Sample Problems for Final Exam

Questions from Math 125a Final Exam, Spring 2006

In the following, you must prove that your answers are correct.

1. Let $\mathbb{R}$ be the structure of the real numbers with 0, 1, $+$, $\times$, and $<$. Show that there are two countable structures $\mathcal{M}_1$ and $\mathcal{M}_2$ with the following properties.

   - $\mathcal{M}_1 \equiv \mathbb{R}$ and $\mathcal{M}_2 \equiv \mathbb{R}$
   - $\mathcal{M}_1$ and $\mathcal{M}_2$ are not isomorphic

   Solution: By the Lowenheim-Skolem Theorem, let $\mathcal{M}_1$ be a countable elementary substructure of $\mathbb{R}$.

   We obtain $\mathcal{M}_2$ by application of the Compactness Theorem. Note that $<$ is definable in $\mathbb{R}$ by $x < y$ if and only if there is a z such that $x + z^2 = y$, so we may use $<$ as part of the language of $\mathbb{R}$. Let Γ be the following set of sentences in the language of $\mathbb{R}$ together with an additional constant symbol c.

   $$\Gamma = \{\varphi : \mathbb{R} \models \varphi\} \cup \{\underbrace{1 + \cdots + 1}_{n \text{ times}} < c : n \in \mathbb{N}\}$$

   Γ is finitely satisfiable, since any finite subset of Γ can be satisfied in $\mathbb{R}$ by interpreting c to be a real number that is sufficiently large. By the Compactness Theorem, Γ is satisfiable. Let $\mathcal{M}_2^*$ be a countable structure which satisfies Γ, and let $\mathcal{M}_2$ be the restriction of $\mathcal{M}_2^*$ to the language of $\mathbb{R}$. $\mathcal{M}_2$ is elementarily equivalent to $\mathbb{R}$ by the definition of Γ.

   In $\mathcal{M}_1$, every element is less than some finite sum of 1. In $\mathcal{M}_2$, that is not the case. Thus, the two structures are not isomorphic.

2. Let $\mathcal{L}_A$ be the language with one unary predicate symbol P. Let $\mathcal{M}$ be the finite structure (M, I) such that $M = \{a, b, c, d, e\}$ and $I(P) = \{a, b\}$. In other words, $\mathcal{M}$ interprets P as holding of a and b and as not holding of c, d, or e.

   (a) Which subsets of M which are definable in $\mathcal{M}$ without parameters?

   (b) Which subsets of M are are definable in $\mathcal{M}$ with parameters?

   Solution: Consider the permutations of M: transpose a and b and leave c, d and e fixed, and transpose two of the elements of c, d and e and leave the other elements of M fixed. Each of these permutations preserves P and so is an automorphism of $\mathcal{M}$. Since definable sets are closed under automorphisms, if A is definable, then if A contains either a or b then it contains the other and if A contains any of c, d or e then it contains the other two. Thus the only subsets of M which are definable in $\mathcal{M}$ are $\emptyset$, $\{a, b\}$, $\{c, d, e\}$ and $\{a, b, c, d, e\}$.

   Every subset of M is definable from parameters. If the elements of A are $m_1, \ldots, m_k$ then

   $$x = m_1 \vee \cdots \vee x = m_k$$

   is a definition of A with parameters $m_1, \ldots, m_k$.

3. Let A be finite, let k be a natural number, and V_k be the set of sentences φ such that for all $\mathcal{L}_A$-structures $\mathcal{M} = (M, I)$, if M has exactly k elements then $\mathcal{M} \models \varphi$. Give an algorithm to determine when given a sentence ψ in $\mathcal{L}_A$ whether $\psi \in V_k$.

   Solution: First, we can effectively list a finite sequence of $\mathcal{L}_A$-structures of size k such that every $\mathcal{L}_A$-structure is isomorphic to one of these. Since isomorphic structures are elementarily equivalent, it is sufficient to check whether the given sentence ψ is satisfied by every structure in the list.

