Subspace: $W \subseteq V$ that is K-vector space itself satisfying

- $w_1, w_2, \in W \implies w_1 + w_2 \in W \qquad \forall c \in K, w \in W \implies cw \in W \qquad O \in W$

Span: $\text{span}(\{v_1, \ldots, v_n\})$ is a subspace of V consisting of all linear combinations of $\{v_1, \ldots v_n\}$

- If $W = \text{span}(\{v_1, \ldots, v_n\})$, then every $w \in W$ is a linear combination of $\{v_1, \ldots, v_n\}$

Linear Independent: occurs when $a_1 v_1 + \cdots + a_n v_n = 0 \implies a_1 = \cdots = a_n = 0$

- $\{v_1, \ldots, v_n\}$ is linearly independent if and only if for each $i, v_i \notin \text{span}(\{v_1, \ldots, v_n\} \setminus \{v_i\})$

Basis: $\{v_1, \ldots, v_n\}$ that spans W and is linearly independent. **Note**: The empty set $\emptyset$ is a basis for $\{O\}$

Shrinking Lemma: Let $X = \{w_1, \ldots, w_m\} \subseteq W$ span W but not be LI. Then $X \setminus \{w_i\}$ still spans W for some $w_i \in X$

- **Shrinking Theorem**: Some $Y \subseteq X$ is a basis of W (must stop eventually when we get $\emptyset$ basis for $\{O\}$)

Enlarging Lemma: let $X = \{w_1, \ldots, w_m\} \subseteq W$ be LI but not span W. Then for any $w \in W \setminus \text{span}(X), X \cup \{w\}$ is still LI

Exchanging Lemma: Let $X = \{v_1, \ldots, v_n\}$ be a basis for W. Take $w \in W$ where $w \in \text{span}(\{v_1, \ldots, v_n\})$. Then for $i < k$, $Y = (X \setminus \{v_i\}) \cup \{w\}$ is still a basis

- Can be used to show that if $\{w_1, \ldots, w_m\} \subseteq W$ is linearly independent, then $m \leq n$. Thus any basis of W has n elements

Finite Dimensional: W with some basis. **Dimension** of W is the number of elements in the basis

- Any set of vectors that spans W, with the correct dimension, is a basis by the Shrinking Theorem
- Any set of vectors that is linearly independent, with the correct dimension, is a basis by the Enlarging Lemma

Direct Sum: $U \oplus W$ such that $U \oplus W = U + W$ AND $U \cap W = \{O\}$

- **Note**: $U \cap W$ and $U + W$ are subspaces of V
- **Theorem**: For subpsace $W \subseteq V$, there exists a subspace $U \subseteq V$ such that $V = U \oplus W$.

$\mathbf{Mat_{m \times n}(K)}$: K-Vector Space of all $m \times n$ matrices with entries in K

- Basis here is $\bigcup E_{ij}$ where E_{ij} has the the ij entry is 1 and all other entries as 0, which clearly has dimension $m \times n$

Symmetric $\mathbf{2 \times 2}$ Matrices come in the form of $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and is a subspace of $\text{Mat}_{2\times 2}(K)$

Image: $F(D) = \{F(x) \mid x \in D\} \subseteq R$ for the mapping $F : D \to R$

- **Onto** if $F(D) = R$ **1-1** if $F(d) = F(e) \implies d = e$ **Bijection** if both onto and 1-1

Inverse Mapping: If $F : D \to R$ is a bijection, then $\exists F^{-1} : R \to D$ such that $\forall r, \in R, F(F^{-1}(r)) = r$ and $\forall d \in D, F^{-1}(F(d)) = d$

Linear Transformation: Function $T : V \to W$ for vector spaces V, W, satisfying

- $\forall v_1, v_2 \in W, T(v_1 + v_2) = T(v_1) + T(v_w) \qquad \forall c \in K, v \in W, T(cv) = cT(v)$

Pull Back: Any set $\{v_1, \ldots, v_m\} \subseteq V$ such that $T(v_1) = w_1, \ldots, T(v_m) = w_m$

- If $\{w_1, \ldots, w_m\} \subseteq \text{Im}(T)$ is a basis, then $\{v_1, \ldots, v_m\} \subseteq V$ is a basis for $\text{span}(\{v_1, \ldots, v_m\})$. Thus $\dim(\text{Im}(T)) \leq \dim(V)$

