Lazy Evaluation in Haskell

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Abstract

Many modern-day programmers generally prefer programming in imperative languages such as python, C, or C++. After learning how to use one language effectively, one can swiftly start working with other languages because the thought process and method of answering problems are very similar. Little do they know that there actually exists another paradigm known as functional programming in which the goal is to bind everything in clean mathematical functions similar to what one would see in a mathematics course. In this way, programs are built with function application and and function evaluation. Then to break it down even more, in languages such as Haskell, everything is an expression, and the goal is to evaluate them in order to solve a problem. In this paper, we will observe the strategies for evaluating these expressions and introduce Haskell's default: lazy evaluation. Further more, we go into a detailed investigation of parsing and the compilation process as a whole. At the end of the report, we implement our own data structures and algorithms using LambdaFun.

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Part I
Lazy Evaluation in Haskell

1 Introduction

1.1 General Remarks

For this paper, I will assume that the reader has a basic familiarity with Haskell, but if you are new to Haskell, do not worry! I will try to explain everything as clear as possible and work with simple examples. The first step before I get into anything is to make sure that you at least have Haskell installed. Haskell can be downloaded using the instructions from Haskell's official webpage. Assuming everything installs smoothly, we can get started, but for any problems be sure to check out this website. In this paper, we will also be using GHCI to create interactive environments to run Haskell functions. Simply type ghci into your terminal and the interactive environment will open up. To familiarize yourself with GHCI, I recommend looking at this website

1.2 Key Points

Imperative languages are usually composed of statements and expressions. An expression such as `int x = 5 * 2` is evaluated and modifies memory via an assignment operator to create a value. Expressions are entities that can be reduced into a value. On the other hand, statements are units of execution such as conditionals and loops that do not return values, but instead compute by using side effects. A sequence of statements determines how to answer a certain problem. However, functional languages have programs that are built with function application and evaluation. In languages such as Haskell, everything is an expression, and the goal is to evaluate them in order to solve a problem. Haskell defaults to a method of evaluation known as lazy evaluation. Put simply, Haskell is lazy. It won't execute functions and calculate things until it's really forced to show you a result [LYAH]. Haskell uses lazy evaluation to evaluate expressions because it comes with several advantages.

- Function arguments are not computed before invocation
- Expressions are computed only if their value is necessary and it avoids repeated evaluation of expressions, thus making algorithms more efficient and reducing computation complexity
- Allows the ability to program with infinite lists
- Promises termination when possible

We will see examples of lazy evaluation towards the end of part 1 in the report. The goal of this section is to understand what lazy evaluation really is, what makes Haskell lazy, and the advantages of this evaluation strategy. First, we will need to do a quick dive into lambda calculus because it is the building block of purely functional programming. Then, we will use lambda calculus as the key to understanding the difference between two evaluation strategies for reducible expressions (redex). Finally, building upon these foundations, we will work on several examples of lazy evaluation.

2 Lambda Calculus

Before looking into how lazy evaluation is used in Haskell and its advantages, we first need to understand the idea behind lambda calculus. The motivation for looking at lambda calculus is that it elegantly demonstrates how expression evaluation can simply be seen as a sequence of reduction steps. This is also popularly known as rewriting, where we reduce/rewrite an expression into a simpler/shorter form. This process is repeated by any number of steps until we reach the normal form. An expression is in normal form when it cannot be reduced anymore. **More concretely, a lambda expression is in normal form if it contains no more possible redexes**. We will see how lambda expressions open up to two potential methods of expression evaluation: Normal-order evaluation and applicative-order evaluation.

2.1 What is Lambda Calculus?

Lambda calculus, also denoted as λ-calculus, is a convenient notation for functions and applications. What is a function? In mathematics, a function is something that relates an input to output. Say we want to write a function that squares an input, we can do this by $f(x) = x^2$. f is the function name, x is just an placeholder for the input, then x^2 is what we need to output. Now lets look at some examples of function in programming languages. In python, a similar function is written as

```
def square(x):
    return x*x
```

Figure 1: *A python function that squares an input*

Similarly, we can create a function that does the same thing in Haskell

```
square :: Int -> Int
square x = x*x
```

Figure 2: *A Haskell function that squares an input*

Remark: Before getting into it, lets go over this Haskell code first. The purpose of the first line

$$\text{square :: Int} \rightarrow \text{Int}$$

is to give our function *square* an explicit type declaration. The :: symbol is read as "has type of". Then the first parameter is the argument type that gets passed into our function, which in this case is an *Int*. The next parameter is the "return type" of the function, which in this case is also an *Int*. Without going into more advanced detail, there is no special distinction between the parameters and the return type [LYAH]. Therefore , the function type declaration can be read as: *function square takes in an argument of type Int for the and outputs an Int.* Declaring the types for your functions is not necessary, but it makes your code easier to read and understand. Also, I chose the argument to be of type *Int* just for demonstrations purposes. I could have chosen another type such as *float*. For more references about various types in Haskell, check out Tutorials Point.

The second line

$$\text{square x = x*x}$$

is the function definition. We define a function *square* that takes an argument *n* and returns the square of that number, *n*n*.

In both languages, the function is given a name, *square*. Both functions take in an input which we call x, but this variable name is just a placeholder and we can realistically call it whatever we want and the function will not change. Now to apply a value to compute with the function, in Haskell we can say: *square 5 reduces to 5*. We will use this notation:

$$\text{square 5} \rightarrow 25$$

Here , we are merely substituting 5 for variable x in function square. You might be wondering, why I am going over such simple high school algebra? Well to understand lambda calculus, we had to break down the basic components of a function. The main difference between what we've just seen and λ-calculus is that we can write functions without giving them explicit names, but the idea of applying a function to an argument and forming functions by abstraction still holds. Woah, that was a lot, let's see how this same function can be expressed in terms of λ-calculus by breaking down a quick example.

$$\begin{aligned}(\lambda x.x * x)5 &= 5 * 5 \quad [reduction] \\ &= 25\end{aligned}$$

In this examples, abstraction serves as the square function, represented by $\lambda x.x * x$ and the application is represented by applying the function $\lambda x.x * x$ to the argument 5. In essence, we are substituting every occurrence of x with 5 in the lambda term. What is this computation of reduction more specifically? Let's define this as the β-reduction rule.

2.2 β-Reduction Rule

Let's look at the general definition for reducing lambda expressions.

- **β-reduction rule** If λx.M is a lambda term and N is another lambda term, then $(\lambda x.M)N \rightarrow M[N/x]$ that for the term M, each occurrence of x is substituted by N.

This idea of using substitution to reduce an expression, is very fundamental in lambda calculus and the theoretical building blocks of functional programming. As stated earlier, we use the β-reduction rule until we read normal form where we cannot reduce the redex any further.

