EE4305 Fuzzy/Neural Systems for Intelligent Robotics

PART II: FUZZY SYSTEMS

Chapter 2: Fuzzy Sets and Crisp Sets

Topics to be Covered…

- Fuzzy sets and crisp sets
- Fuzzy operations, fuzzy relations, fuzzy compositions
- Extension principle, fuzzy numbers
- Approximate reasoning, fuzzy inference
- Multi-Rule Fuzzy Inference
- Fuzzy knowledge based control (FKBC)
- Fuzzy applications -

Crisp Sets and Fuzzy Sets

Crisp Sets: A brief overview

What is crisp set?

Crisp set is the "set" we have learned in the past, which is one of the most fundamental concepts in mathematics.

A set is a gathering together into a whole of definite, distinct objects of our perception or of our thought—which are called elements of the set.

For example, the numbers 0 and 1 are distinct objects when considered separately, but when they are considered collectively they form a single set of size two, written {1, 0}.

Common Universal Sets

- **R** = reals
- **N** = natural numbers = {0,1, 2, 3, . . . }, the *counting* numbers.
- **Z** = all integers = {. . , -3, -2, -1, 0, 1, 2, 3,4, . .}.
- $\mathbf{Z}^+$ ={1,2,3,…},the set of positive integers.
- **Q**={P/q | p ∈ **Z**,q ∈ **Z** and q≠0},the set of rational numbers.
- $\mathbf{Q}^+$={x∈ **R** | x=p/q, for some positive integers p and q}

Crisp Sets and Fuzzy Sets

Crisp Sets: A brief overview

To distinguish between fuzzy sets and classical (nonfuzzy) sets, we refer to the latter as *crisp* sets.

- Let X denotes the *universe of discourse*, or the *universal set. This set contains all the possible elements of concern in each particular application.*
- The set A whose elements are $a_1, a_2, \ldots, a_n$ is written as $A = \{a_1, a_2, \ldots, a_n\}$
- The number of elements that belong to a set A is called the cardinality of the set and is denoted as $|A|$
- The family of sets consisting of all the subsets of a particular set A is referred to as the power set of A and is indicated by $P(A)$
  - What is the power set of $A = \{a, b, c\}$ and the cardinality of this power set?

$$P(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$$

$$|P(A)| = 2^3 = 8$$

→ The cardinality of the power set is $|P(A)| = 2^{|A|}$

Crisp Sets and Fuzzy Sets

What are the methods to describe a set?

The simplest one is to list out all the members (the list method).

- $A = \{a_1, a_2, \ldots, a_n\}$

This method can be used only for finite sets. If a set is not finite, how do we define a set?

A set is defined by a property satisfied by its members (the rule method).

- $A = \{x|P(x)\}$

It is required that the property P be such that for any *x, P(x)* is either *true* or *false*.

For example, $A = \{x|x>0\}$.

Is there any other way to define a set?

Yes. Characteristic function!

♦ Characteristic function declares which elements of X are members of the set and which are not.

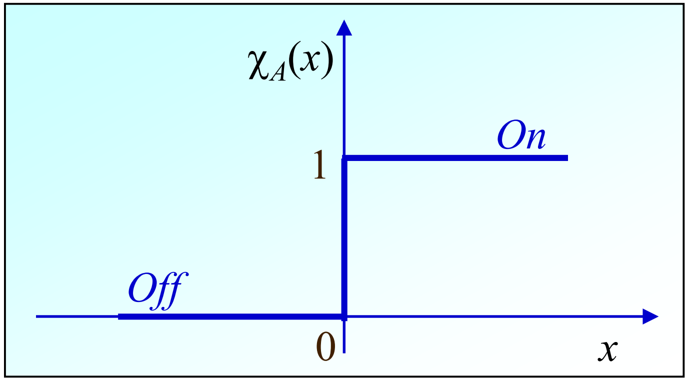
Characteristic function χ_A is defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if and only if } x \in A \\ 0 & \text{if and only if } x \notin A \end{cases}$$

The characteristic function maps elements of X to the elements of the set $\{0,1\}$.

For each $x \in X$, when $\chi_A(x) = 1$, x is declared to be a member of A; when $\chi_A(x) = 0$, x is declared as a non-member of A.

$A = \{x|x>0\}$



Basics of Fuzzy Sets

The characteristic function of a crisp set assigns a value of either 1 or 0 to each individual in the universal set, thereby discriminating between members and non-members. So the membership can only be 1 or 0.

