Assignment 2

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Answer Q. 1.

For eigenvalues of A, we have

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

with

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$
$$= \begin{bmatrix} a - \lambda & b \\ b & -a - \lambda \end{bmatrix}$$

hence

$$\begin{vmatrix} a - \lambda & b \\ b & -a - \lambda \end{vmatrix} = 0$$
$$-(\alpha + \lambda)(\alpha - \lambda) - b^2 = 0 - a^2 + \lambda^2 - b^2 \qquad = 0$$
$$\lambda^2 = a^2 + b^2$$
$$\lambda = \pm\sqrt{a^2 + b^2}$$

For eigenvalue $\sqrt{a^2 + b^2}$ and eigenvector v_1 of $\mathbf{A}$, we have

$$\left(\mathbf{A} - \sqrt{a^2 + b^2}\mathbf{I}\right)\mathbf{v}_1 = 0$$
$$\begin{bmatrix} a + \sqrt{a^2 + b^2} & b \\ b & -a + \sqrt{a^2 + b^2} \end{bmatrix}\mathbf{v}_1 = 0$$

Hence we have

$$\left[\begin{array}{cc|c} a+\sqrt{a^2+b^2} & b & 0 \\ b & -a+\sqrt{a^2+b^2} & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & \frac{b}{a+\sqrt{a^2+b^2}} & 0 \\ 1 & \frac{-a+\sqrt{a^2+b^2}}{b} & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 0 & \frac{b}{a+\sqrt{a^2+b^2}} - \frac{-a+\sqrt{a^2+b^2}}{b} & 0 \\ 2 & \frac{b}{a+\sqrt{a^2+b^2}} + \frac{-a+\sqrt{a^2+b^2}}{b} & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 0 & \frac{b\left(\sqrt{a^2+b^2}-a\right)}{\left(a+\sqrt{a^2+b^2}\right)\left(\sqrt{a^2+b^2}-a\right)} - \frac{-a+\sqrt{a^2+b^2}}{b} & 0 \\ 2 & \frac{b\left(\sqrt{a^2+b^2}-a\right)}{\left(a+\sqrt{a^2+b^2}\right)\left(\sqrt{a^2+b^2}-a\right)} + \frac{-a+\sqrt{a^2+b^2}}{b} & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 0 & \frac{\sqrt{a^2+b^2}-a}{b} - \frac{-a+\sqrt{a^2+b^2}}{b} & 0 \\ 2 & \frac{\sqrt{a^2+b^2}-a}{b} + \frac{-a+\sqrt{a^2+b^2}}{b} & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & \frac{a+\sqrt{a^2+b^2}}{b} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore

$$v_1 = \begin{bmatrix} 1 \\ -\frac{a+\sqrt{a^2+b^2}}{b} \end{bmatrix}$$

For eigenvalue $-\sqrt{a^2+b^2}$ and eigenvector v_2 of $\mathbf{A}$, we have

$$\left(\mathbf{A} + \sqrt{a^2+b^2}\mathbf{I}\right)\mathbf{v}_1 = 0$$

$$\begin{bmatrix} a-\sqrt{a^2+b^2} & b \\ b & -a-\sqrt{a^2+b^2} \end{bmatrix}\mathbf{v}_1 = 0$$

Similar from above, we have

$$\left[\begin{array}{cc|c} a-\sqrt{a^2+b^2} & b & 0 \\ b & -a-\sqrt{a^2+b^2} & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & \frac{-a-\sqrt{a^2+b^2}}{b} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore

$$v_2 = \begin{bmatrix} 1 \\ \frac{a+\sqrt{a^2+b^2}}{b} \end{bmatrix}$$

Eigenpairs of $\mathbf{A}$ are $\left(\sqrt{a^2+b^2}, \begin{bmatrix} 1 \\ -\frac{a+\sqrt{a^2+b^2}}{b} \end{bmatrix}\right)$ and $\left(-\sqrt{a^2+b^2}, \begin{bmatrix} 1 \\ \frac{a+\sqrt{a^2+b^2}}{b} \end{bmatrix}\right)$
Since all the eigenvalues are distinct, $\mathbf{A}$ is diagonizable.

