MATH405: Linear Algebra

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Contents

1 Vector Space

Goals of this course is to discuss

- Vector spaces
- Linear transformations between vector spaces
- Other operations on vector spaces

1.1 Definitions

Definition - Field: A set of numbers containing $0, 1$ that can be added, subtracted, multiplied, and divided (except cannot divide by 0) that satisfy the following **Field Axioms**

1. $a, b \in K \implies a+b, ab \in K$
2. $+, \times$ are commutative so $a+b = b+a$ and $ab = ba$
3. $+, \times$ are associative so $(a+b)+c = a+(b+c)$ and $a(bc) = (ab)c$
4. Distributive Law: $a(b+c) = ab + ac$
5. Additive Identity: $a + 0 = 0 + a = a$
6. Multiplicative Identity: $a \cdot 1 = 1 \cdot a = a$
7. Additive Inverse: $\forall a \in K, \exists b$ such that $a + b = 0$, namely $b = -a$ which is unique
8. Multiplicative Inverse: $\forall a \in K, \exists b$ such that $ab = 1$, name $b = 1/a$ which is unique

Example: R, Q are fields. Z is not a field since there is no multiplicative inverse of 2

Example: $C = \{a + bi \mid a, b \in R\}$, where $i = \sqrt{-1}$ is a field under

- $+ : (a+bi) + (c+di) = (a+c) + (b+d)i$
- $\times : (a+bi)(c+di) = (ac - bd) + (ad + bc)i$

Example: $F_2 = \{0, 1\}$ is a field under

- $+$: where

  $0 + 0 = 0$

  $0 + 1 = 1 + 0 = 1$

  $1 + 1 = 0$

- $\times$: where

  $0 \cdot 0 = 0$

  $0 \cdot 1 = 1 \cdot 0 = 0$

  $1 \cdot 1 = 1$

Example: For a prime p, let $F_p = \{0, \ldots, p-1\}$. Then F_p is a field under

- $+ : a + b \pmod p$
- $\times : ab \pmod p$

Definition - Vector Space: For an arbitrary field K, a K-vector space is a set V, with a distinguished element O, such that any 2 elements in V can be added and scalar multiplied by $c \in K$

- $u, v \in V \implies u + v \in V$
- $c \in K, u \in V \implies cu \in V$

Satisfying the following properties

1. Commutative Addition: $u + v = v + u$
2. Associative Addition: $(u+v)+w = u+(v+w)$

3. Additive Identity: $u + O = u$
4. Additive Inverse: $\forall u \in V, \exists v \in V$ such that $u + v = O$, namely $v = -u$ which is unique
5. Distributive Laws: $\forall a, b \in K, a(u + v) = au + av$ and $(a + b)u = au + bu$
6. Commutative Scalar Multiplication: $(ab)u = a(bu)$
7. Multiplicative Identity: $1 \cdot u = u$

Example: R^3 is an R-vector space defined by the operations

$$R^3 = \{(x, y, z) \mid x, y, z \in R\}$$

- $+$: add componentwise so $(a, b, c) + (d, e, f) = (a + d, b + e, c + f)$
- Scalar $\times$: for $r \in R$, $r(a, b, c) = (ra, rb, rc)$
- Additive Identity is $O = (0, 0, 0)$

Example: For any field K, K^2 is a K-vector space defined by the operations

$$K^2 = \{(x, y) \mid x, y \in K\}$$

- $+$: add componentwise so $(a, b) + (c, d) = (a + c, b + d)$
- Scalar $\times$: for $k \in K$, $k(a, b) = (ka, kb)$
- Additive Identity is $O = (0, 0)$

Example: R is an R-vector space since clearly the necessary properties hold

Example R is a Q-vector space since clearly the necessary properties hold

- Notably, for $q \in Q$ and $r \in R$, we have $qr \in R$. Thus scalar multiplication is closed

Example: For any field K, the set $\{O\}$ is a K-vector space

Example: Let X be any non-empty set and let $\mathcal{F}(X)$ be the set of all functions $f : X \to R$. Then $\mathcal{F}$ is an R-vector space under the operations

- $+$: for $f, g \in \mathcal{F}(X)$, define $f + g := (f + g)(x)$
- Scalar $\times$: let $r \in R$, then define $rf := r(f(x))$
- Additive Identity is $O = f(x) = 0$, the function that takes any x to 0

Example: Take $X = N$ and let $F(X) = \{$ all functions $f : N \to R\}$ is a vector space

- **Note**: $f : N \to R$ is a sequence $(a_0, \ldots, a_n)$ where $a_n = f(n)$

Lemma 1 - Cancellation: For $u, v, w \in V$ and if $u + v = w + v$, then $u = w$

Proof: $v \in V$ has an additive inverse, namely $-v$. Thus we have

$$u + v - v = w + v - v \implies u = w$$

Lemma 2 - Unique Additive Inverse: For all $v \in V$, there is a unique additive inverse, namely $-v$

Proof: Suppose u, w are both additive inverses of v. Then we have

$$v + u = v + w \implies u = w$$

Lemma 3 - 0 Times a Vector: For all $v \in V$, $0v = O$

Proof: $v = 1v = (0+1)v = 0v + 1v = 0v + v \implies 0v = O$

Lemma 4 - (-1)v is the Additive Inverse: For all $v \in v, (-1)v$ is the unique additive inverse of v

