NATIONAL UNIVERSITY OF SINGAPORE

EXAMINATION FOR

(Semester I: 2014/2015)

EE3731C – SIGNAL PROCESSING METHODS

November/December 2014 - Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This paper contains **FOUR (4)** questions and comprises **FIVE (5)** printed pages.

2. Answer all the questions.

3. Questions 1 and 2 carry 25 marks. Question 3 carries 30 marks, while Question 4 carries 20 marks.

4. This is a **CLOSED BOOK** examination.

Q1 Given a set of data points

$$\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}-5\\-5\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}$$

(a) What is the first principal component?

(8 marks)

(b) What is the total reconstruction error after the following two steps:

i) projecting the data points from the original 2D space to the 1D space by the principal component from (a), and
ii) representing the data from i) in the original 2D space.

(Note: the total reconstruction error is denoted as

$$e = \sum_{i=1}^{n} \|\boldsymbol{x}_i - (\boldsymbol{v}^T \boldsymbol{x}_i)\boldsymbol{v}\|^2,$$

where $\boldsymbol{x}_i$ is a 2D data point (mean subtracted), n is the number of data points, and $\boldsymbol{v}$ is the principal component).

(6 marks)

(c) With $1{,}000{,}000$ RGB-colored face images (each with size 600×800),

i) What is the dimensionality of the raw image data, and what is the number of images that can be generated if each pixel is represented by three bytes, each $\epsilon[0{,}255]$?

(4 marks)

ii) What is an Eigenface? Describe how Eigenfaces work to reduce dimensionality.

(4 marks)

iii) Write down the methods, including input, output, and steps, to obtain Eigenfaces.

(3 marks)

Q2 (a) For the difference equation $x_{k+1} = -\frac{1}{2}x_k - \frac{1}{2}$, $x_0 = 0$,

i) Find its analytical solution.

(4 marks)

ii) Find its equilibrium x^*.

(2 marks)

iii) Discuss the stability of x^*.

(2 marks)

(b) For the difference equation $x_{k+1} = (-1)^k x_k$, $x_0 = 1$,

i) Find its analytical solution.

(2 marks)

ii) Prove the analytical solution by induction.

(2 marks)

(c) For the difference equation $x_{k+1} = x_k - \frac{k}{(k+1)!}$, $x_0 = 1$,

i) Find its analytical solution.

(2 marks)

ii) Prove the analytical solution by induction.

(2 marks)

(d) Give an example of spatial domain linear image filters in the form of a 3×3 matrix for

i) Image blurring.

(3 marks)

ii) Image shifting to the right.

(3 marks)

iii) Vertical edge detection.

(3 marks)

Q3 The sub-questions (a)-(e) are independent.

(a) Alice and Bob play the following game everyday. On day n, Alice independently sampled (x_n, y_n) from the joint probability distribution $p(x, y)$. Given observation y_n, Bob guesses x_n to be $\widehat{x_n}$ and pays Alice $e(x_n, \widehat{x_n})$ dollars.

i) Suppose $e(x_n, \widehat{x_n}) = (x_n - \widehat{x_n})^2$. What estimation strategy should Bob use to minimize the payout to Alice on a daily basis (on average)?

(2 marks)

ii) Suppose $e(x_n, \widehat{x_n}) = 0$ if $x_n = \widehat{x_n}$ and $e(x_n, \widehat{x_n}) = 10$ if $x_n \neq \widehat{x_n}$. What estimation strategy should Bob use to minimize the payout to Alice on a daily basis (on average)?

(2 marks)

(b) Given the observation $y = 5$, the MMSE estimate of the random variable x_1 is 3 while the MMSE estimate of the random variable x_2 is 11. Let $z = \frac{x_1 + x_2}{2}$ .What is the MMSE estimate of z given the observation that $y = 5$? Show your intermediate steps.

(2 marks)

(c) Consider a biased coin where q is the probability of getting a head when the coin is tossed. The bias q is generated from the probability distribution function $p(q) = U[0,1]$ (i.e., uniform between 0 and 1). The coin is tossed 10 times independently (q is fixed for all the coin tosses), resulting in 1 head.

i) What is the ML estimate of q? Show your intermediate steps.

(10 marks)

ii) What is the MAP estimate of q? Justify your answer.

