EE4305 Fuzzy/Neural Systems for Intelligent Robotics

PART II: FUZZY SYSTEM

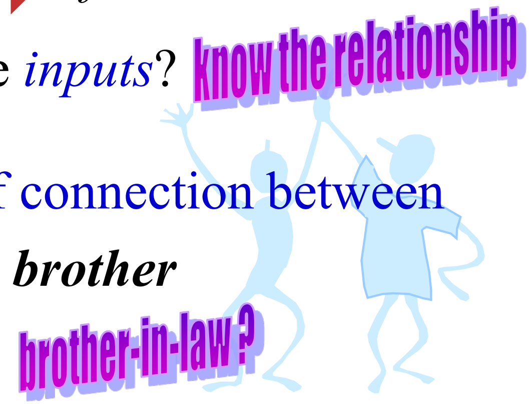
Chapter 4: Fuzzy Relations

Topics to be Covered…

- Fuzzy sets and crisp sets
- Fuzzy operations, fuzzy relations, fuzzy compositions
- Extension principle, fuzzy numbers
- Approximate reasoning, fuzzy inference
- Multi-rule Fuzzy Inference
- Fuzzy knowledge based control (FKBC)
- Fuzzy applications - Inverted pendulum control
  - Fuzzy nonlinear simulations

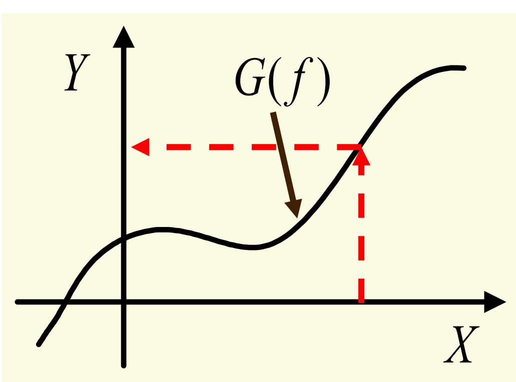
Fuzzy Relations, Fuzzy Compositions

- *If* *speed* is fast *then* *pressure* should be high ⟹ *if A then B*
  - How do we determine the *output* if we know the *inputs*?

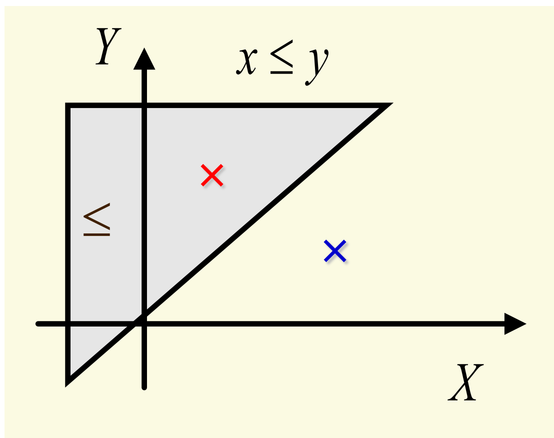


- Crisp relation represents the presence or absence of connection between elements of two or more sets, e.g., (John, Robert) - ***brother***

E.g. given a $f: X \rightarrow Y$,
the graph $G(f)$ is a relation.

