EE4305 Fuzzy/Neural Systems for Intelligent Robotics

PART II: FUZZY SYSTEMS

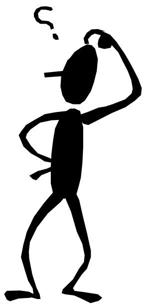
Chapter 5: Fuzzy Numbers

Topics to be Covered…

- Fuzzy sets and crisp sets
- Fuzzy operations, fuzzy relations, fuzzy compositions
- Extension principle, fuzzy numbers
- Approximate reasoning, fuzzy inference
- Multi-rule Fuzzy Inference
- Fuzzy knowledge based control (FKBC)
- Fuzzy applications - Inverted pendulum control
  - Fuzzy nonlinear simulations

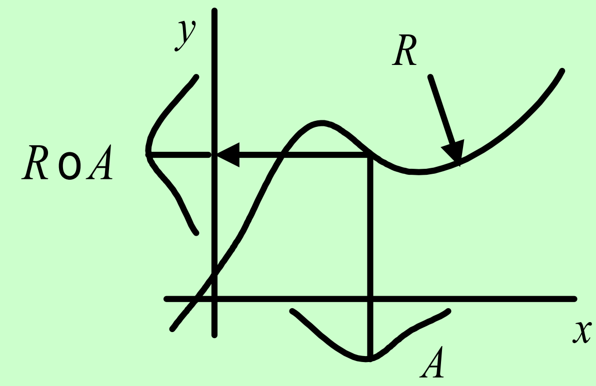
Extension Principle

Extension Principles (EP)



Fuzzy set A → $\boxed{f}$ → Fuzzy set $B = f(A)$

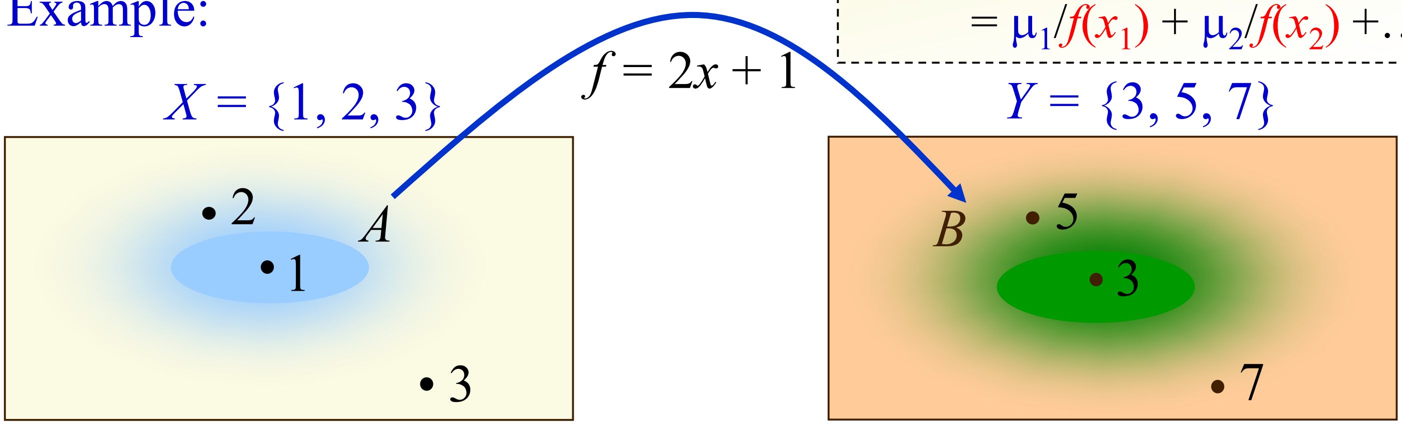
- What is the output for a *fuzzy input* or *fuzzy function* or *both* ?



A, Fuzzy Relation $R \rightarrow R \circ A$

$$f(A) = B = f(\mu_1/x_1 + \mu_2/x_2 + \ldots + \mu_n/x_n)$$
$$= \mu_1/f(x_1) + \mu_2/f(x_2) + \ldots + \mu_n/f(x_n)$$

- Example:



