However, since these probabilities are defined, I will include these columns.

The nth step transition matrix for N_n is thus an $(n+1) \times \infty$ matrix, since the support of N_n is $\{0, 1, 2, \dots, n\}$, where the i,jth entry is $P_{ij} = \mathbb{P}(N_{n+1} = j - 1 \mid N_n = i - 1)$, so that the first row and column correspond to the first state, which is $N_n = 0$. The matrix is defined by

$$P_{ij} = \begin{cases} \frac{1}{6} & j = i + 1 \\ \frac{5}{6} & j = i \\ 0 & \text{else} \end{cases}$$

for all $i \in \{0, 1, 2 \dots, n\}$ and $j \in \{0, 1, 2, \dots\}$.

The nth step transition matrix for C_n is again an $(n+1) \times \infty$ matrix, since the support of C_n is $\{0, 1, 2, \dots, n\}$, where the i,jth entry is $Q_{ij} = \mathbb{P}(C_{n+1} = j - 1 \mid C_n = i - 1)$, so that the first row and column correspond to the first state, which is $C_n = 0$. The matrix is defined by

$$Q_{ij} = \begin{cases} \frac{1}{6} & j = 0 \\ \frac{5}{6} & j = i + 1 \\ 0 & \text{else} \end{cases}$$

for all $i \in \{0, 1, 2 \ldots, n\}$ and $j \in \{0, 1, 2, \ldots\}$.

The transition matrix for B_n is homogeneous. It is an $\infty \times \infty$ matrix where the ijth entry is $R_{ij} = \mathbb{P}(B_{n+1} = j \mid B_n = i)$, since the kth state is $B_n = k$. The matrix is defined by

$$R_{ij} = \begin{cases} 1 & j = i - 1 \text{ and } i > 1 \\ \frac{1}{6}(\frac{5}{6})^{j-1} & i = 1 \\ 0 & \text{else} \end{cases}$$

where $i, j \in \{1, 2, \ldots\}$.

$$P = \begin{pmatrix} 5/6 & 1/6 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 5/6 & 1/6 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & 5/6 & \ddots & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & \cdots & \ddots & 1/6 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & 5/6 & 0 & \cdots \end{pmatrix} \qquad Q = \begin{pmatrix} 1/6 & 5/6 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 1/6 & 0 & 5/6 & 0 & \cdots & 0 & 0 & \cdots \\ 1/6 & 0 & 0 & 5/6 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots \\ 1/6 & 0 & 0 & 0 & \cdots & 5/6 & 0 & \cdots \end{pmatrix}$$

$$R = \begin{pmatrix} \frac{1}{6} & (\frac{5}{6})(\frac{1}{6}) & (\frac{5}{6})^2(\frac{1}{6}) & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

□

Q 5. Let M_n be a martingale. Show that for $0 < n < m$,

$$\mathrm{Cov}(M_{n+1} - M_n, M_{m+1} - M_m) = 0.$$

Proof. First, note that $\mathbb{E}[M_{n+1} - M_n] = \mathbb{E}[M_0] - \mathbb{E}[M_0] = 0$ because M_n is a martingale. So the numerator of $\mathrm{Cov}(M_{n+1} - M_n, M_{m+1} - M_m)$ is

$$\begin{aligned} & \mathbb{E}[(M_{n+1} - M_n)(M_{m+1} - M_m)] - \mathbb{E}[M_{n+1} - M_n]\,\mathbb{E}[M_{m+1} - M_m] \\ = {} & \mathbb{E}[M_{n+1}M_{m+1}] - \mathbb{E}[M_nM_{m+1}] - \mathbb{E}[M_{n+1}M_m] + \mathbb{E}[M_nM_m] - 0 \\ = {} & \mathbb{E}[\mathbb{E}[M_{n+1}M_{m+1} \mid \mathcal{F}_m]] - \mathbb{E}[\mathbb{E}[M_nM_{m+1} \mid \mathcal{F}_m]] - \mathbb{E}[M_{n+1}M_m] + \mathbb{E}[M_nM_m] \\ = {} & \mathbb{E}[M_{n+1}\mathbb{E}[M_m \mid \mathcal{F}_m]] - \mathbb{E}[M_n\mathbb{E}[M_m \mid \mathcal{F}_m]] - \mathbb{E}[M_{n+1}M_m] + \mathbb{E}[M_nM_m] \\ = {} & \mathbb{E}[M_{n+1}M_m] - \mathbb{E}[M_nM_m] - \mathbb{E}[M_{n+1}M_m] + \mathbb{E}[M_nM_m] = 0 \end{aligned}$$

therefore the covariance is 0. □