EE3731C Tutorial - Classical Signal 4

Department of Electrical and Computer Engineering

Questions 2 and 3 are optional and won't be covered in the exams.

1. Recall that in this application

   (a) Convolution (denoted by $*$) assumes (1) padding of zeros on either side of the signal and filter, (2) considering only the valid entries of the convolution ($N+M-1$, where N is the length of the original signal and M is the length of the filter), and (3) dropping the last entry of the convolution output.

   (b) $\downarrow 2$ means keeping only the odd samples (where the first entry of the input signal is considered sample number 0, which is even), e.g., $[2\ \ 3\ \ 4\ \ 5] \to [3\ \ 5]$

   (c) $\uparrow 2$ means inserting 0 between every samples of the original signal and a final 0 at the end, e.g., $[2\ \ 3] \to [2\ \ 0\ \ 3\ \ 0]$

- $x \to (a) : [6\ \ 12\ \ 120\ \ 116] \xrightarrow{\text{circular shift to the right by 1}} [116\ \ 6\ \ 12\ \ 120]$
- $(a) \to (b)$:

$$
\begin{aligned}
&[116\ \ 6\ \ 12\ \ 120] * \left[-\frac{1}{2}\ \ \ \frac{1}{2}\right]\sqrt{2} \\
&= [-\frac{116}{2}\ \ \ \frac{116-6}{2}\ \ \ \frac{6-12}{2}\ \ \ \frac{12-120}{2}\ \ \ \frac{120}{2}]\sqrt{2} \\
&= [-58\ \ 55\ \ -3\ \ -54\ \ 60]\sqrt{2} \\
&\xrightarrow{\text{drop last sample followed by keep odd samples}} [55\ \ -54]\sqrt{2}
\end{aligned}
$$

- $(b) \to (c)$:

$$
\begin{aligned}
\sqrt{2}[55\ \ -54] &\xrightarrow{\text{upsample by two}} \sqrt{2}[55\ \ 0\ \ -54\ \ 0] \\
&\xrightarrow{\text{convolution}} \sqrt{2}\,[55\ \ 0\ \ -54\ \ 0] * \left[\frac{1}{2}\ \ \ -\frac{1}{2}\right]\sqrt{2} \\
&= [55\ \ 0\ \ -54\ \ 0] * [1\ \ -1] \\
&= [55\ \ -55\ \ -54\ \ 54\ \ 0] \\
&\xrightarrow{\text{drop last sample}} [55\ \ -55\ \ -54\ \ 54]
\end{aligned}
$$

- $(a) \to (d)$:

$$
\begin{aligned}
&[116\ \ 6\ \ 12\ \ 120] * \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\end{bmatrix}\sqrt{2} \\
&= [\frac{116}{2} \quad \frac{116+6}{2} \quad \frac{6+12}{2} \quad \frac{12+120}{2} \quad \frac{120}{2}]\sqrt{2} \\
&= [58\ \ 61\ \ 9\ \ 66\ \ 60]\sqrt{2} \\
&\xrightarrow{\text{drop last sample followed by keep odd samples}} [61\ \ 66]\sqrt{2}
\end{aligned}
$$

- $(d) \to (e)$:

$$
\begin{aligned}
&[61\ \ 66]\,\sqrt{2} * \begin{bmatrix}-\frac{1}{2} & \frac{1}{2}\end{bmatrix}\sqrt{2} \\
&= [61\ \ 66] * [-1 \quad 1] \\
&= [-61\ \ -5\ \ 66] \\
&\xrightarrow{\text{drop last sample followed by keep odd samples}} [-5]
\end{aligned}
$$

- $(e) \to (f)$:

$$
\begin{aligned}
&[-5] \xrightarrow{\text{upsample by two}} [-5\ \ 0] \\
&\xrightarrow{\text{convolution}} [-5\ \ 0] * \begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\end{bmatrix}\sqrt{2} \\
&= [-2.5\ \ 2.5\ \ 0]\,\sqrt{2} \\
&\xrightarrow{\text{drop last sample}} [-2.5\ \ 2.5]\,\sqrt{2}
\end{aligned}
$$

