Subspace: $W \subseteq V$ where $\quad w_1, w_2 \in W \implies w_1 + w_2 \in W \quad w_1 \in W, a \in K \implies aw_1 \in W$

$\{v_1, \ldots, v_n\}$ is linearly independent $\iff v_i \notin \text{span}(\{v_1, \ldots, v_n\} \setminus \{v_i\})$

Basis: $\{v_1, \ldots, v_n\}$ spans V and is linearly independent

For a basis, each $v \in V$ is unique with respect to the basis

- Span $\implies$ all $v \in V$ is a linear combination of the basis
- Linear Independence $\implies$ if $v = a_1 + v_1 + \cdots + a_n v_n = b_1 + v_1 + \cdots + b_n v_n$, then $0 = (a_i - b_i)w_i \implies a_i = b_i$
- n spanning vectors $\implies$ it is a basis $\quad$ n linearly independent vectors $\implies$ it is a basis

1.2.6: Show that $\{t, 1/t\}$ is linearly independent

- Suppose $at + b/t = 0 \quad t = 1 \implies a = -b \quad t = 2 \implies a = b = 0$

Direct Sum: For any subspace $W \subseteq V$, there exists a subspace U such that $V = W \oplus U$

- Span: $v = a_1 w_1 + \cdots a_k w_k + b_1 u_1 + \cdots + b_r u_r$
- Linear Independence: $v = a_1 w_1 + \cdots + a_k w_k = b_1 u_1 + \cdots + b_r u_r \implies a_i = b_j = 0$

Onto: $\text{Im}(F) = R \qquad$ **1-1**: $F(d) = F(e) \implies d = e$

Linear Transformation $T : V \to W$: $\quad T(v_1 + v_2) = T(v_1) + T(v_2) \quad T(cv) = cT(v)$

- $T(0_V) = O_W \quad T(a_1 v_1 + \cdots a_n v_n) = a_1 T(v_1) + \cdots + a_n T(v_n)$
- $\text{Im}(T) \subseteq W \quad w_1 + w_1 = T(v_1) + T(v_2) = T(v_1 + v_2) \in \text{Im}(T) \quad cw = cT(w) = T(cw) \in \text{Im}(T)$

Pullback: $T(v_1) = w_1, \ldots, T(v_m) = w_m$

- If $\{w_1, \ldots, w_m\}$ is linearly independent, then $\{v_1, \ldots, v_m\}$ is linearly independent $\quad \dim(\text{Im}(T)) \leq n$
- If $\text{Ker}(F) = 0$ and $\{v_1, \ldots, v_n\}$ is linearly independent, then $\{F(v_1), \ldots, F(v_n)\}$ is linearly independent

$V = \text{Ker}(T) \oplus S$, the pull back of basis of W

- Take $v \in V$ and $s \in S$, then $T(v - s) = 0 \implies v - s \in \text{Ker}(T) \implies v = (v - s) + s \implies V = \text{Ker}(T) + S$
- Take $v \in \text{Ker}(T) \cap S \implies v = a_1 v_1 + \cdots + a_m v_m \implies T(v) = 0 \implies a_i = 0 \implies v = 0 \implies \text{Ker}(T) \cap S = \{0\}$

$\dim(V) = \dim(\text{Ker}(T)) + \dim(\text{Im}(T))$

- $\dim(\text{Im}(T)) < \dim(W) \implies$ NOT onto $\quad \dim(\text{Ker}(T)) > 0 \implies$ NOT 1-1

Isomorphism: Linear Transformation and a Bijection

- T^{-1} is an isomorphism: $T(v_1) = w_1$ and $T(v_2) = w_2 \implies T(v_1 + v_2) = w_1 + w_2 \implies T^{-1}(w_1 + w_2) = v_1 + v_2$

Matrix Linear Mapping: $L_A : R^n \to R^m$ defined by an $m \times n$ matrix

- WRT to standard basis, $A = [T(E_1), \ldots, T(E_n)]$
- WRT to any basis $B = \{v_1, \ldots, v_n\}$, $M^B_{B'}(T) = [T(v_1), \ldots, T(v_n)]$ with respect to basis B'
- Function composition is matrix multiplication $\quad$ Rotation defined by $\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$

Scalar Product: $\langle v, w \rangle = \langle w, v \rangle \quad \langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v, w_2 \rangle \quad \langle v, cw \rangle = c\langle v, w \rangle$