The real-world is complex:

- Uncertainty, fuzziness inherent in our natural language
- A "Tall person?", e.g., > 1.8 m or > 1.7 m
- "Fast", e.g., speed > 50 km/h or speed > 80 km/h

♦ Fuzzy set theory uses linguistic variables, rather than quantitative variables, to represent imprecise concepts:

E.g., "nearly 6-feet tall", "very hot", "rather small"

To deal with the human concepts which do not have sharp boundaries, the characteristic function can be generalized such that the values assigned to the elements fall within a specified range and indicate the membership grade of these elements in the set in question.

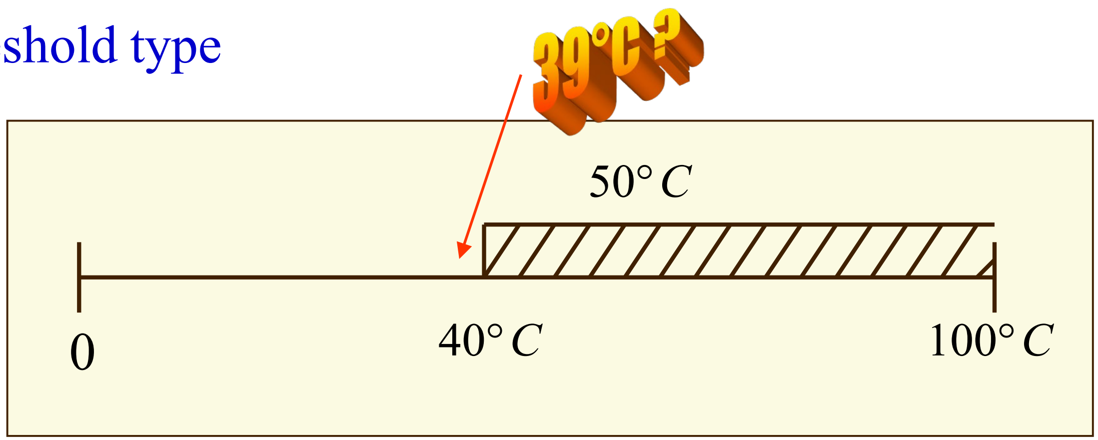
Larger values denote higher degrees of set membership.

- How to represent the concept such as “hot ” so that machines can communicate with humans?

- How strongly do you agree that a given number (Temperature) ← [0, 100] is a “hot” temperature that belongs to “Hot”?

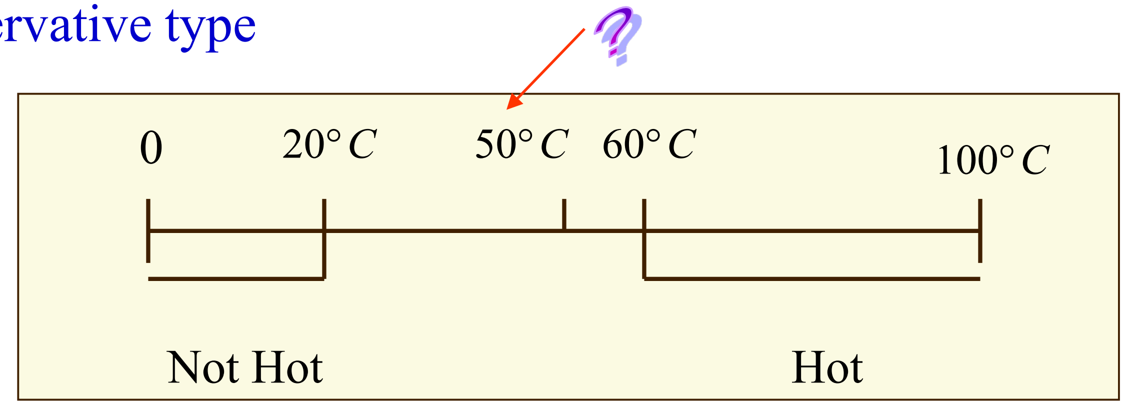
3 Types:

(1) Threshold type

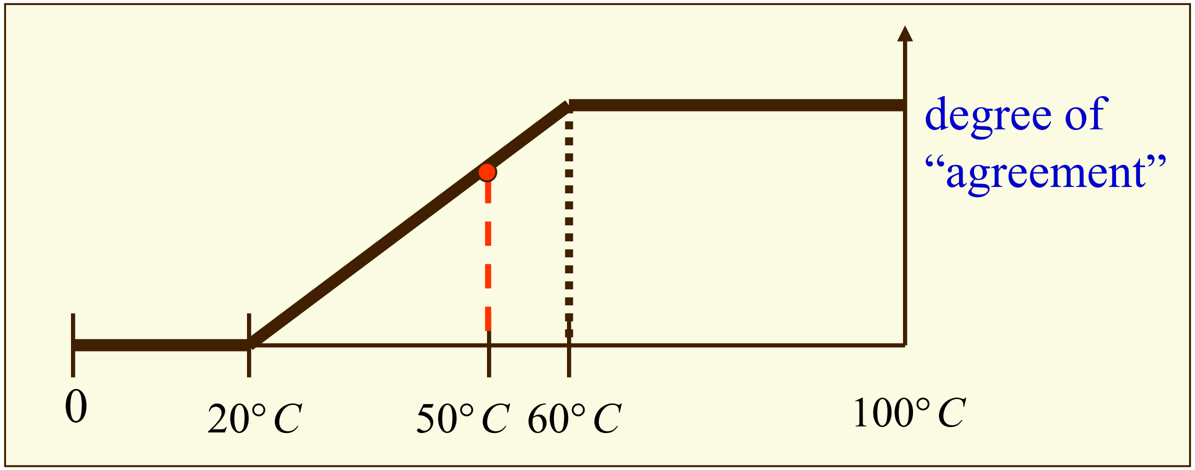


$$\text{Hot} = \{ x \mid x \geq 40°C \}$$

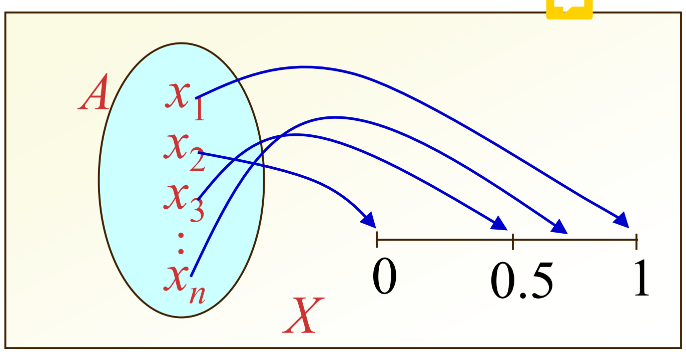
(2) Conservative type

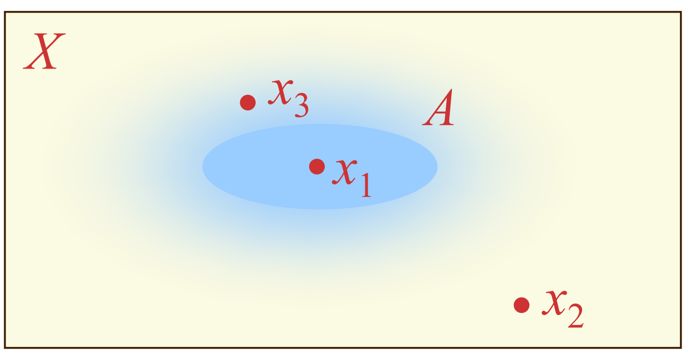


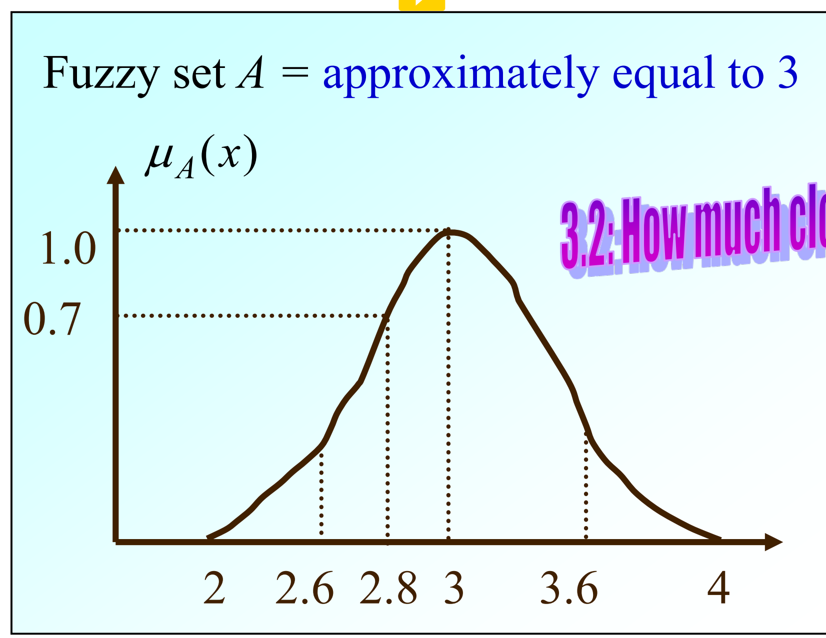
(3) Compromiser type



- A *fuzzy set* is a set containing elements that have varying degrees of memberships in the set.
- Larger values of the membership denotes higher degrees of the membership, and such a function is called a *membership function*.
- The membership function μ_A by which a fuzzy set A is usually defined has the form $\mu_A : X \to [0, 1]$. (Compared to $\chi_A : X \to \{0, 1\}$)







3.2: How much close to 3?