   Given a finite structure $\mathcal{M}$, a formula φ and a sequence $\vec{m}$ of elements of M, we can check whether $\mathcal{M} \models \varphi[\vec{m}]$ by recursion on the length of φ. We can evaluate terms in $\mathcal{M}$ by application of its given operations. This lets us tell whether $\mathcal{M} \models \varphi[\vec{m}]$ when φ is atomic. In the cases of negation or implication, the recursion step is immediate. Finally, when φ is $\forall x\theta$, $\mathcal{M} \models \varphi[\vec{m}]$ if and only if for all $m^* \in M$, $\mathcal{M} \models \varphi[m^*, \vec{m}]$, in which it is understood that x is to be assigned the value m^*. Since M is finite, this too can be checked in finitely many steps.

4. Let $A = \{c_i : i \in \mathbb{N}\}$.

   (a) Give an example of an $\mathcal{L}_A$ structure $\mathcal{M}$ such that

   $$T_{\mathcal{M}} = \{\varphi : \ \varphi \text{ is a sentence and } \mathcal{M} \models \varphi\}$$

   does not have the Henkin property.

   (b) Is there an example $\mathcal{M} = (M, I)$ such that $T_{\mathcal{M}}$ does not have the Henkin property and $\{I(c_i) : i \in \mathbb{N}\}$ is infinite?

   Solution: (1) Let $M = \{m_1, m_2\}$ and let $I(c_i) = m_1$ for all i. Then, $\mathcal{M} \models \exists x(x \neq c_1)$ but there is no constant c such that $\mathcal{M} \models c \neq c_1$. Thus, $T_{\mathcal{M}}$ does not have the Henkin Property.

   (2) No, there is no example $\mathcal{M} = (M, I)$ such that $T_{\mathcal{M}}$ does not have the Henkin property and $\{I(c_i) : i \in \mathbb{N}\}$ is infinite. Suppose that $\mathcal{M}$ is an $\mathcal{L}_A$ structure and $\{I(c_i) : i \in \mathbb{N}\}$ is infinite. Suppose that $\mathcal{M} \models \exists x\varphi$. It could happen that there is a constant c_i such that c_i appears in φ and $\mathcal{M} \models \varphi[x; c_i]$, which would be an instance of the Henkin Property. Otherwise, there is an $m \in M$ such that m is not the interpretation of any constant mentioned in φ and $\mathcal{M} \models \varphi[m]$. But then any permutation of the elements of M which fixes the interpretations of the constants that appear in φ is an automorphism of the structure with universe M and language the constants of φ. Thus, for every m^* in M, if m is not the interpretation of any constant of φ then $\mathcal{M} \models \varphi[m^*]$. Since $\{I(c_i) : i \in \mathbb{N}\}$ is infinite, there is such an m^* and a constant c_j such that $I(c_j) = m^*$. By the Substitution Lemma, $\mathcal{M} \models \varphi[x; c_j]$, as required to verify the Henkin Property.

5. Suppose that $\mathcal{M} \subseteq \mathcal{N}$ are infinite $\mathcal{L}_\emptyset$ structures. Show that $\mathcal{M} \preceq \mathcal{N}$.

   Solution: By Tarski's Criterion, it is sufficient to show that if A is nonempty and definable in $\mathcal{N}$ with parameters from $\mathcal{M}$, then A has an element in $\mathcal{M}$. Suppose that $A = \{n : \mathcal{N} \models \varphi[n, \vec{m}]\}$, where $\vec{m}$ is a sequence of elements from $\mathcal{M}$. If one of the elements of $\vec{m}$ belongs to A then $A \cap \mathcal{M}$ is not empty. Otherwise, by the analogous automorphism observation as in the previous problem, for every element n of $\mathcal{N}$ which is not in $\vec{m}$, $\mathcal{N} \models \varphi[n, \vec{m}]$. Since $\mathcal{M}$ is infinite, it contains infinitely many such elements and $A \cap \mathcal{M}$ is not empty, as required.