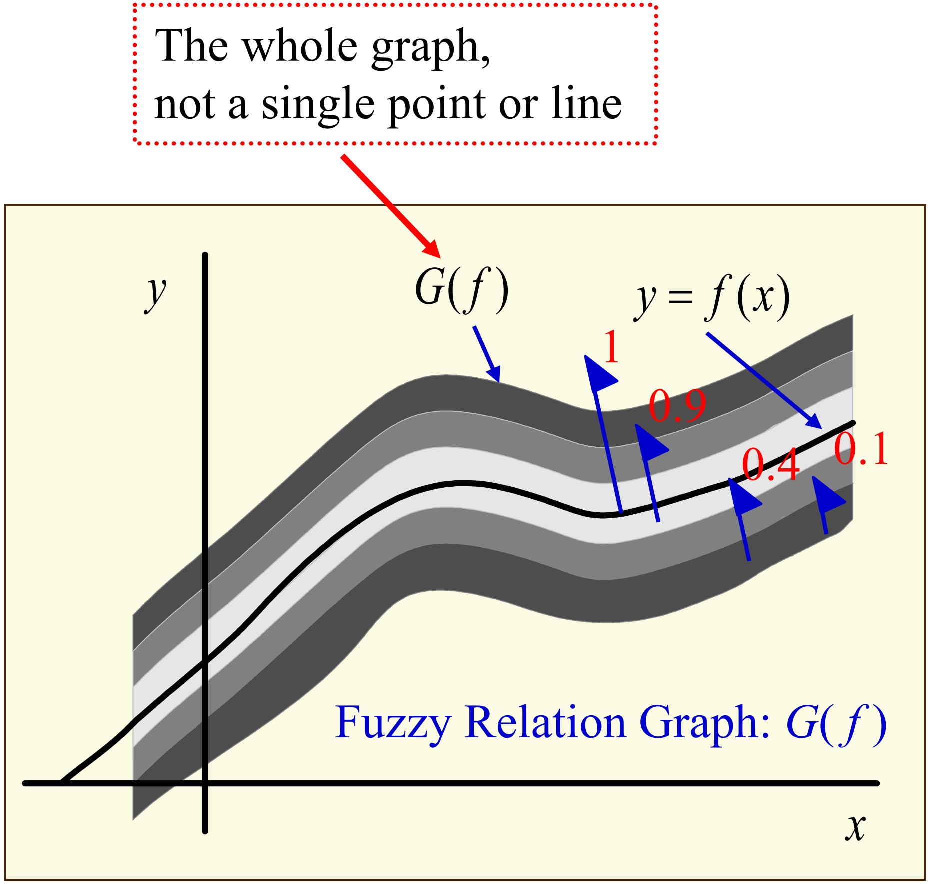


E.g. $\leq$ is also a relation



- Fuzzy relation allows relationship between elements of two or more sets to take an infinite number of relationship degrees.

A fuzzy relation *R*: *y* is approximately equal to *f(x)*



- Both crisp relation and fuzzy relation are defined on Cartesian Product.
- (Def) Cartesian Product:
  - An ordered sequence of n elements, written in the form $(x_1, x_2, x_3, \ldots, x_n)$, is called an ordered n-tuples;
  - For crisp sets, $X_1, X_2, \ldots, X_n$, the set of all n-tuples is called the *Cartesian product* of $X_1, X_2, \ldots, X_n$;
  - E.g. The elements in two sets A and B are given as $A = \{0, 1\}$ and $B = \{a, b, c\}$, various Cartesian products of these two sets are:

$$A \times B = \{(0, a), (0, b), (0, c), (1, a), (1, b), (1, c)\}$$

Crisp relation is a subset, e.g., $R(A, B) = \{(0, c), (1, a)\}$

Fuzzy relation, e.g., $R(A, B) = 0.5/(0, c) + 0.3/(1, a)$

Cartesian product = Universal set for crisp/fuzzy relation

$$B \times A = \{(a, 0), (a, 1), (b, 0), (b, 1), (c, 0), (c, 1)\}$$

$$A \times A = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$$

Crisp Relations

- (Def) A *relation* among crisp sets $X_1 \times X_2 \times \ldots \times X_n$ is a subset of the Cartesian product, which is denoted by $R(X_1 \times X_2 \times \ldots \times X_n)$
  - E.g., Relation between two sets $\rightarrow$ *binary*
  - In general: Relation defined on *n* sets $\rightarrow$ *n-ary*

- The Cartesian product $X_1 \times X_2 \times \ldots \times X_n$ represents the universal set

$$\chi_R(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if and only if } (x_1, x_2, \ldots, x_n) \in R, \\ 0 & \text{otherwise} \end{cases}$$

So for crisp sets, two or more objects are either related (1) or not related (0).

In the real world, the relationship can be strong or weak!

Hence, the crisp relation can be generalized to allow tuples to have different degrees of relationship!

Fuzzy Relations

- (Def) A *fuzzy relation* is a fuzzy set defined on the Cartesian product of crisp sets $X_1, X_2, \ldots, X_n$, where tuples $(x_1, x_2, x_3, \ldots, x_n)$ may have varying degrees of membership $\in [0, 1]$, indicating the strength of the relationship present between the elements of the tuple.

  e.g., x is similar to y;
  x is almost equal to y;
  x is much greater than y.

