MATH405: Linear Algebra

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1 Vector Spaces

1.1 R^n and C^n

Definition - Complex Numbers: ordered pairs (a, b) where $a, b \in R$, denoted $a + bi$ where $i = \sqrt{-1}$

The set of all complex numbers is denoted $C = \{a + bi \mid a, b \in R\}$ - Addition is defined as $(a + bi) + (c + di) = (a + c) + (b + d)i$ - Multiplication is defined as $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

1.3 - Properties of Complex Arithmetic (for $\alpha, \beta, \lambda \in C$):

- **Commutativity**: $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$
- **Associativity**: $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$
- **Identities**: $\lambda + 0 = \lambda$ and $\lambda 1 = \lambda$
- **Additive Inverse**: $\forall \alpha \in C$, there eixsts a unique $\beta \in C$ such that $\alpha + \beta = 0$
- **Multiplicative Inverse**: $\forall \alpha \in C$, with $\alpha \neq 0$, there exists a unique $\beta \in C$ such that $\alpha\beta = 1$
- **Distributive Property**: $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$

1.5 - Additive Inverse, Subtraction, Multiplicative Inverse, Division (for $\alpha, \beta \in C$)

- **Additive Inverse** of α is denoted $-\alpha$, where $\alpha + (-\alpha) = 0$
- **Subtraction** on C is defined by $\beta - \alpha = \beta + (-\alpha)$
- **Multiplicative Inverse** of $\alpha \neq 0$ is denoted $1/\alpha$, where $\alpha(1/\alpha) = 1$
- **Division** on C is defined by $\beta/\alpha = \beta(1/\alpha)$

ASIDE on Fields: both R and C are known as **fields**. Elements of F are called **scalars** and all of the work in linear algebra can be abstracted into dealing with fields

- For $\alpha \in F$ and $m \in Z^+$, $\alpha^m = \underbrace{\alpha \cdots \alpha}_{\text{m times}}$
- $(\alpha^m)n = \alpha^{mn}$
- $(\alpha\beta)^m = \alpha^m\beta^m$

Definition - Lists: A **list** of length n is an ordered collection of n elements that looks like $(x_1, \ldots, x_n)$

- 2 lists are equal if and only if they have the same length and the same elements in the same order

Definition - $\mathbf{F^n}$: The set of all lists of length n of elements of F, denoted $F^n = \{(x_1, \ldots, x_n) \mid x_i \in F\}$

- Here x_i is known as the **ith coordinate** of the list
- Addition is defined by adding corresponding coordinates: $(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$ and short handed as $x + y$

Properties for F^n similar to that of C can be seen:

- Clearly addition in F^n is commutative: $x + y = y + x$

- There is a 0 element whose coordinates are all 0 such that $x + 0 = x$ for all $x \in F^n$
- $\forall x \in F^n$, there exists a unique $-x$ such that $x + (-x) = 0$ known as the **additive inverse**
- **NOTE**: multiplication in F^n between 2 lists is not particular useful. Instead we look at **scalar multiplication**. Take $\lambda \in F$ and vector $x \in F^n$ then
$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

1.2 Definition of Vector Space

Definition - Vector Space: A set V with addition and scalar multiplication on V over F, for $u, v, w \in V$ and $a, b \in F$, satisfying

- **Commutativity**: $u + v = v + u$
- **Associativity**: $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$
- **Additive Identity**: $\exists 0 \in V$ such that $v + 0 = v$ for all $v \in V$
- **Additive Inverse**: $\forall v \in V, \exists w \in V$ such that $v + w = 0$
- **Multiplicative Identity**: $1v = v$ for all $v \in V$
- **Distributive Properties**: $a(u + v) = au + av$ and $(a + b)v = av + bv$

Definition - Vectors: Elements of a vector space

An interesting vector space to consider is $\mathbf{F^S}$: the set of functions from S to F

- For $f, g \in F^S$, $f + g \in F^S$ is defined by $(f + g)(x) = f(x) + g(x)$ for all $x \in S$
- For $f \in F^S$ and $\lambda \in F$, the product $\lambda f \in F^S$ is defined by $(\lambda f)(x) = \lambda f(x)$ for all $x \in S$
- **Example**: If $S = [0, 1]$ and $F = R$, then $R^{[0,1]}$ is the set of real-valued functions on the interval $[0, 1]$
  - Clearly addition and scalar multiplication is well defined for F^S
  - Additive identity of F^S is $0(x) = 0$ for all $x \in S$
  - Additive inverse of $f \in F^S$ is $(-f)(x) = -f(x)$ for all $x \in S$
- **NOTE**: we can treat F^n as $F^{\{1,2,\ldots,n\}}$

1.2.1 Properties of Vector Spaces

1.25 - Unique Additive Identity: Vector spaces have a unique additive identity

Proof: Suppose 0 and $0'$ are both additive identities for a vector space V. Then

$$0' = 0' + 0 = 0 + 0' = 0$$

1.26 - Unique Additive Inverses: Each element in V has a unique additive inverse

Proof: Suppose w and w' are both additive inverses of v. Then

$$w = w + 0 = w + (w' + v) = (w + v) + w' = 0 + w' = w'$$

1.29 - 0 Times a Vector: For every $v \in V$, $0v = 0$ (**note** 0 here is a scalar)

Proof: $0v = (0 + 0)v = 0v + 0v$. Then adding the inverse of $0v$ to both sides, we get $0 = 0v$

1.30 - A Number Times the 0 Vector: For every $a \in F, a0 = 0$ (**note** 0 here is a vector)

Proof: $a0 = a(0 + 0) = a0 + a0$. Then adding the inverse of $a0$ to both sides, we get $0 = a0$

1.31 - -1 Times a Vector: For every $v \in V$, $(-1)v = -v$

Proof: $v + (-1)v = (1 + (-1))v = 0v = 0$. Thus $(-1)v$ is the additive inverse of v

1.3 Subspaces

Definition - Subspace: A subset U of V is also a vector space under the same addition and scalar multiplication of V

- **Example**: $\{(x_1, x_2, 0 \mid x_1, x_2 \in F\}$ is a subspace of F^3

1.34 - Conditions for a Subspace: $U \subseteq V$ is a subspace of V if and only if U satisfies the following conditions

1. **Additive Identity**: $0 \in U$
2. **Closed under Addition**: $u, w \in U \implies u + w \in U$
3. **Closed under Scalar Multiplication**: $a \in F$ and $u \in U \implies au \in U$

Proof: $\implies$ if U is a subspace of V then U satisfies the 3 conditions above by the definition of vector space

