MATH410 Advanced Calculus Midterm 2

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Contents

1 Differentiation

If $x_0 \in (a, b)$, then (a, b) is a **neighborhood** of x_0

Let f be defined for a neighborhood of x_0. Then f is **differentiable** at x_0 if

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists}$$

Example: Let $g(x) = \sqrt{x}$. Show that $g'(x) = \frac{1}{2\sqrt{x}}$ for $x > 0$

Solution: Let $x_0 > 0$. Then

$$g'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} = \lim_{x \to x_0} \frac{1}{\sqrt{x} + \sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}$$

Example: show that $g(x) = \sin(x)$ is differentiable

Solution: For any x_0,

$$\begin{aligned} g'(x_0) &= \lim_{h \to 0} \frac{\sin(x_0 + h) - \sin(x_0)}{h} \\ &= \lim_{h \to 0} \frac{(\sin(x_0)\cos(h) + \sin(h)\cos(x_0)) - \sin(x_0)}{h} \\ &= \lim_{h \to 0} \frac{(\sin(x_0))(\cos(h) - 1)}{h} + \lim_{h \to 0} \frac{\sin(h)\cos(x_0)}{h} \\ &= 0 + (1)(\cos(x_0)) = \cos(x_0) \end{aligned}$$

Assume $f'(x_0)$ exists. Then the **tangent line** to f at $(x_0, f(x_0))$ for all x is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

Where $f'(x_0)$ is the **slope** of the tangent line

Example: Let $f(x) = \sqrt{x}$. Find the tangent line to f at $(4, 2)$

Solution: First we have

$$f'(4) = \lim_{x \to 4} \frac{\sqrt{x} - \sqrt{4}}{x - 4} = \frac{1}{4}$$

Then we have

$$y - f(y) = f'(4)(x - 4) \implies y - 2 = \frac{1}{4}(x - 4)$$

Proposition 4.5: If $f'(x_0)$ exists, then f is continuous at x_0

- **Note**: f is continuous at x_0 if and only if $\lim_{x \to x_0}(f(x) - f(x_0)) = 0$
- **Note**: converse is NOT true. Counterexample: $f(x) = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$.

Differentiation Rules: If $f'(x_0)$ and $g'(x_0)$ exist then

- **Addition**: $(f + g)'(x_0) = f'(x_0) + g'(x_0)$
- **Product**: $(fg)'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0)$
- **Quotient**: if $g(x) \neq 0$ then $(\frac{f}{g})'(x_0) = \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$

Theorem 4.11: Let I be a neighborhood of x_0 and let $f\colon I \to \mathbb{R}$ be continuous and strictly monotone. Assume $f'(x_0)$ exists and $\neq 0$. If $f(x_0) = y_0$ then

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

- **Note**: since $f\colon I \to \mathbb{R}$ is strictly increasing, by Theorem 3.29, f^{-1} exists and is continuous on the interval $f(I)$

Example: Let $g(x) = x^n$ for $x \geq 0, n \geq 1$. Show that $(g^{-1})'(x)$ exists for $x > 0$

Solution: Since g is strictly increasing, g^{-1} must exist and $g^{-1}(y) = x = y^{1/n}$ for $y > 0$. By Theorem 4.11,

$$(g^{-1})'(y) = \frac{1}{g'(x)} = \frac{1}{nx^{n-1}} = \frac{1}{n}y^{\frac{1}{n}-1}$$

Chain Rule: Let $f\colon I \to \mathbb{R}$ and $g\colon J \to \mathbb{R}$ with $f(I) \subseteq J$. Assume that $f'(x_0)$ and $g'(f(x_0))$ exist. Then

$$(g \circ f)'(x_0) = (g'(f(x_0)))(f'(x_0))$$

Example: If $h(x) = (1 - x^2)^{3/2}$, find $h'(x)$

Solution: Let $f(x) = 1 - x^2$ and $g(x) = x^{3/2}$, so $h(x) = g(f(x))$. Then by the Chain Rule

$$h'(x) = g'(f(x))f'(x) = (\frac{3}{2}(1 - x^2)^{1/2})(-2x) = -3x(1 - x^2)^{1/2}$$