3 Reduction Strategies

Reduction strategies can also be thought of as evaluation strategies. This implies that at each step during expression evaluation unless we are in normal form, there may be an expression we can reduce ti by applying a reduction definition. Let's look at a lambda expression that will apply a lambda term to another lambda term. $(\lambda x.x)((\lambda y.y)a)$. This expression can be reduced with two possible strategies.

$$\begin{tikzcd}
& (\lambda x.x)((\lambda y.y)a) \arrow[dl] \arrow[dr] & \\
(\lambda y.y)a \arrow[dr] & \text{or} & (\lambda x.x)a \arrow[dl] \\
& a &
\end{tikzcd}$$

Figure 3: *Possible Reduction Strategies*

On the right-hand branch, we are evaluating the second lambda term

$$(\lambda y.y)a \rightarrow a$$

Then we apply this result, a into

$$\lambda x.x$$

in order to evaluate to the final form a. In programming, this method of evaluation is known as *call by value*, and more generally as "innermost reduction" or "applicative-order reduction".

On the left-hand branch, we are evaluating the first lambda term by substituting the whole right lambda term. We are basically replacing the variable x in

$$\lambda x.x$$

with the lambda term

$$(\lambda y.y)a$$

In programming, this method of evaluation is known as *call by name*, and more generally it is known as "outermost reduction" or "normal-order reduction." This strategy of evaluation is what we are more interested in because this is the essential idea behind lazy evaluation in Haskell.

3.1 Outermost Reduction Strategy

I like to think of this "meta-rule" as the "outermost reduction strategy" instead of the "normal-order reduction rule" because innermost reduction is more conceptual. Notice that I am saying "outermost reduction strategy" not "outermost reduction rule" because it is not a rule of lambda-calculus but rather a "meta-rule", a rule about how to apply the rules. As previously mentioned, a redex is a reducible expression. So if we have an expression composed of two or more redexes, we reduce the leftmost first before reducing the sub-expressions inside of it. This can be seen in the example we had above where we have two possible redexes:

$$(\lambda x.x)((\lambda y.y)a)$$

For outermost reduction, we reduced the leftmost redex $(\lambda x.x)$ by applying/substituting the innermost redex $(\lambda y.y)a$ into the outermost redex. Furthermore, we are delaying the evaluation of the innermost redex (argument for the outermost redex) until we actually need to compute the expression. Functional languages such as Haskell follow this evaluation strategy. This makes Haskell lazy because it does not compute the argument until it finally has to. Remark, that not all functional programming languages evaluate in this fashion. Languages such as ML and OCAML implement call by value.

3.2 Innermost Reduction Rule

Innermost reduction (applicative-order reduction) is basically the opposite of outermost reduction. The innermost redex is reduced first, so essentially function arguments are evaluated first before they are substituted into the function. Functions in languages most imperative languages such as python and C/C++ follow this strategy.

3.3 Comparing Both Reduction Rules

Lets look at an example in Haskell demonstrating these two different possibilities of evaluation. We are going to compute $(2+3)^2$ using a Haskell function name *square*

```haskell
-- square an int
square :: Int -> Int
square n = n * n
```

Outermost	**Innermost**
$square(2+3)$	$square(2+3)$
$= (2+3) * (2+3)$	$= square(5)$
$= 5 * (2+3)$	$= 5 * 5$
$= 5 * 5$	$= 25$
$= 25$	

Now lets look at the first column with outermost evaluation. In this method, we delay the evaluation of the argument. Instead, we reduce the outermost redex, which is the *square* abstraction. Following the definition of *square* above, we substitute $(2+3)$ into the variable n, then perform the squaring step of $n * n$

Next, lets observe the second column with innermost evaluation. In this method, we do not delay the evaluation of the argument. The first step in fact is to compute the expression $(2+3)$. Basically, the argument is reduced before it is substituted into the function. This sort of reduction is what most programmers are used to. It is intuitive and generally preferred.

How can we compare the two methods? The first thing we can notice is that both strategies end up with the same result. This is an important note to make because in functional programming, if there are no side effects, we are guaranteed to end up with the same results from both evaluation methods. Also we can see that innermost evaluation reduces the entire expression in the three steps, compared to the four steps taken by outermost reduction. In outermost reduction, we can see that we end up having to reduce the same expression $(2+3)$ on two separate occasions. That being said, it seems like innermost evaluation is more efficient and doesn't duplicate expressions. We will observe how Haskell implements CBN without duplicating expressions using sharing.

Lets look at another example when outermost reduction terminates while innermost reduction fails to terminate. In the example `false(decr(10), 0)`, with reduction rules `false(a,b) -> b` and `decr(a) -> decr(a-1)` on integers. Innermost reduction fails to terminate because it will evaluate the innermost function first: `decr(10)`. This will be infinite because `decr(10)` will evaluate to `decr(10) -> decr(9) -> decr(8)` and so on. If we approach this expression with outermost reduction, we will evaluate to 0 in a single rewriting step [Gnaedig].

Up until now, I have been building up a connection between outermost reduction and lazy evaluation. The two really sound the same right? Earlier I said that languages such as Haskell that default to lazy evaluation, do not compute function arguments before invocation. This sounds almost exactly the same as the strategy of outermost or normal-order expression evaluation. So this raises the question, if outermost evaluation is more inefficient because it needs to reduce the same, duplicate expression multiple times, then why should we care about it or lazy evaluation? Have I been wasting your time talking about lazy evaluation? The short answer is no... We have been building up how outermost reduction is essentially lazy evaluation, however, we are missing an extra ingredient. This ingredient is known as argument sharing or memoization. Argument sharing stores a value of a function to prevent re-computation. This is an essential ingredient for lazy evaluation because lazy evaluation avoids repeated evaluation.

4 Lazy Evaluation

First lets look at the example in the previous section and incorporate argument sharing.

Outermost

$$\begin{aligned} & square(2+3) \\ &= (2+3)*(2+3) \\ &= 5*5 \\ &= 25 \end{aligned}$$

The example of reductions doesn't perfectly illustrate how re-computation is prevented, but I will try to clarify it. The important difference is found on the 2nd step. Unlike the outermost reduction example in the previous section, once we compute the first $(2+3)$ expression, we can automatically reduce both occurrences of the expression in one step. As a result, we have fully reduced the expression in the same number of steps as inner reduction.

4.1 What is Lazy Evaluation?

Lazy evaluation can be thought of as outermost reduction + argument sharing. Thus by combining these two things, lazy evaluation is an evaluation strategy which holds the evaluation of an expression until its value is needed and it avoids repeated evaluation [TP]. With lazy evaluation, Haskell evaluates a value when it is needed then updates all copies of that expression with the new value.

4.2 How is Haskell Lazy and What Are The Advantages?

The first advantage of lazy evaluation in Haskell is that expressions are computed only if their value is necessary in this context. Lets look at a couple examples regarding this point. We will create a similar function in python and in Haskell to demonstrate this advantage. The first function takes in two arguments and just returns the first one. First we will look at the python code

```python
def func(x, y):
    return x
```

Figure 4: *A python function that returns the first argument*

Now lets write the same function in Haskell:

```
func :: Int -> Int -> Int
func x y = x
```

Figure 5: *A Haskell function that returns the first argument*

Both functions take in two arguments and return the first one. Pretty simple right? It seems as if the function in both languages will always return the same thing, but it turns out they don't.

