5. We will first show that if $p_j^w > \alpha$ then the confidence interval contains 0. To begin it is to be noted we are only worried about $\alpha \in (0,1)$ and thus $\frac{\alpha}{2} \in (0, .5)$

To begin, we assume $p_j^w > \alpha$

$$2\Phi\left(-\left|\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_j}}\right|\right) > \alpha$$

$$\Phi\left(-\left|\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_j}}\right|\right) > \frac{\alpha}{2}$$ We can then take the ~~standard normal quantile~~ inverse normal cdf of both sides because it is an increasing function on our interval

$$-\left|\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_j}}\right| > \Phi^{-1}\left(\frac{\alpha}{2}\right)$$

$$\left|\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_j}}\right| < -\Phi^{-1}\left(\frac{\alpha}{2}\right)$$ We can then rewrite this as the following because we know $\Phi^{-1}\left(\frac{\alpha}{2}\right)$ is negative on our interval so $-\Phi^{-1}\left(\frac{\alpha}{2}\right) = Z_{\frac{\alpha}{2}}$ is always positive

no absolute value

$$-Z_{\frac{\alpha}{2}} < \frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_j}} < Z_{\frac{\alpha}{2}}$$

$$-Z_{\frac{\alpha}{2}}\hat{\sigma}\sqrt{v_j} < \hat{\beta}_j < Z_{\frac{\alpha}{2}}\hat{\sigma}\sqrt{v_j}$$

$$\left(-Z_{\frac{\alpha}{2}}\hat{\sigma}\sqrt{v_j} - \hat{\beta}_j < 0 < Z_{\frac{\alpha}{2}}\hat{\sigma}\sqrt{v_j} - \hat{\beta}_j\right) \cdot -1$$

$$① \quad \hat{\beta}_j + Z_{\frac{\alpha}{2}}\hat{\sigma}\sqrt{v_j} > 0 > \hat{\beta}_j - Z_{\frac{\alpha}{2}}\hat{\sigma}\sqrt{v_j} \quad ②$$

①+② are definition of $100(1-\alpha)\%$ CI

As noted before $Z_{\frac{\alpha}{2}}$ is always positive where we are concerned and $\hat{\sigma}$ is positive because it is the sqrt of a real number and same thing for $\sqrt{v_j}$. Thus the left side of this is always going to be the upper bound of the interval and the right side is always going to be the lower bound thus our confidence interval will ~~always~~ contain 0 if $p_j^w > \alpha$

Now we will show that if the CI contains 0, then $p_j^W > \alpha$

thus by definition if the CI contains 0 then

$$\hat{\beta}_j + Z_{\frac{\alpha}{2}}\hat{\sigma}\sqrt{v_j} > 0 > \hat{\beta}_j - Z_{\frac{\alpha}{2}}\hat{\sigma}\sqrt{v_j}$$

$$-\Phi^{-1}\left(\tfrac{\alpha}{2}\right)\hat{\sigma}\sqrt{v_j} > -\hat{\beta}_j > \Phi^{-1}\left(\tfrac{\alpha}{2}\right)\hat{\sigma}\sqrt{v_j}$$

$$\Phi^{-1}\left(\tfrac{\alpha}{2}\right) < \frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_j}} < -\Phi^{-1}\left(\tfrac{\alpha}{2}\right)$$

$$\left|\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_j}}\right| < -\Phi^{-1}\left(\tfrac{\alpha}{2}\right)$$

$$-\left|\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_j}}\right| > \Phi^{-1}\left(\frac{\alpha}{2}\right)$$

We can then take the norm cdf of both sides since that is strictly increasing on our interval

$$\Phi\left(-\left|\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_j}}\right|\right) > \frac{\alpha}{2}$$

$$2\Phi\left(-\left|\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_j}}\right|\right) > \alpha$$

$$p_j^W > \alpha$$

Therefore if the CI contains 0 then $p_j^W > \alpha$ and we have proven both directions so the Wald test p-value $> \alpha$ iff the corresponding approximate $100(1-\alpha)\%$ CI contains 0