

I worked with Sydney Wong on this assignment.

1. Prove that for any fixed positive integer  $k$ , each positive integer  $n$  has a unique representation in the form

$$n = \binom{b_1}{1} + \binom{b_2}{2} + \cdots + \binom{b_k}{k}$$

where  $0 \leq b_1 < b_2 < \cdots < b_k$ . For example, for  $k = 2$ ,  $5 = \binom{2}{1} + \binom{3}{2}$ ,  $6 = \binom{0}{1} + \binom{4}{2}$ , etc. (Hint: we already did a problem that is about a “unique representation.” Use the same technique.)

*Proof.* First we will show that, if such a representation does exist, then the only possible choice for  $b_k$  is the unique number satisfying

$$\binom{b_k}{k} \leq n < \binom{b_k + 1}{k}.$$

Suppose, for a contradiction, that a choice of  $b_k$  not satisfying this inequality could result in such a representation. Then this choice would have to satisfy

$$\binom{b_k}{k} < \binom{b_k + 1}{k} \leq n.$$

Due to the restriction  $0 \leq b_1 < \cdots < b_k$ , we have  $b_i \leq b_k - (k - i)$  for each  $i$ . However, this results in the contradiction

$$n = \sum_{i=1}^k \binom{b_i}{i} \leq \sum_{i=1}^k \binom{b_k - (k - i)}{i} < \sum_{i=0}^k \binom{(b_k - k) + i}{i} = \binom{b_k + 1}{k} \leq n$$

where the strict inequality comes from decreasing the bottom index to 0 (equivalent to adding 1), and the second to last step is an instance of the “hockey stick identity.” Therefore, if any such representation exists for a given  $k$ , then it must be unique.

We will now show, by induction on  $k$ , that this representation does exist. This is trivial for  $k = 1$ , since  $n = \binom{n}{1}$  gives a representation.

Assume the proposition holds for some  $k - 1 \geq 1$ , and let  $b_k$  be the number that satisfies (1). By the inductive hypothesis, there is a unique choice of  $0 \leq b_1 < \cdots < b_{k-1}$  giving  $n - \binom{b_k}{k} = \binom{b_1}{1} + \binom{b_2}{2} + \cdots + \binom{b_{k-1}}{k-1}$ . Also, we must have  $b_{k-1} < b_k$ , since  $b_{k-1} \geq b_k$  would provide the contradiction

$$n = \binom{b_1}{1} + \cdots + \binom{b_{k-1}}{k-1} + \binom{b_k}{k} \geq \binom{b_k}{k-1} + \binom{b_k}{k} = \binom{b_k + 1}{k} > n.$$

So this choice of  $0 \leq b_1 < \cdots < b_{k-1} < b_k$  gives the desired representation of  $n$ . □

2. Find a closed form for

$$\sum_{k=0}^m \binom{m}{k} / \binom{n}{k}.$$

(Hint: it is bad when the denominator moves with the numerator. In general, try to make the denominator more constant.)

*Proof.* First of all, we can assume  $n \geq m$ , since otherwise some term would contain a 0 in the denominator.

To “make the denominator more constant,” we can look for an identity that relates the numerator to the denominator. Consider a set  $A$  of size  $n$ . The number of ways to pick a size  $m$  subset  $B$  from  $A$ , then choose a size  $k$  subset  $C$  from  $B$  is  $\binom{n}{m}\binom{m}{k}$ . Alternatively, there are  $\binom{n}{k}$  ways to choose  $C$  from  $A$ , then  $\binom{n-k}{m-k}$  ways to choose  $B \setminus C$  from  $A \setminus C$ , which uniquely determines  $B$ . This gives the identity

$$\begin{aligned}\binom{n}{m}\binom{m}{k} &= \binom{n}{k}\binom{n-k}{m-k} \\ \implies \binom{n}{k} &= \frac{\binom{n}{m}\binom{m}{k}}{\binom{n-k}{m-k}}.\end{aligned}$$

