MATH410 Advanced Calculus I

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Contents

1 Chapter 6 Integrals

Basic idea of what an integral represents:
For an integratable $f\colon [a,b] \to \mathbb{R}$ with $f(x) \geq 0$ for all $x \in [a,b]$,
$\int_a^b f$ is the area under f and above the interval $[a,b]$

1.1 Darboux Sums

For reals $a < b$, $n \in \mathbb{N}$, and $a = x_0, \dots x_n = b$,
$P = \{x_0, \dots, x_n\}$ is a **partition** of the interval $[a,b]$.
For an index $i \geq 0$, x_1 is called a **partition point** of P
For $i \geq 1$, $[x_{i-1}, x_i]$ is a **partition interval** of P.

Suppose $f\colon [a,b] \to \mathbb{R}$ is bounded and $P = \{x_0, \dots, x_n\}$ is a partition of $[a,b]$, then for $i \geq 1$:

$$m_i = \inf\{f(x)\colon x \in [x_{i-1}, x_i]\}$$
$$M_i = \sup\{f(x)\colon x \in [x_{i-1}, x_i]\}$$

We use m_i and M_i to define the **Lower and Upper Darboux Sums** for f based on partition P

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

$$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

Since $m_i \leq M_i$ for each $i \geq 1$, we have

$$L(f,P) \leq U(f,P) \text{ for any partition } P \text{ of } [a,b]$$

Also useful to note that for any partition of $[a,b]$:

$$b - a = \sum_{i=1}^{n} (x_i - x_{i-1})$$

Lemma 6.1: if $f\colon [a,b] \to \mathbb{R}$ is bounded and for m, M: $m \leq f(x) \leq M$ for all $x \in [a,b]$ then

$$m(b-a) \leq L(f,P) \text{ and } U(f,P) \leq M(b-a)$$

Proof: m is a lower bound for all m_i as defined above so we have:

$$m(b-a) = \sum_{i=1}^{n} m(x_i - x_{i-1}) \leq \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = L(f,P)$$

Similar proof is applied for $U(f,P) \leq M(b-a)$

Partition P^* is the **refinement** of partition P if each partition pt of P is a partition pt of P^*
Partition pts of P^* that belong to the partition interval $[x_{i-1}, x_i]$ define a partition interval P_i. Observe:

$$\sum_{i=1}^{n} L(f,P_i) = L(f,P^*) \text{ and } \sum_{i=1}^{n} U(f,P_i) = U(f,P^*)$$

Lemma 6.2 Refinement Lemma: Suppose $f\colon [a,b] \to \mathbb{R}$ is bounded and that P is a partition of $[a,b]$. If P^* is a refinement of P then:

$$L(f,P) \leq L(f,P^*) \text{ and } U(f,P^*) \leq U(f,P)$$

Proof: Let $P = \{x_0, \ldots, x_n\}$, for index $i \geq 1$, define m_i the same as above, and let P_i be the partition of $[x_{i-1}, x_i]$ induced by P^*. Applying Lemma 6.1 to $f\colon [x_{i-1}, x_i] \to \mathbb{R}$ gives:

$$m_i(x_i - x_{i-1}) \leq L(f, P_i)$$

Taking the sum of all n sub-partitions gives

$$L(f, P) \leq \sum_{i=1}^{n} L(f, P_i) = L(f, P^*)$$

Similar argument can be done to show $U(f, P^*) \leq U(f, P)$

Given partitions P_1, P_2 of $[a, b]$, a **common refinement** P^* can be formed by taking the union of the partition pts of P_1, P_2

Lemma 6.3: Suppose $f\colon [a, b] \to \mathbb{R}$ is bounded and that P_1, P_2 are partitions of $[a, b]$ then:

$$L(f, P_1) \leq U(f, P_2)$$

Proof: Let P^* be the common refinement of P_1, P_2. By the Refinement Lemma, we have:

$$L(f, P_1) \leq L(f, P^*) \leq U(f, P^*) \leq U(f, P_2)$$

1.2 Lower and Upper Integrals

Suppose $f\colon [a, b] \to \mathbb{R}$ is bounded. We can define the lower and upper integrals on $[a, b]$ as:

$$\underline{\int_a^b} f = \sup\{L(f, P)\colon P \text{ is a partition of } [a, b]\}$$

$$\overline{\int_a^b} f = \inf\{U(f, P)\colon P \text{ is a partition of } [a, b]\}$$

