# Weisfeiler—Leman for Group Isomorphism: Action Compatibility

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## Introduction

- Weisfeiler--Leman (WL) captures essentially all combinatorial algorithms for Graph Isomorphism (GI), and is the key subroutine in state-of-the-art GI algorithms.
- Recently adapted from graphs to groups [BS20].

## Weisfeiler—Leman for Graphs

- k-WL colors the k-tuples of vertices in an isomorphism invariant manner.
- Tuples initially colored according to their *marked* isomorphism type.
- The color assigned to a k-tuple u at round r takes into account:
    - The color of u at round r-1
    - The multiset of colors at round r-1 of nearby k-tuples
- Algorithm terminates when partition on the k-tuples induced by the coloring is not refined

## Motivation

- For which families of groups does Weisfeiler—Leman serve as a polynomial-time isomorphism test?

## Our Results

**Theorem.** Groups in the following families can be identified by O(1)-dimensional Weisfeiler—Leman in O(1) rounds:
- Coprime Extensions $H \ltimes N$, where H is O(1)-generated and N Abelian†
- Groups without Abelian normal subgroups (aka semisimple groups).

- These are two of the few major classes of groups for which polynomial-time algorithms were previously known.

| Groups | Previous Bounds | WL |
|---|---|---|
| Abelian | - O(n) time [Kav07]<br>- L ∩ $TC^0$(FOLL) [CTW11] | - P [BS20]<br>- $TC^0$ [GL21] |
| O(1)-generated | - P (attributed to Tarjan)<br>- L [Tan13] | - P [BS20]<br>- $AC^0$ [GL21] |
| Semisimple | - P [BCQ12] | - $TC^0$ [GL21] |
| †Coprime Extensions | - P [QST11] | - $TC^0$ [GL21] |

## Weisfeiler—Leman for Groups

- For a group G, construct a graph $\Gamma_G$.
- Run k-WL for graphs on $\Gamma_G$.
- Pull back the coloring to k-tuples of group elements.

## Reduction

- For each group element g, we have a vertex in $\Gamma_G$.
- For each pair $g, h \in G$, we have a multiplication gadget M(g, h) to encode the relation g * h.
- For a group G of order n, $\Gamma_G$ has $\Theta(n^2)$ vertices.
- Runtime of WL for Groups is $O(n^{2k+1} \log n)$

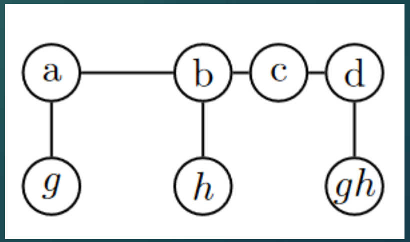


**Figure.** Multiplication Gadget M(g, h) [BS20]

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