

MATH475: Combinatorics and Graph Theory

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1 Chapter 1

1.1 The Basics - Permutations, Combinations, and General Counting

1.1.1 Permutation

Definition - Permutation: A **permutation** of an n -element set is an arrangement of the elements in a specific order

- k -permutation: arrangement of k elements from the set
- The total number of k -permutation of an n -element set is

$$P(n, k) = \frac{n!}{(n-k)!} = n(n-1) \cdots (n-k+1)$$

Example: 10 people can run for office for a committee with a President, Vice President, and Treasurer. The total number of possible committees is $P(10, 3)$

Definition - k th Falling Factorial of n : Let $n \geq k$, both positive integers. Then the **k th Falling Factorial of n** is $(n)_k = n(n-1) \cdots (n-k+1)$

Example - Permutations with Repetitions: How many rearrangements of $AAAABBBCCD$ are there?

- If we were only looking at distinct elements, there would be $10!$ factorial. However, the elements aren't distinct

As an example, look at how we can reorder the A 's

$$A_1 A_2 A_3 A_4 B B B C C D = A_2 A_1 A_3 A_4 B B B C C D = \cdots$$

Thus there are $4!$ of these representations that are equivalent and we divide $10!$ by $4!$ to account for this. Similar idea for the other letters

Thus the number of rearrangements is $\frac{10!}{4!3!2!1!}$

Theorem: Suppose object 1 occurs a_1 times, object 2 occurs a_2 times, ..., object k occurs a_k times. Furthermore, suppose $a_1 + \cdots + a_k = n$

Then the total number of arrangements is $\frac{n!}{a_1! \cdots a_k!} = \binom{n}{a_1, a_2, \dots, a_k}$

- **Definition - Multinomial Coefficients:** The notation above is called a multinomial coefficient

1.1.2 Combinations

Definition - Combinations: The total number of ways to create a k -element subset from $[n] = \{1, 2, \dots, n\}$ is denoted

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{P(n, k)}{k!}$$

- **Note:** the definition comes from $n!$ total possible distinct permutations of which we can permute a k -element subset $k!$ ways

Example: There are 5 cats, 5 dogs, and 4 mice. 3 are chosen at once

- How many total number of ways to get 2 cats, 1 dog?

$\binom{5}{2} \binom{5}{1}$. This comes from choosing 2 cats from 5 cats, and then choosing 1 dog from 5 dogs

- How many total ways to get at least 1 cat?

$\binom{14}{3} - \binom{9}{3}$. This comes from subtracting the possible groupings with no cats from the total number of possible groupings

$= \binom{5}{1}\binom{9}{2} + \binom{5}{2}\binom{9}{1} + \binom{5}{3}$. This comes from enumerating over possible groupings with 1 cat and 2 other animals, 2 cats and 1 other animal, and 3 cats and no other animals

Binomial Theorem: If n is a positive integer then

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

Proof: $(x + y)^n = (x + y) \cdots (x + y)$

The coefficients of $x^k y^{n-k}$ is the product of n terms distributing from k x -terms and $n - k$ y -terms

So out of the n terms that are multiplied together, $\binom{n}{k}$ contribute the x -term, leaving $n - k$ unused terms for y

Thus we have

$$\binom{n}{k} x^k \binom{n-k}{n-k} y^{n-k} = \binom{n}{k} x^k y^{n-k}$$

Multinomial Theorem: For $n \in \mathbb{Z}^+$, we have

$$(x_1 + \cdots + x_k)^n = \sum_{\substack{a_1 + \cdots + a_k = n \\ a_1, \dots, a_k \geq 0}} \binom{n}{a_1, a_2, \dots, a_k} x_1^{a_1} \cdots x_k^{a_k}$$

Proof: $(x_1 + \cdots + x_k)^n = (x_1 + \cdots + x_k) \cdots (x_1 + \cdots + x_k)$

The coefficients of $x_1^{a_1} \cdots x_k^{a_k}$ is the product of n terms resulting from a permutation with repetition of a_1, \dots, a_k

Thus there are $\binom{n}{a_1, \dots, a_k}$ distinct ways of representing the exponents of x_1, \dots, x_k

Worksheet Examples:

1. • How many strictly increasing 6-digit numbers are there (first digit can be 0)

Since the numbers are strictly increasing, they numbers must be distinct

Furthermore, there is only one possible way ordering of these selected digits (in increasing order), so order “doesn’t matter”

Thus we can just count the ways to select 6 digits: $\boxed{\binom{10}{6}}$

- What about when the first digit cannot be zero?

Since 0 can only appear as the first digit, this is identical to the first question, but restricted to $\{1, \dots, 9\}$

$\boxed{\binom{9}{6}}$ ways

2. A quiz has 5 problems with 3 different answers each. How many ways can the quiz be completed?

$\boxed{3^5}$ ways

3. • How many rearrangements of MISSISSIPPI are there?

$$\boxed{\binom{11}{4, 4, 2, 1}}$$

- How many rearrangements do not have all S's together?

Treat all S's as one unit $\Rightarrow \binom{8}{4, 2, 1, 1}$

Thus there are $\boxed{\binom{11}{4, 4, 2, 1} - \binom{8}{4, 2, 1, 1}}$ rearrangements

4. Suppose each digit in a 5-digit code can have any number zero to nine

- How many 5-digit codes are there with no restrictions?

$$\boxed{10^5}$$

- How many 5-digit codes have distinct digits?

$$\boxed{P(10, 5)}$$

- How many codes begin and end with an even number?

$$\boxed{5 * 10^4 * 5}$$

- How many 5-digit codes use exactly 2 different numbers?

$$\binom{5}{2} \text{ ways to choose two digits}$$

2^5 ways to write a five digit number using these two digits. However, we need to subtract the 2 cases where they use the same digit five times

Thus answer is $\boxed{\binom{5}{2}(2^5 - 2)}$

5. Consider a standard 52 card deck with 13 ranks and 4 suits

- Probability of getting a full house (triple of the same rank, pair of another rank)

$$\boxed{\frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}}$$

- Probability of getting 2 pairs of different ranks and a 5th card of a different rank

$$\boxed{\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2} * 44}{\binom{52}{5}}}$$

Logic is to choose 2 types of ranks AT THE SAME TIME and then choosing possible pairs from those ranks. Finally multiply by 44 (the remaining number of possible cards for the 5th card)

Note: the answer below is NOT correct but is added to show an easy mistake to make

$$\frac{\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{2} * 44}{\binom{52}{5}} \text{ is WRONG because we need to consider duplicate pairs (e.g. QQ, KK and KK, QQ)}$$

This solution ends up overcounting since we can choose Queens then Kings versus Kings then Queens, which yields the same hand

6. 15 people want to buy pre-made sandwiches. There are 7 H sandwiches, 4 C sandwiches, and 4 P sandwiches. If each person gets 1 sandwich, how many ways can the sandwiches be distributed?

$$\boxed{\binom{15}{7,4,4}}$$

7. 10 people are sitting at a circular table. How many seating arrangements are there?

$$\frac{10!}{10} = \boxed{9!}$$

1.2 Counting in 2 Ways

Given an equation, we count combinatorially in 2 ways

Example: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

LHS: chooses a k -size subset from $[n]$

RHS: Partitions subsets into 2 cases, containing element 1 or not containing 1

- Containing 1: Pick $k-1$ elements from the remaining $n-1$ elements $\binom{n-1}{k-1}$
- Not containing 1: Pick k from $n-1$ elements $\binom{n-1}{k}$

Thus we have $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

Example: $\sum_{k=0}^n k \binom{n}{k} = n2^{n-1}$

RHS: Picks 1 element to be the leader (n choices)

Then from the remaining $n-1$ people, pick a subset (2^{n-1} ways)

LHS: Picks a group of size k ($\binom{n}{k}$ ways) and then select a leader amongst them (k possible choices)