$A = 1/1 + 0.5/2$

$B = \mu_1/y_1 + \mu_2/y_2 = 1/3 + 0.5/5$

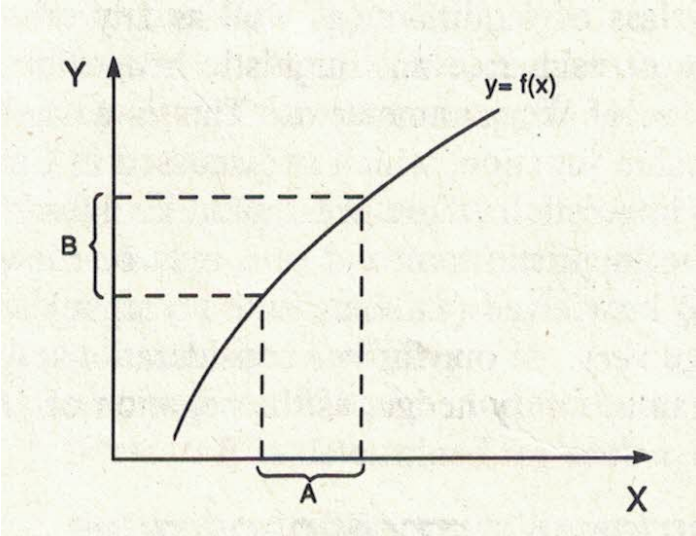
- Consider a crisp function

$$f \ : \ X \rightarrow Y$$

$$y = f(x)$$

Let A be a crisp set in X. Then,

$$B = f(A)$$



is defined to be a crisp subset in Y such that

$$B = \{y \in Y | y = f(x), x \in A\}$$

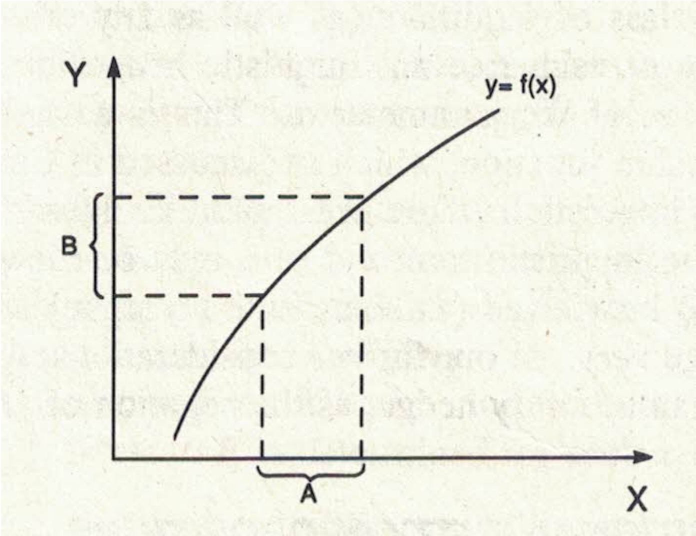
We want to extend the crisp function to the fuzzy function to act on fuzzy set A.

♦ Given a function

$$f : X \to Y$$
$$y = f(x)$$

Let A be a fuzzy set in X. Then, the fuzzy set

$$B = f(A)$$



Given the membership function of fuzzy set A, $\mu_A(x)$,
how to compute the membership function of the output B, $\mu_B(y)$?

In order to answer this question, we need to play the old trick again:

Step one: find out the computations of crisp sets in terms of characteristic functions.

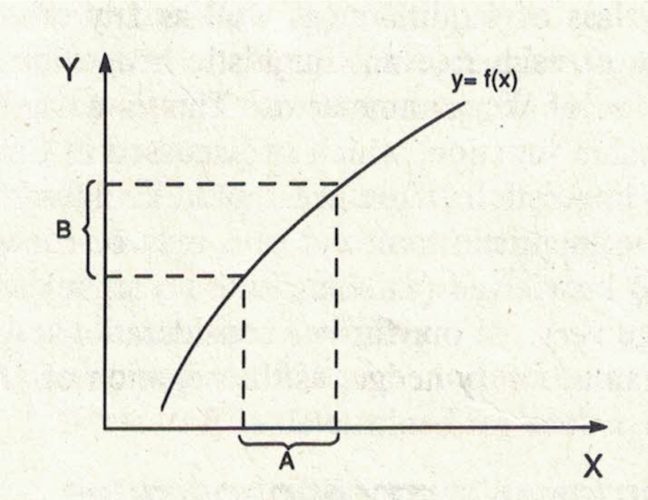
Step two: replace the characteristic functions by membership functions.