$\impliedby$ suppose U satisfies the 3 conditions above

- Associativity and commutativity are automatically satisfied since $U \subseteq V$
- The first condition ensures that the additive identity of V is in U
- The second condition ensures that addition on U makes sense
- The third condition ensures that scalar multiplication makes since on U, helping show that the additive inverse $(-1)u$ and that the distributive properties hold

Definition - Sum of Subsets: Suppose $U_1, \ldots, U_m$ are subsets of V. The **sum** of $U_1, \ldots, U_m$, denoted $U_1 + \cdots + U_m$, is the set of all possible sums of elements of $U_1, \ldots, U_m$

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m \mid u_i \in U_i\}$$

- **Example**: Let U be the set of elements of F^3 whose second and third coordinates are 0, and W be the set of elements of F^3 whose first and third coordinates are 0. Then

$$U = \{(x, 0, 0) \in F^3 \mid x \in F\} \qquad W = \{(0, y, 0) \in F^3 \mid y \in F\} \qquad U + W = \{(x, y, 0) \mid x, y \in F\}$$

1.39 - Sum of Subspaces is the Smallest Containing Subspace: Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$

Proof: Clearly $0 \in U_1 + \cdots + U_m$ and addition is and scalar multiplication is closed. Thus $U_1 + \cdots + U_m$ is a subspace of V

Furthermore, clearly $U_1, \cdots, U_m$ are contained in $U_1 + \cdots + U_m$.

Conversely, all subspaces containing $U_1, \ldots, U_m$ contain $U_1 + \cdots + U_m$ (subspaces contain all finite sums of their elements)

Thus $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$

Suppose $U_1, \ldots, U_m$ are subspaces of V. Then every element of $U_1 + \cdots + U_m$ can be written in the form

$$u_1 + \cdots + u_m \qquad u_j \in U_j$$

Definition - Direct Sum: If each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum of $u_1 + \cdots u_m$ then the sum $U_1 + \cdots + U_m$ is called a **direct sum**. Denoted

$$U_1 \oplus \cdots \oplus U_m$$

- **Example**: Let U be the subspace of F^3 of vectors whose last coordinate is 0 and W be the subspace of F^3 of vectors whose first 2 coordinates are 0

$$U = \{(x, y, 0)\} \qquad W = \{(0, 0, z)\} \qquad F^3 = U \oplus W$$

- **Non-Example**: Let

$$U_1 = \{(x, y, 0)\} \qquad U_2 = \{(0, 0, z)\} \qquad \{(0, y, y)\}$$

Then $U_1 + U_2 + U_3$ is NOT a direct sum since we have

$$(0, 0, 0) = (0, 1, 0) + (0, 0, 1) + (0, -1, -1) = (0, 0, 0) + (0, 0, 0) + (0, 0, 0)$$

1.44 - Condition for a Direct Sum: Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if the only way to write 0 as a sum is by taking each $u_j = 0$

Proof: $\implies$ Suppose $U_1 + \cdots + U_m$ is a direct sum. Then clearly there is a unique way writing 0 as the sum of $u_1 + \cdots + u_m$

$\impliedby$ Suppose that the only way to write 0 as the sum of $u_1 + \cdots + u_m$ is by taking each $u_j = 0$.

By contradiction, to show that $U_1 + \cdots + U_m$ is a direct sum, let $v \in U_1 + \cdots + U_m$ where

$$v = v_1 + \cdots + v_m = w_1 + \cdots + w_m$$

Then we have

$$0 = (v_1 - w_1) + \cdots + (v_m - w_m)$$

Thus $v_j = w_j$ and each vector in $U_1 + \cdots + U_m$ has a unique representation

1.45 - Direct Sum of 2 Subspaces: Suppose U and W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$

Proof: $\implies$ Suppose $U + V$ is a direct sum. If $v \in U \cap W$, then $0 = v + (-v)$.

By the unique representation of 0 as the sum of vectors in U and W, we must have $v = 0$. Thus $U \cap W = \{0\}$

$\impliedby$ Suppose $U \cap W = \{0\}$ and suppose $0 = u + w$. We show that $u = w = 0$

$0 = u + w \implies u = -w \implies u \in W \implies u \in U \cap W$. Thus $u = 0 = w$

2 Finite-Dimensional Vector Spaces

2.1 Span and Linear Independence

Definition - Linear Combination: Let $v_1, \ldots, v_m$ be a list of vectors in V. Then vectors of the form

$$v = a_1 v_1 + \cdots + a_m v_m \qquad a_i \in F$$

are said to be a **linear combination** of the vectors $v_1, \ldots, v_m$

Definition - Span: set of all linear combinations of vectors $v_1, \ldots, v_m$ in V. Denoted

$$\text{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m \mid a_i \in F\}$$

- If $\text{span}(v_1, \ldots, v_m) = V$ then $v_1, \ldots, v_m$ **spans** V

2.7 - Span is the Smallest Containing Subspace: the span of a list of vectors is the smallest subspace of V containing all vectors in the list

Proof: Clearly $\text{span}(v_1, \ldots, v_m)$ is a subspace of V

- $0 = 0v_1 + \cdots + 0v_m \in \text{span}(v_1, \ldots, v_m)$
- $(a_1 v_1 + \cdots + a_m v_m) + (b_1 v_1 \cdots + b_m v_m) = (a_1 + b_1) v_1 + \cdots + (a_m + b_m) v_m$ so closed under addition
- $\lambda(a_1 v_1 + \cdots + a_m v_m) = \lambda a_1 v_1 + \cdots + \lambda a_m v_m$ so closed under scalar multiplication

Clearly each v_j can be written as a linear combination of $v_1, \ldots, v_m$. Thus each $v_j \in \text{span}(v_1, \ldots, v_m)$

Conversely, every subspace containing $v_1, \ldots, v_m$ contains $\text{span}(v_1, \ldots, v_m)$ be closure under addition and scalar multiplication

Thus $\text{span}(v_1, \ldots, v_m)$ is the smallest subspace of V containing all vectors $v_1, \ldots, v_m$

Definition - Finite Dimensional Vector Space: Vector space with a finite list of vectors that span the space

Definition - $\mathcal{P}(\mathbf{F})$: Set of all polynomial functions $p : F \to F$ with coefficients in F

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m \qquad z, a_i \in F$$

- $\mathcal{P}(F)$ is clearly a subspace of F^F (has additive inverse $0(x)$ and is closed under addition and scalar multiplication)
- The coefficients of a polynomial unique determined by the polynomial

Definition - Degree of a Polynomial: A polynomial $p \in \mathcal{P}(F)$ has **degree** m if there exists scalars $a_0, \dots, a_m \in F$ with $a_m \neq 0$ such that