For example, lets set our arguments to these values

$$x = 5$$

$$y = (5/0)$$

Then lets run both function as see what we get. First lets try python

```
>>> def func(x, y):
...     return x
...
>>> func(5, 5/0)
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
ZeroDivisionError: division by zero
```

Figure 6: *Division by zero error in python*

Oh no! What happened here? If you are an experienced programmer, you would no that the result of dividing anything by 0 is undefined, and thus printing an error. What we did was pass (5/0) as the argument for y, and since uses applicative/innermost reduction, it first computes the value of the argument before passing it into the function body. Thus, it computes (5/0) and throws an error before we ever get into the actual function.

Lets now try the same function in Haskell! [1]

```
Prelude> func x y = x
Prelude> func 5 (5/0)
5
```

Figure 7: *Lazy Evaluation example in Haskell with no division by zero error*

WOAH! It works like magic! Why did this not result in a divide by zero error? The reasoning is that in lazy evaluation, the computation of arguments is delayed. Also, expressions are evaluated only if there value is necessary. Turns out that the second argument never got evaluated, since it was not needed in the function body! This example is very basic and simple, but it demonstrates how only necessary expressions are evaluated. This can significantly reduce the time complexity of an algorithm because it won't do unnecessary work. This can be further demonstrated by the next example

[1]If you installed Haskell properly, you can open up an interactive environment that runs Haskell code

```
func :: Int -> Int -> Int
func x y
    | x < 10 = x
    | otherwise = y
```

Figure 8: *Lazy Evaluation example in Haskell*

Now, the function body consists of a notation that we haven't seen before, lets break it down before we analyze what this function does. The pipe characters, | are known as guards. Guards are a way to see whether some expression is true or false. If its true, then the function will return whatever is on the right of the equals sign. It is very similar to if-else statements. Check out this website for more information about guards. So now we can see that this function takes in two integer arguments, then checks to see if the first argument is less than 10; if it is, then it returns the first argument. If the first argument is greater then 10, then it will return the second argument. Lets try this function with various inputs.

$$a = sum[1, 2, 3]$$

$$b = sum[1..]$$

Sum is a built in Haskell function that sums of the values in a list. The second list includes two dots (..) which creates an infinite list from $1 \to \infty$. (Side note, thanks to lazy evaluation, Haskell lists can be infinite)

Ok, now when we pass these arguments to the function $a \to x$ $b \to y$. The function returns 6 almost instantly. Again, since the first argument x is less than 10, we did not have to compute the second argument. The second argument is the sum of an infinite list which is impossible to compute. The function would have never terminated if it wasn't for lazy evaluation. Lets see what happens when we swap arguments

$$a = sum[1..9999999]$$

$$b = sum[1, 2]$$

In this case however, we are not summing from $1 \to \infty$ but instead, to an arbitrarily large number. After running this you will see that there will either be a stack overflow because the computation requires too much memory, or it will take a VERY long time to compute. I highly recommend trying this on your own computer to see how much longer the second case actually runs.

In the previous example, I briefly touched upon lazy evaluation supports programming with infinite list, thus in Haskell, we are able to define infinite lists. This is because using lazy evaluation, expressions are only evaluated as much as required to produce the final result. Therefore, when declaring a list that has infinite length, Haskell doesn't actually create or initialize this list. It really defines a potentially infinite list that only evaluates as much as required by the given context [GHutton]. This is because only the specific components of that list that we need will actually be allocated in memory, not the whole thing.

```
Prelude> func x y | x < 10 = x | otherwise = y
Prelude> a = sum [1,2,3]
Prelude> b = sum [1..]
Prelude> func a b
6
```

Figure 9: *Output of a Haskell function*

5 Conclusions

In this part, we observed the general ideas of functional programming languages. In languages such as Haskell, everything is an expression, and the goal is to evaluate them in order to solve a problem. Haskell defaults to a method of evaluation known as lazy evaluation. Put simply, Haskell is lazy. It won't execute functions and calculate things until it's really forced to show you a result [LYAH]. Haskell uses lazy evaluation to evaluate expressions because it comes with several advantages.

- Function arguments are not computed before invocation
- Expressions are computed only if their value is necessary and it avoids repeated evaluation of expressions, thus making algorithms more efficient and reducing computation complexity
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Part II
Theory

6 Parsing

6.1 Introduction to Parsing

The way all of our computers perform computation is by manipulating 0's and 1's. Every computer handles images, words, data, programs and instructions using only a sequence of 0s and 1s, aka binary sequence. Everything that we see or use on our computers is represented via binary sequences. When you write code using a high-level languages like Haskell or C++, the code is eventually translated into binary. This translation is done by a compiler. You might be wondering what is a compiler? Well if we first look up the definition of compilation, we see that it is *"the action or process of producing something"*. This generic definition is very similar to what a compiler does in your computer. In computing, a compiler is a computer program that translates computer code written in one programming language into another language. The reason we need the compiler is because trying to write programs in binary machine language is almost impossible for a human. Therefore, high-level languages are used because they are more human readable, but this comes at the cost of compiling that high-level language down to a low-level language.

The compiler is split into several phases, called Compilation Phases. The phases can be grouped up into 4 main components: Lexer, parser, Type Checker, and then Code Generator. We will be focusing on parsing, but I will introduce each phase briefly.

- **Lexer**: The lexer is the first step. It reads in the raw string input of characters and converts it into a sequence of characters
- **Parser**: The parser then reads in the sequence of tokens and groups it into a syntax tree. This syntax tree gives a structural representation of input by grouping related parts together.
- **Type Checker**: The Type Checker verifies the types used in the syntax tree.
- **Code Generator**: The Code Generator converts the annotated syntax tree into a list of target code instructions[IPL].

Parsing is the second phase in the compiler. It determines and analyzes if the input tokens conform to the particular language grammar. This grammar is a system of rules describing the language.

6.2 Parsing Example

Languages are defined by their grammars. For programming languages, grammars just give the rules for combining smaller components into expressions, statements, and programs. Lets look at an example of a very basic grammar so that we can learn several things about context free grammars, precedence levels, and parsing. Like we have been doing in class, we will create a grammar using BNFC. BNFC is a language to define context free grammars. Then a parser generator takes in the context free grammar as input and produces an output as a parser. Make in new directory, call it `parsing_examples`. Then in that new directory create another one called `/simple_parse`. In this new directory type the following grammar into a file named `PlusTimes.cf`

```
Plus. Exp ::= Exp "+" Exp1 ;
Times. Exp1 ::= Exp1 "*" Integer ;
EInt. Exp1 ::= Integer ;

coercions Exp 1 ;
```

Figure 10: *Simple grammar for adding and multiplying*

This grammar support the language consisting of expressions; we can add or multiply expressions together. Each line in this grammar is a rule. The first word in each line: Plus, Times, and EInt, are the labels or names for each rule. The right hand side can be thought of as the concrete syntax that consists of terminals, and non-terminals. The terminals are the symbols arithmetic in quotation marks, while the non-terminals are the identifiers, such as Exp and Exp1.

