MPhil Thesis

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Abstract

There are many different ways to manage the interaction between compile-time metaprogramming and effect handlers, but there has been no way to evaluate them against each other. This dissertation introduces such a mechanism, focusing on correctness, expressiveness, and efficiency. Second, we evaluate existing approaches by the aforementioned mechanism, illustrating the trade-offs in the design space. Finally, we introduces a novel approach, a dynamic Best-Effort check, that we show is correct, maximally expressive, and likely more efficient in the common case, as compared to existing checks.

1 Introduction

The promise of an optimising compiler is that the programmer can focus on writing maintainable code, entrusting the responsibility of *optimising* said code to the compiler.

Entrusting optimisation to the compiler is sufficient for many cases, but not all. For reasons that are both theoretical [18] and practical [19], compilers will never be able to identify *all* opportunities for optimisation. What can be done when compiler optimisation is not enough?

Metaprogramming (for example, C++ templates) [1] is one possible solution. In languages that support metaprogramming, programmers can identify what should be executed at compile-time. Importantly, programmers may write **maintainable** code, which when executed by the compiler, generates **efficient** code. For example, assume two implementations of the same function, one that runs faster on Arm, and one on Intel.

The maintainable way to switch between these functions is via a conditional:

OCaml
```
let f () = if (cpu_type == Arm) then
             (* return this program *)
           else
             (* return this program *)
f ()
```

However, it might be expensive to determine processor type at run-time. If efficiency is paramount, programmers might have to duplicate their code, doubling maintenance costs. With metaprogramming, however, we may write

MacoCaml
```
(* macro and $ can be read as annotations which
   indicate that the code should be run at compile-time *)
macro m () = if (cpu_type == Arm) then
               (* generate this program, to be run later *)
             else
               (* gen this program, to be run later *)
$(m ())
```

Importantly, the macro `m` is executed by the compiler, *at compile time*. The Arm and Intel variants are *generated from a single specification*, not manually duplicated.

Importantly, to support metaprogramming, languages have to provide the

ability to build programs *to be run later*. These "suspended programs" are best thought of as abstract syntax trees. The following code constructs the program $(\lambda x.x)1$.

```OCaml
if (cpu_type == Arm) then
  let build_app (f: (int -> int) expr) (v: int expr) = App(f, v)
  let id_ast: (int -> int) expr = Lam(Var(x), Var(x))
  build_app(id_ast, Int(1)) (* int expr *)
else
  (*...*)
```

Effect handlers are a powerful language construct that can simulate many other language features (state, I/O, greenthreading) [17], and have recently been added to `OCaml` [21]. It is thus strategic, and timely, to investigate the interaction between metaprogramming and effect handlers.

Unfortunately, metaprogramming is known to interact unexpectedly, and poorly, with effect handlers, a problem known as *scope extrusion* [11]. Scope extrusion occurs when the programmer accidentally directs the compiler to generate code with unbound variables. In the following program, we store the variable `Var(x)` into a heap cell `(1)`, and retrieve it outside its scope. The program thus evaluates to `Var(x)`, which is unbound.

```OCaml
let l: int expr ref = new(Int(1)) in
let id_ast : (int -> int) expr = Lam(Var(x), l := Var(x); Int(1)) in
!l
```

The problem of scope extrusion has been widely studied, resulting in multitudinous mechanisms for managing the interaction between metaprogramming and effect handlers. Some solutions adapt the type system (Refined Environment Classifiers [13, 8], Closed Types [3]), others insert dynamic checks into the generated code [11]. However, there are two issues.

First, we are a picky lot. Type-based approaches require unfeasible modification of the `OCaml` type-checker, and tend to limit expressiveness, disallowing a wide range of programs that do not lead to scope extrusion. Dynamic solutions are inefficient, reporting scope extrusion long after the error occurred, or unpredictable, allowing some programs, but disallowing morally equivalent programs. Further, many of the dynamic solutions have not been proved correct.

Second, and more importantly, we lacked a way to evaluate solutions fairly and holistically. Our evaluation criteria is three-pronged: correctness, efficiency, and expressiveness. Each solution is described in its own calculus, which made solutions difficult to compare. It is non-trivial to show that the definitions of scope extrusion in differing calculi agree. Hence, one solution may be correct with respect to its definition, but wrong with respect to an-

other. Further, expressiveness and efficiency are inherently comparative criteria.

This thesis will solve both these problems, in decreasing order of priority.

1.1 Contributions

Concretely, our contributions are:

1. A novel "universal" calculus that allows for effect handlers both in the generating and generated code.
2. In the calculus, three precise (but different) definitions of scope extrusion.
3. In the calculus, encodings of the following existing solutions:
   a) Lazy dynamic check (with a proof of correctness)
   b) Eager dynamic check (with a refutation of correctness)
   c) Refined Environment Classifiers
4. A new solution, a Best-Effort dynamic check, that we justify is correct, expressive, and efficient.
5. Implementations of the three dynamic checks in Macocaml.

The scope of this dissertation thus extends to effect systems with **deep** (but **unnamed**) handlers and **multi-shot continuations**.

2 Background

The aim of this dissertation is to evaluate, and propose, policies for mediating the interaction between Macocaml-style **metaprogramming** and **effect handlers**, by addressing the issue of **scope extrusion**.

In this chapter, I provide technical overviews of each of the key concepts: metaprogramming (Section 2.1), effect handlers (Section 2.2), and scope extrusion (Section 2.3).

2.1 Metaprogramming

What is MacoCaml-style metaprogramming? I will provide an answer in two steps. First, I motivate metaprogramming, by illustrating the challenge of writing code that is both fast and maintainable (Section 2.1.1). Second, I will consider the design space of metaprogramming (Section 2.1.2), highlighting decisions made by MacoCaml.

2.1.1 Metaprogramming for Fast and Maintainable Code

Metaprogramming helps programmers write fast and maintainable code. How does one write fast and maintainable code? A naïve answer is "by being a skilled programmer"[1]. Programmer skill is insufficient, because maintainability and efficiency are in constant tension.

I illustrate this tension by considering a concrete problem. Consider computing the gradient of a differentiable function as part of backpropagation over a neural network. More precisely, assume f of type `differentiable`

```OCaml
type differentiable = Sin | Tanh | Sigmoid | ...
                    | Polynomial of float list
                    | Compose of differentiable * differentiable
```

For example, the following expression represents $\sin \circ \tanh$.

```OCaml
Compose(Tanh, Sin)
```

We wish to write a function `grad` such that `grad` $f = f'$. For simplicity, assume the existence of a helper function, `grad_of`, that returns the gradient

[1] Though not the worst answer: "use ChatGPT"

of basic functions. For example, `grad_of Sin 0.0` = `cos 0.0` = `1.0`. `grad_main` (Listing 1) illustrates one way to compute gradients maintainably:

```OCaml
let rec grad_main f x = match f with
  | Sin
  | Tanh
  | Sigmoid
  | ...
  | Polynomial(_) -> grad_of f x
  | Compose(f, g) -> (grad_main f x) * (grad_main g (app f x))
```

Listing 1: A maintainable implementation of `grad`

However, `grad_main` may not be the most efficient implementation. Performing a `match` on every recursive call might result in expensive branches. If `x` is a vector, and the weights of a polynomial are vectors, then `grad_main` could hide opportunities for cache prefetching.

If f is known in advance, for example, f = `Compose(Tanh, Sin)`, we could implement a more efficient `grad_fast` function (Listing 2), whose body is simply a hardcoded equation:

```OCaml
let grad_fast x = (cos x) /. (cosh (sin x) ** 2)
```

Listing 2: A fast implementation of `grad`, assuming f = `Compose(Tanh, Sin)`

Although `grad_fast` only works for a single f, it has eliminated the branching overhead, and enabled opportunities for prefetching. It is thus likely to be faster.

The `grad` example illustrates the trade-off between maintainability and efficiency. `grad_main` is maintainable in part because it parameterises over f. More generally, abstraction centralises implementations, thus reducing maintenance costs. However, `grad_fast` is more efficient because it is more specialised to a specific f (`Compose(Tanh, Sin)`). More generally, many compiler optimisations, like monomorphisation, eliminate abstraction, simplifying functions by applying known arguments in advance.

The tension between maintainability (abstraction) and efficiency (specialisation) has also been observed in regex matching [22], parsing [27], linking [20], statistical modelling [24], and hardware design [23].

A more informed answer might therefore be "by letting the compiler generate optimised versions of my maintainable code". Not quite: for reasons both theoretical and practical, compiler optimisations can be insufficient. In theory, we proposed an optimisation that assumed we would always know f at, or before, compile-time. Is this a reasonable assumption? It is: we assumed that `grad` would perform backpropagation over neural networks. The network over which backpropagation is performed is known at compile-time.

However, notice that this justification appeals to domain-specific knowledge regarding how `grad` will be used. In the general case, `grad` could be applied to a function not known until runtime. It is not feasible to expect a compiler to spot all opportunities for optimisation [18]. In practice, while compiler engineers might have an economic incentive to write optimisations for the machine learning community, this may not be true for less lucrative domains [19]. Even in machine learning, many libraries are built on top of existing languages, like `Python`, which might not perform the desired optimisations.

How does one write maintainable and efficient code, **when one cannot trust the compiler to optimise one's code**?

One answer is metaprogramming, which gives users the ability to perform code-generation. Programmers may thus take matters into their own hands: manually generating optimised code when the compiler may not automatically do so for them. The `grad` function in `JAX`, a Python-based machine learning framework, uses metaprogramming for precisely this purpose [9].

Speeding up exponentiation with Metaprogramming

Metaprogramming allows for code-generation via the creation and manipulation of abstract syntax trees (ASTs). I will now illustrate how metaprogramming works with reference to `MacoCaml`, which implements *compile-time* metaprogramming. While the `grad` example motivated metaprogramming, for pedagogical reasons, I switch to a morally equivalent, but simpler example: raising an integer `x` to an exponent `n`. One maintainable implementation is the `pow` function (Listing 3):

```OCaml
let rec pow (n: int) (x: int) =
  if n == 0 then 1
  else x * pow (n-1) x
```

Listing 3: A maintainable implementation of an exponentiation function

`pow`, which can be applied to any exponent `n`, is analogous to `grad_main` (Listing 1), which could be applied to *any* differentiable function f.

However, should we know the exponent in advance, for example $n = 2$, then a more efficient, but less maintainable implementation, is the `square` function (Listing 4)

```OCaml
let square x = x * x
```

Listing 4: An efficient implementation of exponentiation, assuming $n = 2$

`square` is analogous to `grad_fast` (Listing 2).