Proof: $(-1)v + v = (-1+1)v = 0v = O$. Thus $(-1)v$ is the additive inverse of v, which is unique by Lemma 2

Definition - Subspace: For a K-vector space V and a non-empty subset $W \subseteq V$, W is a **subspace** if it satisfies

- $w_1, w_2, \in W \implies w_1 + w_2 \in W$
- $\forall a \in K, w \in W \implies aw \in W$

Theorem 1: Every subspace of a K-vector space is a K-vector space

Proof: We need to show that $W \subseteq V$ satisfies all the necessary properties of a vector space

1. Verify $O \in W$

   Since W is non-empty and closed under scalar multiplication, take $0w = O \in W$ by Lemma 3
2. $u, v \in W \implies u + v \in W$ and $a \in K, v \in W \implies aw \in W$ by definition of subspace
3. Every $w \in W$ has an additive inverse, namely $-w$

   Since W is closed under scalar multiplication, $(-1)w = -w \in W$ by Lemma 4
4. Other conditions (associative addition, commutative addition, etc.) hold because $u, v, w \in W \implies u, v, w \in V$

   For example, choose $u, v \in W$, then $u + v = v + u$, since $u, v \in V$. Thus commutative addtion is satisfied

Example: Take $(5,3,2) \in R^3$. Then let $W = \{r(5,3,2) \mid r \in R\}$

Then W is an R-vector space. We prove this by showing that W is a subspace of R^3

- $+$: Choose 2 arbitrary elements of W, $r(5,3,2)$ and $s(5,3,2)$ for $r, s \in R$

  Then $r(5,3,2) + s(5,3,2) = (r+s)(5,3,2) \in W$
- $\times$: Choose $r(5,3,2) \in W$ and take $s \in R$

  Then $s(r(5,3,2)) = (sr)(5,3,2) \in W$

Example: Let $U = \{(x,y,z) \in R^3 \mid 2x + 3y = 0\}$. We show that U is a vector space by showing it's a subspace of R^3

- $+$: Take (x_1, y_1, z_1) and $(x_2, y_2, z_2) \in U \implies 2x_1 + 3y_1 = 0$ and $2x_2 + 3y_2 = 0$

  Then $2(x_1 + x_2) + 3(y_1 + y_2) = 0$

  Thus $(x_1 + x_2, y_1 + y_2, z_1 + z_2) \in U$
- $\times$: Let $(x,y,z) \in U$ and $r \in R$

  Then $2x + 3y = 0 \implies r(2x + 3y) = 2rx + 3ry = 0$

  Thus $r(x,y,z) \in U$

Example: Consider $\sin(x), \cos(x) \in \mathcal{F}(R)$ and let $W = \{a\sin(x) + b\cos(x) \mid a, b \in R\}$. Then W is a subspace of $\mathcal{F}(R)$

- $+$: Take $a_1 \sin(x) + b_1 \cos(x)$ and $a_2 \sin(x) + b_2 \cos(x) \in W$. Then $(a_1 + a_2)\sin(x) + (b_1 + b_2)\cos(x) \in W$
- $\times$: Take $r \in R$. Then $r(a\sin(x) + b\cos(x)) = (ra)\sin(x) + (rb)\cos(x) \in W$

1.2 Basis

Definition - Linear Combination: For vectors $\{v_1, \ldots, v_n\} \subseteq V$, a **linear combination** of $\{v_1, \ldots, v_n\}$ is a vector of the form

$$a_1v_1 + \cdots + a_nv_n \qquad a_i \in K$$

Definition - Span: $\text{span}(\{v_1, \ldots, v_n\}) = \{$ all linear combinations of $\{v_1, \ldots, v_n\}\}$

Proposition 1: $W = \text{span}(\{v_1, \ldots, v_n\})$ is a subspace of V and thus is itself a K-Vector Space

Proof: We show that W satisfies the necessary criteria to be a subspace of V

- $+$: Let $a = a_1v_1 + \cdots + a_nv_n \in W$ and $b = b_1v_1 + \cdots + b_nv_n \in W$

  Then $a + b = (a_1 + b_1)v_1 + \cdots + (a_n + b_n)v_n \in W$

  Thus W is closed under addition
- Scalar $\times$: Let $a = a_1v_1 + \cdots + a_nv_n \in W$ and let $c \in K$

  Then $ca = (ca_1)v_1 + \cdots + (ca_n)v_n \in W$

  Thus W is closed under scalar multiplication

Example: Take $(5, 3, 1)$ and $(4, 0, -2) \in R^3$

$\text{span}(\{(5, 3, 1), (4, 0, -2)\})$ is a plane in R^3 passing through $(0, 0, 0)$

Example: Take $(5, 3, 1)$ and $(10, 6, 2) \in R^3$

$\text{span}(\{(5, 3, 1), (10, 6, 2)\})$ is a line in R^3 passing through $(0, 0, 0)$

- **Note**: $(10, 6, 2) = 2(5, 3, 1)$. Thus $\text{span}(\{(5, 3, 1), (10, 6, 2)\}) = a_1(5, 3, 1) + a_2(10, 6, 2) = (a_1 + 2a_2)(5, 3, 1)$

Definition - Linearly Independent: $\{v_1, \cdots, v_n\}$ is **linearly independent** if whenever $a_1v_1 + \cdots + a_nv_n = 0$, then $a_1 = \cdots = a_n = 0$

