

**k-Permutation:** arrangement of  $k$  elements from a set of  $n$  elements  $P(n, k) = \frac{n!}{(n-k)!}$

**Permutation With Repetition:** Can permute each object type  $a_i!$  times  $\binom{n}{a_1, a_2, \dots, a_k} = \frac{n!}{a_1! \dots a_k!}$

**Combination:** Total number of ways to create a  $k$ -element subset of  $[n]$   $\binom{n}{k} = \frac{P(n, k)}{k!}$

- This comes from being able to permute the  $k$ -subset  $k!$  ways

**Binomial Theorem:**  $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$

**Multinomial Theorem:**  $(x_1 + \dots + x_k)^n = \sum_{\substack{a_1 + \dots + a_k = n \\ a_1, \dots, a_k \geq 0}} \binom{n}{a_1, a_2, \dots, a_k} x_1^{a_1} \dots x_k^{a_k}$

**Pigeon Hole Principle:** If  $n$  pigeons are placed into  $k$  holes, then at least one hole has at least  $\lceil \frac{n}{k} \rceil$  (round up)

**Weak Composition:** Ordered  $k$ -tuple  $(a_1, \dots, a_k)$  such that  $a_i \geq 0$  and  $\sum_{i=1}^k a_i = n$   $\binom{n+k-1}{k-1}$

**Compositions:** Ordered  $k$ -tuple  $(a_1, \dots, a_k)$  such that  $a_i \geq 1$  and  $\sum_{i=1}^k a_i = n$   $\binom{n-1}{k-1}$

**Partition of  $[n]$ :**  $\{A_1, \dots, A_k\}$  such that blocks are pairwise disjoint and  $\bigcup_{i=1}^k A_i = X$   $S(n, k) = S(n-1, k-1) + kS(n-1, k)$

**Bell's Number:** Total number of partitions of  $[n]$  into any sized blocks  $B(n) = \sum_{k=1}^n S(n, k) = \sum_{i=1}^n \binom{n-1}{i-1} B(n-i)$

**Partition of  $n$ :**  $(a_1, \dots, a_k)$  such that  $a_1 \geq \dots \geq a_k$  and  $\sum_{i=1}^k a_i = n$  total:  $p(n)$   $k$ -parts:  $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$

- Represented using **Ferrers Diagram:** partial rectangular grid with  $k$  rows, each with  $a_i$  squares (conjugate is also valid)

## Twelvefold Way Counting

- $n$  labeled balls into  $k$  labeled bins:  $k^n$   $k!S(n, k)$   $P(n, k)$
- $n$  unlabeled balls into  $k$  labeled bins:  $\binom{n+k-1}{k-1}$   $\binom{n-1}{k-1}$   $\binom{k}{n}$
- $n$  labeled balls into  $k$  unlabeled bins:  $\sum_{i=1}^k S(n, i)$   $S(n, k)$  1
- $n$  unlabeled balls into  $k$  unlabeled bins:  $\sum_{i=1}^k p_i(n)$   $p_k(n)$  1

**Inclusion-Exclusion Principle:**  $|\bigcup_{i=1}^n A_i| = |X| - \sum_{I \subseteq [n]} (-1)^{|I|} |A_I| = |\bigcap_{i=1}^n \bar{A}_i| = |X| - \sum_{I \subseteq [n]} (-1)^{|I|} |A_I|$

**OGF:**  $F(x) = \sum_{n=0}^{\infty} a_n x^n$   $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$   $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

**Power Series Formulas:**  $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$      $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$      $(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$      $\sum_{n=1}^{\infty} n x^{n-1} = \left( \sum_{n=0}^{\infty} x^n \right)' = \frac{1}{(1-x)^2}$

**OGF:**  $\sum_{n=0}^{\infty} a_n x^n$      $(AB)(x) = \sum_{n=0}^{\infty} \left( \sum_{i=0}^n a_i b_{n-i} \right) x^n$     **EGF:**  $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$      $(AB)(x) = \sum_{n=0}^{\infty} \left( \sum_{i=0}^n \binom{n}{i} a_i b_{n-i} \right) \frac{x^n}{n!}$

• **Even Permutation EGF:**  $\frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{(2n)}}{(2n)!}$     **Odd Permutation EGF:**  $\frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{(2n+1)}}{(2n+1)!}$

