The Basics - Permutations, Combinations, and General Counting

k-Permutation: arrangement of k elments from a set of n elements $\quad P(n,k) = \frac{n!}{(n-k)!}$

Permutation With Repetition: Can permute each object type $a_i!$ times $\quad \binom{n}{a_1, a_2, \dots, a_k} = \frac{n!}{a_1! \cdots a_k!}$

Combination: Total number of ways to create a k-element subset of $[n]$ $\quad \binom{n}{k} = \frac{P(n,k)}{k!}$

- This comes from being able to permute the k-subset $k!$ ways

Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Multinomial Theorem: $(x_1 + \cdots + x_k)^n = \sum_{\substack{a_1+\cdots+a_k=n \\ a_1,\dots,a_k \geq 0}} \binom{n}{a_1, a_2, \dots, a_k} x_1^{a_1} \cdots x_k^{a_k}$

Pigeon Hole Principle: If n pigeons are placed into k holes, then at least one hole has

- At most $\lfloor \frac{n-1}{k} \rfloor$ (round down)
- At least $\lceil \frac{n}{k} \rceil$ (round up)

Weak Composition: Ordered k-tuple $(a_1, \dots, a_k)$ such that $a_i \geq 0$ and $\sum_{i=1}^{k} a_i = n$ $\quad \binom{n+k-1}{k-1}$

Compositions: Ordered k-tuple $(a_1, \dots, a_k)$ such that $a_i \geq 1$ and $\sum_{i=1}^{k} a_i = n$ $\quad \binom{n-1}{k-1}$

Partition of [n]: $\{A_1, \dots, A_k\}$ such that blocks are pairwise disjoint and $\bigcup_{i=1}^{k} A_i = X$ $\quad S(n,k) = S(n-1,k-1) + kS(n-1,k)$

Bell's Number: Total number of partitions of $[n]$ into any sized blocks $\quad B(n) = \sum_{k=1}^{n} S(n,k) = \sum_{i=1}^{n} \binom{n-1}{i-1} B(n-i)$

Partition of n: $(a_1, \dots, a_k)$ such that $a_1 \geq \cdots \geq a_k$ and $\sum_{i=1}^{k} a_i = n$ $\quad$ total: $p(n)$ $\quad$ k-parts: $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$

- Represented using **Ferrers Diagram**: partial rectangular grid with k rows, each with a_i squares (conjugate is also valid)