MATH475: Combinatorics and Graph Theory

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Contents

1 Basic Methods

1.1 Addition and Subtraction

Theorem 1.1 - Addition Principle: If A, B are 2 disjoint finite sets, then $|A \cup B| = |A| + |B|$

Proof: Both sounds count the number of elements in $A \cup B$

LHS directly counts the number of elements whereas RHS counts the number of elements in A and the number of elemnets in B

Since A, B are disjoint, LHS equals RHS

Theorem 1.2 - Generalized Addition Principle: Let $A_1, \ldots, A_n$ be disjoint, finite sets. Then $|A_1 \cup \cdots \cup A_n| = |A_1| + \cdots + |A_n|$

Proof: similar to the proof of Theorem 1.1, both sides count the number of elements in $A_1 \cup \cdots \cup A_n$.

Since these sets are djsoint, LHS equals RHS

Theorem 1.4 - Subtraction Principle: Let A be a finite set and $B \subseteq A$. Then $|A - B| = |A| - |B|$

Proof: First we show that $|A - B| + |B| = |A|$. Note that $A - B, B$ are disjoint and their union is A

Both sides count the number of elements in A

- LHS first counts the elements not in B then those in B
- RHS counts the elements directly

Thus $|A - B| + |B| = |A| \implies |A - B| = |A| - |B|$

- **Note**: We must have $B \subseteq A$ otherwise their union has elements NOT in A

1.2 Multiplication

Theorem 1.6 - Product Principle: Let X, Y be finite sets. The number of pairs (x, y) satisfying $x \in X$ and $y \in Y$ is $|X| \times |Y|$

Proof: There are $|X|$ choices for x, each of which has $|Y|$ choices for y

Theorem 1.8 - Generalized Product Principle: Let $X_1 \dots, X_k$ be finite sets. The number of k-tuples $(x_1, \dots, x_k)$ satisfying $x_i \in X_i$ is $|X_1| \times \cdots \times |X_k|$

Proof by Induction: Base case clearly holds for $k = 1$. Base case for $k = 2$ by Theorem 1.6

IH: Assume the statement holds for $k - 1$

IS: Prove the statement for k

$(x_1, \dots, x_k)$ can be decomposed into an ordered pair $((x_1, \dots x_{k-1}), x_k)$ which has $x_i \in X_i$

The number of elements satisfying the $(k - 1)$ tuple, by IH is $|X_1| \times \cdots \times |X_{k-1}|$.

The number of elements satisfying $x_k \in X_k$ is $|X_k|$.

Thus by the product principle, the number of k-tuples satisfying the condition is $(|X_1| \times \cdots \times |X_{k-1}|) \times |X_k|$

Example: How many 4-digit positive integers both start and end on an even number

- first digit $\in \{2, 4, 6, 8\}$
- second digit $\in \{0, \dots, 9\}$
- third digit $\in \{0, \dots, 9\}$
- foruth digit $\in \{0, 2, 4, 6, 8\}$

Thus answer is $4 * 10 * 10 * 5 = 2000$

Corollary 1.11: The number of k-letter strings over an n-element alphabet A is n^k

Proof: Apply Theorem 1.8 with $X_1 = X_2 = \cdots = A$

Note: Notationwise, $[n] = \{1, 2, \dots, n\}$

Theorem 1.15: For an $n \in Z^+$, the number of ways to arrange all elements of $[n]$ is $n!$

Proof: There are n ways to select the first element, $n - 1$ ways to select the second element, ...

Applying the Product Principle, we get the desired result $n!$

Definition - Permutation: List of each elements in S that appear exactly once

Theorem 1.17: Let $n, k \in Z^+$ such that $n \geq k$. Then the number of ways to make a k-element list from $[n]$ without repeating any elements is

$$(n)_k = (n)(n-1) \cdots (n-k+1)$$

Proof: n choices for the first element, ..., $n - k + 1$ choices for the kth element

Example: If we go north, we can visit 4 out of 10 schools. If we go south, we can visit 5 out of 8 schools. Assuming we can only go one way, how many different itineraries can we set up?