Substituting this expression gives

$$\begin{aligned}\sum_{k=0}^m \binom{m}{k} / \binom{n}{k} &= \sum_{k=0}^m \binom{m}{k} / \frac{\binom{n}{m}\binom{m}{k}}{\binom{n-k}{m-k}} = \frac{1}{\binom{n}{m}} \sum_{k=0}^m \binom{n-k}{m-k} \\ &= \frac{1}{\binom{n}{m}} \sum_{k=0}^m \binom{n-m+k}{k} = \frac{\binom{n+1}{m}}{\binom{n}{m}} \\ &= \frac{m!(n-m)!}{n!} \frac{(n+1)!}{m!(n-m+1)!} \\ &= \frac{n+1}{n-m+1}.\end{aligned}$$

□

3. Consider the integers, laid out in a line. Let us start at 0 at time 0, and at each time unit move from  $i$  to  $i+1$  or  $i-1$  with equal probability. (this is a usual first example of a *random walk*). Come up with a definition of what it means for us to have the event of “we eventually return to 0” and compute its probability.

*Proof.* First, let  $W_n$  be a sequence of random variables where each  $W_i$  is independent of all the others and has distribution  $\mathbb{P}(W_i = -1) = \mathbb{P}(W_i = 1) = \frac{1}{2}$ . Let  $X_n = \sum_{i=1}^n W_i$  be the location of the random walk at time  $n$ . The outcome space can be interpreted to be the set  $\{-1, 1\}^\infty$  of possible sequences of steps, all of which are equally likely. The proper definition of the event “we eventually return to 0” is the event

$$\{(w_n) : \exists n > 0 \text{ such that } \sum_{i=1}^n w_i = 0\}.$$

For convenience, we may divide our analysis into the two symmetric cases, one where  $W_1 = 1$  and one where  $W_1 = -1$ . We will show that the probability of returning home in the first case is 1, since the argument for the other case is completely symmetric. So assume that  $X_1 = W_1 = 1$ .

It should be intuitive that a walker at some state  $s$  cannot go on forever without crossing one of either  $s-r$  or  $s+t$ , but we will still prove this. Let  $k = \max\{r, t\}$ . One simple way for the walker to reach  $s-r$  or  $s+t$  is for the walk to take  $k$  successive steps to the right in a row. The probability

of this happening in a given interval of  $k$  steps is  $\frac{1}{2^k}$ . Let us consider each non-overlapping sequence of  $k$  successive steps in the random walk (after the first step) to be a single trial. If those  $k$  steps are in the same direction, we succeed; otherwise, we fail. Therefore, the number  $T$  of such trials until our first success has a geometric distribution with support  $\mathbb{N}$ , so this number of trials must be finite, thus giving the desired result. Since these facts have not been derived in this course, I will clarify. The distribution of  $T$  is given by

$$\mathbb{P}[T = t] = \begin{cases} \left(1 - \frac{1}{2^k}\right)^{t-1} \left(\frac{1}{2^k}\right) & t > 0 \\ 0 & t \leq 0 \end{cases}.$$

This gives us

$$\begin{aligned} \mathbb{P}[\exists t \in \mathbb{N} : T = t] &= \sum_{t=1}^{\infty} \mathbb{P}[T = t] \\ &= \sum_{t=1}^{\infty} \left(1 - \frac{1}{2^k}\right)^{t-1} \left(\frac{1}{2^k}\right) \\ &= \left(\frac{1}{2^k}\right) \sum_{t=0}^{\infty} \left(1 - \frac{1}{2^k}\right)^t \\ &= \left(\frac{1}{2^k}\right) \left(\frac{1}{1 - (1 - \frac{1}{2^k})}\right) \\ &= 1 \end{aligned}$$

which simply shows that the support of  $T$ 's distribution is the positive integers. Therefore, the probability that the walker eventually takes  $k$  successive steps to the right is 1; so he must eventually reach either  $s - r$  or  $s + t$ , whether by this method or another.

