MATH405: Linear Algebra

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Contents

1 Vector Space

Goals of this course is to discuss

- Vector spaces
- Linear transformations between vector spaces
- Other operations on vector spaces

1.1 Definitions

Definition - Field: A set of numbers containing $0, 1$ that can be added, subtracted, multiplied, and divided (except cannot divide by 0) that satisfy the following **Field Axioms**

1. $a, b \in K \implies a + b, ab \in K$
2. $+, \times$ are commutative so $a + b = b + a$ and $ab = ba$
3. $+, \times$ are associative so $(a + b) + c = a + (b + c)$ and $a(bc) = (ab)c$
4. Distributive Law: $a(b + c) = ab + ac$
5. Additive Identity: $a + 0 = 0 + a = a$
6. Multiplicative Identity: $a \cdot 1 = 1 \cdot a = a$
7. Additive Inverse: $\forall a \in K, \exists b$ such that $a + b = 0$, namely $b = -a$ which is unique
8. Multiplicative Inverse: $\forall a \in K, \exists b$ such that $ab = 1$, name $b = 1/a$ which is unique

- **Example**: R, Q are fields. Z is not a field since there is no multiplicative inverse of 2

Example: $C = \{a + bi \mid a, b \in R\}$, where $i = \sqrt{-1}$ is a field under

- $+ : (a + bi) + (c + di) = (a + c) + (b + d)i$
- $\times : (a + bi)(c + di) = (ac - bd) + (ad + bc)i$

Example: $F_2 = \{0, 1\}$ is a field under

- $+$: where

  $0 + 0 = 0$

  $0 + 1 = 1 + 0 = 1$

  $1 + 1 = 0$

- $\times$: where

  $0 \cdot 0 = 0$

  $0 \cdot 1 = 1 \cdot 0 = 0$

  $1 \cdot 1 = 1$

Example: For a prime p, let $F_p = \{0, \ldots, p - 1\}$. Then F_p is a field under

- $+ : a + b \pmod p$
- $\times : ab \pmod p$

Definition - Vector Space: For an arbitrary field K, a K-vector space is a set V with a distinguished element O such that any 2 elements in V can be added and scalar multiplied by $c \in K$

- $u, v \in V \implies u + v \in V$
- $c \in K, u \in V \implies cu \in V$

Satisfying the following properties

1. Commutative Addition: $u + v = v + u$
2. Associative Addition: $(u + v) + w = u + (v + w)$
3. Additive Identity: $u + O = u$

4. Additive Inverse: $\forall u \in V, \exists v \in V$ such that $u + v = O$, namely $v = -u$ which is unique
5. Distributive Laws: $\forall a, b \in K, a(u + v) = au + av$ and $(a + b)u = au + bu$
6. Commutative Scalar Multiplication: $(ab)u = a(bu)$
7. Multiplicative Identity: $1 \cdot u = u$

Example: R^3 is an R-vector space defined by the operations

$$R^3 = \{(x, y, z) \mid x, y, z \in R\}$$

- $+$: add componentwise so $(a, b, c) + (d, e, f) = (a + d, b + e, c + f)$
- Scalar $\times$: for $r \in R$, $r(a, b, c) = (ra, rb, rc)$
- Additive Identity is $O = (0, 0, 0)$

Example: For any field K, K^2 is a K-vector space defined by the oepartions

$$K^2 = \{(x, y) \mid x, y \in K\}$$

- $+$: add componentwise so $(a, b) + (c, d) = (a + c, b + d)$
- Scalar $\times$: for $k \in K$, $k(a, b) = (ka, kb)$
- Additive Identity is $O = (0, 0)$

Example: R is an R-vector space since clearly the properties hold

Example R is a Q-vector space since clearly the properties hold

- Notably, for $q \in Q$ and $r \in R$, we have $qr \in R$. Thus scalar multiplication is closed

Example: For any field K, the set $\{O\}$ is a K-vector space

Example: Let X be any non-empty set and let $\mathcal{F}(X)$ be the set of all functions $f : X \to R$. Then $\mathcal{F}$ is an R-vector space under the operations

- $+$: for $f, g \in \mathcal{F}(X)$, define $f + g := (f + g)(x)$
- Scalar $\times$: let $r \in R$, then define $rf := r(f(x))$
- Additive Identity is $O = f(x) = 0$, the function that takes any x to 0