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m \qquad z, a_i \in F$$

- **NOTE**: Polynomial that is identically 0 is said to have degree $-\infty$
- $\mathcal{P}_m(F)$ denotes the set of all polynomials with coefficients in F and degree at most m

Definition - Linear Independence: List of vectors $v_1, \dots, v_m$ is **linearly independent** if the only choice of $a_1, \dots, a_m \in F$ that satisfies $a_1 v_1 + \cdots + a_m v_m = 0$ is $a_1 = \cdots = a_m = 0$

- Thus $v_1, \dots, v_m$ is linearly independent if and only if each vector in $\text{span}(v_1, \dots, v_m)$ has only one representation as a linear combination of $v_1, \dots, v_m$
- **NOTE**: If some vectors are removed from a linearly independent list, the remaining list is also linearly independent

Definition - Linear Dependence: List of vectors $v_1, \dots, v_m$ is **linearly dependent** if there exists $a_1, \dots, a_m \in F$, not all 0, such that $a_1 v_1 + \cdots + a_m v_m = 0$

- **NOTE**: If some vector in the list of vectors is a linear combination of the other vectors, the list is linearly dependent
- Every list of vectors containing the 0 vector is linearly dependent

2.21 - Linear Dependence Lemma: Suppose $v_1, \dots, v_m$ is linearly dependent. Then one of the $j \in \{1, \dots, m\}$ satisfies

- $v_j \in \text{span}(v_1, \dots, v_{j-1})$
- If the jth term is removed, the span of the remaining list equals $\text{span}(v_1, \dots, v_m)$

Proof:

- Since $v_1, \dots, v_m$ is linearly dependent, there exists $a_1, \dots, a_m \in F$ not all 0 such that

  $$a_1 v_1 + \cdots + a_m v_m = 0$$

  Let j be the largest in $\{1, \dots, m\}$ such that $a_j \neq 0$ then we have

  $$v_j = -\frac{a_1}{a_j} v - \cdots - \frac{a_{j-1}}{a_j} v_{j-1}$$

- Suppose $u \in \text{span}(v_1, \dots, v_m)$ then there exists $c_1, \dots, c_m \in F$ such that

  $$u = c_1 v_1 + \cdots + c_m v_m$$

  From the previous bullet point, we can replace v_j with a linear combination of $v_1, \dots, v_{j-1}$.

  Thus we have shown that u is the span of the list obtained from removing the jth term from $v_1, \dots, v_m$

2.23 - Length of Linearly Independent List $\leq$ Length of Spanning List

Proof: Suppose $u_1, \dots, u_m$ are linearly independent and suppose $w_1, \dots, w_n$ spans V.

We show that $m \leq n$ through a multistep process where we add one of u's and remove one of the w's

- Step 1: Let B be the list $w_1, \dots, w_n$ that spans V. Adjoining any vectors in V to this list produces a linearly dependent list. Thus by the Linear Dependence Lemma, if we add u_1 to this list, we can remove w_j from the list to form a new list of length n that spans V

- Step j: Let B be the list of length n from $j-1$ that spans V. If we add u_j to B, the list becomes linearly dependent. But since $u_1, \dots, u_{j-1}$ is linearly independent, we need to remove one of the w's to make B independent again

After m steps, we have added all u's to the list and the process stops. Thus we have $m \leq n$

Examples:

- $(1,2,3),(4,5,8),(9,6,7),(-3,2,8)$ is NOT linearly independent in R^3. Since $(1,0,0),(0,1,0),(0,0,1)$ span R^3 and is a list of length 3, no linearly independent list in R^3 has length > 3
- $(1,2,3,5),(4,5,8,1),(4,6,7,-1)$ does NOT span R^4. Since $(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)$ is linearly indepdent in R^4, no list of length < 4 spans R^4

2.26 - Finite-Dimensional Subspaces are Finite-Dimensional: Every subspace of a finite-dimensional vector space is finite-dimensional

Proof: Suppose U is a subspace of a finite-dimensional subspace V. We show U is finite-dimensional by construction

- Step 1: If $U = \{0\}$ then clearly U is finite-dimensional. Otherwisen we choose a nonzero vector $v_1 \in U$
- Step j: If $U = \text{span}(v_1, \dots, v_{j-1})$, then U is finite-dimensional and we are done. Otherwise, take $v_j \notin \text{span}(v_1, \dots, v_j)$ and add it to U

After each step, we have constructed a list of independent vectors that cannot have length longer than the spanning list of V.

Thus the process eventually terminates and U is finite-dimensional

2.2 Bases

Definition - Bases: List of vectors in V that is linearly independent and spans V

- $(1,0,\dots,0),(0,1,0,\dots,0),\dots,(0,\dots,0,1)$ is the **standard basis** of F^n

2.29 - Criterion for Basis: A list $v_1, \dots, v_n$ of vectors is a basis of V if and only if every $v \in V$ can be written uniquely in the form

$$v = a_1 v_1 + \cdots + a_n v_n \qquad a_i \in F$$

Proof: $\implies$ Suppose $v_1, \dots, v_n$ is a basis of V and let $v \in V$.

Since $v_1, \dots, v_n$ spans V, v can be written linear combination of the basis vectors with $a_1, \dots, a_n \in F$

To show this representation is unique, suppose v can also be written as a linear combination using $b_1, \dots, b_n$. Then

$$0 = (a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n$$

Since $v_1, \dots, v_n$ are linearly independent, we have $a_1 = b_1, \dots, a_n = b_n$

$\impliedby$ Suppose every $v \in V$ can be written uniquely as a linear combination using $a_1, \dots, a_n$

By definition this means that $v_1, \dots, v_n$ spans V

Looking at the unique representation of

$$0 = a_1 v_1 + \cdots + a_n v_n$$

We must have $a_1 = \cdots = a_n = 0$. Thus $v_1, \dots, v_n$ are linearly independent and a basis of V

2.31 - Spanning List Contains a Basis: Every spanning list can be reduced to a basis of the vector space

Proof: Suppose $B = v_1, \dots, v_n$ spans V. We can remove vectors using the following steps

- Step 1: if $v_1 = 0$, delete v_1 from B. Otherwise leave B unchanged
- Step j: If $v_j \in \text{span}(v_1, \dots, v_{j-1})$, delete v_j from B. Otherwise leave B unchanged

Once B is of length n, stop. This list B spans V because the original list spanned V and we only discarded extraneous vectors

This process ensures no vector in B is in the span of the previous ones. Thus B is linearly independent

Thus B is a basis of V

2.32 - Basis of Finite-Dimensional Vector Space: Every finite-dimensional vector space has a basis

Proof: Since a finite-dimensional vector space has a spanning list, the previous result shows us the list can be reduced to a basis

2.33 - Linearly independent List Can be Extended to a Basis: Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis

Proof: Suppose $u_1, \dots, u_m$ is linearly independent and $w_1, \dots, w_n$ be a basis of V. Forming the list

$$u_1, \dots, u_m, w_1, \dots, w_n$$

Clearly spans V. Then removing the extraneous vectors from this list produces a basis of V

However, none of the u's will be removed since they are already linearly independent so we only remove w's from the list

2.34 - Every Subspace of V is Part of a Direct Sum Equal to V: Suppose U is a subspace of a finite-dimensional V. Then there is a subspace W of V such that $V = U \oplus W$

Proof: Since V is finite-dimensional, so is U.

