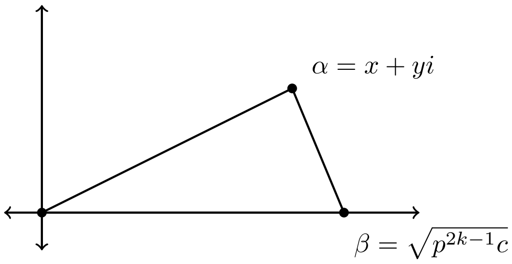


**Lemma 1.** Let $\mathcal{M}$ be a primitive integer norm pointset with characteristic $D$. If $p$ is a rational prime that is also prime in $\mathcal{O}_{-D}$, and $p$ divides the norm of some pairwise difference in $\mathcal{M}$, then $p$ divides it with even multiplicity.

*Proof.* Translate and possibly reflect $\mathcal{M}$ in the plane so that, of the two points whose pairwise distance's norm $p$ divides, one lies at the origin and the other lies on the positive $x$-axis. Denote the latter point by $\beta$.

We will assume, for an contradiction, that $p$ divides $[\beta]$ with odd multiplicity - that is, there exist some positive integers $c$ and $k$ such that $\beta = \sqrt{p^{2k-1}c}$, but $p$ does not divide $c$.

Now let $\alpha$ be any other arbitrary point in $\mathcal{M}$. We will show that $\alpha$ takes the form

$$\alpha = \frac{\delta\sqrt{p}}{2\sqrt{c}} \tag{1}$$

for some $\delta \in \mathcal{O}_{-D}$.

Since $[\alpha], [\beta]$, and $[\alpha - \beta]$ are all integers, we know that

$$[\alpha] - [\beta] - [\alpha - \beta] = \alpha\overline{\beta} + \overline{\alpha}\beta = 2x\sqrt{p^{2k-1}c} = a$$

for some $a \in \mathbb{Z}$ such that $x = \dfrac{a}{2\sqrt{p^{2k-1}c}}$.

The area of the triangle formed by 0, $\alpha$, and $\beta$ is

$$\frac{1}{2}y\sqrt{p^{2k-1}c} = \frac{b}{4}\sqrt{D}$$

for some $b \in \mathbb{Z}$ such that $y = \dfrac{b\sqrt{D}}{2\sqrt{p^{2k-1}c}}$.

Now, letting $\gamma = a + b\sqrt{-D} \in \mathcal{O}_{-D}$, we obtain the following expression for $\alpha$:

$$\alpha = x + y\sqrt{-D} = \frac{a + b\sqrt{-D}}{2\sqrt{p^{2k-1}c}} = \frac{\gamma}{2\sqrt{p^{2k-1}c}}.$$

Since $\alpha$ has integer norm, the norm of its denominator must divide the norm of $\gamma$. So $p^{2k-1} \mid \gamma\overline{\gamma}$. Since $p$ is prime in $\mathcal{O}_{-D}$, we must have either $p^k \mid \gamma$ or $p^k \mid \overline{\gamma}$. Since $p$ is rational, either case implies that $p^k \mid \gamma$. Thus there exists some $\delta \in \mathcal{O}_{-D}$ such that $\gamma = p^k\delta$, hence proving $\alpha$ takes the form given in (1).

First, assume that $p \neq 2$. Since $p \nmid c$, we must have $2c \mid [\delta]$, thus $p \mid [\alpha]$. Now, instead assume $p = 2$. Then $\alpha = \dfrac{\delta}{\sqrt{2c}}$, so $2 \mid \delta\overline{\delta}$, hence $2 \mid \delta$ since we have assumed $p = 2$ is prime. So $\alpha = \dfrac{\delta'\sqrt{p}}{\sqrt{c}}$ for some $\delta' \in \mathcal{O}_{-D}$, and again $p \mid [\alpha]$.

Since

$$[\alpha-\beta]=[\alpha]+[\beta]-\alpha\overline{\beta}-\overline{\alpha}\beta=[\alpha]+[\beta]-\sqrt{p^{2k-1}c}\left(\frac{\delta\sqrt{p}}{2\sqrt{c}}+\overline{\frac{\delta\sqrt{p}}{2\sqrt{c}}}\right)$$

$$=[\alpha]+[\beta]-p^k Re(\delta).$$

Since $p$ divides all three of these terms, it divides $[\alpha-\beta]$.

Now, since $\alpha$ was an arbitrary point in $\mathcal{M}$ other than 0 or $\beta$, we know if we take some other point $\eta\in\mathcal{M}$, $p$ will divide both $[\eta]$ and $[\eta-\beta]$. It will also be of the form $\eta=\dfrac{\varepsilon\sqrt{p}}{2\sqrt{c}}$ given in (1), thus

$$p\mid p\frac{[\delta-\varepsilon]}{4c}=[\alpha-\eta]$$

since $p\nmid c$ and, again, if $p=2$ then $p$ divides both $\delta$ and $\varepsilon$. Therefore, $p$ divides the norms of all pairwise differences in $\mathcal{M}$, contradicting our assumption that $\mathcal{M}$ is primitive.