♦ Given a function

$$f : X \rightarrow Y$$
$$y = f(x)$$

Let A be a crisp set in X. Then, the crisp set

$$B = f(A)$$



$$B = \{y \in Y | y = f(x), x \in A\}$$

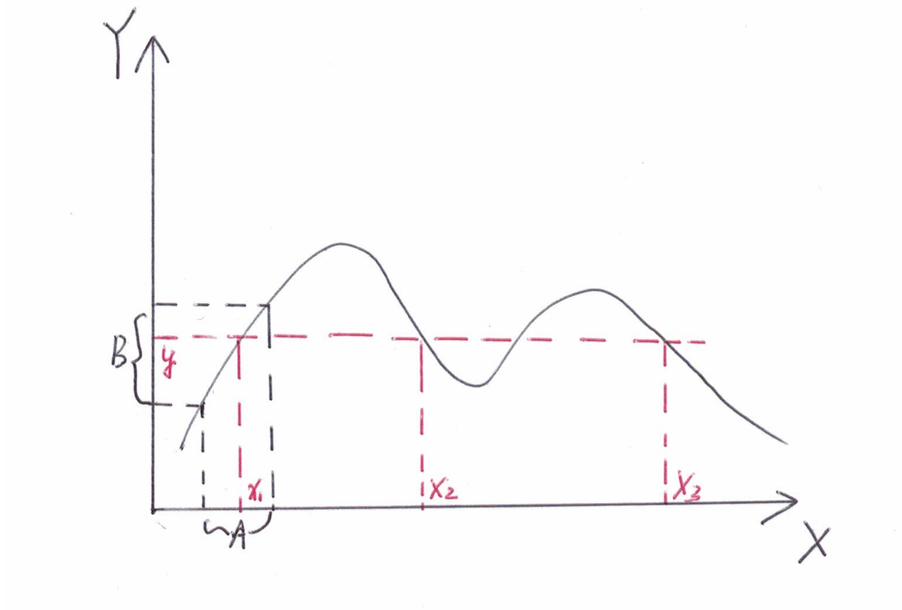
Step one: find out the computations of crisp sets in terms of characteristic functions.

If we analyze the example shown above, we may use the following simple rule:

$$\chi_B(y) = \chi_A(x) \qquad \textbf{where} \qquad x = f^{-1}(y)$$

It means that the membership of the output y is the same as that of the corresponding input $x = f^{-1}(y)$.

But what if there are multiple points that are mapped to the same value, y?



We need to choose the one with the maximal value among all the points mapped to y!

$$\chi_B(y) = \max_{x|y=f(x)} [\chi_A(x)]$$

For fuzzy set, we just replace the characteristic functions by membership functions.

$$\mu_B(y) = \max_{x|y=f(x)} [\mu_A(x)]$$

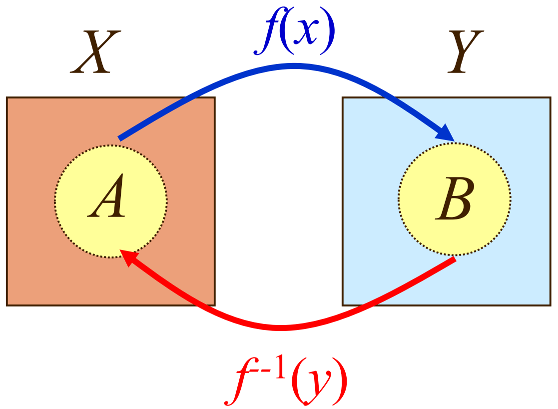
♦ (Def) Extension Principle

$$f \; : \; X \rightarrow Y$$

$$y = f(x)$$

Let A be a fuzzy set in X. Then,

$$B = f(A)$$



is defined to be a fuzzy set in Y with

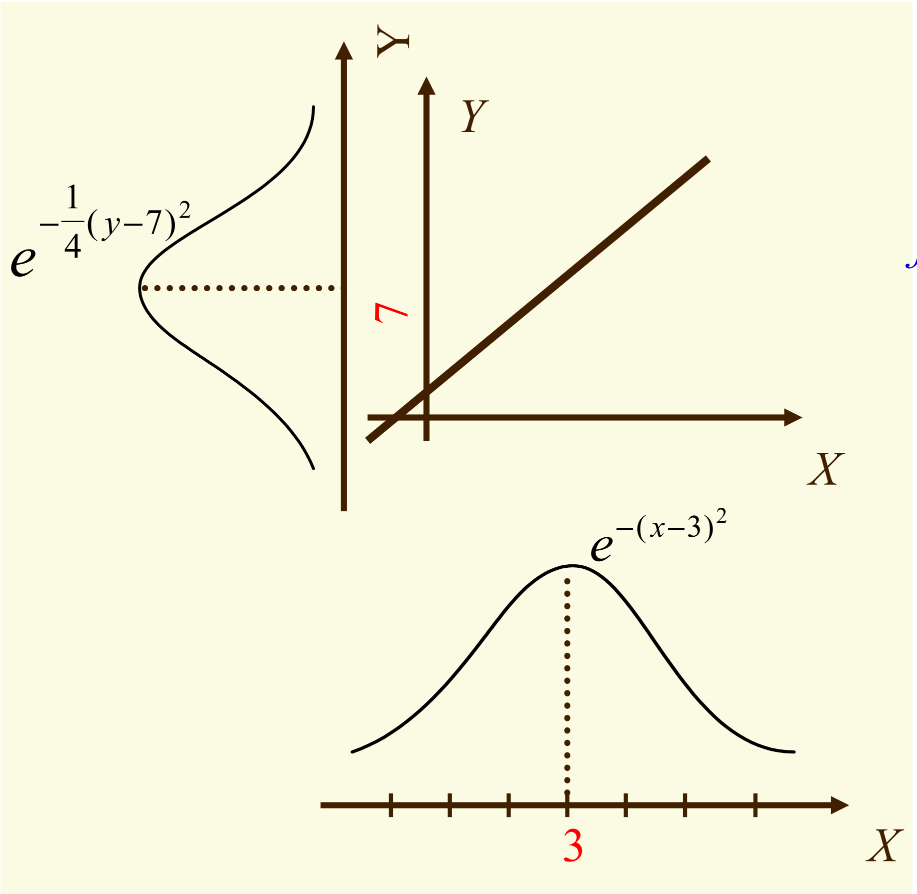
$$\mu_B(y) = \max_{x \mid y = f(x)} [\mu_A(x)]$$

It simply means that the membership of the output y is the maximal membership of its corresponding inputs.