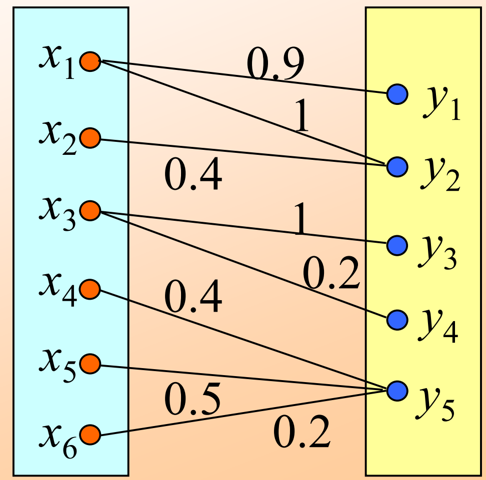
- E.g., Let R be a fuzzy relation between the two sets X = {Beijing, New York City, London} and Y = {New York City, Paris} that represents the relational concept "very far". This relation can be written as,

$R(X,Y)$ = 1/Beijing, NYC + 0/NYC, NYC + 0.6/London, NYC + 0.9/ Beijing, Paris + 0.7/ NYC, Paris + 0.3/London, Paris

$$\Leftrightarrow \quad R(X,Y) = \begin{array}{c c} & \begin{array}{cc} \text{NYC} & \text{Paris} \end{array} \\ \begin{array}{c} \text{Beijing} \\ \text{NYC} \\ \text{London} \end{array} & \begin{bmatrix} 1 & 0.9 \\ 0 & 0.7 \\ 0.6 & 0.3 \end{bmatrix} \end{array}$$

- Other representation of binary relations, e.g., *sagittal diagrams*

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array}
\begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\ \begin{bmatrix} 0.9 & 1 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0.2 \end{bmatrix} \end{array}
\quad \overset{R(X,\,Y)}{\Longleftrightarrow}$$



The application of fuzzy relations

If we know that X and Y are related by R(x,y), we can use it to solve following reference problem:

$$A(X) \longrightarrow \boxed{R(X,Y)} \longrightarrow B(Y)\ ???$$

If A then B, with a fuzzy relation $R(X,Y)$

E.g., R: John is *some what taller* than Albert

$A(X)$: John is *rather tall*

$\Rightarrow$ Albert is *medium*

How to derive a formula for this type of fuzzy inference?

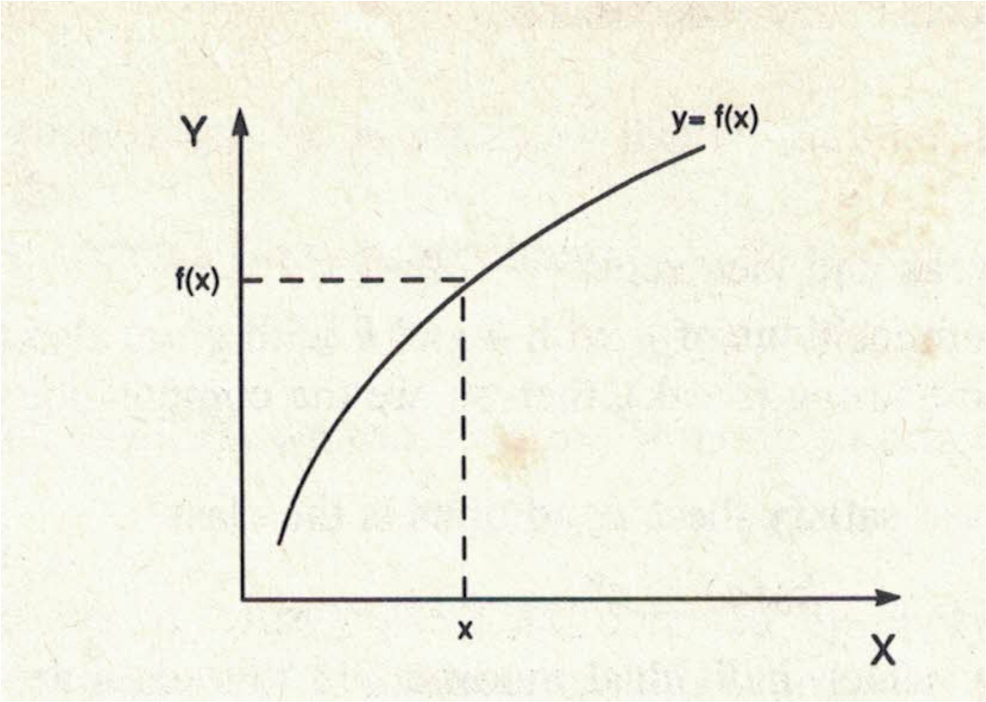
We will follow the same SOP for the operations of set:

Step one: find out the computations of crisp sets in terms of characteristic functions.

