

MATH475: Combinatorics and Graph Theory

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Contents

1	Chapter 1	2
1.1	The Basics - Permutations, Combinations, and General Counting	2
1.1.1	Permutation	2
1.1.2	Combinations	2
1.2	Counting in 2 Ways	5
1.3	Pigeon Hole Principle	6
1.4	Compositions and Set Partitions	6
1.5	Integer Partitions	9
1.6	Twelffold Way	10

1 Chapter 1

1.1 The Basics - Permutations, Combinations, and General Counting

1.1.1 Permutation

Definition - Permutation: A **permutation** of an n -element set is an arrangement of the elements in a specific order

- k -permutation: arrangement of k elements from the set
- The total number of k -permutation of an n -element set is

$$P(n, k) = \frac{n!}{(n-k)!} = n(n-1) \cdots (n-k+1)$$

Example: 10 people can run for office for a committee with a President, Vice President, Treasurer. The total number of possible committees is $P(10, 3)$

Definition - k th Falling Factorial of n : Let $n \geq k$, both positive integers. Then the **k th Falling Factorial of n** is $(n)_k = n(n-1) \cdots (n-k+1)$

Example - Permutations with Repetitions: How many rearrangements of $AAAABBBCCD$ are there?

- If we were only looking at distinct elements, there would be $10!$ factorial. However, the elements aren't distinct

As an example, look at how we can reorder the A 's

$$A_1 A_2 A_3 A_4 B B B C C D = A_2 A_1 A_3 A_4 B B B C C D = \cdots$$

Thus there are $4!$ of these representations that are equivalent and we divide $10!$ by $4!$ to account for this. Similar idea for the other letters

Thus the number of rearrangements is $\frac{10!}{4!3!2!1!}$

Theorem 1: Suppose object 1 occurs a_1 times, object 2 occurs a_2 times, ..., object k occurs a_k times. Furthermore, suppose $a_1 + \cdots + a_k = n$

Then the total number of arrangements is $\frac{n!}{a_1! \cdots a_k!} = \binom{n}{a_1, a_2, \dots, a_k}$

- **Definition - Multinomial Coefficients:** The notation above is called a multinomial coefficient

1.1.2 Combinations

Definition - Combinations: The total number of ways to create a k -element subset from $[n] = \{1, 2, \dots, n\}$ is denoted

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{P(n, k)}{k!}$$

- **Note:** the definition comes from $n!$ total possible distinct permutations of which we can permute a k -element subset $k!$ ways

Example: There are 5 cats, 5 dogs, and 4 mice. 3 are chosen at once

- How many total number of ways to get 2 cats, 1 dog?

$\binom{5}{2} \binom{5}{1}$. This comes from choosing 2 cats from 5 cats, and then choosing 1 dog from 5 dogs

- How many total ways to get at least 1 cat?

$\binom{14}{3} - \binom{9}{3}$. This comes from subtracting the possible groupings with no cats from the total number of possible groupings

$= \binom{5}{1}\binom{9}{2} + \binom{5}{2}\binom{9}{1} + \binom{5}{3}$. This comes from enumerating over possible groupings with 1 cat and 2 other animals, 2 cats and 1 other animal, and 3 cats and no other animals

Binomial Theorem: If n is a positive integer then

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

Proof: $(x + y)^n = (x + y) \cdots (x + y)$

The coefficients of $x^k y^{n-k}$ is the product of n terms distributing from k x -terms and $n - k$ y -terms

So out of the n terms that are multiplied together, $\binom{n}{k}$ contribute the x -term, leaving $n - k$ unused terms for y

Thus we have

$$\binom{n}{k} x^k \binom{n-k}{n-k} y^{n-k} = \binom{n}{k} x^k y^{n-k}$$

Multinomial Theorem: For $n \in \mathbb{Z}^+$, we have

$$(x_1 + \cdots + x_k)^n = \sum_{\substack{a_1 + \cdots + a_k = n \\ a_1, \dots, a_k \geq 0}} \binom{n}{a_1, a_2, \dots, a_k} x_1^{a_1} \cdots x_k^{a_k}$$

Proof: $(x_1 + \cdots + x_k)^n = (x_1 + \cdots + x_k) \cdots (x_1 + \cdots + x_k)$

The coefficients of $x_1^{a_1} \cdots x_k^{a_k}$ is the product of n terms resulting from a permutation with repetition of a_1, \dots, a_k

Thus there are $\binom{n}{a_1, \dots, a_k}$ distinct ways of representing the exponents of x_1, \dots, x_k

Worksheet Examples:

1. • How many strictly increasing 6-digit numbers are there (first digit can be 0)

Since the numbers are strictly increasing, they numbers must be distinct

Furthermore, there is only one possible way ordering of these selected digits (in increasing order), so order “doesn’t matter”

Thus we can just count the ways to select 6 digits: $\boxed{\binom{10}{6}}$

- What about when the first digit cannot be zero?