Thus $\sum_{k=0}^n k \binom{n}{k} = n2^{n-1}$

Example: $\sum_{i=1}^n i(n-i+1) = \binom{n+2}{3}$

LHS: Let a, b, c be numbers with $a < b < c$ from $[n+2]$. Fix $b = i+1$ (the i th $+1$ element). Then there are i choices for the element a and $(n+2) - (i+1) = n-i+1$ choices for the element c

Thus the total ways to create a 3-size subset $\{a, b, c\}$ from $[n+2]$ is $\sum_{i=1}^n i(n-i+1) = \binom{n+2}{3}$

RHS: Counts the number of subsets of size 3 from $[n+2]$

Example: $\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k}^2$

LHS: Count the number of size n subsets of $[2n]$

RHS: Can be written as $\binom{n}{k} \binom{n}{n-k}$

So choose k people from a group of size n and $n - k$ people from another group of size n

Thus $\binom{n}{k} \binom{n}{n-k} = \binom{2n}{n}$

1.3 Pigeon Hole Principle

Theorem - Pigeon Hole Principle: If $n + 1$ pigeons are placed into n pigeonholes, then there is at least one pigeonhole with at least 2 pigeons

Theorem - Generalized Pigeon Hole Principle: If n pigeons are placed into k holes, then at least one hole has

- At most $\lfloor \frac{n-1}{k} \rfloor$ pigeons (round down)
- At least $\lceil \frac{n}{k} \rceil$ pigeons (round up)
- **Note:** the statements above are equivalent

Proof: Suppose all holes have $\leq \lfloor \frac{n-1}{k} \rfloor$ pigeons

Then there are at most $k \lfloor \frac{n-1}{k} \rfloor \leq n-1 < n$ pigeons total. Which is a contradiction

Example: Prove that in a group of 40 people, at least 4 people have a same birthday in the same month

40 pigeons and 12 holes $\implies \lceil \frac{40}{12} \rceil = 4$. Thus by PHP, the statement holds

Example: Prove that if 6 distinct numbers are chosen from $[9]$, then 2 numbers will sum to 10

6 pigeons and 5 holes $(\{(1,9), \dots, (5,5)\} \implies \lceil \frac{6}{5} \rceil = 2$. Thus by PHP, 2 of the numbers belong to the same pair and sum to 10

Example; An athlete wants to work out 45 hours over a 30-day month and will work out at least 1 hour each day. Prove that there is a period of consecutive days such that the cumulative hours he has worked out is exactly 14

Let a_i be the total hours accumulated up to day i

This creates a strictly increasing sequence $a_1, a_2, \dots, a_{30} = 45$ where each a_i is distinct

Define a new sequence $b_i = a_i + 14$

This creates a strictly increasing sequence $b_1, b_2, \dots, b_{30} = 59$ where each b_i is distinct

Since $\{a_i\}_{i=1}^{30}$ is distinct and $\{b_i\}_{i=1}^{30}$ is distinct (only within their own respective sequence), we can have at most 60 possible values, which we treat as pigeons

Furthermore, b_i can take on any of 59 distinct values, which we treat as the holes

Thus by PHP, some $a_j = b_k = a_k + 14$

Thus from day $k + 1$ to day j , the total number of hours worked out is 14

1.4 Compositions and Set Partitions

Definition - Weak Composition: Take $a_1, \dots, a_k \geq 0$ such that $\sum_{i=1}^k a_i = n$. Then the ordered k -tuple (a_1, \dots, a_k) is a **weak composition** of n into k partitions

- **Note:** This can be seen as distributing n unlabelled balls into k labelled boxes

Theorem - Stars and Bars: The number of weak compositions of n into k parts is $\binom{n+k-1}{k-1}$

Proof:

Denote the n objects as stars. We model a distribution as follows:

Place k bars where stars to the right of each bar is how many stars the corresponding part contains

Note: we must begin with a bar. Thus the rest involves n stars and $k - 1$ bars

Thus we can choose $k - 1$ from these $n + k - 1$ objects, resulting in $\binom{n + k - 1}{k - 1}$

Each one of these combination generates a different weak composition

Example: How many ways can we distribute \$100 to 5 people?

- Can also be rephrased as how many non-negative integer solutions are there to $x_1 + \dots + x_5 = 100$

Using the Stars and Bars Theorem, we have $\binom{100 + 5 - 1}{5 - 1} = \binom{104}{4}$ ways

Example: Suppose we want to choose 3 lollipop flavors from 5 orange, 4 strawberry, and 3 grapes. How many ways can we choose 3 lollipops?

Here we want $x_O + x_S + x_G = 3$ where $x_O, x_S, x_G \geq 0$

Thus $\binom{3 + 3 - 1}{3 - 1}$

Definition - Compositions: Same as weak compositions but with the restriction that each $x_i \geq 1$

We can derive this with the formula above but ensuring that each of the k boxes starts with 1 star, and then we distribute the remaining $n - k$ stars in a weak composition fashion. Thus the number of compositions of n into k parts is

$$\binom{n - k + k - 1}{k - 1} = \binom{n - 1}{k - 1}$$

Note: This problem is placing unlabelled balls into labelled bins

Definition - Partition: Let X be a finite set and A_1, \dots, A_k be non-empty subsets of X . Then a **partition** of X , $\{A_1, \dots, A_k\}$ satisfies

- $\bigcup_{i=1}^k A_i = X$
- Each A_i, A_j are pairwise disjoint

Note: This problem can be stated as: for a given k , X is partitioned into k **blocks**

Example: $\{\{1, 2, 4\}, \{3, 5\}, \{6\}\}$ is a partition of $[6]$

The first inner set here can be considered placing balls 1, 2, 4 into its own (unlabelled) bin

Definition - Stirling Number of the 2nd Kind: The number of partitions of $[n]$ into k **blocks**, denoted $S(n, k)$

- $S(0, 0) = 1$ by convention
- $S(n, 0) = 0$ for $n \geq 1$ is a partition of $[6]$

Example: Use counting arguments to compute $S(n, 2)$

For each ball, we decide if it either it goes into block 1 or block 2, resulting in 2^n

However, we require that each block has ≥ 1 elements. Thus we need to subtract by 2 for the cases where block 1 has all balls or block 2 has all balls by 2 for the cases where block 1 has all balls or block 2 has all balls

Finally, we need to divide by 2 since the order of the blocks doesn't matter

$$\frac{2^n - 2}{2}$$

Theorem: For $n \geq k$, we have

$$S(n, k) = S(n-1, k-1) + kS(n-1, k)$$

Proof:

LHS: Counts the number of ways to partition $[n]$ into k blocks

RHS looks at partitions where

- $\{1\}$ is in its own bin. Then we need to partition $[n-1]$ into $k-1$ bins, resulting in $S(n-1, k-1)$
- $\{1\}$ is in a bin with other elements. Then we need to partition $[n-1]$ into k bins, and then choose a bin to place $\{1\}$ into, resulting in $kS(n-1, k)$

Theorem:
$$S(n, k) = \frac{1}{k!} \sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^n$$

Proof: will show later

Definition - Bell Number: Total number of partitions of $[n]$, denoted $B(n) = \sum_{k=1}^n S(n, k)$

- **Note:** $B(0) = 1$ by convention

Example: Take $n = 3$. Then we have

$$\{\{1, 2, 3\}\}, \{\{1\}, \{2\}, \{3\}\}, \{\{1\}, \{2, 3\}\}, \{\{2, 1, 3\}\}, \{\{3\}\{1, 2\}\}$$

Thus $B(3) = 5$

Theorem:
$$B(n) = \sum_{i=1}^n \binom{n-1}{i-1} B(n-i)$$

Proof:

LHS: counts the total number of partitions of $[n]$

RHS:

- Fix the n th element and determine the number of subsets of size i that contain this element: $\binom{n-1}{i-1}$ ways
- Partition the remaining $n-i$ elements: $B(n-i)$ ways

Thus we see that
$$B(n) = \sum_{i=1}^n \binom{n-1}{i-1} B(n-i)$$

Example: Consider $f: A \rightarrow B$ where $|A| = n$ and $|B| = k$ for $n \geq k$. What is the total number of surjective functions?