- $(d) \to (g)$:

$$
\begin{aligned}
&[61\ \ 66]\,\sqrt{2} * \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\end{bmatrix}\sqrt{2} \\
&= [61\ \ 66] * [1 \quad 1] \\
&= [61\ \ 127\ \ 66] \\
&\xrightarrow{\text{drop last sample followed by keep odd samples}} [127]
\end{aligned}
$$

- $(g) \to (h)$:

$$
\begin{aligned}
&[127] \xrightarrow{\text{upsample by two}} [127\ \ 0] \\
&\xrightarrow{\text{convolution}} [127\ \ 0] * \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\end{bmatrix}\sqrt{2} \\
&= [63.5\ \ 63.5\ \ 0]\,\sqrt{2} \\
&\xrightarrow{\text{drop last sample}} [63.5\ \ 63.5]\,\sqrt{2}
\end{aligned}
$$

- $(f) + (h) \to (i)$:

$$[-2.5\ \ 2.5]\sqrt{2} + [63.5\ \ 63.5]\sqrt{2} = [61\ \ 66]\sqrt{2}$$

$$\xrightarrow{\text{upsample by two}} [61\ \ 0\ \ 66\ \ 0]\sqrt{2}$$

$$\xrightarrow{\text{convolution}} [61\ \ 0\ \ 66\ \ 0]\sqrt{2} * \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\end{bmatrix}\sqrt{2}$$

$$= [61\ \ 0\ \ 66\ \ 0] * [1\quad 1]$$

$$= [61\ \ 61\ \ 66\ \ 66\ \ 0]$$

$$\xrightarrow{\text{drop last sample}} [61\ \ 61\ \ 66\ \ 66]$$

- $(c) + (i) \rightarrow (j)$:

$$[55\quad -55\quad -54\ \ 54] + [61\ \ 61\ \ 66\ \ 66] = [116\ \ 6\ \ 12\ \ 120]$$

$$\xrightarrow{\text{circular shift to the left by 1}} [6\ \ 12\ \ 120\ \ 116]$$

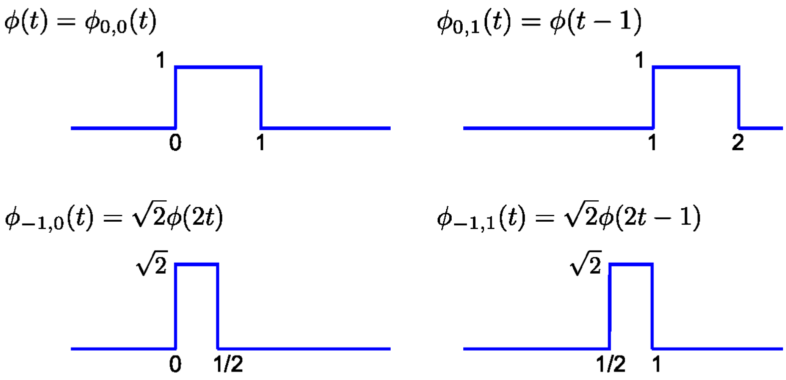


Figure 1: Haar Scaling Functions

2. Note that $\phi_{s,0}(t) = \begin{cases} 2^{-s/2} & 0 \le t \le 2^s \\ 0 & \text{otherwise} \end{cases}$ and $\phi_{s,u}(t) = \begin{cases} 2^{-s/2} & 2^s u \le t \le 2^s + 2^s u \\ 0 & \text{otherwise} \end{cases}$.
   See Figure 1 for illustration.

   (a) See Figure 2 for illustration. For a fixed scale s, suppose $u = u'$, then

$$\int_{-\infty}^{\infty} \phi_{s,u}(t)\phi_{s,u'}(t)dt = \int_{2^s u}^{2^s + 2^s u} 2^{-s/2}2^{-s/2}dt$$
$$= 2^{-s}\int_{2^s u}^{2^s + 2^s u} dt$$
$$= 1$$

   If $u \ne u'$, then the values of t for which $\phi_{s,u}(t)$ is non-zero do not overlap with the values of t for which $\phi_{s,u'}(t)$ is non-zero. Therefore

$$\int_{-\infty}^{\infty} \phi_{s,u}(t)\phi_{s,u'}(t)dt = \int_{-\infty}^{\infty} 0dt = 0$$

Therefore $\int_{-\infty}^{\infty} \phi_{s,u}(t)\phi_{s,u'}(t)dt = \delta(u-u')$ for all $u, u' \in \mathbb{Z}$. We say that $\{\phi_{s,u}(t)\}_{u\in\mathbb{Z}}$ are orthonormal.

