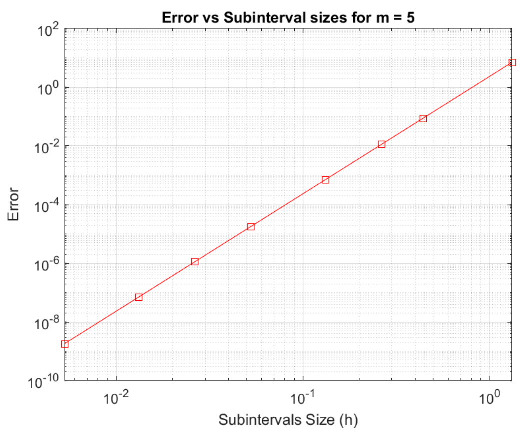
MACM 316 Computing Assignment 5: Convergence Analysis of Simpson's Rule for Numerical Integration

The Simpson's rule is a renowned, and often preferred, method for numerical integration. In this study we use the following:

$$I_m = \int_0^{\pi} f_m(x)\,dx \quad \text{where} \quad f_m(x) = \{x^2 \; for \; x < 0.5;\; x^2 + (x-0.5)^m \; for \; x \geq 0.5\}, \text{ m is a positive integer}$$

The compute the integral and its absolute error, we use Simpson's Composite rule, below, with various m and n values:

$$\int_0^{\pi} f_m(x)\,dx \approx \frac{h}{3}\left[f_m(0) + 4\sum_{i=1,3,5...}^{n-1} f_m(x_i) + 2\sum_{i=2,4,6}^{n-2} f_m(x_i) + f_m(\pi)\right] \text{ where } h = \frac{\pi-0}{n}$$



We use n = [2, 6, 10, 20, 50, 100, 200, 500] for cases m = 3 and m = 5. The absolute error for Simpson's rule is the error between the approximate and exact integral is:

$$|E_s| \leq max|f_m^{(4)}(x)|\frac{\pi-0}{180}h^4$$

We find the particular fourth derivatives of our piecewise function to be:

$$f_m^{(4)}(x) = \{0 \; for \; x < 0.5;\; (m^2 - m)(m-2)(m-3)(x-0.5)^{(m-4)} \; for \; x \geq 0.5\}$$

Part A (m = 3)) The exact integral value in the case of m = 3 is $I_m = 22.5086$. The first figure on the left, with the blue line, is representative of the growth of the error compared to the subinterval size, h. To model the growth of the error, we can use log-log regression; however, it is not required in the case of m = 3. If we analyze the expression for the fourth derivative of our function, we can see that it will always equate to 0 as m = 3. As far as modelling the relationship between the error and the size of the subinterval we obtain $O(h^0)$ or $O(1)$ rate of error growth.
The result presented make sense as for this piecewise function, the fourth derivative of $x^2 \; for \; x < 0.5$ is 0, and as explained above, for when m = 3, the function's fourth derivative, $for \; x \geq 0.5$ also equates to 0 thus the Simpson's error term, $|E_s|$, is also 0, validating the result of $O(1)$ rate of convergence.

Part B (m = 5)) The exact integral value in the case of m = 5 is $I_m = 66.9651$. The second figure on the left, with the red line, represents the growth of the error term compared to the size of the subinterval size, h. Using log-log regression and the power-law relation, we can find the growth of the error as a function of h.
The power-law states that the slope of the log-log regression line is representative of the function's growth rate. Applied here:

$$log(error) = q \cdot log(h) + constant$$

Where the slope, q, is representative of the error growth rate in big O as $O(h^q)$. Applying MATLAB's polyfit() to utilize this principle, we obtain $q = 4$, meaning that the error growth rate is equivalent to $O(h^4)$. Unlike in case A, we do not have any issues with the simpsons absolute error, as we find:

$$f_5^{(4)} = \{0 \; for \; x < 0.5;\; 120(x-0.5) \; for \; x \geq 0.5\}$$

As the error is not 0, we can expect the error to grow in linear fashion according to the Simpson's rule error equation. The red line plot is precisely what we expect to see from this case of m = 5, a linear increase with a slope value of 4.

Part C) The difference between case A and B is the fourth derivative, key in the simpsons error equation. For our given functions, any $m \leq 3$ will have a growth of $O(1)$ as the max of the fourth derivative for our function will always be 0 for all $m \leq 3$. For any $m > 3$ our relation between the error and subinterval size will be > 0. As seen in part B, for $m = 5$ the resulting rate of convergence is $O(h^4)$, expected of Simpson rule. The degree on the fourth derivative is what crucially affects the results.
For the case, $m = 2$, we should expect similar results as in part A, where m = 2, that being a rate of convergence of $O(1)$. The reason behind this is for the similar reason as previously discussed in part A. The fourth derivative is again the culprit that leads us to our conclusive error growth result as when m = 2, the max of the fourth derivative will equal 0. To prove the this concept, we substitute 2 for m in the fourth derivative expression, in which the $m - 2$ term equates to 0, thus making the entire fourth derivative expression to 0. This now attains the same result as part A, an error growth rate of $O(1)$.