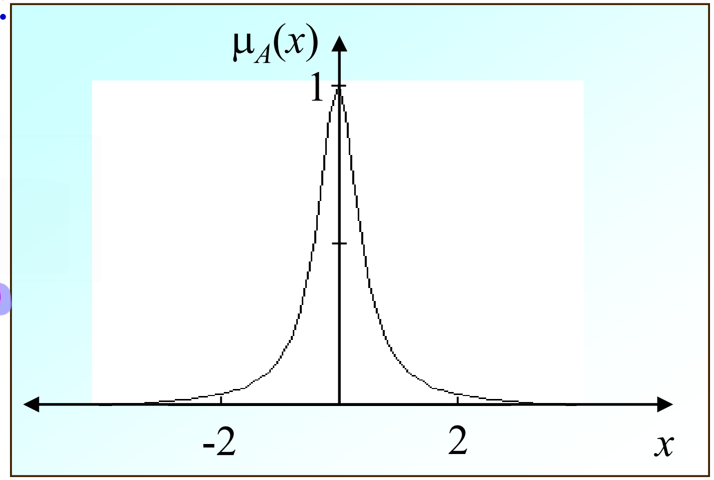
2.8: How much close to 3?

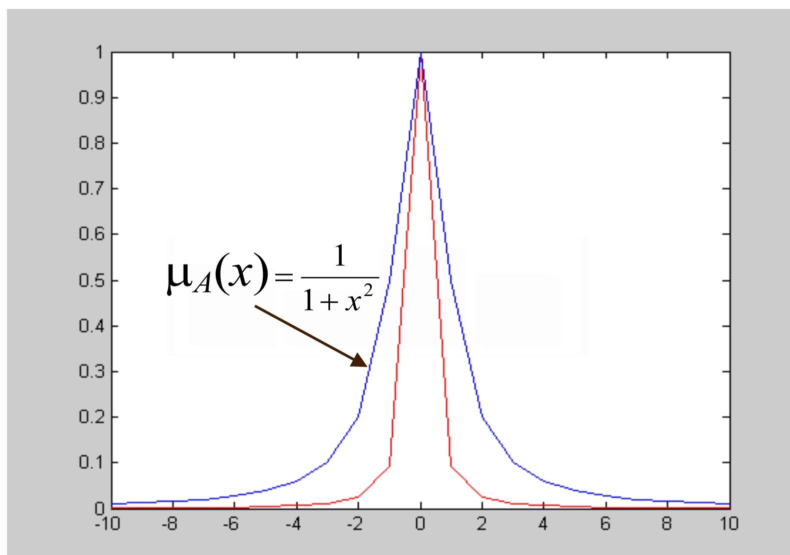
$$\mu_A(2.8) = 0.7$$

- A possible membership function for the fuzzy set of real numbers close to 0 can be defined as:

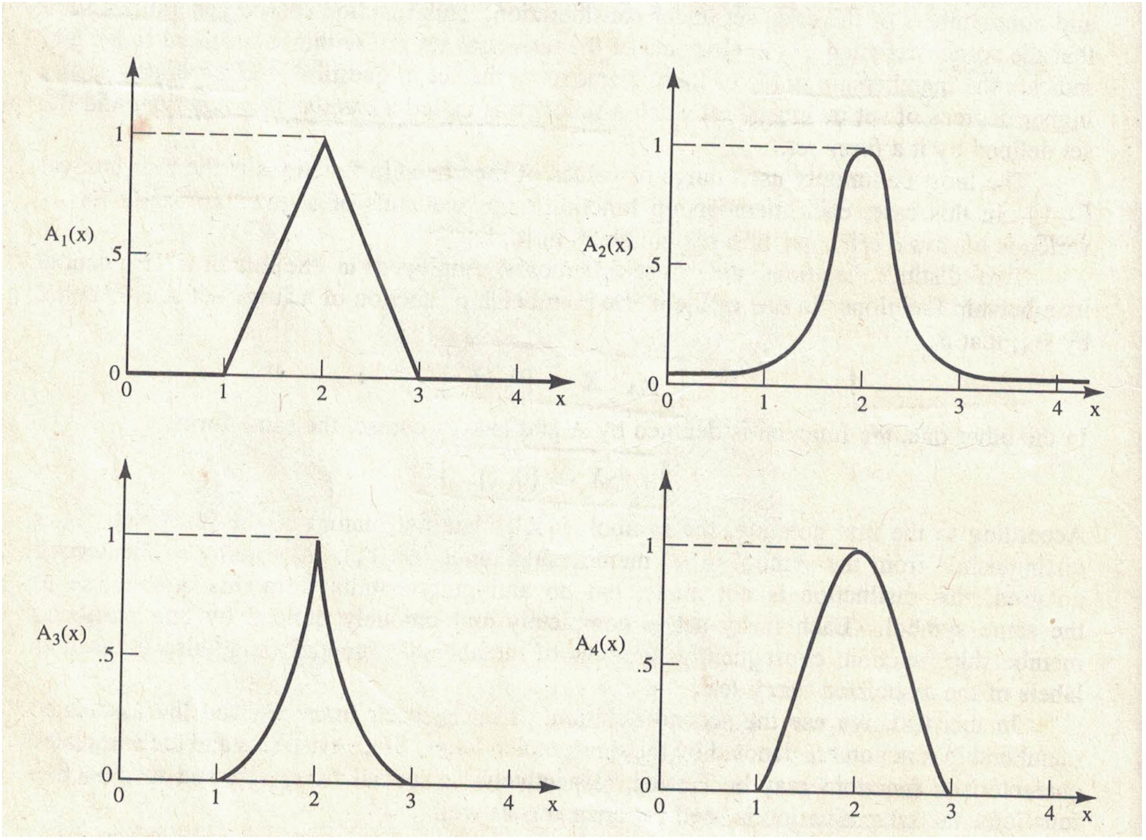
$$\mu_A(x) = \frac{1}{1 + 10x^2}$$

Why 10? not 1?





♦ We may design different membership functions for the same concept. For instance, to express the concept of “ real numbers that are close to 2”, we may have :

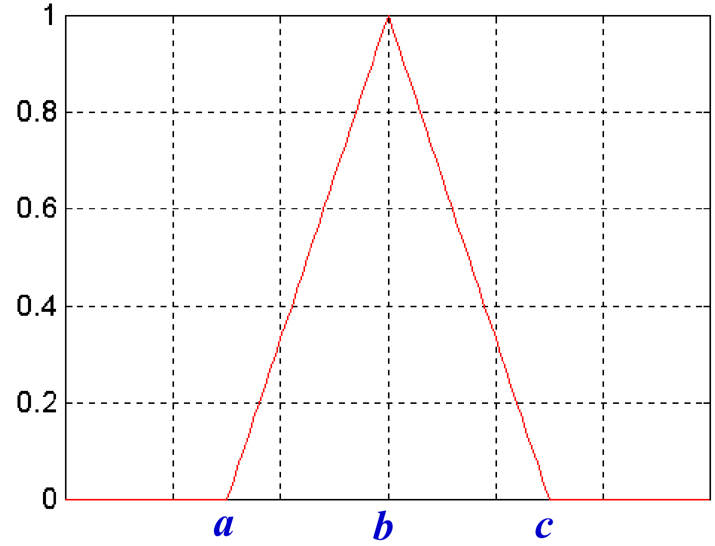


Which one is better?

It turns out that many applications are not sensitive to variations of the shape of the membership function. It is convenient to use a simple shape such as the triangular shape.

♦ Triangular membership function: Depends on three parameters

$$\Delta(x;a,b,c)=\begin{cases}0 & x\le a\\ \dfrac{x-a}{b-a} & a\le x\le b\\ \dfrac{c-x}{c-b} & b\le x\le c\\ 0 & c\le x\end{cases}$$

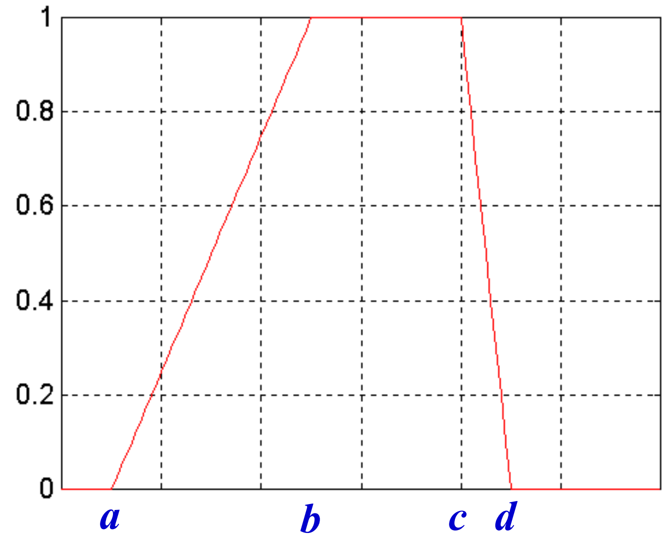


popular due to its simplicity

The parameters *a* and *c* locate the *base* of the triangle and the parameter *b* locates the *peak*.