Lemma 6.4: for a bounded $f\colon [a, b] \to \mathbb{R}$

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f$$

Proof: Let P be a partition of $[a, b]$. By Lemma 6.3, $U(f, P)$ is an upperbound for all Lower Darboux Sums of f. Thus:

$$\underline{\int_a^b} f \leq U(f, P)$$

This inequality implies that $\underline{\int_a^b}$ is a lower bound for all Upper Darboux Sums of f. Thus, by definition of infimum:

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f$$

1.3 Archimedes-Riemann Theorem

Suppose $f\colon [a, b] \to \mathbb{R}$ is bounded. Then f is **integrable** on $[a, b]$ if

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

When this condition is met, the integral of f over $[a, b]$ is defined as

$$\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f$$

Lemma 6.7: For a bounded $f\colon [a,b] \to \mathbb{R}$ and partition P of $[a,b]$,

$$L(f,P) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(f,P)$$

This creates 3 useful inequalities

$$0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le U(f,P) - L(f,P)$$

$$0 \le U(f,P) - \overline{\int_a^b} f \le U(f,P) - L(f,P)$$

$$0 \le \underline{\int_a^b} f - L(f,P) \le U(f,P) - L(f,P)$$

Proof: by definition of lower and upper integrals

$$L(f,P) \le \underline{\int_a^b} f \text{ and } \overline{\int_a^b} \le U(f,P)$$

Using Lemma 6.4 we get:

$$L(f,P) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(f,P)$$

Theorem 6.8 Archimedes-Riemann Theorem: Let $f\colon [a,b] \to \mathbb{R}$ be bounded. Then f is integrable on $[a,b]$ if and only if there is a sequence $\{P_n\}$ of partitions of $[a,b]$ such that

$$\lim_{n\to\infty} [U(f,P_n) - L(f,P_n)] = 0$$

Moreover, for any such sequence of partitions,

$$\lim_{n\to\infty} L(f,P_n) = \int_a^b f \text{ and } \lim_{n\to\infty} U(f,P_n) = \int_a^b f$$

Proof Forward: suppose that such a sequence of partitions exists satisfying the equation. Using Lemma 6.7, for an index n, P_n satisfies the inequality

$$0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le \lim_{n\to\infty} [U(f,P_n) - L(f,P_n)] = 0$$

Thus

$$\underline{\int_a^b} f = \overline{\int_a^b} f \text{ as desired and } f \text{ is integrable over } [a,b]$$

Proof Backwards: fix a natural number n. By definition of lower integral and least upper bound,

$$\left(\underline{\int_a^b} f\right) - 1/n \text{ is not an upper bound for the Lower Darboux Sums of } f$$

Thus for some partition P' of $[a,b]$

$$\underline{\int_a^b} f - 1/n < L(f,P')$$

Similar for upper integral and some partition P'' of $[a,b]$

$$U(f,P'') < \left(\overline{\int_a^b} f\right) + 1/n$$

By the Refinement Lemma, the 2 inequalities above hold for a common refinement, P_n, of P', P''. Thus,

$$0 \leq U(f, P_n) - L(f, P_n) < \left[\left(\overline{\int_a^b} f\right) + 1/n\right] - \left[\left(\underline{\int_a^b} f\right) - 1/n\right] = 2/n$$

