

MATH406: Introduction to Number Theory

Michael Li

Contents

1	Basics	2
2	Divisibility	2
2.1	Divisibility	2
2.2	Euclid's Theorem	2
2.3	The Sieve of Eratosthenes	3
2.4	The Division Algorithm	3
2.5	The Greatest Common Divisor	3
2.6	The Euclidean Algorithm	4
2.6.1	The Extended Euclidean Algorithm	5
2.7	Other Bases	7
2.8	Fermat and Mersenne Numbers	7
3	Linear Diophantine Equation	8
4	Unique Factorization	9
5	Applications of Unique Factorization	11
5.1	A Puzzle	11
5.2	Irrationality Proof	11
5.3	Rational Root Theorem	12
5.4	Pythagorean Triples	12
5.5	Difference of Squares	13
5.6	Prime Factorization of Factorials	14
5.7	Riemann Zeta Function	15
6	Congruences	16
6.1	Definitions and Examples	16
6.2	Modular Exponentiation	17
6.3	Divisibility Tests	18
6.4	Linear Congruences	19
6.5	Chinese Remainder Theorem	21
6.6	Fractions mod m	22
7	Fermat, Euler, and Wilson	23
7.1	Fermat's Theorem	23
7.2	Euler's Theorem	25
7.3	Wilson's Theorem	27
8	RSA	28
8.1	RSA Encryption	28

Notes are based off of *An Introduction to Number Theory with Cryptography* (Second edition), by Washington and Kraft

1 Basics

Well-Ordering Principle: All non-empty subsets of N has a smallest member

- **Note:** This is equivalent to the Principle of Induction

2 Divisibility

2.1 Divisibility

Definition - Divides: Given $a, d \in Z$, for $d \neq 0$, d **divides** a if $\exists c \in Z$ such that $a = cd$

Proposition 2.2: Let $a, b, c \in Z$. If $a \mid b$ and $b \mid c \implies a \mid c$

Proof: $b = ea$ and $c = fb \implies c = (fe)a$

Proposition 2.3: Let $a, b, d, x, y \in Z$. If $d \mid a$ and $d \mid b \implies d \mid ax + by$

Proof: $a = md$ and $b = nd \implies ax + by = d(mx + ny)$

Upshot: Every common divisor of both a, b divides any linear combination of a, b

Corollary 2.4: Let $a, b, d \in Z$. If $d \mid a$ and $d \mid b$, then $d \mid a + b$ and $d \mid a - b$

Proof: Apply Proposition 2.3 using $x = 1, y = 1$, and $x = 1, y = -1$, respectively

Lemma 2.5: Let $d, n \in N$ and $d \mid n$. Then $d \leq n$

Proof: Since $d \mid n$, we have $k \in Z$ such that $dk = n$

Since $d \in N$, we also must have $k \in N$ (otherwise $n \notin N$)

Thus $n = dk \geq d$

2.2 Euclid's Theorem

Definition - Prime: Integer $p \geq 2$ whose divisors are $1, p$

Definition - Composite: Integer $n \geq 2$ not prime such that $n = ab$ for $a, b \in Z$ and $1 < a, b < p$

Lemma 2.6: Every integer greater than 1 is prime or divisible by a prime

Proof 1: If n is NOT prime, then it is divisible by some $a_1 \in Z$ where $1 < a_1 < n$

If a_1 is prime, we are done

Otherwise a_1 is divisible by some $a_2 \in Z$ where $1 < a_2 < a_1 \implies a_2 \mid n$

This creates a decreasing sequence of positive integers, which by the Well Ordering Principle, must have a smallest element a_m

So either some a_i is prime and divides n or we stop at a_m , which is prime. Thus n is divisible by a prime

Proof 2 by Induction: Let $n \in \mathbb{Z}, n \geq 2$, and suppose n is composite. Thus $n = kl$ for $k, l \in \mathbb{Z}$ where $1 < k, l < n$

Base case: we only care about the first composite n , i.e. $n = 4 = 2 \cdot 2$ thus $2 \mid 4$ and 2 is prime

IH: Suppose the Lemma holds for all $i \in \mathbb{N}, i < n$

IS: $n = kl$ where $k < n$. Thus k is either a prime or is divisible by a prime

- If k is prime, we are done since $k \mid n$
- Otherwise $p \mid k$ for some prime $p < k$. Then we have $p \mid k \implies k \mid n \implies p \mid n$

Euclid's Theorem: there are an infinite number of primes

Proof: Assume by contradiction that there are a finite number of primes $2, 3, 5, \dots, p_n$

Let $N = (2 * 3 * 5 * \dots * p_n) + 1$

Since $N > 2p_n + 1 > p_n$, it is composite and thus is divisible by some p_i in the list of primes

Thus $p_i \mid 2 * 3 * 5 * \dots * p_n$ and $p_i \mid N$ (by Lemma 2.6) $\implies p_i \mid N - (2 * 3 * 5 * \dots * p_n) \implies p_i \mid 1$ contradiction since $p_i > 1$

Thus there are an infinite number of primes

2.3 The Sieve of Eratosthenes

Proposition 2.7: If n is composite then n has a prime factor $p \leq \sqrt{n}$

Proof: $n = ab$ where $1 < a \leq b < n \implies a^2 \leq ab = n \implies a \leq \sqrt{n}$

By Lemma 2.6, a has a prime divisor p , where $p \mid a \implies p \leq a \leq \sqrt{n}$

- **Note:** Not all prime factors of n are $\leq \sqrt{n}$. For example, $6 = 2 * 3$ but $3 > \sqrt{6}$

2.4 The Division Algorithm

Division Algorithm: Let $a, b \in \mathbb{Z}$ with $b > 0$. Then there exists unique $q, r \in \mathbb{Z}$ such that $a = bq + r$ with $0 \leq r < b$

Proof: Let $S = \{n \in \mathbb{Z} \mid bn \leq a\}$. Clearly S is non-empty since

- If $a \geq 0$, take $n = -1$
- If $a < 0$, take $n = a$

Since S is bounded above by a/b , it has a largest member, call it q

Thus q is the largest integers $\leq a/b$ such that $q \leq a/b < q + 1$

Then we have $bq \leq a < bq + b \implies 0 \leq a - bq < b$

Setting $r = a - bq$ we see that $0 \leq r < b$ and we have $a = bq + r$ so EXISTENCE is done

To show UNIQUENESS let $a = bq + r = bq_1 + r_1$ for $0 \leq r, r_1 < b$

Then we have $b(q - q_1) = r_1 - r$. Since LHS is a multiple of b , RHS is also a multiple of b

But $0 \leq r, r_1 < b \implies -b < r_1 - r < b \implies r_1 - r = 0$ since $b = 0$ is the only multiple of b that satisfies this inequality

Thus $r_1 = r$ and since $b \neq 0 \implies b(q - q_1) = 0 \implies q = q_1$. So q, r are UNIQUE

2.5 The Greatest Common Divisor

Definition - Relatively Prime: a, b are **relatively prime** if $\gcd(a, b) = 1$

- By definition, we have $\gcd(a, 0) = a$

Proposition 2.10: Let $a, b \in Z$ and $d = \gcd(a, b)$. Then $\gcd(\frac{a}{d}, \frac{b}{d}) = 1$

Proof: Let $c = \gcd(a/d, b/d)$. Then $c \mid (a/d)$ and $c \mid (b/d)$

Thus $a = cd k_1$ and $b = cd k_2$ so cd is a common divisor of a, b

Since d is the greatest common divisor of a, b , we have $d \leq cd \leq d \implies c = 1$

Proposition 2.11: If $a, b \in Z$, not both 0, and $e \in Z^+$. Then $\gcd(ea, eb) = e * \gcd(a, b)$

Proof: Let $d = \gcd(ea, eb)$, we show that $d = e * \gcd(a, b)$

$\gcd(a, b) = ax + by \implies e \gcd(a, b) = eax + eby$. If d is a common divisor of ea and eb , then $d \mid e * \gcd(a, b)$

Thus $d \leq e \gcd(a, b)$. But since $e \gcd(a, b)$ is a common divisor of ea, eb , it is the gcd we desire

Various ways to find $\gcd(a, b)$:

1. List all prime factors of a, b and take the largest factor.

Example: $84 = 2 * 2 * 3 * 7$ and $264 = 2 * 2 * 2 * 3 * 11 \implies \gcd(84, 264) = 2 * 2 * 3 = 12$

2. Take Linear Combination of a, b and find a list of possible factors

Example: $d = \gcd(1005, 500) \implies d \mid (1005 - 2 * 500) \implies d = 1$ or $d = 5$. Clearly $d = 5$

Example: $d = \gcd(2n+3, 3n-7) \implies d \mid 3(2n+3) - 2(3n-7) = 21$ so $d \in \{1, 3, 7, 21\}$. Clearly with $n = 9$, $\gcd(21, 21) = 21$

3. Use Euclidean Algorithm

2.6 The Euclidean Algorithm

Euclidean Algorithm: Let $a, b \in Z$ with $a \geq 0, b > 0$. Then we have

$$\begin{aligned} a &= q_1 b + r_1 & 0 < r_1 < b \\ b &= q_2 r_1 + r_2 & 0 < r_2 < r_1 \\ r_1 &= q_3 r_2 + r_3 & 0 < r_3 < r_2 \\ &\dots \\ r_{n-3} &= q_{n-1} r_{n-2} + r_{n-1} & 0 < r_{n-1} < r_{n-2} \\ r_{n-2} &= q_n r_{n-1} + 0 \end{aligned}$$

Where $r_{n-1} = \gcd(a, b)$

Proof: $r_{n-1} \mid r_{n-2}, r_{n-1} \mid r_{n-3}, \dots, r_{n-1} \mid b, r_{n-1} \mid a$ so clearly r_{n-1} is a common factor of a, b

To show that r_{n-1} is the largest common factor, let d be an arbitrary common divisor of a, b

From the first line, we see that $d \mid r_1$. From the second line, $d \mid r_2$. This continues until $d \mid r_{n-1}$

Thus $d \leq r_{n-1}$ which means that r_{n-1} is the largest divisor and $\gcd(a, b) = r_{n-1}$

NOTE: each common divisor of a, b also divides $\gcd(a, b)$

2.6.1 The Extended Euclidean Algorithm

Extended Euclidean Algorithm: $\gcd(a, b)$ can be expressed as a linear combination of a, b .

