Assignment 2

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Answer Q. 1.

For eigenvalues of A, we have

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0 \tag{1}$$

with

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} a - \lambda & b \\ b & -a - \lambda \end{bmatrix} \tag{3}$$

hence

$$\begin{vmatrix} a - \lambda & b \\ b & -a - \lambda \end{vmatrix} = 0 \tag{4}$$

$$-\left(\alpha + \lambda\right)\left(\alpha - \lambda\right) - b^2 = 0 - a^2 + \lambda^2 - b^2 \qquad = 0 \tag{5}$$

$$\lambda^2 = a^2 + b^2 \tag{6}$$

$$\lambda = \pm\sqrt{a^2 + b^2} \tag{7}$$

For eigenvalue $\sqrt{a^2 + b^2}$ and eigenvector v_1 of $\boldsymbol{A}$, we have

$$\left(\boldsymbol{A} - \sqrt{a^2 + b^2}\boldsymbol{I}\right)\boldsymbol{v}_1 = 0 \tag{8}$$

$$\begin{bmatrix} a + \sqrt{a^2 + b^2} & b \\ b & -a + \sqrt{a^2 + b^2} \end{bmatrix}\boldsymbol{v}_1 = 0 \tag{9}$$

$$\tag{10}$$

Hence we have

$$\left[\begin{array}{cc|c} a+\sqrt{a^2+b^2} & b & 0 \\ b & -a+\sqrt{a^2+b^2} & 0 \end{array}\right] \tag{11}$$

$$\left[\begin{array}{cc|c} 1 & \frac{b}{a+\sqrt{a^2+b^2}} & 0 \\ 1 & \frac{-a+\sqrt{a^2+b^2}}{b} & 0 \end{array}\right] \tag{12}$$

$$\left[\begin{array}{cc|c} 0 & \frac{b}{a+\sqrt{a^2+b^2}} - \frac{-a+\sqrt{a^2+b^2}}{b} & 0 \\ 2 & \frac{b}{a+\sqrt{a^2+b^2}} + \frac{-a+\sqrt{a^2+b^2}}{b} & 0 \end{array}\right] \tag{13}$$

$$\left[\begin{array}{cc|c} 0 & \frac{b\left(\sqrt{a^2+b^2}-a\right)}{\left(a+\sqrt{a^2+b^2}\right)\left(\sqrt{a^2+b^2}-a\right)} - \frac{-a+\sqrt{a^2+b^2}}{b} & 0 \\ 2 & \frac{b\left(\sqrt{a^2+b^2}-a\right)}{\left(a+\sqrt{a^2+b^2}\right)\left(\sqrt{a^2+b^2}-a\right)} + \frac{-a+\sqrt{a^2+b^2}}{b} & 0 \end{array}\right] \tag{14}$$

$$\left[\begin{array}{cc|c} 0 & \frac{\sqrt{a^2+b^2}-a}{b} - \frac{-a+\sqrt{a^2+b^2}}{b} & 0 \\ 2 & \frac{\sqrt{a^2+b^2}-a}{b} + \frac{-a+\sqrt{a^2+b^2}}{b} & 0 \end{array}\right] \tag{15}$$

$$\left[\begin{array}{cc|c} 1 & \frac{a+\sqrt{a^2+b^2}}{b} & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{16}$$

Therefore

$$v_1 = \begin{bmatrix} 1 \\ -\frac{a+\sqrt{a^2+b^2}}{b} \end{bmatrix} \tag{17}$$

For eigenvalue $-\sqrt{a^2+b^2}$ and eigenvector v_2 of $\boldsymbol{A}$, we have

$$\left(\boldsymbol{A}+\sqrt{a^2+b^2}\boldsymbol{I}\right)\boldsymbol{v}_1 = 0 \tag{18}$$

$$\begin{bmatrix} a-\sqrt{a^2+b^2} & b \\ b & -a-\sqrt{a^2+b^2} \end{bmatrix}\boldsymbol{v}_1 = 0 \tag{19}$$

Similar from above, we have

$$\left[\begin{array}{cc|c} a-\sqrt{a^2+b^2} & b & 0 \\ b & -a-\sqrt{a^2+b^2} & 0 \end{array}\right] \tag{20}$$

$$\left[\begin{array}{cc|c} 1 & \frac{-a-\sqrt{a^2+b^2}}{b} & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{21}$$

$$\tag{22}$$

Therefore

$$v_2 = \begin{bmatrix} 1 \\ \frac{a+\sqrt{a^2+b^2}}{b} \end{bmatrix} \tag{23}$$

Eigenpairs of $\boldsymbol{A}$ are $\left(\sqrt{a^2+b^2}, \begin{bmatrix} 1 \\ -\frac{a+\sqrt{a^2+b^2}}{b} \end{bmatrix}\right)$ and $\left(-\sqrt{a^2+b^2}, \begin{bmatrix} 1 \\ \frac{a+\sqrt{a^2+b^2}}{b} \end{bmatrix}\right)$
Since all the eigenvalues are distinct, $\boldsymbol{A}$ is diagonizable.