Thus there is a basis $u_1, \dots, u_m$ of U that can be extended to a basis $u_1, \dots, u_m, w_1, \dots, w_n$ of V

Let $W = \operatorname{span}(w_1, \dots, w_n)$. To show $V = U \oplus W$, we need

$$V = U + W \qquad U \cap W = \{0\}$$

- Since $u_1, \dots, u_m, w_1, \dots, w_n$ spans V, any vector $v \in V$ can be written as

  $$v = u + w \qquad u \in U, w \in W$$

  Thus we have $v \in U + W \implies V = U + W$
- Let $v \in U \cap W$. Then there exists scalars $a_1, \dots, a_m, b_1, \dots, b_n \in F$ such that

  $$v = a_1 u_1 + \cdots + a_m u_m = b_1 w_1 + \cdots + b_n w_n$$

  Then we must have

  $$a_1 u_1 + \cdots + a_m u_m - b_1 w_1 - \cdots - b_n w_n = 0$$

  Since these vectors are linearly independent, we must have $a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0$

  Thus $v = 0$ and $U \cap W = \{0\}$

2.3 Dimension

2.35 - Bases Have the Same Length

Proof: Suppose B_1 and B_2 are bases of a finite-dimensional V

Since both B_1 and B_2 are linearly independent and span V, length $B_1 =$ length B_2

Definition - Dimension $\dim \mathbf{V}$: length of any basis of a vector space

2.38 - Dimension of a Subspace: Let U be a subspace of a finite-dimensional V. Then $\dim U \leq \dim V$

Proof: Basis of U has length $\leq$ the length of the basis of V. Thus $\dim U \leq \dim V$

2.39 - Linearly Independent List of the Correct Length is a Basis

Proof: Suppose $\dim V = n$ and $v_1, \dots, v_n$ is linearly independent.

We know that this list can be extended to a basis of V, but since the basis is of length n, there is nothing to extend

Thus $v_1, \dots, v_n$ is a basis of V

2.42 - Spanning List of the Right Length is a Basis: Every spanning list of vectors in V with length $\dim V$ is a basis of V

Proof: Suppose $v_1, \dots, v_n$ spans V. This list can be reduced to a basis of V.

However, every basis of V has length n, so this reduction is trivial and thus $v_1, \dots, v_n$ is a basis of V

2.43 - Dimension of a Sum: If U_1, U_2 are finite-dimensional vector spaces, then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

Proof: Let $u_1, \dots, u_m$ be a basis of $U_1 \cap U_2$ (thus $\dim(U_1 \cap U_2) = m$)

Note that this list is linearly independent in U_1 and U_2.

Thus we can extend the list to a basis $u_1, \dots, u_m, v_1, \dots, v_j$ of U_1 with $\dim U_1 = m + j$

And extend the list to a basis $u_1, \dots, u_m, w_1, \dots, w_k$ of U_2 with $\dim U_2 = m + k$

Clearly $\operatorname{span}(u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k)$ contains U_1 and U_2, and thus equals $U_1 + U_2$

We now show that this list is linearly independent. We need

$$a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_j v_j + c_1 w_1 + \cdots + c_k w_k = 0$$

This can be rewritten as

$$c_1 w_1 + \cdots + c_k w_k = -a_1 u_1 - \cdots - a_m u_m - b_1 v_1 - \cdots - b_j v_j \in U_1$$

Since all w's are in U_2, this means that $c_1 w_1 + \cdots + c_k w_k \in U_1 \cap U_2$ and thus

$$c_1 w_1 + \cdots c_k w_k = d_1 u_1 + \cdots + d_m u_m$$

However, we know that $u_1, \dots, u_m, w_1, \dots, w_k$ is linearly independent. Thus $c_1 = \cdots = c_k = d_1 = \cdots = d_m = 0$ Thus the original equation becomes

$$a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_j v_j = 0$$

But $u_1, \dots, u_m, v_1, \dots, v_j$ is linearly independent. Thus $a_1 = \cdots = a_m = b_1 = \cdots = b_j = 0$

Thus $\dim(U_1 + U_2) = (m + j) + (m + k) - m = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$

3 Linear Maps

3.1 Vector Space of Linear Maps

Definition - Linear Map: Function $T : V \to W$ satisfying

- **Additivity**: $T(u + v) = Tu + Tv \qquad \forall u, v \in V$
- **Homogeneity**: $T(\lambda v) = \lambda(Tv) \qquad \forall \lambda \in F \quad \forall v \in V$

Definition - $\mathcal{L}(\mathbf{V}, \mathbf{W})$: set of all linear maps from $V \to W$

Examples:

- $0 \in \mathcal{L}(V, W)$: takes any vector in V and maps it to the additivity identity in W. Denoted $0v = 0 \qquad \forall v \in V$
- $I \in \mathcal{L}(V, V)$: identity mapping that takes any vector in V and maps it to itself. Denoted $Iv = v \qquad \forall v \in V$

- $T \in \mathcal{L}(R^3, R^2)$: defined by $T(x, y, z) = (2x - y + 3z, 7x + 5y - 6z)$
- $T \in \mathcal{L}(F^n, F^m)$: define by $T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$ for $A_{j,k} \in F$

3.5 - Linear Maps and Basis of Domain: Let $v_1, \ldots, v_n$ be a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T : V \to W$ such that

$$Tv_j = w_j \qquad j \in \{1, \ldots, n\}$$

Proof: First we show existence of such an equation. Define $T : V \to W$ by

$$T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n \qquad c_i \in F$$

Since $v_1, \ldots, v_n$ is a basis of V, the equation above is a function $T : V \to W$ since each element of V can be uniquely written in the form $c_1v_1 + \cdots + c_nv_n$

For each j, take $c_j = 1$ and all other $c_i = 0$. Then clearly $Tv_j = w_j$

Next we show that this function is a linear map.

