k-Permutation: arrangement of k elments from a set of n elements $P(n,k) = \dfrac{n!}{(n-k)!}$

Permutation With Repetition: Can permute each object type $a_i!$ times $\dbinom{n}{a_1, a_2, \ldots, a_k} = \dfrac{n!}{a_1! \cdots a_k!}$

Combination: Total number of ways to create a k-element subset of $[n]$ $\dbinom{n}{k} = \dfrac{P(n,k)}{k!}$

- This comes from being able to permute the k-subset $k!$ ways

Binomial Theorem: $(x+y)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Multinomial Theorem: $(x_1 + \cdots + x_k)^n = \displaystyle\sum_{\substack{a_1+\cdots+a_k=n \\ a_1,\ldots,a_k \geq 0}} \binom{n}{a_1, a_2, \ldots, a_k} x_1^{a_1} \cdots x_k^{a_k}$

Pigeon Hole Principle: If n pigeons are placed into k holes, then at least one hole has at least $\lceil \dfrac{n}{k} \rceil$ (round up)

Weak Composition: Ordered k-tuple $(a_1, \ldots, a_k)$ such that $a_i \geq 0$ and $\displaystyle\sum_{i=1}^{k} a_i = n$ $\dbinom{n+k-1}{k-1}$

Compositions: Ordered k-tuple $(a_1, \ldots, a_k)$ such that $a_i \geq 1$ and $\displaystyle\sum_{i=1}^{k} a_i = n$ $\dbinom{n-1}{k-1}$

Partition of [n]: $\{A_1, \ldots, A_k\}$ such that blocks are pairwise disjoint and $\displaystyle\bigcup_{i=1}^{k} A_i = X$ $S(n,k) = S(n-1,k-1) + kS(n-1,k)$

Bell's Number: Total number of partitions of $[n]$ into any sized blocks $B(n) = \displaystyle\sum_{k=1}^{n} S(n,k) = \sum_{i=1}^{n} \binom{n-1}{i-1} B(n-i)$

Partition of n: $(a_1, \ldots, a_k)$ such that $a_1 \geq \cdots \geq a_k$ and $\displaystyle\sum_{i=1}^{k} a_i = n$ total: $p(n)$ k-parts: $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$

- Represented using **Ferrers Diagram**: partial rectangular grid with k rows, each with a_i squares (conjugate is also valid)

Twelvefold Way Counting

- n labeled balls into k labeled bins: k^n $k!S(n,k)$ $P(n,k)$
- n unlabeled balls into k labeled bins: $\dbinom{n+k-1}{k-1}$ $\dbinom{n-1}{k-1}$ $\dbinom{k}{n}$
- n labeled balls into k unlabeled bins: $\sum_{i=1}^{k} S(n,i)$ $S(n,k)$ 1
- n unlabeled balls into k unlabeled bins: $\sum_{i=1}^{k} p_i(n)$ $p_k(n)$ 1

Inclusion-Exclusion Principle: $\left|\bigcup_{i=1}^{n} A_i\right| = |X| - \displaystyle\sum_{I \subseteq [n]} (-1)^{|I|} |A_I| = \left|\bigcap_{i=1}^{n} \bar{A}_i\right| = |X| - \sum_{I \subseteq [n]} (-1)^{|I|} |A_I|$

OGF: $F(x) = \sum_{n=0}^{\infty} a_n x^n$ $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

Power Series Formulas: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ $\quad \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ $\quad (1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$ $\quad \sum_{n=1}^{\infty} nx^{n-1} = \left(\sum_{n=0}^{\infty} x^n\right)' = \frac{1}{(1-x)^2}$

OGF: $\sum_{n=0}^{\infty} a_n x^n$ $\quad (AB)(x) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} a_i b_{n-i}\right) x^n$ $\quad$ **EGF**: $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ $\quad (AB)(x) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}\right) \frac{x^n}{n!}$

- **Even Permutation EGF**: $\frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{(2n)}}{(2n)!}$ $\quad$ **Odd Permutation EGF**: $\frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{(2n+1)}}{(2n+1)!}$

Weak Compositions OGF: $\frac{1}{(1-x)^k} = (1 + x + \cdots)(1 + x + \cdots)\cdots = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n$

Stirling Number OGF: $\frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} = \sum_{n=0}^{\infty} S(n,k) x^n$

Partitions OGF: $\frac{1}{(1-x)(1-x^2)\cdots} = \sum_{n=0}^{\infty} p(n) x^n$ $\quad \frac{x^k}{(1-x)(1-x^2)\cdots} = \sum_{n=0}^{\infty} p_k(n) x^n$

Permutations EGF: $(1+x)^m = \sum_{n=0}^{\infty} P(m,n) \frac{x^n}{n!}$

Stirling Number EGF: $\frac{(e^x - 1)^k}{k!} = \sum_{n=0}^{\infty} S(n,k) \frac{x^n}{n!}$ $\quad$ **Bell Number EGF**: $e^{(e^x - 1)} = \sum_{n=0}^{\infty} B(n) \frac{x^n}{n!}$

Catalan Numbers: $C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}$ $\quad$ **OGF Catalan Numbers**: $C_n = \frac{\binom{2n}{n}}{n+1}$ $\quad C_0 = 0$

Theorem: G is bipartite if and only if G has no odd cycles

Theorem: G with size m has $\sum_{v \in V(G)} \deg(v) = 2m$ $\quad$ **Corollary**: G must have an even number of odd degree vertices

Theorem: There exists a d-regular graph on n vertices if and only if at least one of d, n is even

Theorem: For any graph G, there exists a d-regular graph G such that G is an induced subgraph of H