- Recall that f is surjective if for $\forall b \in B$, there exists at least one $a \in A$ such that $f(a) = b$

This problem can be viewed as partitioning A into $\{A_1, \dots, A_k\}$ and determining how they map to B , where $A_i = \{a \in A \mid f(a) = i\}$

Clearly the number of ways to partition $[n]$ into k parts is $S(n, k)$

However, we also need to consider the order of the boxes (since they determine which subsets map to which values of B). Thus we need to multiply by $k!$

Thus the final answer is $k!S(n, k)$

1.5 Integer Partitions

Definition - Partition of Integer n : Positive sequence of integers a_1, \dots, a_k , with $a_1 \geq a_2 \geq \dots \geq a_k$ such that

$$a_1 + a_2 + \dots + a_k = n$$

Total number of such sequences is denoted $p(n)$

- **Note:** Observe that since the values are non-increasing, the order “doesn’t matter” so we don’t need to distinguish $1 + 3 = 4$ from $3 + 1 = 4$
- **Note:** By convention, we have $p(0) = 1$

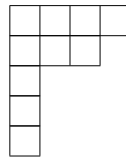
Example: Take $n = 4$. Then we have

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$$

Thus $p(4) = 5$

Definition - Ferrers Diagram: Represents a partition (a_1, \dots, a_k) by taking a partial rectangular grid with k rows whose rows contain a_i dots, such that $a_1 \geq \dots \geq a_k$

Example:

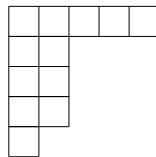


This corresponds to $4 + 3 + 1 + 1 + 1 = 10$

Definition - Conjugate of a Partition: Diagram created by reflecting a Ferrers Diagram along the diagonal

- **Note:** Can be seen as replacing columns as rows and rows as columns

Example: Conjugate of the previous example

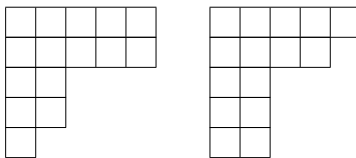


This corresponds to $5 + 2 + 2 + 1 = 10$

Theorem: Let P_1 denote the number of partitions where $a_1 = a_2 \geq a_3 \geq \dots \geq a_k$ (for unfixed k). Let P_2 be the number of partitions whose smallest part is at least 2. Then

$$P_1 = P_2$$

Proof: \implies Take a partition where $a_1 = a_2$. Then take its conjugate.



Clearly every part of the conjugate has length ≥ 2

Adding more dots to the original diagram doesn't affect the minimal lengths of the conjugate

Thus the conjugate assures each row has at least 2 dots \implies each part has at least size 2

\Leftarrow Given a partition with $a_k \geq 2$, the conjugate creates a partition with 2 rows that are the largest and have the same size. Thus $a_1 = a_2$ in the conjugate

Example: Show that the total number of partitions of n containing a part of size 1 is $p(n-1)$

\implies Take partition P of n containing a part of size 1

This part will be the bottom row. Deleting that row yields $p(n-1)$

\Leftarrow Given a partition P_2 of $n-1$, add a dot to the bottom. This yields a partition of n with a part of size 1

Definition - $p_k(n)$: Number of partitions of n into exactly k parts

Theorem: For $1 < k < n$

$$p_k(n) = p_{k-1}(n-1) + p_k(n-k)$$

Proof:

LHS: counts the number of partitions of n into k parts

RHS:

- Count the number of partitions such that there is a partition of size 1: $p_{k-1}(n-1)$
- Count the number of partitions such that every part has size ≥ 1 : $p_k(n-k)$

Note: Each part must have size ≥ 1 by definition

1.6 Twelfold Way

Goal is to distribute n (labelled or unlabelled) balls into k (labelled or unlabelled) bins, depending on if the bins

- Have no restriction
- Have at least 1 ball
- Have at most 1 ball

This results in 12 possible scenarios

1. n Labelled Balls into k Labelled Bins

- No restrictions: For each ball, pick a bin to put it into $\implies k^n$
- At least 1: Can be viewed as the number of surjective functions $\implies k!S(n, k)$
- At most 1:
 - If $n > k \implies 0$
 - Otherwise $n \leq k$ and we pick n bins in a specific order $\implies P(k, n)$

2. n Unlabelled Balls into k Labelled Bins

- No restrictions: Weak composition $\implies \binom{n+k-1}{k-1}$
- At least 1: Composition $\implies \binom{n-1}{k-1}$
- At most 1:

- If $n > k \implies 0$
- Otherwise $n \leq k$ and we just need to choose n bins to use $\implies \binom{k}{n}$

3. n Labelled Balls into k Unlabelled Bins

- No restrictions: Need to count all possible partitions with 0 to k parts $\implies \sum_{i=1}^k S(n, i)$
 - **Note:** This is NOT Bell's Number since we might have $k < n$
- At least 1: Partition $[n]$ into k parts $\implies S(n, k)$
- At most 1:
 - If $n > k \implies 0$
 - Otherwise $n \leq k$ and the bins are indistinguishable $\implies 1$ way

4. n Unlabelled Balls into k Unlabelled Bins

- No restrictions: need to count all possible partitions with 0 to k parts $\implies \sum_{i=1}^k p_i(n)$
- At least 1: Partition n into k parts $\implies p_k(n)$
- At most 1:
 - If $n > k \implies 0$
 - Otherwise $n \leq k$ and the bins are indistinguishable $\implies 1$ way

Definition - Multiset: Set of elements such that elements may repeat

Example: How many multisets of size 5 can be created using [7]?

Weak compositions for $1, 2, \dots, 7$ such that the number of balls in each bin totals up to 5 $\implies \binom{5+7-1}{7-1}$

Example: How many injective functions $f : [n] \rightarrow [k]$ are there?

n labelled balls into k labelled bins where each bin gets at most 1 $\implies P(k, n)$

Example: 20 kids split into 4 study groups. 2 brothers must be in the same group and 2 sisters must be in the same group. What is the total number of ways to split the kids

Treat the brothers and sisters each as one unit then partition $[18]$ into 4 bins $\implies S(18, 4)$

1.7 Inclusion-Exclusion Principle

Theorem: Let $A_1, \dots, A_n \subseteq X$ where X is finite and each A_i is non-empty. Also let

- $I \subseteq [n]$
- $A_I = \bigcap_{i \in I} A_i$ with $A_\emptyset = X$

Then

$$\left| \bigcup_{i=1}^n A_i \right| = |X| - \sum_{I \subseteq [n]} (-1)^{|I|} |A_I|$$

Note: By De Morgan's Law, this is equivalent to

$$\left| \bigcap_{i=1}^n \bar{A}_i \right| = |X| - \sum_{I \subseteq [n]} (-1)^{|I|} |A_I|$$

Example: For $n = 2 \implies I = \emptyset, \{1\}, \{2\}, \{1, 2\}$

$$\begin{aligned}
\left| \bigcup_{i=1}^2 A_i \right| &= |X| - ((-1)^0 |X| + (-1)^1 |A_1| + (-1)^1 |A_2| + (-1)^2 |A_{\{1,2\}}|) \\
&= |A_1| + |A_2| - |A_1 \cap A_2|
\end{aligned}$$

Example: How many integer solutions satisfy

$$x_1 + x_2 + x_3 = 35 \quad \text{for } 0 \leq x_1, x_2, x_3 \leq 15$$

Consider the problem where $X_1 \geq 16$. We can place 16 balls into the first bin then perform weak composition. We now use the negation form of the theorem above

Let X be the set of all weak compositions. Now define

- $A_1 = \{ \text{weak compositions} \mid x_1 \geq 16 \}$
- $A_2 = \{ \text{weak compositions} \mid x_2 \geq 16 \}$
- $A_3 = \{ \text{weak compositions} \mid x_3 \geq 16 \}$

Now we see that

$$\begin{aligned}
|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| &= \sum_{I \subseteq [3]} (-1)^{|I|} |A_I| \\
&= \underbrace{\binom{35+3-1}{3-1}}_{I=\emptyset} - \underbrace{\binom{3}{1} \binom{35-16+3-1}{3-1}}_{I=\{1\}, \{2\}, \{3\}} + \underbrace{\binom{3}{2} \binom{35-32+3-1}{3-1}}_{I=\{1,2\}, \{1,3\}, \{2,3\}} - \underbrace{\binom{3}{3} (0)}_{\text{impossible since } 48 > 35}
\end{aligned}$$