$(10)_4 + (8)_5 = 5040 + 6720 = 11760$

1.3 Division

Definition - d-to-One Function: Let S, T be a finite sets and d be a fixed integer. Then a function $f : T \to S$ is a d-to-one function if for each $s \in S$, there are d elements in T such that $f(t) = s$

Theorem 1.21 - Division Principle: Let S, T be finite sets such that $f : T \to S$ is d-to-one. Then $|S| = \frac{|T|}{d}$

Proof: results from the definition of d-to-one functions

Example: Number of different seatings for n people at a circular table is $(n-1)!$

If the table were linear, then there are $n!$ arrangements.

Let T be the number of arrangements on a linear table and S be the arrangements around a circular table.

Each $s \in S$ corresponds to n different $t \in T$

Clearly $f : T \to S$ is n-to-one so by Division Principle $|S| = \frac{|T|}{n} = (n-1)!$

Theorem 1.23: Let $n \in Z^+$ and $k \leq n$ be non-negative. Then the number of k-element subsets of $[n]$ is

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

Proof: The number of ways to make a k-element list from $[n]$ is $(n)_k$

Since each k-element subset has $k!$ ways of being listed, each k-subset will be counted $k!$ times in $(n)_k$

Thus by Division Principle, the number of k-subsets is $\frac{(n)_k}{k!}$

Definition - Binomial Coefficients: Values of $\binom{n}{k}$

Theorem 1.24 Binomial Theorem: Let $n \in Z^+$. Then $(x+y)^n = \sum_{n=0}^{n} \binom{n}{k} x^k y^{n-k}$

Proof: LHS is the product of $(x+y)$ n times

RHS takes a term (x or y) from each of the n factors and multiplies the selected terms (2^n possibilities) and add all of the 2^n sums

The sum $x^k y^{n-k}$ appears $\binom{n}{k}$ times since we chose x from k factors ($\binom{n}{k}$ ways of doing this) and y receives the remaining factors

Example: Given 110 bus lines and a machine that punches either 2 or 3 holes on a ticket within some of the 9 numbered squares, can a city set up machines such that each line will punch the tickets differently?

$\binom{9}{2} + \binom{9}{3} = 36 + 84 = 120 > 110$. Thus the city can punch the ticket differently for each bus line

1.4 Applications of Basic Counting Principles

1.4.1 Bijection

Definition - Bijection: A map $f : S \to T$ is called a **bijection** if it is one-to-one and onto

Corollary 1.28: Let S, T be finite sets. If a bijection $f : S \to T$ exists, then $|S| = |T|$

Proof: Follows from the Division Principle with $d = 1$

Example: Consider the possible lattice paths from $(0,0)$ to $(6,4)$ moving only eastward and northward.

- Number of ways to reach $X = (6,4)$ is $\binom{10}{6}$

- Number of ways to stop at $Y = (4, 2)$ and then $X = (6, 4)$ is $\binom{6}{4}\binom{4}{2}$
- Number of ways stop at $U = (3, 2)$ and $X = (6, 4)$ or stop at $V = (2, 3)$ and $X = (6, 4)$ is $\binom{5}{3}\binom{5}{3} + \binom{5}{2}\binom{5}{4}$

The calculation above works because there is a bijection between the set S of lattice paths and the set T of six-element subsets of $[10]$

Proposition 1.29: For $n \in Z^+$, the number of divisors of n greater than $\sqrt{n}$ is equal to the number divisors less than $\sqrt{n}$

Proof: Let S be the set of divisors of n larger than $\sqrt{n}$ and T be the set of divisors less than $\sqrt{n}$

Define $f : S \to T$ by $f(s) = n/s$

- For all $s \in S$, $s \cdot f(s) = n \implies f(s) \mid n$ and $f(s) < \sqrt{n} \implies f(s) \in T$. Thus f is a function from S into T
- Show that f is one-to-one
  - For all $t \in T$, there is at least one $s \in S$ such that $f(s) = t$, namely $s = n/t$
  - On the other hand, if $f(s) = t$, there is only one good s since $s \cdot f(s) = n \implies st = n \implies s = n/t$