♦ E.g., Suppose $y = f(x) = 2x + 1$ and $A = approx\_3 = \int_X e^{-(x-3)^2} / x$

Find $f(approx\_3)$?

Solution: $\mu_B(y) = \max_{x|y=f(x)} [\mu_A(x)]$



$$y = 2x + 1 \;\Rightarrow\; x = \frac{y-1}{2}$$

$$f(approx\_3) = \int_Y \mu_A\left(\frac{y-1}{2}\right) / y$$

$$= \int_Y e^{-\frac{1}{4}(y-7)^2} / y$$

- Let f: $X \rightarrow Y$, define $A = \mu_1/x_1 + \mu_2/x_2 + \ldots + \mu_n/x_n$ on universe X, for a function f that maps one element in X to one element in universe Y,

$$f(A) = f(\mu_1/x_1 + \mu_2/x_2 + \ldots + \mu_n/x_n)$$
$$= \mu_1/f(x_1) + \mu_2/f(x_2) + \ldots + \mu_n/f(x_n)$$

- What if e.g., $f(x_1) = f(x_3)$ in the above equation?

  → If more than one element of X is mapped to the same element y in Y by f (many to one mapping), then

$$\mu_{f(A)}(y) = \max_{\substack{x_i \in X \\ f(x_i)=y}} [\mu_A(x_i)]$$

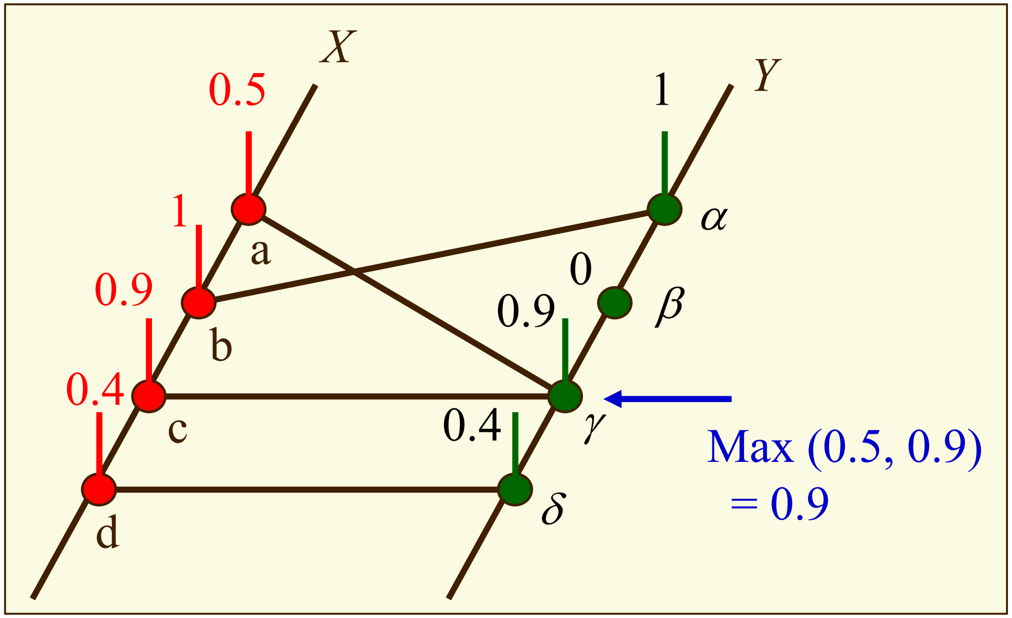
  where x_i are the elements that are mapped to the same y.

