DTFT of Type 1 and Type 2 FIR Filters

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Consider symmetric causal FIR $h[n] = \begin{cases} 0, & n < 0, n > M \\ h[M-n], & 0 \le n \le M \end{cases}$

Type 1: Suppose M is even, then $H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n}$. Let's consider two cases:

- For $n = M/2$, $h[n]$ has no symmetric counterpart because M is even, hence the DTFT term becomes $h[M/2]e^{-j\omega M/2} \triangleq a[0]e^{-j\omega M/2}$

- For $n \neq M/2$, $h[n]$ has a symmetric counterpart $h[M-n] = h[n]$, so we collect both DTFT terms:

$$\begin{aligned}
&h[n]e^{-j\omega n} + h[M-n]e^{-j\omega(M-n)} \\
&= h[n]\left(e^{-j\omega n} + e^{-j\omega(M-n)}\right) \\
&= h[n]e^{-j\omega M/2}\left(e^{j\omega(M/2-n)} + E^{-j\omega(M/2-n)}\right) \\
&= 2h[n]e^{-j\omega M/2}\cos\left[\omega(M/2-n)\right] \\
&= 2h[M/2-k]e^{-j\omega M/2}\cos\omega k \qquad \text{substitute } k = M/2 - n \\
&\triangleq a[k]e^{-j\omega M/2}\cos\omega k
\end{aligned}$$

Therefore we can write $H(e^{j\omega}) = e^{-j\omega M/2}\sum_{k=0}^{M/2} a[k]\cos\omega k$, where

$$\begin{aligned}
a[0] &= h[M/2] \\
a[k] &= 2h[(M/2)-k]
\end{aligned}$$

Type 2: Suppose M is odd, then $H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n}$. Then $h[n]$ has a symmetric counterpart $h[M-n] = h[n]$, so we collect both DTFT terms:

$$\begin{aligned}
&h[n]e^{-j\omega n} + h[M-n]e^{-j\omega(M-n)} \\
&= h[n]\left(e^{-j\omega n} + e^{-j\omega(M-n)}\right) \\
&= h[n]e^{-j\omega M/2}\left(e^{j\omega(M/2-n)} + e^{-j\omega(M/2-n)}\right) \\
&= 2h[n]e^{-j\omega M/2}\cos\left[\omega(M/2-n)\right] \\
&= 2h[(M+1)/2-k]e^{-j\omega M/2}\cos\left[\omega(k-\frac{1}{2})\right] \qquad \text{substitute } k = (M+1)/2 - n \\
&\triangleq b[k]e^{-j\omega M/2}\cos\left[\omega(k-\frac{1}{2})\right]
\end{aligned}$$

Therefore we can write

$$H(e^{j\omega}) = e^{-j\omega M/2} \sum_{k=1}^{(M+1)/2} b[k] \cos\left[\omega(k - \frac{1}{2})\right]$$

where

$$b[k] = 2h[(M+1)/2 - k]$$