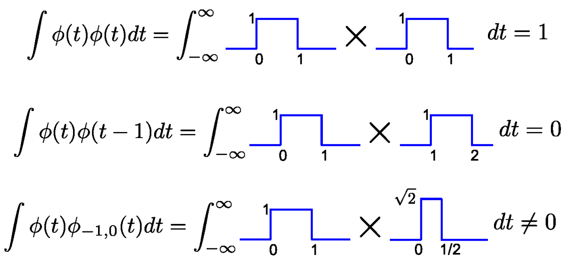


Figure 2: Haar Scaling Orthonormality

(b) According to the question

$$f(t) = \sum_{u\in\mathbb{Z}} \alpha_s[u]\phi_{s,u}(t), \text{where } \alpha_s[u] \in \mathbb{R}$$

Let integrate both side with respect to $\phi_{s,u'}(t)$, then

$$\begin{aligned}
\int_{-\infty}^{\infty} f(t)\phi_{s,u'}(t)dt &= \int_{-\infty}^{\infty} \left(\sum_{u\in\mathbb{Z}} \alpha_s[u]\phi_{s,u}(t)\right) \phi_{s,u'}(t)dt \\
&= \sum_{u\in\mathbb{Z}} \alpha_s[u] \int_{-\infty}^{\infty} \phi_{s,u}(t)\phi_{s,u'}(t)dt \\
&= \sum_{u\in\mathbb{Z}} \alpha_s[u]\delta(u-u') \\
&= \alpha_s[u'] \qquad \text{by the sifting property of } \delta
\end{aligned}$$

(c) See Figure 3 for illustration. $f(t) \in V_0 \implies f(t) = \sum_{u\in\mathbb{Z}} \alpha_0[u]\phi_{0,u}(t) = \sum_{u\in\mathbb{Z}} \alpha_0[u]\phi(t-u)$ for some coefficients $\alpha_0[u]$. Therefore

$$\begin{aligned}
f(2^{-s}t) &= \sum_{u\in\mathbb{Z}} \alpha_0[u]\phi(2^{-s}t-u) \\
&= \sum_{u\in\mathbb{Z}} \alpha_0[u]\frac{2^{-s/2}}{2^{-s/2}}\phi(2^{-s}t-u) \\
&= \sum_{u\in\mathbb{Z}} \frac{\alpha_0[u]}{2^{-s/2}}\phi_{s,u}(t)
\end{aligned}$$

Therefore $f(2^{-s}t) \in V_s$, where the corresponding coefficients are $\alpha_s[u] = \frac{\alpha_0[u]}{2^{-s/2}}$. This property is known as scaling invariance. Notice that we proved this property without using the fact that $\phi(t)$ is the Haar scaling function. We only made use of the fact that $\phi_{s,u}(t)$ are defined as dilated and translated versions of $\phi(t)$.

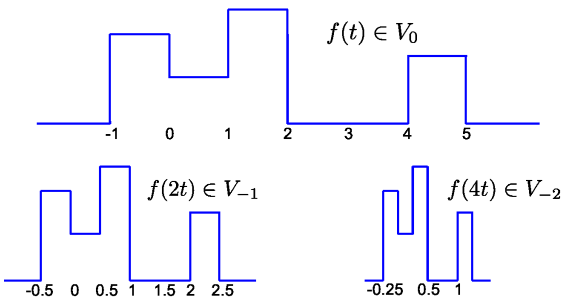


Figure 3: Haar Scaling Invariance

(d) See Figure 4 for illustration. $f(t) \in V_0 \implies f(t) = \sum_{u' \in \mathbb{Z}} \alpha_0[u']\phi_{0,u'}(t) = \sum_{u' \in \mathbb{Z}} \alpha_0[u']\phi(t-u')$ for some coefficients $\alpha_0[u']$. Therefore

$$f(t-u) = \sum_{u' \in \mathbb{Z}} \alpha_0[u']\phi(t-u'-u)$$

Let $u'' = u' + u$, then

$$\begin{aligned} f(t-u) &= \sum_{u'' \in \mathbb{Z}} \alpha_0[u''-u]\phi(t-u'') \\ &= \sum_{u'' \in \mathbb{Z}} \gamma_0[u'']\phi(t-u'') \text{ where } \gamma_0[u''] = \alpha_0[u''-u] \end{aligned}$$

Therefore $f(t-u) \in V_0$. This property is known as translation invariance. Notice that we proved this property without using the fact that $\phi(t)$ is the Haar scaling function. We only made use of the fact that V_0 is defined to be a linear combination of translated versions of $\phi(t)$.