- Trapezoidal membership function: Depends on four parameters

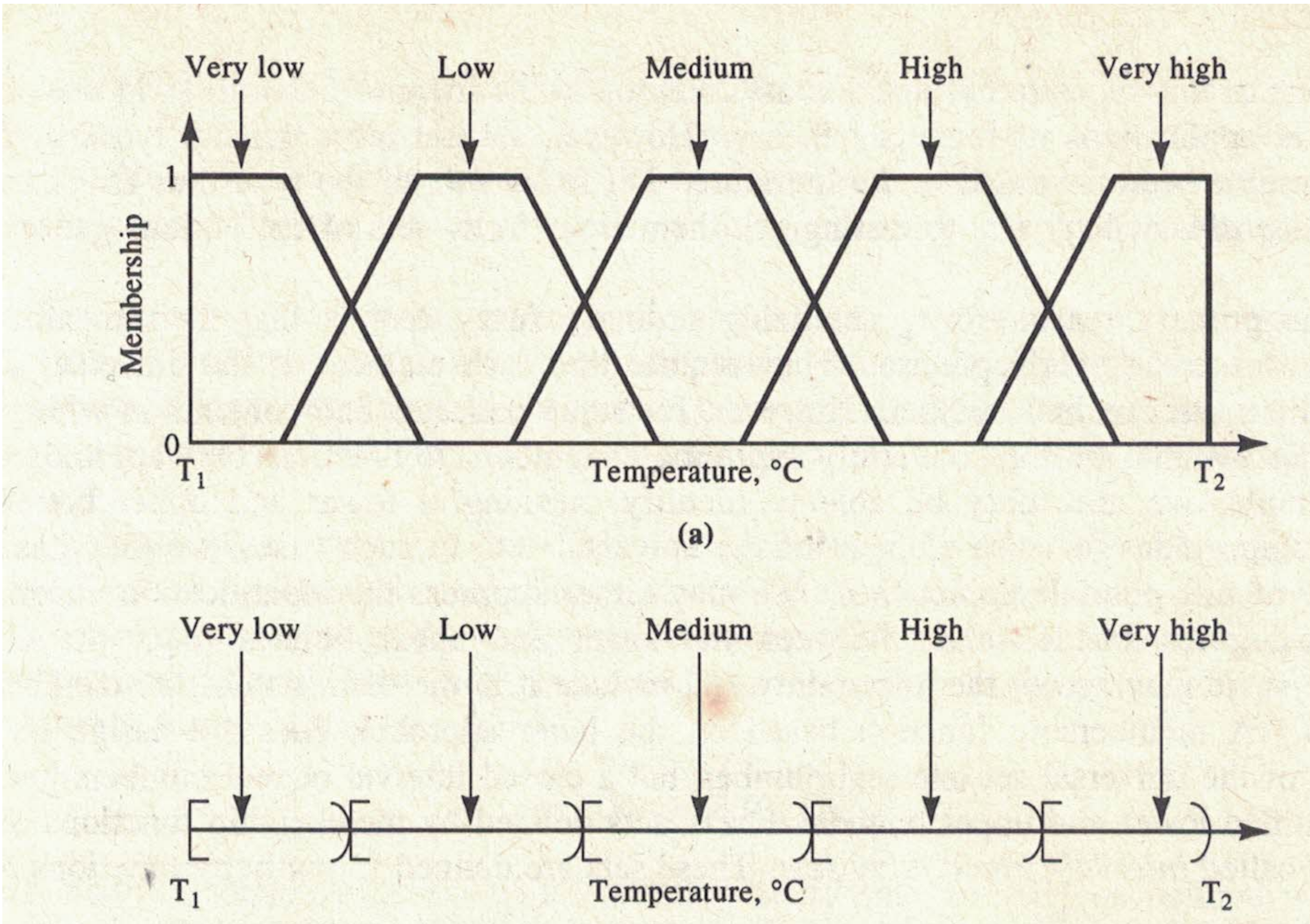
$$\Pi(x;a,b,c,d)=\begin{cases} 0 & x\le a \\ \dfrac{x-a}{b-a} & a\le x\le b \\ 1 & b\le x\le c \\ \dfrac{d-x}{d-c} & c\le x\le d \\ 0 & d\le x \end{cases}$$



- The parameters *a* and *d* locate the *base* of the trapezoid and the parameters *b* and *c* locate the *shoulders*.

- Example: Suggest a membership function for a fuzzy set of *low temperature* with a universe of discourse $X \in [0, 100]$.

Fuzzy sets vs. crisp sets for low, medium and high temperatures.