Positive definite: $\forall v, \langle v, v \rangle \geq 0 \qquad$ **Non-degenerate**: $\forall v, \exists w, \langle v, w \rangle \neq 0 \qquad$ **Non-trivial**: $\exists v, w, \langle v, w \rangle \neq 0$

Trivial: $\forall v, w, \langle v, w \rangle = 0 \qquad$ Any trivial scalar product means that any basis is orthogonal

Lemma: $\langle v, v \rangle = 0 \implies$ scalar product is trivial $\qquad \langle v + w, v + w \rangle = \langle v, v \rangle + 2\langle v, w \rangle + \langle w, w \rangle \implies \langle v, w \rangle = 0$

- **Corollary** $\forall v, \langle v, v \rangle 0 \implies$ has a orthogonal basis

Theorem: For any scalar product, V has a orthogonal basis

- Proof by induction. If $\forall v, \langle v, v \rangle = 0 \implies$ orthogonal by Lemma
- Otherwise, let $V_1 = \text{span}(\{v_1\})$ and show that $V = V_1 \oplus V_1^\perp$ and apply IH to the latter
  - For any $v, \text{proj}_{V_1} v \in V_1$ and $v - \text{proj}_{V_1} v \in V_1^\perp \implies v = \text{proj}_{V_1} v + v - \text{proj}_{V_1} v \implies V = V_1 + V_1^\perp$
  - For $v \in V_1 \cap V_1^\perp$, $\langle v, v \rangle = 0$ and $v = dv_1 \implies d^2\langle v_1, v_1 \rangle = 0$ and since $\langle v_1, \rangle \neq 0 \implies d = 0 \implies v = 0$

Orthogonal: $\langle v, w\rangle = 0 \implies v \perp w$ **Orthogonal Complement**: $W^\perp = \{v \in V \mid \forall w \in W, v \perp w\}$

Length: $\|v\| = \sqrt{\langle v, v\rangle}$ **Projection**: $\text{proj}_w v = \frac{\langle v,w\rangle}{\langle w,w\rangle} w$ $\langle v - cw, w\rangle = 0$

Theorem: $\{w_1, \dots, w_r\}$ is pairwise orthogonal $\implies$ it is linearly independent

$\text{proj}_W v \in W$ $(v - \text{proj}_W v) \in W^\perp$

Gram-Schmidt: $u_1 = \frac{1}{\|v_1\|} v_1$ $p_2 = v_2 - \text{proj}_{u_1} v_2 \implies u_2 = \frac{1}{\|p_2\|} u_2$

- **Upshot**: Any Vector Space of R with positive definite scalar product has a orthonormal basis

Theorem: For a subspace $W \subseteq V$, $V = W \oplus W^\perp$

- Set $w = \text{proj}_W v \in W$ and $v - w \in W^\perp \implies v = (w) + (v - w) \implies V = W + W^\perp$
- For $w \in W \cap W^\perp$, $\langle w, w\rangle = 0 \implies w = 0$ since positive definite
- **Corollary**: $\dim(V) = \dim(W) + \dim(W^\perp)$

5.2.7.a: Let V be the vector space of all $n \times n$ matrices over R, show that $\langle A, B\rangle = \text{Tr}(AB)$ is non-degenerate

- If $\langle A, B\rangle = 0$ for all $B \in V$, then we can choose B_{ij} with 0 in all components except ij coordinate to show that $A = O$

5.2.7.c: Let V be the space of real $n \times n$ symmetric matrices. What is the dimension of W, subspace consisting of matrices with $\text{Tr}(A) = 0$? What is the dimension of $W^\perp$?

- $\dim(W) = \frac{n(n+1)}{2} - 1$ since one of the diagonal entries will be a LC of the other diagonal entries. Thus $\dim(W^\perp) = 1$

Rank: $\dim(R_A) = \dim(C_A)$. For a linear mapping L_A defined by matrix A

- $\text{Im}(L_A) = C_A, \text{Ker}(L_A) = \text{Null}(A) \implies \dim(R^n) = \dim(\text{Im}(L_A)) + \dim(\text{Ker}(L_A)) = \dim(C_A) + \dim(\text{Null}(A))$
- For $v \in \text{Null}(A), A_i \perp v \iff v \in (R_A)^\perp \implies \dim(R^n) = \dim(R_A) + \dim((R_A)^\perp) = \dim(R_A) + \dim(\text{Null}(A))$
- Solution space for homogenous equation is $n - \text{rank}(A)$

5.3.6: Find dimension of solution set of $X \cdot A = P \cdot A$ for vector A and point P

