Permutation: $P(n,k) = \dfrac{n!}{(n-k)!}$ **Permutation w/ Rep**: $\dbinom{n}{a_1, a_2, \dots, a_k} = \dfrac{n!}{a_1! \cdots a_k!}$ **Combination**: $\dbinom{n}{k} = \dfrac{P(n,k)}{k!}$

Binomial Theorem: $(x+y)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ **Multinomial Theorem**: $(x_1+\cdots+x_k)^n = \displaystyle\sum_{\substack{a_1+\cdots+a_k=n \\ a_1,\dots,a_k \geq 0}} \binom{n}{a_1, a_2, \dots, a_k} x_1^{a_1} \cdots x_k^{a_k}$

Weak Composition: $\dbinom{n+k-1}{k-1}$ **Composition**: $\dbinom{n-1}{k-1}$

Stirling Number: $S(n,k) = S(n-1,k-1) + kS(n-1,k)$ **Bell Number**: $B(n) = \sum_{k=1}^{n} S(n,k) = \sum_{i=1}^{n} \dbinom{n-1}{i-1} B(n-i)$

- **Surjective Functions**: $k!S(n,k)$

Partition of n: $(a_1, \dots, a_k)$ such that $a_1 \geq \cdots \geq a_k$ and $\displaystyle\sum_{i=1}^{k} a_i = n$ total: $p(n)$ k-parts: $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$

- Represented using **Ferrers Diagram**: partial rectangular grid with k rows, each with a_i squares (conjugate is also valid)

Twelvefold Way Counting

- n labeled balls into k labeled bins: k^n $k!S(n,k)$ $P(n,k)$
- n unlabeled balls into k labeled bins: $\dbinom{n+k-1}{k-1}$ $\dbinom{n-1}{k-1}$ $\dbinom{k}{n}$
- n labeled balls into k unlabeled bins: $\sum_{i=1}^{k} S(n,i)$ $S(n,k)$ 1
- n unlabeled balls into k unlabeled bins: $\sum_{i=1}^{k} p_i(n)$ $p_k(n)$ 1

Inclusion-Exclusion Principle: $\left|\bigcap_{i=1}^{n} \bar{A}_i\right| = |X| - \sum_{I \subseteq [n]} (-1)^{|I|} |A_I|$

Power Series Formulas: $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ **OGF**: $\sum_{n=0}^{\infty} a_n x^n$

Weak Compositions OGF: $\dfrac{1}{(1-x)^k} = (1+x+\cdots)(1+x+\cdots)\cdots = \displaystyle\sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n$

Stirling Number OGF: $\dfrac{x^k}{(1-x)(1-2x)\cdots(1-kx)} = \displaystyle\sum_{n=0}^{\infty} S(n,k) x^n$

Partitions OGF: $\dfrac{1}{(1-x)(1-x^2)\cdots} = \displaystyle\sum_{n=0}^{\infty} p(n) x^n$ $\dfrac{x^k}{(1-x)(1-x^2)\cdots} = \displaystyle\sum_{n=0}^{\infty} p_k(n) x^n$

Catalan Numbers: $C_n = \displaystyle\sum_{i=0}^{n-1} C_i C_{n-i-1}$ **OGF Catalan Numbers**: $C_n = \dfrac{\binom{2n}{n}}{n+1}$ $C_0 = 0$

Theorem: G is bipartite if and only if G has no odd cycles

Theorem: G with size m has $\displaystyle\sum_{v \in V(G)} \deg(v) = 2m$ **Corollary**: G must have an even number of odd degree vertices

Theorem: There exists a d-regular graph on n vertices if and only if at least one of d, n is even

Theorem: G with degrees $d = d_1, \dots, d_n$ exists if and only if $s_1 = d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n$ is graphical

Theorem: Every tree on 2 or more vertices has at least 2 leaves

Theorem: G is a tree $\iff$ G is connected, acyclic with $n-1$ edges $\iff$ there is a unique path for $u, v \in V(G)$ **Theorem**: A connected graph of order n has at least $n-1$ edges

Theorem: An edge e is a bridge if and only if e isn't in any cycles

Spanning Tree to Code: Delete lowest index leaf and write down vertex adjacent to it. Repeat until only an edge remains

Code to Spanning Tree: Find smallest index b_1 not used and create $a_1 \sim b_1$. Delete a_1 and append b_1. Repeat until $b_1, \dots, b_{n-2}$ then connect missing 2 indices

Theorem: Each Prufer code corresponds to a unique spanning tree. Thus number of spanning trees of K_n is n^{n-2}

