Subgroup Test: $a, b \in H \implies ab \in H$ AND $a^{-1} \in H$

- **Center**: $Z(G) = \{a \in G \mid ax = ax \forall x \in G\}$
- **Centralizer**: $Z(a) = \{g \in G \mid ga = ag\}$
- **3.14**: Let x be a nonidentity element of G and let $b = b = x^{-1}ax$. Then $b^2 = e \implies b = a \implies xa = ax$
- **3.46**: $xa = ax \implies xaaa = axaa \implies xa^3 = a^3x \implies C(a) \subseteq C(a^3)$

  $xa^3 = a^3x \implies xa^3a^3 = a^3xa^3 = xa = ax \implies C(a^3) \subseteq C(a)$

Cyclic Groups: $G = \langle a \rangle$

- $\langle a^k \rangle = \langle a^{\gcd(n,k)} \rangle$
- Number of elements of order d in a finite cyclic group is $\phi(d)$
- $a^k e \implies |a|$ divides k
- $ab = ba \implies |ab|$ divides $|a||b|$

Permutations: $\phi(a) = b$

- Disjoint cycles commute
- $\epsilon = \beta_1 \beta_2 \ldots \beta_r \implies r$ is even
- $|A_n| = n!/2$
- **5.46**: $(1234) = (14)(13)(12)$ is odd but x^2 is even thus no solutions

  Construct 2 solutions for x^3. We see $(1432)^3$ and $((1432)(567))^3$ works
- **5.58**: Show $Z(S_n) = \{\epsilon\}$. Take $\sigma(a) = b$ and $\tau = (bc)$

  Then $\tau\sigma(a) = c$ but $\sigma\tau(a) = b$

Isomorphism: $\phi(ab) = \phi(a)\phi(b)$ where ϕ is 1-1 and onto

- If ϕ is 1-1 and G is finite $\implies \phi$ is onto
- **Automoprhism**: isomorphism that takes $G \to G$. $\text{Aut}(G)$ is the group of automorphisms of G
- $\text{Inn}(G) = \{\phi_a(x) = axa^{-1} \mid a, x \in G\}$
- $\text{Aut}(Z_n) \approx U(n)$
- **6.20**: TODO
- **6.26**: $\phi : Z_{20} \to Z_{20}$ and $\phi(5) = 5$. Determine possible values of $\phi(x)$

  Note $\text{Aut}(Z_{20}) = U(20)$. See that $\phi(5) = 5a = 5 \pmod{20} \implies 20 \mid 5(a-1) \implies a \in \{1, 9, 13, 17\}$
- **6.40**: TODO
- **6.44**: Let G be finite Abelian with no elements of order 2. Show $\phi : g \to g^2$ is an automorphism

  1-1: Assume $a \neq b$ but $\phi(a) = \phi(b) \implies a^2 = b^2 \implies (ab^{-1})^2 = e$ but contradiction since it has order 2. Thus $a = b$

  onto: since G is finite and ϕ is 1-1, ϕ is onto

  $\phi(ab) = (ab)^2 = a^2b^2 = \phi(a)\phi(b)$
- **6.64**: Show Q is not isomorphic to a proper subgroup of itself

  Define $\phi : Q \to H$ which is determined by $\phi(x) = x\phi(1)$ for $x \in Q$

  SInce ϕ is onto, let $x = y/\phi(1)$ then $\phi(x) = x\phi(1) = \frac{y}{\phi(1)}\phi(1) = y$ so $H = Q$ contradiction

Cosets

- **Lagrange's Theorem**: $H \leq G \implies |H|$ divides $|G|$ and the number of distinct left cosets is $|G|/|H|$
- $|HK| = |H||K|/|H \cup K|$
- $|G| = 2p \implies G \approx Z_{2p}$ or $G \approx D_p$
- **7.12**: Let $|G| = 155$ and $a, b \in G$. Show only subgroup of G that contains a, b is G itself

  Let $H \leq G$ and $a, b \in H$. By Lagrange, $|H| = \{1, 5, 31, 155\}$. Let $L = \langle a, b \rangle$. If either has order 155 $\implies L = H = G$

  Otherwise $|a| = 5$ and $|b| = 31 \implies |L| = 5 * 31 = 155 \implies L = H = G$

- **7.26**: Suppose G has no non-trivial proper subgroup, show that $|G|$ is prime.