- We have $(X - P) \cdot A = 0 \implies \dim(A^\perp) = n - 1$

Hermitian Inner Product: $\langle v, w\rangle = \overline{\langle w, v\rangle}$ $\langle v, w_1 + w_2\rangle = \langle v, w_1\rangle + \langle v, w_2\rangle$ $\langle cv, w\rangle = c\langle v, w\rangle$ $\langle v, cw\rangle = \overline{c}\langle v, w\rangle$

- **Note**: Hermitian Inner Product is positive definite

Dual Space: $V^* = L(V, k)$ $\phi : V \to K \in V^*$ Most basic operation: $\phi_i(v_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Theorem: $B' = \{\phi_1, \dots, \phi_n\}$ is a basis for V^*

- Linear Independence: $0 = a_1\phi_1 + \cdots a_n\phi_n \implies 0v_i = 0 \implies a_i\phi_i = 0 \implies a_i = 0$
- Spanning: Take $T \in L(V, K)$ where $T(v_i) = b_i$ and take $\phi^* = b_1\phi_1 + \cdots + b_n\phi_n \in \text{span}(B')$

  Now show that $T = \phi^*$ by applying $Tv_i = \phi^* v_i = b_i \implies T \in \text{span}(B')$
- **Corollary**: $V^* \approx V \implies \dim(V^*) = \dim(V)$
- **Corollary**: There exists a 1-1, onto linear function $F : V \to V^*$ such that $F(v_i) = \phi_i$

5.6.5: For ϕ, ψ non-zero functionals that are not scalar multiples of each other, show that dimension of $\text{Ker}(\psi) \cap \text{Ker}(\phi) = n - 2$

- $v \in \text{Ker}(\psi) \cup \text{Ker}(\phi)$ otherwise $v \notin \text{Ker}(\psi)$ and $v \notin \text{Ker}(\phi) \implies \psi = c\phi$. Contradiction

  Thus $\dim(\text{Ker}(\psi) \cup \text{Ker}(\phi)) = n = (n-1) + (n-1) - \dim(\text{Ker}(\phi) \cap \text{Ker}(\psi)) \implies \dim(\text{Ker}(\phi) \cap \text{Ker}(\psi)) = n - 2$

5.6.6: Show that $v \in V$ gives rise an element $\lambda_v \in V^{**}$ and that $v \to \lambda_v$ gives an isomorphism $V \approx V^{**}$

- $\lambda_v = \phi(v)$ for some $\phi \in V^*$

  For 1-1, take $v \in \text{Ker}(F) \implies \lambda_v = 0 \implies \forall \phi \in V^*, \phi(v) = 0$. BWOC, suppose $v \neq 0 \implies \exists \phi$ such that $\phi(v) \neq 0$

  Since $\dim(V) \approx \dim(V^*) \approx \dim(V^{**}) \implies F$ is onto and thus a bijection

  To show that F is an isomorphism, $F(aw + bv) = \lambda_{aw+bv} = \phi(aw + bv) = a\phi(w) + b\phi(v) = a\lambda_w + b\lambda_v$

5.6.7: Show that $W^{\perp\perp} = W$ for a non-degenerate scalar product

- $W^{\perp\perp} = \{v \in V \mid v \perp W^{\perp}\}$ so for $w \in W, w \perp W^{\perp} \implies W \subseteq W^{\perp}$

  Also $V = W \oplus W^{\perp} = W^{\perp} \oplus W^{\perp\perp} \implies \dim(W) = \dim(W^{\perp\perp}) \implies W = W^{\perp\perp}$

Operators: $T : V \to V$, associated with an $n \times n$ matrix

Multilinear k-form: $V \times \cdots \times V \to K, \omega(v_1, \ldots, (av_i + bw_i), \ldots, v_k) = a\omega(v_1, \ldots, v_i, \ldots, v_k) + b\omega(v_1, \ldots, w_i, \ldots, v_k)$

- **Upshot**: Linear on each coordinate if the other coordinates are fixed
- $\mathrm{ML}_K(V)$: set of all $\omega : V \times \cdots \times V \to K$ is a vector space
  - $\mathrm{ML}_1(V) = \{\omega : V \to K\} = V^*$ $\mathrm{ML}_2(V) = \{\omega \mid V \times V \to K\} \supseteq$ scalar products

Alternating: $\omega : V^K \to K$ such that $v_i = v_j \implies \omega(v_1, \ldots, v_k) = 0$

