MATH475: Combinatorics and Graph Theory

Michael Li

1 Introduction

Combinatorics is concerned with the existence, enumeration, analysis, and optimization of discrete structures (finite sets)

Combinatorics problems can be generalized into 2 types of problems

- *Existence of arrangement*: can we arrange objects of a set such that certain conditions are met?
- *Enumeration/classification of arrangements*: if a special arrangement is possible, there may be several ways of achieving it. Classify them into types

2 Permutations and Combinations

2.1 Basic Counting Principles

Partition: Let S be a set. A **partition** of S is a collection of subsets $S_1, \dots, S_m$ of S such that each element of S is in exactly one subset

$$S = S_1 \cup S_2 \cup \dots \cup S_m$$

Where $S_1, \dots, S_m$ are pairwise disjoint

Addition Principle: Suppose S is partitioned into pairwise disjoint parts $S_1, \dots, S_m$. Then the number of elements in S is determined by the number of elements in each part

$$|S| = |S_1| + |S_2| + \dots + |S_m|$$

Multiplication Principle: Let S be a set of ordered pairs (a, b) where a comes from a set of size p and for each a, there are q choices for b. Then

$$|S| = p \times q$$

This results from the additional principle. Let $a_1, \dots, a_p$ be different choices for a and are each in a partition S_i of S. Then

$$|S_i| = q \implies |S| = |S_1| + \dots + |S_p| = p \times q$$

This can be generalized for ordered pairs with ≥ 2 elements

- **Example**: Determine the number of positive factors of $3^4 * 5^2 * 11^7 * 13^8$

  Solution: number of factors is $5 * 3 * 8 * 9 = 1080$

Subtraction Principle: Let $A \subset U$ and $\bar{A} = U \setminus A$ be the complement of A in U. Then

$$|A| = |U| - |\bar{A}|$$

- **Example**: Suppose passwords compose of 6 symbols from $\{0, \dots, 9, a, \dots, z\}$. How many passwords have repeated symbols?

  Solution: $|U| = 36^6$ and $|\bar{A}| = 36 * 35 * 34 * 33 * 32 * 31 \implies |A| = |U| - |\bar{A}| = 774,372,096$

Division Principle: Let S be a finite set partitioned into k parts such that each part has the same number of elements. Then

$$k = \frac{A}{\text{\# of elements per part}}$$

- **Example**: Suppose we have 740 pigeons and pigeonholes of size 5. How many pigeonholes do we have?

  Solution: $\frac{740}{5} = 148$ pigeonholes

Counting problems can be classified into 2 types of problems

- **Permutation**: count the number of ordered arrangements
  - Without repeating any object
  - With repetition permitted
- **Combination**: count the number of unordered arrangements
  - Without repeating any object
  - With repetition permitted

Example: Find the number of odd numbers between $1000, 9999$ that have distinct digits

Solution: unit digit restricted to $\{1, 3, 5, 7, 9\}$, tens and hundreds place restricted to $\{0, \ldots, 9\}$, thousands place restricted to $\{1, \ldots, 9\}$

We quantify units, thousands, hundreds, tens: $5 * 8 * 8 * 7 = 2240$

- **NOTE**: we start with the most restrictive choice first (unit digit)

Example: Number of integers between 0 and $10,000$ with one digit equal to 5

Solution: First partition S into

- S_1: set of one digits numbers. $|S_1| = 5$
- S_2: set of two digit numbers. $|S_2| = 8 + 9 = 17$
- S_3: set of three digit numbers. $|S_3| = 8 * 9 + 8 * 9 + 9 * 9 = 225$
- S_4: set of four digit numbers. $|S_4| = 8 * 9 * 9 + 8 * 9 * 9 + 8 * 9 * 9 + 9 * 9 * 9 = 2673$

Thus $|S| = 2916$

2.2 Permutation of Sets

Let r be a positive integer. A **r-permutation** of a set S with n elements is an ordered arrangements of r of the n elements

- If $r > n$ then $P(n, r) = 0$
- $P(n, 1) = n$
- $P(n, 0) = 1$

Theorem 2.2.1: $P(n, r) = n * (n - 1) * \cdots * (n - r + 1)$

Proof: We can choose the first item n ways, the second item $n - 1$ ways, ..., the rth item $n - (r - 1)$ ways.

By multiplication principle we get $P(n, r) = n * (n - 1) * \cdots * (n - r + 1)$

Example: the number of ways to order 26 letters such that no 2 vowels occur consecutively

Solution: There are 21 consonants $\implies$ 21! permutations of consonants

Vowels must be in 5 of the spaces before, between, or after consonants $P(22, 5) = \frac{22!}{17!}$

Thus number of arrangements is $21! * \frac{22!}{17!}$

Example: How many 7 digit numbers are there with distinct digits and don't have $5, 6$ appearing consecutively in either order

Solution 1: We count various partitions and sum their orders together

- S_1: neither $5, 6$ appear $\implies P(7, 7) = 7! = 5040$
- S_2: 5 appears but not 6 $\implies 7 * P(7, 6) = 7 * 7! = 35280$
- S_3: 5 appears but not 5 $\implies 7 * P(7, 6) = 7 * 7! = 35280$
- S_4: Both $5, 6$ appear
  - First digit is 5 and second digit is not 6 $\implies 5 * P(7, 5) = 126000$
  - Last digit is 5 and second to last digit is not 6 $\implies 5 * P(7, 5) = 126000$
  - 5 occupies one of the interior 5 places so 6 can appear only in 4 places $\implies 5 * 4 * P(7, 5) = 50400$

Thus answer is 151200

Solution 2: Consider the set of all digits $P(9, 7) = 181400$

Let $\bar{S}$ be the complement of the set we need.