- E.g., Given that $y = f(x) = (x-3)^2 + 2$ and a fuzzy set *Around*-4 for x as *Around-4* = 0.3/2 + 0.6/3 + 1/4 + 0.6/5 + 0.3/6

  Using E.P., f(*Around-4*) = 0.3/3 + 0.6/2 + 1/3 + 0.6/6 + 0.3/11

  = 0.6/2 + 1/3 + 0.6/6 + 0.3/11

- Example:



$$y = f(x)$$

$x \in X$	a	b	c	d
$y \in Y$	γ	α	γ	δ

$$A = 0.5/a + 1/b + 0.9/c + 0.4/d$$

$$B = f(A) = ?$$

$$= 1/\alpha + 0.9/\gamma + 0.4/\delta$$

♦ Fuzzy Number

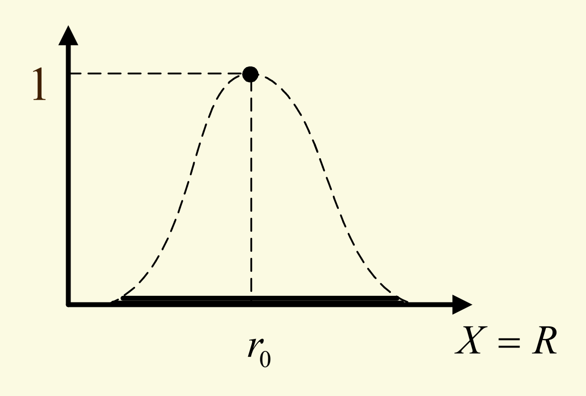
- A fuzzy number is a suitable model of quantitative approximation notions such as approx. 4 tsp, about 7 kg, around 2 o'clock....

- (Def) A fuzzy set A is a fuzzy number if:

  (i) A is normal,

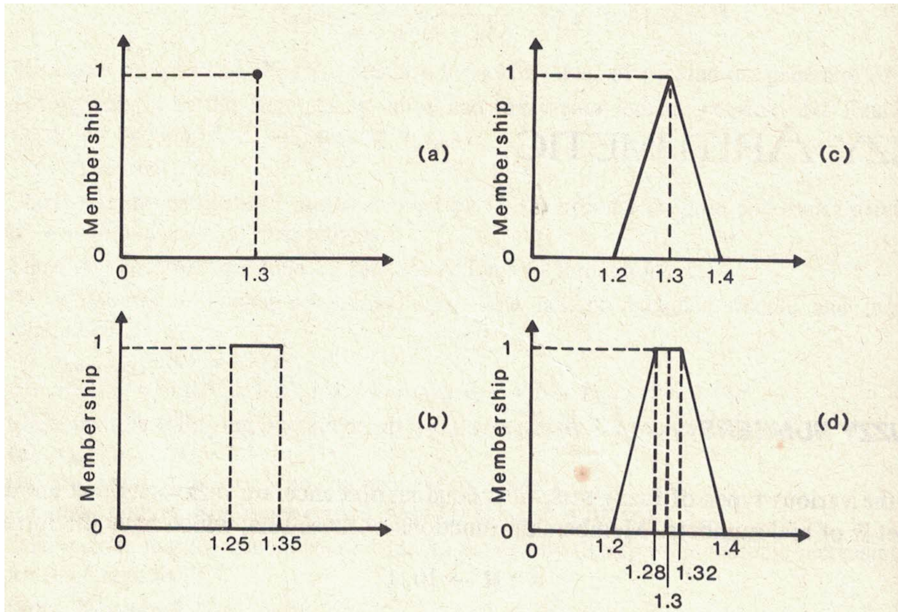
  (ii) All the **α-cut** of A must be a closed interval,

  (iii) A has a bounded support.

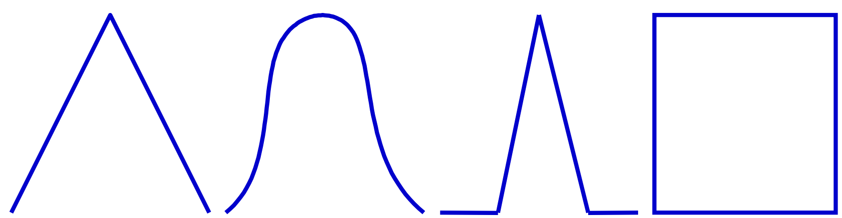


◆ The membership function of **Fuzzy Number**

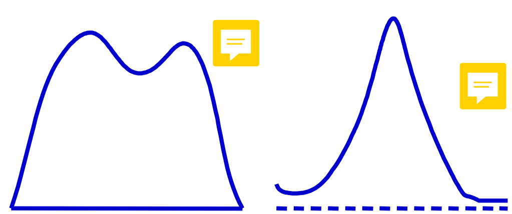
There are various ways to express the fuzzy number like "close to 1.3".