Lets run this file using BNFC: `bnfc -m --haskell PlusTimes.cf`. This command generates lots of files, most of which we don't need to worry about. Then run the command `make` for compiling. Hopefully no errors were thrown. Lets test out out parser to see how it performs. Run the executable `./TestPlusTimes` on an example input $5 + 4 * 3$.

```
$ echo "5 + 4 * 3" | ./TestPlusTimes

Parse Successful!

[Abstract Syntax]

Plus (EInt 5) (Times (EInt 4) 3)

[Linearized tree]

5 + 4 * 3
```

Figure 11: *Simple abstract syntax tree*

We have seen in class how the Abstract Syntax Plus (EInt 5) (Times (EInt 4) 3) in reality is a tree. The parenthesis determine the structure of the tree. This brings us to a very important observation to remember. In essence, parsing just puts parenthesis in their correct positions.

Did the parser parse our example expression correctly? Well, from elementary math we know that with order of operations, our expression can also be seen as $5 + (4 * 3)$ where we are adding 5 to the product of $4 * 3$. Our parser does in fact do this correctly because our rule for plus takes two expressions,

$$\text{(EInt 5)} \quad \text{and} \quad \text{(Times (EInt 4) 3)}$$

So as stated above, our parser knows that we want to add 5 to the product of $4 * 3$ instead of say

Times (Plus (EInt 5) (EInt4)) (EInt 3)

In mathematics that is

$$(5 + 4) * 3$$

How did our parser complete this seemingly hard task? Why did it not add first then multiply? Lets run through what it did step by step. After running the make command, a file named *ParPlusTimes.info* was created. This file is a readable info file where we can see all of our rules in the grammar, terminals, non-terminals, and all of the possible states. These states are what is know as a deterministic finite state machine, which we will cover more in Compiler Construction. Our parser reads the input one token at a time from left to right, like a human would. So how does it know not to add right away and instead wait for multiplication before it has seen it yet? **EDIT**: The trick has to do with pattern matching, the precedence levels in the grammar, and the finite state machine. Our parser is called a LALR(1). This is an abbreviation for *Look-ahead left-to-right parsing, rightmost derivations*. This means that we are parsing from left-to-right and building a stack of results. This stack is then combined afterwards when a grammar rule can be applied to the stack. It is a look-ahead parser because it looks at the next token in the input and decides what it should do given its current state and stack. At first, this process of parsing does sound very hand-wavy, but the parser follows a strict algorithm that we can follow. We will see an example, step by step of how the parser actually reads in the input. The goal of this example is to demonstrate how the parser makes a decision at each and every step. I think seeing this explicitly, helps you understand the inner workings; which will then give you a better picture of what is going on.

Before jumping into the example, open up the *ParPlusTimes.info* file and observe it. For our scope, we will just focus on the **States** section. The parser is a finite state machine that goes from state to state until eventually it **accepts** the input string, or **rejects** if the input is invalid and doesn't follow the grammar. We start at state 0. In the figure below, the left column is the next token that our *Look-ahead* sees. Depending on what that next token is, we complete the rule in the left column. L_integ is the token that BNFC gives for every integer input. The reason for this is because there are infinite number of integers, and our parser cannot make a rule for each unique integer. The action is the same regardless of the magnitude of the integer, so we treat all of them the same.

```
State 0
# NO PREVIOUS INPUT
   '('          shift, and enter state 6
   L_integ      shift, and enter state 3

   Integer      goto state 4
   Exp          goto state 7
   Exp1         goto state 8
```

Figure 12: *State 0*

Next say we are in state 3. A new line exists right beneath the line *State 3*. This line tells us what is on the top of our stack, and what we should reduce to. The token to the left of the period, is what we have already seen, then what is after the period (nothing in this state) is what token is next. The left hand side of the arrow is what we will reduce to given the action we choose.

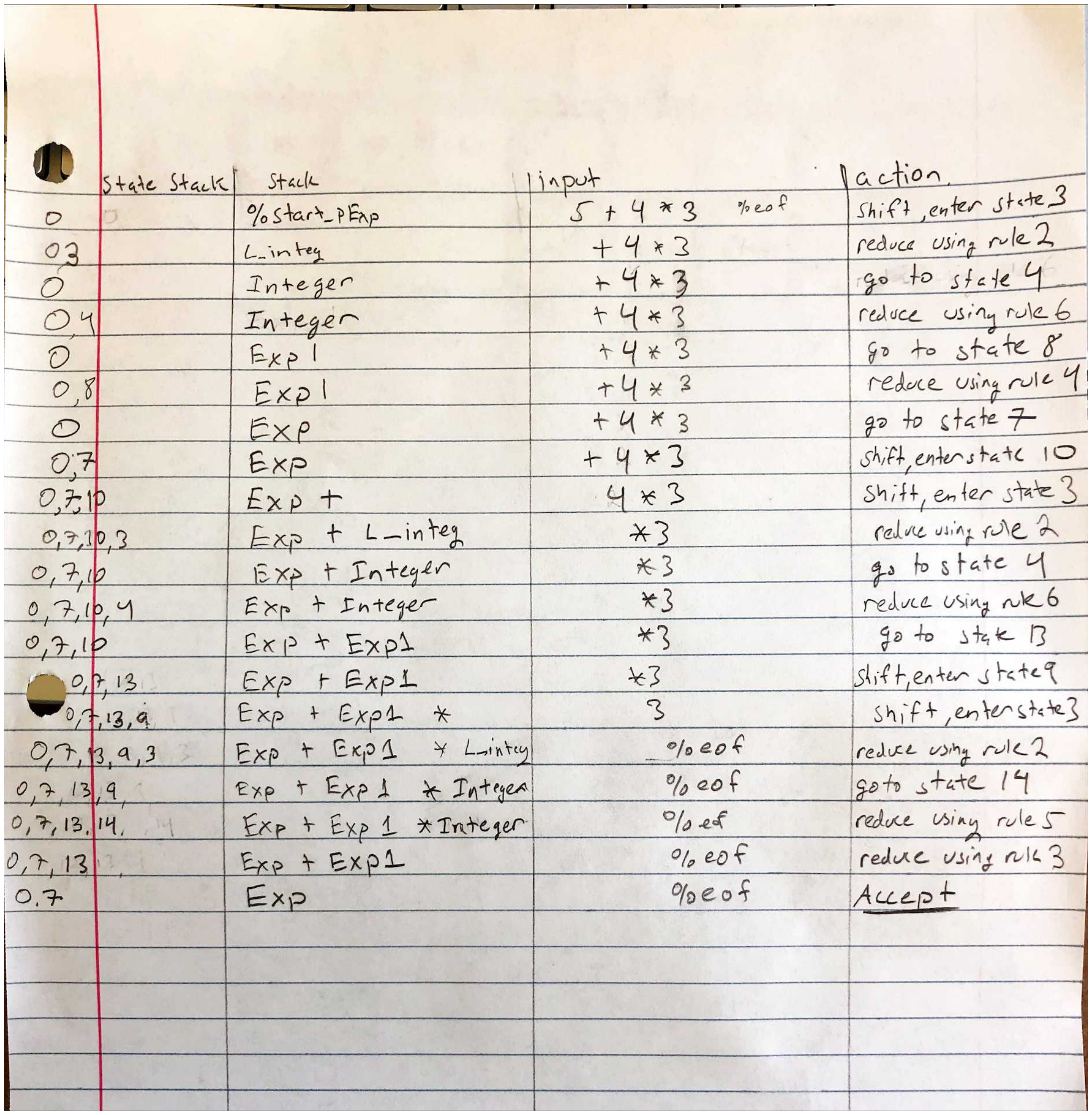
```
State 3

  Integer -> L_integ .                          (rule 2)

  ')'          reduce using rule 2
  '*'          reduce using rule 2
  '+'          reduce using rule 2
  %eof         reduce using rule 2
```

Figure 13: *State 3*

Now since we know the basics, lets run through an example of parsing an input string. In the image below, is the process that the parser follows. The **state stack** is the states that we have visited which we keep on the stack. The **stack** is the built result of what our parser has seen. The **input** is what is left to parse. The **action** describe what action we need to perform on the stack and input. We will attempt to parse the string $5 + 4 * 3$. Keep your .info file open and follow along.



