MATH410 Advanced Calculus

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Contents

1 Foundations

1.1 Law of Induction

1. Given a statement $S(n)$ for $n \geq n_0$
2. Show the base case $S(n_0)$ is valid
3. State the Inductive Hypothesis: assume $S(n)$ is valid for an arbitrary $n \geq n_0$
4. Prove Inductive Step: given $S(n)$ is valid, prove that $S(n+1)$ is valid
5. Then by Law of Induction, $\forall n \geq n_0, S(n)$ is valid

1.2 Proof by Contradiction

If we want $P \implies Q$, assume $\neg Q$ and try to produce $\neg P$.

1.3 $\sqrt{5}$ Irrational Proof

Definition: a rational $q = \frac{p}{q}$ where $p, q \in \mathbb{Z}, q \neq 0,$ and p/q is a reduced fraction.

Proof by contradiction: assume $\sqrt{5}$ is rational.

This implies that $\sqrt{5} = \frac{p}{q}$ where $p, q \in \mathbb{Z}, q \neq 0,$ and p/q is a reduced fraction.

Then $p = \sqrt{5}q \implies p^2 = 5q^2$ which implies $5|p^2 \implies 5|p$.

Thus for some $k \in \mathbb{Z}$, $p = 5k \implies p^2 = 25k^2 = 5q^2$

This implies $5|q^2 \implies 5|q$ which is a contradiction since $5|p$ and $5|q$.

Thus the premise is false and $\sqrt{5}$ is irrational.

2 Properties of $\mathbb{R}$

2.1 Boundness

Definition: if $S \subseteq \mathbb{R}$ is non-empty, then S is **bounded above** if $\exists c \in \mathbb{R}, \forall x \in S, b \geq x$

Definition: if $S \subseteq \mathbb{R}$ is non-empty, then S is **bounded below** if $\exists c \in \mathbb{R}, \forall x \in S, a \leq x$

Definition: if b is an upperbound of S and b is the least upperbound of S, then $b = \sup S$

Definition: if a is a lowerbound of S and b is the greatest lowerbound of S, then $a = \inf S$

2.1.1 Completeness Axiom

The follow properties exist for any set $S \subseteq \mathbb{R}$:

- if S has an upperbound, it has a least upperbound.
- if S has a lowerbound, it has a greatest lowerbound.

2.2 Density in $\mathbb{R}$

2.2.1 Archimedean Property

Following 2 properties are equivalent:

- for an arbitrary $c > 0$, $\exists n \in \mathbb{N}, n > c$
- for an arbitrary $c > 0$, $\exists n \in \mathbb{N}, 0 < \frac{1}{n} < c$

2.2.2 Definition of Density

Definition: a set S is dense in $\mathbb{R}$ if for each non-empty interval (a, b), $\exists x \in S$ in (a, b)

Theorem $\mathbb{Q}$ is dense in $\mathbb{R}$: for any arbitrary a, b where $a < b$, $\exists q \in \mathbb{Q}$ in the interval (a, b)

Proof: by Archimedean property, $\exists n \in \mathbb{Z}$ such that $0 < \frac{1}{n} < \frac{b-a}{2} \implies \frac{2}{n} < b - a$

This then gives the inequality $a < a + \frac{1}{n} < a + \frac{2}{n} < b$

Thus there has to be a $k \in \mathbb{Z}$ such that $\frac{k}{n}$ is in (a, b) and $\frac{k}{n} \in \mathbb{Q}$

Corollary: Irrationals $\mathbb{I}$ is dense in $\mathbb{R}$

Proof: from the theorem above, we know that $\exists r, s \in \mathbb{Q}, a < r < s < b$

Let $t = r + \frac{1}{\sqrt{2}}(s - r)$ thus t is irrational and $a < r < t < s < b$

3 Absolute Values

3.1 Properties of Absolute Value

The following are notable properties:

- $-|x| \leq x \leq |x|$
- if $|x| \leq d$ then $-d \leq x \leq d$
- $|b - a| < d \equiv a - d < b < a + d$

3.2 Triangle Inequality

$|a + b| \leq |a| + |b|$

4 Numerical Formulas

Difference of Powers Formula:

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1}) = (a - b)\sum_{k=0}^{n-1} a^{n-1-k}b^k$$

Geometric Sum Formula:

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

Binomial Formula:

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

5 Sequences

Definition: a **sequence** $\{a_n\}$ is a function f whose domain is $n \in \mathbb{N}$

Definition: a sequence **converges** to a if $\forall \epsilon > 0, \exists N$ such that $\forall n \geq N, |a_n - a| < \epsilon$, or a_n lies in the interval $(a - \epsilon, a + \epsilon)$

Definition: a sequence **diverges** if does not converge

5.1 Comparison Lemma

Assume $\{a_n\}$ converges to a, let $\{b_n\}$ be an arbitrary sequence, and let b by an arbitrary number.