Example: $\gcd(456, 123)$

$$456 = 3 * 123 + 87$$

$$123 = 1 * 87 + 36$$

$$87 = 2 * 36 + 15$$

$$36 = 2 * 15 + 6$$

$$15 = 2 * 6 + 3$$

$$6 = 2 * 3$$

Using the values above, we can create a table

	x	y	
456	1	0	
123	0	1	
87	1	-3	$R_1 - 3R_2$
36	-1	4	$R_2 - R_3$
15	3	-11	$R_3 - 2R_4$
6	-7	26	$R_4 - 2R_5$
3	17	-63	$R_5 - 2R_6$

Thus $3 = 456 * 17 - 123 * 63$

Theorem 2.12 (Bezout's Theorem): For $a, b \in \mathbb{Z}$ with at least one non-zero, $\exists x, y \in \mathbb{Z}$ such that $\gcd(a, b) = ax + by$

Proof: Let S be a set of integers that can be written in the form $ax + by$ for $x, y \in \mathbb{Z}$

Since $a, b, -a, -b \in S$, clearly S contains at least one positive integer.

Using the Well-Ordering Principle, let d be the smallest positive integer in S . Thus $d = ax_0 + by_0$ for $x_0, y_0 \in \mathbb{Z}$

We show that d is a common divisor of a, b

$$a = dq + r \implies r = a - dq = a - (ax_0 + by_0)q = a(1 - x_0q) + b(-y_0q)$$

Thus $r \in S$. But since d is the smallest positive element of S and $0 \leq r < d$, we must have $r = 0$

Thus $d \mid a$. Similarly, $d \mid b$. Thus d is a common divisor of a, b

Next we show that for any common divisor of a, b , call it e , we have $e \leq d$

$e \mid a$ and $e \mid b \implies e \mid ax_0 + by_0 = d$. Thus $e \leq d$ and d is the largest common factor of a, b

Theorem 2.13: Let $n \geq 2$ and $a_1, \dots, a_n \in Z$ with at least one nonzero a_i . Then $\exists x_1, \dots, x_n \in Z$ such that

$$\gcd(a_1, \dots, a_n) = a_1x_1 + \dots + a_nx_n$$

Proof by Induction: By Theorem 2.12, the statement holds for $n = 2$

IH: assume the statement holds for $n = k$. $\gcd(a_1, \dots, a_k) = a_1x_1 + \dots + a_kx_k$

IS: Note that $\gcd(a_1, \dots, a_{k+1}) = \gcd(\gcd(a_1, \dots, a_k), a_{k+1})$

Apply Theorem 2.12 to $a_1x_1 + \dots + a_kx_k$ and a_{k+1} so $\gcd(a_1, \dots, a_{k+1}) = (a_1x_1 + \dots + a_kx_k)y + a_{k+1}x$

But then this satisfies the statement since if we set $y_i = yx_i$ for $1 \leq i \leq k$ and $y_{k+1} = x$

Thus by Induction, $\gcd(a_1, \dots, a_n) = a_1x_1 + \dots + a_nx_n$

Corollary 2.14: If e is a common divisor of a, b then $e \mid \gcd(a, b)$

Proof: $e \mid a$ and $e \mid b \implies e$ divides any linear combination of $a, b \implies e \mid \gcd(a, b) = ax + by$

Proposition 2.15: Let $a, b, c \in Z$ with $\gcd(a, c) = \gcd(b, c) = 1$. Then $\gcd(ab, c) = 1$

Proof: $\gcd(a, c) = 1 \implies ax_1 + cy_1 = 1$

$\gcd(b, c) = 1 \implies bx_2 + cy_2 = 1$

Multiplying these 2 equations we get $1 = (ab)(x_1x_2) + (c)(by_1x_2 + ax_1y_2 + cy_1y_2)$

Thus by Proposition 2.3, any common divisor of ab and c must divide 1 $\implies \gcd(ab, c) = 1$

Proposition 2.16: Let $a, b, c \in Z$ with $a \neq 0$ and $\gcd(a, b) = 1$. Then $a \mid bc \implies a \mid c$

Proof: By Theorem 2.12, $1 = ax + by \implies c = acx + bcy$

Thus by Proposition 2.3, $a \mid a$ and $a \mid bc \implies a \mid acx + bcy = c$

Proposition 2.17: Let $a, b, c \in Z$ with a, b nonzero and $\gcd(a, b) = 1$. Then if $a \mid c$ and $b \mid c \implies ab \mid c$

Proof: By Theorem 2.12, $1 = ax + by \implies c = acx + bcy$

$b \mid c \implies ab \mid ac$

$a \mid c \implies ba \mid bc$

Since c is a linear combination of ac and bc , by Proposition 2.3, we must have that $ab \mid c$

2.7 Other Bases

We can convert a number from base 10 to any other base using the Division Algorithm

Example: Convert 21963_{10} to base 8

$$21963 = 2745 * 8 + 3$$

$$2745 = 343 * 8 + 1$$

$$343 = 42 * 8 + 7$$

$$42 = 5 * 8 + 2$$

$$5 = 0 * 8 + 5$$

Thus $21963_{10} = 52713_8$ This is because

$$5 * 8^4 + 2 * 8^3 + 7 * 8^2 + 1 * 8 + 3 = 52713_8$$

Note: decimal representations in other bases are NOT unique. For $a_k \leq n - 1$

$\sum_{k=1}^{\infty} \frac{a_k}{n^k} \leq \sum_{k=1}^{\infty} \frac{n-1}{n^k}$, which is the geometric series and converges

Thus any sequence $\{a_n\}_{n=1}^{\infty}$ for $0 \leq a_k \leq n - 1$ converges

In particular, for $j > 1$, $\sum_{k=j}^{\infty} \frac{n-1}{n^k} = \frac{1}{n^{j-1}}$

- **Example:** for $n = 10$, we have $1 = 0.\bar{9}$
- **Example:** $0.01_7 = 0.000\bar{6}_7$

2.8 Fermat and Mersenne Numbers

Mersenne Numbers: $M_n = 2^n - 1$ for prime n . Thought to generate prime numbers, but doesn't always work (e.g. $n = 11$ results in a composite number)

Proposition 2.18: If n is composite, then $2^n - 1$ is composite

Proof: Recall that $x^k - 1 = (x - 1)(x^{k-1} + x^{k-2} + \dots + x + 1)$

Since n is composite, $n = ab$. Let $x = 2^a$ and $k = b$

Then $2^{ab} - 1 = (2^a - 1)(2^{a(b-1)} + \dots + 2^a + 1)$

$1 < a < n \implies 1 < 2^a - 1 < 2^n - 1$ so $2^a - 1$ is a nontrivial factor and $2^n - 1$ is composite

Corollary 2.18.1: For $k, n \in \mathbb{N}$, $k \mid n \implies M_k \mid M_n$

Proof: Can be seen from the factorization seen in the previous proposition

Corollary 2.18.2: If M_n is prime, then n is prime

Proof: Follows from the contraposition of Proposition 2.18

Fermat Numbers: $F_n = 2^{2^n} + 1$. Thought to generate prime numbers, but doesn't always work (e.g. $n = 5$ results in a composite number)

Proposition 2.19: If $m > 1$ is not a power of 2 then $2^m + 1$ is composite

Proof: Recall that k is odd then $x^k + 1 = (x + 1)(x^{k-1} - x^{k-2} + x^{k-3} - \dots - x + 1)$

Since m is not a power of 2 it has a nontrivial odd factor $a \geq 3$, so $m = ab$. Let $k = a$ and $x = 2^b$

Then $2^{ab} + 1 = (2^b + 1)(2^{b(a-1)} - 2^{b(a-2)} + \dots - 2^b + 1)$

$1 \leq b < m \implies 1 < 2^b + 1 < 2^m + 1$ so $2^b + 1$ is a nontrivial factor and $2^m + 1$ is composite

Proposition 2.20: A regular n -gon is constructable if and only if $n = 2^a F_{n_1} F_{n_2} \dots F_{n_r}$ for distinct Fermat Primes and $a \geq 0$

3 Linear Diophantine Equation

We look for solutions to $ax + by = c$ for $a, b, c \in \mathbb{Z}$

- If $\gcd(a, b) \nmid c$ then there are NO integer solutions (x, y) . This follows from $\gcd(a, b)$ divides any linear combination of a, b

Theorem 3.1: Let $a, b, c \in \mathbb{Z}$ where a, b are not both 0. Then $ax + by = c$ has a solution if and only if $\gcd(a, b) \mid c$

Furthermore, if it has one solution (x_0, y_0) , then there are an infinite number of solutions of the form

$$x = x_0 + \frac{b}{\gcd(a, b)}t \quad y = y_0 - \frac{a}{\gcd(a, b)}t \quad t \in \mathbb{Z}$$

Proof: Let $d = \gcd(a, b)$

\implies Contraposition: If $d \nmid c$ then clearly no solutions

\Leftarrow If $d \mid c$ then by Theorem 2.12, there exists $r, s \in \mathbb{Z}$ such that $ar + bs = d$

$d \mid c \implies df = c$ for $f \in \mathbb{Z} \implies a(rf) + b(sf) = df = c$

Thus $x_0 = rf$ and $y_0 = sf$ is a solution to $ax + by = c$

To show there are an infinite number of solutions, first let $x = x_0 + \frac{b}{d}t$ and $y = y_0 - \frac{a}{d}t$

Then $ax + by = a(x_0 + \frac{b}{d}t) + b(y_0 - \frac{a}{d}t) = ax_0 + by_0 = c$

Thus there are an infinite number of solutions of this form

To show that every solution has the correct form, fix solutions x_0, y_0 and let u, v be any solution

$au + bv = c = ax_0 + by_0 \implies a(u - x_0) - b(v - y_0) = 0 \implies \frac{a}{d}(u - x_0) = \frac{b}{d}(y_0 - v)$

- The last part follows because $d \mid a$ and $d \mid b \implies \frac{a}{d}, \frac{b}{d} \in \mathbb{Z}$

Thus we have $(a/d) \mid (b/d)(y_0 - v)$

Since, by Proposition 2.10, $\gcd(a/d, b/d) = 1$, we have by Proposition 2.6, $(a/d) \mid (y_0 - v)$

Thus $y_0 - v = \frac{a}{d}t \implies v = y_0 - t\frac{a}{d}$

Furthermore, $\frac{a}{d}(u - x_0) = \frac{b}{d}(\frac{a}{d}t) \implies u = x_0 + \frac{b}{d}t$

