MATH410 Advanced Calculus

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1 Foundations

1.1 Law of Induction

1. Given a statement $S(n)$ for $n \geq n_0$
2. Show the base case $S(n_0)$ is valid
3. State the Inductive Hypothesis: assume $S(n)$ is valid for an arbitrary $n \geq n_0$
4. Prove Inductive Step: given $S(n)$ is valid, prove that $S(n+1)$ is valid
5. Then by Law of Induction, $\forall n \geq n_0, S(n)$ is valid

1.2 Proof by Contradiction

If we want $P \implies Q$, assume $\neg Q$ and try to produce $\neg P$.

1.3 $\sqrt{5}$ Irrational Proof

Definition: a rational $q = \frac{p}{q}$ where $p, q \in \mathbb{Z}, q \neq 0,$ and p/q is a reduced fraction.

Proof by contradiction: assume $\sqrt{5}$ is rational.

This implies that $\sqrt{5} = \frac{p}{q}$ where $p, q \in \mathbb{Z}, q \neq 0,$ and p/q is a reduced fraction.

Then $p = \sqrt{5}q \implies p^2 = 5q^2$ which implies $5|p^2 \implies 5|p$.

Thus for some $k \in \mathbb{Z}$, $p = 5k \implies p^2 = 25k^2 = 5q^2$

This implies $5|q^2 \implies 5|q$ which is a contradiction since $5|p$ and $5|q$.

Thus the premise is false and $\sqrt{5}$ is irrational.

2 Properties of $\mathbb{R}$

2.1 Boundness

Definition: if $S \subseteq \mathbb{R}$ is non-empty, then S is **bounded above** if $\exists c \in \mathbb{R}, \forall x \in S, b \geq x$

Definition: if $S \subseteq \mathbb{R}$ is non-empty, then S is **bounded below** if $\exists c \in \mathbb{R}, \forall x \in S, a \leq x$

Definition: if b is an upperbound of S and b is the least upperbound of S, then $b = \sup S$

Definition: if a is a lowerbound of S and b is the greatest lowerbound of S, then $a = \inf S$

2.1.1 Completeness Axiom

The follow properties exist for any set $S \subseteq \mathbb{R}$:

- if S has an upperbound, it has a least upperbound.
- if S has a lowerbound, it has a greatest lowerbound.

2.2 Density in $\mathbb{R}$

2.2.1 Archimedean Property

Following 2 properties are equivalent:

- for an arbitrary $c > 0$, $\exists n \in \mathbb{N}, n > c$
- for an arbitrary $c > 0$, $\exists n \in mathbbN, 0 < \frac{1}{n} < c$

2.2.2 Definition of Density

Definition: a set S is dense in $\mathbb{R}$ if for each non-empty interval (a, b), $\exists x \in S$ in (a, b)

Theorem $\mathbb{Q}$ is dense in $\mathbb{R}$: for any arbitrary a, b where $a < b$, $\exists q \in \mathbb{Q}$ in the interval (a, b)

Theorem: Irrationals $\mathbb{I}$ is dense in $\mathbb{R}$

3 Absolute Values

3.1 Properties of Absolute Value

The following are notable properties:

- $-|x| \leq x \leq |x|$
- if $|x| \leq d$ then $-d \leq x \leq d$
- $|b - a| < d \equiv a - d < b < a + d$

3.2 Triangle Inequality

$|a + b| \leq |a| + |b|$

4 Numerical Formulas

Difference of Powers Formula:

$$a^n - b^n = (a - b) \sum_{k=0}^{n-1} a^{n-1-k} b^k$$

Geometric Sum Formula:

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

Binomial Formula:

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

5 Sequences

Definition: a **sequence** $\{a_n\}$ is a function f whose domain is $n \in \mathbb{N}$

Definition: a sequence **converges** to a if $\forall \epsilon > 0, \exists N$ such that $\forall n \geq N, |a_n - a| < \epsilon$, or a_n lies in the interval $(a - \epsilon, a + \epsilon)$

Definition: a sequence **diverges** if does not converge

5.1 Comparison Lemma

Assume $\{a_n\}$ converges to a, let $\{b_n\}$ be an arbitrary sequence, and let b by an arbitrary number.

If $\exists c \geq 0, \forall n \geq N, |b_n - b| \leq c|a_n - a|$ then $\{b_n\}$ converges to b

5.2 Sequence Boundness

A sequence $\{a_n\}$ is bounded if $\exists M, \forall n \geq N, |a_n| \leq M$

Theorem: Every convergent sequence is bounded

5.3 Set Density Using Sequences

A set S is dense in $\mathbb{R}$ iff for each $x \in \mathbb{R}$, there is a sequence $\{a_n\} \subseteq S$ such that $\{a_n\}$ converges to x

Definition: a set S is **closed** if whenever $\{a_n\} \subseteq S$ has the property that $\{a_n\}$ converges to a, then $a \in S$

5.4 Monotone Sequences

Definition: a sequence $\{a_n\}$ is **monotone increasing** if $\forall n \geq 1, a_n \leq a_{n+1}$

Definition: a sequence $\{a_n\}$ is **monotone decreasing** if $\forall n \geq 1, a_n \geq a_{n+1}$

5.4.1 Monotone Convergence Theorem

A monotone sequence converges iff it is bounded