Math 185
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Summer '15

Homework Assignment (6)

Due in my office, 853 Evans, Friday, August 7 by noon

(41) Let $\Omega \subseteq \mathbb{C}$ be open. Prove that Ω can be written as the countable disjoint union of open connected sets. [Hint: It will probably be easier to first show that Ω can be written as the disjoint union of connected sets. Then prove each of these is open, then prove there are countably many. For the first part, you will probably need the following generalization of Topology Problem (6): If $\{C_\alpha\}_{\alpha \in a}$ is a collection of connected subsets of a metric space (M, d) such that $\bigcap_{\alpha \in a} C_\alpha \neq \emptyset$, then $\bigcup_{\alpha \in a} C_\alpha$ is connected. [Here, a is an arbitrary set indexing the collection. Nothing is assumed about its cardinality.]]

(42) Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ be open, and let $\gamma_1 : [a,b] \to \Omega_1$, $\gamma_2 : [c,d] \to \Omega_2$ be paths. Let f be a continuous function defined on $\gamma_1 \times \gamma_2$, and define $F_1 : \Omega_1 \to \mathbb{C}$, $F_2 : \Omega_2 \to \mathbb{C}$ by

$$F_1(z) := \int_{\gamma_2} f(z,w)\, dw \quad , \quad F_2(w) := \int_{\gamma_1} f(z,w)\, dz .$$

Prove that F_1 and F_2 are continuous and that

$$\int_{\gamma_1} F_1(z)\, dz = \int_{\gamma_2} F_2(w)\, dw ,$$

or in other words,

$$\int_{\gamma_1} \int_{\gamma_2} f(z,w)\, dw\, dz = \int_{\gamma_2} \int_{\gamma_1} f(z,w)\, dz\, dw .$$

[Hint: Use Fubini's Theorem from real analysis.]

(43) Prove that the Laurent expansion is unique: That is, if f is holomorphic in an annulus $0 \le R_1 < |z-z_0| < R_2 \le \infty$ and

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \sum_{n=-\infty}^{\infty} b_n (z-z_0)^n$$

for all z in the annulus, then each $a_n = b_n$. [Caveat: You cannot use the integral representation of the coefficients as in Thm. III.1.4. The proof of that theorem showed only that the coefficients given by those integrals yield a Laurent expansion. It did not show that those are the only coefficients that do so.] [Remark: This problem was tacitly used throughout section III.2.]

(44) Determine the Laurent expansion of each of the following functions in the regions indicated:

a) $\dfrac{1}{(z-1)(z-2)}$ in the regions $1 < |z| < 2$, $|z| > 2$, and $0 < |z-1| < 1$.

b) $\dfrac{1}{z^2(1-z)}$ in the regions $0 < |z| < 1$, $|z| > 1$, and $0 < |z-1| < 1$.

c) $\dfrac{2z+10}{(1+z)^2(z^2-9)}$ in the region $1 < |z| < 3$.

(45) Let z_0 be an isolated singularity of a holomorphic function f. Suppose that there are $A, \varepsilon > 0$ such that for all z sufficiently close to z_0, $|f(z)| \le \dfrac{A}{|z-z_0|^{1-\varepsilon}}$. Prove that z_0 is a removable singularity. [Thus, there is a type of "escape velocity" involved for holomorphic functions at singularities: If a function tries to go ∞ but doesn't go "fast enough", it never actually gets there.]

(46) Prove that if z_0 is an isolated singularity of f, then it is not a pole of the function e^f.

(47) Let f be meromorphic in all of $\mathbb{C}$, and suppose that for all sufficiently large $|z|$, $|f(z)| \le C \cdot |z|^n$. Prove that f is a rational function. [Hint: Find a way to use HW #(35).]

(48) Let z_0 be an isolated singularity of f. Prove that if $\operatorname{res}_{z_0} f = 0$, then f has a primitive in some deleted neighborhood of z_0.

(49) Let $g, h : \Omega \to \mathbb{C}$ be holomorphic, and suppose h has a simple zero at $z_0 \in \Omega$. Prove that

$$\operatorname{res}_{z_0}\left[\frac{g}{h}\right] = \frac{g(z_0)}{h'(z_0)}.$$

(50) Let f be meromorphic in Ω and not identically zero. Show that each isolated singularity of $\frac{f'}{f}$ is a simple pole, and show that the residue at each pole is an integer. [Hint: Use Lemmas III.2.2 and III.2.4.]

(51) Prove that, for $n \ge 3$, the sum of the residues of all the isolated singularities of

$$\frac{z^n}{1 + z + z^2 + \cdots + z^{n-1}}$$

is 0.

(52) Each of the following functions is defined on all of $\mathbb{C}$ except for isolated singularities. Locate each singularity, classify it as removable, a pole [give the order], or essential, and compute the residue there:

a) $\dfrac{1}{\sin^2 z}$ b) $\sin \dfrac{1}{z}$ c) $\dfrac{z}{e^z - 1}$ d) $\tan z$

e) $\dfrac{\cos z}{1 + z + z^2}$ f) $\dfrac{z^m}{1 - z^n}$ [m, n positive integers]

g) $\dfrac{1}{z^m (1-z)^n}$ [m, n positive integers]