□

**Lemma 2.** Let $\mathcal{M}$ be an integer norm pointset with characteristic $D$, and let $s$ be the greatest common divisor of the norms of all pairwise distances in $M$. The following are equivalent:

1. For every $\alpha\in\mathcal{M}$ there exists an ideal $L\subset\mathcal{O}_{-D}$, not necessarily unique, such that $\langle[\alpha]\rangle=L\overline{L}$
2. Every prime $p$ which is also prime in $\mathcal{O}_{-D}$ divides $s$ with even multiplicity. In particular, this occurs if $\mathcal{M}$ is primitive.

*Proof.* Scale $\mathcal{M}$ down by $\sqrt{s}$ to obtain a primitive integer pointset, and let $\alpha\in\mathcal{M}$. By Lemma 1, every rational prime $p$ which is also prime in $\mathcal{O}_{-D}$ divides $\frac{[\alpha]}{s}$ with even multiplicity, so factoring $\langle[\alpha]\rangle$ into prime ideals gives

$$\begin{aligned}\left\langle\frac{[\alpha]}{s}\right\rangle&=\langle p_1\rangle^{2k_1}\cdots\langle p_k\rangle^{2k_m}L_1\overline{L_1}\cdots L_n\overline{L_n}\\&=\left(\langle p_1\rangle^{k_1}\cdots\langle p_k\rangle^{k_m}L_1\cdots L_n\right)\left(\langle p_1\rangle^{k_1}\cdots\langle p_k\rangle^{k_m}\overline{L_1}\cdots\overline{L_n}\right)\\&=J\overline{J}\end{aligned}$$

where $J=\langle p_1\rangle^{k_1}\cdots\langle p_k\rangle^{k_m}L_1\cdots L_n$. Similarly, factoring $s$ into prime ideals gives

$$\langle[\alpha]\rangle=\langle s\rangle\left\langle\frac{[\alpha]}{s}\right\rangle=\langle q_1\rangle^{j_1}\cdots\langle q_k\rangle^{j_m}H_1\overline{H_1}\cdots H_n\overline{H_n}J\overline{J}.$$

This factorization becomes $L\overline{L}$ for some ideal $L$ if and only if each $j_i$ is even.

□

**Theorem.** $\mathcal{O}_{-D}$ contains an element of norm $[\alpha]$ if and only if $\langle[\alpha]\rangle$ has such a factorization where $L$ is principal for some $\alpha\in\mathcal{M}$.

*Proof.* Suppose assume the norm of some $\alpha \in \mathcal{M}$ has such a factorization where $L$ is principal, and let $\beta$ be a generator of $L$. Then $[\beta] = [\alpha]$.

For the other direction, assume $\beta$ has norm $[\alpha]$. Then $[\langle\alpha\rangle] = \langle\beta\rangle \langle\overline{\beta}\rangle$ gives such a factorization where $L$ is principal.

$\square$

**Corollary.** Suppose an integer norm pointset $\mathcal{M}$ has characteristic $D$, where $D$ is a Heegner number, and let $s$ be the greatest common divisor of the norms of all pairwise distances in $\mathcal{M}$. $\mathcal{M}$ embeds in $\mathcal{O}_{-D}$ if and only if every prime $p$ which is also prime in $\mathcal{O}_{-D}$ divides $s$ with even multiplicity.

In particular, every primitive integer norm pointset with characteristic $D$ embeds in $\mathcal{O}_{-D}$.

*Proof.* If some prime $p$ which is also prime in $\mathcal{O}_{-D}$ does not divide $s$ with even multiplicity, then no such factorization exists. So for every $\alpha \in \mathcal{M}$, $\mathcal{O}_{-D}$ contains no element of norm $[\alpha]$, thus $\mathcal{M}$ cannot embed in $\mathcal{O}_{-D}$.

Now suppose every prime $p$ which is also prime in $\mathcal{O}_{-D}$ does divide $s$ with even multiplicity. Then the norm of some $\alpha \in \mathcal{M}$ has such a factorization, and since $\mathcal{O}_{-D}$ is a principal ideal domain, $L$ is principal. So there exists an element of norm $[\alpha]$ for some $\alpha \in \mathcal{M}$. Thus $\mathcal{M}$ embeds in $\mathbb{Q}(\sqrt{-D})$. Since the square of every ideal of $\mathcal{O}_{-D}$ is principal, $\mathcal{M}$ embeds in $\mathcal{O}_{-D}$.

$\square$

The following proposition, proven by Dantong, is the missing direction of Lemma 3 in the paper.

**Proposition.** If $\mathcal{M}$ is primitive, then $\langle K_1, \ldots, K_n\rangle = \langle 1\rangle$.

*Proof.* Suppose some prime ideal $H \subset \mathcal{O}_{-D}$ divides $K_i$ for all i. We know that $\left\langle \left[\frac{\beta_i}{r}\right] \right\rangle = \langle [K_i]\rangle = K_i\overline{K_i}$ for all $i$. By Lemma 1, $\left\langle \left[\frac{\beta_i}{r}\right] \right\rangle = L_i\overline{L_i}$ for some ideal $L \subset \mathcal{O}_{-D}$ as well. So, for all $i$, $H_i \mid L_i$ or $H_i \mid \overline{L_i}$. In either case, $\langle [H]\rangle = H\overline{H} \mid L_i\overline{L_i} = \left\langle \left[\frac{\beta_i}{r}\right] \right\rangle$ for all $i$. Thus $[H] \mid \left[\frac{\beta_i}{r}\right]$ for all $i$.

$\square$