Step two: replace the characteristic functions by membership functions.

Consider two variables x and y are related by a function y=f(x).

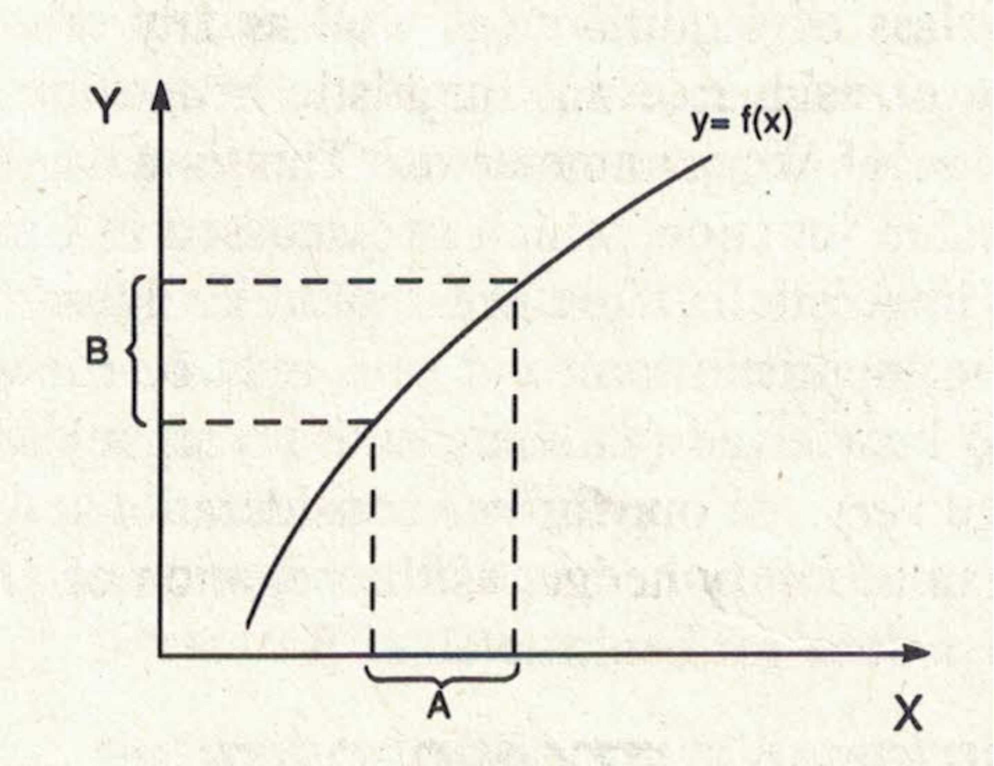
Then given x, we can infer y=f(x). This is trivial.



Now let's take one step further. Knowing that the value of X is in a given crisp set A, what can we infer regarding the values in Y?

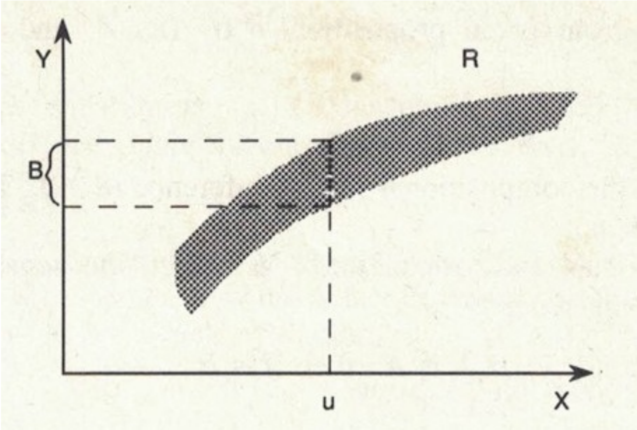
Using the function y=f(x), we can infer that the value of Y is in the set