Fuzzy Numbers



Not Fuzzy Numbers



- Extension principle can be applied to arithmetic operations on fuzzy numbers:

$$(A * B)(z) = \max_{z=x*y} \min[A(x), B(y)]$$

A and *B* are fuzzy numbers;
* denote any of the four arithmetic operations i.e., +, -, ·, /

- Example: Given a fuzzy number *about-1* = 0.2/0 +1/1 + 0.2/2, then by using the E.P. *about-1* plus *about-1* will be given as

about-1 + *about-1* = (0.2/0 +1/1 + 0.2/2) + (0.2/0 +1/1 + 0.2/2)

= min(0.2, 0.2)/0 + max[min(0.2, 1), min(1,0.2)]/1 + max[(min(0.2, 0.2), min(1, 1), min(0.2, 0.2)]/2 + max[min(1, 0.2), min(0.2, 1)]/3 + min(0.2, 0.2)/4

= 0.2/0 + 0.2/1 + 1/2 + 0.2/3 + 0.2/4

= *about-2*

- In the example above, arithmetic operations on fuzzy numbers are performed in the discrete form using the extension principle.
- For a continuous function, we may discretize the function and apply the extension principle as what we did in the example above.
- This however, results in irregular and erroneous output membership function since input variables are discretized for numerical convenience.
- Example:

fuzzy number A = *about-1*

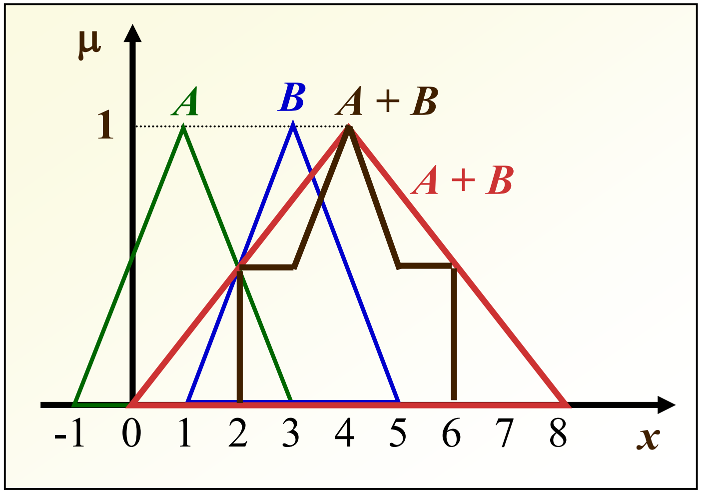
fuzzy number B = *about-3*

$A + B$ = *about-4* (continuous form)

$A + B$ = *about-4* (3 points discrete form)

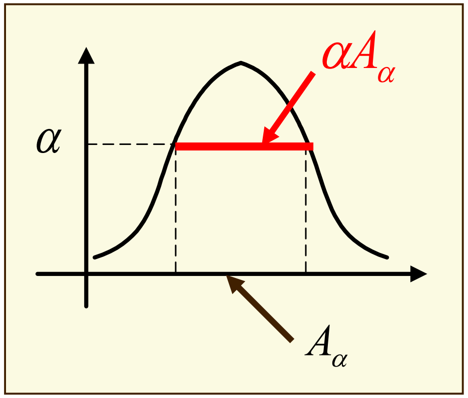
i.e., using $A = (0.5/0 + 1/1 + 0.5/2)$,

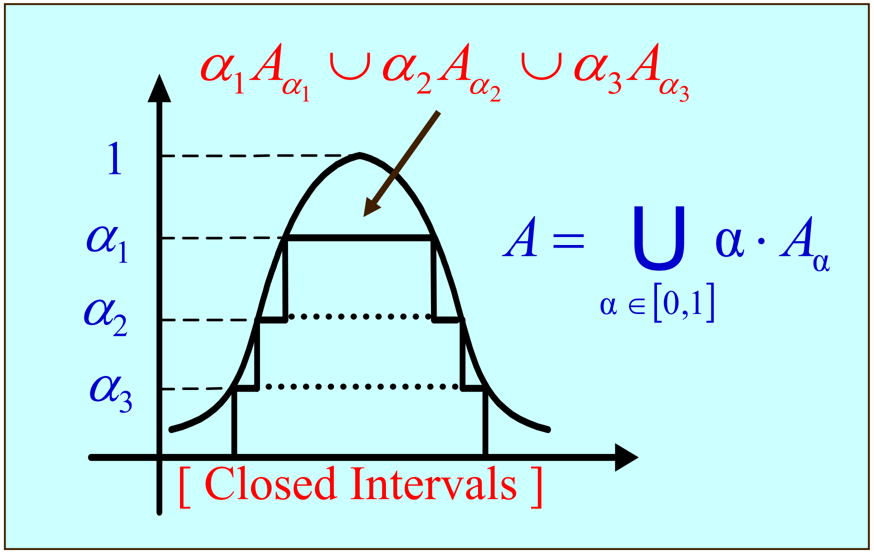
$B = (0.5/2 + 1/3 + 0.5/4)$



Using the α-cut method for arithmetic operations of continuous fuzzy numbers

- Recall the Decomposition Theorem that a fuzzy number can be described by the intervals associated with different levels of α-cuts





