Differentiation

Neighborhood: an open interval $I = (a, b)$ that contains x_0 is called the **neighborhood** of x_0
Slope: for a point $x \in I$ and $x \neq x_0$, the slope of the line containing $(x_0, f(x_0))$ and $(x, f(x))$ is

$$\frac{f(x) - f(x_0)}{x - x_0}$$

Differentiable: Let I be a neighborhood of x_0. $f\colon I \to \mathbb{R}$ is **differentiable** at x_0 if

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{exists}$$

If $f\colon I \to \mathbb{R}$ is differentiable at every pt in I, then f is **differentiable** and $f' \to I \to \mathbb{R}$ is the **derivative** of f
Tangent line to graph of f at $x_0, f(x_0)$ is defined as

$$y = f(x_0) + f'(x_0)(x - x_0)$$

Examples

$f(x) = x^2$ and $f'(x) = 2x$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim \frac{x^2 - x_0^2}{x - x_0} = \lim[x + x_0] = 2x_0$$

$f(x) = |x|$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = 1 \neq \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = -1$$

Since the limit does not exist at 0, f is not differentiable at $x = 0$

Proposition 4.4: For a natural number n and function $f(x) = x^n$, $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable and

$$f'(x) = nx^{n-1}$$

Proof

Using the difference of powers formula, for $x \neq x_0$

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{(x - x_0)(x^{n-1} + x^{n-2}x_0 + \ldots + x_0^{n-1})}{x - x_0} = x^{n-1} + x^{n-2}x_0 + \ldots + x_0^{n-1}$$

Observe that the righthand side has n terms that each have a limit of x_0^{n-1}. Thus by sum property of limits

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = nx_0^{n-1}$$

Proposition 4.5: let I be a neighborhood of x_0 and suppose $f\colon I \to \mathbb{R}$ is differentiable at x_0. Then f is continuous at x_0

Proof

We know that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \quad \text{and} \quad \lim_{x \to x_0} [x - x_0] = 0$$

Thus

$$\lim_{x \to x_0} [f(x) - f(x_0)] = \lim [\frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0)] = f'(x_0) \cdot 0 = 0 \text{ using the product property of limits}$$

Thus $\lim_{x \to x_0} = f(x_0) \implies f$ is continuous at x_0

Theorem 4.6: Let I be a neighborhood of x_0 and suppose that $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ are differentiable at x_0. Then 1. $f + g\colon I \to \mathbb{R}$ is differentiable at x_0 and $(f+g)'(x_0) = f('x_0) + g'(x_0)$

Proof

$$\lim_{x \to x_0} \frac{(f+g)(x) - (f+g)(x_0)}{x - x_0} = \lim \frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0} = f'(x_0) + g'(x_0)$$

Using the sum property of limits and definition of derivatives

2. $fg\colon I \to \mathbb{R}$ is differentiable at x_0 and $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$
   Proof

$$\lim_{x \to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} = \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} = \frac{f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} f(x)[\frac{g(x) - g(x_0)}{x - x_0}] + g(x_0)[\frac{f(x) - f(x_0)}{x - x_0}] = f(x)g'(x_0) + f'(x)g(x_0)$$

If $f'(x)$ exists $\implies f$ is continuous at x, the above expression exists using the using sum and product properties of limits and the definition of derivatives.

3. if $g(x) \neq 0$ for all $x \in I$, $1/g\colon I \to \mathbb{R}$ is differentiable at x_0 and $(\frac{1}{g})'(x_0) = \frac{-g'(x_0)}{(g(x_0))^2}$
   Proof

$$\lim_{x \to x_0} \frac{(1/g)(x) - (1/g)(x_0)}{x - x_0} = \lim \frac{(1)(g(x)) - (1)(g(x_0))}{x - x_0} = \lim \frac{-1}{g(x)g(x_0)} [\frac{g(x) - g(x_0)}{x - x_0}] = \frac{-g'(x_0)}{(g(x_0))^2}$$