$$B = \{y \in Y | y = f(x), x \in A\}$$



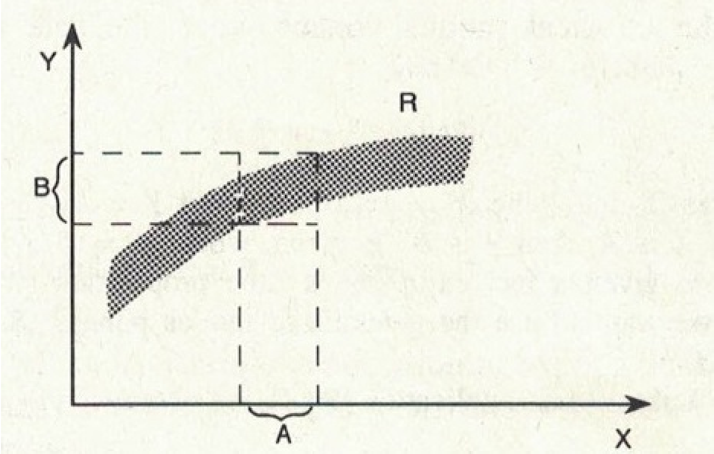
Assuming now that the variables are related by a relation on XxY, not necessary a function.

Then, given x=u, and a relation R, we can infer that $y \in B$, where

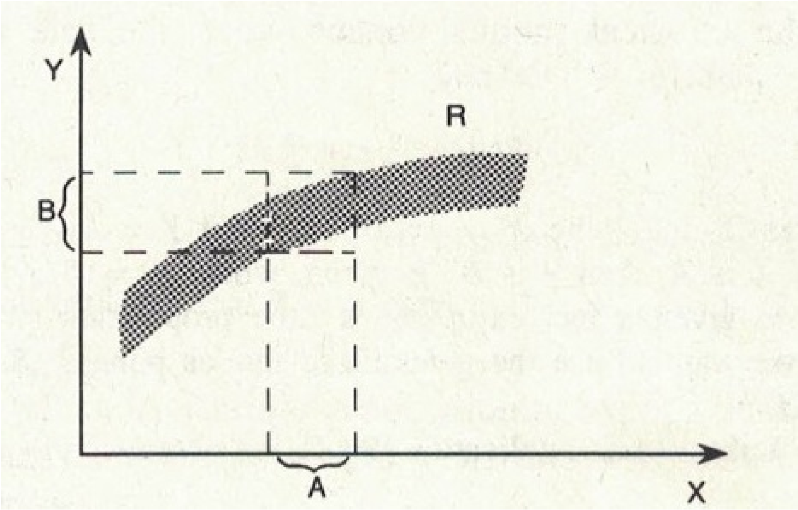


$$B = \{y \in Y | (\mathrm{u}, y) \in R\}$$

Then, given $x \in A$, we can infer that $y \in B$, where



$$B = \{y \in Y | (x, y) \in R, x \in A\}$$



$$B = \{y \in Y | (x, y) \in R, x \in A\}$$

Now let's try to express the characteristic function of this set B in terms of characteristic functions of the set A and the relation R.

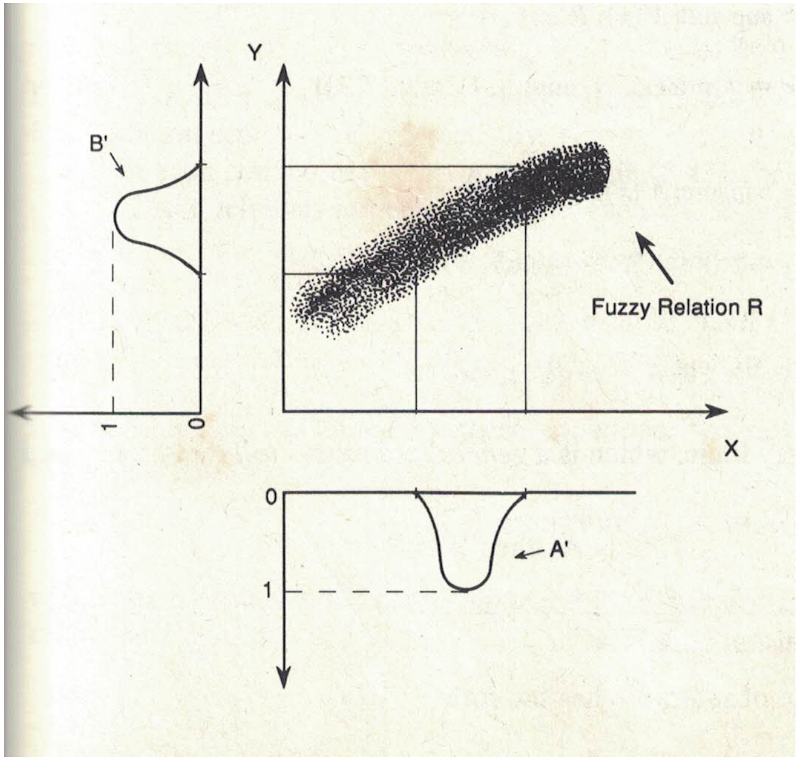
Can we simply use $\chi_B(y) = \max_{x \in X}[\chi_R(x, y)]$ **?**