Example: Count the total number of surjective functions $f : [n] \rightarrow [k]$, where $n \geq k$

Let X be all functions from $[n] \rightarrow [k]$ and let A_i be the set of functions that DO NOT map to $i \in [k]$. Then the total number of surjective functions is

$$\begin{aligned}
\left| \bigcap_{i=1}^k \bar{A}_i \right| &= \sum_{I \subseteq [k]} (-1)^{|I|} |A_I| \\
&= \sum_{i=0}^k \binom{k}{i} (-1)^i (k-i)^n \quad \text{out of } k, \text{ choose a set of image points not mapped to} \\
&= k! S(n, k)
\end{aligned}$$

Example: Consider the Euler Totient Function

$$\phi(n) = |\{i \in \mathbb{Z}^+ \mid 1 \leq i \leq n, \gcd(i, n) = 1\}|$$

Let $p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$, for $e_i \geq 1$ be the prime factorization

Let $A_i = \{a \in [n] \mid p_i \mid a\}$. Then we see value of $\phi(n)$ is

- Note $|A_1| = \frac{n}{p_1}$ how many numbers in $[n]$ have 2 as a factor
- Note $|A_{\{1,2\}}| = \frac{n}{p_1 p_2}$ how many numbers in $[n]$ have 2 AND 3 as factors

$$\begin{aligned}
\left| \bigcap_{i=1}^t \bar{A}_i \right| &= \sum_{I \subseteq [t]} (-1)^{|I|} |A_I| \\
&= \underbrace{n}_{I=\emptyset} - \left(\frac{n}{p_1} + \frac{n}{p_2} + \cdots + \frac{n}{p_t} \right) + \left(\frac{n}{p_1 p_2} + \frac{n}{p_2 p_3} + \cdots \right) - \left(\frac{n}{p_1 p_2 p_3} + \cdots \right) + \cdots \\
&= n \left(1 - \sum_{i=1}^t \frac{1}{p_i} + \sum_{1 \leq i \leq j \leq t} \frac{1}{p_i p_j} - \sum_{1 \leq i \leq j \leq k} \frac{1}{p_i p_j p_k} \right) \\
&= n \left(1 - \frac{1}{p_1} \right) \left(1 - \frac{1}{p_2} \right) \cdots \left(1 - \frac{1}{p_t} \right) \\
&= n \prod_{i=1}^t \left(1 - \frac{1}{p_i} \right)
\end{aligned}$$

Example: How many ways to seat 20 couples in a circular, unlabelled table with 40 seats such that each significant other doesn't sit with each other

Let A_i be the set of seatings where couple 1 is sitting next to each other. Then we see

$$\begin{aligned}
\left| \bigcap_{i=1}^{20} \bar{A}_i \right| &= \sum_{I \subseteq [20]} (-1)^{|I|} |A_I| \\
&= \sum_{i=0}^{20} \underbrace{\binom{20}{i}}_{\text{select couples}} (-1)^i \underbrace{\frac{(40-i)!}{40-i}}_{\text{glue couples}} \overbrace{2^i}^{\text{order of couples}}
\end{aligned}$$

2 Chapter 2

2.1 Review of Power Series

Definition - Formal Power Series: For a sequence $\{a_n\}_{n \geq 0}$, the **Formal Power Series**

$$F(x) = \sum_{n=0}^{\infty} a_n x^n$$

Is the **Ordinary Generating Function** of $\{a_n\}_{n \geq 0}$

Example: Let $\{a_n\}$ be given by

$$\frac{1}{2!}, \frac{1}{3!}, \dots$$

Then the generating function is

$$\begin{aligned}
F(x) &= \frac{1}{2!} + \frac{1}{3!}x + \frac{1}{4!}x^2 + \cdots \\
&= \frac{1}{x^2} \left(\frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \right) \\
&= \frac{1}{x^2} \sum_{n=2}^{\infty} \frac{x^n}{n!} \\
&= \frac{1}{x^2} (e^x - x - 1)
\end{aligned}$$

Same principle can be applied to find a sequence $\{a_n\}_{n \geq 0}$ given an OGF $F(x)$

Example: Find the sequence $\{a_n\}$ corresponding to

$$F(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{-4x + 3}{(1-x)(1-2x)}$$

Using partial fractions we see

$$\frac{A}{1-x} + \frac{B}{1-2x} = -4x + 3 = A(1-2x) + B(1-x) \implies A = 1, B = 2$$

Thus we see that

$$\begin{aligned} F(x) &= \frac{-4x + 3}{(1-x)(1-2x)} \\ &= \frac{1}{1-x} + \frac{2}{1-2x} \\ &= \sum_{n=0}^{\infty} x^n + 2 \underbrace{\sum_{n=0}^{\infty} (2x)^n}_{\sum 2^{n+1} x^n} \\ &= \sum_{n=0}^{\infty} (1 + 2^{n+1}) x^n \end{aligned}$$

Thus $\{a_n\}_{n \geq 0} = \{1 + 2^{n+1}\}$

2.1.1 OGF's and Recurrences

We can replace a_n with recursions for OGFs $\sum a_n x^n$

Example: Consider codes of length n using A, B, C, D

Let a_n be the total codes containing an even number of A's

1. Find a recurrence for a_n in terms of a_{n-1}

$$a_n = 3a_{n-1} + 4^{n-1} - a_{n-1}$$

This comes from

- Code doesn't end in A $\implies a_n = 3a_{n-1}$ (there are 3 possible letters for the last letter)
- Code ends in A $\implies 4n^{n-1} - a_{n-1}$ (All codes of length $n-1$ minus codes of length $n-1$ with an even number of A)

2. Find the OGF for $\{a_n\}$

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n x^n = \underbrace{1}_{a_0} + \sum_{n=1}^{\infty} a_n x^n \\ &= 1 + \sum_{n=1}^{\infty} (2a_{n-1} + 4^{n-1}) x^n \\ &= 1 + \sum_{n=1}^{\infty} 2a_{n-1} x^n + \sum_{n=1}^{\infty} 4^{n-1} x^n \\ &= 1 + 2x \underbrace{\sum_{n=1}^{\infty} a_{n-1} x^{n-1}}_{f(x)} + x \underbrace{\sum_{n=1}^{\infty} (4x)^{n-1}}_{\text{geometric sum}} \\ &= 1 + 2x(f(x) + x(\frac{1}{1-4x})) \end{aligned}$$

Thus $f(x) = \frac{1}{1-2x} + \frac{x}{(1-2x)(1-4x)}$

3. Find the closed form of a_n

By partial fractions, we have

$$\begin{aligned} f(x) &= \frac{1}{1-2x} + \frac{-1}{2(1-2x)} + \frac{1}{2(1-4x)} \\ &= \frac{1}{2} \left(\sum_{n=0}^{\infty} (2x)^n \right) + \frac{1}{2} \sum_{n=0}^{\infty} (4x)^n \\ &= \sum_{n=0}^{\infty} \underbrace{\left(\frac{2^n}{2} + \frac{4^n}{2} \right)}_{a_n} x^n \end{aligned}$$

Thus we have $\{a_n\} = \{2^{n-1} + \frac{4^n}{2}\}$

2.2 OGFs and Counting

Using $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we can differentiate to get

$$\begin{aligned} \frac{1}{(1-x)^2} &= \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + \dots = \sum_{n=0}^{\infty} (n+1) x^n \\ \frac{2}{(1-x)^3} &= \sum_{n=2}^{\infty} n(n-1) x^{n-2} = 1 + 2x + \dots = \sum_{n=0}^{\infty} (n+2)(n+1) x^n \\ &\dots \\ \frac{m!}{(1-x)^{m+1}} &= \sum_{n=0}^{\infty} \underbrace{(n+m)(n+m-1)\dots(n+1)}_{(n+m)_m} x^n = \sum_{n=0}^{\infty} \frac{(n+m)!}{n!} x^n \end{aligned}$$