Step One: Do the operations on the crisp sets, α-cuts $\boldsymbol{A_\alpha}$

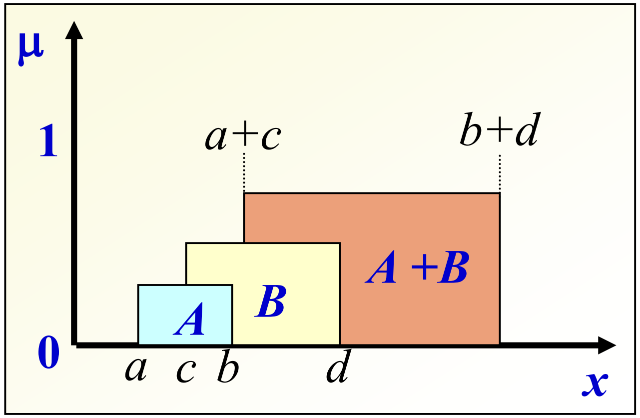
Step Two: Put together (union) all the results on the α-cuts , **and form the fuzzy set.**

♦ Arithmetic Operations on Closed Intervals

Note that fuzzy number is not a single number. The arithmetic of fuzzy number is extension of arithmetic of closed intervals.

How do we compute [a,b]+[c,d]?

We need to include all the possible outcomes from the summation of any point in [a,b] and any other point in [c,d]. The final result is a closed interval.



$$[a, b] + [c, d] = [\min(a+c,\ a+d,\ b+c,\ b+d),\ \max(a+c,\ a+d,\ b+c,\ b+d)]=[a+c, b+d]$$

In general:

- Addition: $[a, b] + [c, d] = [a + c, b + d]$
- Subtraction: $[a, b] - [c, d] = [a - d, b - c]$
- Multiplication: $[a, b] \cdot [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$

Division: $[a, b] / [c, d] = [\min(a/c, a/d, b/c, b/d), \max(a/c, a/d, b/c, b/d)]$

where $0 \notin [c, d]$

- Example: $[-1, 1] \cdot [-2, 0.5] = [\min(-1\cdot(-2), -1\cdot 0.5, 1\cdot(-2), 1\cdot 0.5),$

  $\max(-1\cdot(-2), -1\cdot 0.5, 1\cdot(-2), 1\cdot 0.5)]$

  $= [-2, 2]$

♦ Arithmetic Operations on Fuzzy Numbers (the α-cut method):

For each $\alpha \in (0, 1]$, the α-cut of $A * B$ is given as

$$(A * B)_\alpha = A_\alpha * B_\alpha$$

In this step, the operations are done for crisp sets, the α-cuts of A and B.

$A * B$ can then be expressed by

$$A * B = \bigcup_{\alpha \in [0,1]} \alpha \cdot (A * B)_\alpha$$

- Example: consider the two triangular-shape fuzzy numbers A and B:

$$A(x) = \begin{cases} 0 & \text{for } x \le -1 \text{ and } x > 3 \\ \dfrac{x+1}{2} & \text{for } -1 < x \le 1 \\ \dfrac{3-x}{2} & \text{for } 1 < x \le 3 \end{cases} \qquad B(x) = \begin{cases} 0 & \text{for } x \le 1 \text{ and } x > 5 \\ \dfrac{x-1}{2} & \text{for } 1 < x \le 3 \\ \dfrac{5-x}{2} & \text{for } 3 < x \le 5 \end{cases}$$

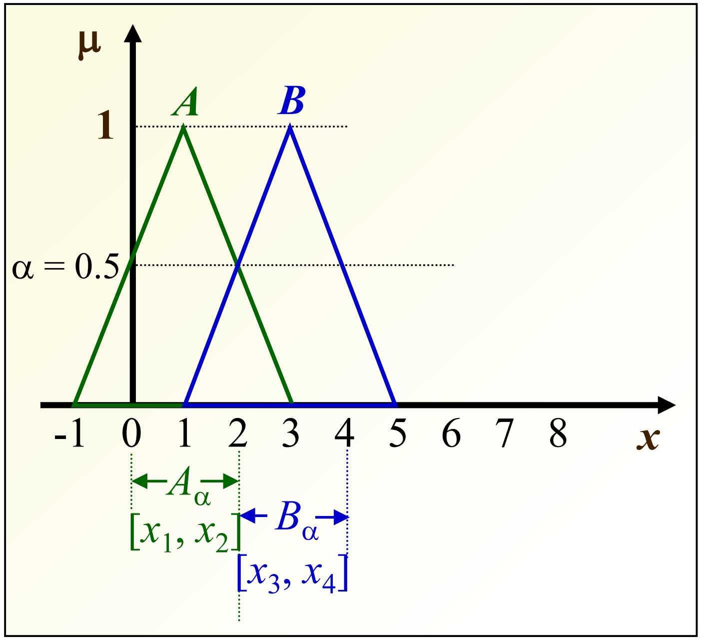
We first need to get the α-cuts of both A and B.