Corollary 3.2: Let $a, b, c \in \mathbb{Z}$ with at least one a, b nonzero. If $\gcd(a, b) = 1$ then $ax + by = c$ has infinite number of solutions

Upshot: If (x_0, y_0) is a particular solution, then all solutions are of the form

$$x = x_0 + bt \quad y = y_0 - at \quad t \in \mathbb{Z}$$

General Steps to Solve Linear Diophantine Equation:

1. Verify $\gcd(a, b) \mid c$
 - If no, then there is no solution
 - If yes, divide the equation by d to get $a'x + b'y = c'$ where $\gcd(a', b') = 1$
2. Then use Extended Euclidean Algorithm to solve for $a'x + b'y = 1$, then multiply the solution by the value of c'
3. If one of the solution variable (e.g. x) is negative, we can perform Extended Euclidean Algorithm with a positive x then flip the sign of x at the end
4. General solutions will be $(x_0 + \frac{b}{d}t, y_0 - \frac{a}{d}t)$

Example: $-17x + 14y = 30 \implies 17x + 14y = 30$ has the solution $(5 * 30, -6 * 30)$ so the desired solution is $(-150, -180)$ and general solution is of the form

$$x = -150 + 14t \quad y = -180 + 17t \quad t \in \mathbb{Z}$$

Proposition 3.3: Let $a, b \in \mathbb{Z}^+$ and relatively prime. Then there are no non-negative $x, y \in \mathbb{Z}$ such that $ax + by = ab - a - b$

Proof: Observe that $a(-1) + b(a-1) = ab - a - b \implies x = -1$ and $y = a-1$ is a solution

Since $\gcd(a, b) = 1$ every solution has the form $x = -1 + bt$ and $y = a-1 - at = a(1-t) - 1$

Note that $x \geq 0$ if and only if $t > 0$ but then we have $1-t \leq 0 \implies y \leq -1$

Thus it is impossible to find a non-negative solution to $ax + by = ab - a - b$

Proposition 3.4: Let $a, b \in \mathbb{Z}^+$ and relatively prime. If $n > ab - a - b$ then there exists non-negative $x, y \in \mathbb{Z}$ such that $ax + by = n$

Proof: First find a pair (x_0, y_0) such that $ax_0 + by_0 = n \geq ab - a - b + 1$. Note (x_0, y_0) may be negative

Solution has the form $x = x_0 + bt$ and $y = y_0 - at$

We find the smallest possible $y \geq 0$ then show that $x \geq 0$

From Division Algorithm and dividing y_0 by a , we have $y_0 = at + y_1$ for $0 \leq y_1 < a$. Let y_1 be our choice of y

Since $y_1 = y_0 - at$, we take $x_1 = x_0 + bt$ as our choice of x . First note that these are a valid solution

$$ax_1 + by_1 = a(x_0 + bt) + b(y_0 - at) = ax_0 + by_0 = n$$

Now we show that $x_1 \geq 0$

Suppose by contradiction that $x_1 \leq -1$, then we have

$$n = ax_1 + by_1 \leq a + by_1 \leq -a + \underbrace{b(a-1)}_{0 \leq y_1 < a}$$

Thus $n = ab - a - b$. Contradiction since we said $n > ab - a - b$

Thus (x_1, y_1) is a non-negative solution

4 Unique Factorization

Theorem 4.1: Let p be prime and $a, b \in \mathbb{Z}$ such that $p \mid ab$. Then $p \mid a$ or $p \mid b$

Proof: Let $d = \gcd(a, p)$. If $d = p$ then $d \mid a \implies p \mid a$

Otherwise applying Extended Euclidean Algorithm, $d = 1 = ax + py \implies b = abx + pby$

$p \mid ab$ and $p \mid p \implies p \mid b$, which is a linear combination of p and ab

- **NOTE:** if n is composite, then we CANNOT conclude $n \mid a$ or $n \mid b$ from $n \mid ab$

Corollary 4.2: Let p be prime and $a_1, a_2, \dots, a_r \in \mathbb{Z}$ such that $p \mid a_1 \cdot a_2 \cdots a_r$. Then $p \mid a_i$ for some i

Proof by Induction: clearly statement holds for $r = 1$

IH: assume statement holds for $r = k$

IS: show statement is true for $r = k + 1$. Let $a = a_1 \cdots a_k$ and $b = a_{k+1}$

We can apply Theorem 4.1 where $p \mid ab \implies$ statement holds for any $r \geq 1$

Lemma 4.3: Every integer can be written as a product of primes

Proof: Assume there exist composite integers that cannot be written as product of primes.

Let S be the set of these integers > 1

Since all $e \in S$ are positive, by Well Ordering Principle, it has a smallest element s

Since s is composite, we have $s = ab$, but $a, b < s \implies a, b \notin S \implies a, b$ can be written as the product of primes

Thus s is also a product of primes and thus S is empty

Fundamental Theorem of Arithmetic: Any positive integer > 1 is either prime or can be factored exactly one way as a product of primes

Proof: Lemma 4.3 shows that any integer > 1 can be written as a product of primes

For uniqueness, suppose that there are 2 ways of factoring an integer. Let n be the smallest of these integers

$$n = p_1 p_2 \cdots p_r = q_1 q_2 \cdots q_s$$

$$p_1 \mid \text{LHS} \implies p_1 \mid \text{RHS} \implies p_1 \mid q_i$$

Rearranging the RHS, we let $p_1 = q_1$ and now we have $n/p_1 = m = p_2 \cdots p_r = q_2 \cdots q_s$

But $m < n$ so it must have a unique factorization but we see that m can be written using 2 different factorization

Thus we have a contradiction and every positive integer > 1 can be unique factored

Proposition 4.4: Let $a, b \in \mathbb{Z}^+$ where $a = 2^{a_2} 3^{a_3} \cdots$ and $b = 2^{b_2} 3^{b_3} \cdots$. Then $a \mid b$ if and only if $a_p \leq b_p$ for all p

Proof: $\implies a \mid b \implies ac = b$ where $c = 2^{c_2} 3^{c_3} \cdots$

Then $2^{a_2+c_2} 3^{a_3+c_3} \cdots = b$

Thus we must have $\forall p, a_p + c_p = b_p \implies a_p \leq b_p$

\Leftarrow suppose $\forall p, a_p \leq b_p$ and let $c_p = b_p - a_p$. Clearly $c_p \geq 0$

Let $c = 2^{c_2} 3^{c_3} \cdots \implies ac = b \implies a \mid b$

Definition - Least Common Multiple: $\text{lcm}(a, b)$ is the smallest positive integer divisible by a, b

Proposition 4.5: Let $a, b \in \mathbb{Z}^+$ where $a = 2^{a_2} 3^{a_3} \cdots$ and $b = 2^{b_2} 3^{b_3} \cdots$. Furthermore, for all p , let $d_p = \min(a_p, b_p)$ and $e_p = \max(a_p, b_p)$. Then $\text{gcd}(a, b) = 2^{d_2} 3^{d_3} \cdots$ and $\text{lcm}(a, b) = 2^{e_2} 3^{e_3} \cdots$

Proof: Let d be any common divisor of a, b such that $d = 2^{d_2} 3^{d_3} \cdots$

$d \mid a \implies d_p \leq a_p$ for all p . Similarly $d \mid b \implies d_p \leq b_p$ for all p

Largest common divisor occurs when $d_p = \min(a_p, b_p)$ for each p

Least common multiple occurs when $e_p = \max(a_p, b_p)$ for each p

Definition - Squarefree: integer whose factors are all distinct (doesn't have a square of a number as a factor)

Proposition 4.7: Let $n \in \mathbb{Z}^+$. Then there exists $r \in \mathbb{Z}, r \geq 1$ and a squarefree integer $s \geq 1$ such that $n = r^2 s$

Proof: Let $n = p_1^{a_1} p_2^{a_2} \dots$.

If a_i is even, write it as $a_i = 2b_i$. Otherwise write $a_i = 2b_i + 1$

Let $r = p_1^{a_1} p_2^{a_2} \dots$ and let $s =$ the product of all primes p_i with odd a_i

Then we have $r^2 s = n$

5 Applications of Unique Factorization

5.1 A Puzzle

Proposition 5.1: Let $k \geq 2$ be an integer and $m \in \mathbb{Z}^+$. Then m is a k th power if and only if all exponents in the prime factorization of m are multiples of k

Proof: \Leftarrow Let $m = 2^{y_2} 3^{y_3} \dots$. If each y_p is a multiple of k then $y_p = kz_p \implies m = (2^{z_2} 3^{z_3} \dots)^k$

\implies If $m = n^k$ where $n = 2^{w_2} 3^{w_3} \dots$, then $2^{y_2} 3^{y_3} \dots = m = n^k = 2^{kw_2} 3^{kw_3} \dots$

By Uniqueness of Factorization, $y_p = kw_p$ for each $p \implies$ each exponent for m is a multiple of k

Example: Find a number A such that $2/3 * A^2$ is a cube

Let $A = 2^a 3^b 5^c \dots$ be the prime factorization of A

We have $2/3 * A^2 = 2^{2a+1} 3^{2b-1} 5^{2c} \dots$ is a cube, so $2a+1, 2b-1, 2c, \dots$ are all multiples of 3

By brute force, we see that $a=1, b=2, c=d=\dots=0$ works and gives us $A=18$

To find the general solution, we note that $3 \mid 2c$ and $\gcd(3, 2) = 1$ so c must be a multiple of 3 $\implies c = 3c'$. Similar for d, e, \dots

Since $2a+1$ is odd and a multiple of 3, we have $2a+1 = 3(2j+1) \implies a = 3j+1$

Since $2b-1$ is odd and a multiple of 3, we have $2b-1 = 3(2k+1) \implies b = 3k+2$

Finally, we see that $A = 2^a 3^b 5^c \dots = 2 * 3^2 (2^j 3^k 5^{c'} \dots)^3 = 18B^3$ for any $B \geq 1$

5.2 Irrationality Proof

Rational: Number that can be expressed as a ratio of 2 integers

Theorem 5.2: $\sqrt{2}$ is irrational

Proof: Suppose by contradiction that $\sqrt{2}$ is rational and $\sqrt{2} = a/b \in \mathbb{Q}$ in reduced form

Then we have $2 = a^2/b^2 \implies 2b^2 = a^2$

Clearly a^2 is even $\implies a$ is even so $a = 2a_1$

But then we have $b^2 = 2a_1$ so b^2 is even $\implies b$ is even. This is a contradiction since we said a/b is in reduced form