Thus we have

$$\frac{1}{(1-x)^{m+1}} = \sum_{n=0}^{\infty} \frac{(n+m)!}{n!m!} x^n = \sum_{n=0}^{\infty} \binom{n+m}{m} x^n$$

For a fixed m , we have

$$\frac{1}{(1-x)^{m+1}} \iff \{a_n\} \quad a_n = \binom{n+m}{m}$$

Which is the OGF for weak compositions into $m+1$ bins. However, we can now interpret

$$\frac{1}{(1-x)^k} = \underbrace{\left(\frac{1}{1-x} \right) \left(\frac{1}{1-x} \right) \dots}_{\text{apply geometric series k times}} = (1+x+x^2+\dots)(1+x+x^2+\dots)\dots$$

Consider the question of getting coefficients of x^7 . This corresponds to the weak compositions of 7 unlabelled balls into k bins. However, now the chosen exponents correspond to the bin sizes. Thus we see that the OGF for weak compositions is

$$F(x) = \frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n$$

To obtain coefficients of a_n , contributions come from x^{n_1} from group 1, x^{n_2} from group 2, etc. such that

$$x^{n_1} x^{n_2} \dots x^{n_k} = x^n \implies n_1 + n_2 + \dots + n_k = n \quad n_i \geq 0$$

Example: 5 apple, 4 pear, and 6 orange lollipops. Find the OGF that would yield the total ways to get 5 lollipops. State the term that we want

$$F(x) = \underbrace{(1 + x + \cdots + x^5)}_{\text{apples}} \underbrace{(1 + x + x^4)}_{\text{pears}} \underbrace{(1 + x + \cdots + x^6)}_{\text{oranges}}$$

We want coefficients of x^5 in $F(x)$. Here the OGF is

$$\frac{1}{(1-x)^k} \longleftrightarrow \left\{ \binom{n+k-1}{k-1} \right\}$$

Theorem: Let k be a fixed positive integer. Then the OGF for Stirling Numbers $S(n, k)$ is

$$F_k(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{x^k}{(1-x)(1-2x) \cdots (1-kx)}$$

Proof: Note that

$$S(0, 0) = 1 \quad S(n, 0) = 0 \quad S(0, k) = 0$$

Also recall the recurrence

$$S(n, k) = S(n-1, k-1) + kS(n-1, k)$$

Then we see that

$$\begin{aligned} F_k(x) &= \sum_{n=0}^{\infty} S(n, k) x^n = 0 + \sum_{n=1}^{\infty} (S(n-1, k-1) + kS(n-1, k)) x^n \\ &= \left(x \sum_{n=1}^{\infty} S(n-1, k-1) x^{n-1} \right) + \left(kx \sum_{n=1}^{\infty} S(n-1, k) x^{n-1} \right) \\ &= xF_{k-1}(x) + kxF_k(x) \end{aligned}$$

Thus $F_k(x) = \frac{x F_{k-1}(x)}{(1-kx)}$ Finally, applying the above equation recursively, we see that

$$\begin{aligned} F_k(x) &= \frac{x}{1-kx} \frac{x}{1-(k-1)x} F_{k-2} = \cdots \\ &= \frac{x^k}{(1-x)(1-2x) \cdots (1-kx)} \end{aligned}$$

Thus the OGF is

$$\prod_{t=1}^k \frac{x}{(1-tx)} \longleftrightarrow \{S(n, k)\}$$

Theorem: Let $p(n)$ denote the total partitions of an integer n . Then the OGF for $p(n)$ is

$$F(x) = \sum_{n=0}^{\infty} p(n) x^n = \frac{1}{(1-x)(1-x^2) \cdots} = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

Proof:

$$\begin{aligned} F(x) &= \frac{1}{1-x} \frac{1}{1-x^2} \frac{1}{1-x^3} \cdots \\ &= \underbrace{(1 + x + x^2 + x^3 + \cdots)}_{\text{how many 1's}} \underbrace{(1 + x^2 + x^4 + x^6 + \cdots)}_{\text{how many 2's}} \underbrace{(1 + x^3 + x^6 + x^9 + \cdots)}_{\text{how many 3's}} \cdots \\ &= \prod_{k=1}^{\infty} (1 + x^k + x^{2k} + \cdots) \end{aligned}$$

Consider the question: how many combinations of x^n ?

For $x^{m(k)*k}$, $m(k)$ is the total number of k 's in the partition. For example, $m(1)$ is the total 1's in the partition

All possible contributions of x^n is exactly the total partitions of integer n

Example: Given an unlimited number of pennies, nickels, dimes, quarters, and half dollars, find the OGF that would count the total ways to make \$1. What is the coefficient?

We want coefficients of x^{100} where

$$F(x) = \underbrace{(1 + x + \cdots + x^{100})}_{\text{number of pennies}} \underbrace{(1 + x^5 + \cdots + x^{100})}_{\text{number of nickels}} \underbrace{(1 + x^{10} + \cdots + x^{100})}_{\text{number of dimes}} \underbrace{(1 + x^{25} + \cdots + x^{100})}_{\text{number of quarters}} \underbrace{(1 + x^{50} + x^{100})}_{\text{number of half dollars}}$$

Expanding (done on computer) yields $292x^{100} \implies 292$ ways

Theorem: Total ways to partition n into distinct parts equals the total ways to partition n into odd sized parts

Proof: Distinct parts OGF is

$$\begin{aligned} F(x) &= (1+x)(1+x^2) + \cdots && \text{this comes from each number only appears once} \\ &= \left(\frac{1-x^2}{1-x}\right)\left(\frac{1-x^4}{1-x^2}\right) \cdots \\ &= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^5}\right) \cdots \\ &= (1+x+x^2+\cdots)(1+x^3+x^6+\cdots) \end{aligned}$$

From above, we see that the even terms get cancelled and we are only left with the odd terms

Thus number of ways to partition n into distinct parts equals the total number of ways to partition n into odd sized parts

Theorem: OGF for $p_k(n)$, the total number of partitions into exactly k parts is

$$F(x) = \sum_{n=0}^{\infty} p_k(n)x^n = \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)} = (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)\cdots(x^k+x^{2k}+\cdots)$$

Proof: Consider the OGF

$$F(x) = \frac{1}{(1-x)(1-x^2)\cdots(1-x^k)} = (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)\cdots(1+x^k+x^{2k}+\cdots)$$

This counts the number of partitions of n where each part has size at most k

Looking at the Ferrer Diagram of this, we see that each row can have at most k dots

Thus the conjugate will have columns of size at most k , which establishes a bijection between counting the total partitions of n with at most k part

To manipulate this into an OGF for $p_k(n)$, we need a exactly k rows in the conjugate, and thus need a part of exactly size k in the original diagram

Thus we can create a new OGF that enforces an x^k term

$$F(x) = \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)} = (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)\cdots(x^k+x^{2k}+\cdots)$$

2.3 Exponential Generating Functions

Definition - EGF: Function for a sequence $\{a_n\}$

$$G(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

Recall that for OGFs $A(x) = \sum a_n x^n$ and $B(x) = \sum b_n x^n$, we have that

$$A(x)B(x) = \sum_{n=0}^{\infty} c_n x^n \quad c_n = \sum_{i=0}^n a_i b_{n-i} \implies \sum_{n=0}^{\infty} c_n x^n = (a_0 + b_0) + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \dots$$

Lemma: For OGFs $F(x) = \sum \bar{a}_n \frac{x^n}{n!}$ and $G(x) = \sum \bar{b}_n \frac{x^n}{n!}$, we have that

$$F(x)G(x) = \sum_{n=0}^{\infty} \bar{c}_n \frac{x^n}{n!} \quad \bar{c}_n = \sum_{i=0}^n \binom{n}{i} \bar{a}_i \bar{b}_{n-i}$$

