EE3731C CA1 Solutions (2019/2020 Sem 1)

Please fill in your NUS ID at the top left hand side of each page. This quiz is worth 20% of your final grade. There are 5 questions (a), (b), (c), (d) and (e) with a total marks of 50. **You can answer them in any order**. You can use the back of the pages if you run out of space.

(a) (10 marks) We wish to use Kaiser window method to design a FIR filter with generalized linear phase that meets the following specifications:

$$\begin{aligned} 0.95 < |H(e^{jw})| < 1.05, &\qquad 0 \le |w| \le 0.2\pi \\ |H(e^{jw})| < 0.1, &\qquad 0.3\pi \le |w| \le 0.6\pi \\ 0.95 < |H(e^{jw})| < 1.05, &\qquad 0.7\pi \le |w| \le \pi \end{aligned}$$

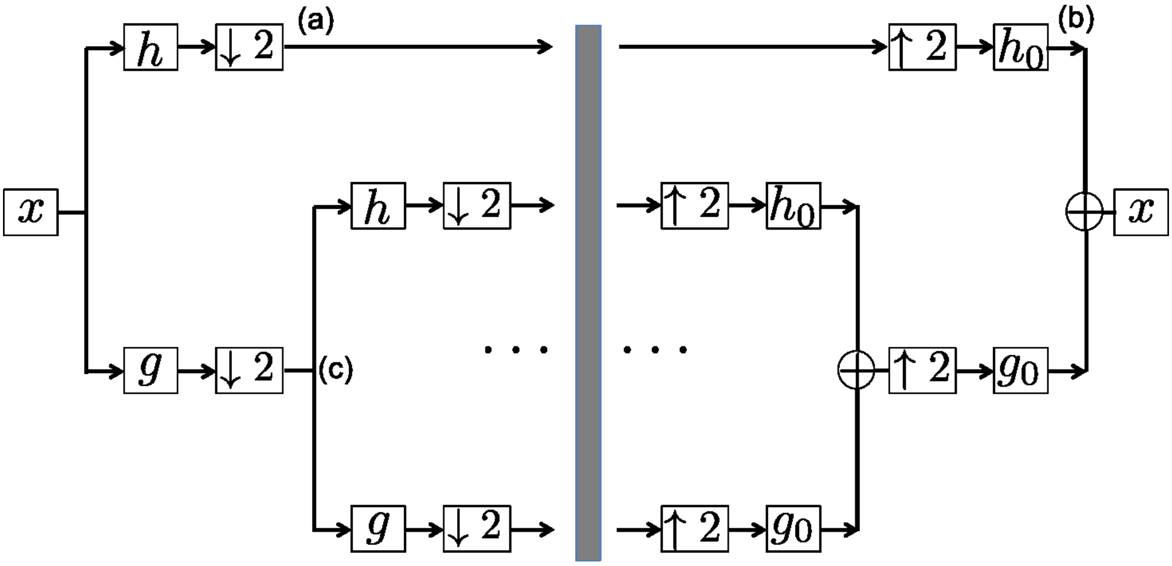
Find the Kaiser window parameters β and M so that the resulting windowed filter satisfies the above criteria. Justify your choice of δ and $\triangle w$. What is the frequency response of the ideal filter?

- δ should be 0.05.
- Reason is because 0.05 is stricter than 0.1 in the stopband
- Therefore $A = -20\log_{10}(0.05) = 26.0206$.
- $\beta = 0.5842(A-21)^{0.4} + 0.07886(A-21) = 1.5099$.
- On the other hand, $\triangle w = 0.3\pi - 0.2\pi = 0.1\pi$ or $\triangle w = 0.7\pi - 0.6\pi = 0.1\pi$. Either is ok.
- Therefore $M = \frac{A-8}{2.285\triangle w} = 25.1$
- We should round this up to 26.
- The ideal frequency response should be

$$\begin{aligned} H_I(e^{jw}) &= \begin{cases} e^{-jwM/2}, & 0 \le |w| \le 0.25\pi \\ 0, & 0.25\pi \le |w| \le 0.65\pi \\ e^{-jwM/2}, & 0.65\pi \le |w| \le \pi \end{cases} \\ &= \begin{cases} e^{-13jw}, & 0 \le |w| \le 0.25\pi \\ 0, & 0.25\pi \le |w| \le 0.65\pi \\ e^{-13jw}, & 0.65\pi \le |w| \le \pi \end{cases} \end{aligned}$$

(b) (12 marks) Consider the filter bank implementation of the discrete wavelet transform and its inverse (show below).

