EE3731C Tutorial - Statistical Signal I

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1. (a) For a discrete random variable x, the probability mass function $p(x)$ is ≥ 0 for all x and $\sum_x p(x) = 1$. Consequently, $p(x) \leq 1$ for all x. To see this, suppose $p(x_0) > 1$ for some x_0, then $\sum_x p(x) > 1$. In class, I might sometimes refer to the probability mass function $p(x)$ as the "probability distribution of x", where I have dropped "mass function" for brevity.

   For a continuous random variable z, the probability distribution function $p(z)$ also has to be ≥ 0 for all z and $\int_z p(z) = 1$. Unlike the discrete case, $p(z)$ can be > 1 for certain z. For example, $p(z) = 2$ for $0 \leq z \leq 0.5$ is a valid probability distribution function. In class, I might refer to the probability distribution function $p(z)$ as the "probability distribution of z", where I have dropped "function" for brevity.

   In this problem, $p(x, y)$ is a valid distribution because

   - $p(x, y) \geq 0$ for all x, y
   - $\int_0^1 \int_0^1 p(x, y) dxdy = \int_0^1 \int_0^1 x + ydxdy = \int_0^1 xdx + \int_0^1 ydy = 1/2 + 1/2 = 1$

   (b)

$$
\begin{aligned}
p(x) &= \int_0^1 p(x, y) dy \text{ for } 0 \leq x \leq 1 \\
&= \int_0^1 x + ydy \\
&= \int_0^1 xdy + \int_0^1 ydy \\
&= x + \int_0^1 ydy \\
&= x + \frac{1}{2}
\end{aligned}
$$

   Note that $p(x) = 0$ for $x < 0$ or $x > 1$. By symmetry,

$$
p(y) = \begin{cases} y + 1/2 & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}
$$

Using the definition of conditional probability

$$\begin{aligned} p(x|y) &= \frac{p(x,y)}{p(y)} \\ &= \begin{cases} \frac{x+y}{y+1/2} & \text{for } 0 \le x, y \le 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(c) $p(x|y) \neq p(x)$. Therefore x and y are not independent.

(d) $E(x) = E(y)$, so we will just calculate $E(x)$

$$\begin{aligned} E(x) &= \int_0^1 xp(x)dx \\ &= \int_0^1 x(x+1/2)dx \\ &= \Big[\frac{x^3}{3} + \frac{x^2}{4}\Big]_0^1 \\ &= \frac{1}{3} + \frac{1}{4} = \frac{7}{12} \end{aligned}$$

(e) $Var(x) = Var(y)$, so we will just calculate $Var(x)$. First let's calculate

$$\begin{aligned} E(x^2) &= \int_0^1 x^2 p(x)dx \\ &= \int_0^1 x^2\Big(x + \frac{1}{2}\Big)dx \\ &= \Big[\frac{x^4}{4} + \frac{x^3}{6}\Big]_0^1 \\ &= \frac{1}{4} + \frac{1}{6} = \frac{5}{12} \end{aligned}$$

Therefore $Var(x) = E(x^2) - (E(x))^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = 11/144$.

(f)

$$\begin{aligned} E(xy) = \int_0^1 \int_0^1 xyp(x,y)dxdy &= \int_0^1 \int_0^1 xy(x+y)dxdy \\ &= \int_0^1 \int_0^1 x^2y + xy^2 dxdy \\ &= \int_0^1 \left[\frac{x^3}{3}y + \frac{x^2y^2}{2}\right]_0^1 dy \\ &= \int_0^1 \frac{y}{3} + \frac{y^2}{2} dy \\ &= \left[\frac{y^2}{6} + \frac{y^3}{6}\right]_0^1 = \frac{1}{6} + \frac{1}{6} = \frac{1}{3} \end{aligned}$$

Therefore $Cov(x,y) = E(xy) - E(x)E(y) = \frac{1}{3} - \frac{7}{12}\frac{7}{12} = -\frac{1}{144}$

(g) $p(x|y=0.5) = \begin{cases} x + 0.5 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

We can verify that $\int_0^1 p(x|y=0.5)dx = 1$ and is therefore a valid probability distribution. Therefore

$$\begin{aligned} E(x^2|y=0.5) &= \int_0^1 x^2 p(x|y=0.5)dx \\ &= \int_0^1 x^3 + 0.5x^2 dx \\ &= \left[\frac{1}{4}x^4 + \frac{1}{6}x^3\right]_0^1 \\ &= \frac{1}{4} + \frac{1}{6} = 5/12 \end{aligned}$$

2.