Proof: let $a_n = \frac{\bar{a}_n}{n!}$ and $b_n = \frac{\bar{b}_n}{n!}$. Then we have

$$F(x)G(x) = \sum_{n=0}^{\infty} c_n x^n \quad c_n = \sum_{i=0}^n a_i b_{n-i} = \frac{1}{n!} \sum_{i=0}^n \binom{n}{i} \bar{a}_i \bar{b}_{n-i}$$

This last part is apparent from

$$\begin{aligned} c_n &= \sum_{i=0}^n \frac{\bar{a}_i}{i!} \frac{\bar{b}_{n-i}}{(n-i)!n} \\ &= \frac{1}{n!} \sum_{i=0}^n \frac{n!}{i!(n-i)!} \bar{a}_i \bar{b}_{n-i} \\ &= \frac{1}{n!} \sum_{i=0}^n \binom{n}{i} \bar{a}_i \bar{b}_{n-i} \end{aligned}$$

Example: For a fixed m , let $\{a_n\} = \left\{ \binom{m}{n} \right\}$. Find the EGF for this

We know that the OGF for $\{a_n\}$ is

$$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n = \sum_{n=0}^{\infty} \frac{m!}{n!(m-n)!} x^n$$

Thus this gives

$$\sum_{n=0}^{\infty} \frac{m!}{(m-n)!} \frac{x^n}{n!} = \sum_{n=0}^{\infty} P(m, n) \frac{x^n}{n!}$$

Thus the EGF is $\{P(m, n)\}$

Example: For a fixed k the EGF for $\{k^n\}$ is

$$\sum_{n=0}^{\infty} k^n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(kx)^n}{n!} = e^{kx}$$

Theorem: For a fixed k , the EGF for Stirling Numbers $S(n, k)$ is

$$G(x) = \sum_{n=0}^{\infty} S(n, k) \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}$$

Corollary: $B(n) = \sum_{k=0}^n S(n, k)$. Then the EGF is

$$G(x) = \sum_{n=0}^{\infty} B(n) \frac{x^n}{n!} = e(e^x - 1)$$

Proof: We know that

$$G(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} S(n, k) \right) \frac{x^n}{n!}$$

- **Note:** If $n < k$, then $S(n, k) = 0$

Applying the previous theorem, we see that

$$\begin{aligned} G(x) &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} S(n, k) \right) \frac{x^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} (e^x - 1)^k \\ &= e^{(e^x - 1)} \quad \text{Note : } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

Consider the EGF of permutations with repetitions of S_1, S_2, S_3

$$\left(\sum_{\substack{S_1 + S_2 + S_3 = n \\ S_1, S_2, S_3 \geq 0}} \frac{n!}{S_1! S_2! S_3!} \right) \frac{x^n}{n!}$$

What is the coefficient of all possible permutations of length n using A, B, C?

Combinatorically, we know that this is 3^n . Thus we see that the EGF is

$$G(x) = \sum_{n=0}^{\infty} 3^n \frac{x^n}{n!} = e^{3x}$$

Example: Find the EGF that counts the total number of permutations of length n containing an odd number of As, even number of Bs, and any number of Cs. Then find a closed form for the value

$$\begin{aligned}
& \underbrace{\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)}_A \underbrace{\left(1 + \frac{x^2}{2!} + \dots\right)}_B \underbrace{\left(1 + x + \frac{x^2}{2!} + \dots\right)}_C \\
&= \left(\left(1 + x + \frac{x^2}{2!} + \dots\right) - \left(1 - x + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \dots\right) \right) / 2 \\
&\quad * \left(\left(1 + x + \frac{x^2}{2!} + \dots\right) - \left(1 - x + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \dots\right) \right) / 2 \\
&\quad * \left(1 + x + \frac{x^2}{2!} + \dots\right) / 2 \\
&= \left(\frac{e^x - e^{-x}}{2} \right) \left(\frac{e^x + e^{-x}}{2} \right) (e^x) \\
&= \frac{e^{3x} - e^{-x}}{4} \\
&= \frac{1}{4} \left(\sum_{n=0}^{\infty} \frac{(3x)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \right) \\
&= \sum_{n=0}^{\infty} \frac{(3^n - (-1)^n) x^n}{4 n!}
\end{aligned}$$

Consider: What about multiplying other functions besides e^x

Recall that if $F(x) = \sum_{n=0}^{\infty} \bar{a}_n \frac{x^n}{n!}$, $G(x) = \sum_{n=0}^{\infty} \bar{b}_n \frac{x^n}{n!}$, the product had coefficients

$$\bar{c}_n = \sum_{i=0}^n \binom{n}{i} \bar{a}_i \bar{b}_{n-i}$$

This can be seen as separating $[n]$ into 2 blocks and assign group 1 to do something in \bar{a}_i ways and $n - i$ to do something in \bar{b}_{n-i} ways

Example: Find EGF and closed form a_n that counts total permutations of length n using A, B, C, D where the total number of A, B used is odd

EGF using odd numbers A, B in total is

$$G(x) = 2 \frac{x}{1!} + 2^3 \frac{x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{(2x)^{2k+1}}{(2k+1)!} = \frac{e^{2x} - e^{-2x}}{2}$$

EGF for permutations of C, D (with no restriction)

$$H(x) = \left(1 + 1 \frac{x}{1} + 1 \frac{x^2}{2!}\right) \left(1 + 1 \frac{x}{1} + 1 \frac{x^2}{2!}\right) = e^x e^x$$

Thus closed form EGF for total permutations is $\frac{e^{2x} - e^{-2x}}{2} e^x e^x = \frac{e^{4x} - 1}{2}$

To find a closed for total ways a_n , we have

$$\begin{aligned}
\frac{1}{2}(e^{4x} - 1) &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{(4x)^n}{n!} \\
&= \sum_{n=1}^{\infty} \frac{1}{2} 4^n \frac{x^n}{n!}
\end{aligned}$$

Thus $a_n = \frac{1}{2}4^n$

For a counting argument, for each good permutation, we can swap A, C to get a bad permutation or swap B, D to get a bad permutation. Thus exactly half of the permutations are good

Example: Place n distinct books into 3 bookshelves. What is the total number of ways to arrange the books such that each shelf has ≥ 1 book?

Counting argument: Look at all $n!$ permutations of books then split them into 3 bins $\implies n! \binom{n-1}{3-1}$

EGF argument: Let a_i be the total number of ways to place i books into shelf 1. Here $a_i = i!$

Thus for a single shelf, the EGF is $\sum_{i=1}^{\infty} a_i \frac{x^i}{i!} = \sum_{i=1}^{\infty} x^i$

To answer the original problem, we want coefficients of $\frac{x^n}{n!}$ in

$$\begin{aligned} \left(\sum_{i=1}^{\infty} x^i\right)^3 &= \left(\frac{1}{1-x} - 1\right)^3 \\ &= \left(\frac{x}{1-x}\right)^3 \\ &= x^3 \sum_{j=0}^{\infty} \underbrace{\binom{j+3-1}{3-1}}_{\text{weak comp into 3 parts OGF}} x^j \\ &= \sum_{j=0}^{\infty} \binom{j+3-1}{3-1} x^{j+3} \\ &= \sum_{n=3}^{\infty} \binom{n-1}{3-1} x^n \\ &= \sum_{n=3}^{\infty} n! \binom{n-1}{2} \frac{x^n}{n!} \end{aligned}$$

Thus the coefficients for the EGF is $n! \binom{n-1}{2}$

Example: EGF of D_n Derangements of $f: [n] \rightarrow [n]$

Let $D(x)$ be the EGF for D_n

$$D(x) = \sum_{n=0}^{\infty} D_n \frac{x^n}{n!}$$

Consider the EGF for all bijections where $f(i) = i$: $\sum_{n=0}^{\infty} 1 \frac{x^n}{n!} = e^x$ (only one way)

Now consider $D(x)e^x = (D_0 + D_1x + D_2\frac{x^2}{2!} + \dots)(1 + x + \frac{x^2}{2!} + \dots)$

- The coefficients of $\frac{x^n}{n!}$ is $\sum_{i=0}^n \binom{n}{i} D_i b_{n-i}$ (choose i to derange and $n-i$ to fix)
- This counts all possible bijections