- **Example**: Determinate with the same column is equal to 0
- Λ: Set of alternating multilinear k forms $\subseteq \mathrm{ML}_K(V)$

Permutation: $\sigma : [n] \to [n]$ **Transposition**: τ swaps 2 entries of $[n]$ so $\tau = \tau^{-1} \implies \tau^2 = \mathrm{id}$

- $\epsilon = \begin{cases} +1 & \sigma \text{ even} \\ -1 & \sigma \text{ odd} \end{cases}$ $\sigma \in S_k$ permutes $\{x_1, \ldots, x_k\} \to \{x_{\sigma(1)}, \ldots, x_{\sigma(k)}\}$
- **Notation**: $(\sigma_\omega) = (x_1, \ldots, x_k) = \omega(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$

Theorem: For $\omega \in \Lambda(V)$ and $\sigma \in S_k$, $(\sigma_\omega) = \epsilon(\sigma)\omega$

- Look at transposition τ and let $\overline{\omega}(x, y) = \omega(v_1, \ldots, x, \ldots, y, \ldots, v_k) \implies \overline{\omega}(x + y, x + y) = 0 \implies \omega(x, y) = -\omega(y, x)$

Theorem: $\{v_1, \ldots, v_k\}$ linearly dependent $\implies \forall \omega \in \Lambda(V), \omega(v_1, \ldots, v_k) = 0$

- **Upshot**: Multilinear forms preserve linear dependence

Big Count: For a basis $B = \{b_1, \ldots, b_n\}$ and $\omega \in \Lambda(V)$, $\omega(v_1, \ldots, v_n) = \omega(b_1, \ldots, b_n) \sum_{\sigma \in S_n} (a_{1\sigma(1)}) \cdots a_{n\sigma(n)} \epsilon(\sigma)$

- **Note**: $\dim(\Lambda(V)) = 1$ if $\dim(V) \geq 1$. Otherwise $\dim(\Lambda(V)) = 0$
- $T : V \to V$, $T^* : \Lambda_n(V) \to \Lambda_n(V)$, define $T^*(\omega) : V^n \to K$ and $T^*(\omega)(u_1, \ldots, u_n) = \omega(T(u_1), \ldots, T(u_n)) \implies T^*(\omega) = d\omega$

Properties of $\det(T)$:

- $T(v) = av \implies T^*(\omega)(u_1, \ldots, u_n) = \omega(T(u_1), \ldots, T(u_n)) = a^n \omega(u_1, \ldots, u_n) \implies \det(T) = a^n$
- $\mathrm{id} \implies \mathrm{id}(v) = v \implies \det(\mathrm{id}) = 1$ $\mathrm{zero} \implies \mathrm{zero}(v) = 0 \implies \det(\mathrm{zero}) = 0$
- $(S \circ T)^* \omega(u_1, \ldots, u_n) = \omega(S \circ T(u_1), \ldots, S \circ T(u_n)) = \det(S)\det(T)\omega(u_1, \ldots, u_n)$
- Invertible: $\det(T^{-1}) = \frac{1}{\det(T)} \iff T$ is invertible $\iff \det(T) \neq 0$

The following are equivalent:

1. T is an isomorphism
2. T is invertible
3. $\mathrm{rank}(T) = n$
4. $\det(T) \neq 0$

For $A \in M_{n \times n}(K), \det(A) = \sum_{\sigma \in S_n} \epsilon(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)} = \epsilon(\mathrm{id}) a_{11} a_{22} + \epsilon(\tau) a_{12} a_{21}$

6.2.1: Show that $D(cA) = c^3 D(A)$

- $D(cA) = D(cA^1, cA^2, cA^3) = c^3 D(A)$

Symmetric Operator: $\langle Av, w \rangle = \langle v, A^t w \rangle$ **Symmetric** $\iff A = A^t \implies \langle Av, w \rangle = \langle v, Aw \rangle$

- $(A + B)^t = A^t + B^t$ $(AB)^t = B^t A^t$ $(cA)^t = cA^t$ $A^{tt} = A$

Hermitian Operator: $\langle Av, w \rangle = \langle v, A^* w \rangle$ **Hermitian** $\iff A^* = \overline{A^t} = A \iff A^t = \overline{A}$

- $(A + B)^* = A^* + B^*$ $(AB)^* = B^* A^*$ $(\alpha A)^* = \overline{\alpha} A^*$ $A^{**} = A$