Since 0 can only appear as the first digit, this is identical to the first question, but restricted to $\{1, \dots, 9\}$

$\boxed{\binom{9}{6}}$ ways

2. A quiz has 5 problems with 3 different answers each. How many ways can the quiz be completed?

$\boxed{3^5}$ ways

3. • How many rearrangements of MISSISSIPPI are there?

$$\boxed{\binom{11}{4, 4, 2, 1}}$$

- How many rearrangements do not have all S's together?

Treat all S's as one unit $\Rightarrow \binom{8}{4, 2, 1, 1}$

Thus there are $\boxed{\binom{11}{4, 4, 2, 1} - \binom{8}{4, 2, 1, 1}}$ rearrangements

4. Suppose each digit in a 5-digit code can have any number zero to nine

- How many 5-digit codes are there with no restrictions?

$$\boxed{10^5}$$

- How many 5-digit codes have distinct digits?

$$\boxed{P(10, 5)}$$

- How many codes begin and end with an even number?

$$\boxed{5 * 10^4 * 5}$$

- How many codes begin and end with an even number?

$$\binom{5}{2} \text{ ways to choose two digits}$$

2^5 ways to write a five digit number using these two digits. However, we need to subtract the 2 cases where they use the same digit five times

Thus answer is $\boxed{\binom{5}{2}(2^5 - 2)}$

5. Consider a standard 52 card deck with 13 ranks and 4 suits

- Probability of getting a full house (triple of the same rank, pair of another rank)

$$\boxed{\frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}}$$

- Probability of getting 2 pairs of different ranks and a 5th card of a different rank

$$\boxed{\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2} * 44}{\binom{52}{5}}}$$

Logic is to choose 2 types of ranks AT THE SAME TIME and then choosing possible pairs from those ranks. Finally multiply by 44 (the remaining number of possible cards for the 5th card)

Note: the answer below is NOT correct but is added to show an easy mistake to make

$$\frac{\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{2} * 44}{\binom{52}{5}} \text{ is WRONG because we need to consider duplicate pairs (e.g. QQ, KK and KK, QQ)}$$

This solution ends up overcounting since we can choose Queens then Kings versus Kings then Queens, which yields the same hand

6. 15 people want to buy pre-made sandwiches. There are 7 H sandwiches, 4 C sandwiches, and 4 P sandwiches. If each person gets 1 sandwich, how many ways can the sandwiches be distributed?

$$\boxed{\binom{15}{7,4,4}}$$

7. 10 people are sitting at a circular table. How many seating arrangements are there?

$$\frac{10!}{10} = \boxed{9!}$$

1.2 Counting in 2 Ways

Given an equation, we count combinatorially in 2 ways

Example: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

LHS: chooses a k -size subset from $[n]$

RHS: Partitions subsets into 2 cases, containing element 1 or not containing 1

- Containing 1: Pick $k-1$ elements from the remaining $n-1$ elements $\binom{n-1}{k-1}$
- Not containing 1: Pick k from $n-1$ elements $\binom{n-1}{k}$

Thus we have $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

Example: $\sum_{k=0}^n k \binom{n}{k} = n2^{n-1}$

RHS: Picks 1 element to be the leader (n choices)

Then from the remaining $n-1$ people, pick a subset (2^{n-1} ways)

LHS: Picks a group of size k ($\binom{n}{k}$ ways) and then select a leader amongst them (k possible choices)

Thus $\sum_{k=0}^n k \binom{n}{k} = n2^{n-1}$

Example: $\sum_{i=1}^n i(n-i+1) = \binom{n+2}{3}$

LHS: Let a, b, c be numbers with $a < b < c$ from $[n+2]$. Fix $b = i+1$ (the i th $+1$ element). Then there are i choices for the element a and $(n+2) - (i+1) = n-i+1$ choices for the element c

Thus the total ways to create a 3-size subset $\{a, b, c\}$ from $[n+2]$ is $\sum_{i=1}^n i(n-i+1) = \binom{n+2}{3}$

