2. a) $\mathbb{P}(D \le d) = 1 - \mathbb{P}(D > d)$

$= 1 - \mathbb{P}(\|x_1\| > d, \dots, \|x_N\| > d)$

$= 1 - (\mathbb{P}(\|x\| > d))^N$

$= 1 - (1 - d^p)^N$

from lecture 2 pg 8 $\mathbb{P}(\|x\| \le d) = d^p$

CDF of D is $1 - (1-d^p)^N$

PDF of D is $\begin{cases} 0, & d \le 0 \text{ and } d > 1 \\ Npd^{(p-1)}(1-d^p)^{N-1}, & 0 < d \le 1 \end{cases}$

$\mathbb{E}[D] = \int_0^1 Npd^p(1-d^p)^{N-1}$ with respect to d

b) Finding $\mathbb{E}[\|pr_a x_i\|^2]$

$\mathbb{E}[\|pr_a x_i\|^2] = \mathbb{E}[\mathbb{E}[\|(a^T x_i)a\|^2 | a]]$

$= \mathbb{E}[\|a\|^2 \mathbb{E}[(a_1 x_1 + \dots + a_p x_p)^2 | a]]$ we know each $x_i \sim N(0,1)$

$= \mathbb{E}[\|a\|^2 \mathbb{E}[a_1^2 x_1^2 + \dots + a_p^2 x_p^2 + a_1 a_2 x_1 x_2 + \dots + a_i a_j x_i x_j + \dots | a]]$

$\mathbb{V}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2$

$1 = \mathbb{E}[x^2]$

$\mathbb{E}[\mathbb{E}[a^2 x^2 | a]]$

$\mathbb{E}[a^2 \mathbb{E}[x^2]]$

$\mathbb{E}[a^2] = a^2$

$\mathbb{E}[\mathbb{E}[a_i a_j x_i x_j | a]]$

$\mathbb{E}[a_i a_j \mathbb{E}[x_i] \mathbb{E}[x_j]]$

$\mathbb{E}[a_i a_j 0]$

0

$= \mathbb{E}[\|a\|^2 (a_1^2 + \dots a_p^2)]$

$= \mathbb{E}[\|a\|^4]$

$= \mathbb{E}[1^4]$

$= 1$

$\|a\| = \left\|\frac{x}{\|x\|}\right\| = \frac{\|x\|}{\|x\|} = 1$

$\mathbb{E}[\|pr_a x_i\|^2] = 1$

Scanned with CamScanner

Finding $\mathbb{E}[\|x\|^2]$

$$\mathbb{E}[\|x\|^2] = \mathbb{E}[x_1^2 + x_2^2 + \dots x_p^2] \quad \text{where } x \sim N(0,1)$$

$$= \mathbb{E}[x_1^2] + \dots + \mathbb{E}[x_p^2] \qquad \mathbb{V}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2$$

$$= p\,\mathbb{E}[x^2] \qquad 1 = \mathbb{E}[x^2]$$

$$= p$$

$$\boxed{\mathbb{E}[\|x\|^2] = p}$$

$\mathbb{E}[\|pr_a X_i\|^2] << \mathbb{E}[\|x\|^2]$ as p is large

$1 << p$