EE3731C Tutorial - Classical Signal 1

Department of Electrical and Computer Engineering

1. (a) $T(x[n]) = g[n]x[n]$

   (i) Stable if $g[n]$ is bounded. Suppose $|g[n]| \leq B_g$ and $|x[n]| \leq B_x$ for all n. Then $|T(x[n])|$ is bounded because $|T(x[n])| = |g[n]x[n]| \leq |g[n]| \times |x[n]| = B_g B_x$.

   (ii) Causal because output at time n only depends on input at time n

   (iii) Linear because $T(ax_1[n] + bx_2[n]) = g[n](ax_1[n] + bx_2[n]) = aT(x_1[n]) + bT(x_2[n])$

   (iv) Time invariant if $g[n] = c$ (a constant). To see this, let $y[n] = T(x[n])$. Then $T(x[n - n_0]) = g[n]x[n - n_0]$. On the other hand, $y[n - n_0] = g[n-n_0]x[n-n_0]$. Therefore for the system to be time invariant, $g[n-n0]$ must be equal to $g[n]$ for all n, which is true only if $g[n]$ is a constant.

   (v) Memoryless because output at time n only depends on input at time n (memoryless is a special case of causal)

   (b) $T(x[n]) = x[n - n_0]$

   (i) Stable because $|T(x[n])| \leq |x[n - n_0]| \leq B_x$ for all n

   (ii) Causal if $n_0 \geq 0$. For example, if $n_0 = -1$, then $y[0] = x[1]$, so the output at time $n = 0$ depends on the input at a future time $n = 1$, so the system is not causal.

   (iii) Linear because $T(ax_1[n]+bx_2[n]) = ax_1[n-n_0]+bx_2[n-n_0] = aT(x_1[n])+bT(x_2[n])$

   (iv) Time invariant because let $y[n] = T(x[n])$. Then $T(x[n - n_1]) = x[n - n_1 - n_0] = y[n - n_1]$

   (v) Memoryless if n_0 is 0

   (c) $T(x[n]) = e^{x[n]}$

   (i) Stable because $|T(x[n])| \leq e^{B_x}$

   (ii) Causal because output at time n only depends on input at time n

   (iii) Not linear because $T(ax_1[n] + bx_2[n]) = e^{ax_1[n]+bx_2[n]} \neq aT(x_1[n]) + bT(x_2[n])$

   (iv) Time invariant because let $y[n] = T(x[n])$. Then $T(x[n-n_0]) = e^{x[n-n_0]} = y[n - n_0]$

(v) Memoryless because output at time n only depends on input at time n (memoryless is a special case of causal)

(d) $T(x[n]) = x[n] + 3u[n+1]$

(i) Stable because $|T(x[n])| \leq |x[n]+3u[n+1]| \leq |x[n]|+3|u[n+1]| \leq B_x+3$

(ii) Causal because output at time n only depends on input at time n

(iii) Not linear because $T(ax_1[n]+bx_2[n]) = ax_1[n]+bx_2[n]+3u[n+1]$, which is not equal to $aT_1(x_1[n]) + bT(x_2[n]) = a(x_1[n] + 3u[n+1]) + b(x_2[n] + 3u[n+1])$

(iv) Not time invariant because let $y[n] = T(x[n])$. Then $T(x[n-n_0]) = x[n-n_0] + 3u[n+1] \neq y[n-n_0]$

(v) Memoryless because output at time n only depends on input at time n (memoryless is a special case of causal)

Here are two more interesting examples:

(I) Suppose $y[n] = x[n] * h[n]$, where $h[n] = 1$ for all n. Then this system is unstable. To see this, suppose $x[n] = 1$ for all n. Then $y[n] = \sum_k h[n-k]x[k] = \sum_k 1 = \infty$. Therefore even though $x[n]$ is bounded ($|x[n]| \leq 1$), $y[n]$ can be unbounded (i.e., blow up).

(II) Suppose $y[n] = x[n] * h[n]$, where $h[n] = \begin{cases} 1/3 & n = -1, 0, 1 \\ 0 & \text{otherwise} \end{cases}$. This is a special case of the moving average filter we discussed in class. This system is not causal. To see this, notice that $y[0] = \frac{1}{3}x[-1] + \frac{1}{3}x[0] + \frac{1}{3}x[1]$. Since the output at time $n = 0$ depends on the input at a future time $n = 1$, therefore the system is not causal.