- Otherwise $\{v_1, \ldots, v_n\}$ is **linearly dependent**

Proposition 2: $\{v_1, \ldots, v_n\}$ is linearly independent if and only if no v_i is a linearly combination of the other $n - 1$ vectors

Proof: $\implies$ Assume $\{v_1, \ldots, v_n\}$ is linearly independent

BWOC, assume some $v_i = a_1v_1 + \cdots + a_nv_n$ for some $v_i \notin \{v_1, \ldots, v_n\}$

Then we have

$$O = a_1v_1 + \cdots + a_nv_n + (-1)v_i$$

Since v_i is a linear combination of $\{v_1, \ldots, v_n\}$, the above equation shows that $\{v_1, \ldots, v_n\}$ is linearly dependent. Contradiction

Thus v_i cannot be written as a linear combination of the other vectors

$\impliedby$ Assume by way of contraposition that $\{v_1, \ldots, v_n\}$ is not linearly independent

Thus choose $a_1, \ldots, a_n \in K$, not all 0 such that

$$a_1v_1 + \cdots + a_nv_n = O$$

WLOG, assume $a_1 \neq 0$. Then $v_2a_2 + \cdots + a_nv_n = a_1v_n$

Since $a_1 \neq 0$ and K is a field, we have

$$v_1 = \frac{a_2}{-a_1}v_2 + \cdots + \frac{a_n}{-a_1}v_n$$

Thus we have shown that v_1 is a linear combination of the other $n-1$ vectors

Corollary 3: $\{v_1, \ldots, v_n\}$ is linearly independent if and only if for each i, $v_i \notin \text{span}(\{v_1, \ldots, v_n\} \setminus \{v_i\})$

Proof: This follows from the previous proposition

Definition - Spans: Let W be a K-Vector Space and $\{v_1, \ldots, v_n\} \subseteq W$. If $\text{span}(\{v_1, \ldots, v_n\}) = W$, then $\{v_1, \ldots, v_n\}$ **spans** W, so every $w \in W$ is a linear combination of $\{v_1, \ldots, v_n\}$

Definition - Basis: $\{v_1, \ldots, v_n\}$ is a **basis** of W if it spans W and is linearly independent

Example: $\{(5,3,1),(4,0,-2)\}$ is a basis for $\text{span}(\{(5,3,1),(4,0,-2)\})$

Example: $\{(5,3,1),(10,6,2)\}$ is not a basis for $\text{span}(\{(5,3,1),(10,6,2)\})$ since it is not linearly independent

Proposition 4: Let $\{v_1, \ldots, v_n\}$ be a basis for W and let $w \in W$ be arbitrary. Then w can be written uniquely as

$$w = a_1v_1 + \cdots + a_nv_n \qquad a_i \in K$$

Proof: Since $\{v_1, \ldots, v_n\}$ spans W, every $w \in W$ is a linear combination of $\{v_1, \ldots, v_n\}$

For uniqueness, suppose

$$w = a_1v_1 + \cdots + a_nv_n = b_1v_1 + \cdots + b_nv_n$$

Then we have

$$O = (b_1 - a_1)v_1 + \cdots (b_n - a_n)$$

Since $\{v_1, \ldots, v_n\}$ is linearly independent, we must have $b_i - a_i = 0$, and thus $b_i = a_i$ for each i

Thus each $w \in W$ can be written uniquely as a linear combination of $\{v_1, \ldots, v_n\}$

Example: Let $W = \text{span}(\{\sin(x), \cos(x)\} = \{a\sin(x) + b\cos(x) \mid a, b \in R\}$

We know that W is an R-Vector Space

$\{\sin(x), \cos(x)\}$ is linearly independent. Otherwise $\sin(x) = r\cos(x)$ for all $x \in X$ and some $r \in R$. However, this cannot hold for when $x = \pi/2$ since $\sin(\pi/2) = 1 \neq r\cos(\pi/2) = r0$

1.3 Dimension

Let $\{v_1, \ldots, v_n\} \subseteq V$ and let $W = \text{span}(\{v_1, \ldots, v_n\})$

Now let $X = \{w_1, \ldots, w_m\} \subseteq W$. Then there are 2 desirable properties of X

- **X is Big**: X spans W if $\text{span}(X) = W$, i.e. all $w \in W$ is a linear combination of elements from X
- **X is Small**: X is linearly independent, i.e. no element in X is a linear combination of the remaining elements

Note: the empty set $\emptyset$ is linearly independent since no element in $\emptyset$ is a linear combination of the others. More notably, $\emptyset$ is a basis for $\{O\}$

Shrinking Lemma: Let $X = \{w_1, \ldots, w_m\} \subseteq W$ and spans W but X is not linearly independent. Then $X \setminus \{w_i\}$ still spans W for some $w_i \in X$

Proof: Since X is not linearly independent, we know that some w_i is a linear combination of elements in $X \setminus \{w_i\}$. Suppose

$$w_i = a_1w_1 + \cdots + a_mw_m \qquad \text{without } w_i \text{ occurring}$$

Then take arbitrary $u \in W$ where

$$u = b_1w_1 + \cdots + b_mw_m$$

Replacing w_i above with the previous equation, we see that u is a linear combination of $X \setminus \{w_i\}$

Thus $X \setminus \{w_i\} = \text{span}(W)$

Shrinking Theorem: Let $X = \{w_1, \dots, w_m\}$ span W. Then for some subset $Y \subseteq X$ is a basis of W

Proof:

Case 0: If X is linearly independent, then X is a basis by definition

Otherwise, apply the shrinking lemma to get $X_1 = X \setminus \{w_i\}$, which spans W

Case 1: If X_1 is linearly independent, then X_1 is a basis

...