- There are 6 ways to position 5 followed by 6
- There are 6 ways to position 6 followed by 5

Thus $|\bar{S}| = 2 * 6 * P(7, 5) = 30240$

Thus $|S| = |U| - |\bar{S}| = 181400 - 30240 = 151200$

Theorem 2.2.2: Number of circular r-permuations is

$$\frac{P(n,r)}{r} = \frac{n!}{r(n-r)!}$$

Proof: Set of linear r-permutations can be partitioned into parts such that 2 linear r-permutations correspond to the same r-permutation if and only if they are in the same part (these are of size r)

This means that the number of circular permuations is the number of parts

Thus by the division principle, the number circular permutations is $\frac{P(n,r)}{r}$

Note: one way to view circular permutations of A, B, C, D, E, F is to fix A and identify the linear permutations of B, c, d, e, f

- In this case, there are 5! circular permutations of A, B, C, D, E, F

Example: Consider seating 10 people around a round table but 2 people don't want to sit next to each other. How many seating arrangements are there?

Solution 1: Let $P_1, \ldots, P_{10}$ be the people and suppose P_1, P_2 don't want to sit next to each other

Consider seating arrangements for 9 people: $X, P_3, \ldots, P_{10}$. There are 8! arrangements for this

If we replace X with P_1, P_2 or P_2, P_1 we get the complement of the set we want

Thus the number of seating arrangements is $9! - 2 * 8! = 282240$

Solution 2: Let P_1 be the head of the table. P_2 cannot be next to P_1

Thus there are 8 choices for P_1's left and 7 choices for P_1's right

Finally we permute the remaining 7 seats

Thus answer is $8 * 7 * 7! = 282240$

2.3 Combinations of Sets

Combination: unordered selection of elements in S

- If $r > n$, $\binom{n}{r} = 0$
- If $r > 0$, $\binom{0}{r} = 0$
- $\binom{n}{0} = \binom{0}{0} = 1$
- $\binom{n}{1} = n$
- $\binom{n}{n} = 1$

Theorem 2.3.1: For $0 \leq r \leq$

$$P(n,r) = r!\binom{n}{r}, \qquad \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Proof: Each r-permutation of S arises exactly one way as a result of

- Choose r elements of S
- Ordering these elements

First step is done by $\binom{n}{r}$

Second step is done by $P(r, r) = r!$

Thus $P(n,r) = r!\binom{n}{r} \implies \binom{n}{r} = \frac{n!}{r!(n-r)!}$

Example: 15 people are enrolled in math class but only 12 students attend class per day. If there are 25 seat, how many ways might the instructor see the students?

Solution: $\binom{15}{12} = \frac{15!}{12!3!} = 455$ combinations of 12 students attending class

For 12 students and 25 seats, there are $P(25, 12)$ ways of seating themselves

Thus there are $\binom{15}{12}P(25, 12)$ ways the instructor might see 12 students

Example: How many 8 letter words are there if each word has $3, 4$, or 5 vowels

Solution:

- $|S_3| = \binom{8}{3} * 5^3 * 21^5$
- $|S_4| = \binom{8}{4} * 5^4 * 21^4$
- $|S_5| = \binom{8}{5} * 5^5 * 21^3$

Thus $|S| = |S_3| + |S_4| + |S_5|$

Corollary 2.3.2: For $0 \leq r \leq n$

$$\binom{n}{r} = \binom{n}{n-r}$$

Theorem 2.3.3 (Pascal's Formula): for integers n, k such that $1 \leq k \leq n-1$

$$\binom{n}{k} = \binom{n}{k} + \binom{n-1}{k-1}$$

Proof: Let $|S| = n$ and take $x \in S$

Consider $S \setminus \{x\}$ and partition set X of k-subsets of S where into

- A: k-subsets without x. This has size $\binom{n-1}{k}$
- B: k-subsets with x. This has size $\binom{n-1}{k-1}$

Thus $|X| = \binom{n}{k} = |A| + |B| = \binom{n}{k} + \binom{n-1}{k-1}$

Theorem 2.3.4: For $n \geq 0$

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

Proof: We use double counting. First, $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$ is the number of subsets of S

Secondly, for a subset of S, we need to decide if $x_1, x_2, \ldots, x_n$ goes into the subset or not. Thus there are 2^n to form a subset of S

Thus the Theorem holds since we counted values on both sides of the equation

Example: Show that the number of 2-subsets of $\{1, 2, \ldots, n\}$ is $\binom{n}{2}$

Solution: Partition the 2-subsets according to the largest largest integer they contain

The number of 2-subsets in which i is the largest integer is $i - 1$

Thus we must have $0 + 1 + \cdots + (n-1) = \binom{n}{2} = \frac{n(n-1)}{2}$