Metaprogramming can be utilised to write a function, `pow_gen`, which resembles `pow` (inheriting its maintainability), but that generates a program

which resembles `square` (inheriting its efficiency). Listing 5 presents the meta-programmed `pow_gen` function. Compilation generates the body of `square y` (line 4), `y * y * 1`. I will now explain the mechanics of generation.

OCaml
```
1 macro rec pow_gen (n: int) (x: int expr) =
2   if n == 0 then <<1>>
3   else <<$x * $(pow_gen (n-1) x)>>
4 let square y = $(pow_gen 2 <<y>>) (*after compile-time: y * y * 1 *)
5 square 3 (*at runtime: 9*)
```

Listing 5: A meta-programmed `pow_gen` function, which resembles `pow` but generates `square`

Recall that (compile-time) metaprogramming gives the programmer the ability to generate programs at compile-time, for use at run-time. We may build this in two steps, by:

1. Deciding on a representation for code values, such that code can be created, and manipulated by programs. Once we have a representation for code values, it is possible to write expressions that return code values. These expressions serve as program generators.

2. Building a mechanism for executing expressions *at compile-time*. We can constrain this mechanism, using types, so only generators can be executed at compile-time.

First, we represent code values as ASTs. Generated programs are ASTs, and program generators are expressions that evaluate to ASTs. We can assume, for clarity, that the language offers, for each program construct, a corresponding AST node: for example, the integer `1` has AST node `Int(1)`. If a program has type `'a`, then its AST node has type `'a expr`. One can now write program generators, that evaluate to ASTs, for example:

MacoCaml
```
1 let rec pow_gen (n: int) (x: int expr) =
2   if n == 0 then Int(1)
3   else Mul(x, (pow_gen (n-1) x))
4 pow_gen 2 Int(3) (*Mul(Int(3), Mul(Int(3), 1))*)
5 pow_gen 2 Var(y) (*Mul(Var(y), Mul(Var(y), 1))*)
6 pow_gen 3 Var(y) (*Mul(Var(y), Mul(Var(y), Mul(Var(y), 1)))*)
```

Second, we need a mechanism to execute expressions at compile-time. In `MacoCaml`, this is the "top-level splice", a splice (`$`) annotation not surrounded by quotes (`<<>>`). For example, in Listing 5, there is only one top-level splice, on line 4: `$(pow_gen 2 <<y>>)`. We may now shift program generators (and only program generators) under top-level splices, to perform generation at

compile time. Note that to access pow_gen at compile-time, we must also move it under the top-level splice.

MacoCaml

```
let square y = $(let rec pow_gen (n: int) (x: int expr) = ...
                 in pow_gen 2 Var(y))
               (*Mul(Var(y), Mul(Var(y), 1))*)
let cube y   = $(let rec pow_gen (n: int) (x: int expr) = ...
                 in pow_gen 3 Var(y))
               (*Mul(Var(y), Mul(Var(y), Mul(Var(y), 1)))*)
```

To allow compile-time functions, like pow_gen, to be re-used across multiple top-level splices, MacoCaml introduces the **macro** (Listing 6)

OCaml

```
macro rec pow_gen (n: int) (x: int expr) =
  if n == 0 then Int(1)
  else Mul(x, (pow_gen (n-1) x))
let square y = $(pow_gen 2 Var(y)) (*Mul(Var(y), Mul(Var(y), 1))*)
let cube y = $(pow_gen 3 Var(y)) (*Mul(Var(y), Mul(Var(y), Mul(Var(y), 1)))*)
```

Listing 6: In MacoCaml, **macro** allows for definitions to be shared across top-level splices

Further, rather than explicit AST constructors, ASTs are created by the <<>> ("quote") and $ annotations. Quotation creates ASTs, by converting a program into its AST representation. For example,

```
<< $x + 0 >> = Plus(Var(x), Int(0))
```

Under a quotation, the $ annotation stops this conversion, allowing for programs that *manipulate* ASTs.

```
<< $x + 0 >> = Plus(x, Int(0))
```

In MacoCaml, the programmer interleaves quotes and splices to perform code generation

```
<< $(add_zero <<1>>) + 0 >> = Plus(add_zero Int(1), Int(0))
```

Notice that $ is overloaded. We must be careful to disambiguate between "top-level splices", which execute programs at compile-time, and splices under quotations, which stop conversion to AST.

Re-writing Listing 6 in this style (being careful about non-top-level splices), we obtain exactly Listing 5.

Applying this technique to the grad example, we obtain

```
MacoCaml
macro rec grad_gen f x = match f with
  | Sin
  | Tanh
  | Sigmoid
  | ...
  | Polynomial(_) -> grad_of f x
  | Compose(f, g) -> <<$(grad_gen f x) * $(grad_gen g (app f x))>>
let grad_fast x = $(grad_gen Compose(Tanh, Sin) <<x>>)
```

where `grad_of` and `app` are appropriately modified.

2.1.2 The Design Space of Metalanguages

Different metalanguages provide slightly different variants of metaprogramming to the user. In this section, I broadly taxonomise these languages by considering three key design decisions:

1. **Homogenous or Heterogenous**
   Do the generated ("object") and generating ("meta") languages agree or differ?

   If the object and meta languages are the same, this is known as homogenous metaprogramming. Otherwise, it is heterogenous [12].

   `MacoCaml` allows for homogenous metaprogramming, where `OCaml` code generates `OCaml` code. In contrast, `MetaHaskell` [15] programs generate `C` code, allowing for heterogenous metaprogramming.

2. **Run-time or Compile-Time**
   When does the generation take place?

   Code generation could take place at compile-time (as with `MacoCaml` programs or `C` macros), or at run-time (as with `MetaOCaml` [11]).

   Run-time and compile-time metaprogramming differ non-trivially. Run-time metaprogramming requires a language construct (`!`, or "run") for explicit invocation of the compiler. Further, in run-time metaprogramming, generated and generating programs may share a heap.

   `MacoCaml` supports compile-time metaprogramming, and we will pay no further attention to run-time metaprogramming.

3. **Two-stage or Multi-stage**
   How many stages of code generation are allowed?

   When introducing `MacoCaml`, I illustrated how one uses top-level splices to perform shift computation from run-time ("level 0") to compile-time ("level -1"). Might it be possible to shift computation from compile-time to a pre-compile-time ("level -2" phase), for example, via a nested splice?

```
1  $($ pow_gen 2 Var(y))                                    MacoCaml
```

In a two-stage system, one is restricted to operating between two levels, so this is disallowed. In contrast, in a multi-stage system, one can operate between any number of levels. Multi-stage metaprogramming is thus strictly more general than two-stage metaprogramming.

Although nested splices are disallowed in `MacoCaml`, it is a multi-stage system, since entire modules may be imported at a decremented level [26].

`MacoCaml` offers homogenous, compile-time, multi-stage metaprogramming. The scope of this dissertation is slightly more restrictive: I focus on two-stage, not multi-stage metaprogramming. This restriction was motivated by a cost-benefit analysis:

1. **Cost**: Since in `MacoCaml`, the module system is the only mechanism for achieving multi-stage programming, investigating multi-stage metaprogramming would require the investigation of module systems, effects, and metaprogramming. The interaction between module systems and metaprogramming is still an ongoing area of research [4].

2. **Benefit**: In practice, "almost all uses" of multi-stage metaprogramming only use two stages [7]. Further, scope extrusion can be observed, and is often studied, in two-stage systems [8, 13].

2.2 Effect Handlers

What is an effect handler? I will first motivate effect handlers by considering the problem of adding resumable exceptions to `OCaml` (Section 2.2.1). Second, I will introduce a calculus for studying the operational behaviour of effect handlers, à lá Pretnar [17] (Section 2.2.2). This calculus will be useful both for precise description of effect handlers, and as a basis for investigating the interaction between metaprogramming and effect handlers (once the calculus has been extended with metaprogramming facilities). Finally, since different design decisions for effect handlers could affect the nature of their interaction with metaprogramming, I will consider the design space of effect handlers (Section 2.2.3).

2.2.1 Composable and Customisable Effects

Effects are a mechanism by which a program interacts with its environment. Examples of effects include state, (resumable) exceptions, non-determinism, and I/O. Effects are typically defined and understood separately, meaning

they are not easily composable. They are also implemented by compiler engineers rather than programmers, meaning they are not customisable. Effect handlers provide a programmable, unifying framework that may be instantiated into different effects. This allows for composable and customisable treatment of effects.

To illustrate the need for effect handlers, consider the following problem, by Kiselyov [10]. Assume a binary search tree of (key, value) pairs. The following code provides two functions. The first `finds` a value `v` associated with key `k`, raising a `NotFound` exception if `k` is not in the tree. The second `updates` the dictionary with a fresh key value pair, overwriting old values.

```OCaml
type ('a, 'b) tree = Lf | Br of 'a * 'b * tree * tree

let rec find (t: tree) (k: 'a) = match t with
 | Lf -> raise NotFound()
 | Br(k', v, l, r) -> if k == k' then v
                      else if k < k' then find l k
                           else find r k

let rec update (t: tree) (k: 'a) (v: 'b) = match t with
  | Lf -> Br(k, v, Lf, Lf)
  | Br(k', v', l, r) -> if k == k' then Br(k, v, l, r)
                        else if k < k' then Br(k', v', update l k v, r)
                             else Br(k', v', l, update r k v)
```

Assume the task is to build a `findOrInsert` function that either finds the value associated with a key, *or* inserts a default value. A naïve approach to writing this function would be

```OCaml
let rec findOrInsert (t: tree) (k: 'a) (default: 'b) =
  try find t k with NotFound -> insert t k default
```

This function is **inefficient**. If a `NotFound` exception is raised, then the `find` function will have raised the exception at the point where the default value should be inserted. The function could be twice as efficient if the exception could be resumed at the point where the exception was raised, in the following style.

```OCaml
let rec findOrInsert (t: tree) (k: 'a) (default: 'b) =
  try find t k with NotFound(p) -> continue p Br(k, default, Lf, Lf)
```

`p` represents the suspended program to be resumed, and is known as a *delimited continuation*.

The aforementioned problem motivates the need for resumable exceptions. To understand the need for effect handlers, consider how one might go about

implementing resumable exceptions. One approach might be to fork the implementation of handlers and tweak it ever-so-slightly. This solution does not scale well. First, the solution may not be **composable**. The intended informal semantics for resumable exceptions is "effectively equivalent to exceptions, with the additional power to resume programs". Resumable exceptions should thus interact with other exceptions in a predictable way, but this is difficult to guarantee, and *continually* guarantee, especially as implementations evolve, and more variants of exceptions are demanded. Second, the solution is not **customisable**. To add resumable exceptions requires a compiler engineer to modify the compiler. With the exception of raising an issue, there is nothing the programmer may do, in the moment, to meet their need.