- For $u, v \in V$ we have

$$\begin{aligned} T(u + v) &= T((a_1 + c_1)v_1 + \cdots + (a_n + c_n)v_n) \\ &= (a_1 + c_1)w_1 + \cdots + (a_n + c_n)w_n \\ &= Tu + Tv \end{aligned}$$

- For $\lambda \in F$ and $v \in V$, we have

$$\begin{aligned} T(\lambda v) &= T(\lambda c_1v_1 + \cdots + \lambda c_nv_n) \\ &= \lambda(c_1w_1 + \cdots + c_nw_n) \\ &= \lambda Tv \end{aligned}$$

Finally we show that this linear mapping is unique. Let $T \in \mathcal{L}(V, W)$ and $Tv_j = w_j$ for $j \in \{1, \ldots, n\}$

- Homogeneity implies that $T(c_jv_j) = c_jw_j$
- Additivity implies $T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n$

Thus T is uniquely determined by $\text{span}(v_1, \ldots, v_n)$. Since $v_1, \ldots, v_n$ is a basis, T is uniquely determined on V

3.2 Algebraic Operations on $\mathcal{L}(\mathbf{V}, \mathbf{W})$

Definition - Addition and Scalar Multiplication on $\mathcal{L}(\mathbf{V}, \mathbf{W})$: Let $S, T \in \mathcal{L}(V, W)$ and $\lambda \in F$. Then

- **Sum**: $(S + T)(v) = Sv + Tv$
- **Product**: $(\lambda T)(v) = \lambda(Tv)$

Clearly both $(S + T)(v)$ and $(\lambda T)(v)$ are linear maps

3.7 - $\mathcal{L}(\mathbf{V}, \mathbf{W})$ is a Vector Space: Under the operations of addition and scalar multiplication defined above, $\mathcal{L}(V, W)$ is a vector space

Proof: Let $S, T, U \in \mathcal{L}(V, W)$ and $a, b \in F$

- **Commutativity**: $S + T = T + S$ holds
- **Associativity**: $(S + T) + U = S + (T + U)$ and $(ab)S = a(bS)$
- **Additivity Identity**: $S + 0 = S$
- **Multiplicative Identity** $1S = S$
- **Distributive Property**: $a(S + T) = aS + aT)$ and $(a + b)S = aS + bS$

Definition - Product of Linear Maps: For $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, the **product** $ST \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(Tu) \qquad u \in U$$

- **NOTE**: this is identical to the usual composition of functions $S \circ T$
- This product is also a linear map

3.9 - Algebraic Properties of Products of Linear Maps: For products of linear maps $T, T_1, T_2, T_3, S, S_1, S_2$ where the domains of the mappings make sense, the following properties hold

- **Associativity**: $(T_1T_2)T_3 = T_1(T_2T_3)$
- **Identity**: $TI = IT = T$
  - If $T \in \mathcal{L}(V, W)$, first I is the identity map on V and second I is the identity map on W
- **Distributive Properties**: $(S_1 + S_2)T = S_1T + S_2T$ and $S(T_1 + T_2) = ST_1 + ST_2$

NOTE: Product of linear maps aren't commutative, so it's not always true that $ST = TS$

Example: Let $D \in \mathcal{L}(\mathcal{P}(R), \mathcal{P}(R))$ be the differentiation map and $T \in \mathcal{L}(\mathcal{P}(R), \mathcal{P}(R))$ be the multiplication by x^2 map

Clearly $((TD)p)(x) = x^2p'(x) \neq x^2p'(x) + 2xp(x) = ((DT)p)(x)$

3.11 - Linear Maps Take 0 to 0: Let T be a linear map from V to W. Then $T(0) = 0$

Proof: By additivity, we have that $T(0) = T(0 + 0) = T(0) + T(0)$

Then adding the additive inverse of $T(0)$ to both sides gives us $T(0) = 0$

3.3 Null Spaces and Ranges

3.3.1 Null Space and Injectivity

Definition - Null Space: For $T \in \mathcal{L}(V, W)$, null T, is a subset of V consisting of vectors that T maps to 0

$$\operatorname{null} T = \{v \in V \mid Tv = 0\}$$

Examples

- If T is the zero map from V to W, then $\operatorname{null} T = V$
- Let $D \in \mathcal{L}(\mathcal{P}(R), \mathcal{P}(R))$ be the differentiation map defined by $Dp = p'$. Then $\operatorname{null} D$ is the set of constant functions
- Let $T \in \mathcal{L}(\mathcal{P}(R), \mathcal{P}(R))$ be the multiplication by x^2 map defined by $(Tp)(x) = x^2p(x)$. Then $\operatorname{null} T = \{0\}$

3.14 - Null Space is a Subspace: For $T \in \mathcal{L}(V, W)$, $\operatorname{null} T$ is a subspace of V

Proof: Since T is a linear mapping, we have $T(0) = 0 \implies 0 \in \operatorname{null} T$

If we take $u, v \in \operatorname{null} T$, then $T(u + v) = Tu + Tv = 0$. Thus $\operatorname{null} T$ is closed under addition

If we take $\lambda \in F$ and $v \in \operatorname{null} T$, then $T(\lambda u) = \lambda Tu = \lambda 0 = 0$. Thus $\operatorname{null} T$ is closed under scalar multiplication

Thus we have satisfied all criterion for $\operatorname{null} T$ to be a subspace of V

Definition - Injective: A function $T : V \to W$ is **injective** if $Tu = Tv \implies u = v$

3.16 Injectivity is Equivalent if Null Space is {0}: If $T \in \mathcal{L}(V, W)$, then T is injective if and only if $\operatorname{null} T = \{0\}$

Proof: $\implies$ Suppose T is injective. Clearly $0 \subseteq \operatorname{null} T$. To show the other way around, take $v \in \operatorname{null} T \implies T(v) = 0 = T(0)$

Since T is injective, we must have $v = 0 \implies \operatorname{null} T = \{0\}$

$\impliedby$ Suppose $\operatorname{null} T = \{0\}$ and suppose there are $u, v \in V$ such that $Tu = Tv$

Then we have $0 = Tu - Tv = T(u - v) \implies u - v \in \operatorname{null} T = \{0\} \implies u = v$

3.4 Range and Surjectivity

Definition - Range: For $T \in \mathcal{L}(V, W)$, range T is a subset of W such that

$$\operatorname{range} T = \{Tv \mid v \in V\}$$

Examples:

- If T is the zero map from V to W, so $\forall v \in V$, $Tv = 0$, then range $T = \{0\}$
- Let $T \in \mathcal{L}(R^2, R^3)$ for $T(x, y) = (2x, 5y, x + y)$. Then range $T = \{2x, 5y, x + y \mid x, y \in R\}$
- Let $D \in \mathcal{L}(\mathcal{P}(R), \mathcal{P}(R))$ be the differentiation map defined by $Dp = p'$. Then range $D = \mathcal{P}(R)$