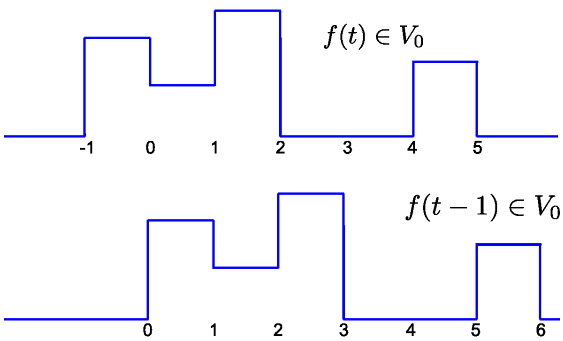


Figure 4: Haar Translation Invariance

(e) $f(t) \in V_0 \implies f(t) = \sum_{u \in \mathbb{Z}} \alpha_0[u]\phi(t-u)$ for some coefficients $\alpha_0[u]$. Furthermore, observe that for the Haar scaling function (see Figure 5 for illustration):

$$\phi(t) = \frac{1}{\sqrt{2}}\phi_{-1,0}(t) + \frac{1}{\sqrt{2}}\phi_{-1,1}(t)$$

The above is known as a <u>two-scale relation</u> because it relates scale 0 with scale -1. Note that it is not an accident that the coefficients $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ corresponds to the low pass filter g in the Haar discrete wavelet transform we learned in lecture. We will come back to this point in the next lecture. Therefore

$$f(t) = \sum_{u \in \mathbb{Z}} \alpha_{-1}[u] \phi_{-1,u}(t-u),$$

where

$$\alpha_{-1}[u] = \begin{cases} \frac{1}{\sqrt{2}} \alpha_0 \left[\frac{u}{2}\right] & \text{if } u \text{ is even} \\ \frac{1}{\sqrt{2}} \alpha_0 \left[\frac{u-1}{2}\right] & \text{if } u \text{ is odd} \end{cases}$$

Therefore $f(t) \in V_{-1}$. Here we did make explicit use of the fact that we are dealing with the Haar scaling function. Using the same argument, we can show that $\cdots, V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2}, \cdots$

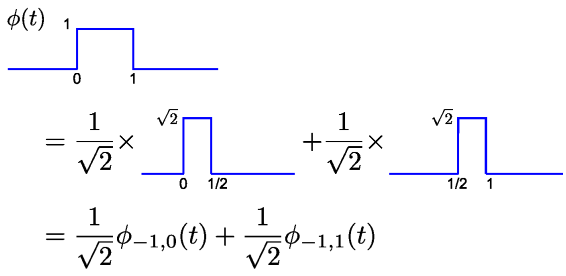


Figure 5: $\cdots, V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2}, \cdots$

There are two other important properties of the Haar scaling functions that we will not show. The first is known as upward completeness: $V_{-\infty} = L^2(\mathbb{R})$, where a function $f(t)$ is said to be be in $L^2(\mathbb{R})$ if $\int_{-\infty}^{\infty} f^2(t)dt < \infty$. In other words, as we utilize finer and finer Haar scaling functions (as $s \to -\infty$), we can approximate any function $f \in L^2(\mathbb{R})$.

The second property is known as downward completeness: $V_\infty = \{0\}$. In other words, we are saying that as we utilize fatter and fatter Haar scaling functions (as $s \to \infty$), the only function $f(t) \in L^2(\mathbb{R})$ we can approximate is $f(t) = 0$.

$\cdots, V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2}, \cdots$ is known as a multi-resolution analysis if and only if it satisfies the above five properties, i.e., (1) orthonormality of scaling function, (2) upward and downward completeness, and (3) scale and translation invariance.

