EE3731C Tutorial - Classical Signal 2

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1. From lecture, the DTFT of $a^n u[n]$ is $\frac{1}{1-ae^{-j\omega}}$. Since $h[n] = 5(-1/2)^n u[n]$, then $H(e^{j\omega}) = \frac{5}{1+\frac{1}{2}e^{-j\omega}}$. Since $x[n] = (1/3)^n u[n]$, then $X(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$. The output

$$\begin{aligned} Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) \\ &= \left(\frac{5}{1+\frac{1}{2}e^{-j\omega}}\right)\left(\frac{1}{1-\frac{1}{3}e^{-j\omega}}\right) \\ &= \frac{3}{1+\frac{1}{2}e^{-j\omega}} + \frac{2}{1-\frac{1}{3}e^{-j\omega}}, \end{aligned}$$

where we have used the result that $\frac{1}{(1-ae^{-j\omega})(1-be^{-j\omega})} = \frac{a/(a-b)}{1-ae^{-j\omega}} - \frac{b/(a-b)}{1-be^{-j\omega}}$, with $a = -1/2$ and $b = 1/3$.

Taking the inverse DTFT of $Y(e^{j\omega})$, we get by linearity of DTFT

$$y[n] = 3(-1/2)^n u[n] + 2(1/3)^n u[n]$$

2.

$$\begin{aligned} H(e^{j\omega}) &= \sum_n \left(\frac{1}{M_1+M_2+1}\sum_{k=-M_1}^{M_2} \delta[n-k]\right)e^{-j\omega n} \\ &= \frac{1}{M_1+M_2+1}\sum_{k=-M_1}^{M_2}\left(\sum_n \delta[n-k]e^{-j\omega n}\right) = \frac{1}{M_1+M_2+1}\sum_{k=-M_1}^{M_2} e^{-j\omega k} \end{aligned}$$

Using the following geometric series formula $\sum_{k=N_1}^{N_2} a^k = \frac{a^{N_1}-a^{N_2+1}}{1-a}$ for $N_2 \geq N_1$, we get

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{M_1+M_2+1}\frac{e^{j\omega M_1} - e^{-j\omega(M_2+1)}}{1-e^{-j\omega}} \\ &= \frac{1}{M_1+M_2+1}\frac{e^{j\omega(M_1+M_2+1)/2} - e^{-j\omega(M_1+M_2+1)/2}}{1-e^{-j\omega}}e^{-j\omega(M_2-M_1+1)/2} \\ &= \frac{1}{M_1+M_2+1}\frac{e^{j\omega(M_1+M_2+1)/2} - e^{-j\omega(M_1+M_2+1)/2}}{e^{j\omega/2}-e^{-j\omega/2}}e^{-j\omega(M_2-M_1)/2} \\ &= \frac{1}{M_1+M_2+1}\frac{\sin[\omega(M_1+M_2+1)/2]}{\sin(\omega/2)}e^{-j\omega(M_2-M_1)/2}, \end{aligned}$$

where we have exploited the fact that $\sin x = \frac{e^{jx}-e^{-jx}}{2j}$. Therefore $|H(e^{j\omega})| = \left|\frac{1}{M_1+M_2+1}\frac{\sin\left(\omega(M_1+M_2+1)/2\right)}{\sin(\omega/2)}\right|$ and $\sphericalangle H(e^{j\omega}) = -\omega(M_2 - M_1)/2 + \{0,\pi\}$, where $+\{0,\pi\}$ comes from $\frac{\sin\left(\omega(M_1+M_2+1)/2\right)}{\sin(\omega/2)}$ switching signs, i.e., we have $+0$ if $\frac{\sin\left(\omega(M_1+M_2+1)/2\right)}{\sin(\omega/2)}$ is positive and $+\pi$ if $\frac{\sin\left(\omega(M_1+M_2+1)/2\right)}{\sin(\omega/2)}$ is negative (because $-1 = e^{j\pi}$). This "$+\pi$" explains the sudden jumps in the phase plots shown in class.

3. A LTI system is causal if and only if $h[n] = 0$ for $n < 0$. Therefore

   (a) $h[n] = 0$ for $n < 1$, hence system is causal

   (b) $h[n] \neq 0$ for $n < 0$, hence system is not causal.

   (c) $h[-1] = 1$, hence system is not causal

4. In general, a LTI is stable if and only if $\sum_k |h[k]| < \infty$ (see previous tutorial answers)

   (a) $\sum_n |h[n]| = \sum_{n=-\infty}^{-1} 3^n = \sum_{n=1}^{\infty}(1/3)^n$, which is bounded. Hence system is stable.

   (b) $\sum_n |h[n]| = \sum_{n=0}^{\infty} |\sin(\pi n/3)|$, which is unbounded. Hence system is not stable.

   (c) $\sum_n |h[n]| = \sum_{n=-\infty}^{\infty} |(3/4)^{|n|}\cos(\pi n/4 + \pi/4)| \leq \sum_{n=-\infty}^{\infty}(3/4)^{|n|}$, which is bounded. Hence system is stable.

5.

$$\begin{aligned}
H(e^{j\omega}) &= \sum_n h[n]e^{-j\omega n} \\
&= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}e^{-j\omega} \\
&= \frac{\sqrt{2}}{2}(-1 + \cos\omega) - \frac{\sqrt{2}}{2}j\sin\omega
\end{aligned}$$

$$\begin{aligned}
|H(e^{j\omega})| &= \sqrt{\frac{1}{2}\left[(-1+\cos\omega)^2 + \sin^2\omega\right]} \\
&= \sqrt{\frac{1}{2}\left[1 + \cos^2\omega - 2\cos\omega + \sin^2\omega\right]} \\
&= \sqrt{1 - \cos\omega}
\end{aligned}$$

Therefore $|H(e^{j\omega})|$ is equal to 0 at $\omega = 0$, and monotonically increases to $\sqrt{2}$ at $\omega = \pi$. Therefore $h[n]$ is a high-pass filter.