Thus we have a contradiction and $\sqrt{2}$ is irrational

Theorem 5.3: Let $k \in \mathbb{Z}$ and $k \geq 2$. Let $n \in \mathbb{Z}^+$ that is not a perfect k th power. Then $\sqrt[k]{n}$ is irrational

Proof: We show the contrapositive that if $\sqrt[k]{n}$ is rational then n is a perfect k th power

Suppose $\sqrt[k]{n} = a/b \implies nb^k = a^k$

We can prime factorize n, b to get $n = 2^{x_2}3^{x_3} \dots$ and $b = 2^{z_2}3^{z_3} \dots$

Thus we have $nb^k = 2^{x_2+kz_2}3^{x_3+kz_3} \dots$

Let $a = 2^{y_2}3^{y_3} \dots$. Since a^k is a perfect power, by Proposition 5.1, every exponent in the prime factorization is a multiple of k

Thus $x_p + kz_p = ky_p \implies x_p = k(y_p - z_p) \implies n$ is a perfect k th power

5.3 Rational Root Theorem

Theorem 5.4 (Rational Root Theorem): let $P(X) = a_nX^n + \dots + a_1X + a_0$ where $a_i \in \mathbb{Z}$ such that $a_n \neq 0$ and $a_0 \neq 0$

If $r = u/v \in \mathbb{Q}$ with $\gcd(u, v) = 1$ and $P(u/v) = 0$ then $u \mid a_0$ and $v \mid a_n$

Proof: $P(u/v) = 0 \implies a_n(u/v)^n + \dots + a_0 = 0 \implies a_nu^n + \dots + a_0v^n = 0$

$a_{n-1}vu^{n-1} + \dots + a_0v^n = -a_nu^n \implies v \mid a_nu^n$. But $\gcd(u, v) = 1 \implies v \mid a_n$

$a_nu^n + \dots + a_1v^{n-1}u = -a_0v^n \implies u \mid a_0v^n$. But $\gcd(u, v) = 1 \implies u \mid a_0$

5.4 Pythagorean Triples

Definition - Pythagorean Triples: positive integers (a, b, c) where $a^2 + b^2 = c^2$

Definition - Primitive Pythagorean Triples: Pythagorean triples where $\gcd(a, b, c) = 1$

Example: A primitive way of generating Pythagorean Triples is using odd numbers

$$(2n+1)^2 = 4n^2 + 4n + 1 = (2n^2 + 2n) + (2n^2 + 2n + 1) \implies (2n+1)^2 = (2n^2 + 2n)^2 + (2n^2 + 2n + 1)^2$$

Lemma 5.6: Let $k \in \mathbb{Z}, k \geq 2$ and let a, b relatively prime integers such that $ab = n^k$. Then a, b are each k th powers of integers

Proof: Let $n = 2^{x_2}3^{x_3} \dots$. Then $ab = n^k = 2^{kx_2}3^{kx_3} \dots$

Let p be a prime in the prime factorization of a and p^c be the exact power of p in the factorization of a

Since $\gcd(a, b) = 1$, p doesn't occur in the factorization of b , so p^c occurs in ab and n^k has p^{kx_p} as the power of p

Since prime factorization is unique, we have $c = kx_p \implies$ every prime in factorization of a occurs with a power of a multiple of k

Thus a is a k th power integer. Similar for b

Lemma 5.7: The square of an odd integer is 1 more than a multiple of 8. The square of an even integer is a multiple of 4

Proof: Let n be even then $n = 2k \implies n^2 = 4k^2 \implies 4 \mid n^2$

Let n be odd $\implies n = 2k + 1 \implies n^2 = 4k(k+1) + 1$

Since k or $k+1$ is even, we have $4k(k+1)$ is a multiple of 8. Thus n^2 is 1 more than a multiple of 8

Theorem 5.5: Let (a, b, c) be a Primitive Pythagorean triple. Then c is odd and exactly one of a, b is even and the other is odd. Assume b is even, then there are relatively prime integers m, n such that $m < n$ and one odd and the other even such that

$$a = n^2 - m^2 \quad b = 2mn \quad c = m^2 + n^2$$

Proof: Let $a^2 + b^2 = c^2$ and $\gcd(a, b, c) = 1$

Suppose by contradiction that both a, b are odd, then by Lemma 5.7, $a^2 + b^2$ is 2 more than a multiple of 8

Thus $a^2 + b^2$ is not a multiple of 4 so by Lemma 5.7, $a^2 + b^2$ cannot be a square. Thus at least one of a, b is even

Suppose by contradiction that both a, b are even. Then $c^2 = a^2 + b^2$ is even so c is even.

But then 2 is common divisor of a, b, c but we have $\gcd(a, b, c) = 1$. Contradiction

Thus one of a, b is even and the other is odd. WLOG let a be odd and b be even

Then we have $a^2 + b^2 = c^2$ is odd.

Let $b = 2b_1$ so we have $c^2 - a^2 = (c + a)(c - a) = b^2 = 4b_1^2$

Thus we have $(\frac{c+a}{2})(\frac{c-a}{2}) = b_1^2$. Since c, a are odd we must have $\frac{c+a}{2}$ and $\frac{c-a}{2} \in \mathbb{Z}$

Let $d = \gcd(\frac{c+a}{2}, \frac{c-a}{2})$ and suppose by contradiction $d > 1$. Then let p be a prime dividing d

Then $c = \frac{c+a}{2} + \frac{c-a}{2}$ and $a = \frac{c+a}{2} - \frac{c-a}{2}$ are multiples of p

Thus $c^2 - a^2 = b^2$ is a multiple of $p \implies p \mid b$ so p is a common divisor of a, b, c , contradicting that $\gcd(a, b, c) = 1$. Thus $d = 1$

Thus we have two relatively prime integers: $\frac{c+a}{2}$ and $\frac{c-a}{2}$ whose product is a square

By Lemma 5.6, each factor is a square so $\frac{c-a}{2} = m^2$ and $\frac{c+a}{2} = n^2$

Thus $c = \frac{c+a}{2} + \frac{c-a}{2} = n^2 + m^2$ and $a = \frac{c+a}{2} - \frac{c-a}{2} = n^2 - m^2$

Thus $b^2 = c^2 - a^2 = (n^2 + m^2)^2 - (n^2 - m^2)^2 = 4m^2n^2 \implies b = 2mn$

Since $\frac{c-a}{2} = m^2$ and $\frac{c+a}{2} = n^2$ are relatively prime, then $\gcd(n, m) = 1$

Finally since $m^2 + n^2 = c$ is odd, one of m, n is odd and the other is even

5.5 Difference of Squares

Theorem 5.8: Let $m \in \mathbb{Z}^+$. Then m is a difference of 2 squares if and only if either m is odd or m is a multiple of 4

Proof: \Leftarrow Let m be odd then $m = 2n + 1 = (n + 1)^2 - n^2$.

Otherwise let m be a multiple of 4 then $m = 4n = (n + 1)^2 - (n - 1)^2$

\implies Suppose $m = x^2 - y^2 = (x + y)(x - y)$. Since $x + y, x - y$ differ by $2y$ (even) they are either both even or both odd

- If they are both even, then $m = (x + y)(x - y)$ is the product of 2 even numbers and is thus a multiple of 4
- If both are odd, then m is clearly odd

As an aside, suppose $m = uv$ where u, v have the same parity and $u \geq v$

If we let $x = \frac{(u+v)}{2}$ and $y = \frac{(u-v)}{2}$ then clearly $x, y \in \mathbb{Z}$ since u, v have the same parity

And we have $x^2 - y^2 = \frac{(u+v)^2}{4} - \frac{(u-v)^2}{4} = uv = m$

Upshot: Writing m as a difference of 2 squares corresponds to factorizing m into 2 factors of the same parity

Example: $m = 15 \implies 15 * 1 = 8^2 - 7^2$ where $8 + 7 = 15$ and $8 - 7 = 1$

$m = 15 \implies 5 * 3 = 4^2 - 1^2$ where $4 + 1 = 5$ and $4 - 1 = 3$

Example: $m = 60 \implies 30 * 2 = 16^2 - 14^2$

$m = 60 \implies 10 * 6 = 8^2 - 2^2$

5.6 Prime Factorization of Factorials

Theorem 5.9: Let $n \geq 1$ and p be a prime. If we write $n! = p^b c$ with $p \nmid c$, then

$$b = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \cdots$$

Proof: write $n = qp + r$ for $0 \leq r < p$. Clearly multiples of p up to n are $p, 2p, \dots, qp$

but we see that $\left\lfloor \frac{n}{p} \right\rfloor = \left\lfloor q + (r/p) \right\rfloor = q$ so there are $\left\lfloor \frac{n}{p} \right\rfloor$ multiples of p up to n

Similarly, there are $\left\lfloor \frac{n}{p^j} \right\rfloor$ multiples of p^j up to n

Thus we can write $b = (\# \text{ of multiples of } p \text{ up to } n) + (\# \text{ of multiples of } p^2 \text{ up to } n) + \cdots$

Take m such that $1 \leq m \leq n$ and $m = p^k m_1$ with $p \nmid m_1$.

