Homework 2

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Exercise 1. Show that S^n and $T^n = S^1 \times \cdots \times S^1$ are not diffeomorphic, for $n \geq 2$.

Proof. First, as a lemma, we will show that diffeomorphisms preserve mod 2 intersection numbers. Suppose $f : Y \to W$ is a diffeomorphism and that $I_2(X, Z)$ is defined. We may assume $X \pitchfork Z$. If not, then we may deform X via a map homotopic to the inclusion map on Z, and simply redefine X to be the resulting submanifold. If $x \in X \cap Z$, then $x \in X$ and $x \in Z$ so $f(x) \in f(X)$ and $f(z) \in f(Z)$. So $f(x) \in f(X) \cap f(Z)$. Similarly, if $y \in f(X) \cap f(Z) = f(X \cap Z)$ (equality holds because f is a diffeomorphism), then $f^{-1}(y) \in f^{-1}(f(X \cap Z)) = X \cap Z$ (again, equality holds because f is a diffeomorphism). Therefore, f is a bijection between $X \cap Z$ and its image, so these submanifolds have the same cardinality and hence the same mod 2 intersection number.

Now, notice that the submanifolds $U = \left\{\frac{1}{2}\right\} \times (S^1)^{n-1}$ and $V = S^1 \times \left\{\frac{1}{2}\right\}^{n-1}$ of T^n have mod 2 intersection number of 1, since their only point of intersection is $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$, at which they are transversal. Also, it is clear that U is diffeomorphic to $(S^1)^{n-1}$ and V is diffeomorphic to S^1.

Therefore, it suffices to show that any two submanifolds of S^n which are diffeomorphic to $(S^1)^{n-1}$ and S^1, respectively, will necessarily have a mod 2 intersection number of 0. Since diffeomorphisms preserve mod 2 intersection numbers, this will mean that any two submanifolds of the torus which are diffeomorphic to $(S^1)^{n-1}$ and S^1 will have to have a mod 2 intersection number of 0, contradicting the example we have just presented.

Let $X \cong (S^1)^{n-1}$ and $Y \cong S^1$ be submanifolds of the n-sphere. By a corollary to Sard's Theorem, we know that the set of points that are simultaneously regular values of any two maps is dense. So let p be any element of this dense set of points which are regular values for the inclusion maps $X \xrightarrow{i} S^n$ and $Y \xrightarrow{i} S^n$. Since $\dim(X) = n-1 < n$ and $\dim(Y) = 1 < n$, neither inclusion can be a submersion at any point in its image. So $p \notin X \cap Y$.

Now, let $\phi : S^n \to \mathbb{R}^n$ be the stereographic projection map that uses the point p as its pole, so that both X and Y are in the domain of ϕ. Thus, $\phi(X)$ and $\phi(Y)$ are submanifolds of $\mathbb{R}^n$ that are diffeomorphic to S^{n-1} and S^1, respectively. So both are compact and connected.

Since $\phi(X)$ also has dimension $n-1$, we may apply the Jordan-Brouwer Separation Theorem to assert that $\phi(X)$ is the boundary of a compact submanifold W of $\mathbb{R}^n$. Since the inclusion map $i : \phi(X) \to \mathbb{R}^n$ extends to all of W, and $\phi(Y)$ is closed and has complementary dimension to $\phi(X)$, we derive from the Boundary Theorem that the mod 2 intersection number of $\phi(X)$ and $\phi(Y)$ must be 0.

Since ϕ preserves mod 2 intersection numbers, then, X and Y also have a mod 2 intersection number of 0. Since X and Y were arbitrary submanifolds of S^n diffeomorphic to S^{n-1} and S^1, this is true of all such submanifolds. Therefore, if S^n and T^n were diffeomorphic, then the two submanifolds of T^n described in the first paragraph, which were diffeomorphic to S^{n-1} and S^1 yet had intersection number 1, would have diffeomorphic images in S^n with intersection number 0, a contradiction.