For eigenvalues of B, we have

$$|\mathbf{B} - \lambda \mathbf{I}| = 0$$

with

$$\begin{aligned}
&\begin{bmatrix} a & -b \\ b & a \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\
&= \begin{bmatrix} a-\lambda & -b \\ b & a-\lambda \end{bmatrix}
\end{aligned}$$

hence

$$\begin{aligned}
b^2 + (a-\lambda)^2 &= 0 \\
(a-\lambda)^2 &= -b^2 \\
a - \lambda &= \pm bi \\
\lambda &= a \pm bi
\end{aligned}$$

For eigenvalue $a + bi$ and eigenvector u_1 for $\mathbf{B}$, we have

$$\begin{aligned}
\left[\mathbf{B} - (a+bi)\,\mathbf{I}\right] u_1 &= 0 \\
b \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} u_1 &= 0
\end{aligned}$$

$$\begin{aligned}
&\left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right] \\
&\left[\begin{array}{cc|c} i+1 & -1+i & 0 \\ 1 & i & 0 \end{array}\right] \\
&\left[\begin{array}{cc|c} (i+1)(i-1) & (-1+i)(-1+i) & 0 \\ 1 & i & 0 \end{array}\right] \\
&\left[\begin{array}{cc|c} \sqrt{a^2+b^2} - 1 - 1 & -2i & 0 \\ 1 & i & 0 \end{array}\right] \\
&\left[\begin{array}{cc|c} 2 & 2i & 0 \\ 1 & i & 0 \end{array}\right] \\
&\left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right] \\
\therefore u_1 &= \begin{bmatrix} 1 \\ -i \end{bmatrix}
\end{aligned}$$

For eigenvalue $a - bi$ and eigenvector u_2 for $\mathbf{B}$, we have

$$\begin{aligned}
\left[\mathbf{B} - (a - bi)\,\mathbf{I}\right] u_2 &= 0 \\
b \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} u_2 &= 0
\end{aligned}$$

Using similar method from above, we will have

$$u_2 = \begin{bmatrix} -1 \\ i \end{bmatrix}$$

Eigenpairs of $\mathbf{B}$ are $\left(a + bi, \begin{bmatrix} 1 \\ -i \end{bmatrix}\right)$ and $\left(a - bi, \begin{bmatrix} -1 \\ i \end{bmatrix}\right)$
Since all the eigenvalues are distinct, $\mathbf{B}$ is diagonizable.

Answer Q. 2.

Let $\mathbf{A}$ be

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}$$

then considering that

$$(A - \lambda_1)\mathbf{v}_1 = 0 \tag{1}$$
$$(A - \lambda_2)\mathbf{v}_2 = 0 \tag{2}$$

hence

$$\begin{bmatrix} x_1 - x_2 - 1 \\ x_4 - x_5 + 1 \\ x_7 - x_8 \end{bmatrix} = 0$$
$$\begin{bmatrix} 2x_1 + x_3 + 2 \\ 2x_4 + x_6 \\ 2x_7 + x_9 + 1 \end{bmatrix} = 0$$

by solving

$$\begin{aligned}
x_1 - x_2 &= 1 \\
2x_1 + x_3 &= -2
\end{aligned}$$

$$\begin{aligned}
x_4 - x_5 &= -1 \\
2x_4 + x_6 &= 0
\end{aligned}$$

$$\begin{aligned}
x_7 - x_8 &= 0 \\
2x_7 + x_9 &= -1
\end{aligned}$$

we have

$$\left(-\frac{x_3}{2}-1, \quad -\frac{x_3}{2}-2, \quad x_3, \quad -\frac{x_6}{2}, \quad -\frac{x_6}{2}+1, \quad x_6, \quad -\frac{x_9}{2}-\frac{1}{2}, \quad -\frac{x_9}{2}-\frac{1}{2}, \quad x_9\right)$$

hence we can express $\mathbf{A}$ as

$$\begin{bmatrix} -\frac{x_3}{2}-1 & -\frac{x_3}{2}-2 & x_3 \\ -\frac{x_6}{2} & -\frac{x_6}{2}+1 & x_6 \\ -\frac{x_9}{2}-\frac{1}{2} & -\frac{x_9}{2}-\frac{1}{2} & x_9 \end{bmatrix}$$

Since any values of x_3, x_6 and x_9 will meet the condition (1) and (2)