More generally, a LTI is stable if and only if $\sum_k |h[k]| < \infty$. To see this, suppose $|x[n]| \leq B_x$ for all n (i.e., x is bounded), then

$$|y[n]| \leq |\sum_k h[k]x[n-k]| \leq \sum_k |h[k]||x[n-k]| \leq B_x \sum_k |h[k]|$$

Therefore $|y[n]|$ is bounded if $\sum_k |h[k]|$ is bounded. Note that the equality sign in the above equation, so the inequality is tight. In other words, if $\sum_k |h[k]|$ is unbounded, then there exists input sequences, where $|y[n]|$ is not bounded. For example, suppose $x[n] = \begin{cases} \frac{h^*[-n]}{|h[-n]|}, & h[n] \neq 0 \\ 0, & h[n] = 0 \end{cases}$. Then

$$y[0] = \sum_k x[-k]h[k] = \sum_k |h[k]|$$

Therefore if $\sum_k |h[k]|$ is not bounded, we can input a sequence that causes $y[0]$ to "blow up".

2. For this question, we will make use of the following geometric series result: $b + ba + ba^2 + ba^3 + \cdots = \frac{b}{1-a}$ if $|a| < 1$.

   First, we can write $h[n]$ as follows $h[n] = \begin{cases} a^{-n}, & n \leq 0 \\ 0 & n > 0 \end{cases}$ and $u[k] = \begin{cases} 0, & k < 0 \\ 1, & k \geq 0 \end{cases}$

   We want to compute $y[n] = u[n] * h[n] = \sum_k u[k]h[n-k]$.

$$h[-k] = \begin{cases} a^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

   For $n \geq 0$, the non-zero values of $h[n-k]$ overlaps completely with all the 1s of $u[k]$. Therefore for $n \geq 0$, we get

$$y[n] = \sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$$

   For $n < 0$, the first $|n|$ non-zero values of $h[n-k]$ does not overlap with the 1s of $u[k]$. Therefore for $n < 0$, we get

$$y[n] = \sum_{k=|n|}^{\infty} a^k = \frac{a^{|n|}}{1-a}$$

   Therefore we have

$$y[n] = \begin{cases} \frac{a^{|n|}}{1-a}, & n < 0 \\ \frac{1}{1-a}, & n \geq 0 \end{cases}$$

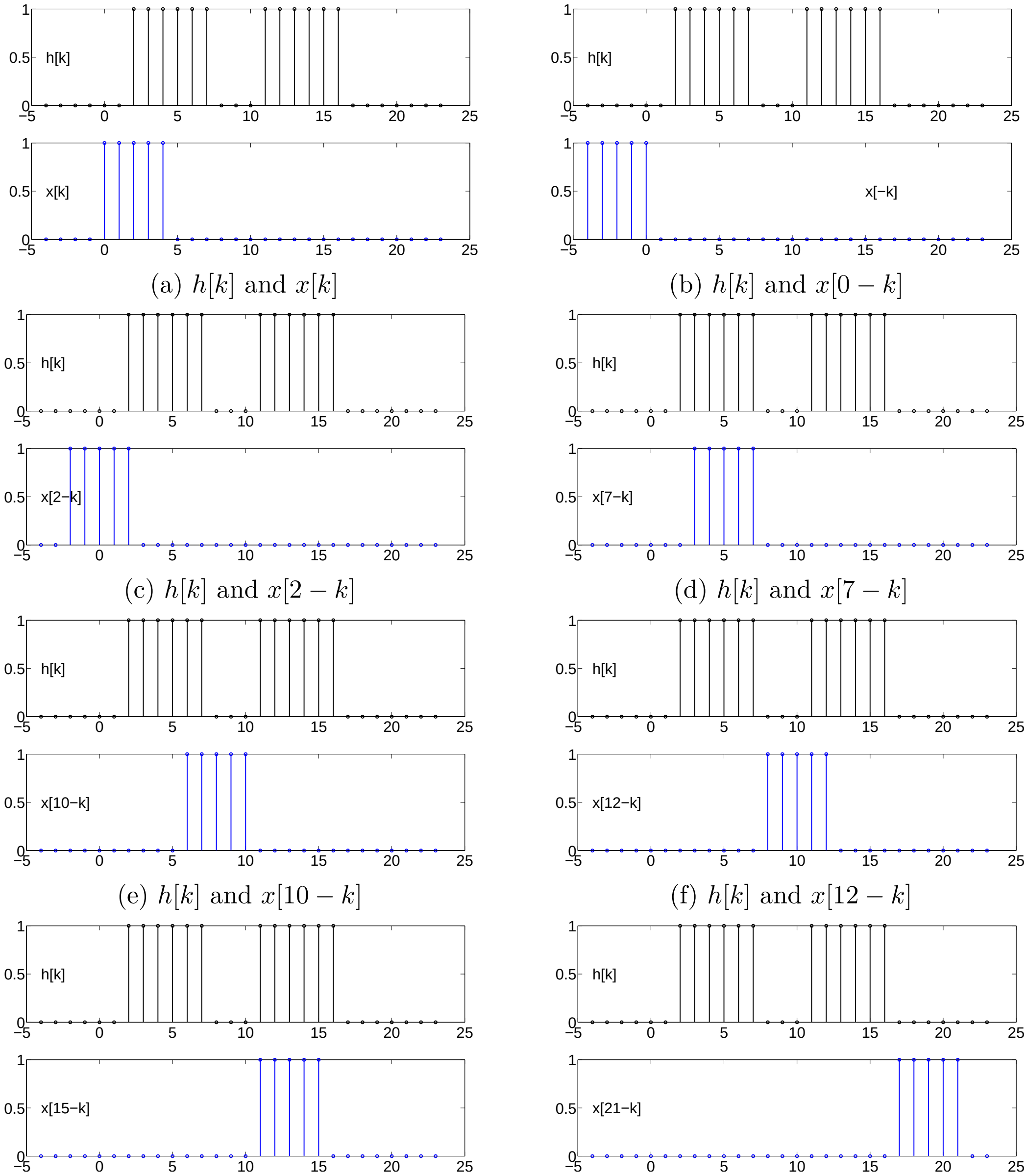
3. $h[k]$ and $x[k]$ are shown in Figure 1a. $h[k]$ and $x[-k]$ are shown in Figure 1b. The overlap between the non-zero values of $h[k]$ and $x[n-k]$ vary as a function of n. Therefore it is useful to consider different intervals of n.