Thus,

$$\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = 0$$

The sequence of partitions $\{P_n\}$ that satisfies the Archimedes-Riemann Theorem is called the **Archimedean sequence of partitions**, satisfying:

$$\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = 0$$

For a $n \in \mathbb{N}$, the partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ is called the **regular partition** of $[a, b]$ into n partition intervals (of length $(b - a)/n$) if:

$$x_i = a + i\frac{b-a}{n} \text{ for } 0 \leq i \leq n$$

For partition P of $[a, b]$, the **gap** of P, denoted gap P, is the length of the largest partition interval:

$$\text{gap } P = \max_{i \leq i \leq n}[x_i - x_{i-1}]$$

Example 6.9: A montonically increasing $f: [a, b] \to \mathbb{R}$ is integrable. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. Since f is monotonically decreasing, we can define for any index $i \geq 1$:

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i] = f(x_{i-1})$$

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i] = f(x_i)$$

Divide P into regular partitions and for $n \in \mathbb{N}$, take P_n. Then

$$U(f, P_n) - L(f, P_n) = \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) = \sum_{i=1}^{n}(M_i - m_i)\frac{b-a}{n} = \frac{b-a}{n}(f(b) - f(a))$$

Thus

$$\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = \lim_{n\to\infty}\frac{(f(b) - f(a))(b-a)}{n} = 0$$

Example 6.11: Let $f(x) = x^2$ for $x \in [0, 1]$. Show that

$$\int_0^1 x^2 dx = 1/3$$

Since $f(x)$ is monotonically increasing, it is integrable on $[0, 1]$ (shown in Example 6.9)
Define a regular partition P_n of $[0, 1]$. Since $\{P_n\}$ is an Archimedean sequence of partitions for f on $[0, 1]$ (Example 6.9), by Archimedes-Rieman Theorem:

$$\int_0^1 x^2 dx = \lim_{n\to\infty} U(f, P_n)$$

For index $i \geq 1$, we have

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\} = f(x_i) = i^2/n^2$$

$$x_i - x_{i-1} = 1/n$$

Thus,

$$M_i(x - x_{i-1}) = i^2/n^3$$

Using the sum of squares we get

$$U(f,P_n)=\sum_{i=1}^{n}M_i(x_i-x_{i-1})=(1/n^3)\left(\sum_{i=1}^{n}i^2\right)=\frac{n(n+1)(2n+1)}{6n^3}$$

Therefore,

$$\int_0^1 x^2dx=\lim_{n\to\infty}U(f,P_n)=\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}=1/3$$

1.4 Additivity, Monotonicity, Linearity

Theorem 6.12 Additivity over Intervals: Let $f\colon[a,b]\to\mathbb{R}$ be integrable over $[a,b]$ and let $c\in(a,b)$. Then f is integrable over $[a,c]$ and $[c,b]$, and

$$\int_a^b f=\int_a^c f+\int_c^b f$$

Proof: since f is integrable on $[a,b]$, by the Archimedes-Riemann Theorem, there is an Archimedean sequence of partitions $\{P_n\}$ for f on $[a,b]$ such that

$$\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)]=0$$

and

$$\lim_{n\to\infty}U(f,P_n)=\int_a^b f$$

Since c belongs to each partition P_n, let P'_n be the partition induced on $[a,c]$ and P''_n be the partition induced on $[c,b]$. Then from the definition of Darboux Sums

$$U(f,P_n)=U(f,P'_n)+U(f,P''_n)$$

$$L(f,P_n)=L(f,P'_n)+L(f,P''_n)$$

Thus

$$U(f,P_n)-L(f,P_n)=[U(f,P'_n)-L(f,P'_n)]+[U(f,P''_n)-L(f,P''_n)]$$

Since P_n is an Archiemedean sequence of partitions, the limits of the two terms in brackets is 0. Thus, by Archimedes-Riemann Theorem, f is integrable on $[a,c]$ and on $[c,b]$, and

$$\lim_{n\to}U(f,P'_n)=\int_a^c f \text{ and } \lim_{n\to\infty}U(f,P''_n)=\int_c^b f$$