Unitary: $\langle Av, Aw \rangle = \langle v, w \rangle$ **Real Unitary** $\iff A^t A = I$ **Complex Unitary** $\iff A^* A = i \iff \overline{A^t} A = I$

7.1.1.c: Show that for a skew-symmetric matrix A, $\det(A) = 0$ if A is an $n \times n$ matrix where n is odd

- $\det(A) = \det(A^t) = \det(-A) = (-1)^n \det(A) \implies \det(A) = 0$ when n is odd

7.1.2: If A is an invertible symmetric matrix, show that A^{-1} is symmetric

- $AA^{-1} = I \implies (A^{-1})^t A^t = I \implies (A^{-1})^t A = I \implies A^{-1} = (A^{-1})^t$

7.1.7: For a non-degenerate $\langle , \rangle$ and linear map $A : V \to V$, show that the image of A^t is the orthogonal space to the kernel of A

- Take $y \in \text{Im}(A^t) \implies A^t x = y$ and take $z \in \text{Ker}(A)$, then $\langle y, z \rangle = \langle Ax, z \rangle = \langle x, Az \rangle = 0$

7.1.10: Take positive definite $\langle , \rangle$ over R and suppose that $V = W + W^\perp$. Let P be the projection on W. Show that P is symmetric and semipositive ($\langle Av, v \rangle \geq 0$)

- Show symmetric, take $a = a_1 + a_2$ and $b = b_1 + b_2$, then $\langle Pa, b \text{range} = \langle a_1, b_1 \rangle = \langle a, Pb \rangle$
- Show semipositive, take $a = a_1 + a_2$, then $\langle Pa, a \rangle = \langle a_1, a_1 \rangle \geq 0$ since positive definite

7.3.1.a: Take a positive definite scalar product over R and let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ be orthonormal bases. Take $A : V \to V$ such that $Av_i = w_i$. Show that A is real unitary.

- $\langle v_i, w_j \rangle = \langle v_i, a_1 v_1 + \cdots + a_n v_n \rangle = a_i$ and $\langle Av_i, Aw_j \rangle = a_i \langle w_i, w_i \rangle = a_i$

7.3.3: Show that A^t is unitary, A^{-1} exists and is unitary, if B is real unitary, then AB and $B^{-1}AB$ are unitary

- $\langle A^t v, A^t w \rangle = \langle v, AA^t w \rangle = \langle v, w \rangle$
- $AA^{-1} = AA^t = I \implies A^{-1} = A^t$ is unitary
- $\langle ABv, ABw \rangle = \langle v, B^t A^t ABw \rangle = \langle v, w \rangle$ and $\langle B^{-1}ABv, B^{-1}AB \rangle = \langle v, B^t A^t BB^{-1}ABw \rangle = \langle v, w \rangle$

Eigenvalue: λ such that $Av = \lambda v$ for some $v \in V$ **Eigenvector**: v such that $Av = \lambda v$ for some $\lambda \in K$

Theorem: If $\{v_1, \ldots, v_n\}$ are eigenvectors of distinct $\lambda_1, \ldots, \lambda_m$, then it is linearly independent

- By Induction: $\lambda_1(c_1 v_1 + \cdots + c_m v_m) = 0$ and $A(c_1 v_1 + \cdots + c_m v_m) = 0 \implies c_2(\lambda_2 - \lambda_1)v_2 + \cdots + c_m(\lambda_m - \lambda_1)v_m = 0 \implies c_1 = \cdots = c_m = 0$

Diagonalization: Basis $\{v_1, \ldots, v_n\}$ is diagonalizes $L : V \to V$ if each v_i is an eigenvector such that $Lv_i = c_i v_i$

- Transformation matrix WRT to basis is diagonal with c_i

Characteristic Polynomial: $P_A(t) = \det(tI - A)$ λ is an eigenvalue $\iff P_A(\lambda) = 0$

Theorem: $A : V \to V$ is symmetric, then A has a non-zero eigenvector

Invariant: $A(W) \subseteq W \iff \forall u \in W, Au \in W$

Theorem: For symmetric $A : V \to V$ and eigenvector v, $w \in V$ and $w \perp v \implies Aw \perp v$

- $\langle Aw, v \rangle = \langle w, Av \rangle = \lambda \langle w, v \rangle = 0 \implies Aw \in W^\perp \implies W^\perp$ is invariant A

Spectral Theorem: For symmetric $A : V \to V$, V has an orthonormal basis of eigenvectors

- Since A is symmetric, take an eigenvector v_1 and define $V_1 = \text{span}(\{v_1\})$, then $V = V_1 \oplus V_1^\perp$, both invariant under A

