Problem Set 2, Math 172 Spring '15

This problem set is due Tuesday, February 3rd, 2015 at **the beginning of class**. All class guide rules apply. "VLW" refers to Van Lint and Wilson, our textbook. **All questions, when graded, are worth an equal number of points unless stated otherwise, though not all questions would end up being graded.**

1. Let S be a set with special subsets $E_1, \ldots, E_n$, as in the setup of inclusion-exclusion. Let f_k denote the number of elements in S that are in **exactly** k of the sets. Show that
$$f_k = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} h_i,$$
where
$$h_i = \sum_{\{a_1,\ldots,a_i\} \subset [n]} |E_{a_1} \cap E_{a_2} \cap \cdots \cap E_{a_i}|.$$
Then give an analogous formula for f'_k, which is the number of elements in S that are in **at least** k of the sets.

2. Find the "best" estimate for the following (give a justification for each answer, but no need to prove why the answer you selected is better than the others):
   (a) $\binom{n}{2}$;
   (b) The sum of the first n positive integers;
   (c) the number of ways to have a set of n total red, white, and blue indistinguishable balls.

   Your answers should be "simple", such as $O(\log^k(n)), O(n^k)$, or $O(k^n)$ for specific k.

3. Show, **with combinatorics and not algebra/number theory**, Fermat's Little Theorem: that $a^p - a$ is divisible by p for any prime p and positive integers a. (hint: you probably want to create a set S with $a^p - a$ elements; also, think quotient principle.)

4. Show that **both** of the following are $O(\log(n))$, where the base of the logarithm can be taken to be any number (say e for natural log):
   (a) The number of digits of n written in base 10.
   (b) The number of digits of n written in base 2.

5. (VLW 10D) Define $\mu(d)$ to be
   (a) 1 if d is a product of an even number of **distinct** primes,
   (b) -1 if it is a product of an odd number of distinct primes, and
   (c) 0 otherwise (in particular, if the square of any prime divides d, you should get 0).

The Riemann ζ-fuction is defined as $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Show that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \mu(n) n^{-s}.$$

Hints: you don't need to touch s at all (which is actually a complex number!) just figure out how to make sense of what the question **means**. You are also free to use the following result without proof: $\sum_{d|n} \mu(d) = 1$ if $n = 1$ and 0 otherwise. (this is Theorem 10.3 in VLW, which is short and easy. Optional: try to prove it without looking).

6. The rules of "172 Craps" is similar to Craps, but slightly different: you have 2 normal 6-sided dice, and you roll. If you get a 2, 3, 11, or 12, you lose immediately. If you roll a 7 you win. After this first turn, you remember the results of your first roll (which must not be 7 otherwise you would have won already) and continue to roll until you get a 7 (in which case you **lose**) or your first roll (in which case you win). What is your probability of winning? (Hint: it is really useful to understand the following baby situation: suppose $p_1 + p_2 + p_3 = 1$ and you have a game where you win with probability p_1, lose with probability p_2, and replay the game with probability p_3. What is the probability that you win?)

7. (optional) What does Inclusion-Exclusion look like for multi-sets (sets where an identical element can occur multiple times)? Design your theorem.

8. How much time did you spend on this problem set? What comments do you have of the problems? (difficulty, type, enjoyment, fairness, etc.)