Suppose $g = g_0 = \left[\frac{\sqrt{2}}{2}\ \frac{\sqrt{2}}{2}\right]$, $h = \left[-\frac{\sqrt{2}}{2}\ \frac{\sqrt{2}}{2}\right]$ and $h_0 = \left[\frac{\sqrt{2}}{2}\ -\frac{\sqrt{2}}{2}\right]$ (i.e., haar wavelet). Given the input $x = [2\ \ 6\ \ 8\ \ 4]$, what are the intermediate coefficients at locations (a), (b) and (c) labeled in the above figure? Show your steps.



- a: $[2\ \ 6\ \ 8\ \ 4] * \left[-\frac{\sqrt{2}}{2}\ \ \frac{\sqrt{2}}{2}\right] = [-1\ \ -2\ \ -1\ \ 2]\sqrt{2}$
  $[-1\ \ -2\ \ -1\ \ 2]\sqrt{2} \overset{downsample}{\longrightarrow} [-2\ \ 2]\ \sqrt{2}$

  faster way: $\left[\frac{2-6}{2}\quad \frac{8-4}{2}\right]\sqrt{2} = [-2\ \ 2]\ \sqrt{2}$ (will give full credits)

- b: $[-2\ \ 2]\ \sqrt{2} \overset{upsample}{\longrightarrow} [-2\quad 0\quad 2\quad 0]\ \sqrt{2}$
  $[-2\quad 0\quad 2\quad 0]\ \sqrt{2} * \left[\frac{1}{2}\quad -\frac{1}{2}\right]\sqrt{2} = [-2\quad 2\quad 2\quad -2]$

- c: $[2\ \ 6\ \ 8\ \ 4] * \left[\frac{\sqrt{2}}{2}\ \ \frac{\sqrt{2}}{2}\right] = [1\ \ 4\ \ 7\ \ 6]\sqrt{2}$
  $[1\ \ 4\ \ 7\ \ 6]\sqrt{2} \overset{downsample}{\longrightarrow} [4\ \ 6]\ \sqrt{2}$

  faster way: $\left[\frac{2+6}{2}\quad \frac{8+4}{2}\right]\sqrt{2} = [4\quad 6]\ \sqrt{2}$ (will give full credits)

(c) (6 marks) Let $x[n]$ be of length 2 and $y[n]$ be of length 3. Describe a procedure using DFT to compute the <u>normal</u> convolution between x and y to obtain $z[n]$. Your procedure should include the minimum number of padded zeros.

- The length of the output sequence is $3+2-1=4$, so we should pad 0 until $x[n]$ and $y[n]$ are of length 4
  $$x_p[n] = \begin{cases} x[n] & 0 \le n \le 1 \\ 0 & 2 \le n \le 3 \end{cases}$$
  $$y_p[n] = \begin{cases} y[n] & 0 \le n \le 2 \\ 0 & 3 \le n \le 3 \end{cases}$$
- Perform DFT of $x_p[n]$ and $y_p[n]$ to get $X_p[k]$ and $Y_p[k]$ respectively.

- Multiply $X_p[k]$ and $Y_p[k]$ together to get $Z[k] = X_p[k]Y_p[k]$
- Perform inverse DFT of Z to get $z[n] = \text{Inverse-DFT}(Z)$

(d) (12 marks) An LTI system has impulse response $h[n] = (1/4)^n u[n]$. Use DTFT to find the output of this system when the input is $x[n] = (1/2)^n u[n-3]$. You can use the fact that the DTFT of $a^n u[n]$ is $\frac{1}{1-ae^{-j\omega}}$. You can also use the identity that $\frac{1}{(1-ae^{-j\omega})(1-be^{-j\omega})} = \frac{a/(a-b)}{1-ae^{-j\omega}} - \frac{b/(a-b)}{1-be^{-j\omega}}$.