Effect handlers resolve both composability and customisability issues. Much like how exception handlers allow users to create custom exceptions with custom semantics, effect handlers provide a general framework for creating custom effects with custom semantics. The interaction between effect handlers is described abstractly, parameterising over the exact semantics of the effect. Hence, implementing effects (in the earlier example, resumable exceptions, but more generally, state, I/O, greenthreading, non-determinism, and more) as effect handlers ensures composability by design.

With effect handlers, we can re-write the previous example to obtain the behaviour of resumable exceptions, even if the `OCaml` compiler does not support it, with the guarantee that **`NotFound`** will interact predictably with other defined effects.

OCaml
```
type _ Effect.t += NotFound: unit -> tree t

let rec find (t: tree) (k: 'a) = match t with
  | Lf -> NotFound()
  | Br(k', v, l, r) -> if k == k' then v
                       else if k < k' then find l k
                            else find r k

let rec findOrInsert (t: tree) (k: 'a) (default: 'b) =
  match find t k with NotFound(p) with
  | v -> v
  | effect NotFound k -> continue p Br(k, default, Lf, Lf)
```

Since effect handlers may be instantiated into a range of different effects, considering the interaction of metaprogramming with effect handlers is an exercise in killing many birds with a single stone. Additionally, effect handlers were recently added to `OCaml` [21], making their interaction a timely problem.

λ_{op}

Values	v	:=	$x \mid n \mid \lambda x.c \mid \kappa x.c$
Computations	c	:=	$v_1 v_2 \mid \textbf{return}\ v \mid \textbf{do}\ x \leftarrow c_1\ \textbf{in}\ c_2 \mid$
			$\textbf{op}(v) \mid \textbf{handle}\ c\ \textbf{with}\ \{h\} \mid \textbf{continue}\ v_1\ v_2$
Handlers	h	:=	$\textbf{return}(x) \mapsto c \mid h; \textbf{op}(x, k) \mapsto c$

Figure 2.1: The syntax of λ_{op}. Terms are syntactically divided into values v, computations c, and handlers h

2.2.2 λ_{op}: A Calculus for Effect Handlers

Having motivated effect handlers, I will now describe a calculus, which I call λ_{op}, for reasoning about their operational behaviour. λ_{op} is a slight variant of the calculus described by Pretnar [17]. Understanding λ_{op} will be useful for two reasons. First, a precise description of the operational behaviour of effects will aid reasoning about their interaction with metaprogramming. Second, my universal calculus will be described by extending λ_{op}. Throughout this section, I will use the λ_{op} program in Listing 7 as a running example.

λ_{op}

```
handle
  do x ← print(1); return 1 in do y ← print(2); return 2 in x + y
with
  {return(x) ↦ return (x, "");
   print(x, k) ↦ do (v, s) ← continue k () in return (v, f"{x};" ^ s)}
```

```
return (3, "1;2")
```

Listing 7: An λ_{op} program that returns `(3, "1;2")`. It will be used as a running example throughout this section.

Figure 2.1 collates the base syntax of λ_{op}. In addition to this base syntax, in this section, I will assume λ_{op} is extended with the following language extensions: a unit value `()`, pairs `(1, 2)` which can be destructured `do (x, y) ← return ("Revolution", f"{9}") in x ^ y`, strings `"Hello"`, format strings `f"{1}"`, and string concatenation `^`. For example, the following code evaluates to `"Revolution 9"`.

λ_{op}

```
do (x, y) ← return ("Revolution", f"{9}") in x ^ y
```

```
return "Revolution 9"
```

Further, I use $c_1; c_2$ as syntactic sugar for $\textbf{do}\ \_ \leftarrow c_1\ \textbf{in}\ c_2$. I will explain key language constructs in turn.

Sequencing computations: `do` and `return`

Effects force us to carefully consider the order of evaluation. For example, consider the following `OCaml` programs

```
OCaml
let pure      = (1+0) + (2+0)
let effectful = let l = new 0 in (l := 1; 1) + (l := 2; 2)
```

The result of `pure`, which has no effects, is independent of the evaluation order. In contrast, the result of `effectful` *is* dependent on the evaluation order. If terms are evaluated left-to-right, the value of `!l` is `2`, otherwise, it is `1`.

In order to be precise about the order of evaluation, λ_{op} terms are stratified into two syntactic categories, "inert values" v and "potentially effectful computations" c [17]. **return** v lifts values into computations, and is also the result of fully evaluating a computation. **do** $x \leftarrow c_1$ **in** c_2 sequences computations, forcing programmers to be explicit the order of evaluation. First, c_1 is fully evaluated to obtain some **return** v. The value v is then bound to x, and finally c_2 is evaluated.

For example, extending λ_{op} with a `plus` function, what is the order of evaluation of `plus 1 2`? Do we evaluate both arguments before applying them, or interleave evaluation and application? The syntax forces programmers to choose explicitly. We can either fully evaluate both arguments before applying them in turn,

λ_{op}
$$\textbf{do}\ x \leftarrow \textbf{return}\ 1\ \textbf{in}\ (\textbf{do}\ y \leftarrow \textbf{return}\ 2\ \textbf{in}\ (\textbf{do}\ f \leftarrow \texttt{plus}\ x\ \textbf{in}\ f y))$$

or alternatively, evaluate `1`, apply it, then evaluate `2`

λ_{op}
$$\textbf{do}\ x \leftarrow \textbf{return}\ 1\ \textbf{in}\ (\textbf{do}\ f \leftarrow \texttt{plus}\ x\ \textbf{in}\ \textbf{do}\ y \leftarrow \textbf{return}\ 2\ \textbf{in}\ f y)$$

Both choices are valid, but the programmer must choose. For clarity, where the ordering cannot affect the result (both of the aforementioned choices evaluate to **return** 3), I will abuse notation and write (for instance) $1 + 2$.

Performing effects: `op`, `handle`, and `continue`

Having made explicit the order of operation, we may now add effect handlers. Recall that effect handlers allow users to register custom effects with custom semantics. I will now illustrate how this is supported by λ_{op}.

For simplicity, λ_{op} assumes that the effects have been registered in advanced, parameterising over them with the placeholder **op**(v). Assume that the user has declared the effects **print** and **read_int** in advance. This would allow the user to write programs like

```
do x ← print(1); return 1 in do y ← print(2); return 2 in x + y
```
λ_{op}

In the program fragment above, we know that **print** is an effect, but we do not know its semantics. Effect handlers, which comprise a **return handler** and zero or more **operation handlers**, specify how effects interact with their environment, and thus may be used to give effects meaning. I will define an effect handler that accumulates print statements in a string (some "stdout"). For example, the aforementioned program should return (3, "1; 2").

We begin by considering how to handle the case where there are no calls to **print**. For example, in the program **return** 3. We may wish to return both the value, and the empty string (empty stdout) to the environment: in this case, (3, ""). We can achieve this by specifying a *return handler*.

$$\textbf{return}(x) \mapsto c$$

In this case, we set c to **return** $(x, \texttt{""})$. All effect handlers must specify a return handler. In many cases, the return handler is simply the identity (c is set to **return** x): for brevity and clarity, if the return handler is the identity, I may drop it.

Next, we consider how to handle a call to **print**. We use an operation handler of the form

$$\textbf{print}(x, k) \mapsto c$$

Where c is the user-defined semantics for **print**. Concretely, one instance of c is

$$\textbf{print}(x, k) \mapsto \textbf{do}\ (v, s) \leftarrow \textbf{continue}\ k\ ()\ \textbf{in}\ \textbf{return}\ (v, \texttt{f"\{}x\texttt{\};"} \mathbin{\hat{}} s)$$

In the definition of c, the programmer may refer to x and k, which I will now explain. x allows programs to send values (for example, values to be printed) to their environment. k is a delimited continuation representing a suspended program, awaiting a value from the environment. Effects also allow programs to receive data from their environment, as in

$$1 + \textbf{get\_int\_from\_user}()$$

Note that the program is suspended until the value is receive. We may write the suspended program as $1 + [-]$, where $[-]$ indicates an as-yet-unknown value. This suspended program is represented by the continuation k. The expression

$$\textbf{continue}\ k\ v$$

is used to resume the suspended program with value v.

We are now able to interpret the concrete operation handler c: we resume the suspended program, supplying a unit value, since **print** effects do not receive values from their environment. This returns a value v and some partially accumulated stdout s. We prepend the printed value, x, onto s.

Having defined the semantics for **print**, the user may now interpret the earlier example with their semantics, using the **handle** e **with** $\{h\}$ construct. Doing so results in the program in Listing 7.

Notice that multiple effects may be handled by the same handler, and the same effect might be handled by multiple handlers, potentially with different semantics.

Operational Semantics

Having described informally the desired semantics of λ_{op}, we may now make our intuitions precise, by means of an operational semantics. The operational semantics is collated in Figure 2.2.

λ_{op}

Auxiliary Definitions

$$
\begin{array}{lrcl}
\text{Evaluation Frame} & F & ::= & \textbf{do}\ x \leftarrow [-]\ \textbf{in}\ c_2 \mid \textbf{handle}\ [-]\ \textbf{with}\ \{h\} \\
\text{Evaluation Context} & E & ::= & [-] \mid E[F]
\end{array}
$$

$$
\begin{array}{lrcl}
\text{Domain of Handler} & \mathsf{dom}(h) & \triangleq & \mathsf{dom}(\textbf{return}(x) \mapsto c) = \emptyset, \\
 & & & \mathsf{dom}(h; \textbf{op}(x,k) \mapsto c) = \mathsf{dom}(h) \cup \{\textbf{op}\} \\
\text{Handled Effects} & \mathsf{handled}(E) & \triangleq & \mathsf{handled}([-]) = \emptyset, \\
 & & & \mathsf{handled}(E[\textbf{do}\ x \leftarrow [-]\ \textbf{in}\ c_2]) = \mathsf{handled}(E), \\
 & & & \mathsf{handled}(E[\textbf{handle}\ [-]\ \textbf{with}\ \{h\}]) = \mathsf{handled}(E) \cup \mathsf{dom}(h),
\end{array}
$$

Operational Semantics

$$
\begin{array}{lrcl}
(\textsc{Red-App}) & (\lambda x.c)v; E & \to & c[v/x]; E \\
(\textsc{Red-Seq}) & \textbf{do}\ x \leftarrow \textbf{return}\ v\ \textbf{in}\ c; E & \to & c[v/x]; E \\
(\textsc{Red-Hdl}) & \textbf{handle}\ \textbf{return}\ v\ \textbf{with}\ \{h\}; E & \to & c[v/x]; E \quad (\text{where } \textbf{return}(x) \mapsto c \in h) \\
 & & & \\
(\textsc{Cng-Psh}) & F[c]; E & \to & c; E[F] \\
(\textsc{Cng-Pop}) & \textbf{return}\ v; E[F] & \to & F[\textbf{return}\ v]; E \\
 & & & \\
(\textsc{Eff-Op}) & \textbf{op}(v); E_1[\textbf{handle}\ E_2\ \textbf{with}\ \{h\}] & \to & c[v/x, \kappa x.\, \textbf{handle}\ E_2[\textbf{return}\ x]\ \textbf{with}\ \{h\}/k]; E_1 \\
 & & & (\text{where } \textbf{op}(x,k) \mapsto c \in h \text{ and } \textbf{op} \notin \mathsf{handled}(E_2)) \\
(\textsc{Eff-Cnt}) & \textbf{continue}\ E_2\ v; E_1 & \to & \textbf{return}\ v; E_1[E_2]
\end{array}
$$

Figure 2.2: The operational semantics of λ_{op}. The semantics is given on configurations of the form $\langle c, E \rangle$, with the brackets dropped for clarity. Rules are divided into three classes: reduction rules Red-X, which perform computation, congruence rules Cng-Y which manipulate the evaluation context, and effect rules Eff-Z that are special to λ_{op}

The operational semantics is given on configurations of the form $\langle c, E \rangle$, where c is a term and E is an evaluation context, in the style of Felleisen and Friedman [5]. Evaluation contexts are represented as a stack of evaluation frames F, à lá Kiselyov [10]. Most of the rules are standard. We will focus on two rules: Eff-Op, the mechanism for giving effects custom semantics, and Eff-Cnt, the mechanism for resuming programs.

