EE3731C Tutorial - Pattern Recognition 1

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1. First we compute the determinant of the matrix $A - \lambda I = \begin{pmatrix} 5-\lambda & 3 \\ 1 & 3-\lambda \end{pmatrix}$ and solve for λ.

$$\begin{aligned} \det \begin{pmatrix} 5-\lambda & 3 \\ 1 & 3-\lambda \end{pmatrix} &= 0 \\ (5-\lambda)(3-\lambda) - 3 &= 0 \\ \lambda^2 - 8\lambda + 12 &= 0 \\ (\lambda - 4)^2 - 4 &= 0 \\ \lambda &= 4 \pm 2 \end{aligned}$$

Therefore the matrix A has two eigenvalues 2 and 6. Next, we solve for $\lambda = 2$,

$$\begin{aligned} (A - \lambda I) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Therefore $x_1 + x_2 = 0$ or $x_1 = -x_2$. A vector that satisfies the equation is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, which we can normalize to unit length as $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$. We can also solve for $\lambda = 6$,

$$\begin{aligned} (A - \lambda I) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Therefore $-x_1 + 3x_2 = 0$ or $x_1 = 3x_2$. A vector that satisfies the equation is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, which we can normalize to unit length as $\begin{pmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix}$.

To summarize, the eigenvalues of A are 2 and 6 with corresponding unit eigenvectors $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$ and $\begin{pmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix}$.

2.

$$\det\begin{pmatrix} 2-\lambda & a \\ a & 2-\lambda \end{pmatrix} = 0$$
$$(2-\lambda)^2 - a^2 = 0$$
$$\lambda = 2 \pm a$$

The necessary and sufficient conditions for a covariance matrix is symmetry and positive semidefinite. Therefore $-2 \leq a \leq 2$, so that the eigenvalues are guaranteed to be at least 0 (positive semidefinite).

To see why the necessary conditions for a covariance matrix is symmetry and positive semidefinite, let x be a d-dimensional random vector with 0 mean. Then the covariance matrix of x is $E(xx^T)$, which is symmetric. In addition, let y be any non-zero vector of length d. Then $y^T E(xx^T)y = E(y^T xx^T y) = E((y^T x)^2) \geq 0$, and therefore $E(xx^T)$ is positive semidefinite. Therefore any covariance matrix must be symmetric and positive semidefinite.

To see why the sufficient conditions for a covariance matrix is symmetry and positive semidefinite, let M be a symmetric and positive semidefinite matrix. Since M is symmetric and positive semidefinite, by the eigendecomposition of such matrices, there is a symmetric square root matrix $M^{1/2}$, where $M^{1/2}M^{1/2} = M$. Let y be a d-dimensional random vector with 0 mean and whose covariance matrix is the identity matrix, i.e., $E(yy^T) = I$. Let x be a random vector, where $x = M^{1/2}y$. Then $E(xx^T) = E(M^{1/2}yy^T M^{1/2}) = M^{1/2}E(yy^T)M^{1/2} = M^{1/2}IM^{1/2} = M$. In other words, M is the covariance matrix of x. Therefore any symmetric and positive semidefinite matrix can be a covariance matrix of some random vector.

3. (a) The original data are

$$\begin{pmatrix} -2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

First we compute the mean vector

$$\mu = \frac{1}{4}\left[\begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Subtracting the mean vector, we denote the four data points as

$$X = \left[\begin{pmatrix} -2 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix}\right]$$

The empirical covariance matrix is

$$C = \frac{1}{4}XX^T = \begin{pmatrix} 2.5 & -1.5 \\ -1.5 & 2.5 \end{pmatrix},$$

Solving for the eigenvalues, we have

$$\begin{aligned}
\det\begin{pmatrix} 2.5-\lambda & -1.5 \\ -1.5 & 2.5-\lambda \end{pmatrix} &= 0 \\
(2.5-\lambda)^2 - 1.5^2 &= 0 \\
\lambda &= 2.5 \pm 1.5
\end{aligned}$$

The largest eigenvalue is 4 with corresponding eigenvector:

$$\begin{aligned}
(C-\lambda I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} -1.5 & -1.5 \\ -1.5 & -1.5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}$$

Therefore $x_1 = -x_2$, and the largest unit length principal axis $v_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

(b) First principal component of x is equal to $v_1^T(x-\mu)$

$$\left[\frac{1}{\sqrt{2}} \quad -\frac{1}{\sqrt{2}}\right]\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \left[\frac{1}{\sqrt{2}} \quad -\frac{1}{\sqrt{2}}\right]\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \sqrt{2}$$

(c) The representation error is given by

$$\begin{aligned}
\left\| x - \left(\mu + \sqrt{2}v_1\right) \right\|^2 &= \left\| \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sqrt{2}\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}\right] \right\|^2 \\
&= \left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|^2 = 1^2 + 1^2 = 2
\end{aligned}$$

4. $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 16 & 20 & 24 \\ 50 & 61 & 72 \end{pmatrix}$. On the other hand,

$$\begin{aligned}
&\begin{pmatrix} 1 \\ 2 \end{pmatrix}(1\ 2\ 3) + \begin{pmatrix} 2 \\ 5 \end{pmatrix}(4\ 5\ 6) + \begin{pmatrix} 1 \\ 4 \end{pmatrix}(7\ 8\ 9) \\
&= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} + \begin{pmatrix} 8 & 10 & 12 \\ 20 & 25 & 30 \end{pmatrix} + \begin{pmatrix} 7 & 8 & 9 \\ 28 & 32 & 36 \end{pmatrix} \\
&= \begin{pmatrix} 16 & 20 & 24 \\ 50 & 61 & 72 \end{pmatrix}
\end{aligned}$$