If the walker is at some state  $s$ , then the probability of him reaching state  $s + t$  before reaching  $s - t$  is  $\frac{1}{2}$ : by the previous paragraph, the probability of him reaching  $s + t$  first plus the probability of him reaching  $s - t$  first must be one, since it is impossible for neither to occur; thus, by symmetry, this probability must be  $\frac{1}{2}$ . Letting  $E_k$  be the event that the walker reaches  $2^k$  before he returns to 0, we know that  $\mathbb{P}[E_{k+1} \mid E_k] = \frac{1}{2}$ , since this is the situation where  $s = t = 2^k$ . We will show by induction that  $\mathbb{P}[E_k] = \frac{1}{2^k}$ . This is clear for  $k = 0$ , since we have assumed the walker begins at state  $1 = 2^0$ . Now, given that this is true for some  $k$ , we have  $\mathbb{P}[E_{k+1}] = \mathbb{P}[E_{k+1} \mid E_k] \mathbb{P}[E_k] = \frac{1}{2} \cdot \frac{1}{2^k} = \frac{1}{2^{k+1}}$ .

The random walk goes on forever without returning to 0 if and only if, for every  $k$ , it reaches  $2^k$  before it reaches 0. Therefore,

$$\mathbb{P}[E] = 1 - \mathbb{P}[\bigwedge_{k=1}^{\infty} E_k] \geq \mathbb{P}[E_k] = 1 - \frac{1}{2^k}$$

for every  $k$ . Since  $\lim(1 - \frac{1}{2^k}) = 1$ , we know  $\mathbb{P}[E] = 1$ .

□

4. In Atlanta they do things slightly differently from New York. Every club has an even number of people and every two clubs must share an odd number of people. What can we say about the relationship between  $n$  (the number of people) and  $m$  (the number of clubs)?

$$n \geq \begin{cases} m & \text{if } m \text{ is odd} \\ m - 1 & \text{if } m \text{ is even} \end{cases}.$$

*Proof.* Let  $v_{i,j} \in \mathbb{F}_2^n$  be 1 if the  $j$ th person is a member of the  $i$ th club and 0 otherwise. Let  $v_i$  denote the vector  $(v_{i,1}, \dots, v_{i,n})$ . We know that

$$\langle v_i, v_j \rangle = \begin{cases} 0 & i = j \\ 1 & i \neq j \end{cases}.$$

Now, let  $A$  be the  $m \times n$  matrix whose  $i$ th row is  $v_i$ . Then  $AA^T$  is an  $m \times m$  matrix with zeros on the diagonal and ones everywhere else.

We want to calculate the rank of  $AA^T$ . First, suppose  $m$  is even. Then the sum of the rows of  $AA^T$  is the length  $m$  vector of all ones. But the sum of this vector and the  $i$ th row of  $AA^T$  is the  $i$ th standard basis vector (which has a one in the  $i$ th component and 0s elsewhere). Therefore, the row space of  $AA^T$  contains a basis for  $\mathbb{F}_2^m$ , so it has rank  $m$ .

Next, suppose that  $m$  is odd. In this case, the sum of the rows is the 0 vector, thus  $AA^T$  cannot have full rank. However, the submatrix obtained by deleting the last row and last column is of the form discussed in the previous paragraph, thus this submatrix has full rank  $m - 1$ . This means  $m - 1 \leq \text{rank}(AA^T) < m$ , so  $\text{rank}(AA^T) = m - 1$ .

We always have that  $\text{rank}(XY) \leq \text{rank}(X)$ , thus  $\text{rank}(AA^T) \leq \text{rank}(A)$ . Also, the rank of an  $n \times m$  matrix can exceed neither  $m$  nor  $n$ . Therefore,

$$n \geq \text{rank}(A) \geq \text{rank}(AA^T) = \begin{cases} m & \text{if } m \text{ is odd} \\ m - 1 & \text{if } m \text{ is even} \end{cases}.$$

□

5. Let  $G$  be a graph with  $n$  vertices where each vertex has degree at least  $n/2$ . Show that  $G$  is connected.

*Proof.* Let  $x, y \in G$ , and let  $V_x$  and  $V_y$  be the sets of vertices adjacent to  $x$  and  $y$ , respectively. We know that  $|V_x| \geq n/2$  and  $|V_y| \geq n/2$ . If some vertex  $z$  is in the intersection of these two sets, then there is a path  $(x, z, y)$  from  $x$  to  $y$ . If no such vertex exists, then these sets are disjoint, thus their union has size  $n$ . We are assuming the graph has no loops, so this requires  $x \in V_y$  and  $y \in V_x$ , thus  $x$  and  $y$  are adjacent (otherwise, the graph would have at least  $n + 2$  vertices). So the graph is connected. □

