MATH410 Advanced Calculus I

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1 Chapter 6 Integrals

Basic idea of what an integral represents:
For an integratable $f\colon [a,b] \to \mathbb{R}$ with $f(x) \geq 0$ for all $x \in [a,b]$,
$\int_a^b f$ is the area under f and above the interval $[a,b]$

1.1 Darboux Sums

For reals $a < b$, $n \in \mathbb{N}$, and $a = x_0, \ldots x_n = b$,
$P = \{x_0, \ldots, x_n\}$ is a **partition** of the interval $[a,b]$.
For an index $i \geq 0$, x_1 is called a **partition point** of P
For $i \geq 1$, $[x_{i-1}, x_i]$ is a **partition interval** of P.

Suppose $f\colon [a,b] \to \mathbb{R}$ is bounded and $P = \{x_0, \ldots, x_n\}$ is a partition of $[a,b]$, then for $i \geq 1$:

$$m_i = \inf\{f(x)\colon x \in [x_{i-1}, x_i]\}$$
$$M_i = \sup\{f(x)\colon x \in [x_{i-1}, x_i]\}$$

We use m_i and M_i to define the **Lower and Upper Darboux Sums** for f based on partition P

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

$$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

Since $m_i \leq M_i$ for each $i \geq 1$, we have

$$L(f,P) \leq U(f,P) \text{ for any partition } P \text{ of } [a,b]$$

Also useful to note that for any partition of $[a,b]$:

$$b - a = \sum_{i=1}^{n} (x_i - x_{i-1})$$

Lemma 6.1: if $f\colon [a,b] \to \mathbb{R}$ is bounded and for $m, M\colon$ $m \leq f(x) \leq M$ for all $x \in [a,b]$ then

$$m(b-a) \leq L(f,P) \text{ and } U(f,P) \leq M(b-a)$$

Proof: m is a lower bound for all m_i as defined above so we have:

$$m(b-a) = \sum_{i=1}^{n} m(b-a) \leq \sum_{i=1}^{n} m_i(b-a) = L(f,P)$$

Similar proof is applied for $U(f,P) \leq M(b-a)$

Partition P^* is the **refinement** of partition P if each partition pt of P is a partition pt of P^*
Partition pts of P^* that belong to the partition interval $[x_{i-1}, x_i]$ define a sub-partition P_i. Observe:

$$\sum_{i=1}^{n} L(f, P_i) = L(f, P^*) \text{ and } \sum_{i=1}^{n} U(f, P_i) = U(f, P^*)$$

Lemma 6.2 Refinement Lemma: Suppose $f\colon [a,b] \to \mathbb{R}$ is bounded and that P is a partition of $[a,b]$. If P^* is a refinement of P then:

$$L(f,P) \leq L(f,P^*) \text{ and } U(f,P^*) \leq U(f,P)$$

Proof: Let $P = \{x_0, \ldots, x_n\}$, for index $i \geq 1$, define m_i the same as above, and let P_i be the partition of $[x_{i-1}, x_i]$ induced by P^*. Applying Lemma 6.1 to $f\colon [x_{i-1}, x_i] \to \mathbb{R}$ gives:

$$m_i(x_i - x_{i-1}) \leq L(f, P_i)$$

Taking the sum of all n sub-partitions gives

$$L(f, P) \leq \sum_{i=1}^{n} L(f, P_i) = L(f, P^*)$$

Similar argument can be done to show $U(f, P^*) \leq U(f, P)$

Given partitions P_1, P_2 of $[a, b]$, a **common refinement** P^* can be formed by taking the union of the partition pts of P_1, P_2
Lemma 6.3: Suppose $f\colon [a, b] \to \mathbb{R}$ is bounded and that P_1, P_2 are partitions of $[a, b]$ then:

$$L(f, P_1) \leq U(f, P_2)$$

Proof: Let P^* be the common refinement of P_1, P_2. By the Refinement Lemma, we have:

$$L(f, P_1) \leq L(f, P^*) \leq U(f, P^*) \leq U(f, P_2)$$

1.2 Lower and Upper Integrals

Suppose $f\colon [a, b] \to \mathbb{R}$ is bounded. We can define the lower and upper integrals on $[a, b]$ as:

$$\underline{\int_a^b} f = \sup\{L(f, P)\colon P \text{ is a partition of } [a, b]\}$$

$$\overline{\int_a^b} f = \inf\{L(f, P)\colon P \text{ is a partition of } [a, b]\}$$

Lemma 6.4: for a bounded $f\colon [a, b] \to \mathbb{R}$

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f$$

Proof: Let P be a partition of $[a, b]$. By Lemma 6.3, $U(f, P)$ is an upperbound for all Lower Darboux Sums of f. Thus:

$$\underline{\int_a^b} f \leq U(f, P)$$

This inequality implies that $\underline{\int_a^b}$ is a lower bound for all Upper Darboux Sums of f. Thus, by definition of infimum:

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f$$

1.3 Archimedes-Riemann Theorem

Suppose $f\colon [a, b] \to \mathbb{R}$ is bounded. Then f is **integrable** on $[a, b]$ if