Thus we have shown f is a bijection and thus $|S| = |T|$

Upshot: Steps to show a bijection exists

1. Define a function f from S into T
2. Show that for all $s \in S$, $f(s) \in T$ holds
3. Show that for all $t \in T$, there is only one $s \in S$ that satisfies $f(s) = t$
   - Show there at least one s satisfying $f(s) = t$
   - Show that there is at most one s satisfying $f(s) = t$

Lemma 1.32: The number of lattice paths from $(0, 0)$ to (n, n) that never go above the line $x = y$ is equal to the number of ways to fill a $2 \times n$ (Standard Young Tableaux) grid such that each row and column is increasing (right, down)

Proof: Let S be the set of all lattice paths from $(0, 0)$ to (n, n) that do not go above the line $y = x$ and T be the set of all Standard Young Tableaux

- Take $s \in S$. Let $e_1, e_2, \ldots, e_n$ denote the positions of the n east steps of s and $n_1, n_2, \ldots, n_n$ denote the n north steps of s

  The ith east/north step always occur before the (i+1)th east/north step. Thus rows are horizontally increasing.

  We also have $n_j < e_j$ since otherwise we would be above the main diagonal. Thus columns are also increasing.

  Thus we have shown that for all $s \in S, f(s) \in T$ holds
- Take $t \in T$ If there is an $s \in S$ such that $f(s) = t$, then s has to be the lattice path whose east steps correspond to the first row of t and whose north steps correspond to the second row of t

  On the other hand, this lattice path s never goes above the main diagonal by the increasing property of the columns

Thus we have shown that f is a bijection

1.4.2 Binomial Coefficients

Proposition 1.35: For $k \leq n$, $\binom{n}{k} = \binom{n}{n-k}$

Proof 1: $\binom{n}{k}$ can be looked as all possible lattice paths from $(0, 0)$ to $P = (k, n - k)$

Similarly, $\binom{n}{n-k}$ can be looked as all possible lattice paths from $(0, 0)$ to $Q = (n - k, k)$

We can define a bijection using a reflection over $y = x$

This will map OP paths to OQ paths in a bijective fashion

Proof 2: LHS counts the number of k-element subsets of $[n]$, and RHS counts the number of $n - k$-element subsets of $[n]$

Let S be the set of k-subsets and T be the set of $n - k$-subsets

We can define the mapping $f(A) = A^c$ which takes the complement of $A \in S$

For all $A \in S$, clearly $f(A)$ is a $n-k$ element subset. Thus $f(A) \in T$

Furthermore, if $B \in T$, then there is exactly one $A \in S$ satisfying $f(A) = B$, namely $A = B^c$

Thus f is a bijection from S to T. Thus $|S| = |T|$

Note: $2^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k}$

Note: $\binom{n}{k}$ form the nth row of Pascal's Triangle

Theorem 1.36: $\dbinom{n}{k} + \dbinom{n}{k+1} = \dbinom{n+1}{k}$

Proof 1: RHS counts the number of lattice paths to $R = (k+1, n-k)$

LHS counts the number of paths to R via $U = (k, n-k)$ or $V = (k+1, n-k-1)$

Proof 2: RHS counts the number of $k+1$-element subsets of $[n+1]$

LHS counts the number of $k+1$-element subsets WITH and WITHOUT the new element $n+1$. The remaining k come from $[n]$

Example 1.38: $\displaystyle\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$

Solution: We can look at this problem as trying to form a committee of n people from n partners and n associates, with a president who is on the committee and a partner

RHS selects one president from n partners (n ways of doing this). Then selects the remaining $n-1$ members on the committee ($\binom{2n-1}{n-1}$ ways of doing this)

LHS selects k partners ($\binom{n}{k}$ ways of doing this), selects a president from these selected partners (k ways of doing this), and then selects the remaining $n-k$ associates ($\binom{n}{n-k} = \binom{n}{k}$ ways of doing this)

Example 1.39: For $n \geq 2, n(n-1)2^{n-2} = \displaystyle\sum_{k=2}^{n} \binom{n}{k} k(k-1)$