State Stack	Stack	input	action
0	%start_pExp	5 + 4 * 3 %eof	Shift, enter state 3
0,3	L_integ	+ 4 * 3	reduce using rule 2
0	Integer	+ 4 * 3	go to state 4
0,4	Integer	+ 4 * 3	reduce using rule 6
0	Exp1	+ 4 * 3	go to state 8
0,8	Exp1	+ 4 * 3	reduce using rule 4
0	Exp	+ 4 * 3	go to state 7
0,7	Exp	+ 4 * 3	shift, enter state 10
0,7,10	Exp +	4 * 3	shift, enter state 3
0,7,10,3	Exp + L_integ	*3	reduce using rule 2
0,7,10	Exp + Integer	*3	go to state 4
0,7,10,4	Exp + Integer	*3	reduce using rule 6
0,7,10	Exp + Exp1	*3	go to state 13
0,7,13	Exp + Exp1	*3	shift, enter state 9
0,7,13,9	Exp + Exp1 *	3	shift, enter state 3
0,7,13,9,3	Exp + Exp1 * L_integ	%eof	reduce using rule 2
0,7,13,9,	Exp + Exp1 * Integer	%eof	goto state 14
0,7,13,14,	Exp + Exp1 *Integer	%eof	reduce using rule 5
0,7,13	Exp + Exp1	%eof	reduce using rule 3
0.7	Exp	%eof	Accept

Figure 14: An image parsing performed on 5 + 4 * 3

We start at state 0 and slowly build up our stack until we are at the end of the input. Then we can accept and the parsing was a success! How great! We continue reading off the input until we are at the end - %eof. Then at this point we will accept. You can see here how such a simple input string takes about 20 steps to parse. The parsing time complexity is on the order of $O(n^3)$. Think of what it would be like if we had to parse a whole file with thousands, or even millions of lines. For one, it would be ridiculously long and inefficient. However, many programming languages attempt to parse in a linear time complexity[IPL].

6.3 Unambiguous Grammars

When creating grammars, the goal is to make them unambiguous. We do not want grammars to have ambiguity because it will lead to inefficiency, and non-determinism. We want our algorithm to be deterministic and not produce different results. We want to guarantee uniqueness. Also, we want to make sure that the parser actually uses the correct rules so that when the code is executed (arithmetic) the correct result is produced. How can we tell if our grammars have ambiguity? We can tell there is ambiguity when we have conflicts. I wont go into too much detail into shift-reduce conflicts or reduce-reduce conflicts, but what they are telling us is that the algorithm has several actions it can take at a certain step. It has several options that it could choose from because each one seems correct, given the grammar. Lets change up our grammar to make it ambiguous. Lets make every value-category and Exp non-terminal have precedence level, Exp1

```
   Plus. Exp1 ::= Exp1 "+" Exp1 ;
   Times. Exp1 ::= Exp1 "*" Exp1 ;
   EInt. Exp1 ::= Integer ;

   coercions Exp 1;
```

Figure 15: *Ambiguous grammar for adding and multiplying*

Look at the .info file and observe these conflicts in state 13, state 14.

```
State 13

  Exp1 -> Exp1 . '+' Exp1                       (rule 3)
  Exp1 -> Exp1 '+' Exp1 .                       (rule 3)
  Exp1 -> Exp1 . '*' Exp1                       (rule 4)
  ')'          reduce using rule 3
  '*'          shift, and enter state 9
      (reduce using rule 3)
  '+'          shift, and enter state 10
      (reduce using rule 3)
  %eof         reduce using rule 3

State 14

  Exp1 -> Exp1 . '+' Exp1                       (rule 3)
  Exp1 -> Exp1 . '*' Exp1                       (rule 4)
  Exp1 -> Exp1 '*' Exp1 .                       (rule 4)
  ')'          reduce using rule 4
  '*'          shift, and enter state 9
      (reduce using rule 4)
  '+'          shift, and enter state 10
      (reduce using rule 4)
  %eof         reduce using rule 4
```

Figure 16: *Ambiguous grammar for adding and multiplying*

The parser has several options to choose from. Either it wants to shift depending on the next terminal, or it wants to reduce the whole expression. Why is our grammar ambiguous now? Think about what we changed, what could have caused this? In our case, the answer is in precedence levels. Usually when we have an ambiguous grammar and we want to make it unambiguous, then we either have too many rules or not enough. But in our case, precedence levels are an important part when building an unambiguous grammar. Precedence levels help distinguish which expressions have higher precedence over the other. In our simple arithmetic examples, precedence levels can be thought of as order of operation. From grade school, we know that multiplication is performed before addition. That is built into our brains now so much that we do not have to think about it. How can we make sure our parser follows the same order of operations? How do we force the parser to to perform multiplication before addition, even if multiplication comes after the addition in the input. We need to precedence levels. Precedence levels are the digits attached to category symbols, such as Exp1 in our example[IPL]. Higher precedence levels for expression tell us that we need to perform the higher level first. They regulate the order of parsing, and associativity.

Precedence levels consist of three main principles

1. All precedence variants of a nonterminal denote the same type in abstract syntax
2. Expression of higher level can always be used on lower levels
3. Any expression can be lifted to highest level if you add parenthesis[IPL]

The second principle is very important, and we have seen it in action in the parsing example up above. When we couldn't match a pattern for Exp1, we reduce Exp1 to Exp.

