Homework 5

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The Schröder-Cantor-Bernstein Theorem states the following.

Theorem 0.1 (Schröder-Cantor-Bernstein)**.** *Let X, Y be sets, and suppose that there exist injections $f : X \to Y$ and $g : Y \to X$. Then there exists a bijection $h : X \to Y$.*

For this problem set, you will work through two proofs of the the Schröder-Cantor-Bernstein Theorem.

1 Proof 1

(Recommended) Problem 1. Suppose that $f : X \to Y$ is an injection. Show that the map $f_* : 2^A \to 2^B$ given by:

$$f^*(S) = \{f(s) : s \in S\}$$

is an injection.

(Recommended) Problem 2. Let $f : X \to Y$ and $g : Y \to X$ be injections, as in the statement of the Schröder-Cantor-Bernstein Theorem. Let $f_* : 2^A \to 2^B$ and $g_* : 2^B \to 2^A$ be the corresponding induced maps, as we saw in the previous problem. Define the map $\sigma : 2^A \to 2^A$ by:

$$\sigma(S) = A - g_*(B - f_*(S)).$$

Show that σ is monotone. That is, if $T \subset S$, then $\sigma(T) \subset \sigma(S)$.

(Recommended) Problem 3. Let $X \neq \emptyset$, and let $\varphi : 2^X \to 2^X$ be a monotone increasing function. That is, if $T \subset S$, then $\varphi(T) \subset \varphi(S)$. Denote:

$$\mathcal{U} := \{T \in 2^X : \varphi(T) \subset T\},$$

and let:

$$A := \bigcap_{T \in \mathcal{U}} T.$$

Show the following:

(a) Show that $\varphi(A) \subset A$.

(b) Show that $\varphi(\varphi(A)) \subset \varphi(A)$. Deduce that $\varphi(A) \in \mathcal{U}$.

(c) Conclude that $A \subset \varphi(A)$. Thus, $\varphi(A) = A$.

(Recommended) Problem 4. Let σ be the function defined in Problem 2. Show that σ has a fixed point.

(Recommended) Problem 5. Show that $X - A \subset \mathrm{Im}(g)$. [**Hint:** Recall that $\sigma(A) = A - g_*(B - f_*(A))$. What can you conclude about $g_*(B - f_*(A))$?]

(Recommended) Problem 6. Define $h : X \to Y$ by:

$$h(x) = \begin{cases} f(x) : & x \in A, \\ g^{-1}(x) : & x \notin A. \end{cases}$$

Do the following.

(a) Show that h is a well-defined function.

(b) Show that h is a bijection.

2 Proof 2

For our second proof, we consider a bipartite graph G with vertex set $V = X \cup Y$. Here, we assume X and Y are disjoint. Now we have edges, which we color red that are of the form $\{x, f(x)\}$, and blue edges that are of the form $\{y, g(y)\}$. If G has a perfect matching, then there exists a bijection $h : X \to Y$. Our goal is to construct a perfect matching.

(Recommended) Problem 7. Argue that for any vertex v, $\deg(v) \in \{1, 2\}$. Conclude that G is decomposed into connected components, which are either paths or cycles. Note that while the cycles are finite, the paths are not.

(Recommended) Problem 8. Let C be a connected component of G that is a cycle. Prove that C has a perfect matching. You may use the fact that any cycle in a bipartite graph has an even number of vertices.

(Recommended) Problem 9. Let C be a connected component of G. We say that C is *doubly-infinite* if all vertices of C have degree 2. Argue that the red edges of C form a perfect matching.

(Recommended) Problem 10. Let C be a connected component of G. We say that C is *doubly-infinite* if all vertices of C have degree 2, except for a single vertex of degree 1. This vertex of degree 1 is our initial vertex, which we label z.

(a) Describe how to construct a perfect matching of C if $z \in X$.

(b) Describe how to construct a perfect matching of C if $z \in Y$.

Note that your constructions in both cases will be quite similar. Thinking about these cases will be useful when defining our bijection later.

(Recommended) Problem 11. Denote condition $(*)$ to indicate that $x \in X$ belongs to a cycle, doubly-infinite path, or singly-infinite path where the initial vertex is in X. Let $h : X \to Y$ be the map given by:

$$h(x) = \begin{cases} f(x) & : \text{condition (*) holds for } x, \\ g^{-1}(x) & : \text{otherwise.} \end{cases}$$

Show that h is a well-defined function. Problems (8)-(10) provide that h is a bijection.