1. a. $\hat{\beta} = \text{argmin}_\beta \, RSS(\beta)$

$RSS(\beta) = \sum_{i=1}^{N} (y_i - f_\beta(x_i))^2$ We can rewrite this as the following where y_{k_i} denotes the i^{th} output of ξ_k and we are still finding the residual of each data point and summing the squares

$$= \sum_{k=1}^{K} \sum_{i=1}^{n_k} (y_{k_i} - f_\beta(\xi_k))^2$$

$$= \sum_{k=1}^{K} \sum_{i=1}^{n_k} (y_{k_i}^2 - 2f_\beta(\xi_k) y_{k_i} + f_\beta(\xi_k)^2)$$

$$= \sum_{k=1}^{K} \left((y_{k_1}^2 + \dots + y_{k_{n_k}}^2) - 2f_\beta(\xi_k)(y_{k_1} + \dots + y_{k_{n_k}}) + n_k f_\beta(\xi_k)^2 \right)$$

$$= \sum_{k=1}^{K} n_k \left(\frac{(y_{k_1}^2 + \dots y_{k_{n_k}}^2)}{n_k} - 2f_\beta(\xi_k) \frac{(y_{k_1} + \dots + y_{k_{n_k}})}{n_k} + f_\beta(\xi_k)^2 \right)$$

$$= \sum_{k=1}^{K} n_k \left(\left(\frac{y_{k_1} + \dots y_{k_{n_k}}}{n_k} \right)^2 - 2f_\beta(\xi_k) \frac{(y_{k_1} + \dots y_{k_{n_k}})}{n_k} + f_\beta(\xi_k)^2 - \left(\frac{\overset{y_{k_1} + \dots + y_{k_{n_k}}}{y_{k_1} + y_{k_{n_k}}}}{n_k} \right)^2 + \frac{(y_{k_1}^2 + \dots y_{k_{n_k}}^2)}{n_k} \right)$$

$$= \sum_{k=1}^{K} n_k \left((\bar{y}_k - f_\beta(\xi_k))^2 - \left(\frac{y_{k_1} + \dots + y_{k_{n_k}}}{n_k} \right)^2 + \frac{(y_{k_1}^2 + \dots y_{k_{n_k}}^2)}{n_k} \right)$$

these are constants

$$= \sum_{k=1}^{K} \left(n_k (\bar{y}_k - f_\beta(\xi_k))^2 - \frac{(y_{k_1} + \dots y_{k_{n_k}})^2}{n_k} + (y_{k_1}^2 + \dots + y_{k_{n_k}}^2) \right)$$

thus minimizing $RSS(\beta)$ is the same as minimizing the first term since the latter two terms are simply constants

$$RSS(\beta) = \sum_{k=1}^{K} n_k (\bar{y}_k - f_\beta(\xi_k))^2 - \frac{\left(\sum_{k=1}^{K} \sum_{i=1}^{n_k} y_{k_i} \right)^2}{n_k} + \sum_{k=1}^{K} \sum_{i=1}^{n_k} y_{k_i}^2$$

$$\text{arg min}_\beta \, RSS(\beta) = \sum_{k=1}^{K} n_k (\bar{y}_k - f_\beta(\xi_k))^2 \to \min$$

therefore

$$\boxed{\hat{\beta} = \text{argmin}_\beta \sum_{k=1}^{K} n_k (\bar{y}_k - f_\beta(\xi_k))^2}$$

b. $\hat{\beta} = \underset{\beta}{\arg\min} \sum_{k=1}^{K} n_k (\bar{y}_k - f_\beta(\xi_k))^2$

$= \underset{\beta}{\arg\min} \sum_{k=1}^{K} n_k (\bar{y}_k - \xi_k^T \beta)^2$

$RSS(\beta) = (\bar{y} - \Xi\beta)^T W (\bar{y} - \Xi\beta) + \ldots$ terms independent of β

$= (\bar{y}^T W - \beta^T \Xi^T W)(\bar{y} - \Xi\beta) + \ldots$

$= \bar{y}^T W \bar{y} - \beta^T \Xi^T W \bar{y} - \bar{y}^T W \Xi \beta + \beta^T \Xi^T W \Xi \beta$