Thus EGF of all bijections is

$$F(x) = \sum_{n=0}^{\infty} n! \frac{x^n}{n!} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Thus $\frac{1}{1-x} = D(x)e^x \implies D(x) = e^{-x} \frac{1}{1-x}$

To get the sequence a_n we have

$$\begin{aligned}
 D(x) &= e^{-x} \frac{1}{1-x} \\
 &= \left(\sum_{j=0}^{\infty} (-1)^j \frac{x^j}{j!} \right) \left(\sum_{i=0}^{\infty} x^i \right) \\
 &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \right) x^n \\
 &= \sum_{n=0}^{\infty} n! \sum_{k=0}^n \frac{(-1)^k}{k!} = D_n
 \end{aligned}$$

TODO LOOK AT THIS AGAIN

Example: How many ways can ≥ 1 people be split into any number of non-empty groups such that each group sits at an unlabelled circular table

EGF for 1 table with n people is

$$\begin{aligned}
 G(x) &= \sum_{n=1}^{\infty} a_n \frac{x^n}{n!} \\
 &= \sum_{n=1}^{\infty} (n-1)! \frac{x^n}{n!} \\
 &= \sum_{n=1}^{\infty} \frac{x^n}{n} \\
 &= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \\
 &= \int \sum_{x=0}^{\infty} x^n \\
 &= \int \frac{1}{1-x} dx \\
 &= \ln\left(\frac{1}{1-x}\right)
 \end{aligned}$$

Thus EGF for k tables is

$$\frac{\ln\left(\frac{1}{1-x}\right) \ln\left(\frac{1}{1-x}\right) \cdots \ln\left(\frac{1}{1-x}\right)}{k!} \quad (\text{need to consider overcounting since tables are unlabelled})$$

For any number of tables, we need to sum over k and want the coefficient of $\frac{x^n}{n!}$

$$\sum_{k=0}^{\infty} \frac{\left(\ln\left(\frac{1}{1-x}\right)\right)^k}{k!} e^{\ln\left(\frac{1}{1-x}\right)} = \frac{1}{1-x}$$

Thus we see that

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} n! \frac{x^n}{n!}$$

Thus there are $n!$ ways to split n people into any number of tables

TODO LOOK AT THIS AGAIN

2.3.1 Compositions of EGF

We want to partition $[n]$ into nonempty blocks and do task 1 to each block, then task 2 to the set of blocks

If $F(x)$ and $G(x)$ are the EGFs for each respective task, provided $F(x) = \sum_{n=1}^{\infty} a_n \frac{x^n}{n!}$, the total number of ways to partition $[n]$ into any number of blocks with task 1 on each block and task 2 on the set of blocks is

$$H(x) = G(F(x)) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!}$$

Example What if $F(x), G(x)$ are trivial, i.e. total ways to partition $[n]$ such that each block is represented in 1 way and the set of blocks is seen in 1 way

$$F(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1$$

$$G(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

$$H(x) = e^{e^x - 1}$$

Example: Given n people, count the ways to seat them at any number of tables where each table is served red or white wine
To seat i people at a table, the EGF was

$$F(x) = \sum_{n=1}^{\infty} (i-1) \frac{x^i}{i!} = \ln\left(\frac{1}{1-x}\right)$$

Now given k tables, EGF for ways to serve red or white wine is

$$G(x) = \sum_{k=0}^{\infty} 2^k \frac{x^k}{k!} = e^{2x}$$

Thus we see that

$$H(x) = G(F(x)) = e^{2 \ln(\frac{1}{1-x})} = \frac{1}{(1-x)^2} = \frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{d}{dx} \left(\sum_{n=0}^{\infty} x^n \right) = \sum_{n=0}^{\infty} (n+1) n! \frac{x^n}{n!}$$

Thus there are $(n+1)!$ total ways

2.4 Catalan Numbers

Given a $m \times n$ lattice, how many UP, RIGHT paths from $(0,0)$ to (m,n) are there?

- This requires $m+n$ total steps $\implies \binom{n+m}{m} = \binom{n+m}{n}$

Now consider $n \times n$ case that don't cross the diagonal

Let C_n be the total number of paths that don't cross the diagonal

Partition these into classes where the last time the diagonal is touched is (i,i) for $0 \leq i \leq n-1$

Once we hit (i,i) , we have C_{n-i-1} paths left. Thus the total paths is

$$c_n = \sum_{i=0}^{n-1} C_i C_{n-i-1} \quad c_0 = 1$$

Theorem: If C_n denotes the total UP/RIGHT paths from $(0,0)$ to (n,n) that don't go over the diagonal is

$$C_n = \sum_{n=0}^{\infty} C_i C_{n-i-1}$$

Moreover, the OGF is

$$F(x) = \frac{1 - \sqrt{1 - 4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n \quad C_n = \frac{\binom{2n}{n}}{n+1}$$

Proof:

$$\begin{aligned} F(x) &= 1 + \sum_{n=1}^{\infty} C_n x^n = 1 + \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} C_i C_{n-i-1} \right) x^n \\ &= 1 + x \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} C_i C_{n-i-1} \right) x^{n-1} \\ &= 1 + x(F(x))^2 \\ &= \frac{1 \pm \sqrt{1 - 4x}}{2x} \end{aligned}$$

For the algebra above, recall that for $f = \sum a_n x^n$ and $g = \sum b_n x^n$, we have

$$fg = \sum d_n x^n \quad d_n = \sum a_i b_{n-i}$$

Thus we see that

$$fg = \sum_{n=1}^{\infty} d_{n-1} x^{n-1} \implies \sum_{i=0}^{n-1} C_i C_{n-i-1} = F(x)^2$$

Additional algebra shows that

$$F(x) = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{n+1} x^n$$

2.4.1 Combinatoric Proof of Catalan Numbers

We know that the total number of UP/RIGHT paths to (n, n) is $\binom{2n}{n}$. We now remove the bad paths

Observe that the first time we touch a bad diagonal (diagonal above main diagonal) is after i RIGHTS and $i+1$ UPS

- This results in $n-i$ RIGHTS and $n-(i+1)$ UPS to reach (n, n)

We alter the remaining path after touching the bad diagonal by reflecting it over the bad diagonal (so all RIGHTS are now UPS)

Now beyond the diagonal, we go $n-i-1$ RIGHTS and $n-i$ UPS

Thus total RIGHTS is $i + (n-i-1) = n-1$ and total UPS is $i+1 + (n-i) = n+1$

Consider which paths to $(n-1, n+1)$ will touch the bad diagonal? All of them

Thus any bad path to (n, n) that goes over the main diagonal corresponds to a path reaching $(n-1, n+1)$, establishing a bijection

$$\binom{(n-1) + n + 1}{n-1} = \binom{2n}{n-1}$$

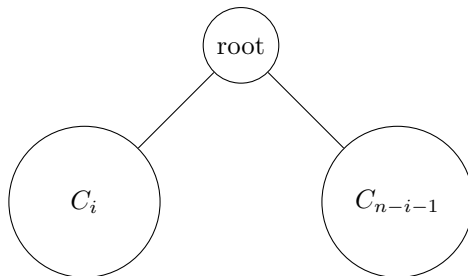
Thus the total paths (n, n) that don't go over the main diagonal is

$$\binom{2n}{n} - \binom{2n}{n-1} = \frac{\binom{2n}{n}}{n+1}$$

Example: Total ways to correctly group n pairs of parentheses

Treat (as RIGHT and) as UP. Then we have the Catalan number C_n

Example: Total number of binary trees with n internal nodes and $n + 1$ leaves is



The root is ignored and we recurse over the subcases. Thus there are

$$C_i = \sum_{i=0}^{n-1} C_i C_{n-i-1}$$

total trees

Example: Total ways to triangulate a regular (labelled) polygon with $n + 2$ sides i.e. total ways to draw $n - 1$ lines connecting 2 vertices such that no lines cross

To triangulate a polygon with $n + 2$ sides, we create a triangle using the line joining vertex $n + 1$ and $n + 2$ as the base