  $|G|$ is finite because otherwise $G = \langle g \rangle$ (no proper subgroup) and $g^2 \leq G$. Contradiction

  Thus $|G|$ must be prime, otherwise there would be a subgroup of order of a divisor of $|G|$

Externel Direct Product: $G \oplus H$ cyclic if and only if $|G|, |H|$ are relatively prime

- $|(g_1, g_2, \ldots)| = \text{lcm}(|g_1|, |g_2|, \ldots)$
- **8.22**: Determine number of elements of order 15 and number of cyclic groups of order 15 in $Z_{30} \oplus Z_{20}$

  $|a| = 15$ and $|b| = 1$ or $5 \implies 8 * 5 = 40$ AND $|a| = 3$ and $|b| = 5 \implies 2 * 4 = 8$. SO 48 elements of order 15

  $\phi(15) = 8$ so each cyclic subgroup of 15 has 8 unique generators $\implies 48/8 = 6$ cyclic subgroups

Normal Subgroup: $H \trianglelefteq G \implies aH = Ha \forall a \in G$

- **Normal Subgroup Test**: $xHx^{-1} \subseteq H \forall x \in G$
- $H \trianglelefteq G, K \leq G \implies HK \leq G$
- $|gHg^{-1}| = |H|$
- $G/Z(G)$ cyclic $\implies G$ Abelian
- **Cauchy Theorem**: p divides order of $G \implies G$ has an element of order p
- $H, K \trianglelefteq G, G = HK$, and $H \cap K = \{e\} \implies G = H \times K$
- $|G| = p^2 \implies G \approx Z_{p^2}$ OR $G \approx Z_p \oplus Z_p$. ALSO G is Abelian
- **9.22**: Determine order of $(Z \oplus Z)/\langle (2, 2) \rangle$. Is it cyclic?

  Since $((1, 0) + (2, 2))^m \notin (2, 2)$ order is infinite.

  Any cyclic group with infinite order cannot have an element of finite order but we see $((1, 1) + (2, 2))^2 \in (2, 2)$
- **9.48**: If $|G : Z(G)| = 4$, show that $G/Z(G) \approx Z_2 \oplus Z_2$

  $|G/Z(G)| = 4$ so either $\approx Z_4$ or $\approx Z_2 \oplus Z_2$. Cannot be cyclic since otherwise G is Abelian but we see $Z(G) \neq G$
- **9.52**: Let G be Abelian and H be a subgroup with with all elements of G with finite order. Show that every nonidentity element of G/H has infinite order

  Take $g \notin H$ and $|gH| < n$. Then $g^n \in H$. Contradiction since g as finite order
- **9.72**: Let G be a group and H be an odd-order subgroup of G of index 2. Show H contains every element of odd order

  Take $g \in G$ with odd order. Since $\gcd(2, |g|) = 1$, g^2 generates $\langle g \rangle$. But since $|G : H| = 2 \implies g^2 \in H \implies g \in H$

Group Homomorphism: $\phi(ab) = \phi(a)\phi(b)$

- $\text{Ker}(\phi) = \{x \in G \mid \phi(x) = e\} \trianglelefteq G$
- **First Isomorphism Theorem**: $G/\operatorname{Ker}(\phi) \approx \phi(G)$ defined by $g \operatorname{Ker}(\phi) \to \phi(g)$
- **10.24**: $Z_{50} \to Z_{15}$ with $\phi(7) = 6$

  $7k = 6 \pmod{15} \implies k = 3 \implies \phi(x) = 3x$

  Image of ϕ is $\langle 3 \rangle$

  $\text{Ker}(\phi) = \langle 5$

  $\phi^{-1}(3) = 1 + \langle 5 \rangle$
- **10.42**: If $M, N \trianglelefteq G$ and $N \leq M$ show $(G/N)/(M/N) \approx G/M$

  Define $\phi : G/N \to G/M$ with $\phi(gN) = gM$

  $\text{Ker}(\phi) = \{gN \mid g \in M\} = M/N$ and then apply first isomorphism theorem
- **10.62**: Determine homomorphisms Z ONTO S_3 and all homomorphisms from Z TO S_3

  No ONTO because Z is cyclic but S_3 not cyclic. To is determined by $\phi(1) = \tau \in S_3$

Conjugacy Class: $cl(a) = \{xax^{-1} \mid x \in G\}$ partitions G into equivalence classes

- $|cl(a)| = |G : C(a)|$
- $|cl(a)|$ divides $|G|$
- $|G| = \sum |G : C(a)|$
- p-groups ($|G| = p^k$) have non trivial center $Z(G)$

Sylow Theorem 1: p^k divides $|G| \implies G$ has at least 1 subgroup of order p^k

Sylow Theorem 3: $|G| = p^k m \implies n_p \equiv 1 \pmod{p}$ AND $n_p \mid m$ AND any 2 Sylow p-subfroup are conjugate

- $|G| = pq, q \nmid p - 1 \implies G$ is cyclic AND $G \approx Z_{pq}$
- **24.22**: Show $|G| = 56 = 2^3 * 7$ has a nontrivial normal subgroup

  $n_7 = 1$ or 8. If 1 done. Otherwise $n_7 = 8$. These comprise of $6 * 8 = 48$ of the elements. So we have unique 2-Sylow subgroup
- **24.40**: TODO
- **24.60**: Determine $|G| = 45 = 3^2 * 5$

  $n_5 = 1$ and $n_3 = 1$ Thus $G \approx Z_9 \times Z_5 \approx Z_3 \times Z_3 \approx Z_5$