To illustrate the operation of Eff-Op and Eff-Cnt, consider the evaluation of the running example in Listing 7, beginning with an empty context. Let h be the handler body

$$\{\textbf{return}(x) \mapsto \textbf{return}\ (x, \texttt{""});\\ \ \textbf{print}(x, k) \mapsto \textbf{do}\ (v, s) \leftarrow \textbf{continue}\ k\ ()\ \textbf{in}\ \textbf{return}\ (v, \texttt{f"\{}x\texttt{\};" \^{}}\ s)\}$$

After several applications of Cng-Pop, we obtain the configuration

$$\langle\ \texttt{print(1)}\ ;\ \textbf{handle}\\ \quad\quad \textbf{do}\ x \leftarrow [-]; \texttt{return 1}\ \textbf{in}\ \textbf{do}\ y \leftarrow \texttt{print(2); return 2}\ \textbf{in}\ x + y\\ \quad \textbf{with}\ \{h\}\ \rangle$$

Let $E = \textbf{do}\ x \leftarrow \texttt{return}\ u; \texttt{return 1}\ \textbf{in}\ \textbf{do}\ y \leftarrow \texttt{print(2); return 2}\ \textbf{in}\ x + y$. Applying Eff-Op, we can suspend the program, find the handler h with the user's semantics for **print**, and give the **print** effect the desired semantics

$$\langle\ \textbf{do}\ (v, s) \leftarrow \textbf{continue}\ (\kappa u.\ \textbf{handle}\ E\ \textbf{with}\ \{h\})\ ()\ \textbf{in}\ \textbf{return}\ (v, \texttt{f"\{1\};" \^{}}\ s)\ \ ;\ \ [-]\rangle$$

Applying Cng-Pop,

$$\langle\ \textbf{continue}\ (\kappa u.\ \textbf{handle}\ E\ \textbf{with}\ \{h\})\ ()\ \ ;\ \ \textbf{do}\ (v, s) \leftarrow [-]\ \textbf{in}\ \textbf{return}\ (v, \texttt{f"\{1\};" \^{}}\ s)\ \rangle$$

Applying Eff-Cnt, we can resume the program that was suspended

$$\langle\ \textbf{return}\ ()\ \ ;\ \ \textbf{do}(v, s) \leftarrow\\ \quad\quad \textbf{handle}\\ \quad\quad\quad \textbf{do}\ x \leftarrow ([-]; \texttt{return 1})\textbf{in}\\ \quad\quad\quad \textbf{do}\ y \leftarrow (\texttt{print(2); return 2})\\ \quad\quad\quad \textbf{in}\ x + y\\ \quad\quad \textbf{with}\ \{h\}\\ \quad \textbf{in}\ \textbf{return}\ (v, \texttt{f"\{1\};" \^{}}\ s)\ \rangle$$

The side-condition on Eff-Op is needed because the user may define multiple handlers with different semantics for the same effect. The side-condition resolves any ambiguity by using the *latest* handler. For example, the following program has a **read** effect that is given two definitions: it could read either 1 or 2. The ambiguity is resolved by choosing the latest handler: in this case, 1.

λ_{op}

```
handle
  handle read() with {return(y) ↦ return y; read(x, k) ↦ continue k 1}
with
  {return(y) ↦ return y; read(x, k) ↦ continue k 2}
```

```
return 1
```

λ_{op}

Effects row	$\Delta ::= \emptyset \mid \Delta \cup \{\mathsf{op}_i\}$	
Value type	$T ::= \mathbb{N}$	
	$\mid T_1 \xrightarrow{\Delta} T_2$	functions
	$\mid T_1 \xrightarrow{\Delta}\!\!\!\!\rightarrow T_2$	continuations
Computation type	$T \,!\, \Delta$	
Handler type	$T_1 \,!\, \Delta \Longrightarrow T_2 \,!\, \Delta'$	

Figure 2.3: λ_{op} types. Notice that, just as terms are divided into values, computations, and handlers, types are divided into value types (T), computation types ($T \,!\, \Delta$), and handler types ($T_1 \,!\, \Delta \Longrightarrow T_2 \,!\, \Delta'$)

Type-and-Effect System

We now give a type-and-effect system to λ_{op}. Figure 2.3 collates the syntax of λ_{op} types, which I will now briefly describe.

Just like terms, types are divided into value types (for example, $\mathbb{N}$), computation types ($\mathbb{N} \,!\, \{\mathbf{print}\}$), and handler types ($\mathbb{N} \,!\, \{\mathbf{print}\} \Longrightarrow \mathbb{N} \,!\, \emptyset$). Since computations may have effects, computation types track unhandled effects using an effects row (Δ), which in this system is simply a set. This type-and-effect system allows us to distinguish between values, computations that return values, and computations that return values and additionally have some unhandled side effects.

Term	**Type**
3	$\mathbb{N}$
$\mathbf{do}\ x \leftarrow \mathbf{return}\ 1\ \mathbf{in}\ \mathbf{do}\ y \leftarrow \mathbf{return}\ 2\ \mathbf{in}\ x + y$	$\mathbb{N} \,!\, \emptyset$
$\mathbf{do}\ x \leftarrow \mathbf{print}(1); \mathbf{return}\ 1\ \mathbf{in}\ \mathbf{do}\ y \leftarrow \mathbf{return}\ 2\ \mathbf{in}\ x + y$	$\mathbb{N} \,!\, \{\mathbf{print}\}$

Functions are values, and are applied to other values, but produce computations on application. For example, the function

$$\lambda x : \mathbb{N}.\, \mathbf{print}(x); \mathbf{return}\ x$$

is a value that accepts a value of type $\mathbb{N}$ and returns a computation of type $\mathbb{N} \,!\, \{\mathbf{print}\}$. We thus say functions have suspended effects, which we write $T_1 \xrightarrow{\Delta} T_2$. In this case, the function has type $\mathbb{N} \xrightarrow{\{\mathbf{print}\}} \mathbb{N}$. For technical reasons, continuations and functions need to be distinguished, but in most cases they may be treated equivalently.

Handlers transform computations of one type to computations of another type. This happens in two ways: first, by handling effects, and thus removing them from the effects row (which recall represents unhandled effects). Second, by modifying the return type of computations. To reflect both abilities,

λ_{op}

(Nat)
$$\frac{}{\Gamma \vdash n : \mathbb{N}}$$

(Var)
$$\frac{\Gamma(x) = T}{\Gamma \vdash x : T}$$

(Lambda)
$$\frac{\Gamma, x : T_1 \vdash c : T_2 \,!\, \Delta}{\Gamma \vdash \lambda x.c : T_1 \xrightarrow{\Delta} T_2}$$

(Continuation)
$$\frac{\Gamma, x : T_1 \vdash c : T_2 \,!\, \Delta}{\Gamma \vdash \kappa x.c : T_1 \xrightarrow{\Delta}\!\!\!\!\rightarrow T_2}$$

(App)
$$\frac{\Gamma \vdash v_1 : T_1 \xrightarrow{\Delta} T_2 \qquad \Gamma \vdash v_2 : T_1}{\Gamma \vdash v_1\, v_2 : T_2 \,!\, \Delta}$$

(Continue)
$$\frac{\Gamma \vdash v_1 : T_1 \xrightarrow{\Delta}\!\!\!\!\rightarrow T_2 \qquad \Gamma \vdash v_2 : T_1}{\Gamma \vdash \textbf{continue}\, v_1\, v_2 : T_2 \,!\, \Delta}$$

(Return)
$$\frac{\Gamma \vdash v : T}{\Gamma \vdash \textbf{return}\ v : T \,!\, \Delta}$$

(Do)
$$\frac{\Gamma \vdash c_1 : T_1 \,!\, \Delta \qquad \Gamma, x : T_1 \vdash c_2 : T_2 \,!\, \Delta}{\Gamma \vdash \textbf{do}\ x \leftarrow c_1\ \textbf{in}\ c_2 : T_2 \,!\, \Delta}$$

(Op)
$$\frac{\Gamma \vdash v : T_1 \qquad \mathsf{op} : T_1 \to T_2 \in \Sigma \qquad \mathsf{op} \in \Delta}{\Gamma \vdash \textbf{op}(v) : T_2 \,!\, \Delta}$$

(Handle)
$$\frac{\begin{array}{c}\Gamma \vdash c : T_1 \,!\, \Delta \\ \Gamma \vdash h : T_1 \,!\, \Delta \Longrightarrow T_2 \,!\, \Delta' \\ \forall \mathsf{op} \in \Delta \setminus \Delta'.\mathsf{op} \in \mathsf{dom}(h)\end{array}}{\Gamma \vdash \textbf{handle}\ c\ \textbf{with}\ \{h\} : T_2 \,!\, \Delta'}$$

(Ret-Handler)
$$\frac{\Gamma, x : T_1 \vdash c : T_2 \,!\, \Delta'}{\Gamma \vdash \textbf{return}(x) \mapsto c : T_1 \,!\, \Delta \Longrightarrow T_2 \,!\, \Delta'}$$

(Op-Handler)
$$\frac{\begin{array}{c}\mathsf{op} : A \to B \in \Sigma \\ \Gamma \vdash h : T_1 \,!\, \Delta \Longrightarrow T_2 \,!\, \Delta' \\ \Gamma, x : A, k : B \xrightarrow{\Delta'}\!\!\!\!\rightarrow T_2 \vdash c : T_2 \,!\, \Delta' \qquad \Delta' \subseteq \Delta \setminus \{\mathsf{op}\} \qquad \textbf{op}(x', k') \mapsto c' \notin h\end{array}}{\Gamma \vdash h; \textbf{op}(x, k) \mapsto c : T_1 \,!\, \Delta \Longrightarrow T_2 \,!\, \Delta'}$$

Figure 2.4: Typing rules for λ_{op} terms

handlers are given a type of the form $T_1 \,!\, \Delta \Longrightarrow T_2 \,!\, \Delta'$. For example, a handler of the form

$$\begin{array}{l}\{\textbf{return}(x) \mapsto \textbf{return}\ (x, \texttt{""}); \\ \ \textbf{print}(x, k) \mapsto \textbf{do}\ (v, s) \leftarrow \textbf{continue}\, k\, ()\ \textbf{in}\ \textbf{return}\ (v, \texttt{f"\{}x\texttt{\};" ^ }s)\}\end{array}$$

may be given type $\mathbb{N} \,!\, \{\textbf{print}\} \Longrightarrow (\mathbb{N} \times \mathsf{String}) \,!\, \emptyset$, reflecting both the handling of the **print** effect and the transformation of the return type to include the collated print statements.