- Define: $A_\alpha = [x_1, x_2]$

then, $A(x_1) = \dfrac{(x_1+1)}{2} = \alpha$

$A(x_2) = \dfrac{(3-x_2)}{2} = \alpha$

α-cut-set is: $A_\alpha = [2\alpha - 1, 3 - 2\alpha]$



- Define: $B_\alpha = [x_3, x_4]$

then, $B(x_3) = \dfrac{(x_3-1)}{2} = \alpha$

$B(x_4) = \dfrac{(5-x_4)}{2} = \alpha$

α-cut-set is: $B_\alpha = [2\alpha + 1, 5 - 2\alpha]$

$A_\alpha = [2\alpha - 1, 3 - 2\alpha]$ $B_\alpha = [2\alpha + 1, 5 - 2\alpha]$

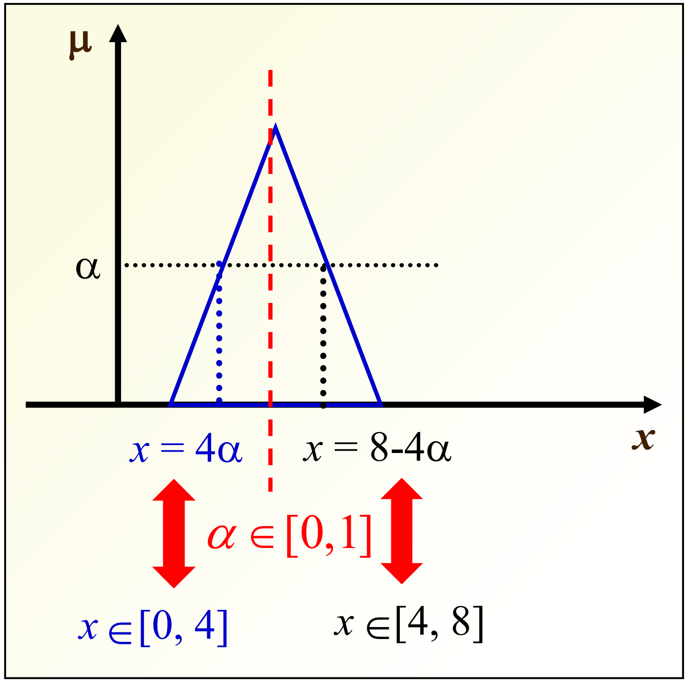
Then do a summation of the crisp intervals A_α and B_α , we have

$$(A+B)_\alpha = (A_\alpha + B_\alpha) = [4\alpha, 8 - 4\alpha] \quad \text{for } \alpha \in [0, 1]$$

Now let's do a union on all the α-cuts

$$A + B = \bigcup_{\alpha \in [0,1]} \alpha \cdot (A + B)_\alpha$$

$$= \bigcup_{\alpha \in [0,1]} \alpha \cdot [4\alpha, 8 - 4\alpha]$$



(i) $4\alpha = x$ for $x \in [0, 4]$, i.e.,

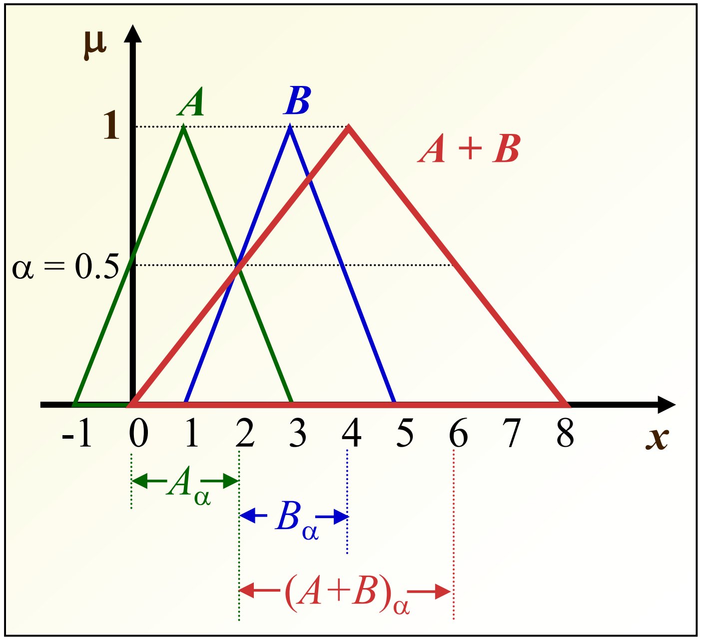
$$\alpha = \frac{x}{4} \quad \text{for } x \in [0, 4]$$

(ii) $8 - 4\alpha = x$ for $x \in [4, 8]$, i.e.,

$$\alpha = \frac{8 - x}{4} \quad \text{for } x \in [4, 8]$$

Therefore,

$$(A+B)(x)=\begin{cases} 0 & \text{otherwise} \\ \dfrac{x}{4} & \text{for } 0<x\leq 4 \\ \dfrac{8-x}{4} & \text{for } 4<x\leq 8 \end{cases}$$



A=1, B=3, so, A+B=1+3=4!