Thus, combining the limits we have

$$\int_a^b f=\lim_{n\to\infty}U(f,P_n)=\int_a^c f+\int_c^b f$$

Theorem 6.13 Monotonicity of Integrals: let $f\colon[a,b]\to\mathbb{R}$ and $g\colon[a,b]\to\mathbb{R}$ be integrable and $f(x)\le g(x)$ for all $x\in[a,b]$. Then

$$\int_a^b f\le\int_a^b g$$

Proof: By the Archimedes-Riemann Theorem and Refinement Theorem, there exists a sequence of partitions $\{P_n\}$ such that $\lim_{n\to\infty}U(f,P_n)=\int_a^b f$ and $=\int_a^b g$
Since $f(x)\le g(x)$ by hypothesis, by the definition of Upper Darboux Sums,

$$U(f,P_n)\le U(g,P_n) \text{ for each index } n$$

Thus

$$\int_a^b f = \lim_{n\to\infty} U(f,P_n) \le \lim_{n\to\infty} U(g,P_n) = \int_a^b g$$

Lemma 6.14: let $f\colon [a,b]\to\mathbb{R}$ and $g\colon [a,b]\to\mathbb{R}$ by bounded functions and let P be a partition of $[a,b]$. Then

$$L(f,P)+L(g,P)\le L(f+g,P) \text{ and } U(f+g,P)\le U(f,P)+U(g,P)$$

$$U(\alpha f,P)=\alpha U(f,P) \text{ and } L(\alpha f,P)=\alpha L(f,P) \text{ if } \alpha\ge 0$$

$$U(\alpha f,P)=\alpha L(f,P) \text{ and } L(\alpha f,P)=\alpha U(f,P) \text{ if } \alpha\le 0$$

Proof: Choose an interval I_i of partition P and for a bounded function $h\colon [a,b]\to\mathbb{R}$ define

$$M_i(h)=\sup\{h(x)\colon x\in I_i\} \text{ and } m_i(h)=\inf\{h(x)\colon x\in I_i\}$$

Thus for any $x\in I_i$,

$$f(x)+g(x)\le M_i(f)+M_i(g)$$

By definition of supremum, we have

$$M_i(f+g)\le M_i(f)+M_i(g)$$

Finally, multiplying this inequality by the length of I_i and summing over all intervals of P yields the first part of the lemma
To prove the second part, notice that

$$M_i(\alpha f)=\alpha M_i(f) \text{ and } m_i(\alpha f)=\alpha m_i(f) \text{ if } \alpha\ge 0$$

$$M_i(\alpha f)=\alpha m_i(f) \text{ and } m_i(\alpha f)=\alpha M_i(f) \text{ if } \alpha\le 0$$

Then summing over all intervals, we get the second part of the lemma

Theorem 6.15 Linearity of Integrals: Let $f\colon [a,b]\to\mathbb{R}$ and $g\colon [a,b]\to\mathbb{R}$ by integrable. Then for any numbers α,β the function $\alpha f+\beta g\colon [a,b[\to\mathbb{R}$ is integrable and

$$\int_a^b [\alpha f+\beta g]=\alpha\int_a^b f+\beta\int_a^b g$$

Proof: By Archimedes-Riemann Theorem and the Refinement Lemma, there is a sequence $\{P_n\}$ of partitions of $[a,b]$ that is both an Archimedean sequence of partitions of f and for g on $[a,b]$
Case 1: $\beta=0$. For an index n, we have

$$U(\alpha f,P_n)-L(\alpha f,P_n)=|\alpha|U(f,P_n)-L(f,P_n)]$$

Since $\{P_n\}$ is an Archimedean sequence of partitions for f on $[a,b]$ it is also an Archmidean sequence of partitions for αf on $[a,b]$. By Lemma 6.14

$$U(\alpha f,P_n)=\begin{cases}\alpha U(f,P_n) & \alpha\ge 0\\ \alpha L(f,P_n) & \alpha\le 0\end{cases}$$

However, by Archimedean-Riemann Theorem, for $\alpha\ge 0$

$$\int_a^b \alpha f=\lim_{n\to\infty}U(\alpha f,P_n)=\alpha\lim_{n\to\infty}U(f<P_n)=\alpha\int_a^b f$$

If $\alpha\le 0$

$$\int_a^b \alpha f=\lim_{n\to\infty}U(\alpha f,P_n)=\alpha\lim_{n\to\infty}L(f<P_n)=\alpha\int_a^b f$$