$\nabla_\beta RSS(\beta) = -\nabla_\beta \beta^T \Xi^T W \bar{y} - \nabla_\beta \bar{y}^T W \Xi \beta + \nabla_\beta \beta^T \Xi^T W \Xi \beta$

$\beta^T \Xi^T W \bar{y} = \sum_{j=0}^{p} \beta_j (\Xi^T W \bar{y})_j \Rightarrow \nabla_\beta \beta^T \Xi^T W \bar{y} = \Xi^T W \bar{y}$

∇f is a column vector

$\bar{y}^T W \Xi \beta = \sum_{j=0}^{p} (\bar{y}^T W \Xi)_j \beta_j \Rightarrow \nabla_\beta \bar{y}^T W \Xi \beta = (\bar{y}^T W \Xi)^T = \Xi^T W \bar{y}$

Let $\Xi^T W \Xi = A$ ← $(p+1)\times(p+1)$ matrix $\Rightarrow (\nabla_\beta \beta^T \Xi^T W \Xi \beta) = \frac{\partial}{\partial \beta_k}\left(\sum_{i,j=0}^{p} a_{ij}\beta_i\beta_j\right)$

$= \sum_{i,j=0}^{p} a_{ij} \underbrace{\frac{\partial \beta_i}{\partial \beta_k}}_{\delta_{ik}=\begin{cases}1 & i=k\\ 0 & i\neq k\end{cases}} \beta_j + \sum_{i,j=0}^{p} a_{ij}\beta_i \underbrace{\frac{\partial \beta_j}{\partial \beta_k}}_{\delta_{jk}=\begin{cases}1 & j=k\\ 0 & j\neq k\end{cases}} = \sum_{j=0}^{p} a_{kj}\beta_j + \sum_{i=0}^{p} a_{ik}\beta_i$ a is positive definite

$= 2\sum_{j=0}^{p} a_{kj}\beta_j = 2(A\beta)_k = 2(\Xi^T W \Xi \beta)_k \Rightarrow \nabla_\beta \beta^T \Xi^T W \Xi \beta = 2\Xi^T W \Xi \beta$

$\nabla_\beta RSS(\beta) = -2\Xi^T W \bar{y} + 2\Xi^T W \Xi \beta$

$0 = -2\Xi^T W \bar{y} + 2\Xi^T W \Xi \beta$

$-2\Xi^T W \Xi \beta = -2\Xi^T W \bar{y}$

$$\boxed{\hat{\beta} = (\Xi^T W \Xi)^{-1} \Xi^T W \bar{y}}$$

2. $\overline{err} = \frac{1}{N}\sum_{i=1}^{N}(y_i - x_i^T\hat{\beta})^2$ $\qquad$ $\overline{Err} = \frac{1}{M}\sum_{i=1}^{M}(\tilde{y}_i - \tilde{x}_i^T\hat{\beta})^2$

$$\mathbb{E}[\overline{Err}] = \mathbb{E}\left[\frac{1}{M}\sum_{i=1}^{M}(\tilde{y}_i - \tilde{x}_i^T\hat{\beta})^2\right]$$
$$= \frac{1}{M}\sum_{i=1}^{M}\mathbb{E}[(\tilde{y}_i - \tilde{x}_i^T\hat{\beta})^2]$$
$$= \mathbb{E}[(\tilde{y}_i - \tilde{x}_i^T\hat{\beta})^2] \text{ since each } \tilde{y}_i - \tilde{x}_i^T\hat{\beta} \text{ is i.i.d}$$

This means that $\mathbb{E}[\overline{Err}]$ is independent of M so we can set $M = N$, $\mathbb{E}[\overline{Err}] = \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}(\tilde{y}_i - \tilde{x}_i^T\hat{\beta})^2\right]$

Then given $((\tilde{x}_1,\tilde{y}_1),\dots(\tilde{x}_N,\tilde{y}_N))$ we will define new OLS estimate for this set $\hat{\hat{\beta}} = \underset{\beta}{\operatorname{argmin}} \sum_{i=1}^{N}(\tilde{y}_i - \tilde{x}_i^T\beta)^2$

