Assignment 2

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Answer Q. 1.

For eigenvalues of A, we have

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \tag{1}$$

with

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} a - \lambda & b \\ b & -a - \lambda \end{bmatrix} \tag{3}$$

hence

$$\begin{vmatrix} a - \lambda & b \\ b & -a - \lambda \end{vmatrix} = 0 \tag{4}$$

$$-(\alpha + \lambda)(\alpha - \lambda) - b^2 = 0 - a^2 + \lambda^2 - b^2 \qquad = 0 \tag{5}$$

$$\lambda^2 = a^2 + b^2 \tag{6}$$

$$\lambda = \pm\sqrt{a^2 + b^2} \tag{7}$$

For eigenvalue $\sqrt{a^2 + b^2}$ and eigenvector v_1 of $\mathbf{A}$, we have

$$\left(\mathbf{A} - \sqrt{a^2 + b^2}\mathbf{I}\right) \mathbf{v}_1 = 0 \tag{8}$$

$$\begin{bmatrix} a + \sqrt{a^2 + b^2} & b \\ b & -a + \sqrt{a^2 + b^2} \end{bmatrix} \mathbf{v}_1 = 0 \tag{9}$$

$$\tag{10}$$

Hence we have

$$\left[\begin{array}{cc|c} a+\sqrt{a^2+b^2} & b & 0 \\ b & -a+\sqrt{a^2+b^2} & 0 \end{array}\right] \tag{11}$$

$$\left[\begin{array}{cc|c} 1 & \frac{b}{a+\sqrt{a^2+b^2}} & 0 \\ 1 & \frac{-a+\sqrt{a^2+b^2}}{b} & 0 \end{array}\right] \tag{12}$$

$$\left[\begin{array}{cc|c} 0 & \frac{b}{a+\sqrt{a^2+b^2}} - \frac{-a+\sqrt{a^2+b^2}}{b} & 0 \\ 2 & \frac{b}{a+\sqrt{a^2+b^2}} + \frac{-a+\sqrt{a^2+b^2}}{b} & 0 \end{array}\right] \tag{13}$$

$$\left[\begin{array}{cc|c} 0 & \frac{b\left(\sqrt{a^2+b^2}-a\right)}{\left(a+\sqrt{a^2+b^2}\right)\left(\sqrt{a^2+b^2}-a\right)} - \frac{-a+\sqrt{a^2+b^2}}{b} & 0 \\ 2 & \frac{b\left(\sqrt{a^2+b^2}-a\right)}{\left(a+\sqrt{a^2+b^2}\right)\left(\sqrt{a^2+b^2}-a\right)} + \frac{-a+\sqrt{a^2+b^2}}{b} & 0 \end{array}\right] \tag{14}$$

$$\left[\begin{array}{cc|c} 0 & \frac{\sqrt{a^2+b^2}-a}{b} - \frac{-a+\sqrt{a^2+b^2}}{b} & 0 \\ 2 & \frac{\sqrt{a^2+b^2}-a}{b} + \frac{-a+\sqrt{a^2+b^2}}{b} & 0 \end{array}\right] \tag{15}$$

$$\left[\begin{array}{cc|c} 1 & \frac{a+\sqrt{a^2+b^2}}{b} & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{16}$$

Therefore

$$v_1 = \begin{bmatrix} 1 \\ -\frac{a+\sqrt{a^2+b^2}}{b} \end{bmatrix} \tag{17}$$

For eigenvalue $-\sqrt{a^2+b^2}$ and eigenvector v_2 of $\mathbf{A}$, we have

$$\left(\mathbf{A} + \sqrt{a^2+b^2}\mathbf{I}\right)\mathbf{v}_1 = 0 \tag{18}$$

$$\begin{bmatrix} a-\sqrt{a^2+b^2} & b \\ b & -a-\sqrt{a^2+b^2} \end{bmatrix}\mathbf{v}_1 = 0 \tag{19}$$

Similar from above, we have

$$\left[\begin{array}{cc|c} a-\sqrt{a^2+b^2} & b & 0 \\ b & -a-\sqrt{a^2+b^2} & 0 \end{array}\right] \tag{20}$$

$$\left[\begin{array}{cc|c} 1 & \frac{-a-\sqrt{a^2+b^2}}{b} & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{21}$$

$$\tag{22}$$

Therefore

$$v_2 = \begin{bmatrix} 1 \\ \frac{a+\sqrt{a^2+b^2}}{b} \end{bmatrix} \tag{23}$$

Eigenpairs of $\mathbf{A}$ are $\left(\sqrt{a^2+b^2}, \begin{bmatrix} 1 \\ -\frac{a+\sqrt{a^2+b^2}}{b} \end{bmatrix}\right)$ and $\left(-\sqrt{a^2+b^2}, \begin{bmatrix} 1 \\ \frac{a+\sqrt{a^2+b^2}}{b} \end{bmatrix}\right)$
Since all the eigenvalues are distinct, $\mathbf{A}$ is diagonizable.