If $\exists c \geq 0, \forall n \geq N, |b_n - b| \leq c|a_n - a|$ then $\{b_n\}$ converges to b

5.2 Sequence Boundness

A sequence $\{a_n\}$ is bounded if $\exists M, \forall n \geq N, |a_n| \leq M$

Theorem: Every convergent sequence is bounded

Proof: let $\epsilon > 0$ by arbitrary. $\exists N$ such that for $n \geq N, |a_n - a| < \epsilon$ by definition of convergence.

Let $M = \max(|a_1|, |a_2|, \ldots, |a_N|, |a_N| + \epsilon)$ then $\forall n \geq 1, |a_n| \leq M$

5.3 Set Density Using Sequences

Definition: A set S is **dense** in $\mathbb{R}$ iff for each $x \in \mathbb{R}$, there is a sequence $\{a_n\} \subseteq S$ such that $\{a_n\}$ converges to x

Definition: a set S is **closed** if whenever $\{a_n\} \subseteq S$ has the property that $\{a_n\}$ converges to a, then $a \in S$

5.4 Monotone Sequences

Definition: a sequence $\{a_n\}$ is **monotone increasing** if $\forall n \geq 1, a_n \leq a_{n+1}$

Definition: a sequence $\{a_n\}$ is **monotone decreasing** if $\forall n \geq 1, a_n \geq a_{n+1}$

5.4.1 Monotone Convergence Theorem

A monotone sequence converges iff it is bounded

How to use: prove that a sequence is bounded and is either monotonically increasing or decreasing. Then apply MCT to say the sequence must converge.

5.4.2 Nested Interval Theorem

Let $I_n = [a_n, b_n]$ with $I_{n+1} \subseteq I_n$ for $n \geq 1$. Assume $\lim_{n\to\infty}(b_n - a_b) = 0$. Then there is a unique x in each I_n and $\lim_{n\to\infty} a_n = x = \lim_{n\to\infty} b_n$

Proof: Since $I_{n+1} \subseteq I_n$, $\{a_n\}$ is monotonically increasing and $\{b_n\}$ is monotonically decreasing. However, both $\{a_n\}$ and $\{b_n\}$ bound each other. Thus by Monotone Convergence Theorem, $\{a_n\}$ converges to some a and $\{b_n\}$ converges to some b.

However, because $\lim_{n\to\infty}(b_n - a_n) = 0$, we conclude that $a = \lim_{a\to\infty} a_n = \lim_{n\to\infty} b_n = b = x$

5.5 Subsequences

Definition: let given a sequence $\{a_n\}$ and a sequence of indices $n_k, n_{k+1}, \ldots$.

If $b_k = a_{n_k}$ for all $k \geq 1$ then $\{b_k\}$ is a **subsequence** of $\{a_n\}$

Definition: for a sequence $\{a_n\}$, if there is an index m such that $\forall n \geq m, a_m \geq a_n$, then m is a **peak index**

Theorem: every sequence $\{a_n\}$ has a monotone subsequence.

Proof: case 1: there are infinitely many peak indices then $\{a_n\}$ is monotonically decreasing and so is $\{a_{n_k}\}$

case 2: there are finitely many peak indices. Then there must a index n_{k_0} such that no peak indices are bigger than it. We can define a subsequence for indices $n > n_{k_0}$ and this subsequence is monotonically increasing.

Corollary: every bounded sequence has a convergent subsequence

Proof: by the previous theorem, the bounded sequence must have a monotone subsequence. By Monotone Convergence Theorem, this subsequence must be convergent.

Definition: a set S is **sequentially compact** if every sequence $\{a_n\} \subseteq S$ has a subsequence converging on an element of S.