3. Note that $\psi_{s,0}(t) = \begin{cases} 2^{-s/2} & 0 \leq t < 2^{s-1} \\ -2^{-s/2} & 2^{s-1} \leq t \leq 2^s \\ 0 & \text{otherwise} \end{cases}$

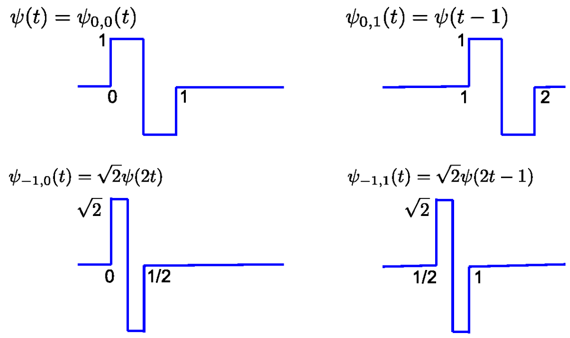


Figure 6: Haar Wavelet Functions

and

$$\psi_{s,u}(t) = \begin{cases} 2^{-s/2} & 2^s u \le t < 2^{s-1} + 2^s u \\ -2^{-s/2} & 2^{s-1} + 2^s u \le t \le 2^s + 2^s u \\ 0 & \text{otherwise} \end{cases}$$

See Figure 6 for illustration.

(a) Suppose $s = s', u = u'$, then

$$\begin{aligned} \int_{-\infty}^{\infty} \psi_{s,u}(t)\psi_{s',u'}(t)dt &= \int_{2^s u}^{2^{s-1}+2^s u} 2^{-s/2}2^{-s/2}dt + \int_{2^{s-1}+2^s u}^{2^s+2^s u} (-2^{-s/2})(-2^{-s/2})dt \\ &= 2^{-s} \int_{2^s u}^{2^s+2^s u} dt \\ &= 1 \end{aligned}$$

At a given scale s, if $u \neq u'$, then the values of t for which $\psi_{s,u}(t)$ is non-zero do not overlap with the values of t for which $\psi_{s,u'}(t)$ is non-zero. Therefore (see first row of Figure 7 for illustration)

$$\int_{-\infty}^{\infty} \psi_{s,u}(t)\psi_{s,u'}(t)dt = \int_{-\infty}^{\infty} 0dt = 0 \quad \text{if } u \neq u'$$

Suppose $s < s'$, then $\psi_{s',u'}(t)$ is a "fatter" function than $\psi_{s,u}(t)$. There are two possible cases here. The first case is that the values of t for which $\psi_{s',u'}(t)$ is non-zero do not overlap with the values of t for which $\psi_{s,u}(t)$ is non-zero (see second row of Figure 7 for illustration). In the second case, the values of t for which $\psi_{s',u'}(t)$ is non-zero overlaps with the values of t for which $\psi_{s,u}(t)$ is non-zero. However, since $u, u' \in \mathbb{Z}$, therefore $\psi_{s',u'}(t)$ is constant value over the interval where $\psi_{s,u}(t)$ is non-zero (see third row of Figure 7 for illustration). Therefore

$$\int_{-\infty}^{\infty} \psi_{s',u'}(t)\psi_{s,u}(t)dt = \int_{-\infty}^{\infty} 0dt = 0 \quad \text{if } s < s'$$

Putting all the above cases together, we get

$$\int_{-\infty}^{\infty} \psi_{s,u}(t)\psi_{s',u'}(t)dt = \delta(s-s', u-u')$$

We say that $\{\psi_{s,u}(t)\}_{s,u\in\mathbb{Z}}$ are orthonormal. Unlike the scaling functions, the wavelet functions are orthonormal across both scale and translation.