For eigenvalues of B, we have

$$|\mathbf{B} - \lambda\mathbf{I}| = 0 \tag{24}$$

with

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \tag{25}$$

$$= \begin{bmatrix} a-\lambda & -b \\ b & a-\lambda \end{bmatrix} \tag{26}$$

hence

$$b^2 + (a-\lambda)^2 = 0 \tag{27}$$

$$(a-\lambda)^2 = -b^2 \tag{28}$$

$$a - \lambda = \pm bi \tag{29}$$

$$\lambda = a \pm bi \tag{30}$$

For eigenvalue $a + bi$ and eigenvector u_1 for $\mathbf{B}$, we have

$$\left[\mathbf{B} - (a+bi)\,\mathbf{I}\right] u_1 = 0 \tag{31}$$

$$b\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} u_1 = 0 \tag{32}$$

$$\tag{33}$$

$$\left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right] \tag{34}$$

$$\left[\begin{array}{cc|c} i+1 & -1+i & 0 \\ 1 & i & 0 \end{array}\right] \tag{35}$$

$$\left[\begin{array}{cc|c} (i+1)(i-1) & (-1+i)(-1+i) & 0 \\ 1 & i & 0 \end{array}\right] \tag{36}$$

$$\left[\begin{array}{cc|c} \sqrt{a^2+b^2} - 1 - 1 & -2i & 0 \\ 1 & i & 0 \end{array}\right] \tag{37}$$

$$\left[\begin{array}{cc|c} 2 & 2i & 0 \\ 1 & i & 0 \end{array}\right] \tag{38}$$

$$\left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{39}$$

$$\therefore u_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \tag{40}$$

For eigenvalue $a - bi$ and eigenvector u_2 for $\mathbf{B}$, we have

$$\left[\mathbf{B} - (a-bi)\,\mathbf{I}\right] u_2 = 0 \tag{41}$$

$$b\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} u_2 = 0 \tag{42}$$

$$\tag{43}$$

Using similar method from above, we will have

$$u_2 = \begin{bmatrix} -1 \\ i \end{bmatrix} \tag{44}$$

Eigenpairs of $\mathbf{B}$ are $\left(a+bi, \begin{bmatrix} 1 \\ -i \end{bmatrix}\right)$ and $\left(a-bi, \begin{bmatrix} -1 \\ i \end{bmatrix}\right)$
Since all the eigenvalues are distinct, $\mathbf{B}$ is diagonizable.

Answer Q. 2.

Let $\mathbf{A}$ be

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} \tag{45}$$

then considering that

$$(A - \lambda_1)\mathbf{v}_1 = 0 \tag{46}$$
$$(A - \lambda_2)\mathbf{v}_2 = 0 \tag{47}$$

hence

$$\begin{bmatrix} x_1 - x_2 - 1 \\ x_4 - x_5 + 1 \\ x_7 - x_8 \end{bmatrix} = 0 \tag{48}$$
$$\begin{bmatrix} 2x_1 + x_3 + 2 \\ 2x_4 + x_6 \\ 2x_7 + x_9 + 1 \end{bmatrix} = 0 \tag{49}$$

by solving

$$x_1 - x_2 = 1 \tag{50}$$
$$2x_1 + x_3 = -2 \tag{51}$$
$$\tag{52}$$
$$x_4 - x_5 = -1 \tag{53}$$
$$2x_4 + x_6 = 0 \tag{54}$$
$$\tag{55}$$
$$x_7 - x_8 = 0 \tag{56}$$
$$2x_7 + x_9 = -1 \tag{57}$$

we have

$$\left(-\frac{x_3}{2} - 1, \quad -\frac{x_3}{2} - 2, \quad x_3, \quad -\frac{x_6}{2}, \quad -\frac{x_6}{2} + 1, \quad x_6, \quad -\frac{x_9}{2} - \frac{1}{2}, \quad -\frac{x_9}{2} - \frac{1}{2}, \quad x_9\right) \tag{58}$$

hence we can express $\mathbf{A}$ as

$$\begin{bmatrix} -\frac{x_3}{2}-1 & -\frac{x_3}{2}-2 & x_3 \\ -\frac{x_6}{2} & -\frac{x_6}{2}+1 & x_6 \\ -\frac{x_9}{2}-\frac{1}{2} & -\frac{x_9}{2}-\frac{1}{2} & x_9 \end{bmatrix} \tag{59}$$

