MATH475: Combinatorics and Graph Theory

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1 Chapter 1

1.1 The Basics- Permutations, Combinations, and General Counting

1.1.1 Permutation

Definition - Permutation: A **permutation** of an n-element set is an arrangement of the elements in a specific order

- k-permutation: arrangement of k elements from the set
- The total number of k-permutation of an n-element set is

$$P(n,k) = \frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1)$$

Example: 10 people can run for office for a committee with a President, Vice President, Treasurer. The total number of possible committees is $P(10,3)$

Definition - kth Falling Factorial of n: Let $n \geq k$, both positive integers. Then the **kth Falling Factorial of n** is $(n)_k = n(n-1)\cdots(n-k+1)$

Example - Permutations with Repetitions: How many rearrangements of $AAAABBBCCD$ are there?

- If we were only looking at distinct elements, there would be 10! factorial. However, the elements aren't distinct

As an example, look at how we can reorder the A's

$$A_1A_2A_3A_4BBBCCD = A_2A_1A_3A_4BBBCCD = \cdots$$

Thus there are 4! of these representations that are equivalent and we divide 10! by 4! to account for this. Similar idea for the other letters

Thus the number of rearrangements is $\dfrac{10!}{4!3!2!1!}$

Theorem 1: Suppose object 1 occurs a_1 times, object 2 occurs a_2 times, ..., object k occurs a_k times. Furthermore, suppose $a_1 + \cdots + a_k = n$

Then the total number of arrangements if $\dfrac{n!}{a_1!\cdots a_k!} = \dbinom{n}{a_1, a_2, \ldots, a_k}$

- **Definition - Multinomial Coefficients**: The notation above is called a multinomial coefficient

1.1.2 Combinations

Definition - Combinations: The total number of ways to create a k-element subset from $[n] = \{1, 2, \ldots, n\}$ is denoted

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{P(n,k)}{k!}$$

- **Note**: the definition comes from $n!$ total possible distinct permutations of which we can permute a k-element subset $k!$ ways

Example: There are 5 cats, 5 dogs, and 4 mice . 3 are chosen at once

- How many total number of ways to get 2 cats, 1 dog?

  $\dbinom{5}{2}\dbinom{5}{1}$. This comes from choosing 2 cats from 5 cats, and then choosing 1 dog from 5 dogs

- How many total ways to get at least 1 cat?

  $\binom{14}{3} - \binom{9}{3}$. This comes from subtracting the possible groupings with no cats from the total number of possible groupings

  $= \binom{5}{1}\binom{9}{2} + \binom{5}{2}\binom{9}{1} + \binom{5}{3}$. This comes from enumerating over possible groupings with 1 cat and 2 other animals, 2 cats and 1 other animal, and 3 cats and no other animals

Binomial Theorem: If n is a positive integer then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Proof: $(x+y)^n = (x+y) \cdots + (x+y)$

The coefficients of $x^k y^{n-k}$ is the product of n terms distributing from k x-terms and $n-k$ y-terms

So out of the n terms that are multiplied together, $\binom{n}{k}$ contribute the x-term, leaving $n-k$ unused terms for y

Thus we have

$$\binom{n}{k} x^k \binom{n-k}{n-k} y^{n-k} = \binom{n}{k} x^k y^{n-k}$$