Thus by definition we know that $\hat{\hat{\beta}}$ minimizes $\sum_{i=1}^{N}(\tilde{y}_i - \tilde{x}_i^T\beta)^2$ so

$$\sum_{i=1}^{N}(\tilde{y}_i - \tilde{x}_i^T\hat{\hat{\beta}})^2 \le \sum_{i=1}^{N}(\tilde{y}_i - \tilde{x}_i^T\beta)^2 \text{ for any } \beta \text{ so we choose } \hat{\beta} \text{ for this}$$

$$\sum_{i=1}^{N}(\tilde{y}_i - \tilde{x}_i^T\hat{\hat{\beta}})^2 \le \sum_{i=1}^{N}(\tilde{y}_i - \tilde{x}_i^T\hat{\beta})^2$$

$$\frac{1}{N}\sum_{i=1}^{N}(\tilde{y}_i - \tilde{x}_i^T\hat{\hat{\beta}})^2 \le \frac{1}{N}\sum_{i=1}^{N}(\tilde{y}_i - \tilde{x}_i^T\hat{\beta})^2$$

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}(\tilde{y}_i - \tilde{x}_i^T\hat{\hat{\beta}})^2\right] \le \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}(\tilde{y}_i - \tilde{x}_i^T\hat{\beta})^2\right]$$

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}(\tilde{y}_i - \tilde{x}_i^T\hat{\hat{\beta}})^2\right] \le \mathbb{E}[\overline{Err}]$$

$\frac{1}{N}\sum_{i=1}^{N}(\tilde{y}_i - \tilde{x}_i^T\hat{\hat{\beta}})^2$ and $\frac{1}{N}\sum_{i=1}^{N}(y_i - x_i^T\hat{\beta})^2$ are identically distributed so their expectations are equal

$$\mathbb{E}[\overline{err}] \le \mathbb{E}[\overline{Err}]$$

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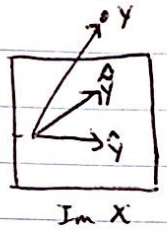
3.

$$f_j = z_j^2$$

$$\frac{\left(RSS(\hat{\tilde{\beta}}) - RSS(\hat{\beta})\right) / \overbrace{(p-\tilde{p})}^{1}}{RSS(\hat{\beta}) / (N-p-1)} = \left(\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_j}}\right)^2$$

$$\frac{\left(RSS(\hat{\tilde{\beta}}) - RSS(\hat{\beta})\right)}{\hat{\sigma}^2} = \frac{\hat{\beta}_j^2}{\hat{\sigma}^2 v_j}$$

$$\left(RSS(\hat{\tilde{\beta}}) - RSS(\hat{\beta})\right) = \frac{\hat{\beta}_j^2}{((X^T X)^{-1})_{jj}}$$



$$\|\hat{y} - \hat{\tilde{y}}\|^2 = \frac{\hat{\beta}_j^2}{((X^T X)^{-1})_{jj}}$$

$A = X(X^T X)^{-1}$ $\quad$ $X^T A = I_{p+1}$

? ? ?

ps2_problem4

May 6, 2020

0.1 IDS/ACM/CS 158: Fundamentals of Statistical Learning

0.1.1 PS2, Problem 4: Linear Regression Analysis of the Prostate Cancer Data

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Notes: Please use python 3.6

You are required to properly comment and organize your code.

- Helper functions (add/remove part label according to the specific question requirements)