Lemma 4.16: Let I be a neighborhood of x_0 and define $f\colon I \to \mathbb{R}$ with $f'(x_0)$ exists If x_0 is a maximizer or minimizer, then $f'(x_0) = 0$

Rolle's Theorem: Let $f\colon [a, b] \to \mathbb{R}$ be continuous, and let f be differentiable on (a, b) with $f(a) = f(b)$. Then there exists a point $x_0 \in (a, b)$ such that $f'(x_0) = 0$

Mean Value Theorem: Let $f\colon [a, b] \to \mathbb{R}$ be continuous, and let f be differentiable on (a, b). Then there is a point $x_0 \in (a, b)$ such that

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}$$

Example: let $f(x) = x^3 + ax^2 + bx + c$. Show that $f(x) = 0$ has ≤ 3 solutions

Solution: By Rolle's Theorem, between any 2 solutions of f, there is a solution of f'. We have

$$f'(x) = 3x^2 + 2ax + b$$

which has at most 2 solutions. Thus f has at most 3 solutions.

Example: Show that $f(x) = x^3 + ax^2 + bx + c = 0$ has at least 1 solution.

Solution: Since $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$, and since f is continuous, by IVT, there has to be one solution to $f(x) = 0$.

Lemma 4.19: Let $f\colon [a, b] \to \mathbb{R}$ be continuous and let $f'(x) = 0$ for $a < x < b$. Then f is a constant function.

Identity Criterion: Let I be an open interval and let $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ be differentiable on I. Then $f' = g'$ if and only iff there exists a constant C such that $f(x) = g(x) + C$

Example: Suppose $f'(x) = 4x^3 - 6x^2 + 2x - 3$ and $f(1) = 2$. Find $f(x)$

Solution: since $\frac{d}{dx}x^n = nx^{n-1}$, we can process $f'(x)$ term by term and use the Identity Criterion to get that for some constant C

$$f(x) = x^4 - 2x^3 + x^2 - 3x + C$$

Since $f(1) = 2$, we have that $C = 5$ Thus

$$f(x) = x^4 - 2x^3 + x^2 - 3x + 5$$

Corollary 4.21: If $f' > 0$ on an open interval I, then f is strictly increasing on I.

- **Note**: if f is continuous on $[a, b]$ and $f' > 0$ on (a, b) then f is strictly increasing on $[a, b]$
- **Note**: If f is continuous on $[a, b]$ and $f' > 0$ except at a finite number of points $x_1, \dots, x_n$, then f is strictly increasing on $[a, b]$

Example: $f(x) = x^3$ is strictly increasing on $(-\infty, 0]$ and $[0, \infty)$. Therefore it is strictly increasing on $(-\infty, \infty)$

Example: Show that $g(x) = x + \sin(x)$ is strictly increasing

Solution: $g'(x) = 1 + \cos(x) > 0$ except at $x = \pi + 2n\pi$. Therefore g is strictly increasing on $(-\infty, \infty)$

Let $f\colon D \to \mathbb{R}$. $x_0 \in D$ is a **local maximizer** of f is there is a neighborhood $U \subseteq D$ such that for $x \in U, f(x) \leq f(x_0)$. Similar definition for **local minimizer**.

2nd Derivative Test: Let x be a point in an open interval I and define $f\colon I \to \mathbb{R}$. Assume f' and f'' exist on I with $f'(x_0) = 0$. Then

- $f''(x_0) < 0 \implies x_0$ is a local maximizer
- $f''(x_0) > 0 \implies x_0$ is a local minimizer

Following properties of 2nd derivatives on **open intervals**:

- $f'' < 0 \implies f$ is **concave down**
- $f'' > 0 \implies f$ is **concave up**
- If $''(x_0)$ changes signs at $x_0 \implies (x_0, f(x_0))$ is an **inflection point**

Cauchy Mean Value Theorem: Let $f\colon [a, b] \to \mathbb{R}$ and $g\colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $g'(x) \neq 0$ for $a < x < b$ and $g(a) \neq g(b)$. Then there is an $x_0 \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}$$

- **Note**: Cauchy MVT is a generalization of the regular MVT where $g(x) = x$

2 Integration

For reals $a < b$, $n \in \mathbb{N}$, and $a = x_0, \dots x_n = b$, $P = \{x_0, \dots, x_n\}$ is a **partition** of the interval $[a, b]$.

