25. Let $f : \Omega \to \mathbb{C}$ be holomorphic, and suppose in addition that f' is continuous. Use Green's Theorem to show that whenever $T \subseteq \Omega$ is a triangular path, the inside of which is also in Ω, we have $\int_T f(z)dz = 0$.

Proof. Let D be the interior of the triangle. Then $f = u + iv$ is defined on the open region Ω containing D. Since f' is continuous, the partial derivatives of u and v are continuous. So Green's Theorem implies that

$$\int_T f(z)dz = \int_T (udx - vdy) + i\int_T (vdx + udy) = \iint_D -\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dxdy + \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy.$$

However, since f is holomorphic, the Cauchy-Riemann equations state that $\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$. Therefore, $\int_T f(z)dz = 0$. $\square$

26. For any $r > 0$ and $n \in \mathbb{Z}$, compute $\int_{C_r} z^n dz$ and $\int_{C_r} \overline{z^n} dz$, where C_r is the circle of radius r centered at the origin with positive orientation.

Proof. If $n \neq -1$, then $\frac{d}{dz}[\frac{z^{n+1}}{n+1}] = z^n$, thus z^n has a primitive. So, by Corollary II.1.4, the first integral must be 0. If $n = -1$, then we computed this integral in an example following this corollary:

$$\int_{C_r} z^{-1}dz = \int_0^{2\pi} e^{-it}e^{it}idt = 2\pi i.$$

For the second integral, notice that $|z^n| = |z|^n = r$ on C_r, so $\overline{z^n} = \frac{|z^n|^2}{z^n} = \frac{r^2}{z^n}$. Thus,

$$\int_{C_r} \overline{z^n}dz = \int_{C_r} \frac{r^2}{z^n}dz = r^2\int_{C_r} z^{-n}dz.$$

Therefore, by the previous paragraph, this integral is 0 if $n \neq 1$ and is $2\pi r^2 i$ if $n = 1$. $\square$

27. Compute $\int_{C_r} \mathrm{Re}(z)dz$.

Proof.

$$\int_{C_r} \mathrm{Re}(z)dz = \int_{C_r} \frac{z + \overline{z}}{2}dz = \frac{1}{2}\left(\int_{C_r} zdz + \int_{C_r} \overline{z}dz\right) = \frac{0 + 2\pi r^2 i}{2} = \pi r^2 i$$

$\square$

28. Show that, for any $n \in \mathbb{N}$,

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^{2n} tdt = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

Proof. Let $\gamma(t) = e^{it}$ be a parametrization of the unit circle. If we can find a function $f : \gamma \to \mathbb{C}$ such that $f(\gamma(t))\gamma'(t) = \cos^{2n} t$, then we will have $\int_0^{2\pi} \cos^{2n} tdt = \int_\gamma f(z)dz$. Differentiating γ gives $\gamma'(t) = ie^{it}$; substituting and solving shows that $f(z) = -i(\frac{z+z^{-1}}{2})^{2n}z^{-1}$ suffices, as we will verify:

$$f(\gamma(t))\gamma'(t) = -i\left(\frac{e^{it} + e^{-it}}{2}\right)^{2n} e^{-it}(ie^{it}) = \left(\frac{e^{it} + e^{-it}}{2}\right)^{2n} = \cos^{2n}(t).$$

This implies

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \cos^{2n} tdt &= \frac{1}{2\pi}\int_\gamma -i\left(\frac{z + z^{-1}}{2}\right)^{2n} z^{-1}dt \\
&= -\frac{i}{2^{2n+1}\pi}\int_\gamma z^{-(2n+1)}\left(z^2+1\right)^{2n} dt \\
&= -\frac{i}{2^{2n+1}\pi}\int_\gamma z^{-(2n+1)}\sum_{k=0}^{2n}\binom{2n}{k}z^{2k}dt \\
&= -\frac{i}{2^{2n+1}\pi}\sum_{k=0}^{2n}\binom{2n}{k}\int_\gamma z^{2(k-n)-1}dt.
\end{aligned}$$

From exercise 26, we know that this integral is 0 whenever $2(k-n)-1 \neq -1$, and is $2\pi i$ otherwise; but $2(k-n)-1=-1$ if and only if $k=n$. Therefore, all terms of the summation are 0 except for the term where $k=n$. So this reduces the integral to

$$-\frac{i}{2^{2n+1}\pi}\binom{2n}{n}2\pi i = \binom{2n}{n}\frac{1}{2^{2n}} = \frac{(2n)!}{n!n!}\frac{1}{2^n} = \frac{(2n)!/(n!2^n)}{n!2^n} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}.$$