- $\mathcal{F}(h[n]) = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$
- $\mathcal{F}((1/2)^n u[n]) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$
- $\mathcal{F}((1/2)^{n-3} u[n-3]) = \frac{e^{-3j\omega}}{1-\frac{1}{2}e^{-j\omega}}$
- $\mathcal{F}(x[n]) = \mathcal{F}((1/2)^3(1/2)^{n-3}u[n-3]) = (\frac{1}{2})^3 \frac{e^{-3j\omega}}{1-\frac{1}{2}e^{-j\omega}}$
- The output

$$
\begin{aligned}
Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) \\
&= \left(\frac{1}{1-\frac{1}{4}e^{-j\omega}}\right)\left(\left(\frac{1}{2}\right)^3 \frac{e^{-3j\omega}}{1-\frac{1}{2}e^{-j\omega}}\right) \\
&= \left(\frac{\frac{1/4}{1/4-1/2}}{1-\frac{1}{4}e^{-j\omega}} - \frac{\frac{1/2}{1/4-1/2}}{1-\frac{1}{2}e^{-j\omega}}\right)\left(\frac{1}{2}\right)^3 e^{-3j\omega} \quad \text{using } a = 1/4, b = 1/2 \\
&= \left(-\frac{1}{1-\frac{1}{4}e^{-j\omega}} + \frac{2}{1-\frac{1}{2}e^{-j\omega}}\right)\left(\frac{1}{2}\right)^3 e^{-3j\omega}
\end{aligned}
$$

- $\mathcal{F}^{-1}\left(\frac{e^{-3j\omega}}{1-\frac{1}{4}e^{-j\omega}}\right) = (\frac{1}{4})^{n-3}u[n-3]$
- $\mathcal{F}^{-1}\left(\frac{2e^{-3j\omega}}{1-\frac{1}{2}e^{-j\omega}}\right) = 2(\frac{1}{2})^{n-3}u[n-3]$
-

$$
\begin{aligned}
y[n] &= \left(-\left(\frac{1}{4}\right)^{n-3} u[n-3] + 2\left(\frac{1}{2}\right)^{n-3} u[n-3]\right)\left(\frac{1}{2}\right)^3 \\
&= -\left(\frac{1}{4}\right)^{n-3}\left(\frac{1}{2}\right)^3 u[n-3] + 2\left(\frac{1}{2}\right)^n u[n-3]
\end{aligned}
$$

(e) (10 marks) Consider the following system T. Given an input $x[n]$, the output is $y[n] = T(x[n]) = \sin(x[n]) + x[n-1]$. Is the system (i) stable, (ii) causal, (iii) linear, (iv) time invariant, (v) memoryless? Please explain your reasoning for full credits.

- The system is stable. Because if $|x[n]| \leq M$, then $|y[n]| \leq 1 + M$. So bounded $x[n]$ implies bounded $y[n]$.
- The system is causal since the output at time n depends on the input at time $n-1$ and n (i.e., does not depend on input from future timepoints).

- The system is nonlinear. Let $y_1[n] = T(x_1[n])$ and $y_2[n] = T(x_2[n])$. Then $T(cx_1[n] + dx_2[n]) = \sin(cx_1[n] + dx_2[n]) + cx_1[n-1] + dx_2[n-1] \neq c\Big(\sin(x_1[n]) + x_1[n-1]\Big) + d\Big(\sin(x_2[n]) + x_2[n-1]\Big)$.
- The system is time invariant, since $T(x[n-n_0]) = \sin x[n-n_0] + x[n-n_0-1] = y[n-n_0]$
- The system is not memoryless because $y[n]$ does not just depend on current timepoint (but also previous timepoint) of x.