Kernel: $\text{Ker}(T) = \{v \in V \mid T(v) = O_W\}$, which can be shown to be a subspace of V

- **Proposition** $V = \text{Ker}(T) \oplus \text{span}(\{v_1, \ldots, v_m\})$ for any pullback $\{v_1, \ldots, v_m\} \subseteq V$
- **Theorem**: $\dim(V) = \dim(\text{Ker}(T)) + \dim(\text{Im}(T))$. Comes from $V = \text{Ker}(T) \oplus S \implies \dim(V) = \dim(\text{Ker}(T)) + \dim(S)$

Upshot: $\dim(\text{Ker}(T)) > 0 \implies T$ is NOT 1-1 $\qquad \dim(\text{Im}(T)) < \dim(W) \implies T$ is NOT onto

Isomorphism: $T : V \to W$ such that T is a linear transformation and a bijection

- If $\dim(V) = \dim(W)$ and $T : V \to W$ is a linear transformation and is 1-1 $\implies$ onto OR is onto $\implies$ is 1-1

Inverse Mapping/Transformation: An isomorphism $T^{-1} : W \to V$ where $T^{-1}(w)$ is the unique $v \in V$ such that $T(v) = w$

Linear Map/Matrix: Matrix L_A that determines the LT $R^n \to R^m$, and is itself a LT (from logic of dot products)

- Transformation $T : V \to W$ WRT to bases $B = \{v_1, \ldots, v_m\} \subseteq V$ and $B' = \{w_1, \ldots, w_m\} \subseteq W$ is given by $M^B_{B'} = \begin{bmatrix} T(v_1) & T(v_2) & \cdots & T(v_n) \end{bmatrix}$ where v_1 is WRT to B and the result is written in terms of coordinates of B'

  Upshot: Any vector, written in B coordinates, when multiplied by this matrix, yields an answer in B' coordinates

Change of Basis: $M^B_{B'}(\text{id}) = \begin{bmatrix} \text{id}(v_1) & \text{id}(v_2) & \cdots \text{id}(v_n) \end{bmatrix}$ with respect to bases B, B' of the same vector space V

Scalar Product: $V \times V \to K$ satisfying $\langle v, w\rangle = \langle w, v\rangle \qquad \langle v, c(w_1 + w_2)\rangle = c\langle v, w_1\rangle + c\langle v, w_2\rangle$

- **Positive Definite**: For $v \neq O, \langle v, v\rangle > 0$
- **Non-Degenerate**: For $v \neq O, \exists w \in W, \langle v, w\rangle \neq 0$
- **Non-Trivial**: $\exists v, w \in V$ such that $\langle v, w\rangle \neq 0$
- **Trivial**: $\forall v, w \in V, \langle v, w\rangle = 0$

Orthogonal: $v \perp w \implies \langle v, w\rangle = 0$ **Orthogonal Complement**: $W^{\perp} = \{v \in V \mid \forall w \in W, v \perp w\} \subseteq V$

Length: $\|v\| = \sqrt{\langle v, v\rangle}$ **Projection**: For $w \in V$ and any $v \in V, \exists c \in K$ such that $v - cw \perp w \implies \operatorname{proj}_w v = \frac{\langle v,w\rangle}{\langle w,w\rangle} w$

Pythagoras Theorem: $v \perp w \implies \|v + w\|^2 = \|v\|^2 + \|w\|^2$ **Parallelogram Law**: $\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$

Schwartz Inequality: $|\langle v, w\rangle| \leq \|v\|\|w\|$ **Triangle Inequality**: $\|v + w\| \leq \|v\| + \|w\|$

Proposition: $\{w_1, \ldots, w_r\}$ pairwise orthogonal $\implies \{w_1, \ldots, w_r\}$ is linearly independent

Projection onto Subspace: $\operatorname{proj}_W v = \sum_{i=1}^{r} \operatorname{proj}_{w_i} w_i = \sum_{i=1}^{r} c_i w_i$. Clearly $\operatorname{proj}_W v \in W$

Proposition: $(v - \sum_{j=1}^{r} c_j w_j) \perp w_i$ for all i **Corollary**: $(v - \sum_{j=1}^{r} c_j w_j) \perp w$ for all $w \in W$

