Subspace: $W \subseteq V$ that is K-vector space itself satisfying

- $w_1, w_2, \in W \implies w_1 + w_2 \in W \qquad \forall c \in K, w \in W \implies cw \in W \qquad O \in W$

Span: $\text{span}(\{v_1, \ldots, v_n\})$ is a subspace of V consisting of all linear combinations of $\{v_1, \ldots v_n\}$

- If $W = \text{span}(\{v_1, \ldots, v_n\})$, then every $w \in W$ is a linear combination of $\{v_1, \ldots, v_n\}$

Linear Independent: occurs when $a_1v_1 + \cdots + a_nv_n = 0 \implies a_1 = \cdots = a_n = 0$

- $\{v_1, \ldots, v_n\}$ is linearly independent if and only if for each $i, v_i \notin \text{span}(\{v_1, \ldots, v_n\} \setminus \{v_i\})$

Basis: $\{v_1, \ldots, v_n\}$ that spans W and is linearly independent. **Note**: The empty set $\emptyset$ is a basis for $\{O\}$

Shrinking Lemma: Let $X = \{w_1, \ldots, w_m\} \subseteq W$ span W but not be LI. Then $X \setminus \{w_i\}$ still spans W for some $w_i \in X$

- **Shrinking Theorem**: Some $Y \subseteq X$ is a basis of W (must stop eventually when we get $\emptyset$ basis for $\{O\}$)

Enlarging Lemma: let $X = \{w_1, \ldots, w_m\} \subseteq W$ be LI but not span W. Then for any $w \in W \setminus \text{span}(X), X \cup \{w\}$ is still LI

Exchanging Lemma: Let $X = \{v_1, \ldots, v_n\}$ be a basis for W. Take $w \in W$ where $w \in \text{span}(\{v_1, \ldots, v_n\})$. Then for $i < k$, $Y = (X \setminus \{v_i\}) \cup \{w\}$ is still a basis

- Can be used to show that if $\{w_1, \ldots, w_m\} \subseteq W$ is linearly independent, then $m \leq n$. Thus any basis of W has n elements

Finite Dimensional: W with some basis. **Dimension** of W is the number of elements in the basis

- Any set of vectors that spans W, with the correct dimension, is a basis by the Shrinking Theorem
- Any set of vectors that is linearly independent, with the correct dimension, is a basis by the Enlarging Lemma

Direct Sum: $U \oplus W$ such that $U \oplus W = U + W$ AND $U \cap W = \{O\}$

- **Note**: $U \cap W$ and $U + W$ are subspaces of V
- **Theorem**: For subpsace $W \subseteq V$, there exists a subspace $U \subseteq V$ such that $V = U \oplus W$.

$\mathbf{Mat_{m \times n}(K)}$: K-Vector Space of all $m \times n$ matrices with entries in K

- Basis here is $\bigcup E_{ij}$ where E_{ij} has the the ij entry is 1 and all other entries as 0, which clearly has dimension $m \times n$

Symmetric $\mathbf{2 \times 2}$ Matrices come in the form of $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and is a subspace of $\text{Mat}_{2\times 2}(K)$

Image: $F(D) = \{F(x) \mid x \in D\} \subseteq R$ for the mapping $F : D \to R$

- **Onto** if $F(D) = R \qquad$ **1-1** if $F(d) = F(e) \implies d = e \qquad$ **Bijection** if both onto and 1-1

Inverse Mapping: If $F : D \to R$ is a bijection, then $\exists F^{-1} : R \to D$ such that $\forall r, \in R, F(F^{-1}(r)) = r$ and $\forall d \in D, F^{-1}(F(d)) = d$

Linear Transformation: Function $T : V \to W$ for vector spaces V, W, satisfying

- $\forall v_1, v_2 \in W, T(v_1 + v_2) = T(v_1) + T(v_w) \qquad \forall c \in K, v \in W, T(cv) = cT(v)$

Pull Back: Any set $\{v_1, \ldots, v_m\} \subseteq V$ such that $T(v_1) = w_1, \ldots, T(v_m) = w_m$

- If $\{w_1, \ldots, w_m\} \subseteq \text{Im}(T)$ is a basis, then $\{v_1, \ldots, v_m\} \subseteq V$ is a basis for $\text{span}(\{v_1, \ldots, v_m\})$. Thus $\dim(\text{Im}(T)) \leq \dim(V)$

Kernel: $\text{Ker}(T) = \{v \in V \mid T(v) = O_W\}$, which can be shown to be a subspace of V

- **Proposition** $V = \text{Ker}(T) \oplus \text{span}(\{v_1, \ldots, v_m\})$ for any pullback $\{v_1, \ldots, v_m\} \subseteq V$
- **Theorem**: $\dim(V) = \dim(\text{Ker}(T)) + \dim(\text{Im}(T))$. Comes from $V = \text{Ker}(T) \oplus S \implies \dim(V) = \dim(\text{Ker}(T)) + \dim(S)$

Upshot: $\dim(\text{Ker}(T)) > 0 \implies T$ is NOT 1-1 $\qquad \dim(\text{Im}(T)) < \dim(W) \implies T$ is NOT onto

Isomorphism: $T : V \to W$ such that T is a linear transformation and a bijection

- If $\dim(V) = \dim(W)$ and $T : V \to W$ is a linear transformation and is 1-1 $\implies$ onto OR is onto $\implies$ is 1-1

Inverse Mapping/Transformation: An isomorphism $T^{-1} : W \to V$ where $T^{-1}(w)$ is the unique $v \in V$ such that $T(v) = w$

Linear Map/Matrix: Matrix L_A that determines the LT $R^n \to R^m$, and is itself a LT (from logic of dot products)

- Transformation $T : V \to W$ WRT to bases $B = \{v_1, \ldots, v_m\} \subseteq V$ and $B' = \{w_1, \ldots, w_m\} \subseteq W$ is given by $M_{B'}^{B} = \begin{bmatrix} T(v_1) & T(v_2) & \cdots & T(v_n) \end{bmatrix}$ where v_1 is WRT to B and the result is written in terms of coordinates of B'

  Upshot: Any vector, written in B coordinates, when multiplied by this matrix, yields an answer in B' coordinates

Change of Basis: $M_{B'}^{B}(\text{id}) = \begin{bmatrix} \text{id}(v_1) & \text{id}(v_2) & \cdots \text{id}(v_n) \end{bmatrix}$ with respect to bases B, B' of the same vector space V