Example: Take $X = N$ and let $F(X) = \{$ all functions $f : N \to R\}$ is a vector space

- **Note**: $f : N \to R$ is a sequence $(a_0, \ldots, a_n)$ where $a_n = f(n)$

Lemma 1 - Cancellation: For $u, v, w \in V$ and if $u + v = w + v$, then $u = w$

Proof: $v \in V$ has an additive inverse, namely $-v$. Thus we have

$$u + v - v = w + v - v \implies u = w$$

Lemma 2 - Unique Additive Inverse: For all $v \in V$, there is a unique additive inverse, namely $-v$

Proof: Suppose u, w are both additive inverses of v. Then we have

$$v + u = v + w \implies u = w$$

Lemma 3 - 0 Times a Vector: For all $v \in V$, $0v = O$

Proof: $v = 1v = (0+1)v = 0v + 1v = 0v + v \implies 0v = O$

Lemma 4 - (-1)v is the Additive Inverse: For all $v \in v, (-1)v$ is the unique additive inverse of v

Proof: $(-1)v + v = (-1+1)v = 0v = O$. Thus $(-1)v$ is the additive inverse of v, which is unique by Lemma 2

Definition - Subspace: For a K-vector space V and a non-empty subset $W \subseteq V$, W is a **subspace** if it satisfies

- $w_1, w_2, \in W \implies w_1 + w_2 \in W$
- $\forall a \in K, w \in W \implies aw \in W$

Theorem 1: Every subspace of a K-vector space is a K-vector space

Proof: We need to show that $W \subseteq V$ satisfies all the necessary properties of a vector space

1. Verify $O \in W$

   Since W is non-empty and closed under scalar multiplication, take $0w = O \in W$ by Lemma 3
2. $u, v \in W \implies u + v \in W$ and $a \in K, v \in W \implies aw \in W$ by definition of subspace
3. Every $w \in W$ has an additive inverse, namely $-w$

   Since W is closed under scalar multiplication, $(-1)w = -w \in W$ by Lemma 4
4. Other conditions (e.g. associative addition, commutative addition, etc.) hold because $u, v, w \in V \implies u, v, w \in W$

   For example, choose $u, v \in V$, then $u + v = v + u$, which also holds under W. Thus commutative addtion is satisfied

Example: Take $(5,3,2) \in R^3$. Then let $W = \{r(5,3,2) \mid r \in R\}$

Then W is an R-vector space. We prove this by showing that W is a subspace of R^3

- $+$: Choose 2 arbitrary elements of W, $r(5,3,2)$ and $s(5,3,2)$ for $r, s \in R$

  Then $r(5,3,2) + s(5,3,2) = (r+s)(5,3,2) \in W$
- $\times$: Choose $r(5,3,2) \in W$ and take $s \in R$

  Then $s(r(5,3,2)) = (sr)(5,3,2) \in W$

Example: Let $U = \{(x,y,z) \in R^3 \mid 2x + 3y = 0\}$. We show that U is a vector space by showing it's a subspace of R^3

- $+$: Take (x_1, y_1, z_1) and $(x_2, y_2, z_2) \in U \implies 2x_1 + 3y_1 = 0$ and $2x_2 + 3y_2 = 0$

  Then $2(x_1 + x_2) + 3(y_1 + y_2) = 0$

  Thus $(x_1 + x_2, y_1 + y_2, z_1 + z_2) \in U$
- $\times$: Let $(x,y,z) \in U$ and $r \in R$

  Then $2x + 3y = 0 \implies r(2x+3y)2rx + 3ry = 0$

  Thus $r(x,y,z) \in U$

Example: Consider $\sin(x), \cos(x) \in \mathcal{F}(R)$ and let $W = \{a\sin(x) + b\cos(x) \mid a, b \in R\}$. Then W is a subspace of $\mathcal{F}(R)$

- $+$: Take $a_1 \sin(x) + b_1 \cos(x)$ and $a_2 \sin(x) + b_2 \cos(x) \in W$. Then $(a_1 + a_2)\sin(x) + (b_1 + b_2)\cos(x) \in W$
- $\times$: Take $r \in R$. Then $r(a\sin(x) + b\cos(x)) = (ra)\sin(x) + (rb)\cos(x) \in W$