4. if $g(x) \neq 0$ for all $x \in I$, $f/g\colon I \to \mathbb{R}$ is differentiable at x_0 and $(\frac{f}{g})'(x_0) = \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}$
   Proof

$$\frac{f(x)}{g(x)} = \frac{1}{g(x)} \cdot f(x)$$

So use the quotient and product properties above to prove differentiability

Theorem 4.11: Let I be a neighborhood of x_0 and let $f\colon I \to \mathbb{R}$ be strictly monotone and continuous. Suppose f is differentiable at x_0 and that $f'(x_0) \neq 0$. Define $J = f(I)$. Then the inverse $f^{-1}\colon J \to \mathbb{R}$ is differentiable at the point $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

Proof

By the IVT, we know that J is a neighborhood of $y_0 = f(x_0)$ so for a point $y \in J, y \neq y_0$ we can define $x = f^{-1}(y)$ such that

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = 1/\frac{f(x) - f(x_0)}{x - x_0}$$

Since f^{-1} is continuous, we have

$$\lim_{y \to y_0} f^{-1}(y) = f^{-1}(y_0) = x_0$$

Thus by quotient property of limits and the definition of differentiability, we have

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim 1/\frac{f(x) - f(x_0)}{x - x_0} = \frac{1}{f'(x_0)}$$

Corollary 4.12: Let I be an open interval and suppose $f\colon I \to \mathbb{R}$ is strictly monotone and is differentiable with a nonzero derivative at each point in I. Define $J = f(I)$. Then the inverse function $f^{-1}\colon J \to \mathbb{R}$ is differentiable and for all $x \in J$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Proof

Since differentiability $\implies$ continuity, $f\colon I \to \mathbb{R}$ is continuous and by applying Theorem 4.11 at $x \in J$, where $x = f(f^{-1}(x))$ and $f^{-1}(x)$ plays the role of x_0.

Theorem 4.14 The Chain Rule: Let I by a neighborhood of x_0 and let $f\colon I \to \mathbb{R}$ be differentiable at x_0. Let J be an open interval such that $f(I) \subseteq J$ and let $g\colon J \to \mathbb{R}$ be differentiable at $f(x_0)$. Then $g \circ f\colon I \to \mathbb{R}$ is differentiable at x_0 and

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$$

Proof

Let $y_0 = f(x_0)$ and $y = f(x)$. Then we have

$$\frac{f(x) - f(x_0)}{y - y_0} = 1$$

Using this we have

$$\frac{g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = \frac{g(y) - g(y_0)}{x - x_0} = \frac{g(y) - g(y_0)}{y - y_0} * \frac{f(x) - f(x_0)}{x - x_0}$$

Provided that $y - y_0 = f(x) - f(x_0) \neq 0$. If there is no open interval containing x_0 such that $f(x) - f(x_0)$ we can define an auxiliary function $h\colon J \to \mathbb{R}$ such that

$$h(y) = \begin{cases} [g(y) - g(y_0)]/[y - y_0] & y \in J, y \neq y_0 \\ g'(y_0) & y = y_0 \end{cases}$$

This gives us

$$\lim_{x \to x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = \lim h(f(x))[\frac{f(x) - f(x_0)}{x - x_0}]$$

$$= h(f(x_0))f'(x_0) = g'(f(x_0))f'(x_0)$$

Lemma 4.16: Let I be a neighborhood of x_0 and suppose $f\colon I \to \mathbb{R}$ is differentiable at x_0. If x_0 is a maximizer or a minimizer of f, then $f'(x_0) = 0$

Proof

Suppose x_0 is a maximizer. Then for $x < x_0, x \in I$

$$f'(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

For $x > x_0, x \in I$, we have

$$f'(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \leq 0$$

Thus $f'(x_0) = 0$

Theorem 4.17 Rolle's Theorem: Suppor $f\colon [a, b] \to \mathbb{R}$ is continuous and that f restricted to the open interval (a, b) is differentiable. Also assume that

$$f(a) = f(b)$$

Then there is a point x_0 in the open interval (a, b) such that

$$f'(x_0) = 0$$

Proof

By the Extreme Value Theorem, f attains both a maximum and a minimum value on $[a, b]$.
Case 1: $f(a) = f(b)$ is the maximum or minimum so f is a constant function and $f'(x) = 0$
Case 2: f has some other maximum or minimum point where the derivative is 0, based on Lemma 4.16.