**Weak Compositions OGF:**  $\frac{1}{(1-x)^k} = (1+x+\dots)(1+x+\dots)\dots = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n$

**Stirling Number OGF:**  $\frac{x^k}{(1-x)(1-2x)\dots(1-kx)} = \sum_{n=0}^{\infty} S(n, k) x^n$

**Partitions OGF:**  $\frac{1}{(1-x)(1-x^2)\dots} = \sum_{n=0}^{\infty} p(n) x^n$      $\frac{x^k}{(1-x)(1-x^2)\dots} = \sum_{n=0}^{\infty} p_k(n) x^n$

**Permutations EGF:**  $(1+x)^m = \sum_{n=0}^{\infty} P(m, n) \frac{x^n}{n!}$

**Stirling Number EGF:**  $\frac{(e^x - 1)^k}{k!} = \sum_{n=0}^{\infty} S(n, k) \frac{x^n}{n!}$     **Bell Number EGF:**  $e^{(e^x - 1)} = \sum_{n=0}^{\infty} B(n) \frac{x^n}{n!}$

**Catalan Numbers:**  $C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}$     **OGF Catalan Numbers:**  $C_n = \frac{\binom{2n}{n}}{n+1}$      $C_0 = 0$

**Vertex Induced Subgraph:** When  $u, v \in V(H)$  and  $u \sim v \in E(G)$ , then  $u \sim v \in E(H)$

**Complete Graph:**  $K_n$  has  $E(K_n) = \{\{v_i, v_j\} \mid 1 \leq i \neq j \leq n\}$  has  $\binom{n}{2}$  edges

**Complete Bipartite Graph:**  $K_{a,b}$  where partites  $A, B$  have sizes  $a, b$  and every vertex in  $A$  is adjacent to a vertex in  $B$

**Theorem:**  $G$  is bipartite if and only if  $G$  has no odd cycles

**Theorem:**  $G$  with size  $m$  has  $\sum_{v \in V(G)} \deg(v) = 2m$     **Corollary:**  $G$  must have an even number of odd degree vertices

**Theorem:** There exists a  $d$ -regular graph on  $n$  vertices if and only if at least one of  $d, n$  is even

**Theorem:** For any graph  $G$ , there exists a  $d$ -regular graph  $G$  such that  $G$  is an induced subgraph of  $H$

**Degree Sequence:** Non-increasing sequence of length  $n$  whose  $i$ th term is the degree of vertex  $i$

**Theorem:**  $G$  with degrees  $d = d_1, \dots, d_n$  exists if and only if  $s_1 = d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n$  is graphical

**Theorem:** Every tree on 2 or more vertices has at least 2 leaves

**Theorem:**  $G$  is a tree  $\iff G$  is connected, acyclic with  $n - 1$  edges  $\iff$  there is a unique path for  $u, v \in V(G)$

**Theorem:** A graph of order  $n$  has at least  $n - 1$  edges

**Theorem:** An edge  $e$  is a bridge if and only if  $e$  isn't in any cycles

**Spanning Tree:** Tree  $T$  such that  $V(T) = V(G)$  and  $E(T) \subseteq E(G)$

**Spanning Tree to Code:** Delete lowest index leaf and write down vertex adjacent to it. Repeat until only an edge remains

**Code to Spanning Tree:** Find smallest index  $b_1$  not used and create  $a_1 \sim b_1$ . Delete  $a_1$  and append  $b_1$ . Repeat until  $b_1, \dots, b_{n-2}$  then connect missing 2 indices

**Theorem:** Each Prufer code corresponds to a unique tree. Thus number of spanning trees of  $K_n$  is  $n^{n-2}$

**Corollary:** Total trees such that vertex  $i$  has degree  $d_i$  is  $\binom{n-2}{d_1-1, d_2-1, \dots}$

**Rooted Plane Tree:** Tree with a root vertex, left/right ordering, but vertices are NOT labeled

- Clockwise walk around border of the tree reveals that the number of rooted plane trees on  $n + 1$  vertices is  $C_n = \frac{\binom{2n}{n}}{n + 1}$

**Rooted Forest:** Forest where each tree component has a distinguishable root vertex

**Theorem:** Number of labeled rooted forests on  $n$  vertices is  $(n + 1)^{n-1}$