6. Give the proof to show that if, for every set of formulas Γ

$$\Gamma \text{ is consistent iff } \Gamma \text{ is satisfiable}$$

then, for every set of formulas Γ and every formula φ

$$\Gamma \vdash \varphi \text{ iff every } (\mathcal{M}, \nu) \text{ which satisfies } \Gamma \text{ also satisfies } \varphi.$$

Solution: Assume that for every set of formulas, that set is consistent iff it is satisfiable. Suppose that Γ is a fixed set of formulas.

First, suppose that $\Gamma \vdash \varphi$. Then,$\Gamma \cup \{(\neg\varphi)\}$ is not consistent. Thus,$\Gamma \cup \{(\neg\varphi)\}$ is not satisfiable. Hence, every $(\mathcal{M}, \nu)$ which satisfies Γ does not satisfy $(\neg\varphi)$, and thereby satisfies φ.

Second, suppose that every $(\mathcal{M}, \nu)$ which satisfies Γ also satisfies φ. Then, $\Gamma \cup \{(\neg\varphi)\}$ is not satisfiable. Hence, by assumption, $\Gamma \cup \{(\neg\varphi)\}$ is not consistent. Let θ be one of the logical axioms. Since an inconsistent set proves every formula,$\Gamma \cup \{(\neg\varphi)\} \vdash (\neg\theta)$. By the Deduction Theorem, $\Gamma \vdash (\neg\varphi) \to (\neg\theta)$. By the completeness theorem for propositional logic, Γ proves the contrapositive: $\Gamma \vdash \theta \to \varphi$. By noting that θ is an axiom and applying modus ponens, $\Gamma \vdash \varphi$.

Extra credit. Suppose that T is a set of sentences and that there is an $\mathcal{N} = (N, J)$ such that $\mathcal{N} \models T$ and N is infinite. Show that there is an $\mathcal{M} = (M, I)$ and an element a of M such that $\mathcal{M} \models T$ and a is not definable in $\mathcal{M}$ without parameters.

Solution: As in the proof of the Completeness Theorem, we may assume that there are infinitely many constant symbols which are not in the language of $\mathcal{N}$. By the Lowenheim-Skolem Theorem, we may assume that $\mathcal{N}$ is countable. Obtain $\mathcal{N}^* = (N, J^*)$ by augmenting $\mathcal{N}$ so that every element of N is the interpretation of some constant symbol and so that the constant symbol c is not used in the language of $\mathcal{N}^*$. Let Γ be as follows:

$$\Gamma = \{\varphi : \mathcal{N}^* \models \varphi\} \cup \{c \neq c_i : c_i \text{ a constant symbol interpreted in } \mathcal{N}^*.,\}$$

nn where φ is a sentence in the language of $\mathcal{N}$.

That Γ is finitely satisfiable follows from $\mathcal{N}$'s being infinite. Let $\mathcal{M}^*$ satisfy Γ. By definition of Γ, $\mathcal{M}^*$ and $\mathcal{N}^*$ are elementarily equivalent.

Let m be the interpretation of c in $\mathcal{M}^*$, and let $\mathcal{M}$ be the reduction of $\mathcal{M}^*$ to the language of $\mathcal{N}$ (same universe as $\mathcal{M}^*$ but interpreting only the symbols in the language of $\mathcal{N}$). Similarly, $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent.

Suppose that $\mathcal{M} \models \varphi[m]$. Then, $\mathcal{M} \models \exists x\varphi$ and so $\mathcal{N} \models \exists x\varphi$. Let n be an element of N such that $\mathcal{N} \models \varphi[n]$. Let c_j be such that n is the interpretation of c_j in $\mathcal{N}^*$. Then $\mathcal{N}^* \models \varphi[x; c_j]$, and so $\mathcal{M}^* \models \varphi[x; c_j]$. Since $\mathcal{M}^* \models c \neq c_i$, $\mathcal{M} \models \exists x_{j_1} \exists x_{j_2} (\varphi[x; x_{j_1}] \wedge \varphi[x; x_{j_2}] \wedge x_{j_1} \neq x_{j_2}$, for any x_{j_1} and x_{j_2} which are substitutable for x in φ. Thus, φ does not define $\{m\}$ in $\mathcal{M}$.