6. If the edges of a complete graph  $K_n$  can be partitioned into  $m$  sets, each of which is the edges of some complete bipartite subgraph, show that  $m \geq n - 1$ . (for a dumb example, a single edge is the edge set of a  $K_{1,1}$ , so one possible  $m$  is just the number of edges, which is  $\binom{n}{2}$ , which happens to be at least  $(n - 1)!$ ) (Hint: the rank of the sum of two matrices is never more than the sum of their ranks.)

Note: The following proof of the Graham-Pollak Theorem is based a paper by Michael Tait.

*Proof.* The adjacency matrix for  $K_n$  is  $J - I$ , where  $J$  is the  $n \times n$  matrix of all ones and  $I$  is the  $n \times n$  identity matrix. The nullity of  $J$  is  $n - 1$ , therefore it has an eigenvalue of 0 with multiplicity  $n - 1$ . So  $u \in \text{null}(J)$  if and only if  $(J - I)u = 0 - u = (-1)u$ , meaning that the adjacency matrix of  $K_n$  has an eigenvalue of  $-1$  with multiplicity  $n - 1$ .

Now, suppose we have partitioned the edges of  $K_n$  into  $m$  sets, each of which is the edge-set of some complete bipartite subgraph, and let the left and right partition sets of the  $k$ th complete bipartite graph be  $A_k$  and  $B_k$ . Let  $a_k$  be a vector whose  $i$ th component is 1 if the  $i$ th vertex of  $K_n$  is in  $A_k$ , and 0 otherwise. Define  $b_k$  similarly for  $B_k$ . This means that  $a_k b_k^T$  is an  $n \times n$  matrix whose  $ij$ th component is 1 if the  $i$ th vertex is in  $A_k$  and the  $j$ th vertex is in  $B_k$ , and 0 otherwise. Since the subgraph formed by  $A_k$  and  $B_k$  is bipartite, this means that its adjacency matrix is precisely  $a_k b_k^T + b_k a_k^T$ , since a bipartite graph contains an edge between any two vertices from different sets. This means that the adjacency matrix for  $K_n$  is

$$J - I = \sum_{k=1}^m a_k b_k^T + b_k a_k^T.$$

Now, recall that  $\text{null}(J)$  is the eigenspace for  $J - I$  associated with the eigenvalue  $-1$ . Define  $V$  to be the span of  $\{a_1, \dots, a_m\}$ . If  $\{a_1, \dots, a_m\}$  is linearly independent, then  $V$  has dimension  $m$ . Otherwise, it has dimension less than  $m$ . So  $\dim(V) \leq m$ , and equivalently  $\dim(V^\perp) \geq n - m$ .

Now, assume that  $u \in \text{null}(J) \cap V^\perp$ . Then either  $u$  is the zero vector, or  $u$  is an eigenvector of  $J - I$  with eigenvalue  $-1$  which is also orthogonal to each  $a_k$ . Therefore,

$$\begin{aligned} -\|u\|^2 &= u^T(-u) = u^T(J - I)u = u^T \left( \sum_{k=1}^m a_k b_k^T + b_k a_k^T \right) u \\ &= \sum_{k=1}^m (a_k^T u)^T (b_k^T u) + (u^T b_k) (a_k^T u) = \sum_{k=1}^m (0)^T (b_k^T u) + (u^T b_k) (0) = 0 \end{aligned}$$

so  $u$  must be the zero vector. Therefore,  $\text{null}(J)$  and  $V^\perp$  have trivial intersection. This means that

$$\begin{aligned} n &\geq \dim(\text{null}(J) + V^\perp) = \dim(\text{null}(J)) + \dim(V^\perp) - \dim(\text{null}(J) \cap V^\perp) \\ &= \dim(\text{null}(J)) + \dim(V^\perp) - 0 \\ &\geq (n - 1) + (n - m). \end{aligned}$$

Adding  $m - n$  to both sides of this resulting inequality gives the desired result:  $m \geq n - 1$ .  $\square$