Then m contributes p^k to $n!$ and contributes k to the exponent b since m is a multiple of p^j for $1 \leq j \leq k$

Example: $n = 30, p = 5 \implies \left\lfloor \frac{30}{5} \right\rfloor + \left\lfloor \frac{30}{25} \right\rfloor = 6 + 1 \implies 5^7$ is the power of 5 in $30!$

Example: $n = 30, p = 2 \implies \left\lfloor \frac{30}{2} \right\rfloor + \left\lfloor \frac{30}{4} \right\rfloor + \left\lfloor \frac{30}{8} \right\rfloor + \left\lfloor \frac{30}{16} \right\rfloor = 15 + 7 + 3 + 1 = 26 \implies 2^{26}$ is the power of 2 in $30!$

Thus $2^{26} 5^7 = 2^{19} 10^7 \implies 30!$ has 7 zeros at the end

5.7 Riemann Zeta Function

Definition - Riemann Zeta Function: For a real number $s > 1$, we define the **Riemann zeta function** as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Theorem 5.10: If $s > 1$, then

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \quad \text{for all primes } p$$

Proof:

Note that the geometric series $1 + r + r^2 + \dots = \frac{1}{1-r} = (1-r)^{-1}$ for $|r| < 1$

Letting $r = p^{-1}$, we get

$$1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots = (1 - p^{-s})^{-1}$$

As an example, consider the product

$$\begin{aligned} (1 - 2^{-s})^{-1}(1 - 3^{-s})^{-1} &= (1 + \frac{1}{2^s} + \frac{1}{4^s} + \dots)(1 + \frac{1}{3^s} + \frac{1}{9^s} + \dots) \\ &= (1 + \frac{1}{2^s} + \frac{1}{4^s} + \dots) + (\frac{1}{3^s} + \frac{1}{2^s 3^s} + \frac{1}{4^s 3^s} + \dots) + (\frac{1}{9^s} + \frac{1}{2^s 9^s} + \frac{1}{4^s 9^s} + \dots) \\ &= \sum_{n \in S(2,3)} \frac{1}{n^s} \quad S(p, q) \text{ are all integers whose prime factorizations only use } p, q \end{aligned}$$

Now consider using m primes

$$(1 - 2^{-s})^{-1}(1 - 3^{-s})^{-1} \dots (1 - p_m^{-s})^{-1} = \sum_{n \in S(2,3,\dots,p_m)} \frac{1}{n^s}$$

The LHS converges to the product over all primes. Since every positive integer has a prime factorization, each n lies in $S(2,3,\dots,p_m)$. Thus RHS converges to the sum over all positive integers n

Infinite Primes Proof: BWOC suppose there are only a finite number of primes. Then

$$\lim_{s \rightarrow 1^+} \prod_p (1 - p^{-s})^{-1} = \prod_p (1 - p^{-1})^{-1}$$

is a finite product and thus must itself be finite

Furthermore, since each of the functions used in the product is continuous at $s = 1$, we have that for $n > 1, x \geq n, s > 1$

$$x^s \geq n^s \implies \frac{1}{n^s} \geq \frac{1}{x^s} \implies \int_n^{n+1} \frac{1}{n^s} dx \geq \int_n^{n+1} \frac{1}{x^s} dx$$

Thus we have

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \geq \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^s} = \int_1^{\infty} \frac{1}{x^s} dx = \frac{1}{s-1}$$

Thus $\zeta(s) \geq \frac{1}{s-1}$ diverges as $s \rightarrow 1^+$. Contradiction since we showed that $\prod_p (1 - p^{-s})^{-1}$ converges

Thus there are an infinite number of primes

6 Congruences

6.1 Definitions and Examples

Definition - Congruence: $a \equiv b \pmod{m}$ if $a - b$ is a multiple of m

Proposition 6.2: $a \equiv b \pmod{m}$ if and only if $a = b + km$ for some $k \in \mathbb{Z}$

Proof: $a \equiv b \pmod{m}$ if and only if $a - b$ is a multiple of m . Thus $a - b = km \implies a = b + km$

Looking at integers mod m , we get m **congruent classes**. Each integer is only in one congruent class mod m

Proposition 6.3: Let $a \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$ then $\exists! r$, with $0 \leq r \leq m - 1$ such that $a \equiv r \pmod{m}$

Proof: By division algorithm, we have \exists unique q, r such that $a = mq + r$ with $0 \leq r \leq m - 1$

Thus from the previous proposition, $a \equiv r \pmod{m}$

Proposition 6.4: Let $a, b, c \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$. Then

- $a \equiv a \pmod{m}$
- $a \equiv b \pmod{m} \implies b \equiv a \pmod{m}$
- $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m} \implies a \equiv c \pmod{m}$

Proof:

- $a = a + 0 * m \implies a \equiv a \pmod{m}$
- $a \equiv b \pmod{m} \implies a = b + km \implies b = a + (-k)m \implies b \equiv a \pmod{m}$
- $a - c = (a - b) + (b - c) = (k_1 + k_2)m \implies a \equiv c \pmod{m}$

Proposition 6.5: Let $a, b, c, d \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then

- $a + c \equiv b + d \pmod{m}$
- $a - c \equiv b - d \pmod{m}$
- $ac \equiv bd \pmod{m}$

Proof: $a \equiv b \pmod{m} \implies a = b + k_1m$ and $c \equiv d \pmod{m} \implies c = d + k_2m$

- $a + c = (b + d) + (k_1 + k_2)m \implies a + c \equiv b + d \pmod{m}$
- $a - c = (b - d) + (k_1 - k_2)m \implies a - c \equiv b - d \pmod{m}$
- $ac = (b + k_1m)(d + k_2m) = bd + (bk_2 + dk_1 + k_1k_2m)m \implies ac \equiv bd \pmod{m}$

Corollary 6.6: $a \equiv b \pmod{m} \implies a^n \equiv b^n \pmod{m}$ for $n \in \mathbb{Z}^+$

Proof: By the previous proposition, $a \equiv b \pmod{m} \implies a^2 \equiv b^2 \pmod{m}$. Repeated multiplication yields $a^n \equiv b^n \pmod{m}$

Proposition 6.7: $ac \equiv bc \pmod{m}$ and $\gcd(c, m) = 1 \implies a \equiv b \pmod{m}$

$ac \equiv bc \pmod{m} \implies m \mid (ac - bc) \implies m \mid c(a - b)$

If c, m are relatively prime, then we must have $m \mid a - b \implies a \equiv b \pmod{m}$

Proposition 6.8: $ac \equiv bc \pmod{m}$ and $\gcd(c, m) = d \implies a \equiv b \pmod{\frac{m}{d}}$ and $a = b + (\frac{m}{d})k$ with $0 \leq k \leq d - 1$

Proof: $ac \equiv bc \pmod{m} \implies m \mid c(a - b) \implies \frac{m}{d} \mid \frac{c}{d}(a - b)$

Since $\gcd(c, m) = d$, we must have $\gcd(\frac{m}{d}, \frac{c}{d}) = 1 \implies \frac{m}{d} \mid a - b \implies a \equiv b \pmod{\frac{m}{d}}$

Furthermore, $a - b = m(\frac{d}{k})$ where $\frac{d}{k} \in \mathbb{Z} \implies 0 \leq k \leq d - 1$

Various ways to solve equations of the form $ax \equiv b \pmod{m}$:

- Add m to b until we find an easy factor of a

Example: $2c \equiv 7 \pmod{9} \equiv 16 \pmod{9} \implies c = 8$

- Use Proposition 6.8 and divide a, b by a common factor c and m by $\gcd(c, m)$

Example: $6c \equiv 18 \pmod{21} \implies c \equiv 3 \pmod{7}$.

Note: Answer is in terms of mod 7

- Divide a, b, m by a common factor. Then solve the reduced congruence

Example: $15x \equiv 25 \pmod{55} \implies 3x \equiv 5 \pmod{11} \implies x \equiv 9 \pmod{11}$

Proposition 6.9: Let $n \in \mathbb{Z}^+$ and $a, b \in \mathbb{Z}$. Then $a \equiv b \pmod{n} \implies \gcd(a, n) = \gcd(b, n)$

Proof: $a \equiv b \pmod{n} \implies a = b + nk$. Let d be a divisor of b, n . Then $d \mid a$ since a is a linear combination of b, n

We also must have $b = a - nk \implies$ any common divisor of a, n is also a divisor of b

Thus the set of common divisors for a, n is the same as the set of common divisors of b, n . Thus $\gcd(a, n) = \gcd(b, n)$

Example: $\gcd(1234, 10) = \gcd(4, 10)$ since $1234 \equiv 4 \pmod{10}$

Proposition 6.10: If p is a prime and $ab \equiv 0 \pmod{p}$. Then $a \equiv 0 \pmod{p}$ or $b \equiv 0 \pmod{p}$

Proof: $ab \equiv 0 \pmod{p} \implies p \mid ab$. Thus by theorem, $p \mid a$ or $p \mid b \implies a \equiv 0 \pmod{p}$ or $b \equiv 0 \pmod{p}$, respectively

Corollary 6.11: Let p be a prime. Then $x^2 \equiv 1 \pmod{p}$ has only solutions $x \equiv \pm 1 \pmod{p}$

Proof: $x^2 \equiv 1 \pmod{p} \iff x^2 - 1 \equiv 0 \pmod{p} \iff (x - 1)(x + 1) \equiv 0 \pmod{p}$

By the previous Proposition, this only happens when $x - 1 \equiv 0 \pmod{p}$ or $x + 1 \equiv 0 \pmod{p}$

Thus the only possible solutions are $x \equiv \pm 1 \pmod{p}$

Exercise 6.34

6.2 Modular Exponentiation

Consider $3^{385} \pmod{479}$

Using **repeated squaring**, we see that

$$\begin{aligned} 3^2 &\equiv 9 \pmod{479} \\ 3^4 &\equiv 81 \pmod{479} \\ 3^8 &\equiv 81^2 \equiv 334 \pmod{479} \\ 3^{16} &\equiv 334^2 \equiv 428 \pmod{479} \\ 3^{32} &\equiv 428^2 \equiv 206 \pmod{479} \\ 3^{64} &\equiv 206^2 \equiv 284 \pmod{479} \\ 3^{128} &\equiv 284^2 \equiv 184 \pmod{479} \\ 3^{256} &\equiv 184^2 \equiv 326 \pmod{479} \end{aligned}$$

Thus we see that

$$3^{385} \equiv 3^{256} 3^{128} 3^1 \equiv 326 * 184 * 3 \equiv 327 \pmod{479}$$

6.3 Divisibility Tests

For $a \in N$, we can express a in base 10 as

$$a = a_0 + 10^1 a_1 + \cdots + 10^k a_k \quad 0 \leq a_i \leq 9$$

Axiom: $2 \mid a$ if and only if $2 \mid a_0 \implies a \equiv a_0 \pmod{2}$

Proposition 6.12: $10 \mid a$ if and only if $a_0 = 0$ AND $5 \mid a$ if and only if $a_0 = 0$ or $a_0 = 5$

Proof:

Let $a = a_0 + 10a_1 + \cdots + 10^k a_k \quad 0 \leq a_i \leq 9$

- \implies Suppose $10 \mid a \implies 10 \mid a_0 \implies a_0 = 0$ since $0 \leq a_0 \leq 9$

- \Leftarrow Suppose $a_0 = 0 \implies a = 10a_1 + \cdots + 10^k a_k \implies 10 \mid a$

- We prove that $a \equiv a_0 \pmod{5}$

$$a = a_0 + 10(a_1 + 10a_2 + \cdots + 10^{k-1} a_k) \implies a \equiv a_0 \pmod{5}$$

Thus it follows that $5 \mid a$ if and only if $a_0 \equiv 0 \pmod{10} \implies a_0 = 0$ or $a_0 = 5$