3.19 - Range is a Subspace: For $T \in \mathcal{L}(V, W)$, we have range T is a subspace of W

Proof: Clearly $T(0) = 0 \implies 0 \in \operatorname{range} T$

For $w_1, w_2 \in \operatorname{range} T$, there exists $v_1, v_2 \in V$ such that $Tv_1 = w_1$ and $Tv_2 = w_2$. Thus

$$T(v_1 + v_2) = Tv_1 + Tv_2 = w_1 + w_2 \in \operatorname{range} T$$

Thus range T is closed under addition

For $w \in \operatorname{range} T$ and $\lambda \in F$, there exists $v \in V$ such that $Tv = w$. Thus

$$T(\lambda v) = \lambda Tv = \lambda w \in \operatorname{range} T$$

Thus range T is closed under scalar multiplication

Definition - Surjective: $T : V \to W$ is **surjective** if range $T = W$

3.4.1 Fundamental Theorem of Linear Maps

3.22 - Fundamental Theorem of Linear Maps: Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then range T is finite-dimensional and

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

Proof: Let $u_1, \dots, u_m$ be a basis of null $T \implies \dim \operatorname{null} T = m$

This list can be extended into a basis of V: $u_1, \dots, u_m, v_1, \dots, v_n \implies \dim V = m + n$

We show that $\dim \operatorname{range} T = n$ by proving that $Tv_1, \dots, Tv_n$ is a basis of range T

Take $v \in V$. Since $u_1, \dots, u_m, v_1, \dots, v_n$ spans V, we can write

$$v = a_1 u_1 = \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n \qquad a_i, b_i \in F$$

Applying T to both sides gives

$$Tv = b_1 T v_2 + \cdots + b_n T v_n$$

Which implies that $Tv_1, \dots Tv_n$ spans range T.

To show $Tv_1, \dots, Tv_n$ is linearly independent, suppose $c_1, \dots c_n \in F$ such that

$$c_1 T v_1 + \cdots + c_n T v_n = T(c_1 v_1 + \cdots + c_n + v_n) = 0 \implies c_1 v_1 + \cdots + c_n v_n \in \operatorname{null} T$$

Since $u_1, \dots, u_m$ spans null T, we have

$$c_1 v_1 + \cdots + c_n v_n = d_1 u_1 + \cdots d_m u_m \qquad d_i \in F$$

Since $u_1, \dots, u_m, v_1, \dots v_n$ is linearly independent, we must have $c_j = d_i = 0$ and thus $Tv_1, \dots, Tv_n$ is linearly independent

Thus we must have $Tv_1, \dots, Tv_n$ is a basis of range T and clearly $\dim \operatorname{range} T = n$

Thus $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$

3.23 - Map to a Smaller Dimensional Space is not Injective: Suppose V, W are finite-dimensional vector spaces where $\dim V > \dim W$. Then no linear map from $V \to W$ is injective

Proof: Let $T \in \mathcal{L}(V, W)$. Then

$$\begin{aligned}\dim \operatorname{null} T &= \dim V - \dim \operatorname{range} T \\ &\geq \dim V - \dim W \\ &> 0\end{aligned}$$

Thus $\operatorname{null} T$ contains vectors other than 0 and T is not injective

3.24 - Map to a Larger Dimensional Space is not Surjective: Suppose V, W are finite-dimensional vector spaces where $\dim V < \dim W$. Then no linear map from $V \to W$ is surjective

Proof: Let $T \in \mathcal{L}(V, W)$. Then

$$\begin{aligned}\dim \operatorname{range} T &= \dim V - \dim \operatorname{null} T \\ &\leq \dim V \\ &< \dim W\end{aligned}$$

Thus $\operatorname{range} T$ cannot equal W and T is not surjective

3.26 Homogeneous System of Linear Equations: Homogeneous system of equations with more variables than equations has nonzero solutions

Proof: Define $T : F^n \to F^m$ by

$$T(x_1, \ldots, x_n) = (\sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} A_{m,k} x_k)$$

Since $n > m$, clearly T is not injective and thus the homogenous system of equations has nonzero solutions

3.29 - Inhomogenous System of Linear Equations: Inhomogenous system of equations with more equations than variables has no solutions for some choice of constant terms

Proof: Define $T : F^n \to F^m$ by

$$T(x_1, \ldots, x_n) = (\sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} A_{m,k} x_k)$$

Since $n < m$, clearly T is not surjective and thus the homogenous system of equations has no solution for some choice of constant terms

3.5 Matrices

Definition - Matrix of a Linear Map $\mathcal{M}(\mathbf{T})$: Let $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ be a basis of V and $w_1, \ldots, w_m$ be a basis of W. Then the **matrix** of T, denoted $\mathcal{M}(T)$, has entries $A_{j,k}$ defined by

$$Tv_k = A_{1,k} w_1 + \cdots + A_{m,k} w_m$$

- Here $A_{1,k}, \ldots, A_{m,k}$ are the column scalars of the matrix A that are needed to write Tv_k as a linear combination of $w_1, \ldots, w_m$
- **NOTE**: Unless stated otherwise, assume the bases between $F^n \to F^m$ are dealing with standard bases
- **UPSHOT**: $\mathcal{M}(T)$ determines T since $\mathcal{M}(T)$ tells what T does to each v_k in the basis. But by linearity, T is determined by what it does to a basis

Example: Suppose $T \in \mathcal{L}(F^2, F^3)$ is defined by $T(x, y) = (x + 3y, 2x + 5y, 7x + 9y)$.

Since $T(1,0) = (1,2,7)$ and $T(0,1) = (3,5,9)$, we have $\mathcal{M}(T) = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{bmatrix}$

3.36 Matrix of Sum of Linear Maps: Suppose $S, T \in \mathcal{L}(V,W)$. Then $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$

Proof: Follows from matrix addition

3.38 Matrix of Scalar Times a Linear Map: Suppose $\lambda \in F$ and $T \in \mathcal{L}(V,W)$. Then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$

Proof: Follows from matrix scalar multiplication

Definition - $\mathbf{F^{m,n}}$: Set of all m-by-n matrices with entries in F

3.40 - $\dim \mathbf{F^{m,n}} = \mathbf{mn}$

Proof: First show that $F^{m,n}$ is a vector space

- Commutativity: Matrix addition is commutative
- Associativity: Matrix addition and scalar multiplication is associative
- Additivity Identity: Matrix with all zeros is the additive identity
- Multiplicative Identity $1 \in F$ is the multiplicative identity
- Distributive Property: Scalar multiplication clearly distributes over matrix addition

Next we show that the basis of $F^{m,n}$ is of length mn.