$n \leq 1$: As illustrated in $n = 0$ (Figure 1b) and $n = 2$ (Figure 1c), there is no overlap between the non-zero values for $n \leq 1$. Therefore $y[n] = 0$.

$2 \leq n \leq 6$: As illustrated in $n = 2$ (Figure 1c) and $n = 7$ (Figure 1d), there is increasing amount of overlap between the two sequences, which reaches a plateau at $n = 6$. Therefore $y[n] = n - 1$.

$7 \leq n \leq 10$: As illustrated in $n = 7$ (Figure 1d) and $n = 10$ (Figure 1e), the overlap reaches a maximum at $n = 7$, and then the overlap decreases until $n = 10$ (just before $x[n-k]$ starts overlapping with the second set of impulse functions in $h[k]$). Therefore $y[n] = 5 - (n-7) = 12 - n$

$11 \leq n \leq 12$: As illustrated in $n = 10$ (Figure 1e) and $n = 12$ (Figure 1f), the overlap stays constant as $x[n-k]$ overlaps less with the first set of impulse functions in $h[k]$ and start overlapping more with the second set of impulse functions in $h[k]$. Therefore $y[n] = 2$.



(a) $h[k]$ and $x[k]$

(b) $h[k]$ and $x[0-k]$

(c) $h[k]$ and $x[2-k]$

(d) $h[k]$ and $x[7-k]$

(e) $h[k]$ and $x[10-k]$

(f) $h[k]$ and $x[12-k]$

(g) $h[k]$ and $x[15-k]$

(h) $h[k]$ and $x[21-k]$

Figure 1: Illustration of convolution

$13 \leq n \leq 15$: As illustrated in $n = 12$ (Figure 1f) and $n = 15$ (Figure 1g), there is increasing amount of overlap between the two sequences, which reaches a plateau at $n = 15$. Therefore $y[n] = n - 10$.

$16 \leq n \leq 21$: As illustrated in $n = 15$ (Figure 1g) and $n = 21$ (Figure 1h), the overlap reaches a maximum at $n = 16$, and then the overlap decreases until $n = 21$.

Therefore $y[n] = 21 - n$.

$n \geq 22$: There is no overlap so $y[n] = 0$.

The final output $y[n]$ is sketched below:

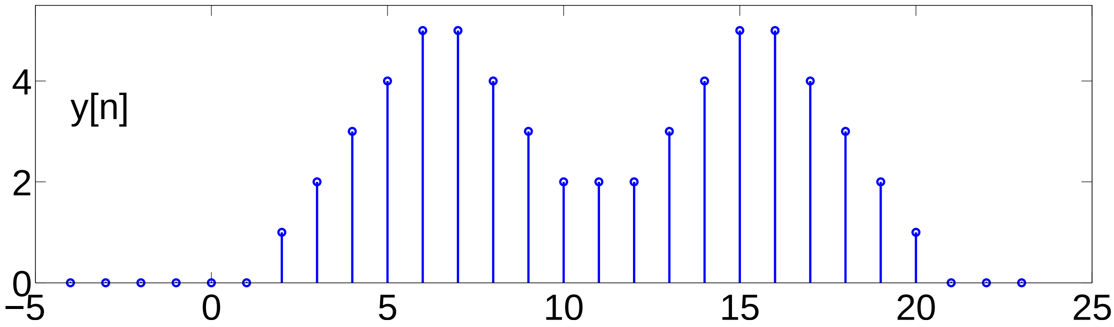


Figure 2: $y[n]$