Solution: We can view this problem as selecting a committee of at least 2 people (including a president and a vice president) from a group of n people

LHS selects the president and the vice president ($n(n-1)$ ways of doing this), and then looks at the possible subsets of people from a group of $n-2$ people (2^{n-2} possible subsets)

RHS selects a committee of k people ($\binom{n}{k}$ ways of doing this), and then selects a president and vice president from this committee ($k(k-1)$ ways of doing this)

1.4.3 Permutation with Repetition

Theorem 1.41: Suppose we want to arrange n objects in a line with k different types of objects that are indistinguishable from each other. Let a_i be the number of objects of type i. Then the total number of arrangements is

$$\frac{n!}{a_1!a_2!\cdots a_k!}$$

Proof 1: If we ignore the object types, then there are $n!$ ways to arrange n objects

We can permute each object amongst its own type. Thus for any arrangement, there are $a_1!a_2!\cdots a_k!$ identical arrangements

Thus using the division principle, we see that the total number of indistinguishable arrangements is $\frac{n!}{a_1!a_2!\cdots a_k!}$

Proof 2: The arrangement of n objects with k different types is determined by the positions of a_1 objects of type 1, a_2 objects of type 2, ...

There are $\binom{n}{a_1}$ choices for positioning objects of type 1, $\binom{n-a_1}{a_2}$ choices for positioning objects of type 2, ..., $\binom{a_k}{a_k} = 1$ choices for positioning objects of type k

Thus we see

$$\binom{n}{a_1}\binom{n-a_1}{a_2}\cdots\binom{a_k}{a_k} = \frac{n!}{a_1!a_2!\cdots a_k!}$$

Definitino - Multinomial coefficient: $\displaystyle\binom{n}{a_1, a_2, \ldots, a_k} = \frac{n!}{a_1!a_2!\cdots a_k!}$

- **NOTE**: $\binom{n}{a_1,a_2} = \binom{n}{a_1} = \binom{n}{a_2}$

Example: Suppose a person wants to visit 4 factories A, B, C, D twice over 8 days. How many different orders can the person visit, with the restriction that he doesn't visit factory A on 2 consecutive days

Solution: Ignoring the restriction on factory A, we have $\displaystyle\binom{8}{2,2,2,2} = 2520$

To consider the restriction on A, let's treat A and A' as one single unit. Thus we have $\displaystyle\binom{7}{1,2,2,2} = 630$

Thus there are $2520 - 630 = 1890$ possible orderings that meet the criteria

1.5 Pigeonhole Principle

Theorem 1.44 Pigeonhole Principle: Let $A_1, A_2, \ldots, A_k$ be pairwise disjoint finite sets and $|A_1 \cup A_2 \cup \cdots \cup A_k| > kr$. Then there exists at least one index i such that $|A_i| > r$

Proof by Contradiction: Assume that for all i, $|A_i| \leq r$. Then we have $|A_1 \cup A_2 \cup \cdots \cup A_k| = |A_1| + |A_2| + \cdots + |A_k| \leq kr$ which contradicts are assumption

Thus we must have at least one index i such that $|A_i| > r$

Example: Consider the sequence $a_i = 2^i - 1$ and let q be an odd integer. Then the sequence has an element divisible by q

Solution: Consider the first q elements of the sequence. If one is divisible by q, we are done

Otherwise consider the remainder of these elements mod q

$$a_i = d_i q + r_i$$

Since $r_i \in (0, q)$, by PHP, there are at least 2 elements a_n, a_m that share the same remainder

Thus $a_n - a_m = (d_n - d_m)q = (2^n - 1) - (2^m - 1) = 2^m(2^{n-m} - 1) = 2^m a_{n-m}$

Since 2^m and q are relatively prime, we must have that $q \mid a_{n-m}$

Example: Suppose we select $n+1$ distinct integers from $[2n]$. Then

- There is at least one pair that has sum $2n+1$
- There is at least one pair that has difference of n

Solution:

- Split $[2n]$ into n subsets $\{i, 2n+1-i\}$. Since we chose $n+1$ elements, by PHP 2 of the elements must lie in the same subset. Thus clearly $i + 2n + 1 - i = 2n + 1$
- Split $[2n]$ into n subsets $\{i, n+i\}$. Since we chose $n+1$ elements, by PHP 2 of the elements must lie in the same subset. Thus clearly $n + i - i = n$

2 Application of Basic Methods

2.1 Multiset/Composition

Definition - Multiset: collection that allows for repetition of elements

- A multiset is determined by the multiplicities of each element

Definition - Weak Composition: Let $a_1, a_2, \ldots, a_k \geq 0$ such that $\displaystyle\sum_{i=1}^{k} a_i = n$. Then the ordered tuple $(a_1, a_2, \ldots, a_k)$ is a **weak composition** of n into k parts

- **Note**: There is a bijection between weak compositions of $[n]$ into k parts AND n-element multisets over a k-element set

Theorem 2.2: The number of weak compositions of $[n]$ into k parts is $\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$

Proof: Consider n balls distributed across k boxes. Each distribution is equivalent to a weak composition

To count the ways of distributing balls, we can insert a wall between each box i and $i+1$

- Each distribution corresponds with arrangement of n balls and $k-1$ walls
- Conversely, each arrangement corresponds to a unique distribution of balls

Thus the number of ways to arrange n balls and $k-1$ walls is $\binom{n+k-1}{k-1}$

$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$ follows from Proposition 1.35

Definition - Composition: Let $a_1, a_2, \ldots, a_k \in Z^+$ such that $\displaystyle\sum_{i=1}^{k} a_i = n$. Then the ordered tuple $(a_1, a_2, \ldots, a_k)$ is a **composition** of n into k parts

- **Note**: compositions involve only positive integers, whereas weak compositions allow numbers to be 0

Corollary 2.5: The number of compositions of n into k parts is $\binom{n-1}{k-1}$

Proof: We show a bijection exists from W, the set of weak compositions of $n-k$ into k parts, into C, the set of compositions of n into k parts.

Simply add an additional element to each part, assuring that each part will have a positive size. Thus the bijection shows $|W| = |C|$

From Theorem 2.2, we see that $|W| = \binom{n-k+k-1}{k-1} = \binom{n-1}{k-1} = |C|$

2.2 Set Partitions

Definition - Blocks: Let $k \leq n$ and let $B = \{B_1, \ldots, B_k\}$ where $B_i \subseteq [n]$ and each B_i are non-empty and pairwise disjoint where $\bigcup_{i=1}^{k} B_i = [n]$. Then B is a partition of $[n]$ into k **blocks**

Example: Find the number of partitions of [5] into 3 blocks

Solution: Possible block sizes are 3-1-1 or 2-2-1

- Possibilities of selecting 3 blocks is $\binom{5}{3} = 10$ ways. The last 2 singleton blocks are automatically determined. Thus this results in 10 possible arrangements
- 5 choices for singleton blocks, and $\binom{4}{2}$ choices for the first doubleton. The last doubleton is automatically determined but we need to divide by 2 to consider double counting. Thus this results in 15 possible arrangements

Thus we have total number of arrangements is 25

Definition - Stirling Number of the Second Kind: Number of partitions of $[n]$ into k blocks denoted $S(n,k)$

- If $k > n$ then $S(n, k) = 0$
- If $n > 0$ then $S(n, 0) = 0$
- $S(0, 0) = 1$
- $S(n, 1) = S(n, n) = 1$

Theorem 2.10: For $n \geq k, S(n, k) = S(n-1, k-1) + kS(n-1, k)$

Proof: LHS counts all partitions of $[n]$ into k blocks

RHS counts 2 classes of arrangements

- The element n is by itself. Then we look at $S(n-1, k-1)$, the number of partitions of $[n-1]$ into $k-1$ blocks
- The element n is NOT by itself. The remaining $n-1$ elements are partitioned in k blocks ($S(n-1, k)$). And then there are k ways to place the element n

Let $h(n, k)$ be the sum of all $\binom{n-1}{k-1}$ products that consist of $n-k$ factors such that all of these factors are elements of $[k]$