Since any values of x_3, x_6 and x_9 will meet the condition (48) and (49), hence we can put $x_3 = 2, x_6 = 2$ and $x_9 = 2$, getting

$$\mathbf{A} = \begin{bmatrix} -2 & -3 & 2 \\ -1 & 0 & 2 \\ -\frac{3}{2} & -\frac{3}{2} & 2 \end{bmatrix} \tag{60}$$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -\lambda-2 & -3 & 2 \\ -1 & -\lambda & 2 \\ -\frac{3}{2} & -\frac{3}{2} & -\lambda+2 \end{bmatrix} \tag{61}$$

$$|\mathbf{A} - \lambda\mathbf{I}| = -\lambda\left(-\lambda-2\right)\left(-\lambda+2\right) - 3\lambda \tag{62}$$

$$= -\lambda(\lambda-1)(\lambda+1) \tag{63}$$

hence with $|\mathbf{A} - \lambda\mathbf{I}| = 0$, we can know that the third eigenvalue is 0

For eigenvalue 0 and eigenvectors v_3 for $\mathbf{A}$, we have

$$\left(A - 0\mathbf{I}\right) v_3 = 0 \tag{64}$$

$$\begin{bmatrix} -2 & -3 & 2 \\ -1 & 0 & 2 \\ -\frac{3}{2} & -\frac{3}{2} & 2 \end{bmatrix} v_3 = 0 \tag{65}$$

hence by solving linear equation

$$\left[\begin{array}{ccc|c} -2 & -3 & 2 & 0 \\ -1 & 0 & 2 & 0 \\ -\frac{3}{2} & -\frac{3}{2} & 2 & 0 \end{array}\right] \tag{66}$$

we have

$$v_3 = k \begin{bmatrix} 2 \\ -\frac{2}{3} \\ 1 \end{bmatrix} \tag{67}$$

Taking $v_3 = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$ for simplicity

Therefore, we can digaonize $\mathbf{A}$ as

$$\mathbf{V} = \begin{bmatrix} v_1 v_3 v_2 \end{bmatrix} \tag{68}$$

$$= \begin{bmatrix} 2 & 6 & -1 \\ 0 & -2 & 1 \\ 1 & 3 & 0 \end{bmatrix} \tag{69}$$

$$\mathbf{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{70}$$

$$|\mathbf{V}| = -2 \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} - 1 \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \tag{71}$$

$$= -2 - 1(0) \tag{72}$$

$$= -2 \tag{73}$$

$$\mathrm{cof}(\mathbf{V}) = \begin{bmatrix} -3 & 1 & 2 \\ -3 & 1 & 0 \\ 4 & -2 & -4 \end{bmatrix} \tag{74}$$

$$\mathrm{adj}(\mathbf{V}) = \mathrm{coj}(\mathbf{V})^T \tag{75}$$

$$= \begin{bmatrix} -3 & -3 & 4 \\ 1 & 1 & -2 \\ 2 & 0 & -4 \end{bmatrix} \tag{76}$$

$$\mathbf{V}^{-1} = -\frac{1}{2} \begin{bmatrix} -3 & -3 & 4 \\ 1 & 1 & -2 \\ 2 & 0 & -4 \end{bmatrix} \tag{77}$$

With

$$\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{D} \tag{78}$$

$$
\begin{aligned}
\mathbf{V}^{-1}\mathbf{A}^{999}\mathbf{V} &= \mathbf{D}^{999} && (79)\\
\mathbf{A}^{999}\begin{bmatrix}4\\2\\3\end{bmatrix} &= \mathbf{V}\mathbf{D}^{999}\mathbf{V}^{-1}\begin{bmatrix}4\\2\\3\end{bmatrix} && (80)\\
&= \begin{bmatrix}2 & 6 & -1\\0 & -2 & 1\\1 & 3 & 0\end{bmatrix}\begin{bmatrix}-1^{999} & 0 & 0\\0 & 0 & 0\\0 & 0 & 1^{999}\end{bmatrix}\begin{bmatrix}\frac{3}{2} & \frac{3}{2} & -2\\-\frac{1}{2} & -\frac{1}{2} & 1\\-1 & 0 & 2\end{bmatrix}\begin{bmatrix}4\\2\\3\end{bmatrix} && (81)\\
&= \begin{bmatrix}-2 & 0 & -1\\0 & 0 & 1\\-1 & 0 & 0\end{bmatrix}\begin{bmatrix}\frac{3}{2} & \frac{3}{2} & -2\\-\frac{1}{2} & -\frac{1}{2} & 1\\-1 & 0 & 2\end{bmatrix}\begin{bmatrix}4\\2\\3\end{bmatrix} && (82)\\
&= \begin{bmatrix}-2 & -3 & 2\\-1 & 0 & 2\\-\frac{3}{2} & -\frac{3}{2} & 2\end{bmatrix}\begin{bmatrix}4\\2\\3\end{bmatrix} && (83)\\
&= \begin{bmatrix}-8\\2\\-3\end{bmatrix} && (84)
\end{aligned}
$$