  Apply IH to $W^\perp$, with dimension $n - 1$ to create an orthonormal basis $\{e_1, \ldots, e_n\}$ where diagonal matrix of λ_i is the matrix WRT to $\{e_1, \ldots, e_n\}$

Theorem: For symmetric A matrix, there exists real unitary matrix U such that $U^t AU = U^{-1}AU$ is diagonal

- **Upshot**: All symmetric matrices can be written as $A = UDU^t$ where D is diagonal and U is real unitary

8.2.10: Show that eigenvalues of A, A^t are the same

- $\det(tI - A) = \det((tI - A)^t) = \det(tI - A^t)$

8.2.11: Let A be invertible matrix, then $\lambda \neq 0$ and λ^{-1} is an eigenvalue of A^{-1}

- If $\lambda = 0$, then $Av = 0 \implies A$ not invertible since $\text{Ker}(A) \neq \{O\}$
- $A^{-1}Av = A^{-1}\lambda v \implies Iv = A^{-1}v\lambda v \implies A^{-1}v = \lambda^{-1}v$

8.2.12: Does the derivative of $\{\sin t, \cos t\}$ have any non-zero eigenvectors?

- $D(a \cos t + b \sin t) = b \cos t - a \sin t = \lambda(a \cos t + b \sin t) \implies (b - a\lambda) \cos t + (-a - b\lambda) \sin t = 0 \implies \lambda^2 b + \lambda = 0 \implies$ no

8.2.15: Show that eigenvalues of AB are the same as eigenvalues of BA

- $ABv = \lambda v \implies B\lambda v = BABv = (BA)Bv$ so λ is an eigenvalue of AB, BA

  If $\lambda = 0$, then $P_{AB}(t) = \det(-AB) = \det(-BA) = P_{AB}(t)$

8.4.3: For symmetric $A : V \to V$ show that $\langle Av, v \rangle > 0 \implies \lambda > 0$ and $\exists$ symmetric B such that $B^2 = A$ and $BA = AB$, and what are the eigenvalues of B

- $\langle Av, v\rangle = \lambda\langle v, v\rangle \implies \lambda > 0$
- $A = UDU^t = (UD'U^t)(UD'U^t) = B^2$ (symmetric since $B^t = B$) and $(UD'U^t)(UDU^t) = UD'DU^t = (UDU^t)(UD'U^t)$

8.4.12: Show that if symmetric A only has one eigenvalue, every orthogonal basis of V consists of eigenvectors of A

- $D = U^{-1}AU \implies A = UDU^{-1} = U\lambda IU^{-1} = \lambda I$. Thus any vector is an eigenvector

8.4.13: Let $A : V \to$ be symmetric and that there are n distinct eigenvalues of A. Then the eigenvectors form an orthogonal basis of V

- Any v_i, v_j are orthogonal since $\langle Av_i, v_j\rangle = \lambda_i\langle v_i, v_j\rangle = \lambda_j\langle v_i, v_j\rangle = \langle v_i, Av_j\rangle \implies (\lambda_i - \lambda_j)\langle v_i, v_j\rangle = 0 \implies v_i \perp v_j$
- Induction: $\lambda_1(a_1v_1+\cdots+a_nv_n) = A(a_1v_1+\cdots+a_nv_n) \implies (\lambda_1-\lambda_2)a_2v_2+\cdots = 0 \implies a_i = 0$. Thus linearly independent

Polynomial: $f(t) = a_nt^n + \cdots + a_0$ $\quad$ $\deg(fg) = \deg(f) + \deg(g)$ $\quad$ **Root** $\alpha \implies f(\alpha) = 0$

- For complex polynomial, $\exists\alpha_1, \ldots, \alpha_n$ such that $f(t) = (t - \alpha_1)\cdots(t - \alpha_n)$

For $n \times n$ matrix A, exists a polynomial $f \in K[t]$ such that $f(A) = 0$. Follows from owers of A are linearly dependent for $N > n^2$

$$a_NA^n + \cdots + a_0I = 0 \implies f(t) = a_Nt^N + \cdots + a_0$$

- **Note**: this also applies to linear maps A

9.2.2: If A is a symmetric matrix, show that $f(A)$ is also symmetric

- Clearly symmetry holds over + and scalar multiplication. For A^n, use induction: $(A^{k+1})^t = A^t(A^k)^t = A^{k+1}$