Clearly the list of m-by-n matrices with 0 in all entries except for a 1 in one entry form a basis of $F^{m,n}$.

There are mn such matrices. Thus $\dim F^{m,n} = mn$

3.43 Matrix of the Product of Linear Map: Let $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$. Then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$

Proof: Follows from matrix multiplication

Notation - $\mathbf{A_{j,\cdot}}, \mathbf{A_{\cdot,k}}$: For a m-by-n matrix A

- $A_{j,\cdot}$ denotes row j of A
- $A_{\cdot,k}$ denotes column k of A

3.47 - Entry of Matrix Product Equals Row Times Column: Let M be an m-by-n matrix and C be an n-by-p matrix. Then

$$(AC)_{j,k} = A_{j,\cdot}C_{\cdot,k}$$

3.49 - Column of Matrix Product Equals Matrix Times Column: Let M be an m-by-n matrix and C be an n-by-p matrix. Then

$$(AC)_{\cdot,k} = AC_{\cdot,k}$$

Example: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} * \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 19 \\ 31 \end{bmatrix}$

3.49.2 - Row of Matrix Product Equals Row Times Matrix: Let M be an m-by-n matrix and C be an n-by-p matrix. Then

$$(AC)_{j,\cdot} = A_{j,\cdot}C$$

3.52 - Linear Combination of Columns: Suppose A is an m-by-n matrix and c is an m-by-1 matrix. Then

$$Ac = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}$$

Example: $\begin{bmatrix} 8 & 4 & 5 \\ 1 & 9 & 7 \end{bmatrix} * \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 8 \\ 1 \end{bmatrix} + -1 \begin{bmatrix} 4 \\ 9 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 30 \\ 8 \end{bmatrix}$

3.52.2 - Linear Combination of Rows: Suppose a is an 1-by-n matrix and C is an n-by-p matrix. Then

$$aC = a_1 C_{1,\cdot} + \cdots + a_n C_{1,\cdot}$$

- Thus aC is a linear combination of the rows of C and with scalars a_i from a

Example: $\begin{bmatrix} 3 & -1 \end{bmatrix} * \begin{bmatrix} 8 & 4 & 5 \\ 1 & 9 & 7 \end{bmatrix} = 3 \begin{bmatrix} 8 & 4 & 5 \end{bmatrix} + -1 \begin{bmatrix} 1 & 9 & 7 \end{bmatrix} = \begin{bmatrix} 23 & 3 & 38 \end{bmatrix}$

3.6 Invertibility and Isomorphic Vector Spaces

3.6.1 Invertible Linear Maps

Definition - Invertible: $T \in \mathcal{L}(V, W)$ is **invertible** if there exists $S \in \mathcal{L}(W, V)$ such that $ST = I$ on V and $TS = I$ on W

Definition - Inverse: $S \in \mathcal{L}(W, V)$ satisfying $ST = I$ and $TS = I$ is called the **inverse** of T, denoted T^{-1}

3.54 - Inverse is Unique: An invertible linear map has a unique inverse

Proof: Let $T \in \mathcal{L}(V, W)$ and S_1, S_2 be inverses of T. Then we have

$$S_1 = S_1 I = S_1(TS_2) = (S_1 t)S_2 = IS_2 = S_2$$

3.56 - Invertibility is Equivalent to Injectivity and Surjectvitiy: $T \in \mathcal{L}(V, W)$ is invertible if and only if it is injective and surjective

Proof: $\implies$ Suppose T is invertible

- Injective: Let $u, v \in V$ such that $Tu = Tv$. Then $u = T^{-1}(Tu) = T^{-1}(Tv) = v$
- Surjective: Let $w \in W$ where $w = T(T^{-1}w) \implies w \in \operatorname{range} T$. Thus $\operatorname{range} T = W$

$\impliedby$ Assume T is injective and surjective and let $S \in \mathcal{L}(W, V)$.

For each $w \in W$, let Sw to be the unique element of V such that $T(Sw) = w$ (this follows from surjectivity and injectivity of T).

Clearly $T \circ S$ is the identity mapping on W

To show that $S \circ T$ equals the identity mapping on V, take $v \in V$. Then

$$T((S \circ T)v) = (T \circ S)(Tv) = I(Tv) = Tv$$

Thus $(S \circ T)v = v \implies S \circ T$ is the identity mapping on V

Finally we show that S is linear. Suppose $w_1, w_2 \in W$. Then

$$T(S(w_1 + w_2)) = T(Sw_1 + Sw_2) = T(Sw_1) + T(Sw_2) = w_1 + w_2$$

Thus $S(w_1 + w_2) = Sw_1 + Sw_2$

Suppose $w \in W$ and $\lambda \in F$. Then

$$T(S(\lambda w)) = T(\lambda Sw) = \lambda T(Sw) = \lambda w$$

Thus $S(\lambda w) = \lambda Sw$

3.6.2 Isomorphic Vector Spaces

Definition - Isomorphism: An invertible linear map

Definition - Isomorphic: 2 vector spaces are **isomorphic** if there is an isomorphism from one vector space onto the other

3.59 - Dimension Shows Whether Vector Spaces are Isomorphic: 2 finite-dimensional vector spaces over F are isomorphic if and only if they have the same dimension

Proof: $\implies$ Suppose V, W are isomorphic finite-dimensional vector spaces, meaning that there is an isomorphism $T : V \to W$.

Since T is invertible, $\operatorname{null} T = \{0\} \implies \dim \operatorname{null} T = 0$ and $\operatorname{range} T = W \implies \dim \operatorname{range} T = \dim W$

Thus $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = \dim W$

$\impliedby$ Suppose V, W are finite-dimensional vector spaces with the same dimension and let $v_1, \dots w_n$ and $w_1, \dots, w_n$ be bases of V and W. Let $T \in \mathcal{L}(V, W)$ be defined by

$$T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n$$

T is well-defined because $v_1, \dots, v_n$ is a basis of V and thus each $v \in V$ can be uniquely represented as a LC of $v_1, \dots, v_n$

- T is surjective because $w_1, \dots, w_n$ spans W
- $\operatorname{null} T = \{0\}$ since $w_1, \dots, w_n$ is linearly independent. Thus T is injective

Since T is both injective and surjective, T is an isomorphism and V, W are isomorphic

3.60 - $\mathcal{L}(\mathbf{V}, \mathbf{W})$ and $\mathbf{F^{m,n}}$ are Isomorphic: Let $v_1, \dots, v_n$ and $w_1, \dots, w_m$ be bases of V, W respectively. Then $\mathcal{M}$ is an isomorphism between $\mathcal{L}(V, W)$ and $F^{m,n}$

Proof: Note that for each $T \in \mathcal{L}(V, W)$, we have a matrix $\mathcal{T} \in F^{m,n}$. Thus $\mathcal{M}$ is a function from $\mathcal{L}(V, W) \to F^{m,n}$

We know that $\mathcal{M}$ is a linear map since additivity and homogeneity hold.