(2 marks)

(d) Let x_n be a random walk defined as

$$x_0 = 0$$

$$x_N = x_0 + \sum_{n=1}^{N} z_n$$

where z_n is an i.i.d. process with $p(z_n = -1) = p(z_n = 1) = 1/2$. Define the absolute value random process $y_N = |x_N|$. Find $p(y_N = k)$. Show your intermediate steps.

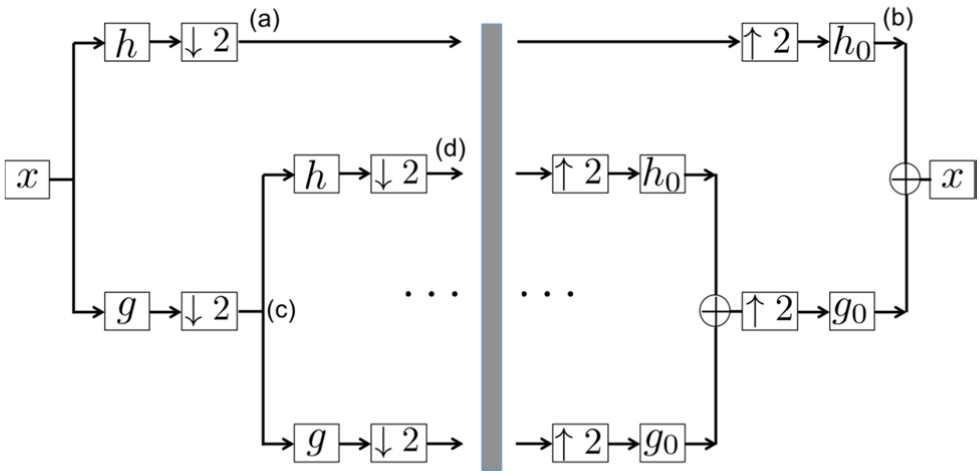
(10 marks)

(e) Given the observation y, suppose we use the Metropolis algorithm to generate five independent samples of x from the posterior distribution $p(x \mid y)$. Let $x_1 = 2, x_2 = 5, x_3 = 3, x_4 = 4$ and $x_5 = 1$ be the five samples. Use the five samples to compute an approximate MMSE estimate of x given y.

(2 marks)

Q4 The sub-questions (a)-(c) are independent.

(a) Consider the filter bank implementation of the discrete wavelet transform and its inverse (shown below). Suppose $g = g_0 = \left[\frac{\sqrt{2}}{2} \;\; \frac{\sqrt{2}}{2}\right]$, $h = \left[-\frac{\sqrt{2}}{2} \;\; \frac{\sqrt{2}}{2}\right]$ and $h_0 = \left[\frac{\sqrt{2}}{2} \;\; -\frac{\sqrt{2}}{2}\right]$ (i.e., Haar wavelet). Given the input $x = [16 \;\; 8 \;\; 20 \;\; 24]$, what are the intermediate coefficients at locations (a), (b), (c) and (d) labeled in the above figure? Show your steps.



(8 marks)

(b) One weakness of the discrete wavelet transform is that it is not translation invariant, i.e., translating the input by one time point can result in very different wavelet coefficients. Draw a filterbank that avoids this issue and explain how the filter bank handles the translation issue.

(6 marks)

(c) Suppose $\phi(t)$ is the scaling function of a multi-resolution analysis, and $\psi(t)$ is the corresponding wavelet function. Let $\phi_{s,u}(t) = 2^{-s/2}\phi(2^{-s}t - u)$ and $\psi_{s,u}(t) = 2^{-s/2}\psi(2^{-s}t - u)$. Let V_s and W_s be the sets of all functions that are linear combinations of $\phi_{s,u}(t)$ and $\psi_{s,u}(t)$ respectively (for a fixed scale s). <u>Explain</u> whether the following statements are true or false.

i) $\int_{-\infty}^{\infty} \phi_{s,u}(t)\phi_{s',u'}(t)dt = \delta(s - s', u - u')$ for all $s, s', u, u' \in \mathbb{Z}$

ii) If $f(t) \in V_2$, then $f(2^{-s}t - u) \in V_s$ for all $s, u \in \mathbb{Z}$

iii) $W_2 \subset W_1 \subset W_0 \subset W_{-1} \subset W_{-2}$

iv) The wavelet coefficients at the next coarser resolution can be obtained by convolving the wavelet coefficients at the current resolution with the wavelet filter h and downsample by 2.

(6 marks)

END OF PAPER