Theorem 2.18: Every convergent sequence is bounded:
Proof: let $\epsilon > 0$ be arbitrary. Then $\exists N$ such that $\forall n \geq N, |a_n - a| < \epsilon$
Let $M = \max(|a_1|, |a_2|, \dots, |a_N|, |a_N| + \epsilon)$, then $\forall n \geq 1, M \geq |a_n|$
Monotone Convergence Theorem: a **monotone** sequence **converges if and only if** it is **bounded**. It will converge to * $\sup\{a_n \mid n \in \mathbb{N}\}$ if the sequence is monotonically increasing * $\inf\{a_n \mid n \in \mathbb{N}\}$ if the sequence is monotonically decreasing

Proof: Without loss of generality, assume the sequence is monotonically increasing and let $S = \{a_n\}$ that is bounded. By the Completeness Axiom, S has a least upper bound $l = \sup S$. Thus $|a_n - l| < \epsilon$ and we have

$$l - \epsilon < a_n < l + \epsilon$$

For the right side, $a_n \leq l < l + \epsilon$ as desired.
For the left side, $l - \epsilon$ is not an upper bound so $\exists N, l - \epsilon < a_N$ but $\{a_n\}$ is monotonically increasing, thus $\forall n \geq N, l - \epsilon < a_n \leq a_n$ as desired.

Nested Interval Theorem: let $a_n < b_n$ and $I = [a_n, b_n]$. Assume $I_{n+1} \subseteq I_n$ and $\lim_{n\to\infty}[b_n - a_n] = 0$. There is a single point x in I_n that $\{a_n\}$ and $\{b_n\}$ converge to.
Proof: We know that $a_n \leq a_{n+1} < b_{n+1} \leq b_n$. Since $\{a_n\}$ is monotonically increasing and is bounded by b_n, it must converge to a. Similarly, $\{b_n\}$ must converge to b.
Since $\lim_{n\to\infty}[b_n - a_]= b - a = 0$, $x = b = a$

Note: If $\{a_n\} \to a$, then every subsequence $\{a_{n_k}\} \to a$

Theorem 2.32: Every sequence has a monotonic subsequence (peek index proof)
Theorem 2.33: Every bounded sequence has a convergent subsequence.
Proof: Every sequence has a monotone subsequence. Since the sequence is bounded, so is the subsequence and by Monotone Convergence Theorem, that subsequence must converge.

Sequentially Compactness: A set S is **sequentially compact** if every sequence in S has a subsequence that converges to a point in S.
Sequential Compactness Theorem: Any closed interval $[a, b]$ is sequentially compact.
Proof: we need 2 conditions 1. Sequence in $[a, b]$ has a convergent subsequence (using Theorem 2.33) 2. The limit of any sequence in a bounded, closed interval is in that interval. This works for subsequences as well.

Continuity: $f\colon D \to \mathbb{R}$ is **continuous at** $x_0 \in D$ if whenever $\{x_n\} \subseteq D \to x_0$, then $\{f(x)\} \to f(x_0)$.

Extreme Value Theorem: a continuous function on a closed, bounded interval $f\colon [a, b] \to \mathbb{R}$ has a minimum and a maximum value.
Proof: 1. Show $f\colon [a, b] \to \mathbb{R}$ is bounded above.
Proof by contradiction: $\exists x \in [a, b], f(x) > n$. Define a sequence $\{x_n\}$ such that $\forall n \geq 1, f(x_n) > n$
By the Sequential Compactness Theorem, there is a subsequence $\{x_{n_k}\} \to x_0 \in [a, b]$. Since f is continuous, $\{f(x_{n_k})\} \to f(x_0)$.
Since $\{f(x_{n_k})\}$ converges, it is bounded. Thus contradiction is reached since

$$\forall k, f(x_{n_k}) > n_k \geq k$$

2. $\sup f(D)$ is a functional value:
Let $M = \sup\{f(x) \mid a \leq x \leq b\}$. This means that there is a sequence $\{f(x_n)\} \to M$
There is also a subsequence $\{x_{n_k}\} \to x_0 \in [a, b]$. Since f is continuous we have

$$\{f(x_{n_k})\} \to f(x_0) = M$$

Intermediate Value Theorem: Let $f\colon [a, b] \to \mathbb{R}$ is continuous. If there is a c strictly between $f(a)$ and $f(b)$ then there is a point x_0 in (a, b) such that $f(x_0) = c$

Uniform Continuity: for arbitrary sequences $\{u_n\}$ and $\{v_n\}$,

$$\lim_{n\to\infty}[u_n - v_n] = 0 \implies \lim_{n\to\infty}[f(u_n) - f(v_n)] = 0$$

Note: neither sequence needs to convergent

Theorem 3.17: A continuous function on a closed bounded interval $f\colon [a,b] \to \mathbb{R}$ is uniform continuous
Proof: Proof by contradiction so for an arbitrary $\epsilon > 0$ and arbitrary sequences $\{u_n\}$ and $\{v_n\}$

$$\lim_{n\to\infty}[f(u_n) - f(v_n)] \geq \epsilon$$

By the Sequential Compactness Theorem, $\{u_{n_k}\} \to x_0 \in [a,b]$. Since $\lim u_n = \lim v_n$, we can conclude that $\{v_{n_k}\} \to x_0$. However, since f is continuous, we have

$$|f(u_{n_k}) - f(v_{v_k})| \leq |f(u_{n_k}) - f(x_0)| + |f(v_{n_k}) - f(x_0)| = 0$$

Thus a contradiction is reached

$\epsilon - \delta$ **criterion at a point**: $f\colon D \to \mathbb{R}$ satisifies $\epsilon - \delta$ criterion at $x_0 \in D$ if for each $\epsilon > 0, \exists\delta > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$$

Following are equivalent: * $f\colon D \to \mathbb{R}$ is uniform continuous if $\lim[u_n - v_n] = 0 \implies \lim[f(u_n) - f(v_n)] = 0$ * f satisfies $\epsilon - \delta$ on domain D if $|u - v| < \delta \implies |f(u) - f(v) < \epsilon$

Theorem 3.23: if $f\colon D \to \mathbb{R}$ is monotone and $f(D)$ is an interval, then f is continuous
Corollary 3.25: Let $f\colon I \to \mathbb{R}$ be monotone. Then f is continuous iff $f(I)$ is an interval.

$f\colon D \to \mathbb{R}$ is **1-1** if each point $y \in f(D)$ has exactly one $x \in D$ such that $f(x) = y$

Inverse properties * $f^{-1}(f(x)) = x$ * f(f^{-1}(y)) = y$ * Inverse of strictly monotone function is strictly monotone

Theorem 3.29: if $f\colon I \to \mathbb{R}$ is strictly monotone then $f^{-1}\colon f(I) \to \mathbb{R}$ is continuous

Completeness Axiom: if a non empty set S of real number is bounded above, then S has a least upper bound
Archimedean Property: for any positive number c, there is a natural number n such that $n > c$. Also for any positive ϵ, there is a natural n such that $1/n < \epsilon$

Difference of Powers Formula:

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1})$$

Geometric Sum Formula:

$$1 + r + r^2 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

Binomial Formula:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$