Suppose $f\colon [a, b] \to \mathbb{R}$ is bounded and $P = \{x_0, \dots, x_n\}$ is a partition of $[a, b]$, then for $i \geq 1$:

$$m_i = \inf\{f(x)\colon x \in [x_{i-1}, x_i]\}$$

$$M_i = \sup\{f(x)\colon x \in [x_{i-1}, x_i]\}$$

We use m_i and M_i to define the **Lower and Upper Darboux Sums** for f based on partition P

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

$$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

Since $m_i \leq M_i$ for each $i \geq 1$, we have

$$L(f,P) \leq U(f,P) \text{ for any partition } P \text{ of } [a,b]$$

Also useful to note that for any partition of $[a,b]$:

$$b - a = \sum_{i=1}^{n} (x_i - x_{i-1})$$

Partition P^* is the **refinement** of partition P if each partition pt of P is a partition pt of P^*

Lemma 6.3: Suppose $f\colon [a,b] \to \mathbb{R}$ is bounded and that P, Q are partitions of $[a,b]$ then:

$$L(f,P) \leq U(f,Q)$$

Suppose $f\colon [a,b] \to \mathbb{R}$ is bounded. We can define the lower and upper integrals on $[a,b]$ as:

$$\underline{\int_a^b} f = \sup\{L(f,P)\colon P \text{ is a partition of } [a,b]\}$$

$$\overline{\int_a^b} f = \inf\{U(f,P)\colon P \text{ is a partition of } [a,b]\}$$

Lemma 6.4: for a bounded $f\colon [a,b] \to \mathbb{R}$

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f$$

Suppose $f\colon [a,b] \to \mathbb{R}$ is bounded. Then f is **integrable** on $[a,b]$ if

$$\underline{\int_a^b} f = \int_a^b f = \overline{\int_a^b} f$$

Theorem 6.8 Archimedes-Riemann Theorem: Let $f\colon [a,b] \to \mathbb{R}$ be bounded. Then f is integrable on $[a,b]$ if and only if there is a sequence $\{P_n\}$ of partitions of $[a,b]$ such that

$$\lim_{n\to\infty} [U(f,P_n) - L(f,P_n)] = 0$$

Moreover, for any such sequence of partitions,

$$\lim_{n\to\infty} L(f,P_n) = \int_a^b f \text{ and } \lim_{n\to\infty} U(f,P_n) = \int_a^b f$$

Example 6.9: A montonically increasing $f\colon [a,b] \to \mathbb{R}$ is integrable. Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a,b]$. Since f is monotonically increasing, we can define for any index $i \geq 1$:

$$m_i = \inf\{f(x)\colon x \in [x_{i-1}, x_i] = f(x_{i-1})$$

$$M_i = \sup\{f(x)\colon x \in [x_{i-1}, x_i] = f(x_i)$$

Let P_n be a regular partition of $[a,b]$. Then

$$U(f, P_n) - L(f, P_n) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} (M_i - m_i)\frac{b-a}{n} = \frac{b-a}{n}(f(b) - f(a))$$

Thus

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = \lim_{n\to\infty} \frac{(f(b) - f(a))(b-a)}{n} = 0$$

For a $n \in \mathbb{N}$, the partition $P = \{x_0, \dots, x_n\}$ of $[a,b]$ is called the **regular partition** of $[a,b]$ into n partition intervals (of length $(b-a)/n$) if:

$$x_i = a + i\frac{b-a}{n} \text{ for } 0 \leq i \leq n$$

For partition P of $[a,b]$, the **gap** of P, denoted gap P, is the length of the largest partition interval:

$$\text{gap } P = \max_{i \leq i \leq n} [x_i - x_{i-1}]$$

Theorem 6.12 Additivity over Intervals: Let $f\colon [a,b] \to \mathbb{R}$ be integrable over $[a,b]$ and let $c \in (a,b)$. Then f is integrable over $[a,c]$ and $[c,b]$, and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Theorem 6.13 Monotonicity of Integrals: let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be integrable and $f(x) \leq g(x)$ for all $x \in [a,b]$. Then