RHS: Counts the number of subsets of size 3 from $[n+2]$

Example: $\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k}^2$

LHS: Count the number of size n subsets of $[2n]$

RHS: Can be written as $\binom{n}{k} \binom{n}{n-k}$

So choose k people from a group of size n and $n - k$ people from another group of size n

Thus $\binom{n}{k} \binom{n}{n-k} = \binom{2n}{n}$

1.3 Pigeon Hole Principle

Theorem - Pigeon Hole Principle: If $n + 1$ pigeons are placed into n pigeonholes, then there is at least one pigeonhole with at least 2 pigeons

Theorem - Generalized Pigeon Hole Principle: If n pigeons are placed into k holes, then at least one hole has

- At most $\lfloor \frac{n-1}{k} \rfloor$ pigeons (round down)
- At least $\lceil \frac{n}{k} \rceil$ pigeons (round up)
- **Note:** the statements above are equivalent

Proof: Suppose all holes have $\leq \lfloor \frac{n-1}{k} \rfloor$ pigeons

Then there are at most $k \lfloor \frac{n-1}{k} \rfloor \leq n-1 < n$ pigeons total. Which is a contradiction

Example: Prove that in a group of 40 people, at least 4 people have a same birthday in the same month

40 pigeons and 12 holes $\implies \lceil \frac{40}{12} \rceil = 4$. Thus by PHP, the statement holds

Example: Prove that if 6 distinct numbers are chosen from $[9]$, then 2 numbers will sum to 10

6 pigeons and 5 holes $(\{(1,9), \dots, (5,5)\} \implies \lceil \frac{6}{5} \rceil = 2$. Thus by PHP, 2 of the numbers belong to the same pair and sum to 10

Example; An athlete wants to work out 45 hours over a 30-day month and will work out at least 1 hour each day. Prove that there is a period of consecutive days such that the cumulative hours he has worked out is exactly 14

Let a_i be the total hours accumulated up to day i

This creates a strictly increasing sequence $a_1, a_2, \dots, a_{30} = 45$ where each a_i is distinct

Define a new sequence $b_i = a_i + 14$

This creates a strictly increasing sequence $b_1, b_2, \dots, b_{30} = 59$ where each b_i is distinct

Since $\{a_i\}_{i=1}^{30}$ is distinct and $\{b_i\}_{i=1}^{30}$ is distinct (only within their own respective sequence), we can have at most 60 possible values, which we treat as pigeons

Furthermore, b_i can take on any of 59 distinct values, which we treat as the holes

Thus by PHP, some $a_j = b_k = a_k + 14$

Thus from day $k + 1$ to day j , the total number of hours worked out is 14

1.4 Compositions and Set Partitions

Definition - Weak Composition: Take $a_1, \dots, a_k \geq 0$ such that $\sum_{i=1}^k a_i = n$. Then the ordered k -tuple (a_1, \dots, a_k) is a **weak composition** of n into k partitions

- **Note:** This can be seen as distributing n unlabelled balls into k labelled boxes

Theorem - Stars and Bars: The number of weak compositions of n into k parts is $\binom{n+k-1}{k-1}$

Proof:

Denote the n objects as stars. We model a distribution as follows:

Place k bars where stars to the right of each bar is how many stars the corresponding part contains

Note: we must begin with a bar. Thus the rest involves n stars and $k - 1$ bars

Thus we can choose $k - 1$ from these $n + k - 1$ objects, resulting in $\binom{n + k - 1}{k - 1}$

Each one of these combination generates a different weak composition

Example: How many ways can we distribute \$100 to 5 people?

- Can also be rephrased as how many non-negative integer solutions are there to $x_1 + \dots + x_5 = 100$

Using the Stars and Bars Theorem, we have $\binom{100 + 5 - 1}{5 - 1} = \binom{104}{4}$ ways

Example: Suppose we want to choose 3 lollipop flavors from 5 orange, 4 strawberry, and 3 grapes. How many ways can we choose 3 lollipops?