I now consider the typing rules for terms, which are collated in Figure 2.4. Most rules are standard, but a few are worth paying attention to.

First, the Return and Do rules. In the Return rule, we are allowed to assign

the term **return** v any set of effects. For example, we could write:

$$\frac{}{\Gamma \vdash \textbf{return}\ \texttt{0} : \mathbb{N}\,!\,\{\textbf{print}\}}$$

This flexibility is important, because to type **do** $x \leftarrow c_1$ **in** c_2, the Do rule requires both c_1 and c_2 to have the same effects. For example, without this flexibility, we would not be able to complete the following typing derivation

$$\frac{\vdots}{\Gamma \vdash \textbf{do}\ x \leftarrow \textbf{print}(\texttt{0})\ \textbf{in}\ \textbf{return}\ \texttt{0} : \mathbb{N}\,!\,\{\textbf{print}\}}$$

A valid alternative would be to forbid this flexibility and add explicit subtyping. However, such an approach would no longer be syntax directed.

Second, the Op rule. Previously, we assumed that the user declared their effects in advance. We also assume that they declare the types of their effects in advance, and that we we store the mapping from effects to types in Σ. For example, we might assume $\Sigma = \{\textbf{print} : \mathbb{N} \to 1\}$. In `OCaml`, this would correspond to writing:

```OCaml
type _ Effect.t += Print: nat -> unit
```

Note further the $\mathsf{op} \in \Delta$ restriction – flexibility allows us to over-approximate the effects in a term, but never underapproximate them.

Third, the Ret-Handler and Op-Handler rules, which are used to type handlers, which I will explain by means of an example. Assume we are trying to type the handler

$$\begin{array}{l}\{\textbf{return}(x) \mapsto \textbf{return}\ (x, \texttt{""});\\ \ \textbf{print}(x, k) \mapsto \textbf{do}\ (v, s) \leftarrow \textbf{continue}\ k\ ()\ \textbf{in}\ \textbf{return}\ (v, \texttt{f"\{}x\texttt{\};"} \mathbin{\hat{}} s)\}\end{array}$$

with the type $\mathbb{N}\,!\,\{\textbf{print}\} \Longrightarrow (\mathbb{N} \times \mathsf{String})\,!\,\emptyset$. We apply the Op-Handler rule, which is transcribed below. Preconditions are numbered for reference.

(Op-Handler)

$$\frac{\begin{array}{c}(1)\ \mathsf{op} : A \to B \in \Sigma\\ (2)\ \Gamma \vdash h : T_1\,!\,\Delta \Longrightarrow T_2\,!\,\Delta' \qquad (3)\ \Gamma, x : A, k : B \xrightarrow{\Delta'\twoheadrightarrow} T_2 \vdash c : T_2\,!\,\Delta'\\ (4)\ \Delta' \subseteq \Delta \setminus \{\mathsf{op}\} \qquad (5)\ \textbf{op}(x', k') \mapsto c' \notin h\end{array}}{\Gamma \vdash h; \textbf{op}(x, k) \mapsto c : T_1\,!\,\Delta \Longrightarrow T_2\,!\,\Delta'}$$

The preconditions of the Op-Handler rule direct us to check, in turn:

(1) **print** $: \mathbb{N} \to 1 \in \Sigma$, which is true by assumption

(2) Recursively check the rest of the handler $h = \textbf{return}(x) \mapsto \textbf{return}\ (x, \texttt{""})$, ensuring it has type $\mathbb{N}\,!\,\{\textbf{print}\} \Longrightarrow (\mathbb{N} \times \mathsf{String})\,!\,\emptyset$. This follows from a trivial application of the Ret-Handler rule.

(3) Assuming x has type $\mathbb{N}$ and k has type $1 \xrightarrow{\emptyset\!\!\twoheadrightarrow} (\mathbb{N} \times \mathsf{String})$, the body

$$\textbf{do}\ (v, s) \leftarrow \textbf{continue}\ k\ ()\ \textbf{in}\ \textbf{return}\ (v, \mathtt{f"\{}x\mathtt{\};"} \mathbin{\hat{}} s)$$

has type $(\mathbb{N} \times \mathsf{String})\,!\,\emptyset$. This is easy to show.

(4) That the handler *only* removes **print** from the effects row, and no other effects. This check passes, but would fail if we tried to type the handler with, for example, $\mathbb{N}\,!\,\{\textbf{print}, \textbf{get}\} \Longrightarrow (\mathbb{N} \times \mathsf{String})\,!\,\emptyset$

(5) That there are no other handlers for **print** in h.

A full typing derivation may be found in the appendix.

Metatheory

I will build not only on λ_{op}, but on metatheoretic properties of λ_{op}, which are proven by Bauer and Pretnar [2]. We first state the standard progress and preservation properties.

Theorem 2.2.1 (Progress) *If $\Gamma \vdash E[c] : T\,!\,\Delta$ and then either*

1. *c of the form $\textbf{return}\ v$ and $E = [-]$,*
2. *c of the form $\textbf{op}(v)$ for some $\mathsf{op} \in \Delta$,*
3. *$\exists. E', c'$ such that $\langle c; E\rangle \to \langle c'; E'\rangle$*

Theorem 2.2.2 (Preservation) *If $\Gamma \vdash E[c] : T\,!\,\Delta$ and $\langle c; E\rangle \to \langle c'; E'\rangle$, then $\Gamma \vdash E'[c'] : T\,!\,\Delta$*

As a corollary, we obtain type safety.

Corollary 2.2.1 (Type Safety) *If $\cdot \vdash E[c] : T\,!\,\Delta$ and then either*

1. *c of the form $\textbf{return}\ v$ and $E = [-]$ for $\cdot \vdash v : T$*
2. *c of the form $\textbf{op}(v)$ for some $\mathsf{op} \in \Delta$,*
3. *$\exists. E', c'$ such that $\langle c; E\rangle \to \langle c'; E'\rangle$ and $\cdot \vdash E'[c'] : T\,!\,\Delta$*

Finally, λ_{op} is a *fine-grained call-by-value* [14] approach. We first define a notion of contextual equivalence $\cong_{ctx}$. Informally, two computations/values are contextually equivalent if they behave the same in all "relevant" contexts and at all ground (first-order) types (in this case, just $\mathbb{N}$).

Definition 2.2.1 (Contextual Equivalence) *c and c' are contextually equivalent at context Γ and type $T \,!\, \Delta$, written $\Gamma \vdash c \cong_{\text{ctx}} c' :' T \,!\, \Delta$ if*

1. *$\Gamma \vdash c : T \,!\, \Delta$ and $\Gamma \vdash c' : T \,!\, \Delta$*
2. *For all E such that $\cdot \vdash E[c] : \mathbb{N} \,!\, \emptyset$ and $\cdot \vdash E[c'] : \mathbb{N} \,!\, \emptyset$,*

$$\langle c; E\rangle \;\to^*\; \langle \mathbf{return}\ v; [-]\rangle \iff \langle c'; E\rangle \;\to^*\; \langle \mathbf{return}\ v; [-]\rangle$$

When the context Γ is empty, we write $c \cong_{\text{ctx}} c' : T \,!\, \Delta$, and when the type is unimportant, we write $c \cong_{\text{ctx}} c'$

Theorem 2.2.3 (Fine-Grained CBV) *λ_{op} is a fine-grained call-by-value language, meaning in particular that the following equations hold (notationally, I use c for computations, v for values, and f for function values):*

1. $\mathbf{do}\ x \leftarrow \mathbf{return}\ v\ \mathbf{in}\ c \;\cong_{\text{ctx}}\; c[v/x]$
2. $c \;\cong_{\text{ctx}}\; \mathbf{do}\ x \leftarrow c\ \mathbf{in}\ \mathbf{return}\ x$
3. $\mathbf{do}\ x \leftarrow c_1\ \mathbf{in}\ (\mathbf{do}\ y \leftarrow c_2\ \mathbf{in}\ c_3) \;\cong_{\text{ctx}}\; \mathbf{do}\ y \leftarrow (\mathbf{do}\ x \leftarrow c_1\ \mathbf{in}\ c_2)\ \mathbf{in}\ c_3$
4. $(\lambda x.c)v \;\cong_{\text{ctx}}\; c[v/x]$
5. $f \;\cong_{\text{ctx}}\; \lambda x.fx$

In the third equation, we assume x not free in c_3.

2.2.3 The Design Space of Effect Handlers

The design space of effect handlers is large. I consider three key design decisions made by different systems.

1. **Named or Unnamed Handlers**
   Can I invoke a specific handler for an operation?

   In λ_{op}, when there are multiple handlers for the same effect, we invoke the "nearest" or "most recent" handler for that effect. An alternative approach is **named handlers** [25], where, by associating each handler with a *name*, we can more easily specify which handler should be invoked.

   While named handlers can be more ergonomic, they do not provide greater expressiveness than using distinct effects [25]. I do not consider named handlers in this thesis.

2. **Deep, Shallow, or Sheep Handlers**
   Are multiple instances of the same effect handled by the same handler? In λ_{op}, continuations reinstate handlers (Eff-Cnt) and thus multiple instances of the same effect are handled by the same handler. For example, in the following example, the effect **addn** is handled by the same handler, adding one each time. We say these handlers are **deep**.