[1]:
```
import numpy as np
import numpy.matlib
import scipy.stats
import pandas as pd

def standardize_col(column):
    """
    column - an np array of values from a population

    returns the standardized column with mean 0 and std = 1
    """
    mean = np.mean(column)
    std = np.std(column)

    return (column - mean) / std

def find_beta(data):
    """
    data - a matrix where each row corresponds to the
           p predictors in the first p columns and
           the observed output y in the final column

    returns the OLS estimate of the regression parameter
    """
    x = data[:,:-1]
    y = data[:,-1]
```

```python
    # add bias term to training data
    bias = np.matlib.repmat(1, len(x), 1)
    x = np.concatenate((bias, x), axis=1)

    # calculate beta
    intermediate = np.matmul(x.transpose(), x)
    inverse_intermediate = np.linalg.inv(np.array(intermediate))
    pseudo_x = np.matmul(inverse_intermediate, x.transpose())

    return np.matmul(pseudo_x, y), inverse_intermediate

def predict(ols, data):
    """
    ols - ols estimate of the regression parameter
    data - a matrix where each row corresponds to the
           p predictors in the first p columns and
           the observed output y in the final column

    returns the predictions for the observations in data
    """
    x_with_bias_term = np.concatenate((np.matlib.repmat(1, len(data), 1), data[:
 ↪,:-1]), axis=1)
    return np.matmul(x_with_bias_term, ols)

def rss(data, preds):
    """
    data - a matrix where each row corresponds to the
           p predictors in the first p columns and
           the observed output y in the final column
    preds - the predictions for the observations in data

    returns the residual sum of squares for the values
    """
    return np.sum((data[:,-1] - preds)**2)

def find_sigma(data, preds):
    """
    data - a matrix where each row corresponds to the
           p predictors in the first p columns and
           the observed output y in the final column
    preds - the predictions for the observations in data

    returns sigma hat for the values
    """
    coef = 1 / (len(data) - len(data[0]))
    tot = rss(data, preds)
```

```
    return np.sqrt(coef * tot)

def l2_loss(data, preds):
    """
    data - a matrix where each row corresponds to the
           p predictors in the first p columns and
           the observed output y in the final column
    preds - the predictions for the observations in data

    returns the L2 loss of the values
    """
    return np.mean((data[:,-1] - preds)**2)
```

- Part A

[2]:
```
data = np.genfromtxt('prostate_cancer.csv', delimiter=',', skip_header=1)

standardized_data = data.copy()

for i in range(len(data[0])-2):
    standardized_data[:,i] = standardize_col(data[:,i])

# split the data into train and test
train_data = np.array([observation[:-1] for observation in standardized_data if␣
 ↪observation[-1] == 1])
test_data = np.array([observation[:-1] for observation in standardized_data if␣
 ↪observation[-1] == 0])

# find the OLS estimate and keep track of the inverse for calculations later
ols_full_model, full_model_inverse_intermediate = find_beta(train_data)
```

[3]:
```
ols_full_model
```

[3]:
```
array([ 2.46493292,  0.67601634,  0.26169361, -0.14073374,  0.20906052,
        0.30362332, -0.28700184, -0.02119493,  0.26557614])
```

- Part B

[4]:
```
full_model_training_preds = predict(ols_full_model, train_data)
full_model_sigma = find_sigma(train_data, full_model_training_preds)

# All the values of this part are summarized at bottom of the file in table
```

[5]:
```
z_scores = []

for i in range(len(ols_full_model)):
    z_scores.append(ols_full_model[i] / (full_model_sigma * np.
 ↪sqrt(full_model_inverse_intermediate[i][i])))
```

[6]:
```python
z_scores
```

[6]:
```
[27.598203120218404,
 5.366290456150523,
 2.7507893898693854,
 -1.3959089818189607,
 2.055845625930907,
 2.4692551777938245,
 -1.8669126353948005,
 -0.14668120644372185,
 1.737839719569918]
```

[7]:
```python
wald_test = []

for i in range(len(ols_full_model)):
    wald_test.append(2*scipy.stats.norm.cdf(-1 * np.abs(z_scores[i])))
```

[8]:
```python
wald_test
```

[8]:
```
[1.1693547957255616e-167,
 8.037247566881759e-08,
 0.005945185311053233,
 0.16274190557571133,
 0.039797398582855026,
 0.013539463005511015,
 0.06191378907134302,
 0.8833836532246512,
 0.08223905974843178]
```

[9]:
```python
t_test = []

for i in range(len(ols_full_model)):
    t_test.append(2*scipy.stats.t(len(train_data) - len(train_data[0])).cdf(-1␣
 ↪* np.abs(z_scores[i])))
```

[10]:
```python
t_test
```

[10]:
```
[4.761696772938845e-35,
 1.4694149583757016e-06,
 0.007917894909336934,
 0.16806259017049052,
 0.044307842021366985,
 0.01650538687470883,
 0.06697084708906915,
 0.8838923143371643,
 0.0875462787480178]
```