To show that $\mathcal{M}$ is invertible

- If $\mathcal{M}(T) = 0$, then $Tv_k = 0$ for $k \in \{1, \dots, n\}$. Since $v_1, \dots, v_n$ is a basis of V, we must have $T = 0$. Thus $\mathcal{M}$ is injective
- Let $A \in F^{m,n}$ and $Tv_k = \displaystyle\sum_{j=1}^{m} A_{j,k} w_j$. Clearly $\mathcal{M}(T) = A$. Thus $\operatorname{range} \mathcal{M} = F^{m,n}$

3.61 - $\dim \mathcal{L}(\mathbf{V}, \mathbf{W}) = (\dim \mathbf{V})(\dim \mathbf{W})$: Let V, W be finite-dimensional. Then $\mathcal{L}(V, W)$ is finite-dimensional and

$$\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$$

Proof: We know that $\mathcal{L}(V, W)$ is isomorphic to $F^{m,n}$. Thus they must have the same dimension

We know that $F^{m,n}$ has dimension $mn = (\dim V)(\dim W)$

3.6.3 Linear Maps as Matrix Multiplication

Definition - Matrix of a Vector $\mathcal{M}(\mathbf{v})$: Suppose $u \in V$ and $v_1, \dots, v_n$ is a basis of V. Then the **matrix** of u with respect to this basis is

$$\mathcal{M}(u) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Where $c_1, \dots, c_n$ are scalars such that

$$u = c_1 v_1 + \cdots + c_n v_n$$

- **NOTE**: Once a basis is chosen, the function $\mathcal{M}$ that takes $v \in V$ to $\mathcal{M}(v)$ is an isomorphism $V \to F^{n,1}$

3.64 - $\mathcal{M}(\mathbf{T})_{\cdot,\mathbf{k}} = \mathcal{M}(\mathbf{v_k})$: Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then the kth column of $\mathcal{M}(T)$ is equal to $\mathcal{M}(v_k)$

3.65 - Linear Maps Act Like Matrix Multiplication: Suppose $T \in \mathcal{L}(V, W)$ and $v \in V$, and suppose $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ are bases of V and W respectively. Then

$$\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v)$$

Proof: Suppose $v = c_1 v_1 + \cdots + c_n v_n$ Then

$$Tv = c_1 Tv_1 + \cdots + c_n Tv_n$$

Thus we have

$$\begin{aligned}\mathcal{M}(Tv) &= c_1\mathcal{M}(Tv_1) + \cdots + c_n\mathcal{M}(T_n v_n)\\ &= c_1\mathcal{M}(T)_{\cdot,1} + \cdots + c_n\mathcal{M}(T)_{\cdot,n}\\ &= \mathcal{M}(T)\mathcal{M}(v)\end{aligned}$$

3.6.4 Operators

Definition - Operator $\mathcal{L}(\mathbf{V})$: Linear map from a vector space to itself

3.69 - Injectivity is Equivalent to Surjectivity in Finite Dimensions: Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then the following are equivalent:

- T is invertible
- T is injective
- T is surjective

Proof: Clearly T invertible $\implies T$ is injective

Now suppose T is injective (thus $\operatorname{null} T = \{0\}$) and thus we have

$$\begin{aligned}\dim \operatorname{range} T &= \dim V - \dim \operatorname{null} T\\ &= \dim V\end{aligned}$$

Thus $\operatorname{range} T = V$ and thus T is surjective

Finally, suppose that T is surjective, meaning that $\operatorname{range} T = V$. Then we have

$$\begin{aligned}\dim T &= \dim V - \dim \operatorname{range} T\\ &= 0\end{aligned}$$

Thus $\operatorname{null} T = \{0\}$ and thus T is injective and surjective, meaning T is invertible

3.7 Products and Quotients of Vector Spaces

3.7.1 Products of Vector Spaces

Definition - Product of Vector Spaces: Suppose $V_1, \ldots, V_m$ are vector spaces over F. The **product** $V_1 \times \cdots \times V_m$ is defined by

$$V_1 \times \cdots \times V_m = \{(v_1, \ldots, v_m) \mid v_1 \ldots, v_m \in V_m\}$$

- Addition is defined by
$$(u_1, \ldots, u_m) + (v_1, \ldots, v_m) = (u_1 + v_1, \ldots, u_m + v_m)$$
- Scalar multiplication is defined by
$$\lambda(v_1, \ldots, v_m) = (\lambda v_1, \ldots, \lambda v_m)$$

3.73 - Product of Vector Spaces is a Vector Space: Suppose $V_1, \ldots, V_m$ are vector spaces over F. Then $V_1 \times \cdots \times V_m$ is a vector space over F

Proof: Necessary properties of commutative, associativity, identities, additive inverse, and distributivity all hold

3.76 - Dimension of a Product is the Sum of Dimensions: Suppose $V_1, \ldots, V_m$ are vector spaces over F. Then $V_1 \times \cdots \times V_m$ is finite-dimensional and

$$\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m$$

Proof: Choose a basis of each V_j. For each basis vector of each V_j, consider the element of $V_1 \times \cdots \times V_m$ such that

- has the appropriate basis vector in the jth slot
- 0 in all other slots

This list is clearly linearly independent and spans $V_1 \times \cdots \times V_m$ and thus is a basis of $V_1 \times \cdots \times V_m$

The length of this basis is $\dim V_1 + \cdots + \dim V_m$

3.7.2 Products and Direct Sums

3.77 - Products and Direct Sums: Suppose $U_1, \ldots, U_m$ are subspaces of V. Define a linear map $\Gamma : U_1 \times \cdots \times U_m \to U_1 + \cdots + U_m$ by

$$\Gamma(u_1 \ldots, u_m) = u_1 + \cdots + u_m$$

Then $U_1 + \cdots + U_m$ is a direct sum if and only if Γ is injective

Proof: Γ is injective if and only if the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each $u_j \in U_j$, is by taking each $u_j = 0$.

Thus matches the requirement of $U_1 + \cdots U_m$ is a direct sum if and only if the only way to write 0 is to take each $u_j = 0$

3.78 - Sum is a Direct Sum If and Only If Dimensions Add Up: Let V be finite-dimensional and $U_1, \ldots, U_m$ be subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if

$$\dim(U_1 + \cdots + U_m) = \dim U_1 + \cdots + \dim U_m$$