No, that will include the points that are related to points outside A.

We should only include the points that are related to the points inside A!

$$\chi_B(y) = \max_{x \in X} \min[\chi_A(x), \chi_R(x, y)]$$

If *R* is a fuzzy relation on XxY , and *A(x)* and *B(y)* are fuzzy sets on X and Y. Then , if *R* and *A(x)* are given, we can obtain *B(y)* by replacing the characteristic functions with the membership functions.



$$\chi_B(y) = \max_{x \in X} \min[\chi_A(x), \chi_R(x, y)]$$

$$\mu_B(y) = \max_{x \in X} \min[\mu_A(x), \mu_R(x, y)]$$

$$\mu_B(y) = \max_{x \in X} \min[\mu_A(x), \mu_R(x,y)]$$

- Example:

$$1/10 + 0.6/20 + 0.2/30 \quad \circ \quad \begin{array}{c|ccc} & 10 & 20 & 30 \\ \hline 10 & 1 & 0.4 & 0 \\ 20 & 0.5 & 1 & 0.4 \\ 30 & 0 & 0.4 & 1 \end{array} \quad = \quad ??$$

John is young

John and Albert are approximately same age

Albert is also young

$$\mu_B(10) = \max_{x \in X} \min[\mu_A(x), \mu_R(x,y)]$$

=max [min[1,1], min[0.6, 0.5], min[0.2, 0]]

=max[1, 0.5, 0]=1

$$\mu_B(20) = \max_{x \in X} \min[\mu_A(x), \mu_R(x,y)]$$

=max [min[1,0.4], min[0.6, 1], min[0.2, 0.4]]

=max[0.4, 0.6, 0.2]=0.6

$$\mu_B(30) = \max_{x \in X} \min[\mu_A(x), \mu_R(x,y)]$$

=max [min[1,0], min[0.6, 0.4], min[0.2, 1]]

=max[0, 0.4, 0.2]=0.4

♦ Example:

$$1/10+0.6/20+0.2/30 \;\circ\; \begin{array}{c} \\ 10 \\ 20 \\ 30 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 10 & 20 & 30 \end{array} \\ \begin{bmatrix} 1 & 0.4 & 0 \\ 0.5 & 1 & 0.4 \\ 0 & 0.4 & 1 \end{bmatrix} \end{array} = 1/10+0.6/20+0.4/30$$

John is young	John and Albert are approximately same age	Albert is also young

To simplify the calculation, let's define a composition operator of two vectors:

Let

$$x = [x_1, x_2, \cdots, x_n]$$
$$y = [y_1, y_2, \cdots, y_n]^T$$

Then

$$z = x \circ y$$
$$= \max_k \min[x_k, y_k]$$
$$= \max[\min[x_1, y_1], \min[x_2, y_2], \cdots, \min[x_n, y_n]]$$

♦ Example:

$$1/10 + 0.6/20 + 0.2/30 \quad \circ \quad \begin{array}{c|ccc} & 10 & 20 & 30 \\ \hline 10 & 1 & 0.4 & 0 \\ 20 & 0.5 & 1 & 0.4 \\ 30 & 0 & 0.4 & 1 \end{array} \quad = \quad 1/10 + 0.6/20 + 0.4/30$$

John is young	John and Albert are approximately same age	Albert is also young

In order to use composition operator , let's use the matrix and vector.

$$B = A \circ R$$

$$= \begin{bmatrix} 1 & 0.6 & 0.2 \end{bmatrix} \circ \begin{bmatrix} 1 & 0.4 & 0 \\ 0.5 & 1 & 0.4 \\ 0 & 0.4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.6 & 0.4 \end{bmatrix}$$