When making rules in our grammar, usually the higher precedence is to the right of the terminal. Also, usually it is left associative were the left non-terminal should be the same as the value-category. This makes because think back to the principles of precedence levels, *Expression of higher level can always be used on lower levels*. If there are no patterns for the higher level, then we can drop a level down and see what new patterns we are able to match. Also, this plays an important part in the order of operations in our grammar. Our addition rule has the higher level non-terminal Exp1 on the right.

```
  Plus. Exp ::= Exp "+" Exp1 ;
```

Say we had to parse our example $5+4*3$. Before reducing our addition expression to just an Exp (adding 5 + 4), it looks if there are any rules for Exp1. It will perform those actions first because it has a higher level. So if there are any rules with the value-category Exp1, our parser will attempt those first, before dropping the precedence level down to Exp. Once it is at Exp, it can match the rule for addition again. So from this explanation, it is clearer that that higher levels of Exp, have higher precedence over the other ones. With the multiplication rule:

```
  Times. Exp1 ::= Exp1 "*" Integer ;
```

our parser knows to first look for the possibility of multiplication after the 4. Since there is a multiplication symbol, it will reduce the multiplication terms first before going back to the addition. This is also analogous to putting parenthesis the their correct place. $5+4*3$ is also $5+(4*3)$

6.4 Calculator Grammar

Now that we have seen the basics of grammars and precedence levels, lets look at a more complex example. This grammar builds on the simple arithmetic that we have already done, and adds several more operations to build an almost complete calculator. By adding more operations, we will need to add more rules into our grammar. Adding more rules will undoubtedly add more complexity and thus increase the risk of having conflicts. We need to make sure we add rules correctly, and thoroughly. Below is the grammar for our Calculator, there is though once small issue that is hard to spot, and can cause problems.

```
Plus. Exp ::= Exp "+" Exp1 ;
Minus. Exp ::= Exp "-" Exp1 ;
Times. Exp1 ::= Exp1 "*" Exp2 ;
Divide. Exp1 ::= Exp1 "/" Exp2 ;
Mod. Exp2 ::= Exp2 "mod" Exp3 ;
Pow. Exp3 ::= Exp3 "^" Exp4 ;
Neg. Exp4 ::= "neg" Exp4 ;
Sqrt. Exp5 ::= "sqrt" Exp5 ;

Num. Exp5 ::= Integer ;

coercions Exp 5 ;
```

Figure 17: *Small Issue in Calculator Grammar*

Lets look at a couple examples and see how it parses.

```
$ echo "3*3^3" | ./TestNumbers
Times (Num 3) (Pow (Num 3) (Num 3))
[Linearized tree]
3 * 3 ^ 3
```

Figure 18: *Parsing* 3×3^3 *correctly*

The parser correctly parses this example. We are first evaluating the power expression 3^3 then multiplying that to the first 3. We can check that it does that order of operations correctly by seeing which the parenthesis drop because of the parenthesis. If we try the same expression with parenthesis, we get the same result.

```
$ echo "3*(3^3)" | ./TestNumbers
Times (Num 3) (Pow (Num 3) (Num 3))
[Linearized tree]
3 * 3 ^ 3
```

Figure 19: *Parsing* $3 \times (3^3)$ *correctly*

The abstract syntax looks exactly the same, and the linearized tree is exactly the same, even though the input was slightly different. Because of the convention of our precedence levels for **Pow** are higher than **Times**, the expression is equivalent regardless of the parenthesis. Therefore, the parenthesis can be dropped. Now lets look at when the linearized tree cannot drop the parenthesis, because their positioning is important.

```
$ echo "(3*3)^3" | ./TestNumbers
Pow (Times (Num 3) (Num 3)) (Num 3)
[Linearized tree]
(3 * 3) ^ 3
```

Figure 20: *Small Issue in Calculator Grammar*

The parenthesis stay around the multiplication because the parenthesis have the highest precedence and they are necessary to make sure that we evaluate the multiplication first before the power. Now lets look at an issue in our calculator.

```
# ex 1
echo "3^3^3" | ./TestNumbers
Pow (Pow (Num 3) (Num 3)) (Num 3)
[Linearized tree]
3 ^ 3 ^ 3

# ex 2
echo "(3^3)^3" | ./TestNumbers
Pow (Pow (Num 3) (Num 3)) (Num 3)
[Linearized tree]
3 ^ 3 ^ 3

# ex 3
echo "3^(3^3)" | ./TestNumbers
Pow (Num 3) (Pow (Num 3) (Num 3))
[Linearized tree]
3 ^ (3 ^ 3)
```

Figure 21: *Three examples of pow*

Which of these examples is incorrect? Well, we can see the issue by looking at the abstract syntax tree for ex 1. Notice how it applies the first power, then applies the second power to that result. Compare this with the syntax tree and linearized tree of ex 2, they are EXACTLY the same! This doesn't look right! With exponents, we know that we need to evaluate the power first before evaluating the base. This is a tricky example of order of operations that our parser does incorrectly. Based on the convention of our precedence levels, it drops the parenthesis when it really shouldn't have! Ex 2 shows us how it reads 3^{3^3} as $(3^3)^3$. If we reduce the equation, we get $(3^3)^3 = 3^9$ which is NOT EQUAL to 3^{3^3}. The correct example is ex 3. From our knowledge of mathematics, what we were aiming for in ex 1 was 3^{3^3} which is equivalent to ex 3: $3^{(3^3)}$. Therefore we need our parser to parse ex 1 as it would ex 3, even without the parenthesis.

How did this issue come up and what can we do to fix it? Well lets look at the rule for pow

```
Pow. Exp3 ::= Exp3 "^" Exp4 ;
```

Figure 22: *Pow Rule*

Notice how it is left-associative just like the other operations. Again, left-associative means that the left non-terminal should be the same as the value-category. This is common because operators such addition are left associative, and therefore, left recursive. In order to build an Exp, the parser first needs to build an Exp, and so on[IPL]. This allows us to drop the parenthesis for addition terms in this convention: $(3 + 3) + 3 = 3 + 3 + 3$.

However pow should not follow the same convention. We do NOT want (3^3)^3 = 3^3^3 , we actually want 3^(3^3) = 3^3^3

What can we do to fix this. We need to make this rule right-associative (right-recursive). If a power term follows a power term, we need group (build) the second term, before applying it to the first. Lets change the pow rule to:

```
Pow. Exp3 ::= Exp4 "^" Exp3 ;
```

Figure 23: *Correct Pow Rule*

Now when we parse the same input, it correctly groups the terms together and drops the correct parenthesis.

```
$ echo "3^(3^3)" | ./TestNumbers
Parse Successful!
[Abstract Syntax]
Pow (Num 3) (Pow (Num 3) (Num 3))
[Linearized tree]
3 ^ 3 ^ 3

$ echo "3^3^3" | ./Calculator
7625597484987
```

Figure 24: *pow rule*

The parser creates the correct abstract syntax and successfully parses the input string. Also if we evaluate the expression, we end up with the correct answer.

7 Conclusion

In this part of the report, we went into great detail regarding parsing. We looked at when parsing comes into play during the compilation phases and we also learned about the importance of parsing. After going over a detailed example of parsing an input, we saw what the actual parsing algorithm does under the hood. Looking under the hood give us a better understanding of what is really happening and what the limitations are. We looked at how to make grammars non-ambiguous. Also after analyzing the Calculator grammar by seeing the incorrect parse trees produced, we changed the precedence levels in order correctly parse our input expressions. From the example we saw how changing difference precedence levels corresponds to different conventions about which parentheses can be dropped. In the next section, we change topics and begin implementing data structures and algorithms.