Corollary: Total spanning trees such that vertex i has degree d_i is $\binom{n-2}{d_1 - 1, d_2 - 1, \dots}$

Rooted Plane Tree: Tree with a root vertex, left/right ordering, but vertices are NOT labeled

- Clockwise walk around border of the tree reveals that the number of rooted plane trees on $n+1$ vertices is $C_n = \frac{\binom{2n}{n}}{n+1}$

Rooted Forest: Forest where each tree component has a distinguishable root vertex

Theorem: Number of labeled rooted forests on n vertices is $(n+1)^{n-1}$

Matching: Set of edges with no shared endpoints

Hall's Theorem: A bipartite graph G has a matching that saturates A if and only if for all $S \subseteq A, |N(S)| \geq |S|$

k-factor: Spanning k-regular subgraph **k-factorable**: Exists factors $F_1, \dots, F_k$ that decompose $E(G)$ into disjoint sets

- **Note**: G is 1-factorable if and only if G has a perfect matching 2-factor is just a union of cycles
- **Note**: Any k-regular bipartite graph has a perfect matching

SDR: Given sets $A_1, \dots, A_n$ (not necessarily distinct), it is an **SDR** if there are n distinct elements such that $a_i \in A_i$

Theorem: Let G be a connected graph. Then G is Eulerian if and only if every vertex has an even degree

Corollary: Graph G has a **Eulerian Trail** $\iff$ 2 vertices have odd degree, which are the start and end of the trail

Hamiltonian Cycle: A cycle (vertices are used only once except the start/end) that contains all vertices of G

t-tough: A graph is t-tough if $t \leq \frac{|S|}{c(G \setminus S)}$ where S runs through all subsets of vertices that disconnect G

Theorem: If G is Hamiltonian, then $t(G) = \frac{|S|}{c(G \setminus S)} \geq 1$, i.e. $\forall$ disconnecting set S, $|S| \geq c(G \setminus S)$

- **Corollary**: If there exists a subset of vertices where $|S| < c(G \setminus S)$, then the graph is not Hamiltonian

Theorem (Ore): Let G have order $n \geq 3$. If $\deg(u) + \deg(v) \geq n$ for any 2 non-adjacent vertices, then G is Hamiltonian

- **Note**: This is a sufficient but NOT necessary condition. Consider C_n, non-adjacent vertices have degree $\deg(u) + \deg(v) < n$

Euler's Identity: If G is a connected, planar graph on n vertices, m edges, and f faces, then $n - m + f = 2$

Theorem: For a connected planar graph of order ≥ 3, we have $m \leq 3n - 6$

- **Corollary**: If G is planar, then there is a vertex of degree ≤ 5

Keratowski Theorem: G is planar if and only if G doesn't contain K_5 or $K_{3,3}$, or a subdivision of either

Independence Number: Size of largest independet set, denoted $\alpha(G)$

- **Note**: k-chromatic $\implies$ $V(G)$ can be partitioned into k independent sets (color classes)

Theorem: $\chi(G) = 2$ if and only if G is non-empty bipartite graph **Corollary**: If G has an odd cycle, then $\chi(G) \geq 3$

Theorem: $\chi(G) \geq \omega(G)$ $\chi(G) \geq \frac{n}{\alpha(G)}$

Theorem: $\chi(G) \leq \Delta(G) + 1$ **Brooks' Theorem**: For a connected graph not equal to odd C_n or K_n, $\chi(G) \leq \Delta(G)$

Mycielski Construction: Maintain $\omega(G) = 2$ while arbitrarily grow $\chi(G)$ by adding $n+1$ vertices $(w, u_1, \dots, u_n)$

- Join w to all u_i such that u_i's are not adjacent to each other. Join u_i with v_j where $v_j \sim v_i$

Ramsey Number: $R(s,t) = N$, we must show that

1. K_{N-1} has the existence of a coloring with no blue K_s or red K_t
   - Showing the existence of a color α with no red K_s and blue K_t in K_m shows that $R(s,t) > m$

2. For any edge coloring of K_N, there is always a red K_s or blue K_t

    - Showing that all colorings have a red K_s or blue K_t in K_m shows that $R(s,t) \leq m$

Mantel Theorem: G on n vertices with m edges and no K_3 satisfies $m \leq \lfloor \frac{n^2}{4} \rfloor$

Turan Graph: r-partite graph such that each partite set has nearly the same size

Turan Theorem: Turan graph $T_{n,r}$ doesn't contain a K_{r+1} subgraph and has the max number of edges