For eigenvalues of B, we have

$$|\boldsymbol{B} - \lambda \boldsymbol{I}| = 0 \tag{24}$$

with

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \tag{25}$$

$$= \begin{bmatrix} a-\lambda & -b \\ b & a-\lambda \end{bmatrix} \tag{26}$$

hence

$$b^2 + (a-\lambda)^2 = 0 \tag{27}$$

$$(a-\lambda)^2 = -b^2 \tag{28}$$

$$a - \lambda = \pm bi \tag{29}$$

$$\lambda = a \pm bi \tag{30}$$

For eigenvalue $a+bi$ and eigenvector u_1 for $\boldsymbol{B}$, we have

$$[\boldsymbol{B} - (a+bi)\, \boldsymbol{I}]\, u_1 = 0 \tag{31}$$

$$b \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} u_1 = 0 \tag{32}$$

$$\tag{33}$$

$$\left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right] \tag{34}$$

$$\left[\begin{array}{cc|c} i+1 & -1+i & 0 \\ 1 & i & 0 \end{array}\right] \tag{35}$$

$$\left[\begin{array}{cc|c} (i+1)(i-1) & (-1+i)(-1+i) & 0 \\ 1 & i & 0 \end{array}\right] \tag{36}$$

$$\left[\begin{array}{cc|c} \sqrt{a^2+b^2} - 1 - 1 & -2i & 0 \\ 1 & i & 0 \end{array}\right] \tag{37}$$

$$\left[\begin{array}{cc|c} 2 & 2i & 0 \\ 1 & i & 0 \end{array}\right] \tag{38}$$

$$\left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{39}$$

$$\therefore u_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \tag{40}$$

For eigenvalue $a-bi$ and eigenvector u_2 for $\boldsymbol{B}$, we have

$$[\boldsymbol{B} - (a-bi)\, \boldsymbol{I}]\, u_2 = 0 \tag{41}$$

$$b \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} u_2 = 0 \tag{42}$$

$$\tag{43}$$

Using similar method from above, we will have

$$u_2 = \begin{bmatrix} -1 \\ i \end{bmatrix} \tag{44}$$

Eigenpairs of $\boldsymbol{B}$ are $\left(a+bi, \begin{bmatrix} 1 \\ -i \end{bmatrix}\right)$ and $\left(a-bi, \begin{bmatrix} -1 \\ i \end{bmatrix}\right)$
Since all the eigenvalues are distinct, $\boldsymbol{B}$ is diagonizable.

Answer Q. 2.

Let $\boldsymbol{v}$ be the vector provided, consider that

$$\boldsymbol{v} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} \tag{45}$$

$$= -2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \tag{46}$$

$$= -2\boldsymbol{v}_1 + 3\boldsymbol{v}_2 \tag{47}$$

Hence we have

$$\boldsymbol{A}^{999} \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} \tag{48}$$

$$=\boldsymbol{A}^{999}\boldsymbol{v} \tag{49}$$

$$= -2\boldsymbol{A}^{999}\boldsymbol{v}_1 + 3\boldsymbol{A}^{999}\boldsymbol{v}_2 \tag{50}$$

$$= -2\lambda_1^{999}\boldsymbol{v}_1 + 3\lambda_2^{999}\boldsymbol{v}_2 \tag{51}$$

$$= -21^{999}\boldsymbol{v}_1 + 3(-1)^{999}\boldsymbol{v}_2 \tag{52}$$

$$= -2\boldsymbol{v}_1 - 3\boldsymbol{v}_2 \tag{53}$$

$$= -2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \tag{54}$$

$$= \begin{bmatrix} -8 \\ 2 \\ -3 \end{bmatrix} \tag{55}$$

Answer Q. 3.

