NATIONAL UNIVERSITY OF SINGAPORE

EXAMINATION FOR

(Semester I : 2015/2016)

EE3731C – SIGNAL PROCESSING METHODS

Nov 2015 – Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This paper contains **FOUR (4)** questions and comprises **FIVE (5)** printed pages.
2. All questions are compulsory. Answer **ALL** questions.
3. This is a **CLOSED BOOK** examination. One A4-size formula sheet is allowed.
4. Programmable calculators are not allowed.

Q1 The sub-questions (a), (b) and (c) can be answered independently.

(a) The impulse response $h[n]$ of a linear time-invariant system is 0, except in the interval $N_0 \le n \le N_1$. The input $x[n]$ is 0, except in the interval $N_2 \le n \le N_3$. Therefore the output is 0, except in the interval $N_4 \le n \le N_5$. Determine N_4 and N_5 in terms of N_0, N_1, N_2, N_3.

(6 marks)

(b) Let $x_1[n] = n + 1$ for $0 \le n \le 5$ and $x_2[n] = \begin{cases} 2 & for\ n = 3 \\ 0 & for\ n = 0, 1, 2, 4, 5 \end{cases}$

Let $y[n]$ be the circular convolution $x_1[n] \otimes x_2[n]$. What is $y[n]$ for $0 \le n \le 5$?

(9 marks)

(c) Consider an ideal high-pass filter with 0 delay and frequency response

$$H_{hp}(e^{j\omega}) = \begin{cases} 0 & for\ |\omega| < \omega_{hp} \\ 0.5 & for\ |\omega_{hp}| < |\omega| < \pi \end{cases}$$

Consider an ideal low-pass filter with 0 delay and frequency response

$$H_{lp}(e^{j\omega}) = \begin{cases} 1 & for\ |\omega| < \omega_{lp} \\ 0 & for\ |\omega_{lp}| < |\omega| < \pi \end{cases}$$

whose impulse response is $h_{lp}[n] = \frac{\sin \omega_{lp} n}{\pi n}$

(i) To obtain the above high-pass filter, we can scale the above low-pass filter (with appropriate ω_{lp}) by 1/2 and translate by π in the frequency domain. What is the resulting impulse response of the ideal high-pass filter in terms of ω_{hp}?

(4 marks)

(ii) Modify the impulse response from (i) so that the resulting high-pass filter has a linear phase of $M/2$.

(1 mark)

(iii) We wish to use the Kaiser window method to design a FIR filter with generalized linear phase that meets the following specifications:

$$|H(e^{j\omega})| < 0.03 \quad |\omega| < 0.575\pi$$
$$0.45 < |H(e^{j\omega})| < 0.55 \quad 0.625\pi \le |\omega| \le \pi$$

The Kaiser window will be applied to the ideal impulse response from (ii) with $\omega_{hp} = 0.6\pi$. Find the Kaiser window parameters β and M so that the resulting windowed filter satisfies the above criteria.

(5 marks)

Q2 The sub-questions (a), (b), (c) and (d) can be answered independently.

(a) Alice and Bob play the same game everyday. On day n, Alice independently sampled (x_n, y_n) from the joint probability distribution $p(x, y)$. Given observation y_n, Bob guesses x_n to be $\hat{x}_n$ and pays Alice $e(x_n, \hat{x}_n)$ dollars.

(i) Suppose $e(x_n, \widehat{x_n}) = (x_n - \widehat{x_n})^2$. What estimation strategy should Bob use to minimize the payout to Alice on a daily basis (on average)?

(1 mark)

(ii) Suppose $e(x_n, \widehat{x_n}) = 0$ if $x_n = \widehat{x_n}$ and $e(x_n, \widehat{x_n}) = 10$ if $x_n \neq \widehat{x_n}$. What estimation strategy should Bob use to minimize the payout to Alice on a daily basis (on average)?

(1 mark)

(b) Given the observation y, suppose we use Metropolis algorithm to generate ten independent samples of x from the posterior distribution $p(x \mid y)$. Let $x_1 = 2, x_2 = 5$, $x_3 = 3$, $x_4 = 4$, $x_5 = 1$, $x_6 = 2$, $x_7 = 2$, $x_8 = 5$, $x_9 = 3$ and $x_{10} = 1$ be the ten samples. Use the ten samples to compute an approximate MAP estimate of x given y.