Fuzzy Composition

$A(X) \longrightarrow \boxed{R(X,Y)} \longrightarrow B(Y) = A \circ R$

Composition

If A then B, with a fuzzy relation $R(X,Y)$

E.g., R: John is *some what taller* than Albert
$A(X)$: John is *rather tall*

➡ Albert is *medium*

- Rules of Composition Operation (e.g., a bit like matrix multiplication):

Fuzz set	o	Fuzzy relation	=	Fuzz set
$1 \times n$		$n \times m$		$1 \times m$
$A(X)$		$R(X,Y)$		$B(Y) = A \circ R$

or

Fuzzy relation	o	Fuzzy relation	=	Fuzzy relation
$n \times m$		$m \times k$		$n \times k$
$P(X,Y)$		$R(Y,Z)$		$Q(X,Z) = P \circ R$

$$R(X, Y) \circ S(Y, Z) = \max_{y \in Y} \min[\mu_R(x, y), \mu_S(y, z)]$$

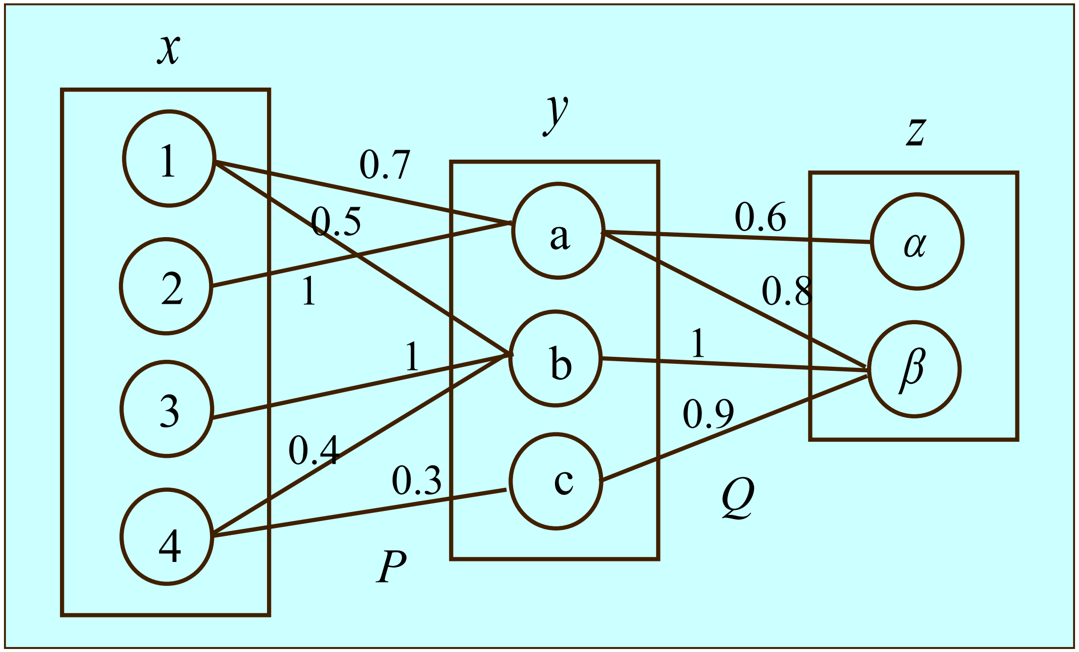
Example:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad AB = \begin{bmatrix} 4 & 6 \\ 4 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.1 \end{bmatrix} \quad B = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.4 \end{bmatrix} \quad \overset{\max\{\min\}}{A \circ B} = \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0.2 \end{bmatrix}$$

$$\max\{\min(.1, .1), \min(.2, .3)\} = \max\{.1, .2\} = .2$$

♦ Example:



Composition : $R = P \circ Q$		
x	z	$\mu_{R(x,z)}$
1	α	0.6
1	β	0.7
2	α	0.6
2	β	0.8
3	β	1
4	β	0.4

$$R(x,z) = R(x,y) \circ R(y,z)$$

$$= \begin{bmatrix} 0.7 & 0.5 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.4 & 0.3 \end{bmatrix} \circ \begin{bmatrix} 0.6 & 0.8 \\ 0 & 1 \\ 0 & 0.9 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.7 \\ 0.6 & 0.8 \\ 0 & 1 \\ 0 & 0.4 \end{bmatrix}$$