Fan: A **fan** of operator A is sequence of subspace $\{V_1, \ldots, V_n\}$ such that $V_i \subset V_{i+1}$ where each V_i is A-invariant

Fan Basis: $\{v_1, \ldots, v_n\}$ such that $\{v_1, \ldots, v_i\}$ is a basis for V_i

Theorem: If $\{v_1, \ldots, v_n\}$ is a fan basis for A, the matrix associated with A is upper triangular

- Since $AV_i \subset V_i, \exists a_{ij}$ such that $Av_i = a_{1i}v_1 + \cdots + a_{ii}v_n$, which creates an upper triangular matrix
- **Note**: Converse holds since an upper triangular matrix A has column unit vectors that form a fan basis for A

Trianglular: Operator $A : V \to V$ has a basis with an associated matrix of A that is **triangular**

10.1.4: Show that the inverse of an invertible triangular matrix is also triangular, having the same fan as A, $\{V_1, \ldots, V_n\}$

- $A : V_i \to V_i \implies A^{-1} : V_i \to V_i$ so V_i is A^{-1} invariant. Also each $V_i \subset V_{i+1}$

Theorem: For V over the complex, fan of A exists

For char polynomial $P(A) = (A - \lambda_1I)v_1\cdots(A - \lambda_nI)v_i = 0 \implies P(A) = 0$ for vectors of a basis of eigenvectors $\{v_1, \ldots, v_n\}$

Theorem: Let V be a complex vector space and $A : V \to V$, let P be the char polynomila, then $P(A) = 0$

- **Corollary**: For a $n \times n$ matrix of complex numbers A, $P(A) = 0$

Euclidean Algorithm: $f(t) = q(t)g(t) = r(t)$ where $\deg(r) < \deg(g)$

- **Corollary**: $f(\alpha) = 0 \implies f(t) = (t - \alpha)q(t)$

Ideal: $J \subseteq K[t]$ satisfying: $0 \in J$ $\quad$ $f, g \in J \implies f + g \in J$ $\quad$ $f \in J \implies \forall g \in K[t], gf \in J$

- 1 is the **generator** of $K[t]$, also 1 is called the **unit ideal**

Theorem: For an ideal J, there exists a generator g where g is the smallest degree

Greatest Common Divisor: $g = \gcd(f_1, f_2) \implies g \mid f_1, g \mid f_2$ and $h \mid f_1$ and $h \mid f_2 \implies h \mid g$

Theorem If g is the generator of the ideal created by f_1, f_2, then $\gcd(f_1, f_2) = g$

- **Corollary**: Can be generalized to n polynomials $f_1, \ldots, f_n$

Relatively Prime: $\gcd(f_1, \ldots, f_n) = 1$ $\quad$ **Irreducible** if $p = fg \implies f$ or g has degree 0

Theorem: Every $f \in K[t]$ can be expressed as a product of irreducibles p_i

Multiplicity: $f = p^mg \implies m$ is the mulitplicity of p in f

Take $f(t) \in K[t]$ and $w = W = \mathrm{Ker}(f(A))$, then W is invariant under A

- For $v \in W$, we have $tf(t) = f(t)t \implies Af(A)v = f(A)Av = 0 \implies Av \in W$

Note: $f(A)g(A) = g(A)f(A)$

Theorem: For $f \in K[t]$ where $f = f_1 f_2$ where f_1, f_2 have degree $\geq$ and gcd $= 1$, $f(A) = 0$, and $W_1 = \text{Ker}(f_1(A)), W_2 = \text{Ker}(f_2(A))$, we have that $V = W_1 + W_2$

- $g_1(A)f_1(A)v + g_2(A)f_2(A)v = v \implies V = W_2 + W_1$ since left component in W_2 and right component in W_1
- Set $v = w_1 + w_2 \implies g_1 f_1 v = g_1 f_1 w_2 \implies w_2 = g_1 f_1 v$. Similar for $w_1 = g_2 f_2 v$. This comes from $g_1 f_1 w_2 + g_2 f_2 w_2 = w_2$
- **Corollary**: Theorem also applies for several product of factors

Theorem: Vector Space over C and an $A : V \to A$ such that $P(A) = 0$ where $P(t) = (t - \alpha_1)^{m_1} \cdots (t - \alpha_r)^{m_r}$. Set $W_i = \text{Ker}(A - \alpha_i I)^{m_i}$, then $V = \oplus W_1 \oplus \cdots \oplus W_r$