Here we want $x_O + x_S + x_G = 3$ where $x_O, x_S, x_G \geq 0$

Thus $\binom{3 + 3 - 1}{3 - 1}$

Definition - Compositions: Same as weak compositions but with the restriction that each $x_i \geq 1$

We can derive this with the formula above but ensuring that each of the k boxes starts with 1 star, and then we distribute the remaining $n - k$ stars in a weak composition fashion. Thus the number of compositions of n into k parts is

$$\binom{n - k + k - 1}{k - 1} = \binom{n - 1}{k - 1}$$

Note: This problem is placing unlabelled balls into labelled bins

Definition - Partition: Let X be a finite set and A_1, \dots, A_k be non-empty subsets of X . Then a **partition** of X , $\{A_1, \dots, A_k\}$ satisfies

- $\bigcup_{i=1}^k A_i = X$
- Each A_i, A_j are pairwise disjoint

Note: This problem can be stated as: for a given k , X is partitioned into k **blocks**

Example: $\{\{1, 2, 4\}, \{3, 5\}, \{6\}\}$ is a partition of $[6]$

The first inner set here can be considered placing balls 1, 2, 4 into its own (unlabelled) bin

Definition - Stirling Number of the 2nd Kind: The number of partitions of $[n]$ into k **blocks**, denoted $S(n, k)$

- $S(0, 0) = 1$ by convention
- $S(n, 0) = 0$ for $n \geq 1$ is a partition of $[6]$

Example: Use counting arguments to compute $S(n, 2)$

For each ball, we decide if it either it goes into block 1 or block 2, resulting in 2^n

However, we require that each block has ≥ 1 elements. Thus we need to subtract by 2 for the cases where block 1 has all balls or block 2 has all balls by 2 for the cases where block 1 has all balls or block 2 has all balls

Finally, we need to divide by 2 since the order of the blocks doesn't matter

$$\frac{2^n - 2}{2}$$

Theorem: For $n \geq k$, we have

$$S(n, k) = S(n-1, k-1) + kS(n-1, k)$$

Proof:

LHS: Counts the number of ways to partition $[n]$ into k blocks

RHS looks at partitions where

- $\{1\}$ is in its own bin. Then we need to partition $[n-1]$ into $k-1$ bins, resulting in $S(n-1, k-1)$
- $\{1\}$ is in a bin with other elements. Then we need to partition $[n-1]$ into k bins, and then choose a bin to place $\{1\}$ into, resulting in $kS(n-1, k)$

Theorem:
$$S(n, k) = \frac{1}{k!} \sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^n$$

Proof: will show later

Definition - Bell Number: Total number of partitions of $[n]$, denoted $B(n) = \sum_{k=1}^n S(n, k)$

- **Note:** $B(0) = 1$ by convention

Example: Take $n = 3$. Then we have

$$\{\{1, 2, 3\}\}, \{\{1\}, \{2\}, \{3\}\}, \{\{1\}, \{2, 3\}\}, \{\{2, 1, 3\}\}, \{\{3\}\{1, 2\}\}$$

Thus $B(3) = 5$

Theorem:
$$B(n) = \sum_{i=1}^n \binom{n-1}{i-1} B(n-i)$$

Proof:

LHS: counts the total number of partitions of $[n]$

RHS:

- Fix the n th element and determine the number of subsets of size i that contain this element: $\binom{n-1}{i-1}$ ways
- Partition the remaining $n-i$ elements: $B(n-i)$ ways

Thus we see that
$$B(n) = \sum_{i=1}^n \binom{n-1}{i-1} B(n-i)$$

Example: Consider $f : A \rightarrow B$ where $|A| = n$ and $|B| = k$ for $n \geq k$. What is the total number of surjective functions?

- Recall that f is surjective if for $\forall b \in B$, there exists at least one $a \in A$ such that $f(a) = b$

This problem can be viewed as partitioning A into $\{A_1, \dots, A_k\}$ and determining how they map to B , where $A_i = \{a \in A \mid f(a) = i\}$

Clearly the number of ways to partition $[n]$ into k parts is $S(n, k)$

However, we also need to consider the order of the boxes (since they determine which subsets map to which values of B). Thus we need to multiply by $k!$

Thus the final answer is $k!S(n, k)$

1.5 Integer Partitions

Definition - Partition of Integer n : Positive sequence of integers a_1, \dots, a_k , with $a_1 \geq a_2 \geq \dots \geq a_k$ such that

$$a_1 + a_2 + \dots + a_k = n$$

Total number of such sequences is denoted $p(n)$

- **Note:** Observe that since the values are non-increasing, the order “doesn’t matter” so we don’t need to distinguish $1 + 3 = 4$ from $3 + 1 = 4$
- **Note:** By convention, we have $p(0) = 1$