Case 2: $\alpha = \beta = 1$. For an index n, we have

$$L(f, P_n) + L(g, P_n) \le L(f+g, P_n) \le U(f+g, P_n) \le U(f, P_n) + U(g, P_n)$$

From Archimedes-Riemann Theorem, the sequence of partitions satisfy

$$\lim_{n\to\infty} L(f+g, P_n) = \lim_{n\to\infty} U(f+g, P_n) = \int_a^b f + \int_a^b g$$

Thus we have

$$\int_a^b [f+g] = \int_a^b f + \int_a^b g$$

General case: follow by some combination of case 1 and case 2

Corollary 6.16: Let $f\colon [a,b] \to \mathbb{R}$ and $|f|\colon [a,b] \to \mathbb{R}$ be integrable. Then

$$\left| \int_a^b f(x)dx \right| \le \int_a^b |f(x)|dx$$

Proof: For all $x \in [a,b]$,

$$-|f(x)| \le f(x) \le |f(x)|$$

Using the monotonicity and linearity of integration, we have

$$-\int_a^b |f(x)|dx \le \int_a^b f(x)dx \le \int_a^b |f(x)dx$$

Thus by definition of absolute value, the corollary holds.

1.5 Continuity and Integrability

Lemma 6.17: Let $f\colon [a,b] \to \mathbb{R}$ be continuous and let P be a partition of $[a,b]$. Then there is a partition interval of P that contains the points u, v where

$$0 \le U(f,P) - L(f,P) \le [f(v) - f(u)][b-a]$$

Proof: Let $P = \{x_0, \dots, x_n\}$. For index $i \ge 1$, since f is continuous on the closed bounded partition interval $[x_{i-1}, x_i]$, by the Extreme Value Theorem, this partition interval has a max value and a min value. Thus there are points u_i and v_i in $[x_{i-1}, x_i]$ such that

$$f(u_i) = m_i = \inf\{f(x)\colon x \in [x_{i-1}, x_i]\}$$
$$f(v_i) = M_i = \sup\{f(x)\colon x \in [x_{i-1}, x_i]\}$$

Choosing an index i_0 such that

$$M_{i_0} - m_{i_0} = \max_{1\le i\le n} [M_i - m_i]$$

and defining $u = u_{i_0}$ and $v = v_{i_0}$ we have

$$M_i - m_i \le M_{i_0} - m_{i_0} = f(v) - f(u) \text{ for } i \le i_0 \le n$$

Thus we have

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} [M_i - m_i][x_i - x_{i-1}] \le \sum_{i=1}^{n} [f(v) - f(u)][x_i - xi - 1] = [f(v) - f(u)][b-a]$$

Theorem 6.18: A continuous function on a closed bounded interval is integrable

Proof: Let $\{P_n\}$ be any sequence of partitions of $[a, b]$ such that $\lim_{n\to\infty} \text{gap } P_n = 0$
Since u_n and v_n belong to the common partition interval P_n

$$|v_n - u_n| \leq \text{ gap } P_n$$

Furthermore, since the function is continuous we know that

$$\lim_{n\to\infty}[u_n - v_n] = 0$$

Since f is continuous on a closed bounded interval, by Theorem 3.17, it is uniformly continuous and thus

$$\lim_{n\to\infty}[f(u_n) - f(v_n)] = 0$$

Using Lemma 6.17, we have

$$0 \leq \lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] \leq \lim_{n\to\infty}[f(v_n) - f(u_n)][b - a] = 0$$

Thus the sequence $\{P_n\}$ is an Archimedean sequence of partitions for f on $[a, b]$. Thus by the Archimedes-Riemann Theorem, f is integrable on $[a, b]$

Theorem 6.19: If $f\colon [a, b] \to \mathbb{R}$ is bounded an on the closed interval $[a, b]$ and is continuous on the open interval $(a, b0$ then f is integrable on $[a, b]$ and the value of $\int_a^b f$ does not depend on the values of f at the endpoints of the interval.