λ_{op}

```
handle
  addn(1) + addn(2)
with
  {return(x) ↦ return x;
   addn(y, k) ↦ continue k (y + 1)}
```

```
return 5
```

We may also choose *not* to reinstate the handler, in an approach known as **shallow** handlers [6]. The example above would be stuck, since the second **addn** would not be handled.

Finally, we could choose to modify the interface for **continue** such that it accepts a handler

$$\textbf{continue}\ k\ v\ h$$

This would allow multiple effects to be handled by different handlers. That is, we could add one the first time **addn** is performed, and two the second time. These handlers behave as a hybrid of shallow and deep handlers, and are thus termed **sheep** handlers [16].

`OCaml` allows the programmer to choose between shallow and deep handlers. Since most prior work on scope extrusion focuses on deep handlers [8], we focus on those.

3. **One-Shot or Multi-Shot Continuations**
   How many times can one resume the same continuation?

   In λ_{op}, continuations may be resumed multiple times. For example, we can write

λ_{op}

```
handle
  performTwice(1)
with
  {return(x) ↦ return x;
   performTwice(y, k) ↦ (continue k y) + (continue k y)}
```

```
return 2
```

We say the effect system permits **multi-shot continuations**. Multi-shot continuations are useful for simulating certain effects, like non-determinism [16].

In other systems, like `OCaml` and `WasmFX` [16], this is not allowed: continuations are only allowed to be resumed once. These systems permit **one-shot continuations**.

Although continuations in `OCaml` are one-shot, due to the utility of multi-shot continuations, I believe it is worthwhile to study effect systems with multi-shot continuations.

2.3 Scope Extrusion

I now turn my attention to scope extrusion, which arises from the unexpected interaction of effects and metaprogramming. To illustrate scope extrusion, I will first extend λ_{op} (Page 16) with AST constructors $\texttt{Var}(x_T)$, $\texttt{Nat}(n)$, `Lam`, and `Plus`. For example, we may generate the AST of $\lambda x : \mathbb{N}.\, x + 0$ as follows:

λ_{op}

$$\textbf{return}\ \texttt{Lam}(\texttt{Var}(x_{\mathbb{N}}), \texttt{Plus}(\texttt{Var}(x_{\mathbb{N}}), \texttt{Nat}(0)))$$

Listing 8 illustrates the problem of scope extrusion. The program constructs the AST of $\lambda x : \mathbb{N}.\, x$, but additionally performs an effect, **`extrude`**, with type $\mathbb{N}\,\texttt{expr} \to \mathbb{N}\,\texttt{expr}$. The handler for **`extrude`** discards the continuation, simply returning the value it was given: $\texttt{Var}(x_{\mathbb{N}})$. The entire program evaluates to $\texttt{Var}(x_{\mathbb{N}})$, and the generated AST is ill-scoped. We say that the result of evaluation demonstrates scope extrusion.

λ_{op}

```
handle
  do body ← extrude(Var(x_ℕ)) in return Lam(Var(x_ℕ), body)
with
  {return(u) ↦ return Nat(0);
   extrude(y, k) ↦ return y}
```

```
return Var(x_ℕ)
```

Listing 8: A λ_{op} program that evaluates to the $\texttt{Var}(x_{\mathbb{N}})$. The AST is ill-scoped, and thus exhibits scope extrusion. It will be used as a running example.

It is difficult to give a precise definition to scope extrusion, because there are multiple competing definitions [11, 13], and many are given informally. For example, is scope extrusion a property of the *result* of evaluation [13], as in Listing 8, or is it a property of *intermediate* configurations [11]? We can, for example, build ASTs with extruded variables, that are bound at some future

point. In Listing 11, we produce the intermediate AST $\texttt{Plus}(\texttt{Nat}(0), \texttt{Var}(x_{\mathbb{N}}))$, which is not well scoped. However, the result of evaluation is well scoped: $\texttt{Lam}(\texttt{Var}(x_{\mathbb{N}}), \texttt{Plus}(\texttt{Nat}(0), \texttt{Var}(x_{\mathbb{N}})))$. Does Listing 11 exhibit scope extrusion?

Nevertheless, all definitions agree on the example in Listing 8. Making precise the competing definitions of scope extrusion these competing definitions, and their relation to one another, is a contribution of this dissertation.

λ_{op}

```
handle
  do body ← extrude(Var(x_ℕ)); return Var(x_ℕ) in return Lam(Var(x_ℕ), body)
with
  {return(u) ↦ return u;
   extrude(y, k) ↦ do z ← return y in continue k z}
```

```
return Lam(Var(x_ℕ), Plus(Nat(0), Var(x_ℕ)))
```

Listing 9: A λ_{op} program that is a slight variation of Listing 11, but that (unlike Listing 11) passes the Eager Dynamic Check.

2.3.1 Existing Solutions to the Scope Extrusion Problem

There are multiple solutions to the problem of scope extrusion. The solution space can be broadly divided into two types of approaches: static (type-based) and dynamic. I will now survey two dynamic approaches, which I term the lazy and eager checks, and one static approach, the method of refined environment classifiers [13, 8].

Lazy Dynamic Check

Scope extrusion, at least, of the kind in Listing 12, may seem trivial to resolve: evaluate the program to completion, and check that the resulting AST is well-scoped [11]. I term this the Lazy Dynamic Check. This approach, while clearly correct and maximally expressive, is not ideal for efficiency and error reporting reasons.

To illustrate the inefficiency of this approach, consider a slight variation of Listing 8, Listing 10. In Listing 10, we can, in theory, report a warning as soon scope extrusion is detected. However, waiting for the result of the program can be much more inefficient.

In terms of error reporting, note that, in waiting for the result of execution, we lose information about *which program fragment* was responsible for scope extrusion, reducing the informativeness of reported errors [11].

λ_{op}

```
do x ← handle
         do body ← extrude(Var(x_ℕ)) in return Lam(Var(x_ℕ), body)
       with
         {return(u) ↦ return Nat(0);
          extrude(y, k) ↦ return y}
in some very long program; return x
```

```
return Var(x_ℕ)
```

Listing 10: A λ_{op} program that evaluates to the $\texttt{Var}(x_{\mathbb{N}})$. Executing the entire program to determine if it exhibits scope extrusion is inefficient.

Eager Dynamic Check

A second dynamic check, motivated by the problems with the Lazy Dynamic Check, adopts a stricter definition of scope extrusion. During the code generation process, one inserts checks into the running program, reporting errors when one encounters unbound free variables in intermediate ASTs. Hence, the Eager Dynamic Check would classify the program in Listing 11 as exhibiting scope extrusion. The Eager Dynamic Check has been adopted by BER MetaOCaml, and offers better efficiency and error reporting guarantees over the Lazy Dynamic Check [11].

However, the Eager Dynamic Check is not without issue. The problem relates to the manner in which checks are inserted, which we will make precise later. To illustrate the problem, consider Listing 9, a slight variation of Listing 11 in which we replace the program fragment $\texttt{Plus}(\texttt{Nat}(0), y)$ with y.

λ_{op}

```
handle
  do body ← extrude(Var(x_ℕ)); return Var(x_ℕ) in return Lam(Var(x_ℕ), body)
with
  {return(u) ↦ return u;
   extrude(y, k) ↦ do z ← return Plus(Nat(0), y) in continue k z}
```

```
return Lam(Var(x_ℕ), Plus(Nat(0), Var(x_ℕ)))
```

Listing 11: A λ_{op} program that may, or may not demonstrate scope extrusion, depending on one's definition. The final result of the program is well-scoped, but not all intermediate results are well-scoped. If scope extrusion is a property of the resulting AST, then this does not display scope extrusion. If, instead, it is a property of intermediate ASTs, then this does display scope extrusion.

While the program in Listing 9 produces an intermediate AST with ex-

truded variables, because of the mechanism for inserting checks, it passes the Eager Dynamic Check. I assert that this behaviour is unintuitive, and exposes too much of the internals to the programmer.

Refined Environment Classifiers

Refined Environment Classifiers are a static check that uses the type system to prevent scope extrusion. Recall that metaprogramming involves the *creation* and *manipulation* of ASTs. Refined environment classifiers prevent scope extrusion by checking:

1. *Created* ASTs are well-scoped
2. *Manipulating* ASTs preserves well-scopedness

We shall first ignore the ability to manipulate ASTs, and consider how to ensure *created* ASTs are well-scoped. What does it mean to be well-scoped? Consider Figure 2.5, the AST of

$$(\lambda f.\lambda x.fx)(\lambda y.y)$$

Informally, a scope represents a set of variables that are permitted to be free. In the example, there are four scopes: one where no variables are free, one where only f is free, one where f and x are free, and one where y is free. An AST is well-scoped *at a scope* if it is well-typed, and where the only free variables are those permitted by the scope.

Refined environment classifiers make this notion precise. Each classifier represents a scope. The AST has four scopes, corresponding to four classifiers:

$\gamma_\bot$ The top-level, where no free variables are permitted

γ_f Only $\mathtt{Var}(f)$ permitted to be free

γ_{fx} Only $\mathtt{Var}(f)$ and $\mathtt{Var}(x)$ permitted to be free

γ_y Only $\mathtt{Var}(y)$ permitted to be free

As every variable binder creates a scope, we may refer to the classifier created by $\mathtt{Var}(\alpha)$, $\mathsf{classifier}(\mathtt{Var}(\alpha))$. For example, $\gamma_{fx} = \mathsf{classifier}(\mathtt{Var}(x))$.

With classifiers, we may be precise about "the variables permitted to be free (within the scope)". As illustrated by the nesting in Figure 2.5, scopes are related to other scopes. For example, since the scope γ_{fx} is created within scope γ_f, any variable tagged with γ_f may be safely used in the scope γ_{fx}. We say γ_f is compatible with γ_{fx}, and write

$$\gamma_f \sqsubseteq \gamma_{fx}$$

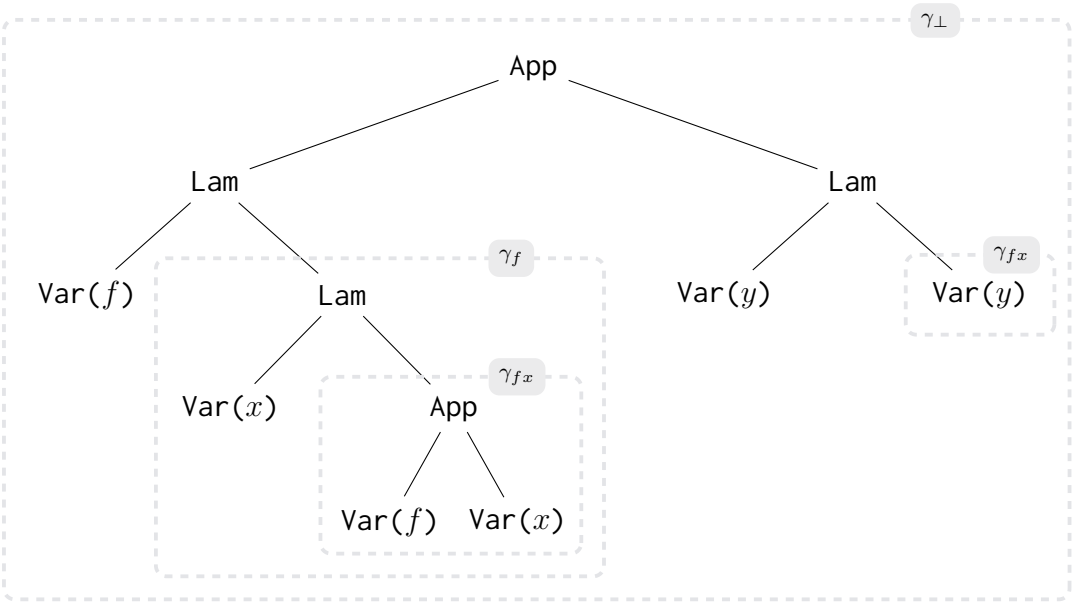

Figure 2.5: The AST of $(\lambda f.\lambda x.fx)(\lambda y.y)$, where each scope is labelled with the corresponding environment classifier.