Representation of Fuzzy Sets

- Fuzzy set A for a discrete and finite universe of discourse X is

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \ldots + \mu_A(x_n)/x_n$$

$$= \sum_{i=1}^{n} \mu_A(x_i)/x_i$$

$1/2 \quad \text{VS} \quad \frac{1}{2} = 0.5$

E.g. Integer close to 10

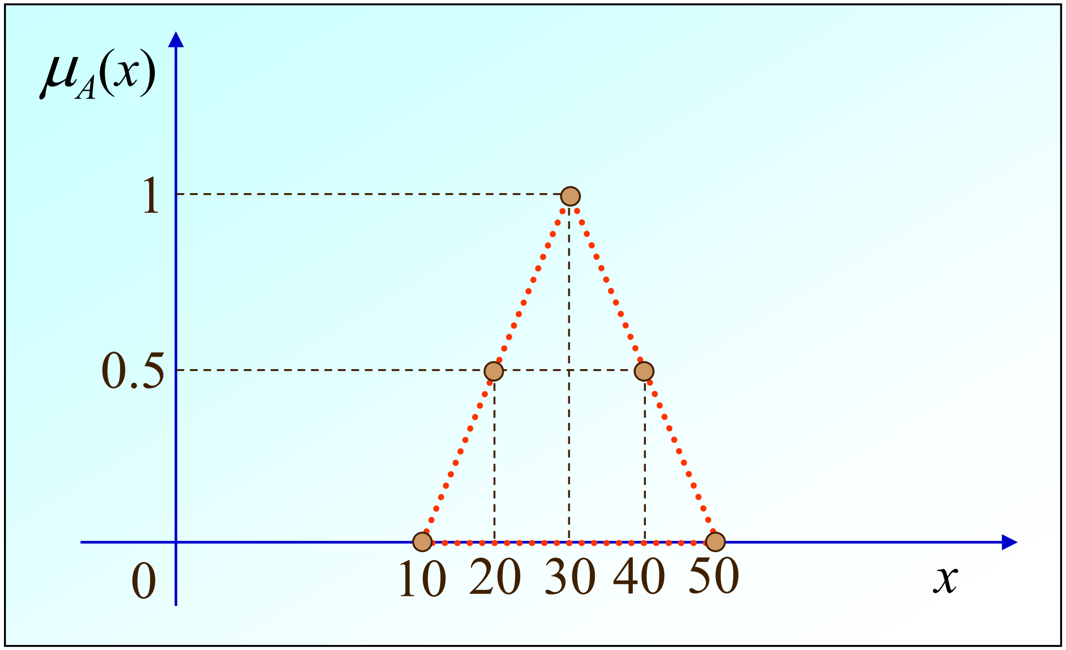
$$= 0.1/7 + 0.5/8 + 0.8/9 + 1/10 + 0.8/11 + 0.5/12 + 0.1/13$$

- For continuous and infinite set, the fuzzy set is: $A = \int_x \mu_A(x)/x$

E.g. Fuzzy set *Old* $\quad \mu_A(x) = \begin{cases} 0 \quad , & 0 < x < 40 \\ \left(1 + \left(\frac{x-40}{5}\right)^{-2}\right)^{-1}, & 40 < x < 120 \end{cases}$

continuous form

Example: What is the fuzzy set A?



$$A = 0.5/20 + 1/30 + 0.5/40$$

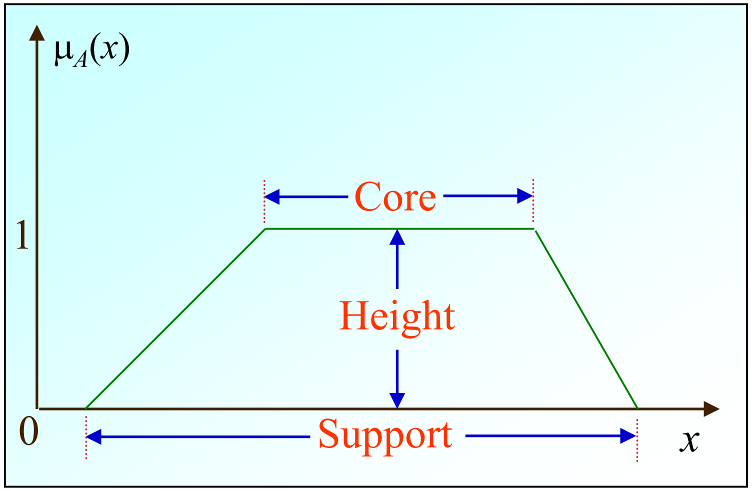
- (Def) *Height* of a fuzzy set A is $hgt(A)$ = the maximum value of the membership function, i.e., supreme$\{\mu_A(x)\}$ = sup$\{\mu_A(x)\}$ = max$\{\mu_A(x)\}$