Since X is finite (it has m elements), we will stop eventually. Either

- Some X_i is linearly independent. Thus X_i is a basis for W
- Otherwise if we hit case m: $X_m = \emptyset$, which is linearly independent, and thus X_m spans $W = \{O\}$

Corollary: If $W = \text{span}(\{v_1, \dots, v_n\})$, then some subset of $\{v_1, \dots, v_n\}$ is a basis

- **Note**: In particular, W has to have a basis

Enlarging Lemma: Suppose $X = \{w_1, \dots, w_m\} \subseteq W$ and is linearly independent but doesn't span W. Then for any $w \in W \setminus \text{span}(X)$, $X \cup \{w\}$ is still linearly independent

Proof: Suppose $a_1 w_1 + \cdots + a_m w_m + bw = O$. We show that $a_1 = \cdots = a_m = b = 0$

Suppose BWOC, $b \neq 0$, then we can solve for w

$$w = \frac{-a_1}{b} w_1 + \cdots + \frac{-a_m}{b} w_m$$

Which means that w is a linear combination of $X \implies w \in \text{span}(X)$. Contradiction

Thus $b = 0$. This gives

$$a_1 w_1 + \cdots + a_m w_m + 0w = O$$

Since $X = \{w_1, \dots, w_m\}$ is linearly independent, we also have $a_1 = \cdots = a_m = 0$

Thus $X \cup \{w\}$ is linearly independent

Main Question: does the enlarging process above terminate? After some steps, do we get a set $\{w_1, \dots, w_m\}$ that spans W?

Exchanging Lemma: Let $X = \{v_1, \dots, v_n\}$ be any basis for W. Choose any $w \in W$ but $w \notin \text{span}(\{v_k, \dots, v_n\})$. Then $\exists v_i, i < k$, such that $Y = (X \setminus \{v_i\}) \cup \{w\}$ is still a basis

- **Note**: If $k > n$, then $\{v_k, \dots, v_n\} = \emptyset$

Proof: First we show that $\text{span}(Y) = W$. Since X spans W, we can write

$$w = a_1 v_1 + \cdots + a_n v_n \implies v_1 = \frac{1}{a_1} w + \frac{-a_2}{a_1} v_2 + \cdots + \frac{-a_m}{a_1} v_m$$

Since $w \notin \text{span}(\{v_k, \dots, v_n\})$, we must have $a_i \neq 0$ for some $i < k$

WLOG, let $a_1 \neq 0$. We show that Y spans W

Since X spans W, for arbitrary $u \in W$, we have

$$u = d_1 v_1 + \cdots + d_n v_n$$

Replacing v_1 above with the previous equation, we see that u is a linear combination of elements of Y and thus $u \in \text{span}(Y)$

Thus $\text{span}(Y) = W$

Next we show that Y is linearly independent

Suppose we have

$$cw + b_2v_2 + b_nv_n = O$$

We show that $c = b_2 = \cdots = b_n = 0$

- If $c = 0 \implies b_2 = \cdots = b_n = 0$ since $\{b_2, \ldots, b_n\}$ is linearly independent
- Otherwise suppose $c \neq 0$, then we can solve for w

$$w = \frac{-b_2}{c}v_2 + \cdots + \frac{-b_n}{c}v_n \implies v_1 = \frac{1}{a_1}(\frac{-b_2}{c}v_2 + \cdots + \frac{-b_n}{c}v_n) + \frac{-a_1}{a_1}v_2 + \cdots + \frac{-a_m}{a_1v_m}$$

Thus v_1 is a linear combination of $\{v_2, \ldots, v_n\}$. Contradiction since we said X was linearly independent. Thus $c = 0$

Theorem: Let $X = \{v_1, \ldots, v_n\}$ be a basis for W, and let $\{w_1, \ldots, w_m\} \subseteq W$ be linearly independent. Then $m \leq n$

Proof: If $m < n$, we are done

Now assume $m \geq n$, we show that $m = n$

Since $\{w_1, \ldots, w_m\}$ is linearly independent, we have that $w_1 \neq O = \text{span}(\emptyset)$

Now apply the Exchanging Lemma to the basis X, with $k > n$ and w_1 Then $\exists v_i$ such that $X_1 = (X \setminus \{v_i\}) \cup \{w_1\}$ is a basis

After reindexing, we see that X_1 has $n-1$ vectors from X and 1 vector from w_1

Now take $k = n$. Since $\{w_1, \ldots, w_m\}$ is linearly independent, $w_2 \notin \text{span}(\{w_1\})$

Thus applying the Exchanging Lemma again, there exists $j < k = n$ such that $X_2 = (X_1 \setminus \{v_j\}) \cup \{w_2\}$ is a basis

Reindexing again, we get that $X_2 = \{v_1, \ldots, v_{n-2}, w_1, w_2\}$ is a basis

After n steps, X_n has no elements from X and $X_n = \{w_1, \ldots, w_n\}$ is a basis