Corollary 6.12.1: $a \equiv a_0 \pmod{10}$

Proposition 6.13: $4 \mid a$ if and only if $4 \mid 10a_1 + a_0$ AND $8 \mid a$ if and only if $8 \mid 100a_2 + 10a_1 + a_0$

Proof:

- Note that $4 \mid 10^j$ for $j \geq 2$. Thus $a \equiv 10a_1 + a_0 \pmod{4} \implies 4 \mid a$ if and only if $4 \mid 10a_1 + a_0$

- Note that $8 \mid 10^j$ for $j \geq 3$. Thus $a \equiv 100a_2 + 10a_1 + a_0 \pmod{8} \implies 8 \mid a$ if and only if $8 \mid 100a_2 + 10a_1 + a_0$

Proposition 6.14: An integer mod 3 (respectively, mod 9) is congruent to the sum of its digits mod 3 (respectively, mod 9)

Proof: Clearly $10 \equiv 1 \pmod{3}$. Since $1^k = 1$ for all integers k , we have

$$10^k \equiv 1^k \equiv 1 \pmod{3}$$

Thus when we look at n expanded in its base 10 form mod 3, we get

$$n = a_m 10^m + \cdots + a_1 10 + a_0 \equiv a_m + \cdots + a_1 + a_0 \pmod{3}$$

Identical for mod 9

Corollary 6.15: An integer n is divisible by 3 if and only if the sum of its digits are divisible by 3. It is divisible by 9 if and only if the sum of its digits is divisible by 9

Example: $8675309 \equiv 38 \pmod{9} \equiv 11 \pmod{9} \equiv 2 \pmod{9}$

Proposition 6.15.1: $6 \mid a$ if and only if $2 \mid a$ and $3 \mid a$

Proof: \implies Suppose $6 \mid a$. Then any factor of 6 also divides a

\Leftarrow Suppose $2 \mid a$ and $3 \mid a$. Then by the unique prime factorization of a , we know that $6 \mid a$

Corollary 6.15.2: $a \equiv 0 \pmod{6}$ if and only if $a_0 \equiv 0 \pmod{2}$ AND $\sum_{n=0}^k a_n \equiv 0 \pmod{3}$

Proposition 6.16: $a \equiv a_0 + a_1 + a_2 + \cdots + (-1)^k a_k \pmod{11}$

Proof: Note that $10 \equiv -1 \pmod{11} \implies 10^k \equiv (-1)^k \pmod{11}$

Thus when we look at n expanded in its base 10 form mod 11, we get

$$n = a_m 10^m + \cdots + a_1 10 + a_0 \equiv a_0 - a_1 + \cdots + (-1)^m a_m \pmod{11}$$

Corollary 6.17: An integer n is divisible 11 if and only if the alternating sum of its digits is divisible by 11

Proposition 6.17.1: To test if $7 \mid a$, take a , truncate the last digit and subtract the rest of the digit by $2 * a_0$. Repeat until we reach one digit and it is 0 or 7. Then $7 \mid a$. Otherwise $7 \nmid a$

Proof:

$$\begin{aligned} a &= a_0 + 10(a_1 + 10a_2 + \cdots + 10^{k-1}a_k) \\ &\equiv (-20)a_0 + 10(a_1 + \cdots + 10^{k-1}a_k) \pmod{7} \\ &\equiv 10(-2a_0 + a_1 + 10a_2 + \cdots + 10^{k-1}a_k) \pmod{7} \end{aligned}$$

Thus $7 \mid a \implies 7 \mid (-2a_0 + a_1 + 10a_2 + \cdots + 10^{k-1}a_k)$, which is the recursion we created above

Example: Consider $n = 42735$

$$\begin{aligned} 4273 - 2(5) &= 4263 \\ 426 - 2(3) &= 420 \\ 42 - 2(0) &= 42 \\ 4 - 2(2) &= 0 \end{aligned}$$

Thus $7 \mid 42735$

6.4 Linear Congruences

Theorem 6.18: Let $m \in \mathbb{Z}^+$ and $a \neq 0$. Then $ax \equiv b \pmod{m}$ has a solution if and only if $d = \gcd(a, m)$ divides b . If $d \mid b$, then there are exactly d solutions distinct mod m . Let x_0 be a solution, then the other solutions are of the form

$$x = x_0 + \left(\frac{m}{d}\right)k \quad 0 \leq k \leq d$$

Where x_0 can be found by satisfying

$$\left(\frac{a}{d}\right)x_0 \equiv \left(\frac{b}{d}\right) \pmod{\frac{m}{d}}$$

Proof: $ax \equiv b \pmod{m} \iff ax = b + my \iff -my + ax = b$. This is a Diophantine problem with $(-m, a, b)$

Let $d = \gcd(a, m)$. If $d \nmid b$, then there are no solutions

Otherwise let $d \mid b \implies$ solutions are of the form

$$x = x_0 + \left(\frac{m}{d}\right)k \quad y = y_0 + \left(\frac{a}{d}\right)k$$

Which implies that $x \equiv x_0 \pmod{\frac{m}{d}}$

To show that these solutions are distinct mod m , let $x_1 = x_0 + \left(\frac{m}{d}\right)k_1$ and $x_2 = x_0 + \left(\frac{m}{d}\right)k_2$ be distinct solutions and suppose $x_1 \equiv x_2 \pmod{m}$

Then $x_1 - x_2 = mk_3 \iff \left(\frac{m}{d}\right)(k_1 - k_2) = mk_3 \iff k_1 - k_2 = dk_3 \implies k_1 \equiv k_2 \pmod{d}$

• **Note** that $0 \leq k \leq d - 1$

Finally, to show that x_0 arises from solving $\left(\frac{a}{d}\right)x_0 \equiv \frac{b}{d} \pmod{\frac{m}{d}}$,

Note that $\left(\frac{a}{d}\right)x_0 = \frac{b}{d} + \left(\frac{m}{d}\right)z \implies ax_0 = b + mz \implies ax_0 \equiv b \pmod{m}$

Thus x_0 is a solution we desire

Corollary 6.19: If $\gcd(a, m) = 1$, then $ax = b \pmod{m}$ has exactly 1 solution mod m

Proof: Let $d = 1$ and apply Theorem 6.18. Then $d \mid b \implies$ there is only 1 solution

Example: $6x \equiv 7 \pmod{15}$ has no solutions because $\gcd(6, 15) = 3$ but $3 \nmid 7$

Example: $5x \equiv 6 \pmod{11} \implies x = 10$ is a unique solution since $\gcd(5, 11) = 1$

Example: $9x \equiv 6 \pmod{15}$ has $\gcd(9, 15) = 3$ solutions mod 15

Reducing the equation, we get $3x \equiv 2 \pmod{5} \implies x_0 = 4 \implies$ solutions are $\{4, 4 + \frac{15}{3}, 4 + 2 * \frac{15}{3}\} = \{4, 9, 14\}$

We can also solve linear congruence problems using Extended Euclidean Algorithm

Example: $183x \equiv 15 \pmod{31} \implies 28x \equiv 15 \pmod{31}$

Converting it into a Linear Diophantine problem, we get $28x - 31y = 15$. Now we find $\gcd(28, 31)$

$$\begin{aligned} 31 &= 1 * 28 + 3 \\ 28 &= 9 * 3 + 1 \\ 3 &= 3 * 1 \end{aligned}$$

Thus $\gcd(28, 31) = 1$. Now we write it as a linear combination of 28, 31

$$\begin{aligned} 31 &= 1 * 31 + 0 * 28 \\ 28 &= 0 * 31 + 1 * 28 \\ 3 &= 1 * 31 - 1 * 28 \\ 1 &= 1 * 28 - 9 * 3 = -9 * 31 + 10 * 28 \end{aligned}$$

Thus $28(10) + 31(-9) = 1 \implies 28(150) + 31(-135) = 15 \implies 28(150) \equiv 15 \pmod{31} \implies x \equiv 150 \equiv 26 \pmod{31}$

Definitino - Multiplicative Inverse: a has a **multiplicative inverse** b if $ab \equiv 1 \pmod{m}$

Corollary 6.21: a has an inverse mod m if and only if $\gcd(a, m) = 1$

Proof: From Theorem 6.18, $ax = 1 \pmod{m}$ has a solution if and only if $\gcd(a, m) \mid 1 \iff \gcd(a, m) = 1$

Example: $7x \equiv 4 \pmod{19}$ where $7^{-1} \equiv 11 \pmod{19}$

$$77x \equiv 44 \pmod{19} \implies x \equiv 6 \pmod{19}$$

Steps to solve $ax \equiv b \pmod{m}$ where $\gcd(a, m) = 1$

1. Convert the problem into Linear Diophantine problem $ax - my = b$
2. Use Extended Euclidean Algorithm to find x_0, y_0 such that $ax_0 - my_0 = 1$
3. Compute $x = bx_0$

Steps to find an inverse of $a \pmod{m}$ with $\gcd(a, m) = 1$

1. Convert the problem into Linear Diophantine problem $ax - my = 1$
2. Use Extended Euclidean Algorithm to find x_0, y_0 such that $ax_0 - my_0 = 1$
3. $x_0 \pmod{m}$ is the inverse of $a \pmod{m}$

6.5 Chinese Remainder Theorem

Theorem 6.22: Let m, n be relatively prime. Then the system of congruences

$$\begin{aligned} x &\equiv a \pmod{m} \\ x &\equiv b \pmod{n} \end{aligned}$$

Has a unique solution mod mn

Existence Proof: $x \equiv a \pmod{m} \implies x = a + mt \equiv b \pmod{n} \implies mt \equiv (b - a) \pmod{n}$

Since m, n are relatively prime, there is a unique solution (call it t_0). Clearly $x = a + mt_0$ is a solution to both congruences

- $x = a + mt_0 \equiv a \pmod{m}$
- $x = a + mt_0 \equiv a + (b - a) \equiv b \pmod{n}$

Uniqueness Proof: Let x_1, x_2 be 2 different solutions. Then we must have

$$\begin{aligned} x_1 &\equiv a \pmod{m} & x_1 &\equiv b \pmod{n} \\ x_2 &\equiv a \pmod{m} & x_2 &\equiv b \pmod{n} \end{aligned}$$

Thus $x_1 \equiv x_2 \pmod{m}$ and $x_1 \equiv x_2 \pmod{n} \implies m \mid (x_1 - x_2)$ and $n \mid (x_1 - x_2) \implies x_1 - x_2$ is multiple of m, n