Examples:

- For $n = 4, k = 2$, h(4, 2) = 1 *1 + 1*2 + 2 *2 = 7 = S(4, 2)$
- For $n = 4, k = 3, h(4, 3) = 1 + 2 + 3 = 6 = S(4, 3)$

Lemma 2.14: $h(n, k) = h(n-1, k-1) + kh(n-1, k)$

Proof: LHS is the sum of all $(n-k)$-factor products from $[k]$

RHS splits into 2 classes

- Those that contain the factor k. These also contain an $(n-k-1)$-factor product over the set $[k]$. Summing all of these products, we get $kh(n-1, k)$
- Those that do not contain k. This is handled by $h(n-1, k-1)$

Example: For $n = 4, k = 2, h(4, 2) = 1 * 1 + 2 * 1 + 2 * 2 = 1 * 1 + 2(1 + 2) = h(3, 1) + 2 * h(3, 2)$

Theorem 2.11: $h(n, k) = S(n, k)$

Proof by Induction:

Base case: $n + k = 0 \implies h(0, 0) = S(0, 0) = 1$ is clearly true

IH: Assume if $n + k \leq m$, then $S(n, k) = h(n, k)$

IS: Show for $n + k = m + 1$. Then we have

$$\begin{aligned} S(n, k) &= S(n-1, k-1) + kS(n-1, k) \\ &= h(n-1, k-1) + h(n-1, k) \\ &= h(n, k) \end{aligned}$$

Theorem 2.16: For $n \geq k, S(n+1, k) = \sum_{i=0}^{n} \binom{n}{i} S(n-i, k-1)$

Proof: LHS counts the number of partitions of $[n+1]$ into k blocks

RHS counts the number of partitions of $[n+1]$ into k blocks where the element $n+1$ is in a block of size $i+1$

- $\binom{n}{i}$ ways to choose i elements to share a block with the element $n+1$
- Then there are $S(n-i, k-1)$ ways to partition the remaining $k-1$ blocks

Bell Number: Number of all partitions of $[n]$, denote $B(n)$

n	$B(n)$
0	1
1	1
2	2
3	5
4	15
5	52
5	52
6	203
7	877
8	4140

Theorem 2.18: $B(n+1) = \sum_{k=0}^{n} B(k)\binom{n}{k}$

2.3 Partitions of Integers

Partition: finite sequence $(a_1, a_2, \ldots, a_k)$ of positive integers such that $a_1 \geq a_2 \geq \cdots \geq a_k$ and $a_1 + a_2 + \cdots + a_k = k$. Then this sequence is called a **partition** of the integer n

- Number of partitions of n is denoted $p(n)$

Example: For $n = 4$. There are 5 partitions

- $(4), (3,1), (2,2), (2,1,1), (1,1,1,1) \implies p(4) = 5$

Theorem 2.21: As $n \to \infty, p(n)$ satisfies

$$p(n) \approx \frac{1}{4\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$$

SKIPPING FERRERS SHAPES and EULER PENTAGONAL NUMBER THEOREM

2.4 Inclusion-Exclusion Principle

Lemma 2.32: $|A \cup B| = |A| + |B| - |A \cap B|$

Proof: LHS counts the number of elements in $|A \cup B|$

RHS also does the same but $|A| + |B|$ ends up double counting elements in both A AND B. Subtracting by $|A \cap B|$ corrects this anomaly

Example: Find the number of positive integers ≤ 300 that are divisible by 2 or 3

Solution: Number of eligible integers divisible by 2 is $300/2 = 150$

Number of eligible integers divisible by 3 is $300/3 = 100$

Number of eligible integers that were double counted is $300/6 = 50$

Thus answer is $|A \cup B| = 150 + 100 - 50 = 200$

Example: Suppose there are 30 guests that we want to split into 3 groups of 10. However, guest U cannot be in the same group as guest V, and guest X cannot be in the same group as guest Y. How many possible arrangements are there?

Solution: First count the bad partitions of [30] into 3 blocks. When $1, 2$ are in the same block, when $3, 4$ are in the same block, or when both events occur