Furthermore, we see that $w_m \in \text{span}(\{w_1, \ldots, w_n\})$, contradicting that $\{w_1, \ldots, w_m\}$ is linearly independent

Thus $m = n$

Corollary: If W is any K-vector space and some basis of W has n elements, then every basis of W has n elements

Definition - Finite Dimensional: Let W be a K-vector space. Then W is **finite dimensional** if some basis for W is finite

Definition - Dimension: Number of elements in any basis for a vector space W

Corollary: Suppose $\dim(W) = n$ and $X = \{w_1, \ldots, w_n\}$ are any n-vectors

1. If X spans W, then X is a basis for W
2. If X is linearly independent, then X is a basis for W

Proof:

1. By Shrinking Theorem, there exists a basis $Y \subseteq X$

   However, $|Y| < n$ contradicts that $\dim(W) = n$

   Thus $Y = X$, i.e. X is a basis
2. By Expansion Theorem, we can expand X to a basis Y

   However, $|Y| > n$ contradicts that $\dim(W) = n$

   Thus $Y = X$, i.e. X is a basis

1.3.1 Toolbox Corollaries and Results

The following are useful corollaries that can be used to prove additional interesting results

Let V be a K-Vector Space with $\dim(V) = n$, i.e. V has some basis with n elements

1. Every basis for V has n elements
2. If $X \supseteq V$ and $\text{span}(X) = V$, then X has at least n elements and some subset $Y \subseteq X$ is a basis for V
3. If $Z \subseteq V$ is linearly independent, then Z has at most n elements and Z can be extended to a basis $Y \supseteq Z$ for V

Example: Let $V = R^3$. Since $\dim(V) = 3$, V has a basis with 3 elements

- Consider the **Standard Basis**: $B = \{(1,0,0),(0,1,0),(0,0,1)\}$

Suppose $X = \{v_1, v_2, v_3\} \subseteq V$ for arbitrary vectors

- If $\text{span}(X) = V$ then X is a basis
- If X is linearly independent, since $|X| = 3$, X is a basis for V

Example: Describe all subspaces $W \subseteq R^3$

Note: Since $\dim(V) = 3$, we must have $\dim(W) \leq \dim(V) = 3$

- Case 0: $\dim(W) = 0$

  Clearly $W = \{O\}$
- Case 1: $\dim(W) = 1$

  W is a line going through $(0,0,0)$

  Thus a basis for W will be $\{w\}$ for any nonzero $w \in W$
- Case 2: $\dim(W) = 2$

  W is a plane containing $(0,0,0)$

  Thus a basis for W will be any 2 element set $\{w_1, w_2\} \subseteq W$ such that

  - Neither element is O
  - w_2 is not a scalar multiple of w_1
- Case 3: $\dim(W) = 3$

  Only possibility is $W = V = R^3$

Examples: Consider subspaces of $\mathcal{F}(R)$ and look at small subspaces

- $W = \text{span}(\{e^x\}) = \{re^x \mid r \in R\}$

  This can be thought of as a 1-dimensional subpsace of $\mathcal{F}(R)$
- $V = \text{span}(\{\sin(x), \cos(x)\}) = \{a\sin(x) + b\cos(x) \mid a, b \in R\}$

  Clearly $\dim(V) = 2$

  Consider $f(x) = \sin(x) \quad g(x) = \cos(x) \quad h(x) = 3\sin(x) - 2\cos(x)$

  Since $h = 3f + (-2)g$, $\{f, g, h\}$ is not linearly independent

  Thus $\text{span}(\{f, g, h\}) = \text{span}(\{f, g\})$

1.4 Direct Sums

Let V be a K-Vector Space with $\dim(V) = n$. Let $W \subseteq V$ be a subspace of V. Then $\dim(W) \leq n$

Now choose another subspace $U \subseteq V$

Note: $W \cap U \neq \emptyset$ since both must contain O

Thus the smallest we can make $W \cap U$ is $\{O\}$

Furthermore, it can be shown that both $U \cap W$ and $U + W$ are both subspaces of V

Definition - Direct Sum: $U \oplus W$ is called a **direct sum** if

- $U \oplus W = U + W$
- $U \cap W = \{O\}$

We often look at cases where $V = U \oplus W$

Example: Consider R^3 and let W be any plane containing $(0, 0, 0)$

If U is any line through $(0, 0, 0)$ such that $U \notin W$, then $R^3 = W \oplus U$

Theorem: Let V be a K-Vector Space with $\dim(V) = n$. Let $W \subseteq V$ be any subspace of V. Then there exists a subspace $U \subseteq V$ such that

$$V = U \oplus W$$

Proof: Choose any basis $Z = \{w_1, \ldots, w_m\}$ of W (we know that $m \leq n$)

Now extend Z to $Y = Z \cup \{u_1, \ldots, u_r\}$, which is a basis for V

Let $U = \text{span}(\{u_1, \ldots, u_r\})$. Then U is a subspace of V and $\{u_1, \ldots, u_r\}$ is a basis for U

- Show that $U \cap W = \{O\}$

  Choose $v \in U \cap W$

  Then we have $v = a_1u_1 + \cdots + a_ru_r = b_1w_1 + \cdots + b_mw_m$

  Since Y is a basis for V, then $\{u_1, \ldots, u_r, b_1, \ldots, b_m\}$ is linearly independent

  Thus $v - v = a_1u_1 + \cdots + a_ru_r - b_1w_1 - \cdots - b_mw_m = O \implies a_1 = \cdots = a_3 = b_1 = \cdots = b_m = 0$

