Problem Set 6, Math 172 Spring '15

This problem set is due Tuesday, March 3rd, 2015 at **the beginning of class**. All class guide rules apply. "VLW" refers to Van Lint and Wilson, our textbook. **All questions, when graded, are worth an equal number of points unless stated otherwise, though not all questions would end up being graded.**

The theme for this problem set is "midterm review," as such, the length/difficulty is different from usual, focusing more on fundamentals. You should be in good shape if these are easy for you.

1. Show, *with generating functions*, that there is exactly one way to represent each non-negative integer as a sum of *distinct* powers of 2. (i.e. there is a unique binary expansion of each integer) Hint: you should probably write an expression $F(x)$ that captures the idea of choosing distinct powers of 2 such that every such choice contributes to x^n exactly once. If so, this expression should equal $1 + x + x^2 + x^3 + \cdots$, which has coefficient 1 for every x^n, completing your proof.

2. Find a closed-form formula for the Lucas numbers, defined by $L_1 = 2$, $L_2 = 1$ and the recursive formula $L_{n+2} = L_{n+1} + L_n$.

3. Practice your big O's!

   (a) Find the problem with the following argument: "Since $kn = O(n)$ for all fixed k, $\sum_{k=1}^{n} kn = \sum_{k=1}^{n} O(n) = O(n^2)$."

   (b) Find your best big O approximation of the number of perfect matchings on K_{2n} (here we don't have bipartiteness, so I just mean matchings that use all the vertices exactly once).

4. Fix positive integers n and k, and let $S = [n]$. Find the number of k-lists $(T_1, T_2, \ldots, T_k)$ of subsets T_i of S such that for every $i < j$, $T_i \subset T_j$.

5. Find the number of ways to sit n couples (which makes $2n$ people) at a round table (with distinguishable seats) such that (a) the two people in every couple sit next to each other; (b) the two people in no couple sit next to each other.

6. Let G be a graph such that all its vertices have degree 2. Prove that G is a union of pairwise disjoint cycles.