Part III
Mini Project

8 Goals for the Project

In this final section of the report, we will be building a mini project together. The goal is to implement data structures and algorithms using LambdaFun. We will build off from what was covered in class and implement some more advanced ideas. Follow the instructions to make sure LambdaFun runs on your computer. I will be including snippets of code throughout this section. All of the code can be found in this link to my github. I recommend that you follow along and write code in your own test file. Inside of your `LambdaFun/test` folder, create a `.lc` file.

Now that it is up and running we will begin building two new data structures: a stack and a binary search tree. After implementing both we will combine the stack and binary search tree to construct a preorder traversal algorithm. The aim is to learn the fundamentals behind important data structures and algorithms by implementing them from scratch.

9 Stack

Lets start with building a stack. A stack is an ordered collections of items that follows the paradigm of last in first out (LIFO). To learn more about stacks, I recommend watching this great video by Professor David Brailsford. The idea is to build the stack using a linked list. We will pick up where we left off in round-robin, Assignment 3, but add some adjustments.

Instead of treating the stack as just a linked list, we are going to treat the stack as an object with elements inside of it. Now, we will initialize a stack by defining a stack "object". We can append to it, pop from it and peek from the top. The stack will still follow a similar structure to a linked list, but we will abstract several nuances.

9.1 Initializing the Stack

Each element in the stack will have a value, and also a pointer to the next element in the stack. We create a dummy head and a dummy tail. This is a popular technique in algorithms involving linked lists because it allows for less boundary conditions and special edge cases. These two dummy nodes act as the border of the list and allow us to treat the stack as an object. The dummy head is the top of the stack and is analogous to the stack.

```
val newStack = \e.
    let val nullNode = new [] in
    let val dummyHead = new [] in
    dummyHead := ["FRONT", nullNode] ;
    nullNode := ["NULL", nullNode];
    dummyHead
;;
```

Figure 25: *initializing stack*

NOTE, the parameter `e` is just a place holder. Without it we run into some issues when initializing several stacks in one environment. We make the head node point to the tail. This pointer is what will be changing once we start appending elements into the stack. Then the tail node, `nullNode` is the final node and it points to itself. It points to itself because it represents the bottom of the stack so we cannot iterate past it. Lets run this code snippet. Start LambdaFun with `stack exec lamfun`. Load in your file, in this case its in `test/miniProject.lc`. Then run command `:env` and notice how our heap memory is empty.

Lets initialize a stack: `val stack = newStack 0 ;;` Again the argument 0 doesn't mean anything. Now look at the environment (note in the output below I am not including the whole output, just what we care about for this example in order to save space). The front node points to the last node. Then the last node, "NULL" points to itself.

```
>>> val stack = newStack 0 ;;
>>> :env
Env:
stack = <address 1>

Memory:
0 -> ["NULL", <address 0>]
1 -> ["FRONT", <address 0>]
```

Figure 26: *Initializing stack, memory model*

The variable name `stack` (on the immutable stack memory) is a name for address 0 (in the mutable heap). At that memory address 0, sits a list representing a linked list element. The variable name stack, now points to the front of the stack. The tail node or the bottom of the stack is a "NULL" node. The importance of this will be seen in the next section. Furthermore, its should be noted that the stack only takes in positive integer values.

9.2 Implementing Important Stack Methods

Before implementing the usual stack methods: insert, pop, and peek, we first need to create some helper functions to make the code more readable.

```
val next = \a.
   head(tail !a)
;;
val get = \a.
   head(!a)
;;
```

Figure 27: *Next and Get Helper Functions*

The `next` function returns the memory address of the next element in the stack. The `get` function returns the value of the element. Now that we have our helper functions, lets move on to implementing the core stack functions: peek, append, and pop.

```
val peek = \stack.
   get(next stack)
;;
val append = \e. \stack.
   let val b = new [] in
   b := [e,next(stack)];
   stack := [get stack, b]
;;
val pop = \stack.
   let val ptr = new [] in
   ptr := next stack;
   stack := [get stack,head (tail (!(next stack)))] ;
   !ptr
;;
```

Figure 28: *Peek, Append, and Pop*

For the `peek` function, we return the value of the top element in the stack. This is done by accessing the memory address of the top element and returning its value. Append inserts a node between the `dummyHead` and the current top element, by switching two pointers around. After you insert an element, the `dummyHead` node now points to the newly inserted element, and the inserted element points to the `nullNode`. The function pop removes the element from the stack and returns the address of that linked list element. Lets try out some examples in our console.

```
>>> val stack = newStack 0 ;;
>>> append 1 stack ;;
>>> append 2 stack ;;
>>> peek stack ;;
2
>>> :env
Memory:
0 -> ["NULL", <address 0>]
1 -> ["FRONT", <address 3>]
2 -> [1, <address 0>]
3 -> [2, <address 2>]
```

Figure 29: *Stack Append and Peek Example*

In the example above, we first initialize a stack, append 1 then 2. So element with value 2 is sitting on the top of stack at the moment. When we apply ”peek stack”, the output turns out to be 2. SUCCESS!

Now when we look at the memory model we notice how the front points to the actual top element, 2, then this points to element with value 1, which then points to the end of the stack (”NULL”) which has address 0 and points to itself. Now, in the same environment, try running pop.

```
>>> pop stack ;;
>> :env
Memory:
0 -> ["NULL", <address 0>]
1 -> ["FRONT", <address 2>]
2 -> [1, <address 0>]
3 -> [2, <address 2>]
4 -> <address 3>
```

Figure 30: *Stack Pop*

After popping, we see how the head/front element does not point to address 3 anymore, where value 2 was sitting. It now points to address 2 node with value 1. In essence, we removed the link to address 3 in our stack but didn't actually delete the element. Now our stack only has 1 element other than the dummy nodes.

9.3 Finished Stack

We have finished up writing the core functions for a stack, so we should now have a fully functioning stack. Lets create a `printStack` function takes in a stack and prints the contents of the stack.

```
val printStack = \stack.
    let val ptr = new [] in
    ptr := next stack;
    while (get (!ptr)) != "NULL" do
        (println (get (!ptr)); ptr := next (!ptr) )
;;
```

Figure 31: *Print Stack Function*

In the `printStack` function, we initialize a pointer, `ptr`, that will act as a "running pointer". It starts at the top/front of the stack, prints the value, then moves to the next element in the stack until it reaches the end. Lets look of an example of it in action:

```
>>> val stack = newStack 0 ;;
>>> append 1 stack ;;
>>> append 2 stack ;;
>>> append 3 stack ;;
>>> printStack stack ;;
3
2
1
```

Figure 32: *Print Stack output*

The functions successfully prints the values inside of the stack, nice! Congrats, we have created a functional stack! Lets move now build a Binary Search Tree.

10 Binary Search Tree (BST)

What is a Binary Search Tree (BST)? A BST is a specific type of a binary tree where each node can have at most two children. BSTs are rooted, meaning each tree has a root at the top. What are the properties of a binary search tree? Each `bstNode` has an integer `nodeVal`, a `leftChild` node, and a `rightChild` node. A node is said to be a valid `bstNode` if and only if it satisfies the BST property; its `nodeVal` is strictly greater than the values of every node to its left; its `nodeVal` is less than or equal to the values of every node to its right; and its children nodes are either valid BST nodes themselves or `"NULL"` [AlgoExpert].