□

29. Show that $\int_0^\infty \sin x^2 dx = \int_0^\infty \cos x^2 dx = \frac{\sqrt{2\pi}}{4}$.

Proof. Let $\gamma_1(t)=t$ for $0\leq t\leq R$, $\gamma_2(t)=Re^{it}$ for $0\leq t\leq \frac{\pi}{4}$, and $\gamma_3 = te^{i\frac{\pi}{4}}$ for $0\leq t\leq R$. Let $\Gamma=\gamma_1+\gamma_2+(-\gamma_3)$. The region bounded by the closed curve Γ is convex and e^{iz^2} is holomorphic, so $\int_\Gamma e^{iz^2}dz = \int_{\gamma_1} e^{iz^2}dz + \int_{\gamma_2} e^{iz^2}dz + \int_{-\gamma_3} e^{iz^2}dz = 0$.

Computing the third term gives

$$\begin{aligned}\int_{-\gamma_3} e^{iz^2}dz &= -\int_{\gamma_3} e^{iz^2}dz\\ &= -\int_0^R e^{it^2e^{i\frac{\pi}{2}}}(e^{i\frac{\pi}{4}})dt\\ &= -e^{i\frac{\pi}{4}}\int_0^R e^{i^2t^2}dt\\ &= -\frac{1+i}{\sqrt{2}}\int_0^R e^{-t^2}dt.\end{aligned}$$

As $R\to\infty$, the limit of this integral is $-(\frac{1+i}{\sqrt{2}})\frac{\sqrt{\pi}}{2} = -\frac{\sqrt{2\pi}}{4} - i\frac{\sqrt{2\pi}}{4}$.

Computing the second term gives

$$\int_{\gamma_2} e^{iz^2}dz = \int_0^{\frac{\pi}{4}} e^{iR^2e^{2it}}(iRe^{it})dt = iR\int_0^{\frac{\pi}{4}} e^{i(R^2e^{2it}+t)}dt.$$

We know that

$$\begin{aligned}\left|iR\int_0^{\frac{\pi}{4}} e^{i(R^2e^{2it}+t)}dt\right| &\leq |iR|\int_0^{\frac{\pi}{4}}\left|iRe^{i(R^2e^{2it}+t)}dt\right| && (1)\\ &= R\int_0^{\frac{\pi}{4}}\left|Re^{i(R^2e^{2it}+t)}\right|dt && (2)\\ &= R\int_0^{\frac{\pi}{4}}\left|e^{i(R^2(\cos(2t)+i\sin(2t))+t)}\right|dt && (3)\\ &= R\int_0^{\frac{\pi}{4}}\left|e^{i(R^2\cos(2t)+t)}\right|\left|e^{-R^2\sin(2t)}\right|dt && (4)\\ &= R\int_0^{\frac{\pi}{4}}\left|e^{-R^2\sin(2t)}\right|dt && (5)\\ &\leq R\int_0^{\frac{\pi}{4}}\left|e^{-R^2t}\right|dt && (6)\\ &= R\int_0^{\frac{\pi}{4}} e^{-R^2t}dt && (7)\\ &= \frac{1-e^{-R^2\frac{\pi}{4}}}{R} && (8)\end{aligned}$$

A quick explanation: the factor that disappears between lines 4 and 5 was an exponential with purely imaginary argument, thus it had modulus 1. From line 5 to line 6, we used the fact that $\sin(2x) \geq x$ for all $x \in [0, \frac{\pi}{4}]$, which we will justify now:

The derivative of $g(t) = \sin(2t) - t$ is $g'(t) = 2\cos(2t) - 1$, which, on the interval $(0, \frac{\pi}{4})$, has its only zero at $t = \frac{\pi}{6}$. Since $g'(0) = 2\cos(2 \cdot 0) - 1 = 1$, $g(t)$ is increasing on this interval. Therefore, $g(t) \geq g(0) = 0$ for all $t \in [0, \frac{\pi}{6}]$, so $g(t)$ is positive on this interval. $g'(t)$ has no zeros on $(\frac{\pi}{6}, \frac{\pi}{4}]$, and $g'(\frac{\pi}{4}) = 2\cos(2\frac{\pi}{4}) - 1 = -1$. Thus, $g(t)$ is decreasing on this interval. So, for all $t \in (\frac{\pi}{6}, \frac{\pi}{4})$, $g(t) > g(\frac{\pi}{4}) = 1 - \frac{\pi}{4} = \frac{4-\pi}{4} > 0$. Therefore, $g(t)$ is nonnegative on this interval as well, hence $\sin(2t) \geq t$ on $[0, \frac{\pi}{4}]$.