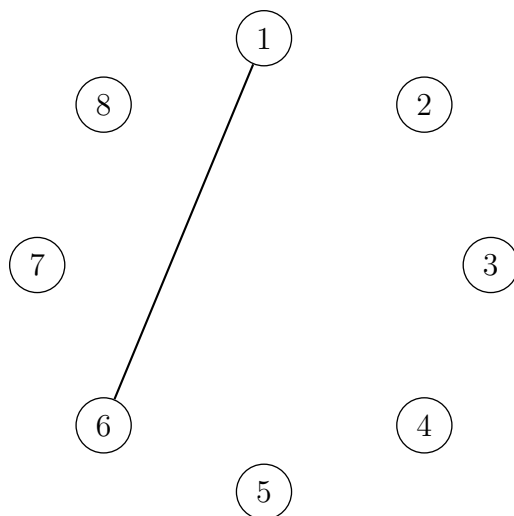
7.  $2n$  points are given on a circle. With uniform probability we split them into  $n$  pairs and join each pair with a segment. Find the probability that that no two of these segments intersect.

*Proof.* Label the vertices  $1, \dots, 2n$ . The total number of ways to pair up the vertices is

$$\frac{(2n)!}{2^n n!}.$$

To see this, first pick a permutation of the  $2n$  vertices, then pair up 1 with 2, 3 with 4, etc. However, any combination of swapping the two vertices in any of the  $n$  pairs (which gives  $2^n$  choices) and rearranging the pairs ( $n!$  choices) actually leads to the same pairing on the graph.

Let  $s_n$  be the number of pairings, without intersections, on such a graph with  $2n$  vertices labeled  $1, \dots, 2n$  (note that there are 0 pairings on a graph with an odd number of vertices). Let  $t_{n,k}$  be the number of such pairings in which vertex 1 is joined with vertex  $2k$  (if vertex 1 were joined with an



odd-labeled vertex, then there would be no way to pair the remaining vertices without intersecting the edge from 1 to that vertex). So  $s_n = \sum_{k=1}^n t_{n,k}$ . Also,  $t_{n,k}$  satisfies the recurrence

$$t_{n,k} = s_{k-1}s_{n-k}$$

To see this, look at the example of the graph on 8 vertices. After pairing 1 with  $2k = 6$  in this example, we are left with a circle of  $2(k-1) = 4$  vertices on the right and a circle of  $2(n-k) = 2$  vertices on the left. The number of ways to complete this pairing is the product of the numbers of ways to successfully pair those two circles. This also shows why 1 must be paired with an even-labeled vertex, since otherwise each of the two smaller circles would have an odd number of vertices, meaning that the number of ways to pair it would be 0. Therefore,

$$\begin{aligned} s_n &= \sum_{k=1}^n t_{n,k} = \sum_{k=1}^n s_{k-1}s_{n-k} \\ \implies s_{n+1} &= \sum_{k=0}^n s_k s_{n-k}. \end{aligned}$$

So  $s_n$  satisfies the same recurrence as the Catalan numbers. Since there is 1 successful way to pair a graph with no vertices,  $s_0 = 1$ . Therefore,  $s_n$  must be the Catalan numbers, so the probability of a successful pairing is

$$\frac{\frac{1}{n+1} \binom{2n}{n}}{\frac{(2n)!}{2^n n!}} = \frac{(2n)! n! 2^n}{(2n)! (n+1) n!^2} = \frac{2^n}{(n+1)!}.$$

□

8. How much time did you spend on this problem set? What comments do you have on the problems? (difficulty, type, enjoyment, fairness, etc.)

I spent a lot of time on this problem set, but I always spend a lot of time on these problem sets. On problem 4, I believe there is a slightly tighter bound (equality cannot hold), but I didn't prove it.

I could not find a generating function solution for problem 1, and I would not have thought of a solution to 6 on my own. I did problem 7 by looking at small cases first and recognizing the Catalan numbers. Problem 5 was easy, and problem 3 was easy except annoying to formalize. On problem 2, I just tried to think of identities that would relate the numerator to the denominator. This was a fun problem set, overall.