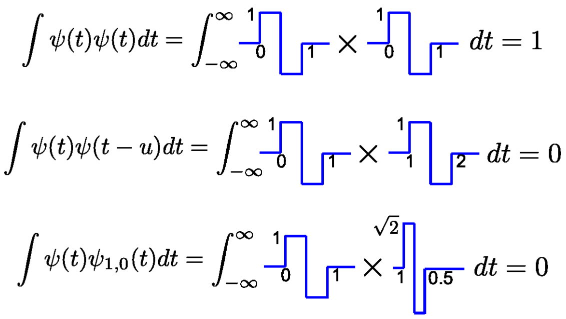


Figure 7: Haar Wavelet Orthonormality

(b) According to the question

$$f(t) = \sum_{u\in\mathbb{Z}} \beta_s[u]\psi_{s,u}(t), \text{where } \alpha_s[u] \in \mathbb{R}$$

Let integrate both side with respect to $\psi_{s,u'}(t)$, then

$$\begin{aligned}
\int_{-\infty}^{\infty} f(t)\psi_{s,u'}(t)dt &= \int_{-\infty}^{\infty} \left(\sum_{u\in\mathbb{Z}} \beta_s[u]\psi_{s,u}(t)\right) \psi_{s,u'}(t)dt \\
&= \sum_{u\in\mathbb{Z}} \beta_s[u] \int_{-\infty}^{\infty} \psi_{s,u}(t)\psi_{s,u'}(t)dt \\
&= \sum_{u\in\mathbb{Z}} \beta_s[u]\delta(0, u-u') \\
&= \beta_s[u']
\end{aligned}$$

(c) We first show that $\phi(t-u)$ and $\psi(t-u')$ are orthogonal for all $u, u' \in \mathbb{Z}$, i.e., $\int_{-\infty}^{\infty} \phi(t-u)\psi(t-u')dt = 0$ for all $u, u' \in \mathbb{Z}$. Suppose $u = u'$, then

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(t-u)\psi(t-u')dt &= \int_{u}^{u+1} \phi(t-u)\psi(t-u)dt \\
&= \int_{u}^{u+1} \psi(t-u)dt = 0
\end{aligned}$$

Suppose $u \neq u'$, then

$$\int_{-\infty}^{\infty} \phi(t-u)\psi(t-u')dt = \int_{-\infty}^{\infty} 0dt = 0$$

Now, suppose we have a function $f(t)$, such that $f(t) \in V_0$ and $f(t) \in W_0$. Then we can write

$$f(t) = \sum_{u \in \mathbb{Z}} \alpha_0[u]\phi(t-u) = \sum_{u' \in \mathbb{Z}} \beta_0[u]\psi(t-u') \tag{1}$$

Multiplying both sides of Eq. (1) by $\phi(t-u'')$ and integrating, the left side is

$$\int_{-\infty}^{\infty} \sum_{u \in \mathbb{Z}} \alpha_0[u]\phi(t-u)\phi(t-u'')dt = \alpha_0[u'']$$

and the right hand side is

$$\int_{-\infty}^{\infty} \sum_{u' \in \mathbb{Z}} \beta_0[u]\psi(t-u')\phi(t-u'')dt = 0$$

Therefore, $\alpha_0[u] = 0$ for all $u \in \mathbb{Z}$. Similarly, multiplying both sides of Eq. (1) by $\psi(t-u'')$ and integrating, we get $\beta_0[u] = 0$ for all $u \in \mathbb{Z}$. Therefore $f(t) = 0$. Therefore V_0 and W_0 are non-overlapping except for the zero function. In fact, this property is true for any fixed scale, i.e., $V_s \cap W_s = \{0\}$ for any fixed scale s.

(d) Our strategy is to show that (i) $V_1 \oplus W_1 \subset V_0$ and (ii) $V_0 \subset V_1 \oplus W_1$. Given (i) and (ii), this would mean that $V_0 = V_1 \oplus W_1$.

(i) We first show that $V_1 \oplus W_1 \subset V_0$. First, $f(t) \in V_1 \oplus W_1 \implies f(t) = \sum_{u \in \mathbb{Z}} \alpha_1[u]\phi_{1,u}(t) + \sum_{u \in \mathbb{Z}} \beta_1[u]\psi_{1,u}(t)$ for some coefficients $\alpha_1[u]$ and $\beta_1[u]$. Observe that

$$\phi_{1,0}(t) = \frac{1}{\sqrt{2}}\phi(t) + \frac{1}{\sqrt{2}}\phi(t-1)$$

This is the same <u>two-scale relation</u> for scaling functions that was mentioned in Q2e. In addition,