The compatibility relation ($\sqsubseteq$) is a partial order, meaning $\sqsubseteq$ is reflexive, anti-symmetric, and transitive, and identifies a smallest classifier. Reflexivity expresses that $\mathtt{Var}(\alpha)$ may be used within the scope it creates

$$\forall \gamma.\, \gamma \sqsubseteq \gamma$$

anti-symmetric, since nesting only proceeds in one direction

$$\forall \gamma_1, \gamma_2.\, \gamma_1 \sqsubseteq \gamma_2 \wedge \gamma_2 \sqsubseteq \gamma_1 \implies \gamma_1 = \gamma_2$$

and transitive, since we should count nestings within nestings

$$\forall \gamma_1, \gamma_2, \gamma_3.\, \gamma_1 \sqsubseteq \gamma_2 \wedge \gamma_2 \sqsubseteq \gamma_3 \implies \gamma_1 \sqsubseteq \gamma_3$$

$\gamma_\perp$ acts as the least element of this partial order

$$\forall \gamma.\, \gamma_\perp \sqsubseteq \gamma$$

Given a classifier γ, we may now define the variables permitted to be free in γ, written $\mathsf{permitted}(\gamma)$

$$\mathsf{permitted}(\gamma) \triangleq \{\mathtt{Var}(\alpha) \mid \mathsf{classifier}(\mathtt{Var}(\alpha)) \sqsubseteq \gamma\}$$

For example, $\mathsf{permitted}(\gamma_{fx}) = \{\mathtt{Var}(f), \mathtt{Var}(x)\}$

We now say that an AST n is well-scoped at type T and scope γ if it is well-typed at T, and all free variables in n are in $\mathsf{permitted}(\gamma)$. We write

$$\Gamma \vdash^{\gamma} n : T$$

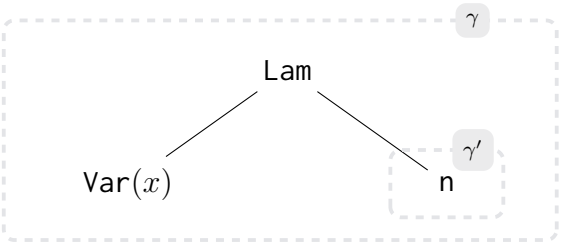

Figure 2.6: Visual depiction of the (C-Abs) typing rule.

Ensuring that created ASTs are well-scoped is the responsibility of the type system. The key rule is the C-Abs rule, which, **assuming we know that we are creating an AST**, has roughly the following shape[2]:

$$\frac{\gamma \in \Gamma \qquad (2)\ \gamma' \text{fresh} \qquad (3)\ \Gamma, \gamma', \gamma \sqsubseteq \gamma', (x : T_1)^{\gamma'} \vdash^{\gamma'} n : T_2}{(1)\ \Gamma \vdash^{\gamma} \lambda x.n : T_1 \longrightarrow T_2}\ \text{(C-Abs)}$$

The premises and conclusions have been numbered for reference, and many technical details have been simplified for clarity. Figure 2.6 visually depicts the typing rule.

(1) The goal of the typing rule is to ensure the function is well-scoped at type $T_1 \to T_2$ and scope γ.

(2) Since the function introduces a new variable binder, $\mathtt{Var}(x)$, one has to create a new scope. This is achieved by picking a fresh classifier γ'.

(3) We record the following:

   (a) Since γ' is created within the scope of γ, $\gamma \sqsubseteq \gamma'$.

   (b) $\mathsf{classifier}(\mathtt{Var}(x)) = \gamma'$ (as a shorthand $(x : T_1)^{\gamma'}$)

   With this added knowledge, we ensure that the function body is well-scoped at type T_2 and γ'.

The above example focused on **creating** ASTs, and had no compile-time executable code. We now consider how to maintain well-scopedness while **manipulating** ASTs. We consider the "AST" of the scope extrusion example, Listing 8 (Figure 2.7). Notice that in place of AST nodes, we may now have compile-time executable code that *evaluate* to AST nodes. Thus, both code and AST nodes reside within scopes. We have two classifiers: $\gamma_\perp$ and γ_α, with $\mathsf{classifier}(\mathtt{Var}(\alpha)) = \gamma_\alpha$.

[2] I will revisit this assumption when introducing the type system for my calculus

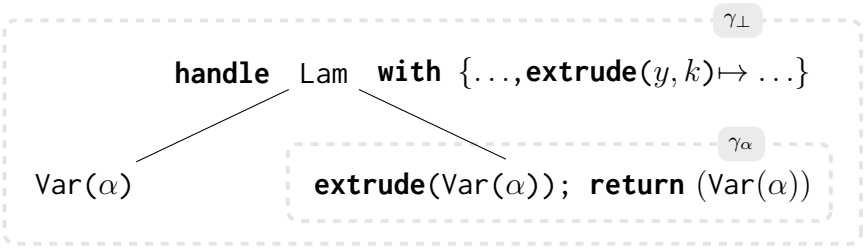

Figure 2.7: The "AST" of the scope extrusion example, Listing 8. Notice that in place of AST nodes, we may now have compile-time executable code that *evaluate* to AST nodes.

Key to the prevention of scope extrusion is the typing of handlers and operations, like **extrude**, that manipulate ASTs. Since these rules are complex, we describe them informally. The handle expression

$$\textbf{handle}\ e\ \textbf{with}\ \{h\}$$

is in scope $\gamma_\perp$. Therefore, for each operation handled by h, such as **extrude**, the argument to the operation must either not be an AST, or be an AST that is well-scoped at some $\gamma \sqsubseteq \gamma_\perp$. However, $\texttt{Var}(\alpha)$ is typed at γ_α, and clearly, $\gamma_\alpha \not\sqsubseteq \gamma_\perp$. There is thus no way to type the scope extrusion example in Listing 8.

Note that the analysis was independent of the *body* of the handler. Therefore, the examples in Listings 9 and 11 are *also* not well-typed. Perhaps somewhat surprisingly, so too is Listing 12 (which would pass both the Eager and Lazy Dynamic Checks). Refined environment classifiers statically prevent variables ($\texttt{Var}(\alpha)$) from becoming *available* in program fragments ($\textbf{op}(y, k) \mapsto \ldots$) where, *if misused*, *might* result in scope extrusion. This is, of course, an over-approximation. It means that the refined environment classifiers check prevents not only both types of scope extrusion, but even more benign examples, such as that in Listing 12.

λ_{op}

```
handle
  do body ← extrude(Var(x_ℕ)); return Var(x_ℕ) in return Lam(Var(x_ℕ), body)
with
  {return(u) ↦ return App(u, Nat(1));
   extrude(y, k) ↦ return Nat(0)}
---
return Nat(0)
```

Listing 12: A λ_{op} program that passes the Eager and Lazy Dynamic Checks, but is not well-typed under the Refined Environment Classifiers type system.

Listing 12 thus illustrates one of the key drawbacks of Refined Environment Classifiers: the check is too stringent, and restricts expressiveness.

3 Calculus

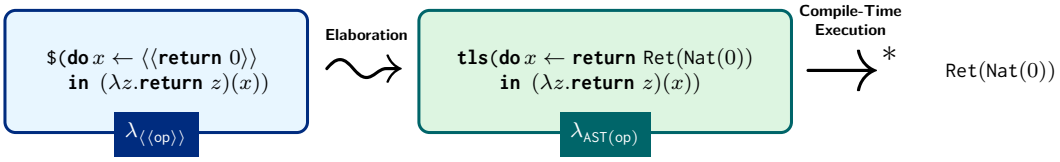

Figure 3.1: $\lambda_{\langle\langle \mathsf{op} \rangle\rangle}$ is first elaborated into $\lambda_{\mathsf{AST}(\mathsf{op})}$, which is then executed **at compile-time** to obtain the AST of a run-time program.

To re-iterate, I am considering the interaction between homogenous, compile-time, two-stage **metaprogramming** (Page 7), and an **effect system** with deep handlers and multi-shot continuations (Page 13). In this chapter, I describe a calculus, $\lambda_{\langle\langle \mathsf{op} \rangle\rangle}$, for studying said interaction. $\lambda_{\langle\langle \mathsf{op} \rangle\rangle}$ will have both metaprogramming and effect handlers. To the best of my knowledge, this is the first calculus in which one may write effectful compile-time code that generates effectful run-time code. However, $\lambda_{\langle\langle \mathsf{op} \rangle\rangle}$ will not mediate the interaction between metaprogramming and effects: scope extrusion prevention is not a language feature. Rather, the aim will be to implement different scope extrusion checks as algorithms in $\lambda_{\langle\langle \mathsf{op} \rangle\rangle}$, such that the checks may be evaluated in a comparative fashion.

Programs written in $\lambda_{\langle\langle \mathsf{op} \rangle\rangle}$ cannot be directly executed. Rather, following the style of Xie et al. [26], one must first elaborate (or compile) from $\lambda_{\langle\langle \mathsf{op} \rangle\rangle}$ (the "source" language) to a "core" language, $\lambda_{\mathsf{AST}(\mathsf{op})}$. Programs written in $\lambda_{\mathsf{AST}(\mathsf{op})}$ may then be executed, to obtain the AST of a run-time program. This process is summarised in Figure 3.1. Elaboration is necessary, since it is the point at which dynamic checks may be inserted.