  : 'Normal' if $hgt(A) = 1$

  : 'Subnormal' if $hgt(A) < 1$

- (Def) *Support* of a fuzzy set A, $Supp(A)$ is the crisp set of all $x \in X$ such that $\mu_A(x) > 0$, $Supp(A) = \{x \in X \mid \mu_A(x) > 0\}$

- (Def) *Core* of a fuzzy set A, $Core(A)$ is the crisp set of all $x \in X$ such that $\mu_A(x) = 1$, $Core(A) = \{x \in X \mid \mu_A(x) = 1\}$



For crisp set, since an individual x, is either a member or not a member of a set A, we can use $x \in A$ to denote that x is a member of A.

Can we use the same notation for fuzzy set?

No, for fuzzy set, we can only use the membership function, $\mu_A(x)$ (or A(x)), to indicate its membership grade.

How do we find out whether two fuzzy sets A and B are the same?

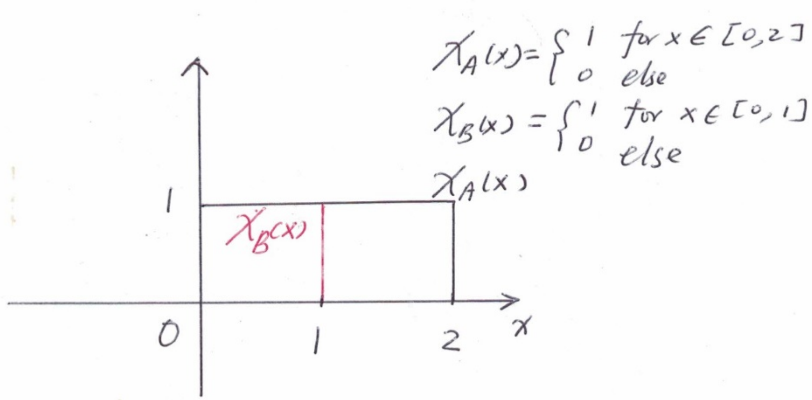
$$A{=}B \text{ if } \mu_A(x) = \mu_B(x)$$

We need to re-define all the concepts and operations associated with the crisp sets using the membership function.

How do we find out whether a fuzzy set *A* is a part of fuzzy set *B*? In other words, how do we define the subset?

To answer this question, let's see how we do that for crisp sets using the characteristic functions first.

Let A=[0,2], B=[0, 1]. Clearly B is a subset of A. Their characteristic functions are



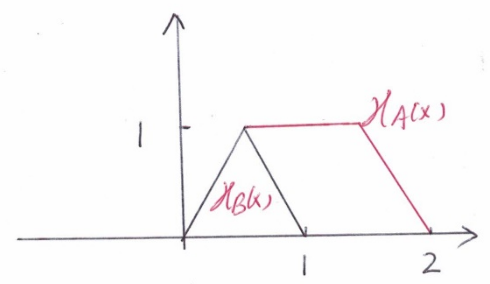
How do we relate these two characteristic functions?

$$\chi_B(x) \leq \chi_A(x)$$

How do we find out whether a fuzzy set *A* is a part of fuzzy set *B*?

$$B \subseteq A \quad \text{iff} \quad \mu_B(x) \le \mu_A(x)$$

For example,



$B \subseteq A$

How about this?

B is not a subset of A!

From this exercise, we have a systematic way (SOP) to extend the concepts and operations from crisp sets to fuzzy sets.

First, try to find out the definition of the operations for crisp sets in terms of the characteristic functions.

Since the characteristic functions are special cases of the membership functions, we just extend the definition by replacing the characteristic functions with the membership functions.

We are going to follow this procedure to define other concepts and operations of fuzzy sets.

What is the cardinality of the crisp set? $|A| = \sum_{x \in X} \chi_A(x)$

- (Def) Scalar Cardinality

$$SC(A) = |A| = \sum_{x \in X} \mu_A(x)$$

- (Def) Relative Cardinality

$$RC(A) = \frac{SC(A)}{|\text{Universal Set}|} = \frac{|A|}{|X|}$$

Example

(ages)	*old*
5	0
10	0
20	0.1
30	0.2
40	0.4
50	0.6
60	0.8
70	1
80	1

- Scalar Cardinality $SC(A)$

$$|old| = 0 + 0 + 0.1 + 0.2 + 0.4 + 0.6 + 0.8 + 1 + 1 = 4.1$$

- Relative Cardinality $RC(A)$

$$|old| = \frac{4.1}{9} = 0.4556$$