[11]:
```python
confidence_intervals = []

for i in range(len(ols_full_model)):
    factor = 2 * full_model_sigma * np.
 ↪sqrt(full_model_inverse_intermediate[i][i])
    interval = [round(ols_full_model[i]-factor, 2),␣
 ↪round(ols_full_model[i]+factor, 2)]
    confidence_intervals.append(interval)
```

[12]:
```python
confidence_intervals
```

[12]:
```
[[2.29, 2.64],
 [0.42, 0.93],
 [0.07, 0.45],
 [-0.34, 0.06],
 [0.01, 0.41],
 [0.06, 0.55],
 [-0.59, 0.02],
 [-0.31, 0.27],
 [-0.04, 0.57]]
```

- Part C

[13]:
```python
# find the indexes of the coefficients that are insignificant
insignificant_coefficients = (np.where(np.abs(z_scores) < 2)[0] & np.where(np.
 ↪array(wald_test) > .05)[0]) - 1

# reduce the dataset and find new OLS estimate and predictions
reduced_train_data = np.delete(train_data, insignificant_coefficients, 1)
reduced_test_data = np.delete(test_data, insignificant_coefficients, 1)
ols_reduced, _ = find_beta(reduced_train_data)
reduced_training_preds = predict(ols_reduced, reduced_train_data)

# calculate rss for both models
rss_h0 = rss(reduced_train_data, reduced_training_preds)
rss_h1 = rss(train_data, full_model_training_preds)
p = len(train_data[0])
p_reduced = len(reduced_train_data[0])

# calculate f and then find p value
f = ((rss_h0 - rss_h1) / (p - p_reduced)) / (rss_h1 / (len(train_data)- p))
f_test_p_val = 1 - scipy.stats.f(p-p_reduced, (len(train_data)- p)).cdf(f)
```

[14]:
```python
f_test_p_val
```

[14]:
```
0.16933707265225229
```

- Part D

[15]:

```python
# Base model
b0 = np.mean(train_data[:,-1])
base_err = l2_loss(test_data, b0)

# full model
full_model_testing_preds = predict(ols_full_model, test_data)
full_err = l2_loss(test_data, full_model_testing_preds)

# reduced model
reduced_test_preds = predict(ols_reduced, reduced_test_data)
reduced_err = l2_loss(test_data, reduced_test_preds)

print("Base Model Average Test Error: {}".format(base_err))
print("Full Model Average Test Error: {}".format(full_err))
print("Reduced Model Average Test Error: {}".format(reduced_err))
```

```
Base Model Average Test Error: 1.0567332280603818
Full Model Average Test Error: 0.5212740055076003
Reduced Model Average Test Error: 0.45633212204016255
```

[16]:

```python
params = ['1', 'lcavol', 'lweight', 'age', 'lbph', 'svi', 'lcp', 'gleason',␣
 ↪'pgg45']
pd.DataFrame(data={'OLS estimate': ols_full_model,
                   'z-score': z_scores,
                   'wald test p val': wald_test,
                   't test p val': t_test,
                   '95% CI': confidence_intervals},
             index=['1', 'lcavol', 'lweight', 'age', 'lbph', 'svi', 'lcp',␣
 ↪'gleason', 'pgg45'])
```

[16]:

	OLS estimate	z-score	wald test p val	t test p val	95% CI
1	2.464933	27.598203	1.169355e-167	4.761697e-35	[2.29, 2.64]
lcavol	0.676016	5.366290	8.037248e-08	1.469415e-06	[0.42, 0.93]
lweight	0.261694	2.750789	5.945185e-03	7.917895e-03	[0.07, 0.45]
age	-0.140734	-1.395909	1.627419e-01	1.680626e-01	[-0.34, 0.06]
lbph	0.209061	2.055846	3.979740e-02	4.430784e-02	[0.01, 0.41]
svi	0.303623	2.469255	1.353946e-02	1.650539e-02	[0.06, 0.55]
lcp	-0.287002	-1.866913	6.191379e-02	6.697085e-02	[-0.59, 0.02]
gleason	-0.021195	-0.146681	8.833837e-01	8.838923e-01	[-0.31, 0.27]
pgg45	0.265576	1.737840	8.223906e-02	8.754628e-02	[-0.04, 0.57]