Theorem: each closed, bounded interval $[a, b]$ is sequentially compact

Proof: since a bounded sequence has a converge subsequence, the subsequence must converge to some x in $[a, b]$.

6 Continuous Functions

Definition: $f : D \to R$ is **continuous** at x_0 in D if for every sequence $\{x_n\} \subseteq D$, if $x_n \to x_0$, then $f(x_n) \to f(x_0)$. Furthermore, f is a **continuous function** if it is continuous at each point in its domain.

Sum, product, quotient properties of continuous functions

6.1 Extreme Value Theorem

Definition: for a function $f : D \to \mathbb{R}$, the set of **images** of f is

$$f(D) = \{y \mid y = f(x), \text{ for some } x \in D\}$$

Definition: f attains a **maximum** provided that $f(D)$ has a maximum, meaning that there is some x_0 such that

$$\forall x \in D, f(x) \leq f(x_0)$$

Definition: such an x_0 is called the **maximizer** of f.

Similar definitions can be provided for **minimum** and **minimizer**.

A nonempty set has a maximum if it is both

1. bounded above

2. contains its supremum

Extreme Value Theorem: a continuous function on a closed bounded interval attains both a minimum and a maximum.

$$f : [a, b] \to \mathbb{R}$$

Proof:

1. The image $f(D)$ is bounded above: assume the contrary and that $f(x)$ is unbounded. We can define a sequence $\{x_n\}$ in $[a, b]$ and a subsequence $\{x_{n_k}\}$. By Sequential Compactness Theorem, $\{x_n n_k\}$ must converge to a point $x_0 \in [a, b]$ thus $\{f(x_{n_k}\}$ converges to $f(x_0)$. However, a convergent sequence is bounded contradiction is reached and the image of f is bounded.

2. Show that $\sup f(D)$ is a functional value: let $S = f([a, b])$ and $c = \sup S$. This means that $c - 1/n$ is not an upper bound for S, so we have $\forall n > N, c - 1/n < f(x_n) \leq c$ and $\{f(x_n)\}$ converges to c. Thus f contains the value c.
   Similar proof can be applied to $-f$ to find the minimum of f.

6.2 Intermediate Value Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous, and there is a number c such that

$$f(a) < c < f(b) \text{ or } f(b) < c < f(a)$$

Then there is a point x_0 in (a, b) such that $f(x_0) = c$

Proof: Bisection Method
Consider the case $f(a) < c < f(b)$. Define a sequence of nested, closed subintervals of $[a, b]$ whose endpoints converge to a point in $[a, b]$
Consider the midpoint $m_n = (a_n + b_n)/2$

- If $f(m_n) \leq c$, define $a_{n+1} = m_n$ and $b_{n+1} = b_n$
- If $f(m_n) > c$, define $a_{n+1} = a_n$ and $b_{n+1} = m_n$

This gives $a \leq a_n \leq a_{n+1} < b_{n+1} \leq b_n \leq b$ and by Nested Interval Theorem the sequences must converge to a unique c.

6.3 Uniform Continuity

Definition: a function $f : D \to \mathbb{R}$ is **uniformly continuous** if whenever sequence $\{u_n\}$ and $\{v_n\}$ of D such that

$$\lim_{n\to\infty} [u_n - v_n] = 0$$

then

$$lim_{n\to\infty}[f(u_n) - f(v_n)] = 0$$

Intuition is that $f(u) - f(v)$ becomes arbitrarily small for any two points u and v that are sufficiently close to each other.
INSERT PROOF that uniformly continuous implies continuous p67

Theorem: a continuous function on a closed boundary interval is uniformly continuous

$$f : [a, b] \to \mathbb{R}$$

Proof: by contradiction, suppose $\{f(u_n) - f(v_n)\}$ does not converge to 0. Then for an arbitrary ϵ, $|f(u_n(-f(v_n)| \geq \epsilon$ for every index n. However, by Sequential Compactness Theorem, $\{u_{n_k}\}$

and $\{v_{n_k}\}$ much converge to an x_0 in $[a, b]$. The 2 subsequences must converge to the same value because by hypothesis, $\lim_{n\to\infty} [u_n - v_n] = 0$. Thus

$$\lim_{k\infty} [f(u_{n_k}) - f(u_{n_k})] = \lim_{k\infty} [f(x_0) - f(x_0)] = 0$$

Thus a contradiction is reached.