**k-Permutation**: arrangement of $k$ elments from a set of $n$ elements $\qquad P(n,k) = \dfrac{n!}{(n-k)!}$

**Permutation With Repetition**: Can permute each object type $a_i!$ times $\qquad \dbinom{n}{a_1, a_2, \ldots, a_k} = \dfrac{n!}{a_1! \cdots a_k!}$

**Combination**: Total number of ways to create a $k$-element subset of $[n]$ $\qquad \dbinom{n}{k} = \dfrac{P(n,k)}{k!}$

- This comes from being able to permute the $k$-subset $k!$ ways

**Binomial Theorem**: $(x+y)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

**Multinomial Theorem**: $(x_1 + \cdots + x_k)^n = \displaystyle\sum_{\substack{a_1+\cdots+a_k=n \\ a_1,\ldots,a_k \geq 0}} \binom{n}{a_1, a_2, \ldots, a_k} x_1^{a_1} \cdots x_k^{a_k}$

**Pigeon Hole Principle**: If $n$ pigeons are placed into $k$ holes, then at least one hole has at least $\lceil \dfrac{n}{k} \rceil$ (round up)

**Weak Composition**: Ordered $k$-tuple $(a_1, \ldots, a_k)$ such that $a_i \geq 0$ and $\displaystyle\sum_{i=1}^{k} a_i = n$ $\qquad \dbinom{n+k-1}{k-1}$

**Compositions**: Ordered $k$-tuple $(a_1, \ldots, a_k)$ such that $a_i \geq 1$ and $\displaystyle\sum_{i=1}^{k} a_i = n$ $\qquad \dbinom{n-1}{k-1}$

**Partition of [n]**: $\{A_1, \ldots, A_k\}$ such that blocks are pairwise disjoint and $\displaystyle\bigcup_{i=1}^{k} A_i = X$ $\qquad S(n,k) = S(n-1,k-1) + kS(n-1,k)$

**Bell's Number**: Total number of partitions of $[n]$ into any sized blocks $\qquad B(n) = \displaystyle\sum_{k=1}^{n} S(n,k) = \sum_{i=1}^{n} \binom{n-1}{i-1} B(n-i)$

**Partition of n**: $(a_1, \ldots, a_k)$ such that $a_1 \geq \cdots \geq a_k$ and $\displaystyle\sum_{i=1}^{k} a_i = n$ $\qquad$ total: $p(n)$ $\qquad$ $k$-parts: $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$

- Represented using **Ferrers Diagram**: partial rectangular grid with $k$ rows, each with $a_i$ squares (conjugate is also valid)

**Twelvefold Way Counting**

- $n$ labelled balls into $k$ labelled bins: $\quad k^n \qquad k!S(n,k) \qquad P(n,k)$
- $n$ unlabelled balls into $k$ labelled bins: $\dbinom{n+k-1}{k-1} \qquad \dbinom{n-1}{k-1} \qquad \dbinom{k}{n}$
- $n$ labelled balls into $k$ unlabelled bins: $\sum_{i=1}^{k} S(n,i) \qquad S(n,k) \qquad 1$
- $n$ unlabelled balls into $k$ unlabelled bins: $\sum_{i=1}^{k} p_i(n) \qquad p_k(n) \qquad 1$

**Inclusion-Exclusion Principle**: $\left|\bigcup_{i=1}^{n} A_i\right| = |X| - \displaystyle\sum_{I \subseteq [n]} (-1)^{|I|} |A_I| = \left|\bigcap_{i=1}^{n} \bar{A}_i\right| = |X| - \sum_{I \subseteq [n]} (-1)^{|I|} |A_I|$

**OGF**: $F(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

**Power Series Formulas**: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ $\quad \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ $\quad (1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$ $\quad \sum_{n=1}^{\infty} n x^{n-1} = \left(\sum_{n=0}^{\infty} x^n\right)' = \frac{1}{(1-x)^2}$

**OGF**: $\sum_{n=0}^{\infty} a_n x^n$ $\quad (AB)(x) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} a_i b_{n-i}\right) x^n$ $\quad$ **EGF**: $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ $\quad (AB)(x) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}\right) \frac{x^n}{n!}$

- **Even Permutation EGF**: $\frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{(2n)}}{(2n)!}$ $\quad$ **Odd Permutation EGF**: $\frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{(2n+1)}}{(2n+1)!}$