The structure of each node is somewhat similar to a linked list, but now we need to add an extra pointer. A node has a value, a pointer to the left child, and a pointer to the right child.

10.1 Initializing a BST

We consider each node of a BST, to also be a BST. We aren't creating a single object that represents the tree. The tree is just a group of nodes that are linked to each other abstractly. We will only have a single reference to the root of the tree, which we can then access any other element by traversing the tree.

```
val bstNode = \e.
    let val node = new [] in
    node := [e,"NULL", "NULL"];
    node
;;
val nodeVal = \node.
    head(!node)
;;
val leftChild = \node.
    (head (tail(!node)))
;;
val rightChild = \node.
    (head (tail(tail(!node))))
;;
```

Figure 33: *Initializing a BST*

In the above code, we initialize a BST node and also create several helper functions that allow us to access the nodes value and also its children. Remember, that the `leftChild` and `righChild` are pointers to the childrens nodes. So, the `nodeVal` function returns the value of the node, `leftChild` and `rightChild` return the addresses of the left child node and right child node respectively. Lets try an example in the console.

```
>>> val root = bstNode 5 ;;
>>> :env
ENV:
root = <address 0>
Memory:
0 -> [5, "NULL", "NULL"]

>>> nodeVal root ;;
5
>>> leftChild root ;;
"NULL"
>>> rightChild root ;;
"NULL"
```

Figure 34: *Testing BST*

In the above code, we create a root node with value 5 then we check to make sure the the **nodeVal**, **leftChild** and **rightChild** methods work properly.

10.2 Inserting into the BST

The next step in building a BST is to insert elements starting at the root. What this means is that we first initialize a BST node, then we begin inserting values. We determine where the new nodes go using the BST property. The code for the insert function is relatively long, so I will include a link to it. If you go here to line 93, you see the beginning of the **insertNode** function. The main idea of the function follows: we need to insert a value e into the tree. To do this, we need to find the right spot in the tree so that the BST property still holds. If the value is less than the current node, then the new node should be inserted in the left subtree. Otherwise, the new node should be inserted into the right subtree. We keep doing this until we get to a ”NULL” node. Once we find the correct spot, we set make the parent node point to the new node, thus ”inserting” it into the tree. In the code from the link, there are comments explaining each portion of the algorithm.

Now lets test out the **insertNode** function in the console. In the code below, we will initialize a BST with 7 nodes. If you do the same on your machine, inserting in the same order, you should see a similar memory model. Also, I included a visual in Figure 36 representing the BST that we just created.

```
>>> val root = bstNode 4 ;;
>>> insertNode 2 root ;;
>>> insertNode 1 root ;;
>>> insertNode 3 root ;;
>>> insertNode 6 root ;;
>>> insertNode 5 root ;;
>>> insertNode 7 root ;;
>>> :env
Memory:
0 -> [4, <address 1>, <address 10>]
1 -> [2, <address 4>, <address 7>]
4 -> [1, "NULL", "NULL"]
7 -> [3, "NULL", "NULL"]
10 -> [6, <address 13>, <address 16>]
13 -> [5, "NULL", "NULL"]
16 -> [7, "NULL", "NULL"]
```

Figure 35: *Initializing a BST with 7 nodes*

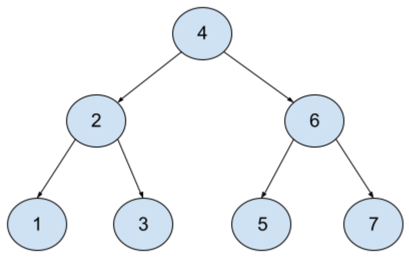


Figure 36: Visualization of the BST

Again, in the code output above, I am not including many irrelevant lines from the memory model in `:env`. If you compare the tree visualization with the memory model, we see that they are a perfect match. The root node, 4 has children, points to memory addresses, `<address> 1 <address> 10`. These children nodes are 2 and 6 respectively. So, we have successfully build an insert function that creates a correct BST. Look at the pointers and addresses of each node and it will become clear that the structure is the same.

11 BST Preorder Traversal Using Stack

So far we have created two different data structures, a stack and a BST. We will now combine these data structures to build a tree traversal algorithm. Unlike linear data structures (such as arrays, linked lists, stacks) which only have one logical way to traverse them, trees can be traversed in several different ways [GFG]. The two main categories of tree traversals are depth-first and breadth-first. We will build a depth-first preorder traversal for our BST. The idea for a preorder traversal is this: First we visit the node, then we traverse the left subtree, then we traverse the right subtree. This algorithm can be implemented with recursion, or iteratively with a stack. For this project, we will implement it with a stack so that we can take advantage of our fabulous stack that we have already built.

The code for the preorder traversal function is relatively long, so I will include a link to it. If you go here to line 138, you see the beginning of the `preorderTraverse` function. The algorithm starts by inserting the root node into a stack. As long as the stack is not empty, we will continue traversing the tree and print out each node. Once we have visited every node, then the stack will be empty, thus we can end the while loop and finish the function. Inside of the while loop, we pop from the stack, print the value of that node then append the right and left children to the stack. The reason we append the left child second is because we are using a stack and need to traverse the left subtree first, thus we need to have the left child at the top of the stack. After you have implemented the function on your machine, try testing it out on the same tree that we have built up above. In the code below, we will run the algorithm on the root node of the tree.

```
>>> preorderTraverse root ;;
4
2
1
3
6
5
7
```

Figure 37: *Output of the Preorder Traversal*

In the above console output, we see the order in which nodes are visited in a preorder traversal. It is helpful to draw out the tree yourself and traverse the tree, then you can compare your results. Lucky for you, I included a visualization below to demonstrate that the algorithm successfully executes a preorder traversal. GREAT!

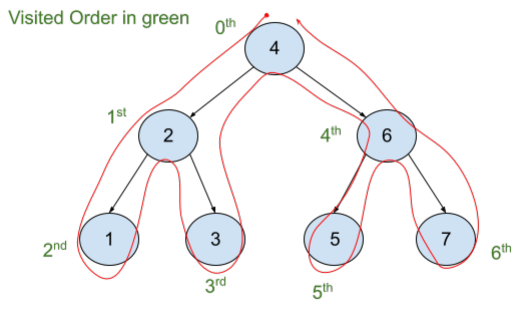


Figure 38: Visualization of Preorder Tree Traversal

12 Conclusion and Future Directions

In this final section of the report, we explored more advanced topics from what was covered in class. We built a stack, a binary search tree, and a preorder tree traversal algorithm. Implementing these data structures and algorithms from scratch was fun because it forced me to be creative and solve complex problems. I enjoyed working on this mini-project and I am happy with what I learned. Hopefully, you also learned along the way and found the project interesting.

Future development of this project would be to implement more BST functions. The main ones that would be worthwhile for the reader to implement are searching and deleting functions. They are vital components of a BST and will prove to be effective exercises. Furthermore, it would be good to implement other traversals such as inorder and postorder using a stack. Also, all of these traversals can also be implemented recursively.

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