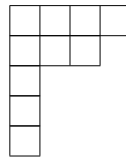
Example: Take $n = 4$. Then we have

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$$

Thus $p(4) = 5$

Definition - Ferrers Diagram: Represents a partition (a_1, \dots, a_k) by taking a partial rectangular grid with k rows whose rows contain a_i dots, such that $a_1 \geq \dots \geq a_k$

Example:

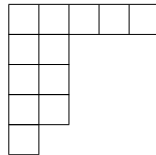


This corresponds to $4 + 3 + 1 + 1 + 1 = 10$

Definition - Conjugate of a Partition: Diagram created by reflecting a Ferrers Diagram along the diagonal

- **Note:** Can be seen as replacing columns as rows and rows as columns

Example: Conjugate of the previous example

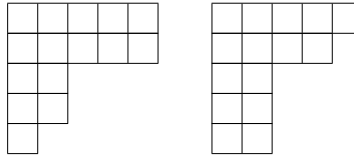


This corresponds to $5 + 2 + 2 + 1 = 10$

Theorem: Let P_1 denote the number of partitions where $a_1 = a_2 \geq a_3 \geq \dots \geq a_k$ (for unfixed k). Let P_2 be the number of partitions whose smallest part is at least 2. Then

$$P_1 = P_2$$

Proof: \implies Take a partition where $a_1 = a_2$. Then take its conjugate.



Clearly every part of the conjugate has length ≥ 2

Adding more dots to the original diagram doesn't affect the minimal lengths of the conjugate

Thus the conjugate assures each row has at least 2 dots \implies each part has at least size 2

\Leftarrow Given a partition with $a_k \geq 2$, the conjugate creates a partition with 2 rows that are the largest and have the same size. Thus $a_1 = a_2$ in the conjugate

Example: Show that the total partition of n containing a part of size 1 is $p(n-1)$

\implies Take partition P of n containing a part of size 1

This part will be the bottom row. Deleting that row yields $p(n-1)$

\Leftarrow Given a partition P_2 of $n-1$, add a dot to the bottom. This yields a partition of n with a part of size 1

Definition - $p_k(n)$: Number of partitions of n into exactly k parts

Theorem: For $1 < k < n$

$$P_k(n) = P_{k-1}(n-1) + P_k(n-k)$$

Proof:

LHS: counts the number of partitions of n into k parts

RHS:

- Count the number of partitions such that there is a partition of size 1: $p_{k-1}(n-1)$
- Count the number of partitions such that every part has size ≥ 1 : $p_k(n-k)$

Note: Each part must have size ≥ 1 by definition

1.6 Twelfold Way

Goal is to distribute n (labelled or unlabelled) balls into k (labelled or unlabelled) bins, depending on if the bins

- Have no restriction
- Have at least 1 ball
- Have at most 1 ball

This results in 12 possible scenarios

1. n Labelled Balls into k Labelled Bins

- No restrictions: For each ball, pick a bin to put it into $\implies k^n$
- At least 1: Can be viewed as the number of surjective functions $\implies k!S(n, k)$
- At most 1:
 - If $n > k \implies 0$
 - Otherwise $n \leq k$ and we pick n bins in a specific order $\implies P(k, n)$

2. n Unlabelled Balls into k Labelled Bins

- No restrictions: Weak composition $\implies \binom{n+k-1}{k-1}$
- At least 1: Composition $\implies \binom{n-1}{k-1}$
- At most 1:

- If $n > k \implies 0$
- Otherwise $n \leq k$ and we just need to choose n bins to use $\implies \binom{k}{n}$

3. n Labelled Balls into k Unlabelled Bins

- No restrictions: Need to count all possible partitions with 0 to k parts $\implies \sum_{i=1}^k S(n, i)$
 - **Note:** This is NOT Bell's Number since we might have $k < n$
- At least 1: Partition $[n]$ into k parts $\implies S(n, k)$
- At most 1:
 - If $n > k \implies 0$
 - Otherwise $n \leq k$ and the bins are indistinguishable $\implies 1$ way

4. n Unlabelled Balls into k Unlabelled Bins

- No restrictions: need to count all possible partitions with 0 to k parts $\implies \sum_{i=1}^k p_i(n)$
- At least 1: Partition n into k parts $\implies p_k(n)$
- At most 1:
 - If $n > k \implies 0$
 - Otherwise $n \leq k$ and the bins are indistinguishable $\implies 1$ way