(2 marks)

(c) Let x_n be a random walk defined as

$$x_0 = 0$$

$$x_N = x_0 + \sum_{n=1}^{N} z_n$$

where z_n, $n \geq 1$ is a discrete time white Gaussian noise process, i.e. $z_1, z_2, \ldots$ are i.i.d $\mathcal{N}(0, 1)$. Find the MMSE estimate of x_{20} given $x_1, \ldots, x_{10}$

(7 marks)

(d) Suppose $\phi(t)$ is the scaling function of a multi-resolution wavelet analysis. Let $\phi_{s,u}(t) = 2^{-s/2}\phi(2^{-s}t - u)$. In class, we mentioned that there exists $g_0[u]$, such that the two-scale relationship $\phi(t) = \sum_{u=-\infty}^{\infty} g_0[u]\phi_{-1,u}(t)$ holds.

(i) Show that $\|g_0\|^2 = \sum_u (g_0[u])^2 = 1$.

(5 marks)

(ii) Show that $g_0[u] = \int_{-\infty}^{\infty} \phi(t)\phi_{-1,u}(t)dt$.

(4 marks)

(iii) Suppose $\phi(t)$ is the Haar scaling function. Show that

$$g_0[u] = \begin{cases} \frac{1}{\sqrt{2}} & u = 0 \; or \; 1 \\ 0 & otherwise \end{cases}$$

(5 marks)

Q3 The sub-questions (a), (b), (c) can be answered independently.

(a) Perform PCA on the following set of data: $\binom{-1}{2}$, $\binom{3}{5}$, $\binom{-1}{-3}$ and $\binom{3}{0}$.
What is the first principal axis? What is the first principal component of the data point $x = \binom{2}{1}$?

(14 marks)

(b) Suppose we project D-dimensional data points $x_1 = [x_1^1, \dots, x_1^D]^T$ and $x_2 = [x_2^1, \dots, x_2^D]^T$ onto K principal axes, resulting in principal components $a_1 = [a_1^1, \dots, a_1^K]^T$ and $a_2 = [a_2^1, \dots, a_2^K]^T$ respectively. Let's denote the reconstructed x's as $\hat{x}_1$ and $\hat{x}_2$. Show that $\|\hat{x}_1 - \hat{x}_2\| \triangleq \sqrt{\sum_{i=1}^{D}(\hat{x}_1^i - \hat{x}_2^i)^2}$ is equal to $\sqrt{\sum_{i=1}^{K}(a_1^i - a_2^i)^2}$. In other words, Euclidean distance between reconstructed data points is the same as Euclidean distance of principal components.

(6 marks)

(c) Consider training data $x_1 = \binom{1}{0}, x_2 = \binom{1}{1}, x_3 = \binom{0}{0}, x_4 = \binom{-1}{0}$ with corresponding class labels $y_1 = 0, y_2 = 0, y_3 = 1, y_4 = 1$. What is the 3-NN estimate of the class label posterior probabilities of data points $x_5 = \binom{0.5}{0.5}$ and $x_6 = \binom{-0.5}{0}$, where the distance metric used is the Euclidean distance? What are the MAP classifications of data points x_5 and x_6?

(5 marks)

Q4 Sub-questions (a) and (b) can be answered independently. Sub-question (c) depends on sub-questions (a) and (b), but you might be able to answer it without fully answering sub-questions (a) and (b).

(a) Consider a biased coin where θ is the probability of getting a head when the coin is tossed. The coin is tossed N times independently (θ is fixed for all the coin tosses), resulting in N_1 heads. Consider a uniform prior where $p_1(\theta) \sim U[0,1]$. What is the MAP estimate of θ in terms of N_1 and N. Show your steps.

(10 marks)

(b) Consider the same coin from part (2a). Now consider the following prior, that believes the coin is fair, or is slightly biased towards tails:

$$p_2(\theta) = \begin{cases} 0.5 & for\ \theta = 0.5 \\ 0.5 & for\ \theta = 0.4 \\ 0 & otherwise \end{cases}$$

Derive the MAP estimate under the prior as a function of N_1 and N.

(10 marks)

(c) Suppose the true parameter is $\theta = 0.41$. Does the uniform prior (sub-question 4a) or the biased coin prior (sub-question 4b) lead to a better estimate when N is small? Which leads to a better estimate when N is large?

(5 marks)

END OF PAPER