Assume that these vertices join with a vertex k for $1 \leq k \leq n$. This creates 2 sub-polygons

- Polygon formed by $n + 2, 1, 2, \dots, k \implies k + 1$ vertices $\implies C_{k-1}$
- Polygon formed by $k, k + 1, \dots, n + 1 \implies n - k + 2$ vertices $\implies C_{n-k}$

Thus total number of triangulations is

$$\sum_{k=1}^n C_{k-1} C_{n-k} = \sum_{n=0}^n C_k C_{n-k-1} = C_n$$

3 Graph Theory

3.1 Introduction to Graph Theory

Definition - (Simple) Graph G: ordered pair (V, E) where $V = V(G)$ is a set of **vertices** and $E = E(G)$ is a set of **edges** (2-size subsets of V)

Definition - Adjacent: Two vertices $v_1, v_2 \in V$ are **adjacent** if $\{v_1, v_2\} \in E$, denoted as $v_1 \sim v_2$

Definition - Incident: An edge $e = \{v_1, v_2\}$ is **incident** to a vertex v if v is one of the endpoints of e

Definition - Subgraph: Graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$

- **Spanning Subgraph** if $V(H) = V(G)$
- **Vertex Induced Subgraph** if whenever $u, v \in V(H)$ and $u \sim v \in E(G)$, then $u \sim v \in E(H)$
- **Subgraph Induced by S** if $S = V(H) \subseteq V(G)$, denoted $G[S]$

Definition - Walk: Sequence of vertices beginning at u and ending at v such that consecutive vertices are adjacent

- **Closed** if $u = v$
- **Length:** total edges traversed

Definition - Trail: Walk with no edge traversed more than once

Definition - Path: Walk with no vertex traversed more than once

- **Note:** path \implies trail \implies walk

Definition - Circuit: Closed trail of length 3 or more

- **Cycle:** Circuit with only the first and last vertex repeated
 - **Even Cycle** if total edges traversed is even
 - **Odd Cycle** if total edges traversed is odd

Definition - Connected Graph: Graph G such that for any distinct vertices u, v there is a path uv

Definition - Distance: Minimum length of a path uv , denoted $d(u, v)$

Definition - Diameter: Distance of the farthest apart vertices, denoted $\text{diam}(G) = \max_{u, v \in V(G)} \{d(u, v)\}$

Definition - Path Graph: Graph on n vertices, denoted P_n , with edge set $E(P_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}\}$

- **Note:** P_n will have $n - 1$ edges

Definition - Cycle Graph: Graph on n vertices, denoted C_n , with edge set $E(C_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_1\}, \{v_n, v_1\}\}$

- **Note:** C_n will have n edges

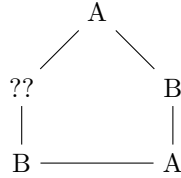
Definition - Complete Graph: Graph on n vertices, denoted K_n with edge set $E(K_n) = \{\{v_i, v_j\} \mid 1 \leq i \neq j \leq n\}$

- **Note:** $E(K_n)$ has size $\binom{n}{2}$

Definition - Complement: The **complement** of graph G , denoted \overline{G} , has vertex set $V(\overline{G}) = V(G)$ and edge set $E(\overline{G}) = \{v_i, v_j \mid v_i, v_j \notin E(G)\}$

Definition - Bipartite: G is **bipartite** if its vertex set can be partitioned into 2 subsets A, B , called **partite sets**: such that every edge of G has one endpoint in A and the other in B

Example



Above graph cannot be bipartite since the neighbors of ?? lie in A and B

Theorem: G is bipartite if and only if G contains no odd cycles

Proof: \implies By contraposition, assume G has an odd cycle $v_1, v_2, \dots, v_{2k-1}, v_1$

Then we have $v_1 \in A, v_2 \in B, \dots, v_{2k-1} \in A, v_1 \in A$. Thus G is not bipartite

\Leftarrow Let $u \in V(G)$ and assume G has no odd cycles

Define $A = \{w \mid d(u, w) \text{ is even}\}$ and $B = \{w \mid d(u, w) \text{ is odd}\}$

BWOC, let $w_1, w_2 \in B$ such that $w_1 \sim w_2$ (similar idea can be shown for A)

Consider the paths $u = v_0, v_1, \dots, v_{2s+1} = w_1$ and $u = u_0, u_1, \dots, u_{2t+1} = w_2$

Let x be the last vertex shared by the 2 paths

Necessarily, we must have that $x = v_i = u_i$. Otherwise a shorter path exists

But then we have $x = v_i, v_{i+1}, \dots, v_{2s+1} = w_1 \sim w_2$
 $\underbrace{\hspace{10em}}_{2s+1-i \text{ steps}} \quad \underbrace{\hspace{1em}}_{1 \text{ step}}$

Which yields a cycle of length $2(s + t - i + 1) + 1$, which is an odd cycle. Contradiction

TODO LOOK AT PROOF AGAIN

Definition - Complete Bipartite Graph with partites A, B of size a, b is denoted $K_{a,b}$ where every vertex in A is adjacent to every vertex in B

3.2 Vertex Degree

Definition - Vertex Degree: Total edges incident to v , denoted $\deg(v)$

Definition - Neighborhood: Set of vertices adjacent to v , denoted $N(v)$

Definition - Min [Max] Degree: Lowestest [highest] degree of G , denoted $\delta(G) [\Delta(G)]$

- **Note:** An isolated vertex has degree 0

Theorem: A Graph G with size m (m edges) has $\sum_{v \in V(G)} \deg(v) = 2m$

Proof: Summing over all vertices, every edge gets counted twice by the 2 endpoints

Corollary: Graph G must have an even number of vertices with odd degree

Definition - d-regular: Graph G where every vertex has degree d

Example: K_4 is 3-regular with 4 vertices



Non-Example: A 3-regular graph with 5 vertices doesn't exist by the Corollary above

Theorem: There exists a d -regular graph on n vertices if and only if at least one of d, n is even

Proof: \implies By Contraposition. If d, n are both odd, then a d -regular graph doesn't exist by the corollary

\impliedby If $d = 2k$, arrange the vertices in a cycle. Then v_i joins to k preceding and k succeeding vertices

Otherwise assume d is odd and n is even. We join edges as before, along with v_i joining to $v_{i+t \pmod n}$ where $n = 2t$ (the opposite vertex)

Example: TODO DRAW EXAMPLE OF INDUCING A NON-REGULAR GRAPH INSIDE A REGULAR GRAPH

Theorem: For a given graph G , there exists a d -regular graph H such that G is an induced subgraph of H

Proof: Shown in the example above

Definition - Degree Sequence: Given a graph G with n vertices, its **degree sequence** is a non-increasing sequence of length n whose i th term is the degree of vertex i

Theorem: Let d_1, \dots, d_n be a non-increasing sequence. There exists a graphical graph G with these degrees if and only if the sequence

$$d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, \underbrace{d_{d_1+2}, \dots, d_n}_{\text{no minus term}}$$

is graphical. That is, delete d_1 then delete 1 edge from the next d_1 terms (may require reordering the sequence)

Proof: \Leftarrow Given $d_2 - 1, \dots, d_n$ is graphical, we can add a new vertex v_1 and join v_1 to the first d_1 vertices in the sequence

\implies BWOC assume there is no v_1 with $\deg(v_1) = d_1$ that is adjacent to all vertices of the next highest degrees $d_2, d_3, \dots, d_{d_1+1}$

Among all graphs with the degree sequence d_1, \dots, d_n , take the one where the sum degree of vertices adjacent to v_1 is maximized

Necessarily, there is a vertex $v_s, s \notin \{2, 3, \dots, d_1 + 1\}$ adjacent to v_1 such that $\deg(v_s) < \deg(v_r)$ for $r \in \{2, 3, \dots, d_1 + 1\}$

But then some v_r must be adjacent to a vertex v_t that is NOT adjacent to v_1

Example: $(3, 3, 2, 1) \leftrightarrow (2, 2, 1, 1) \leftrightarrow (1, 0, 1) = (1, 1, 0)$