$$\int_a^b f \leq \int_a^b g$$

Theorem 6.15 Linearity of Integrals: Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ by integrable. Then for any numbers α, β the function $\alpha f + \beta g\colon [a, b[\to \mathbb{R}$ is integrable and

$$\int_a^b [\alpha f + \beta g] = \alpha \int_a^b f + \beta \int_a^b g$$

Theorem 6.18: A continuous function on a closed bounded interval is integrable

- **Note**: if $f\colon [a,b] \to \mathbb{R}$ is bounded and continuous on $(a,b]$, then $\int_a^b f$ exists
- **Note**: Similarly if it is continuous on (a,b). This allows for $g(x) = \begin{cases} 0 & x = 0 \\ \sin(1/x) & 0 < x \leq 1 \end{cases}$ to be integrable on the interval $[0,1]$

Theorem 6.28: Let $f\colon [a,b] \to \mathbb{R}$ be continuous and let $F(x) = \int_a^x f(t)\,dt$ for $a \le x \le b$. Then

$$F'(x) = f(x) \text{ for } a < x < b \text{ and } F \text{ is continuous on } [a,b]$$

Second Fundamental Theorem: Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then

$$\frac{d}{dx}\int_a^x f = f(x) \text{ for } x \in (a,b)$$

Corollary 6.32: Let I, J be open intervals and $f\colon I \to \mathbb{R}$ and $\phi\colon J \to \mathbb{R}$ and let f, ϕ be differentiable and $\phi(J) \subseteq I$. Then

$$\frac{d}{dx}\int_a^{\phi(x)} f(t)\,dt = f(\phi(x))\phi'(x) \text{ for } x \in J$$

Example: $G = \int_{x^2}^{\sin(x)} e^t\,dt$. Find G'

Solution:

$G = \left(-\int_0^{x^2} e^t\,dt\right) + \int_0^{\sin(x)} e^t\,dt$

$G' = -xe^{x^2} + \cos(x)e^{\sin(x)}$

- **Note**: for $a \le b$, $\int_b^a f = -\int_a^b f$

Example: $G = \int_0^x \sin(x+t)\,dt$. Find G'

Solution:

$G = \left(\int_0^x \sin(x)\cos(t)\,dt\right) + \int_0^x \cos(x)\sin(t)\,dt$

$G' = \left(\cos(x)\int_0^x \cos(t)\right) + \sin(x)\cos(x) - \left(\sin(x)\int_0^x \sin(t)\right) + \cos(x)\sin(x)$

First Fundamental Theorem: Let $F\colon [a,b] \to \mathbb{R}$ be continuous and let $F' = f$ on (a,b) with f continuous on (a,b) and bounded on $[a,b]$. Then

$$\int_a^b f(t)\,dt = \int_a^b F'(t)\,dt = F(b) - F(a)$$

Key Notes about Fundamental Theorems:

- First Fundamental Theorem says that the integral of a derivative of F is $F + C$.
- Second Fundamental Theorem says that the derivative of the integral of f is f.

Example: Let $G(x) = \int_0^x (x-t)f(t)\,dt$. Show that $G''(x) = f(x)$

Solution:

$G(x) = \left(x\int_0^x f(t)\,dt\right) - \int_0^x tf(t)\,dt$

$G'(x) = \left(\int_0^x f(t)\,dt\right) + xf(x) - xf(x)$

$G''(x) = f(x)$

Mean Value Theorem of Integrals: Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then there is an $x_0 \in [a,b]$ such that

$$f(x_0) = \frac{1}{b-a}\int_a^b f(t)\,dt$$

Improper Integrals: If f is continuous on $[a, b)$ with f unbounded near b and if $\lim_{c\to b^-} \int_a^c f(x)\,dx$ exists as a number, then the integrals converges and

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$$

Example: Find $\int_1^2 \frac{1}{(x-1)^{4/3}}\,dx$

Solution:

$$= \lim_{a\to 1^+} \int_a^2 \frac{1}{(x-1)^{4/3}}$$
$$= -3 + \lim_{a\to 1^+} \frac{3}{(a-1)^{1/3}}$$

Thus the integral diverges

Example: Find $\int_2^\infty \frac{x}{1+x^4}$.