$$\begin{aligned} & E_{p(x)}(x-a)^2 \\ & = E_{p(x)}(x^2 + a^2 - 2ax) \\ & = E_{p(x)}(x^2) + E_{p(x)}(a^2) - E_{p(x)}(2ax) \qquad \text{linearity of expectation} \\ & = E_{p(x)}(x^2) + a^2 - 2a\mu_x \qquad \text{more linearity of expectation} \\ & = E_{p(x)}(x^2) + (a - \mu_x)^2 - \mu_x^2 \qquad \text{completing the square} \\ & = Var(x) + (a - \mu_x)^2 \\ & \ge Var(x) \text{ with equality achieved when } a = \mu_x \end{aligned}$$

3.

$$\begin{aligned} Var(cx) &= E((cx)^2) - (E(cx))^2 && \text{Definition of variance} \\ &= E(c^2x^2) - (cE(x))^2 && \text{Expand first term and linearity of expectation second term} \\ &= c^2E(x^2) - c^2(E(x))^2 && \text{Linearity of expectation first term and expand second term} \\ &= c^2(E(x^2) - (E(x))^2) && \text{collect terms} \\ &= c^2Var(x) && \text{Definition of variance} \end{aligned}$$

4. (a) By linearity of expectation, $E(2x+3y) = 2E(x) + 3E(y) = 2 \times 5 + 3 \times 3 = 19$
   (b) Using $Q3$ and the fact that $Var(x) = E(x^2) - (E(x))^2$, we get $Var(2x) = 4Var(x) = 4[E(x^2) - (E(x))^2] = 4(30 - 5^2) = 20$
   (c) $Cov(x,y) = E(xy) - E(x)E(y) = 4 - (5 \times 3) = -11$
   (d) $Cov(x,y) \neq 0 \implies x$ and y are not independent

5. Consider two random variables x and y with joint distribution $p(x,y)$. Let

$$\begin{array}{ll} p(5,5) = 2/5 & p(-5,-5) = 2/5 \\ p(-10,10) = 1/10 & p(10,-10) = 1/10 \\ p(x,y) = 0, \text{otherwise} & \end{array}$$

Clearly x and y are not independent, since knowing one completely determines the other. However,

$$E(XY) = \sum_x \sum_y xyp(x,y) = 25 \times \frac{2}{5} + 25 \times \frac{2}{5} - 100 \times \frac{1}{10} - 100 \times \frac{1}{10} = 0$$

Furthermore, observe that $p(x) = p(y)$, where $p(x = -5) = p(x = 5) = 2/5$ and $p(x = -10) = p(x = 10) = 1/10$. Therefore

$$E(x) = \sum_x xp(x) = 5 \times \frac{2}{5} - 5 \times \frac{2}{5} - 10 \times \frac{1}{10} + 10 \times \frac{1}{10} = 0$$
$$E(Y) = \sum_y yp(y) = 5 \times \frac{2}{5} - 5 \times \frac{2}{5} + 10 \times \frac{1}{10} - 10 \times \frac{1}{10} = 0$$

Therefore $Cov(X,Y) = E(XY) - E(X)E(Y) = 0$.

The webpage (http://en.wikipedia.org/wiki/Correlation_and_dependence) shows various examples of continuous distributions where $Cov(x,y) = 0$, but x and y are not independent. Here's a more formal example. Let $x \sim U[-1/2, 1/2]$ (uniform distribution between $-1/2$ and $1/2$).

Let $y = x^2 + \epsilon$, where $\epsilon \sim \mathcal{N}(0,1)$, so we can think of y as equal to x^2 plus some Gaussian noise. Then $p(y|x) \sim \mathcal{N}(x^2, 1)$. Therefore x and y are not independent. Let's compute $E(xy)$

$$\begin{aligned}
E(xy) &= \int_{-0.5}^{0.5} \int_{-\infty}^{\infty} xyp(x)p(y|x)dydx \\
&= \int_{-0.5}^{0.5} \int_{-\infty}^{\infty} xy\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-x^2)^2}{2}}dydx \\
&= \int_{-0.5}^{0.5} x \int_{-\infty}^{\infty} y\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-x^2)^2}{2}}dydx \\
&= \int_{-0.5}^{0.5} x \int_{-\infty}^{\infty} (u - x^2)\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}dudx \quad \text{substitute } u = y - x^2 \\
&= -\int_{-0.5}^{0.5} x^3 dx \\
&= 0
\end{aligned}$$

Note that $E(x) = 0$. Therefore $Cov(x,y) = E(xy) - E(x)E(y) = 0$.