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

When this condition is met, the integral of f over $[a, b]$ is defined as

$$\underline{\int_a^b} f = \int_a^b = \overline{\int_a^b} f$$

Lemma 6.7: For a bounded $f\colon [a,b] \to \mathbb{R}$ and partition P of $[a,b]$,

$$L(f,P) \leq \underline{\int}_a^b f \leq \overline{\int}_a^b f \leq U(f,P)$$

This creates 3 useful inequalities

$$0 \leq \overline{\int}_a^b f - \underline{\int}_a^b f \leq U(f,P) - L(f,P)$$

$$0 \leq U(f,P) - \overline{\int}_a^b f \leq U(f,P) - L(f,P)$$

$$0 \leq \underline{\int}_a^b f - L(f,P) \leq U(f,P) - L(f,P)$$

Proof: by definition of lower and upper integrals

$$L(f,P) \leq \underline{\int}_a^b f \text{ and } \overline{\int}_a^b \leq U(f,P)$$

Using Lemma 6.4 we get:

$$L(f,P) \leq \underline{\int}_a^b f \leq \overline{\int}_a^b f \leq U(f,P)$$

Theorem 6.8 Archimedes-Riemann Theorem: Let $f\colon [a,b] \to \mathbb{R}$ be bounded. Then f is integrable on $[a,b]$ if and only if there is a sequence $\{P_n\}$ of partitions of $[a,b]$ such that

$$\lim_{n\to\infty}[U(f,P_n) - L(f,P_n)] = 0$$

Moreover, for any such sequence of partitions,

$$\lim_{n\to\infty} L(f,P_n) = \int_a^b f \text{ and} \lim_{n\to\infty} U(f,P_n) = \int_a^b f$$

Proof Forward: suppose that such a sequence of partitions exists satisfying the equation. Using Lemma 6.7, for an index n, P_n satisfies the inequality

$$0 \leq \overline{\int}_a^b f - \underline{\int}_a^b f \leq \lim_{n\to\infty}[U(f,P_n) - L(f,P_n)] = 0$$

Thus

$$\underline{\int}_a^b f = \overline{\int}_a^b f \text{ as desired and } f \text{ is integrable over } [a,b]$$

Proof Backwards: fix a natural number n. By definition of lower integral and least upper bound,

$$\left(\underline{\int}_a^b f\right) - 1/n \text{ is not an upper bound for the Lower Darboux Sums of } f$$

Thus for some partition P' of $[a,b]$

$$\underline{\int}_a^b f - 1/n < L(f,P')$$

Similar for upper integral and some partition P'' of $[a,b]$

$$U(f,P'') < \left(\overline{\int}_a^b f\right) + 1/n$$

By the Refinement Lemma, the 2 inequalities above hold for a common refinement, P_n, of P', P''. Thus,

$$0 \leq U(f, P_n) - L(f, P_n) < \left[\left(\overline{\int_a^b} f\right) + 1/n\right] - \left[\left(\underline{\int_a^b} f\right) - 1/n\right] = 2/n$$

Thus,

$$\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = 0$$

The sequence of partitions $\{P_n\}$ that satisfies the Archimedes-Riemann Theorem is called the **Archimedean sequence of partitions**, satisfying:

$$\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = 0$$

For a $n \in \mathbb{N}$, the partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ is called the **regular partition** of $[a, b]$ into n partition intervals (of length $(b - a)/n$) if:

$$x_i = a + i\frac{b-a}{n} \text{ for } 0 \leq i \leq n$$

For partition P of $[a, b]$, the **gap** of P, denoted gap P, is the length of the largest partition interval:

$$\text{gap } P = \max_{i \leq i \leq n}[x_i - x_{i-1}]$$

Example 6.9: A montonically increasing $f\colon [a, b] \to \mathbb{R}$ is integrable. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. Since f is monotonically decreasing, we can define for any index $i \geq 1$:

$$m_i = \inf\{f(x)\colon x \in [x_{i-1}, x_i] = f(x_{i-1})$$

$$M_i = \sup\{f(x)\colon x \in [x_{i-1}, x_i] = f(x_i)$$

Divide P into regular partitions and for $n \in \mathbb{N}$, take P_n. Then

$$U(f, P_n) - L(f, P_n) = \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) = \sum_{i=1}^{n}(M_i - m_i)\frac{b-a}{n} = \frac{b-a}{n}(f(b) - f(a))$$

Thus

$$\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = \lim_{n\to\infty}\frac{(f(b) - f(a))(b-a)}{n} = 0$$

Example 6.11: Let $f(x) = x^2$ for $x \in [0, 1]$. Show that

$$\int_0^1 x^2 dx = 1/3$$

Since $f(x)$ is monotonically increasing, it is integrable on $[0, 1]$ (shown in Example 6.9)
Define a regular partition P_n of $[0, 1]$. Since $\{P_n\}$ is an Archimedean sequence of partitions for f on $[0, 1]$ (Example 6.9), by Archimedes-Rieman Theorem:

$$\int_0^1 x^2 dx = \lim_{n\to\infty} U(f, P_n)$$

For index $i \geq 1$, we have

$$M_i = \sup\{f(x)\colon x \in [x_{i-1}, x_i]\} = f(x_i) = i^2/n^2$$

$$x_i - x_{i-1} = 1/n$$

Thus,

$$M_i(x - x_{i-1}) = i^2/n^3$$

Using the sum of squares we get

$$U(f, P_n) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = (1/n^3)\left(\sum_{i=1}^{n} i^2\right) = \frac{n(n+1)(2n+1)}{6n^3}$$

Therefore,

$$\int_0^1 x^2 dx = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = 1/3$$