11.4.1: Show that $\text{Im}(f_1(A)) = \text{Ker}(f_2(A))$

- $y \in \text{Im}(f_1(A)) \implies f_1(A)x = y \implies f_2(A)y = 0 \implies \text{Im}(f_1(A)) \subseteq \text{Ker}(f_2(A))$
- $v \in \text{Ker}(f_2(A)) \implies v = g_1 f_1 v + g_2 f_2 v = g_1 f_1 v \in \text{Im}(f_1(A))$

11.4.3: Show that if $P_A(t) = (t - \alpha_1) \cdots (t - \alpha_n)$ that V has a basis consisting of eigenvectors of A

- $P(A) = 0$ and $P = \prod f_i$ where $f_i(t) = (t - \alpha_i)$

  Set $V_i = \text{Ker}(f_i(A))$, then V_i is an eigenspace of A and $V = V_1 \oplus \cdots \oplus V_n$ and V has a basis of eigenvectors

S-invariant: $BW \subseteq W$ for all operators $B \in S$ **Simple S-space**: Only S-invariant subspaces of V are V, O

Example 1: A such that $AB = BA$ for all $B \in S \implies \text{Im}(A), \text{Ker}(A)$ are S-invariant

- $w = Av \implies Bw = BAv = ABv \implies Bw \in \text{Im}(A)$ $u \in \text{Ker}(A) \implies ABu = BAu = 0 \implies Bu \in \text{Ker}(A)$

Theorem: For a simple S-space V and operator $A : V \to V$ such that $AB = BA$ for all $B \in S$, either A is invertible or A is zero map

- Assume $A \neq 0$, by example 1, $\text{Ker}(A) = \{O\}$ and its image is all of $V \implies A$ is invertible

Theorem: For a vector space V over C, a set of operators S, V is S-simple, operator $A : V \to V$ such that $AB = bA$, there exists λ such that $A = \lambda I$

- Take the ideal J such that $f(A) = 0$ and take its generator g. g is irreducible since otherwise $g = h_1 h_2 \implies h_1(A) \neq 0, h_2(A) \neq 0$ are invertible $\implies g = h_1 h_2$ is invertible but $\deg(g) > 0$

  Any irreducible over C has degree 1 thus $g(t) = t - \lambda \implies g(A) = 9 \implies A = \lambda I$

Cyclic: $(A - \lambda I)$ is cyclic if $(A - \lambda I)^r v = O$ **Period** is r such that $(A - \alpha I)^k v \neq 0$ for $0 \leq k < r$

Lemma: $\{v, (A - \alpha I)v, \ldots, (A - \alpha I)^{r-1} v\}$ is linearly independent

- Let $B = A - \alpha I$ and $f(t) = c_0 + \cdots + c_s t^s$ for $s \leq r - 1$, then $f(B)v = 0$

  Take $g = t^r$ and let $h = \gcd(f, g) \implies h = f_1 f + g_1 g \implies h(B)v = 0$

  Then $h \mid g \implies h = t^d$ for $d \leq r - 1$. Contradiction since r is the period

A vector space is **cyclic** if $\{v, Av, \ldots, A^{r-1} v\}$ generates V

- **Corollary**: By the previous lemma, $\{(A - \alpha I)^{r-1} v, \ldots, v\}$ is a basis for V

Jordon Basis for V with respect to A

$$\begin{pmatrix} \alpha & 1 & 0 & \cdots & 0 & 0 \\ 0 & \alpha & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots\ \vdots & \vdots \\ & & & & & 0 \\ 0 & 0 & 0 & \cdots & \alpha & 1 \\ 0 & 0 & 0 & \cdots & 0 & \alpha \end{pmatrix}.$$

Figure 1: Jordan Basis

- **Note**: $(A - \alpha I)^{r-1} v$ is an eigenvector with eigven value α

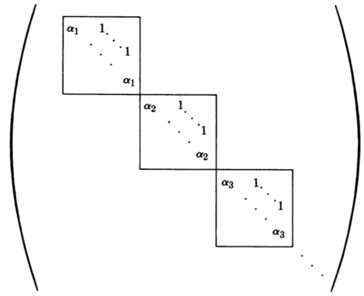

Figure 2: Jordan Normal Form

If V is the direct sum of A-invariant subspaces, $V = V_1 \oplus \cdots \oplus V_m$ and each V_i is cyclic, the then the sequence of Jordan basis for each V_i form a basis for V

Jordan Normal Form:

Theorem: For V over C and operator $A : V \to V$, V can be expressed as a direct sum of A-invariant cyclic subspace