  Thus $v = O$

- Show that $V = U + W$

  Choose any $v \in V$

  Since Y is a basis for V

  $v = \underbrace{a_1u_1 + \cdots + a_ru_r}_{u \in U} + \underbrace{b_1w_1 + \cdots + b_mw_m}_{w \in W}$

  Thus $v = u + w \implies V = U + W$

2 Matrices

Definition - m × n Matrix: Entries $\in K$ of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}$$

Example: $A = \begin{bmatrix} 4 & 0 & 2 \\ -1 & 3 & 6 \end{bmatrix}$ is a 2×3 matrix with entries $\in Q$

Note: Any 2×3 matrices can be added together componentwise or multiplied by a scalar, resulting in a 2×3 matrix

- Here the additive identity is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- Here the additive inverse of A (from previous example) is $-A = \begin{bmatrix} -4 & 0 & -2 \\ 1 & -3 & -6 \end{bmatrix}$

Thus $\text{Mat}_{2\times3}(K)$, the set of all 2×3 matrices with entries in K is a K-Vector Space

Here the basis is $B = \{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\}$

- Clearly spans since any 2×3 matrix $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}$ can be written as a linear combination of elements in B
- Clearly B is linearly independent since the only way to write O is to take each scalar $a_i = 0$

Thus $\dim(\text{Mat}_{2\times3}(K)) = 6$

Upshot: We can generalize the discussion above to show that $\text{Mat}_{m\times n}(K)$ is a K-Vector Space of $\dim = m\times n$

Example: $\{\begin{bmatrix} a & b \\ b & d \end{bmatrix}\}$, **symmetric $\mathbf{2\times2}$ matrices**, is a subspace of $\text{Mat}_{2\times2}(K)$, which has dimension 4

Non-Example: $\text{Mat}(K)$ is NOT a Vector Space since addition between 2×2 and 3×3 matrices is not defined

Notation: $A_i = (a_{i1}, \dots, a_{in})$, the ith row vector, is a $1\times n$ matrix

Notation: $A^j = (a_{1j}, \dots, a_{mj})$, the jth column vector, is a $m\times1$ matrix

Definition - Transpose: Given an $m\times n$ matrix A, the **transpose** ${}^tA$ is an $n\times m$ matrix that swaps the rows and columns, and vice versa

- **Note**: If A is a square $n\times n$ matrix, then ${}^tA$ is also a square $n\times n$ matrix

Example: ${}^t\begin{bmatrix} 4 & 0 & 3 \\ -1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 0 & 3 \\ 2 & 6 \end{bmatrix}$

Definition - Matrix Multiplication: An $m\times n$ matrix A can multiply with an $n\times k$ matrix B where

$$C_{il} = \sum_{d=1}^{n} a_{ij} b_{d,l}$$

- **Note**: If A, B are both $n\times n$ matrices, then AB is an $n\times n$ matrix
- **Upshot**: Square matrices are closed under transposition and matrix multiplication

Example: $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ 4 \end{bmatrix}$

2.1 Linear Equations

Consider

$$5x_1 + 3x_2 - 6x_3 = 8$$
$$x_1 - 2x_2 + x_3 = 4$$

We can represent this using

$$A = \begin{bmatrix} 5 & 3 & -6 \\ 1 & -2 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad B = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \implies AX = B$$

3 Mappings

Definition - Function: Mapping between 2 sets D, R such that for each $x \in D$, there exists a unique $y \in R$ such that $f(x) = y$

$$F : D \to R$$

- **Note**: D here is the **domain** of F and R is the **range** of F

Definition - Image: $F(D) = \{F(x) \mid x \in D\} \subseteq R$

Example: $F : R \to R \qquad F(x) = x^2$

- Domain(F) = Range(F) = R
- Image of $F = \{y \in R \mid y \geq 0\} = [0, \infty)$

Example: $G[0, \infty) \to R \qquad G(x) = \sqrt{x}$

- Image of $G = [0, \infty)$

Example: $\mathcal{F}$ = all functions $F :\to R$

Let S be all "infinitely" differentiable functions

Let $\frac{d}{dx} : S \to S$ where $\frac{d}{dx}(f) = f'$

Thus $\frac{d}{dx}$ is a function

Example: $t : \text{Mat}_{2\times3}(K) \to \text{Mat}_{3\times2}(K)$

Then $t(A) =^t A$ is a function

Definition - Onto: A function $F : D \to R$ is **onto** if Image of $F = R$

Definition - 1-1: A function $F : D \to R$ is **1-1** if different elements from D get mapped to different elements of R

$$F(d) = F(e) \implies d = e$$

Definition - Bijection: A function that is both onto and 1-1

Definition - Inverse Function: If $F : D \to R$ is a bijection, there exists an inverse function $F^{-1} : R \to D$ such that

$$\forall r, \in R, F(F^{-1}(r)) = r$$
$$\forall d, \in D, F^{-1}(F(d)) = d$$

Definition - Linear Transformation: For fixed K-Vector Spaces V, W, a **linear transformation** $T : V \to W$ is a function satisfying

1. $\forall v_1, v_2 \in W$, $T(v_1 + v_2) = T(v_1) + T(v_2)$
2. $\forall c \in K, v \in W$, $T(cv) = cT(v)$