Theorem 4.18: Mean Value Theorem: Suppose that $f\colon [a, b] \to \mathbb{R}$ is continuous and that the restriction of f to the open interval (a, b) is differentiable. Then there is a point x_0 in (a, b) such that

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}$$

Proof

Let $h\colon [a, b] \to \mathbb{R}$ be defined by $h(x) = f(x) - mx$ for $x \in [a, b]$. To apply Rolle's Theorem, we need $h(a) = h(b)$, which happens when

$$m = \frac{f(b) - f(a)}{b - a}$$

Thus by this choice of m and Rolle's Theorem, then there is a point x_0 in (a, b) such that $h'(x_0) = 0$. Since $h'(x_0) = f'(x_0) - m$, we have

$$f'(x_0) = m = \frac{f(b) - f(a)}{b - a}$$

as desired.

Proposition 4.20: Identity Criterion: Let I by an open interval and let $g\colon I \to \mathbb{R}$ and $h\colon I \to \mathbb{R}$ by differentiable. These functions differ by a constant if and only if for all $x \in I$

$$g'(x) = h'(x)$$

Proof

Define $f = g - h\colon I \to \mathbb{R}$. We then have

$$f'(x) = g'(x) - h'(x)$$

Observe that f is constant if and only if g and h differ by a constant. The derivative of any constant is 0. Thus $g'(x) = h'(x)$

Corollary 4.21 Strict Monotonicity Criterion: Let I be an open interval and $f\colon I \to \mathbb{R}$ be differentiable. Suppose $f'(x) > 0$ for all $x \in I$. Then f is strictly increasing.

Proof

Let $u, v \in I$ and $u < v$. Using the Mean Value Theorem, we can restrict f to closed bounded interval $[u, v]$ and choose a point $x_0 \in (a, b)$ such that

$$f'(x_0) = \frac{f(v) - f(u)}{v - u}$$

Since $f'(x_0) > 0$ and $v - u > 0$, we have $f(u) < f(v)$. Thus f is strictly increasing.
Similar proof can be done to prove that $f'(x) < 0$ for all $x \in I \implies f$ is strictly decreasing.

Local Maximizer: A point x_0 in the domain of $f\colon D \to \mathbb{R}$ is a **local maximizer** if there is some $\delta > 0$ such that for all $x \in D$ such that $|x - x_0| < \delta$, we have

$$f(x) \leq f(x_0)$$

Local Minimizer: if $|x - x_0| < \delta$ implies

$$f(x) \geq f(x_0)$$

By Lemma 4.16, if I is a neighborhood of x_0 and $f\colon I \to \mathbb{R}$ is differentiable at x_0, then for x_0 to either be a local minimizer or local maximizer, we need

$$f'(x_0) = 0$$

However, $f'(x_0) = 0$ does NOT imply that it is a local minimizer or local maximizer (e.g. $f(x) = x^3$ and $f'(0) = 0$)

Theorem 4.22: Let I be an open interval containing x_0 and suppose that $f\colon I \to \mathbb{R}$ has a second derivative. Suppose $f'(x_0) = 0$ then * If $f''(x_0) > 0$ then x_0 is a local minimizer * If $f''(x_0) < 0$ then x_0 is a local maximizer

Proof

Suppose $f''(x_0) > 0$. This implies

$$f''(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0} > 0$$

There is an open interval such that for $\delta > 0$, $(x_0 - \delta, x_0 + \delta)$ that is contained in I, we have * $f'(x) > 0$ if $x_0 < x < x_0 + \delta$ * $f'(x) < 0$ if $x_0 - \delta < x < x_0$

Using the Mean Value Theorem, we have if $0 < |x - x_0| < \delta$

$$f(x) > f(x_0)$$

Similar proof for $''(x_0) < 0$