Theorem: G with degrees $d = d_1, \ldots, d_n$ exists if and only if $s_1 = d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$ is graphical

Theorem: Every tree on 2 or more vertices has at least 2 leaves

Theorem: G is a tree $\iff$ G is connected, acyclic with $n-1$ edges $\iff$ there is a unique path for $u, v \in V(G)$

Theorem: A connected graph of order n has at least $n-1$ edges

Theorem: An edge e is a bridge if and only if e isn't in any cycles

Spanning Tree to Code: Delete lowest index leaf and write down vertex adjacent to it. Repeat until only an edge remains

Code to Spanning Tree: Find smallest index b_1 not used and create $a_1 \sim b_1$. Delete a_1 and append b_1. Repeat until $b_1, \ldots, b_{n-2}$ then connect missing 2 indices

Theorem: Each Prufer code corresponds to a unique spanning tree. Thus number of spanning trees of K_n is n^{n-2}

Corollary: Total spanning trees such that vertex i has degree d_i is $\binom{n-2}{d_1 - 1, d_2 - 1, \ldots}$

Rooted Plane Tree: Tree with a root vertex, left/right ordering, but vertices are NOT labeled

- Clockwise walk around border of the tree reveals that the number of rooted plane trees on $n+1$ vertices is $C_n = \frac{\binom{2n}{n}}{n+1}$

Rooted Forest: Forest where each tree component has a distinguishable root vertex

Theorem: Number of labeled rooted forests on n vertices is $(n+1)^{n-1}$

Parking Function: $P(n) = (n+1)^{n-1}$ $\quad$ $P(n+1) = \sum_{i=1}^{n} \binom{n}{i} (i+1)P(i)P(n-i)$

Matching: Set of edges with no shared endpoints

Hall's Theorem: A bipartite graph G has a matching that saturates A if and only if for all $S \subseteq A, |N(S)| \geq |S|$

k-factor: Spanning k-regular subgraph $\quad$ **k-factorable**: Exists factors $F_1, \ldots, F_k$ that decompose $E(G)$ into disjoint sets

- **Note**: G is 1-factorable if and only if G has a perfect matching $\quad$ 2-factor is just a union of cycles
- **Note**: Any k-regular bipartite graph has a perfect matching

SDR: Given sets $A_1, \ldots, A_n$ (not necessarily distinct), it is an **SDR** if there are n distinct elements such that $a_i \in A_i$

Eulerian Circuit: Circuit that traverses all edges exactly once with the same start and end vertices

Theorem: Let G be a connected graph. Then G is Eulerian if and only if every vertex has an even degree

Corollary: Graph G has a **Eulerian Trail** $\iff$ 2 vertices have odd degree, which are the start and end of the trail

Hamiltonian Cycle: A cycle (vertices are used only once except the start/end) that contains all vertices of G

- **Note**: Hamiltonian Cycle $\implies$ Hamiltonian Path (remove an edge) but HP $\not\Longrightarrow$ HC (consider P_n)

t-tough: A graph is t-tough if $t \leq \dfrac{|S|}{c(G \setminus S)}$ where S runs through all subsets of vertices that disconnect G

Theorem: If G is Hamiltonian, then $t(G) = \dfrac{|S|}{c(G \setminus S)} \geq 1$, i.e. $\forall$ disconnecting set S, $|S| \geq c(G \setminus S)$

- **Corollary**: If there exists a subset of vertices where $|S| < c(G \setminus S)$, then the graph is not Hamiltonian

Theorem (Ore): Let G have order $n \geq 3$. If $\deg(u) + \deg(v) \geq n$ for any 2 non-adjacent vertices, then G is Hamiltonian

- **Note**: This is a sufficient but NOT necessary condition. Consider C_n, non-adjacent vertices have degree $\deg(u)+\deg(v) < n$

Planar Graph: Graph can be drawn without edges crossing

Euler's Identity: If G is a connected, planar graph on n vertices, m edges, and f faces, then $n - m + f = 2$

Theorem: For a connected planar graph of order ≥ 3, we have $m \leq 3n - 6$

- **Corollary**: If G is planar, then there is a vertex of degree ≤ 5

Keratowski Theorem: G is planar if and only if G doesn't contain K_5 or $K_{3,3}$, or a subdivision of either

Chromatic Number: Smallest number of colors in any coloring of G, denoted $\chi(X)$

- **k-colorable**: Can color vertices of G using k colors $\quad$ **k-chromatic**: G such that $\chi(G) = k$

Independent Set: Set S of vertices where no two vertices of S are adjacent

Independence Number: Size of largest independet set, denoted $\alpha(G)$

- **Note**: k-chromatic $\implies$ $V(G)$ can be partitioned into k independent sets (color classes)

Theorem: $\chi(G) = 2$ if and only if G is non-empty bipartite graph $\quad$ **Corollary**: If G has an odd cycle, then $\chi(G) \geq 3$

Clique: Complete subgraph of G $\quad$ **Clique Number**: Size of the largest clique of G, denoted $\omega(G)$

Theorem: $\chi(G) \geq \omega(G)$ $\quad$ $\chi(G) \geq \frac{n}{\alpha(G)}$

Theorem: $\chi(G) \leq \Delta(G) + 1$ $\quad$ **Brooks' Theorem**: For a connected graph not equal to odd C_n or K_n, $\chi(G) \leq \Delta(G)$

Mycielski Construction: Maintain $\omega(G) = 2$ while arbitrarily grow $\chi(G)$ by adding $n+1$ vertices $(w, u_1, \ldots, u_n)$

- Join w to all u_i such that u_i's are not adjacent to each other. Join u_i with v_j where $v_j \sim v_i$