Considering that the eigenvector $\boldsymbol{v}_i$ of matrix $\boldsymbol{A}$ with eigenvalue $\boldsymbol{v}_i$ is given by

$$\left(\boldsymbol{A} - \lambda_i \boldsymbol{I}\right) \boldsymbol{v}_i = 0 \tag{56}$$

and a vector $\boldsymbol{v}$ is considered a null space vector of $\boldsymbol{A}$ only if

$$\boldsymbol{A}\boldsymbol{v} = 0 \tag{57}$$

Hence if $\lambda_i = 0$, then we must have

$$(\boldsymbol{A} - 0\boldsymbol{I})\boldsymbol{v}_i = 0 \tag{58}$$
$$\boldsymbol{A}\boldsymbol{v}_i = 0 \tag{59}$$

$\therefore$ $\boldsymbol{A}$ has a nullspace vector v_i if it has a eigenvalue of $\lambda_i = 0$

If there is no such $\lambda_i = 0$, then for all $\boldsymbol{v}_i$

$$\boldsymbol{A}\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i \neq 0 \tag{60}$$

$\therefore$ By (86), no null space vector exists.
$\therefore$ $\boldsymbol{A}$ has a null space vector v_i only if it has a eigenvalue of $\lambda_i = 0$

$\because$ $p(\lambda)$ is of degree 6
$\therefore$ $\boldsymbol{A}$ has a size of 6×6

By rank–nullity theorem,

$$\text{Rank}(\boldsymbol{A}) + \text{Nullity}(\boldsymbol{A}) = \dim \boldsymbol{A} \tag{61}$$

$\because$ No solution of $p(\lambda) = 0$ is 0
$\therefore$ By the above proved statement, A have no null space vector
$\therefore$ $\text{Nullity}(\boldsymbol{A}) = 0$
$\therefore$ By (90), rank of $\boldsymbol{A}$ is given by $\text{Rank}(\boldsymbol{A}) = \dim \boldsymbol{A} = 0$

Answer Q. 4.

Let $\boldsymbol{M}$ be the matrix, hence

$$|\boldsymbol{M} - \lambda \boldsymbol{I}| = \begin{bmatrix} -\lambda & 0 & 0 & 0 \\ 1 & 2-\lambda & 3 & 4 \\ 0 & 0 & -\lambda & 0 \\ 1 & 2 & 3 & 4-\lambda \end{bmatrix} \tag{62}$$
$$= \lambda^2((2-\lambda)(4-\lambda) - 8) \tag{63}$$
$$= \lambda^3(\lambda - 6) \tag{64}$$

Hence by solving $\lambda^3(\lambda - 6) = 0$,
we have eigenvalue 0 of algebraic multiplicity 3
and eigenvalue 6 of algebraic multiplicity 1.

For $\lambda = 0$,

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{array}\right] \tag{65}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{66}$$

Hence eigenvectors of 0 are

$$\left[\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right] \tag{67}$$

Which mean it has linear multiplicity of 3

For $\lambda = 6$,
For a eigenvalue of algebraic multiplicity 1, its linear multiplicity must be larger than 0, since at least 1 eigenvectors exists, and its eigenvalue must be equal or smaller than 1, since linear multiplicity must be smaller or equal to algebraic multiplicity. Hence its linear multiplicity is 1 as well.

Answer Q. 5.

Let $\boldsymbol{v} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ Then we have

$$\boldsymbol{v}\boldsymbol{v}^T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \tag{68}$$

$$= \begin{bmatrix} a_1^2 & a_1a_2 & \dots & a_1a^n \\ a_2a_1 & a_2^2 & \dots & a_2a^n \\ \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & \dots & a_n^2 \end{bmatrix} \tag{69}$$

Therefore $\boldsymbol{v}\boldsymbol{v}^T$, hence $-\frac{2}{||v||^2}\boldsymbol{v}\boldsymbol{v}^T$ are symmetric.
Since $\boldsymbol{I}$ is symmetric as well,

$\boldsymbol{M} = \boldsymbol{I} - \frac{2}{||v||^2}\boldsymbol{v}\boldsymbol{v}^T$ is also symmetric.

$$\begin{aligned}
\boldsymbol{M}^2 =& (\boldsymbol{I} - \frac{2}{||\boldsymbol{v}||^2}\boldsymbol{v}\boldsymbol{v}^T)^2 && (70)\\
=& \boldsymbol{I} - 4\frac{\boldsymbol{v}\boldsymbol{v}^T}{||v||^2} + 4\frac{1}{||\boldsymbol{v}||^4}\boldsymbol{v}(||\boldsymbol{v}||^2)\boldsymbol{v}^T && (71)\\
=& \boldsymbol{I} && (72)\\
\boldsymbol{M} =& \boldsymbol{M}^{-1} && (73)
\end{aligned}$$