Continuity: f is continuous at x_0 iff $\lim_{x\to x_0}(f(x)-f(x_0))=0$

Uniform Continuity: if $\lim_{n\to\infty}[u_n-v_n]=0$ then $\lim[f(u_n)-f(v_n)]=0$

Note: if $f\colon [a,b]\to\mathbb{R}$ is continuous, then f is f is uniformly continuous on $[a,b]$

Identity Criterion: Let I be an open interval and $f\colon I\to\mathbb{R}$, $g\colon I\to\mathbb{R}$ be differentiable on I. Then $f'=g'$ iff $f=g+C$.

Integral: if $f\colon [a,b]\to\mathbb{R}$ is bounded an $\underline{\int_a^b} f = \overline{\int_a^b} f$ then f is integrable on $[a,b]$

Proof: if $f\colon [a,b]\to\mathbb{R}$ is continuous then $\int_a^b f$ exists

Since f is continuous on $[a,b]$, it is uniformly continuous so $\forall\delta>0$ such that $|x-z|<\delta$, $\exists\epsilon$ such that $|f(x)-f(z)|<\epsilon/(b-a)$

By Archimedes Property, there is an n such that $(b-a)/n<\delta$

Let P_n be a regular partition of $[a,b]$ then $U(f,P_n)-L(f,P_n)=\sum_{i=1}^n(M_i-m_i)\frac{b-a}{n}<\sum_{i=1}^n\frac{\epsilon}{b-a}\frac{b-a}{n}=\epsilon$.

Proof: If $f\colon [a,b]\to\mathbb{R}$ is continuous and $F=\int_a^x f(t)$ then $F'=f$ on $a<x<b$ and F is continuous on $[a,b]$

$\lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0}=\lim\frac{\int_a^x f-\int_a^{x_0} f}{x-x_0}=\lim\frac{\int_{x_0}^x f}{x-x_0}\approx\frac{f(x_0)(x-x_0)}{x-x_0}=f(x_0)$ and is continuous on (a,b)

Let $x_n\to a$ with $a<x_n<b$. Then since f is continuous, $|F(x_n)-F(a)|=|\int_{x_n}^a f-\int_a^a f|\approx|f(a)(x_n-a)|\to 0$ thus F continuous at a

First Fundamental Theorem: If $F\colon [a,b]\to\mathbb{R}$ is continuous and $F'=f$ on (a,b) with f continuous on (a,b) and bounded on $[a,b]$, then $\int_a^b f(t)\,dt=\int_a^b F'(t)\,dt=F(b)-F(a)$

Let $G(x)=\int_a^x f(t)$ for $a\le x\le b$. Then by Second Fundamental Theorem, $G'(x)=f(x)$ on (a,b) and G is cont on $[a,b]$

Since $F'=G'=f$, by the Identity Criterion, $G=F+c$ on (a,b)

Since F,G are continuous on $[a,b]$ then $G=f+c$ on $[a,b]$ and $\int_a^b f=G(b)-G(a)=(F(a)+c)-(F(a)+c)$

Integration by Parts: $\int u\,dv=uv-\int v\,du$

Trapezoidal Rule: $\int_a^b f(x)\approx\frac{b-a}{2n}[f(a)+2f(x_1)+2f(x_2)+\dots f(b)]$, $E_n^T\le\frac{M_T}{12n^2}(b-a)^3$ where $M_T=\sup\{|f''(x)|\colon a<x<b\}$

- If f is linear, $=\int$
- If f is concave up, $\ge\int$
- If f is concave down, $\le\int$

Simpson's Rule: $\int_a^b f(x)\approx\frac{b-a}{3n}[f(a)+4f(x_1)+2f(x_2)\dots 4f(x_{n-1})+f(b)]$, $M_S=\sup\{|f^{(4)}|\colon a<x<b\}$

- n must be positive and even

Example: Show $g(x)=\sin(x)$ is differentiable:

$g'(x_0)=\lim_{h\to 0}\frac{\sin(x_0+h)-\sin(x_0)}{h}=\lim\frac{\sin(x_0)\cos(h)+\sin(h)\cos(x_0)-\sin(x_0)}{h}=\lim\frac{\sin(x_0)(\cos(h)-1)}{h}+\lim\frac{\sin(h)\cos(x_0)}{h}=\cos(x_0)$

Example: Show that $g(x)=x+\sin(x)$ is strictly increasing

$g'(x)=1+\cos(x)$ which is increasing except at $x=\pi+2n\pi$ which is fine b/c $g(x)$ is continuous at these points

Example: Let $f\colon [a,b]\to\mathbb{R}$ be monotonically increasing, then $\int_a^b f$ exists

Let $\epsilon>0$ be arbitrary. Then by the Archimedean Property, $0\le\frac{b-a}{n}(f(b)-f(a)\le\epsilon$

Define P_n as a regular partition of $[a,b]$ then since f is increasing on $[a,b]$, $U(f,P_n)-L(f,P_n)=\sum_{i=1}^n\frac{b-a}{n}(f(b)-f(a))\le\epsilon$

Rolle's Theorem requires f to be continuous on $[a,b]$ and differentiable on (a,b)

Gauss sum is $\frac{(n)(n+1)}{2}$

Darboux sum involves i and $i-1$