Examples

1. $F : R \to R, F(x) = x^2$
    - Not onto since x^2 cannot be negative

- Not 1-1 since $1^2 = (-1)^2 = 1$
- Not a linear transformation since $(1+2)^2 = 9 \neq 1^2 + 2^2$

2. $F : [0, \infty) \to R, F(x) = \sqrt{x}$
   - Not onto since x^2 cannot be negative
   - 1-1 since $\sqrt{x} = \sqrt{y} \implies x = y$
   - Not a linear transformation since $[0, \infty)$ isn't a Vector Space
3. Let S be the set of all infinite differentiable functions. Consider $\frac{d}{dx} : S \to S$ where $\frac{d}{dx}(f) = f'$
   - Onto by the Fundamental Theorem of Calculus
   - Not 1-1 since f and $f+5$ share the same derivative
   - Is a linear transformation by addition and scalar multiplication properties of derivatives
4. Let C be the set of continuous functions on $[0, 1]$. Consider $I : C \to R, I(f) = \int_0^1 f(t)\, dt$
   - Onto since we can generate any value of R by taking the integral of the constant function
   - Not 1-1 since the definite integral of 2 functions could yield the same result
   - Is a linear transformation by additional and scalar multiplication properties of integrals
5. $I^* : G \to C, I^*(f) = \int_0^x f(t)\, dt$
   - Not onto since not all functions of $f(0) = 0$
   - 1-1 since indefinite integral yields a unique function
   - Is a linear transformation by additional and scalar multiplication properties of integrals
6. Fix $(4, 0, 2)$ and consider $T_{(4,0,2)} : R^3 \to R^3, T_{(4,0,2)}((x, y, z)) = (x+4, y, z+2)$
   - Clearly onto
   - Clearly 1-1
   - Not a linear transformation since $T_{(4,0,2)}((0, 0, 0) + (1, 1, 1)) = (5, 0, 3) \neq T_{(4,0,2)}((0, 0, 0)) + T_{(4,0,2)}((1, 1, 1))$
7. $E_\pi : R^3 \to R^3, E_\pi((x, y, z)) = (\pi x, \pi y, \pi z)$
   - Clearly onto
   - Clearly 1-1
   - Is a linear transformation since $E_\pi((a, b, c) + (d, e, f)) = (\pi(a+d), \pi(b+e), \pi(c+f)) = E_\pi((a, b, c)) + E_\pi((d, e, f))$

3.1 Consequences of Properties of Linear Transformations

Proposition: For any linear transformation $T : V \to W$, we have that

$$T(O_V) = O_W$$

Proof: Let $w = T(O_V)$

Since $O_V = 0 * O_V$, we have that

$$T(O_V) = T(0 * O_V) = 0 * T(O_V) = 0 * w = O_W$$

Proposition: $T(a_1 v_1 + \cdots + a_n v_n) = a_1 T(v_1) + \cdots + a_n T(v_n)$

Proof: Follows from linearly properties of linear transformations

- **Note**: If $x = \{v_1, \ldots, v_n\}$ is a basis for V and if $w_1, \ldots, w_n$ are arbitrary vectors in W, then there is a unique linear transformation $T : V \to W$ such that

$$T(v_1) = w_1, \ldots, T(v_n) = w_n$$

Lemma: $\text{Im}(T)$ is a subspace of W

Proof: We show the necessary conditions for a subspace

- $+ : w_1, w_2 \in \text{Im}(T) \implies \exists v_1, v_2 \in V$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$

  Then $w_1 + w_2 = T(v_1) + T(v_2) = T(\underbrace{v_1 + v_2}_{\in V}) \in \text{Im}(T)$

- $\times : w \in \text{Im}(T) \implies \exists v \in V$ such that $T(v) = w$

  Then for $c \in K$, we have $cw = c(Tv) = T(\underbrace{cv}_{\in V}) \in \text{Im}(T)$

Definition - Pull Back: Suppose $Y = \{w_1, \dots, w_m\} \subseteq \text{Im}(T)$. Then a **pull-back** is any set $\{v_1, \dots, v_m\} \subseteq V$ such that

$$T(v_1) = w_1, \dots, T(v_m) = w_m$$

Lemma: If $\{w_1, \dots, w_m\}$ is linearly independent in $\text{Im}(T)$ (or in W), then any pull back $\{v_1, \dots, v_m\} \subseteq V$ is linearly independent in V

Proof: Let $a_1v_1 + \cdots + a_mv_m = O_V$

Thus $T(a_1, v_1 + \cdots + a_mv_m = O_V) = a_1w_1 + \cdots + a_mw_m = O_W$

Since $\{w_1, \dots, w_m\}$ is linearly independent, we have $a_1 = \cdots = a_m = 0$ as desired

Pull Back Property: Suppose $\{w_1, \dots, w_m\}$ is a basis for $\text{Im}(T)$, and let $\{v_1, \dots, v_m\} \subseteq V$ be any pull back. Furthermore, let $S = \text{span}(\{v_1, \dots, v_m\}) \subseteq V$ be a subspace. Then $\{v_1, \dots, v_m\}$ is a basis for S

Proof: By the previous lemma, $\{v_1, \dots, v_m\}$ is linearly independent

Furthermore, $\{v_1, \dots, v_m\}$ spans S by definition

Corollary: If $T : V \to W$ is any linearly transformation and if $\dim(V) = n$, then $\dim(\text{Im}(T)) \leq n$

Proof: BWOC, suppose $\dim(\text{Im}(T)) > n$, thus we can create a set of $n + 1$ linearly independent elements in $\text{Im}(T)$.