**Weak Compositions OGF**: $\frac{1}{(1-x)^k} = (1+x+\cdots)(1+x+\cdots)\cdots = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n$

**Stirling Number OGF**: $\frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} = \sum_{n=0}^{\infty} S(n,k) x^n$

**Partitions OGF**: $\frac{1}{(1-x)(1-x^2)\cdots} = \sum_{n=0}^{\infty} p(n) x^n$ $\quad \frac{x^k}{(1-x)(1-x^2)\cdots} = \sum_{n=0}^{\infty} p_k(n) x^n$

**Permutations EGF**: $(1+x)^m = \sum_{n=0}^{\infty} P(m,n) \frac{x^n}{n!}$

**Stirling Number EGF**: $\frac{(e^x-1)^k}{k!} = \sum_{n=0}^{\infty} S(n,k) \frac{x^n}{n!}$ $\quad$ **Bell Number EGF**: $e^{(e^x-1)} = \sum_{n=0}^{\infty} B(n) \frac{x^n}{n!}$

**Catalan Numbers**: $C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}$ $\quad$ **OGF Catalan Numbers**: $C_n = \frac{\binom{2n}{n}}{n+1}$ $\quad C_0 = 0$

**Vertex Induced Subgraph**: When $u, v \in V(H)$ and $u \sim v \in E(G)$, then $u \sim v \in E(H)$

**Path Graph**: $P_n$ has $E(P_n) = \{\{v_1, v_2,\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}\}$ has $n-1$ edges

**Cycle Graph**: $C_n$ has $E(C_n) = \{\{v_1, v_2,\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$ has $n$ edges

**Complete Graph**: $K_n$ has $E(K_n) = \{\{v_i, v_j\} \mid 1 \leq i \neq j \leq n\}$ has $\binom{n}{2}$ edges

**Complete Bipartite Graph**: $K_{a,b}$ where partites $A, B$ have sizes $a, b$ and every vertex in $A$ is adjacent to a vertex in $B$

**Theorem**: $G$ is bipartite if and only if $G$ has no odd cycles

**Theorem**: $G$ with size $m$ has $\sum_{v \in V(G)} \deg(v) = 2m$ $\quad$ **Corollary**: $G$ must have an even number of odd degree vertices

**d-Regular Graph**: Every vertex in $G$ has degree $d$

**Theorem**: There exists a $d$-regular graph on $n$ vertices if and only if at least one of $d, n$ is even

**Theorem**: For any graph $G$, there exists a $d$-regular graph $G$ such that $G$ is an induced subgraph of $H$

**Degree Sequence**: Non-increasing sequence of length $n$ whose $i$th term is the degree of vertex $i$

**Theorem**: $G$ with degrees $d = d_1, \ldots, d_n$ exists if and only if $s_1 = d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$ is graphical

**Theorem**: Every tree on 2 or more vertices has at least 2 leaves

**Theorem**: $G$ is a tree $\iff$ $G$ is connected, acyclic with $n-1$ edges $\iff$ there is a unique path for $u, v \in V(G)$

**Theorem**: An edge $e$ is a bridge if and only if $e$ isn't in any cycles

**Spanning Tree**: Tree $T$ such that $V(T) = V(G)$ and $E(T) \subseteq E(G)$

**Spanning Tree to Code**: Delete lowest index leaf and write down vertex adjacent to it. Repeat until only an edge remains

**Code to Spanning Tree**: Find smallest index $b_1$ not used and create $a_1 \sim b_1$. Delete $a_1$ and append $b_1$. Repeat until $b_1, \ldots, b_{n-2}$ then connect missing 2 indices

**Theorem**: Each Prufer code corresponds to a unique tree. Thus number of spanning trees of $K_n$ is $n^{n-2}$

**Corollary**: Total trees such that vertex $i$ has degree $d_i$ is $\binom{n-2}{d_1 - 1, d_2 - 1, \ldots}$

**Rooted Plane Tree**: Tree with a rot vertex, left/right ordering, but vertices are NOT labelled

- Clockwise walk around border of the tree reveals that the number of rooted plane trees on $n+1$ vertices is $C_n = \dfrac{\binom{2n}{n}}{n+1}$

**Rooted Forest**: Forest where each tree component has a distinguishable root vertex

**Theorem**: Number of labelled rooted forests on $n$ vertices is $(n+1)^{n-1}$