In this chapter, I will first introduce $\lambda_{\langle\langle \mathsf{op} \rangle\rangle}$ (Section 3.1), then $\lambda_{\mathsf{AST}(\mathsf{op})}$ (Section 3.2). Following this, I will describe the elaboration from $\lambda_{\langle\langle \mathsf{op} \rangle\rangle}$ to $\lambda_{\mathsf{AST}(\mathsf{op})}$ (Section 3.3). Finally, I will discuss the metatheoretic properties of $\lambda_{\langle\langle \mathsf{op} \rangle\rangle}$ (Section 3.4).

3.1 The source language: $\lambda_{\langle\langle \mathsf{op} \rangle\rangle}$

$\lambda_{\langle\langle \mathsf{op} \rangle\rangle}$ extends λ_{op} with quotes and splices. Recall that λ_{op}, following a fine-grained call-by-value approach, divides terms into two syntactic categories,

$\lambda_{\langle\langle \mathsf{op} \rangle\rangle}$

Effects Row

Run-Time	$\xi ::= \cdot \mid \xi \cup \{\mathsf{op}_i^0\}$
Compile-Time	$\Delta ::= \cdot \mid \Delta \cup \{\mathsf{op}_i^{-1}\}$

Types

Level 0	Values	$T^0 ::= \mathbb{N}^0$	naturals
		$\mid (T_1^0 \xrightarrow{\xi} T_2^0)^0$	functions
		$\mid (T_1^0 \xrightarrow{\xi}\!\!\!\!\rightarrow T_2^0)^0$	continuations
	Computations	$T^0 \,!\, \xi$	
		$\mid T^0 \,!\, \Delta; \xi$	
	Handlers	$T_1^0 \,!\, \xi \Longrightarrow T_2^0 \,!\, \xi'$	
Level -1	Values	$T^{-1} ::= \mathbb{N}^{-1}$	naturals
		$\mid (T_1^{-1} \xrightarrow{\Delta} T_2^{-1})^{-1}$	functions
		$\mid (T_1^{-1} \xrightarrow{\Delta}\!\!\!\!\rightarrow T_2^{-1})^{-1}$	continuations
		$\mid \mathsf{Code}(T^0 \,!\, \xi)^{-1}$	run-time code
	Computations	$T^{-1} \,!\, \Delta$	
	Handlers	$T_1^{-1} \,!\, \Delta \Longrightarrow T_2^{-1} \,!\, \Delta'$	

Figure 3.2: $\lambda_{\langle\langle \mathsf{op} \rangle\rangle}$ types. I highlight three important elements: first, stratifying types into two levels, 0 and -1. Second, stratifying effects into two levels, ξ (for run-time effects) and Δ for compile-time effects. Third, the Code type at level -1 allows for compile-time programs to manipulate ASTs of run-time code.

values v and computations c. $\lambda_{\langle\langle \mathsf{op} \rangle\rangle}$ is similar, dividing terms into values v and expressions e. We add quote and splice as follows:

$$e ::= \ldots \mid \langle\langle e \rangle\rangle \mid \$e$$

Notice that we cannot quote values: we *must* generate effectful programs.

3.1.1 Type System

I will now introduce the $\lambda_{\langle\langle \mathsf{op} \rangle\rangle}$ type system, by first introducing the types, and then the typing rules. The $\lambda_{\langle\langle \mathsf{op} \rangle\rangle}$ types are summarised in Figure 3.2. I highlight three important details: types are stratified into two levels (-1 for compile-time and 0 for run-time), effect rows are similarly stratified, and run-time code is made available at compile-time via a Code type.

First, **types are stratified into two levels, T^0 (run-time), and T^{-1} (compile-time).**

To motivate this stratification, consider the following question: what is the type of the number 3 in $\lambda_{\langle\langle \mathsf{op} \rangle\rangle}$? Perhaps surprisingly, the answer is not $\mathbb{N}$. Since we are working with a two-stage system, we must be careful to disambiguate between run-time naturals and compile-time naturals, since these are not interchangeable. For example, the following program should not be well-typed, since 3 is a compile-time natural, whereas x is a run-time natural.

$$\lambda x : \mathbb{N}.\, \$(3 + x)$$

However, removing the splice makes the program well-typed

$$\lambda x : \mathbb{N}.\, 3 + x$$

To separate compile-time and run-time programs, we annotate each type with a level: $\mathbb{N}^0$ for run-time naturals, and $\mathbb{N}^{-1}$ for compile-time naturals. The ill-typed example becomes

$$\lambda x : \mathbb{N}^0.\, \$((3 : \mathbb{N}^{-1}) + (x : \mathbb{N}^0))$$

and the well-typed example

$$\lambda x : \mathbb{N}^0.\, (3 : \mathbb{N}^0) + (x : \mathbb{N}^0)$$

We say 0 and -1 are *levels*, defined as follows:

> **Definition 3.1.1 (Level)** *The level of an expression e is calculated by subtracting the number of surrounding splices from the number of surrounding quotations.*

The definition of level generalises to multi-stage languages [26], where negative levels represent compile-time and non-negative levels represent run-time, and separation is even more granular. However, since we only deal with two stages, we only ought to consider two levels, 0 and -1. The definition further implies that the "default" level, in the absence of quotes and splices, is level 0. Intuitively, in the absence of quotes and splices, the programmer is ignoring metaprogramming facilities, and constructing a program to be run at run-time.

Notice that the opening question was slightly devious[1]! We cannot assign a type to *program fragments*, like 3, since without knowledge of the wider context, we cannot know which level we are at: in the ill-typed example, 3 occurs under a splice, but no quotes, so it has type $\mathbb{N}^{-1}$, and in the well-typed example, it has type $\mathbb{N}^0$. When giving program fragments, like 3, I will always assume that the starting level is 0.

[1] sorry

Second, **effect rows are stratified into ξ (run-time) and Δ (compile-time).** In the following example, we print 1 at compile-time, and 2 at run-time. Further, we read an integer at run-time.

$$\$(\texttt{print}(1); \langle\langle \texttt{print}(2); \texttt{readInt}() \rangle\rangle)$$

Hence, $\Delta = \{\texttt{print}\}$ and $\xi = \{\texttt{print}, \texttt{readInt}\}$. We may now disambiguate between different computation types:

T^0 Compile-time value, run-time value (value types)
Examples: 1, $\lambda x.\, \texttt{return}\ x$

$T^0 \,!\, \xi$ Compile-time value, run-time computation
Examples: $\texttt{return}\ 1$, $\texttt{print}(2)$

$T^0 \,!\, \Delta; \xi$ Compile and run-time computation
Example: $\$(\texttt{print(2)}; \langle\langle \texttt{return}\ 1 \rangle\rangle)$

As a result of the type system, we never encounter $T^0 \,!\, \Delta$ (compile-time computations that are run-time values).

Third, **there is an level -1 Code type, representing run-time ASTs.** By stratifying types to two levels, we have ensured that run-time (resp. compile-time) terms only interact with run-time (compile-time) terms. However, to enable meta-programming, run-time terms, *should be available* at compile-time as ASTs. This is exactly the role of the Code type, thus allowing level -1 programs to manipulate ASTs of level 0 terms.

Typing Rules

Having described the types, I now present the type system. The shape of the typing judgement is familiar, though with the addition of level information and compiler modes.

$$\Gamma \vdash^{\text{Level}}_{\text{Mode}} e : T$$

Level. Recall that, when describing the λ_{op} types, I argued that one cannot type a program fragment, like 3, directly. One must also know the *level* (0 or -1), which is accordingly attached to the typing judgement.

Mode. For the purposes of elaboration, it can be useful to keep classify code into three categories:

c Code that is **ambient** and **inert**.
No surrounding quotes or splices

s Code that **manipulates ASTs** at compile-time.
Last surrounding annotation was a splice

q Code that **builds ASTs** to be manipulated at compile time.
Last surrounding annotation was a quote

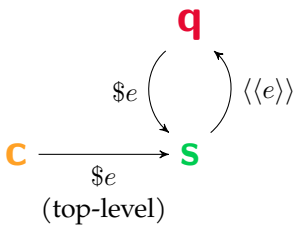

Figure 3.3: Transitions between modes c, s, and q. Top-level splices transition from c to s, quotes transition from s to q, and quotes transition from q to s.

To illustrate the purpose of the modes, consider the following meta-program, which evaluates to the AST of $\lambda x.1+2+3$, where the code has been classified into the three modes.

$$\lambda x.\,\$(\textbf{do}\ f \leftarrow (\lambda y.\langle\langle\$(y)+2\rangle\rangle)\ \textbf{in do}\ a \leftarrow \langle\langle 1\rangle\rangle\ \textbf{in}\ fa)+3$$

c Identifies AST nodes that are **ambient** (within which computation may take place) and **inert** (cannot themselves be manipulated at compile-time).

$$\lambda x.\,\$(\textbf{do}\ f \leftarrow (\lambda y.\langle\langle\$(y)+2\rangle\rangle)\ \textbf{in do}\ a \leftarrow \langle\langle 1\rangle\rangle\ \textbf{in}\ fa)+3$$

s Identifies code that can be executed at run-time to **manipulate ASTs**. Will be fully reduced at compile-time, and will not appear at run-time.

$$\lambda x.\,\$(\textbf{do}\ f \leftarrow (\lambda y.\langle\langle\$(y)+2\rangle\rangle)\ \textbf{in do}\ a \leftarrow \langle\langle 1\rangle\rangle\ \textbf{in}\ fa)+3$$

q Identifies code that **builds ASTs** (like c-mode) that can be manipulated at compile-time (unlike c-mode), to create run-time programs.

$$\lambda x.\,\$(\textbf{do}\ f \leftarrow (\lambda y.\langle\langle\$(y)+2\rangle\rangle)\ \textbf{in do}\ a \leftarrow \langle\langle 1\rangle\rangle\ \textbf{in}\ fa)+3$$

We may also describe how *transitions* between modes occur:

1. Top-level splices ($\$e$) transition from c (outside the splice) to s (within).
2. Quotes ($\langle\langle e\rangle\rangle$) transition from s (outside the quote) to q (within).
3. Splices ($\$e$) transition from q (outside the splice) to s (within).

Transitions between modes are illustrated in Figure 3.3.

Since $\lambda_{\langle\langle \text{op}\rangle\rangle}$ has only two levels, and my type system will ban nested splices and quotations ($\$\$e$ and $\langle\langle\langle\langle e\rangle\rangle\rangle\rangle$ are not valid program fragments), the compiler mode will uniquely identify the level (c and q imply level 0, and s level -1). I thus drop the level from my typing judgement, leaving it implicit.

3.2 The core language: $\lambda_{\mathsf{AST}(\mathsf{op})}$

3.3 Elaboration from $\lambda_{\langle\langle \mathsf{op} \rangle\rangle}$ to $\lambda_{\mathsf{AST}(\mathsf{op})}$

3.4 Metatheory

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