$$\psi_{1,0}(t) = \frac{1}{\sqrt{2}}\phi(t) - \frac{1}{\sqrt{2}}\phi(t-1)$$

This is known as the <u>two-scale relation</u> for wavelet functions because it relates the wavelet function with the scaling functions at the next finer scale. For illustration, see Figure 8. Note that it is not an accident that the coefficients $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$ corresponds to the reflected version of the high pass filter h in the Haar discrete wavelet transform we learned in lecture. We will come back to this point in the next lecture. Therefore if $f(t) \in V_1 \oplus W_1$, we can write

$$f(t) = \sum_{u \in \mathbb{Z}} \alpha_0[u]\phi(t-u)$$

where

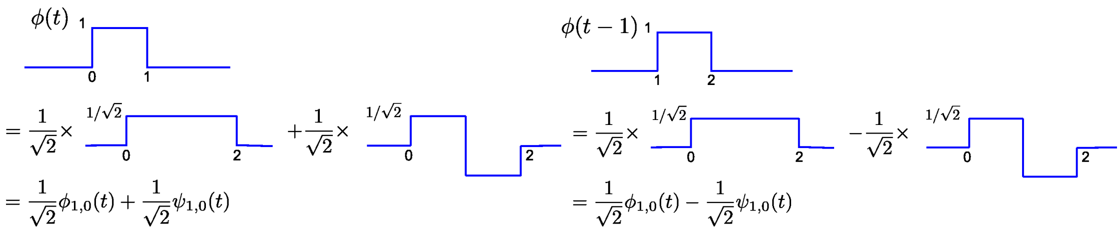
$$\alpha_0[u] = \begin{cases} \frac{1}{\sqrt{2}}\alpha_1\left[\frac{u}{2}\right] + \frac{1}{\sqrt{2}}\beta_1\left[\frac{u}{2}\right] & \text{if } u \text{ is even} \\ \frac{1}{\sqrt{2}}\alpha_1\left[\frac{u-1}{2}\right] - \frac{1}{\sqrt{2}}\beta_1\left[\frac{u-1}{2}\right] & \text{if } u \text{ is odd} \end{cases}$$

Therefore $f(t) \in V_0$.

(ii) Now let's show that $V_0 \subset V_1 \oplus W_1$. First, $f(t) \in V_0 \implies f(t) = \sum_{u\in\mathbb{Z}} \alpha_0[u]\phi(t-u)$ for some coefficients $\alpha_0[u]$. Observe that (see Figure8 for illustration):

$$\phi(t) = \frac{1}{\sqrt{2}}\phi_{1,0}(t) + \frac{1}{\sqrt{2}}\psi_{1,0}(t)$$

$$\phi(t-1) = \frac{1}{\sqrt{2}}\phi_{1,0}(t) - \frac{1}{\sqrt{2}}\psi_{1,0}(t)$$



(a) (b)

Figure 8: Complementary subspaces

Therefore if $f(t) \in V_0$, we can write

$$f(t) = \sum_{u\in\mathbb{Z}} \alpha_1[u]\phi_{1,u}(t) + \sum_{u\in\mathbb{Z}} \beta_1[u]\psi_{1,u}(t)$$

where

$$\alpha_1[u] = \frac{1}{\sqrt{2}}\alpha_0[2u] + \frac{1}{\sqrt{2}}\alpha_0[2u+1]$$

$$\beta_1[u] = \frac{1}{\sqrt{2}}\alpha_0[2u] - \frac{1}{\sqrt{2}}\alpha_0[2u+1]$$

Therefore $f(t) \subset V_1 \oplus W_1$. The above two equations are known as the <u>two-scale relations</u> for scaling and wavelet coefficients, because it relates the scaling coefficients at a finer scale to the scaling and wavelet coefficients at the next coarser scale. As will be discussed in the next lecture, these two-scale relations are the basis of the filterbank implementation of the discrete wavelet transform (also known as at the fast wavelet transform).

(i) and (ii) together implies that $V_0 = W_1 \oplus V_1$. By continuing the same line of argument, we have $V_0 = W_1 \oplus W_2 \oplus V_2 = W_1 \oplus W_2 \oplus W_3 \oplus V_3 = \cdots$. We say that <u>the wavelet spaces are complementary subspaces to the scaling spaces</u>.