Thus $S^n \ncong T^n$ for $n \geq 2$.

$\square$

Exercise 2. *The Smooth Urysohn Theorem.* If A and B are disjoint, smooth, closed subsets of a manifold X, prove that there is a smooth function ϕ on X such that $0 \leq \phi \leq 1$ with $\phi = 0$ on A and $\phi = 1$ on B.

Proof. Since A and B are disjoint and closed, their complements in X, A^C and B^C, form an open cover of X. Thus, by the theorem on pg. 52, a partition of unity $\{\theta_i\}$ exists for which the support of each θ_i is contained either completely within A^C or completely within B^C, and $\sum_i \theta_i(x) = 1$ for all $x \in X$.

Let $I = \{i : \operatorname{supp}(\theta_i) \subset A^C\}$, where $\operatorname{supp}(\theta_i)$ denotes the support of θ_i. This implies that, for $i \in I$, $\theta_i(x) = 0$ for all $x \in A$. Now, let

$$\phi = \sum_{i \in I} \theta_i.$$

For any $x \in A$, we have

$$\phi(x) = \sum_{i \in I} \theta_i(x) = \sum_{i \in I} 0 = 0.$$

The key point to note is that, if $j \notin I$, then $\mathrm{supp}(\theta_j) \subset B^C$. Thus, $\theta_j(x) = 0$ for all $x \in B$.

Thus, for any $x \in B \subset A^C$, we have

$$\phi(x) = \sum_{i \in I} \theta_i(x) = \sum_{i \in I} \theta_i(x) + 0 = \sum_{i \in I} \theta_i(x) + \sum_{j \in \mathbb{N} - I} \theta_j(x) = \sum_{i \in \mathbb{N}} \theta_i(x) = 1$$

for all $x \in B$.

Since ϕ is a sum of smooth functions, it is also smooth. Since each θ_i is bounded between 0 and 1, a sum of some collection of those functions must also be bounded between 0 and 1. Thus $0 \le \phi \le 1$.

$\square$

Exercise 3. *Tubular Neighborhood Theorem.* Prove that there exists a diffeomorphism from an open neighborhood of Z in $N(Z; Y)$ onto an open neighborhood of Z in Y.

Proof. Let $Y^\epsilon \xrightarrow{\pi} Y$ be as in the ϵ-Neighborhood Theorem. Consider the map $h : N(Z; Y) \to \mathbb{R}^M$ defined by $h(z, v) = z + v$. Clearly this map is smooth, since it is the sum of two smooth functions.

Let $W = h^{-1}(Y^\epsilon)$. W is open because h is continuous and Y^ϵ is open by definition. Also, if $(z, 0) \in Z \times \{0\}$ then $h(z, 0) = z + 0 = z \in Y^\epsilon$, therefore h^{-1} contains $Z \times \{0\}$. So W is an open neighborhood of Z in $N(Z; Y)$.

Consider the sequence of maps $W \xrightarrow{h} Y^\epsilon \xrightarrow{\pi} Y$. If $(z, 0) \in Z \times \{0\}$ then $\pi \circ h(z, 0) = \pi(z + 0) = \pi(z) = z$, since $z \in Y$ and π is the identity on Y. Therefore, this sequence of maps iss the natural projection of $Z \times \{0\} \subset N(Z; Y)$ onto $Z \subset Y$. So $\pi \circ h$ maps $Z \times \{0\}$ diffeomorphically onto Z.

Since $h \circ \pi$ is a diffeomorphism on $Z \times \{0\}$, it is locally equivalent to the identity map. Thus, it's derivative at every point of Z is an isomorphism. Therefore, by exercise 14 from section 8, $h \circ \pi$ maps an open neighborhood of $Z \times \{0\}$ in $N(Z; Y)$ diffeomorphically onto an open neighborhood of Z in Y.

$\square$