Geometric Interpertation: $\operatorname{proj}_W v$ is the closest point to v in W: $\|v - \operatorname{proj}_W v\| \leq \|v - w\|$ for any $w \in W$

Orthonormal Basis: $\{w_1, \ldots, w_r\}$ that is pairwise orthogonal and each $\|w_i\| = 1$

Gram-Schmidt Process: $u_1 = v_1 \qquad p_2 = v_2 - \operatorname{proj}_{u_1} v_2 \implies u_2 = \frac{1}{\|p_2\|} p_2 \qquad p_3 = v_3 - \operatorname{proj}_{u_1} v_3 - \operatorname{proj}_{u_2} v_3 \implies u_3 = \frac{1}{\|p_3\|} p_3$

Theorem: $V = W \oplus W^{\perp}$ **Corollary**: $\dim(V) = \dim(W) + \dim(W^{\perp})$

Rank: $\dim(R^n) = \dim(C_A) + \dim(\operatorname{Null}(A)) = \dim(R_A) + \dim((R_A)^{\perp}) \implies \dim(R_A) = \dim(C_A)$

Hermitian Inner Product: For $y, z \in C^n$, $\langle y, z\rangle = y_1\overline{z_1} + \cdots + y_n\overline{z_n}$

- **Proposition**: Positive definite since $\langle v, v\rangle = x_1\overline{x_1} + \cdots x_n\overline{x_n} = \|x_1\|^2 + \cdots + \|x_n\|^2 \geq 0$

Lemma: $\forall v \in V, \langle v, v\rangle = 0 \implies \langle , \rangle$ is trivial **Corollary**: $\forall v \in V, \langle v, v\rangle = 0 \implies$ any basis of V is orthogonal

Theorem: If $\langle , \rangle$ is a scalar product on V, then V has an orthogonal basis

Dual Space: $V^* = \mathcal{L}(V, K)$ containing linear transformations $\phi : V \to K$

- Typically we take ϕ_i where $\phi_i(v_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ **Proposition**: $B' = \{\phi_1, \ldots, \phi_n\}$ is a basis for V^*
- **Corollary** $\dim(V^*) \approx \dim(V)$. Namely $\dim(V^*) = \dim(V)$ and $\exists$ a 1-1, onto linear transformation $F : V \to V^*, F(v_i) = \phi_i$

Annihilator: $\operatorname{Ann}(W) = \{\phi \in V^* \mid \forall w \in W, \phi(w) = 0\}$. The set of linear transformations ϕ in V^* where $W \subseteq \operatorname{Ker}(\phi)$

Annihilator Theorem: $\dim(W) + \dim(\operatorname{Ann}(W)) = \dim(V) = n$

Determinate Formula: $D(A) = (-1)^{i+1} a_{i1} \det(A_{i1}) + \cdots + (-1)^{i+n} a_{in} \det(A_{in})$

- $D(A^1, \ldots, A^n) = \sum_{\sigma} \epsilon(\sigma) a_{\sigma(1),1} \cdots a_{\sigma(n),n}$
- Determinant is linear: $D(A^1, \ldots, C + C', \ldots, A^n) = D(A^1, \ldots, C, \ldots, A^n) + D(A^1, \ldots, C', \ldots, A^n)$
- If 2 columns are equal, i.e. $A^j = A^i$, then $D(A) = 0$
- For the unit matrix I, $D(I) = 1$
- Interchanging columns changes sign: $D(A^1, \ldots, A^i, \ldots, A^j, \ldots) = -D(A^1, \ldots, A^j, \ldots, A^i, \ldots)$
- Adding a scalar multiple of a column to another column doesn't change $D(A) : D(\ldots, A^k + tA^j, \ldots) = D(\ldots, A^k, \ldots)$

Symmetric Operator: $\langle Av, w\rangle = \langle v, Aw\rangle \implies A =^t A$

Hermitian Operator: $\langle Av, w\rangle = \langle v, A^* w\rangle = \langle v, ^t\overline{A} w\rangle \implies A =^t \overline{A}$

Unitary: $\langle Av, Aw\rangle = \langle v, w\rangle$

- $^tAA = I \iff$ real unitary $\qquad A^*A =^t \overline{A}A = I \iff$ complex unitary