Since $\gcd(m, n) = 1$, we must have $mn \mid x_1 - x_2 \implies x_1 \equiv x_2 \pmod{mn}$

Example: $x \equiv 2 \pmod{3}$ $x \equiv 4 \pmod{5}$

$$x \equiv 4 \pmod{5} \implies x = 4 + 5k \equiv 2 \pmod{3} \text{ for some } k \in \mathbb{Z}$$

$$\implies 5k \equiv 1 \pmod{3} \implies -1k \equiv 1 \pmod{3} \implies k \equiv 2 \pmod{3}$$

Thus $x = 4 + 5(2 + 3l)$ for some $l \in \mathbb{Z}$

Thus $x \equiv 14 \pmod{15}$

Theorem 6.23 Chinese Remainder Theorem: Let $m_1, m_2, \dots, m_r \in \mathbb{Z}^+$ and are pairwise relatively prime. Then

$$\begin{aligned} x &\equiv a_1 \pmod{m_1} \\ x &\equiv a_2 \pmod{m_2} \\ &\dots \\ x &\equiv a_r \pmod{m_r} \end{aligned}$$

Has a unique solution $x \pmod{m_1 m_2 \cdots m_r}$

Proof by Induction:

Base Case $r = 2$ is handled by previous Theorem

IH: Suppose that for an arbitrary $k \leq n$, we the CRT holds true

IS: Prove CRT is true for $n + 1$

Consider the first n congruences. By IH, they have a unique solution mod $m_1 m_2 \cdots m_n$. Call the solution x_0

Now we have the system

$$\begin{aligned} x &\equiv a_{n+1} \pmod{m_{n+1}} \\ x &\equiv x_0 \pmod{m_1, \dots, m_n} \end{aligned}$$

This is handled by the previous theorem, thus CRT holds for any $n \geq 2$

Example Let $x \equiv 2 \pmod{3}$ $x \equiv 3 \pmod{5}$ $x \equiv 2 \pmod{7}$

Taking the largest modulus, we have $x = 2 + 7k \equiv 3 \pmod{5} \implies 7k \equiv 1 \pmod{5} \implies k \equiv 3 \pmod{5}$

Thus $k = 3 + 5l \equiv 2 \pmod{3}$. Now plugging this back into the original equation for x , we get

$$x = 2 + 7(3 + 5l) = 23 + 35l \equiv 2 \pmod{3}$$

This implies that $l \equiv 0 \pmod{3} \implies l = 3m$

Thus $x = 23 + 35(3m) \equiv 23 \pmod{105}$

Example: $x^2 \equiv 1 \pmod{275 = 5^2 * 11}$ can be broken down into

$$\begin{aligned} x^2 &\equiv 1 \pmod{25} \implies x \equiv 1, 24 \pmod{25} \\ x^2 &\equiv 1 \pmod{11} \implies x \equiv 1, 10 \pmod{11} \end{aligned}$$

Thus solutions are of the form

$$\begin{aligned} x &\equiv 1 \pmod{25} & x &\equiv 1 \pmod{11} \implies x \equiv 1 \pmod{275} \\ x &\equiv 1 \pmod{25} & x &\equiv 10 \pmod{11} \implies x \equiv 76 \pmod{275} \\ x &\equiv 24 \pmod{25} & x &\equiv 1 \pmod{11} \implies x \equiv 199 \pmod{275} \\ x &\equiv 24 \pmod{25} & x &\equiv 10 \pmod{11} \implies x \equiv 274 \pmod{275} \end{aligned}$$

Thus the solutions are $x \equiv \{1, 76, 199, 274\} \pmod{275}$

Upshot: We can factor composite modulus m into distinct prime powers and then solve the system of congruence mod

6.6 Fractions mod m

We can interpret $\frac{a}{b} \pmod{m}$ as $a(b^{-1}) \pmod{m}$ where b^{-1} comes from $bb^{-1} \equiv 1 \pmod{m}$

- Only works when $\gcd(b, m) = 1$. Since these are the only b 's with a multiplicative inverse mod m
- Here we interpret $\frac{1}{b}$ as the number we need to multiply b by to get $1 \pmod{m}$

Example: Calculate $\frac{2}{7} \pmod{19}$

We see that $7^{-1} \equiv 11 \pmod{19}$. Thus $\frac{2}{7} = 2 * 11 \equiv 3 \pmod{19}$

7 Fermat, Euler, and Wilson

7.1 Fermat's Theorem

Lemma 8.3: For a prime p ,

$$(x + y)^p \equiv x^p + y^p \pmod{p}$$

Proof: Using the binomial theorem, we have that

$$(x + y)^p = \sum_{k=0}^p \binom{p}{k} x^k y^{p-k}$$

Where

$$\binom{p}{k} = \frac{p!}{k!(p-k)!} \implies p! = k!(p-k)! \binom{p}{k}$$

Clearly p divides the LHS and thus p must also divide the RHS.

However, for $0 < k < p$, clearly $p \nmid (p-k)!$ and $p \nmid k!$. Thus $p \mid \binom{p}{k}$

Lemma 8.4: Let $b \not\equiv 0 \pmod{p}$, then the set

$$b, 2b, \dots, (p-1)b \pmod{p}$$

contains each nonzero congruence class mod p exactly once

Proof: let $a \not\equiv 0 \pmod{p}$ be arbitrary and look at the linear congruence

$$bx \equiv a \pmod{p}$$

This must have a solution x where $1 \leq x \leq p-1$

Thus a belongs to one of the congruence classes defined by $\{b, 2b, \dots, (p-1)b\} \pmod{p}$

Since a was arbitrary, every congruence class occurs

To show that each congruence class only occurs once, BWOC suppose that

$$bi \equiv bj \pmod{p} \implies i \equiv j \pmod{p} \quad 1 \leq i < j \leq p-1$$

However, the given bounds on i, j make this impossible.

Thus each nonzero congruence class occurs exactly once among the multiples of b

Example: Let $p = 7$ and $b = 2$

Then the numbers $2, 4, 6, 8, 10, 12 \pmod{7}$ are the same as $2, 4, 6, 1, 3, 5 \pmod{7}$

Thus every nonzero congruence class mod 7 is represented exactly once

Fermat's Theorem: For a prime p , the following hold true

- $\forall b \in \mathbb{Z}, b^p - b \equiv 0 \pmod{p}$
- $b \not\equiv 0 \pmod{p} \implies b^{p-1} \equiv 1 \pmod{p}$

Proof 1 (Using Lemma 8.3): Show that $b^p \equiv b \pmod{p}$ by Induction

Base Case: $b = 0 \implies 0^p \equiv 0 \pmod{p}$ and $b = 1 \implies 1^p \equiv 1 \pmod{p}$

IH: Assume that for any arbitrary b , we have that $b^p \equiv b \pmod{p}$

IS: Show for $b + 1$. From the binomial coefficients formula and Lemma 8.3, we see that

$$(b + 1)^p \equiv b^p + 1 \equiv \underbrace{b + 1}_{\text{by IH}} \pmod{p}$$

The above proves Fermat's Theorem for non-negative integers

Now for negative integers, suppose that $b < 0$. Then for an odd prime p , we have $(-b)^p \equiv -b \pmod{p}$ by the ideas above.

- If p is odd, then $(-1)^p \equiv -1 \pmod{p}$
- If p is 2, then clearly $-b^p \equiv -b \pmod{p} \implies b^p \equiv b \pmod{p}$

Proof 2 (Using Lemma 8.4): Suppose that $b \not\equiv 0 \pmod{p}$.

From Lemma 8.4, we know that

$$\prod_{i=1}^{p-1} i \equiv \prod_{i=1}^{p-1} bi \pmod{p} \implies (p-1)! \equiv b^{p-1}(p-1)!$$

Since $p \nmid (p-1)!$, we have that

$$b^{p-1} \equiv 1 \pmod{p}$$

Multiplying both sides by b gives the other form

$$b^p \equiv b \pmod{p}$$

Note that for the case where $b \equiv 0 \pmod{p}$, we have that $b^p \equiv 0^p \equiv 0 \equiv b \pmod{p}$

Thus the congruence holds for all $b \in \mathbb{Z}$

Example: $2^6 = 64 \equiv 1 \pmod{7}$ and $2^7 \equiv 2 \pmod{7}$

Example: $3^{28} = (3^4)^7 \equiv 1^7 \equiv 1 \pmod{5}$

- This follows from the second claim in Fermat's Theorem (since $3^{5-1} \equiv 1 \pmod{5}$)

Example: Divide 23 into 7^{200} . What is the remainder?

By Fermat's Theorem, we know that $7^{22} \equiv 1 \pmod{23}$

Thus $7^{200} = (7^{22})^9 * 2^2 \equiv 1^9 * 49 \equiv 3 \pmod{23}$

Corollary 8.2: For prime p and $b \not\equiv 0 \pmod{p}$,

$$x \equiv y \pmod{p-1} \implies b^x \equiv b^y \pmod{p}$$

Proof: We know that $x = y + (p - 1)k$ for some $k \in \mathbb{Z}$

Thus we see that $b^x = b^y b^{(p-1)k} \implies b^x \equiv b^y \pmod{p}$ by Fermat's Theorem

Upshot: We can apply the Divisional Algorithm to the exponent of an integer with $p - 1$ to quickly evaluate congruences mod p

Fermat Primality Test: If n is odd, $b \not\equiv 0 \pmod{n}$, and $b^{n-1} \not\equiv 1 \pmod{n}$, then n is not prime

Proof: Using Fermat's Theorem, we see that for an odd prime p , $b^{p-1} \equiv 1 \pmod{p}$

Now by contraposition, suppose that n is odd and that $b^{n-1} \not\equiv 1 \pmod{n}$, we get that n is not prime

Upshot: We can quickly test if a number n is not prime by looking at $2^{n-1} \not\equiv 1 \pmod{n}$

- **Note:** $2^{n-1} \equiv 1 \pmod{n}$ DOES NOT guarantee n is prime

Example: For $n = 77$, we see that

$$2^{n-1} = 2^{76} \equiv 9 \pmod{77} \not\equiv 1 \pmod{77}$$

Thus 77 is not prime

7.2 Euler's Theorem

Definition - Euler Function: $\phi(n)$ is the number of integers $1 \leq j \leq n$ such that $\gcd(j, n) = 1$

Examples:

- $\phi(12) = 4$ this comes from $\{1, 5, 7, 11\}$
- For any prime p , $\phi(p) = p - 1$

Proposition 8.6: For $m, n \in \mathbb{Z}^+$, if $\gcd(m, n) = 1$ then

$$\phi(mn) = \phi(m)\phi(n)$$

Proposition 8.7: For a prime p and $k \geq 1$,

$$\phi(p^k) = p^k - p^{k-1}$$

Proof: For $1 \leq j \leq p^k$, there are p^{k-1} multiples of p , namely $\{p, 2p, \dots, p^k\}$

These multiples are exactly when $\gcd(j, p^k) \neq 1$

Theorem 8.8: Let $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ be the prime factorization of n where each exponent $a_i \geq 1$. Then

$$\phi(n) = \prod_{i=1}^r (p_i^{a_i} - p_i^{a_i-1}) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

Proof: Applying Propositions 8.6 and 8.7, we see that

$$\phi(n) = \prod_{i=1}^r \phi(p_i^{a_i}) = \prod_{i=1}^r (p_i^{a_i} - p_i^{a_i-1})$$

For the second part of the equality of the theorem, note that $p^a - p^{a-1} = p^a(1 - \frac{1}{p})$. Thus we see that

$$\begin{aligned} \prod_{i=1}^r (p_i^{a_i} - p_i^{a_i-1}) &= \prod_{i=1}^r p_i^{a_i} (1 - \frac{1}{p_i}) \\ &= n \prod_{i=1}^r (1 - \frac{1}{p_i}) \\ &= n \prod_{p|n} (1 - \frac{1}{p}) \quad \text{since each } a_i \geq 1 \end{aligned}$$

Example: $\phi(100)$

- Applying Propositions 8.6, 8.7, we get that $\phi(100) = \phi(2^2)\phi(5^2) = (2^2 - 2)(5^2 - 5) = 40$
- Applying Theorem 8.8, we get that $\phi(100) = 100(1 - \frac{1}{2})(1 - \frac{1}{5}) = 40$

Lemma 8.10: Let T be the set of $1 \leq j \leq n$ with $\gcd(j, n) = 1$. Choose any b with $\gcd(b, n) = 1$ and let $bT \pmod n$ be the set of numbers of the form $bt \pmod n$ for $t \in T$. Then each $t \in T$ is congruent to exactly one element of $bT \pmod n$

Proof: Let $t \in T$.

Then $\gcd(t, n) = \gcd(b, n) = 1 \implies \gcd(bt, n) = 1$

Let $bt \equiv c \pmod n$. Then $\gcd(c, n) = 1$

Thus $c \in T$ so $bT \pmod n \subseteq T$

To show that $T = bT \pmod n$, we show that they have the same number of elements

Take $t_1, t_2 \in T$ with $bt_1 \equiv bt_2 \pmod n$

From the second equation, we get that $t_1 \equiv t_2 \pmod n$. However elements of T are distinct mod n . Thus we have a contradiction

Thus distinct elements of T must remain distinct when multiplied by b under mod n

Thus $bT \pmod n$ and T have the same elements and are thus equal

Example: Let $n = 12, b = 5$

Then we have $T = \{1, 5, 7, 11\}$ and $bT = \{5, 25, 35, 55\} \equiv \{5, 1, 11, 7\} \pmod{12} = T$

Euler's Theorem: For any b such that $\gcd(b, n) = 1$, we have that

$$b^{\phi(n)} \equiv 1 \pmod n$$

- **Note:** This generalizes Fermat's Theorem since $\phi(p) = p - 1$

Proof: Consider the set T from Lemma 8.10. Then

$$\prod_{i \in T} i \equiv \prod_{i \in T} bi \equiv b^{\phi(n)} \prod_{i \in T} i \pmod n$$

Lemma 8.10 says that the second product is just a rearrangement of the first product. Thus we get that

$$1 \equiv b^{\phi(n)} \pmod{n}$$

Example: $\phi(10) = 4$ and $\gcd(3, 10) = 1 \implies 3^4 = 81 \equiv 1 \pmod{10}$

Example: $3^{84} \pmod{100}$

We see that $\phi(100) = 40$ so by Euler's Theorem, we have that $3^{40} \equiv 1 \pmod{100}$

Thus $3^{84} = (3^{40})^2 3^4 \equiv 81 \pmod{100}$

Corollary 8.11: Take $b \in \mathbb{Z}$ such that $\gcd(b, n) = 1$. Then

$$x \equiv y \pmod{\phi(n)} \implies b^x \equiv b^y \pmod{n}$$

- **Note:** This also generalizes the Corollary of Fermat's Theorem since $\phi(p) = p - 1$

Proof: We know that $x = y + \phi(n)k$ for some $k \in \mathbb{Z}$

Thus we see that $b^x \equiv b^y (b^{\phi(n)})^k \equiv b^y \pmod{n}$

Example: Let $n = 15$. Then we have $\phi(n) = 8$ and $9 \equiv 1 \pmod{8}$

Thus $2^9 \equiv 2^1 \pmod{15}$

Example: Let $n = 10$. Then $\phi(n) = 4$ and $5 \equiv 1 \pmod{4}$

Thus for any b such that $\gcd(b, 10) = 1$, we have that $b^5 \equiv b \pmod{10}$

Thus b^5 and b have the same last digit for $b \in \{1, 3, 7, 9\}$

Example: Given $m \in \mathbb{Z}$, let $\gcd(m, 77) = 1$ and let $c \equiv m^7 \pmod{77}$. Find $c^{43} \pmod{77}$

$\phi(77) = 60$ and $301 \equiv 1 \pmod{60}$

Thus we see that $c^{43} \equiv (m^7)^{43} \equiv m^{301} \equiv m \pmod{77}$

Example: Find the last digit of 3^{7^5}

First, note that $\phi(4) = 2$ and $5 \equiv 1 \pmod{2}$

Thus $7^5 \equiv 7^1 \equiv 3 \pmod{4}$

Furthermore, we see that $\phi(10) = 4$.

Thus $3^{7^5} \equiv 3^3 \equiv 27 \equiv 7 \pmod{10}$

7.3 Wilson's Theorem

Wilson's Theorem: For a prime p

$$(p-1)! \equiv -1 \pmod{p}$$

Proof: For integers $1 \leq b \leq p-1$, $bx \equiv 1 \pmod{p}$ has a unique solution $1 \leq x \leq p-1$

We pair multiple inverses with each other

- Note that $b^2 \equiv 1 \pmod{p}$ only if $b \equiv \pm 1 \pmod{p}$, so $b \equiv 1$ and $b \equiv p-1 \pmod{p}$ are the only numbers that are paired with themselves

Now rearrange the factors so that each inverse is next to each other. This gives

$$(p-1)! \equiv 1(p-1) \equiv -1 \pmod{p}$$

Example: For $p = 7$, we have $(p-1)! = 6! = 720 \equiv -1 \pmod{7}$

This comes from $6! = (6)(5*3)(4*1)(1) \equiv -1*1*1*1 \equiv -1 \pmod{7}$

Corollary 8.13: For $n \geq 2$, n is prime if and only if $(n-1)! \equiv -1 \pmod{n}$

Proof: \implies If n is prime, then $(n-1)! \equiv -1 \pmod{n}$ by the Wilson's Theorem

\Leftarrow BWOC suppose n is composite. Then $n = ab$ for $a, b \in \mathbb{Z}$ and $1 < a < n$

Thus a is a factor of $(n-1)! \implies (n-1)! \equiv 0 \pmod{a}$.

But we also have that $(n-1)! \equiv -1 \pmod{n} \implies (n-1)! \equiv -1 \pmod{a}$

Contradiction. Thus n must be prime

Example: Let $n = 6$, then $(n-1)! = 5! = 120 \equiv 0 \not\equiv -1 \pmod{6}$

Thus n is not prime

8 RSA

8.1 RSA Encryption

RSA Setup:

1. Alice chooses 2 primes p, q and calculates $n = pq$ and $\phi(n) = (p-1)(q-1)$
2. Alice chooses an encryption key e such that $\gcd(e, \phi(n)) = 1$
3. Alice calculates a decryption key such that $ed \equiv 1 \pmod{\phi(n)}$
4. Alice makes n, e public and d, p, q private

RSA Encryption:

1. Bob looks up Alice's public values n, e
2. Bob writes the message as $m \pmod{n}$
3. Bob computes $c \equiv m^e \pmod{n}$
4. Bob sends c to Alice

RSA Decryption

1. Alice receives c
2. Alice computes $m \equiv c^d \pmod{n}$

Example

Let $p = 3598279$ and $q = 781629$

Then $n = 28122813702491$ $\phi(n) = 28122802288584$ $e = 233$ $d = 27519308677241$

Let $A = 01, B = 02, \dots, Z = 26$ be the alphabet

Suppose Bob wants to send CAR $\implies m = 030118 = 30118$

Then $c \equiv m^e \pmod{n} \equiv 21666077416496 \pmod{n}$

Finally, Alice decrypts the text as $m \equiv c^d \pmod{n}$

Proposition 9.1: Let $n = pq$ for distinct primes p, q , and take e, d satisfying $ed \equiv 1 \pmod{\phi(n)}$. Then for all m , we have

$$m^{ed} \equiv m \pmod{n} \quad c \equiv m^e \pmod{n} \implies m \equiv c^d \pmod{n}$$

Proof: Suppose $\gcd(m, n) = 1$.

Then $ed \equiv 1 \pmod{\phi(n)} \implies ed = 1 + k\phi(n)$ for some $k \in \mathbb{Z}$

Thus using Euler's Theorem, we have

$$m^{ed} \equiv m^{1+k\phi(n)} \equiv m(m^{\phi(n)})^k \equiv m \pmod{n}$$

Otherwise, suppose that $\gcd(m, n) \neq 1$. So possible values are p, q, pq

- $pq \implies m \equiv 0 \pmod{n} \implies m^{ed} \equiv 0 \equiv m \pmod{n}$
- $p \implies m \equiv 0 \pmod{p} \implies m^{ed} \equiv 0 \equiv m \pmod{p}$

However since $q \nmid m$, we have by Fermat Theorem that $m^{q-1} \equiv 1 \pmod{q}$

Thus $m^{ed} \equiv m(m^{q-1})^{k(p-1)} \equiv m \pmod{q}$

Thus $p \mid m^{ed} - m$ and $q \mid m^{ed} - m \implies pq \mid m^{ed} - m \implies m^{ed} \equiv m \pmod{pq}$