5. We will first show that if $p_j^w > \alpha$ then the confidence interval contains 0. To begin it is to be noted we are only worried about $\alpha \in (0,1)$ and thus $\frac{\alpha}{2} \in (0, .5)$

To begin, we assume $p_j^w > \alpha$

$$2\Phi\left(-\left|\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_j}}\right|\right) > \alpha$$

$$\Phi\left(-\left|\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_j}}\right|\right) > \frac{\alpha}{2}$$

, We can then take the ~~standard normal quantile~~ inverse normal cdf of both sides because it is an increasing function on our interval

$$-\left|\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_j}}\right| > \Phi^{-1}\left(\frac{\alpha}{2}\right)$$

$$\left|\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_j}}\right| < -\Phi^{-1}\left(\frac{\alpha}{2}\right)$$

We can then rewrite this as the following because we know $\Phi^{-1}(\frac{\alpha}{2})$ is negative on our interval so $-\Phi^{-1}(\frac{\alpha}{2}) = Z_{\frac{\alpha}{2}}$ is always positive

no absolute value

$$-Z_{\frac{\alpha}{2}} < \frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_j}} < Z_{\frac{\alpha}{2}}$$

$$-Z_{\frac{\alpha}{2}}\hat{\sigma}\sqrt{v_j} < \hat{\beta}_j < Z_{\frac{\alpha}{2}}\hat{\sigma}\sqrt{v_j}$$

$$\left(-Z_{\frac{\alpha}{2}}\hat{\sigma}\sqrt{v_j} - \hat{\beta}_j < 0 < Z_{\frac{\alpha}{2}}\hat{\sigma}\sqrt{v_j} - \hat{\beta}_j\right) \cdot -1$$

$$① \; \hat{\beta}_j + Z_{\frac{\alpha}{2}}\hat{\sigma}\sqrt{v_j} > 0 > \hat{\beta}_j - Z_{\frac{\alpha}{2}}\hat{\sigma}\sqrt{v_j} \; ②$$

①+② are definition of $100(1-\alpha)\%$ CI

As noted before $Z_{\frac{\alpha}{2}}$ is always positive where we are concerned and $\hat{\sigma}$ is positive because it is the sqrt of a real number and same thing for $\sqrt{v_j}$. Thus the left side of this is always going to be the upper bound of the interval and the right side is always going to be the lower bound thus our confidence interval will ~~always~~ contain 0 if $p_j^w > \alpha$

Now we will show that if the CI contains 0, then $p_j^W > \alpha$

thus by definition if the CI contains 0 then

$$\hat{\beta}_j + Z_{\frac{\alpha}{2}} \hat{\sigma}\sqrt{v_j} > 0 > \hat{\beta}_j - Z_{\frac{\alpha}{2}} \hat{\sigma}\sqrt{v_j}$$

$$-\Phi^{-1}\left(\frac{\alpha}{2}\right)\hat{\sigma}\sqrt{v_j} > -\hat{\beta}_j > \Phi^{-1}\left(\frac{\alpha}{2}\right)\hat{\sigma}\sqrt{v_j}$$

$$\Phi^{-1}\left(\frac{\alpha}{2}\right) < \frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_j}} < -\Phi^{-1}\left(\frac{\alpha}{2}\right)$$

$$\left|\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_j}}\right| < -\Phi^{-1}\left(\frac{\alpha}{2}\right)$$

$$-\left|\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_j}}\right| > \Phi^{-1}\left(\frac{\alpha}{2}\right)$$

We can then take the norm cdf of both sides since that is strictly increasing on our interval

$$\Phi\left(-\left|\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_j}}\right|\right) > \frac{\alpha}{2}$$

$$2\Phi\left(-\left|\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_j}}\right|\right) > \alpha$$

$$p_j^W > \alpha$$

Therefore if the CI contains 0 then $p_j^W > \alpha$ and we have proven both directions so the Wald test p-value $> \alpha$ iff the corresponding approximate $100(1-\alpha)\%$ CI contains 0