By the Pull Back Property, this pulls back to $n + 1$ linearly independent elements in V. Contradiction since $n + 1 > n = \dim(V)$

Note: $T : V \to W$ where $T(v) = \{O_W\}$ is a linearly transformation with $\dim(\text{Im}(T)) = 0$, regardless of the value of $\dim(V)$

3.2 Kernel

Definition - Kernel: For $T : V \to W$, the **kernel** $\text{Ker}(T) = \{v \in V \mid T(v) = O_W\}$

Proposition: $\text{Ker}(T)$ is a subspace of V

Proof: Clearly $O_V \in \text{Ker}(T)$

- $+$: For $v_1, v_2 \in \text{Ker}(T)$, we see that $T(v_1 + v_2) = T(v_1) + T(v_2) = O_W + O_W = O_W$. Thus $v_1 + v_2 \in \text{Ker}(T)$
- $\times$: For $c \in K$ and $v \in \text{Ker}(T)$, we see that $T(cv) = cT(v) = O_W$. Thus $cv \in \text{Ker}(V)$

Proposition: Let $T : V \to W$ be any linear transformation. For any basis $B = \{w_1, \dots, w_m\} \subseteq \text{Im}(T)$ and for any pullback $\{v_1, \dots, v_m\} \subseteq V$, we have

$$V = \text{Ker}(T) \oplus S \qquad S = \text{span}(\{v_1, \dots, v_m\})$$

Proof: We need to show $V = \text{Ker}(T) + S$ and $\text{Ker}(T) \cap S = \{O_V\}$

- Take arbitrary $v \in V \implies T(v) \in \text{Im}(T) = a_1w_1 + \cdots + a_mw_m$

  Let $s = a_1v_1 + \cdots a_mv_m \in S$.

  Then $T(s) = T(v) \implies T(v - s) = T(v) - T(s) = O_W \implies v - s \in \text{Ker}(T)$

  Let $u = v - s \in \text{Ker}(T)$

  Thus clearly $v = u + s$ for $u \in \text{Ker}(T)$ and $s \in S$

- Clearly $O_V \in \text{Ker}(T) \cap S$ since both are subspaces of V

  Take any arbitrary $v \in \text{Ker}(T) \cap S$

  $v \in S \implies v = b_1 v_1 + \cdots b_m v_m \implies T(v) = b_1 w_1 + \cdots b_m w_m$

  Since $v \in \text{Ker}(T)$, we have that $T(v) = O_W \implies b_1 = \cdots = b_m = 0$ since $\{w_1, \ldots, w_m\}$ is linearly independent

  Thus we have $v = 0v_1 + \cdots + 0v_m = O_V \implies \text{Ker}(T) \cap S = \{O_V\}$

Thus we have shown the necessary properties for $V = \text{Ker}(T) \oplus S$

Theorem: $\dim(V) = \dim(\text{Ker}(T)) + \dim(\text{Im}(T))$

Proof: Choose a basis $B = \{w_1, \ldots, w_m\}$ for $\text{Im}(T)$ and a pullback $\{v_1, \ldots, v_m\}$

Let $S = \text{span}(\{v_1, \ldots, v_m\})$

Since $V = \text{Ker}(T) \oplus S$, we have $\dim(\text{Ker}(T)) + \dim(S) = \dim(\text{Ker}(T)) + \dim(\text{Im}(T)) = \dim(V)$

3.2.1 Consequences of Kernel

Corollary 1: For linear $T : R^3 \to R^4$, T is NOT onto

Proof: $\dim(\text{Im}(T)) \leq \dim(R^3) = 3 < 4 \implies \text{Im}(T) \neq R^4 \implies T$ is NOT onto

Corollary 2: For linear $T : R^4 \to R^3$, T is NOT 1-1

Proof: $\dim(\text{Ker}(T)) + \underbrace{\dim(\text{Im}(T))}_{\leq 3} = \dim(R^4) = 4 \implies \dim(\text{Ker}(T)) \geq 1$

Thus $\text{Ker}(T)$ has something non-zero mapped to $O_W \implies T$ is NOT 1-1

Definition - Isomorphism: $T : V \to W$ such that T is linear transformation and a bijection

Corollary 3: $\dim(V) = \dim(W)$ and $T : V \to W$ is a linear transformation and 1-1 $\implies T$ is an isomorphism (i.e. T is onto)

Proof: $\dim(\text{Ker}(T)) + \dim(\text{Im}(T)) = \dim(V)$

But we know that $\dim(\text{Ker}(T)) = 0 \implies \dim(\text{Im}(T)) = \dim(V) = \dim(W)$

Furthermore $\text{Im}(T)$ is a subspace of W and $\dim(\text{Im}(T)) = \dim(W) \implies T$ is onto

Corollary 4: $\dim(V) = \dim(W)$ and $T : V \to W$ is a linear transformation and onto $\implies T$ is an isomorphism (i.e. T is 1-1)

Proof: $\dim(\text{Ker}(T)) + \dim(\text{Im}(T)) = \dim(V)$

But we know that $\dim(\text{Im}(T)) = \dim(V) \implies \dim(\text{Ker}(T)) = 0$