Solution:

$$= \lim_{b\to\infty} \int_2^b \frac{x}{1+x^4}\,dx \leq \lim_{b\to\infty} \int_2^b \frac{1}{x^3}$$
$$= \lim_{b\to\infty} \frac{-1}{2b^2} + 1/8 = 1/8$$

Integration by Parts: let $g\colon [a, b] \to \mathbb{R}$ and $h\colon [a, b] \to \mathbb{R}$ be continuous and have continuous derivatives on (a, b). Then

$$\int_a^b h(x)g'(x)\,dx = h(b)g(b) - h(a)g(a) - \int_a^b g(x)h'(x)\,dx$$

- **Note**: $\int u\,dv = uv - \int v\,du$

u-substitution: let $f\colon [a, b] \to \mathbb{R}$ and $f\colon [c, d] \to \mathbb{R}$ be continuous with g' bounded and continuous, and let $g(c, d) \subseteq (a, b)$. Then

$$\int_c^d f(g(x))g'(x)\,dx = \int_{g(c)}^{g(d)} f(u)\,du \text{ where } u = g(x)$$

Trapezoidal Rule: let $f\colon [a, b] \to \mathbb{R}$ be continuous and define a regular partition P_n of $[a, b]$

For an interval $[x_{i-1}, x_i]$, by the Mean Value Theorem of Integrals, there is an $x^* \in [x_{i-1}, x_i]$ such that

$$\int_{x_{i-1}}^{x_i} f(x)\,dx = f(x^*)(x_i - x_{i-1}) \approx \frac{f(x_{i-1}) + f(x_i)}{2}\frac{b-a}{n}$$

Thus the Trapezoidal Rule says that

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}[f(a) + 2f(x_1) + \ldots 2f(x_{n-1}) + f(b)]$$

- **Note**: if f is linear, then Trapezoidal Rule = definite integral

- **Note**: if f is concave up on (a,b) then Trapezoidal Rule $\geq \in 5_a^b f(x)$. Similarly, if f is concave down then it is $\leq \int_a^b f(x)$.

Example: Approximate $\int_0^2 x^3\,dx$ with Trapezoidal Rule with $n = 6$

Solution:

$$\int_0^2 x^3\,dx \approx \frac{2-0}{6}[0^3 + 2(\frac{1}{3})^3 + \ldots + 2(\frac{5}{3})^3 + 2^3]$$

Trapezoidal Error:

$$E_n^T \leq \frac{M_T}{12n^2}(b-a)^3$$

Where $M_T = \sup(\{|f''(x)| \colon a < x < b\}$

Example: Let $f(x) = x^3$ on $[0,2]$. Find the smallest reasonable $n > 0$ so that $E_n^T \leq 100$

Solution: $f''(x) = 6x$ so

$$E_n^T \leq \frac{12(2-0)^3}{12n^2} \leq 1100 \implies n = 29$$

Simpson's Rule: Let $f\colon [a,b] \to \mathbb{R}$ be continuous and P_n be a regular partition of $[a,b]$. Also define

$$p(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{h}(x - x_0) + \frac{f(x_0) - 2f(x_1) + f(x_2)}{2h^2}(x - x_0)(x - x_1)$$

Note that $\int_{x_0}^{x_2} p(x)\,dx = \frac{b-a}{3}[f(x_0) + f(x_1) + f(x_2)]$. If n is even, then we can group the partitions into groups of 3: $\{x_0, x_1, x_2\}, \{x_2, x_3, x_4\}, \ldots$. This gives us

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}[f(a) + 4(x_1) + 2f(x_2) + 4f(x_4) + \ldots 2f(x_{n-2}) + 4f(x_{n-1}) + f(b)]$$

- **Note**: n must be even to use Simpson's Rule
- **Note**: restriction on n is not present for Trapezoidal Rule

Example: Approximate $\int_0^2 x^5$ with Simpson's Rule and $n = 4$

Solution:

$$\int_0^2 x^5 \approx \frac{2}{12}[0^5 + f(\frac{1}{2})^5 + 2(1)^5 + 4(\frac{3}{2})^5 + 2^5] = 10.75$$

Simpson's Error:

$$E_n^S \leq \frac{M_S(b-a)^5}{180n^4}$$

Where $M_S = \sup\{|f^{(4)}(x)| \colon a < x < b\}$