1. Show that the fuzzy set A can be represented by $A = \bigcup_{\alpha \in [0,1]} \alpha A_\alpha$, where αA_α denotes the algebraic product of a scalar α with the α-cut A_α

Ans: $A = 0.2/v + 0.4/w + 0.6/x + 0.8/y + 1/z$

$A_{0.2} = \{v, w, x, y, z\} = 1/v + 1/w + 1/x + 1/y + 1/z$

$A_{0.4} = \{w, x, y, z\} = 1/w + 1/x + 1/y + 1/z$

$A_{0.6} = \{x, y, z\} = 1/x + 1/y + 1/z$

$A_{0.8} = \{y, z\} = 1/y + 1/z$

$A_{1.0} = \{z\} = 1/z$

$0.2A_{0.2} = 0.2/v + 0.2\,/w + 0.2\,/x + 0.2\,/y + 0.2\,/z$

$0.4A_{0.4} = 0.4/w + 0.4\,/x + 0.4\,/y + 0.4\,/z$

$0.6A_{0.6} = 0.6/x + 0.6\,/y + 0.6\,/z$

$0.8A_{0.8} = 0.8/y + 0.8/z$

$1A_{1.0} = 1/z$

$$A = \bigcup_{\alpha \in [0,1]} \alpha A_\alpha = 0.2/v + 0.4/w + 0.6\,/x + 0.8\,/y + 1/z$$

2. Show that $|A| + |B| = |A \cup B| + |A \cap B|$

$$
\begin{aligned}
|A \cup B| + |A \cap B| &= \sum_{x \in X} \left\{ \min\left[\mu_A(x), \mu_B(x)\right] + \max\left[\mu_A(x), \mu_B(x)\right] \right\} \\
&= \sum_{x \in X} \left[\mu_A(x) + \mu_B(x)\right] \\
&= \sum_{x \in X} \mu_A(x) + \sum_{x \in X} \mu_B(x) \\
&= |A| + |B|
\end{aligned}
$$

3(a) Show that the function

$$c(a) = \frac{\alpha^2(1-a)}{a+\alpha^2(1-a)} \quad , \ \forall\, a \in [0, 1], \ \alpha > 0$$

is a fuzzy complement.

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \left(v\tfrac{du}{dx} - u\tfrac{dv}{dx}\right) \Big/ v^2$$

Ans: (i) $c(0) = 1$ and $c(1) = 0$ → *Proves the 1st condition*

(ii) The derivative of c with respect to a is

$$c'(a) = -\frac{\alpha^2}{\left[a+\alpha^2(1-a)\right]^2}$$

Differentiable → *Proves the 3rd condition*

$\forall a \in [0, 1]$. Clearly, $c'(a) < 0, \ \forall\, a \in [0, 1]$.

That is, function c is *strictly decreasing*. → *Proves the 2nd condition*

(iii)

$$c\big(c(a)\big)=\frac{\alpha^2\left[1-\dfrac{\alpha^2(1-a)}{a+\alpha^2(1-a)}\right]}{\left[\dfrac{\alpha^2(1-a)}{a+\alpha^2(1-a)}\right]+\alpha^2\left[1-\dfrac{\alpha^2(1-a)}{a+\alpha^2(1-a)}\right]}$$

$$=\frac{\alpha^2\left[a+\alpha^2(1-a)-\alpha^2(1-a)\right]}{\alpha^2(1-a)+\alpha^2\left[(a+\alpha^2(1-a)-\alpha^2(1-a)\right]}$$

$$=a$$

Proves the 4th condition

Therefore c is involutive, which proves that c is a fuzzy complement

3(b) Find the equilibrium of the fuzzy complement c.

***Ans*:**

The equilibrium of the complement c is $c(a) = a$, which occurs when:

$$c(a) = \frac{\alpha^2(1-a)}{a+\alpha^2(1-a)} = a$$

$$\Rightarrow \alpha^2(1-a) = a^2 + a\alpha^2(1-a)$$

$$\Rightarrow 1 = \frac{a^2}{\alpha^2(1-a)} + a$$

$$\Rightarrow 1-a = \frac{a^2}{\alpha^2(1-a)}$$

$$\Rightarrow (1-a)^2 = \frac{a^2}{\alpha^2}$$

$$\Rightarrow (1-a) = \frac{a}{\alpha}$$

$$\Rightarrow \alpha = a + \alpha a$$

$$\Rightarrow \alpha = a(1+\alpha)$$

$$\Rightarrow a = \frac{\alpha}{(1+\alpha)}$$