The limit of the integral in (8) as $R \to \infty$ is 0, which leaves us with

$$\begin{aligned}\lim_{R\to\infty} \int_\Gamma e^{iz^2} dz &= \int_{\gamma_1} e^{iz^2} dz + \int_{\gamma_2} e^{iz^2} dz + \int_{-\gamma_3} e^{iz^2} dz \\ &= \int_0^R \cos(x^2) + i\sin(x^2) dx + 0 - \frac{\sqrt{2\pi}}{4} - i\frac{\sqrt{2\pi}}{4} \\ &= 0.\end{aligned}$$

Therefore,

$$\int_0^R \cos(x^2) + i\int_0^R \sin(x^2) dx = \frac{\sqrt{2\pi}}{4} + i\frac{\sqrt{2\pi}}{4}$$

Equating the real and imaginary parts gives the desired result. □

30. Compute $\int_C \frac{e^z}{z}$, where C denotes the unit circle with positive orientation.

Proof. Let $f(z) = e^z$. Since f is holomorphic, Cauchy's Integral Theorem tells us that $\int_C \frac{e^z}{z-0} = 2\pi i f(0) = 2\pi i \cdot 1 = 2\pi i$.

□

31. Compute

$$\int_{|z|=2} \frac{dz}{z^2+1}$$

understanding the circle to be oriented positively.

Proof. We can decompose the integrand as

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{a}{z+i} + \frac{b}{z-i}$$

using partial fraction decomposition by solving $a(z-i) + b(z+i) = 1$. Since this equation must hold for any $z \in \mathbb{C}$, we see that substituting $z = i$ yields $2bi = 1$ and substituting $z = -i$ yields $-2ai = 1$. Thus $a = -\frac{1}{2i} = \frac{i}{2}$ and $b = \frac{1}{2i} = -\frac{i}{2}$. Thus,

$$\int_{|z|=2} \frac{dz}{z^2+1} = \int_{|z|=2} \frac{i/2}{z+i} dz - \int_{|z|=2} \frac{i/2}{z-i} dz.$$

Since $\frac{i}{2}$ is holomorphic, we can apply Cauchy's Integral Theorem to see that both of these integrals equal $2\pi i \cdot \frac{i}{2}$. Therefore, the integral is 0. □

32. Let $0 < r < 1$. Compute
$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos\theta + r^2}d\theta.$$

Proof. Let $\gamma(\theta) = e^{i\theta}$. Again, we look for a function $f : \gamma \to \mathbb{C}$ such that $f(\gamma(\theta))\gamma'(\theta) = f(e^{i\theta})(ie^{i\theta}) = \frac{1-r^2}{1-2r\cos\theta + r^2}$. Solving shows that $f(z) = \frac{1-r^2}{i(rz-1)(r-z)}$ suffices:

$$\begin{aligned} f(e^{i\theta}) &= -ie^{-i\theta}\frac{1-r^2}{1-2r\cos\theta + r^2} \\ &= -ie^{-i\theta}\frac{1-r^2}{1-2r\frac{e^{i\theta}+e^{-i\theta}}{2} + r^2} \\ &= -ie^{-i\theta}\frac{1-r^2}{1-re^{i\theta}-re^{-i\theta} + r^2} \\ &= -ie^{-i\theta}\frac{1-r^2}{(re^{i\theta}-1)(re^{-i\theta}-1)} \\ &= \frac{1-r^2}{i(re^{i\theta}-1)(r-e^{i\theta})} \end{aligned}$$

We can use partial fraction decomposition to obtain

$$f(z) = \frac{1-r^2}{i(rz-1)(r-z)} = \frac{r}{i(rz-1)} - \frac{1}{i(r-z)} = \frac{1}{i(z-r^{-1})} + \frac{1}{i(z-r)}$$

which can be checked easily. Therefore,

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos\theta + r^2}d\theta = \frac{1}{2\pi i}\int_C \frac{1}{z-r^{-1}}dz + \frac{1}{2\pi i}\int_C \frac{1}{z-r}dz = 0 + 1 = 1$$

because $\frac{1}{z-r^{-1}}$ is holomorphic on the disk C (since $r^{-1} > 1$), so Cauchy's Theorem states the lefthand integral is 0; and by Cauchy's Integral Theorem, the second integral equals $g(r)$, where $g(z)$ is the constant function 1 (since $0 < r < 1$). $\square$