EE4305 Fuzzy/Neural Systems for Intelligent Robotics

PART II: FUZZY SYSTEMS

Chapter 3: Fuzzy Operations

Topics to be Covered…

- Fuzzy sets and crisp sets
- Fuzzy operations, fuzzy relations, fuzzy compositions
- Extension principle, fuzzy numbers
- Approximate reasoning, fuzzy inference
- Multi-rule Fuzzy Inference
- Fuzzy knowledge based control (FKBC)
- Fuzzy applications - Inverted pendulum control
  - Fuzzy nonlinear simulations

Fuzzy Complement, Union, Intersection

Properties of Fuzzy Sets

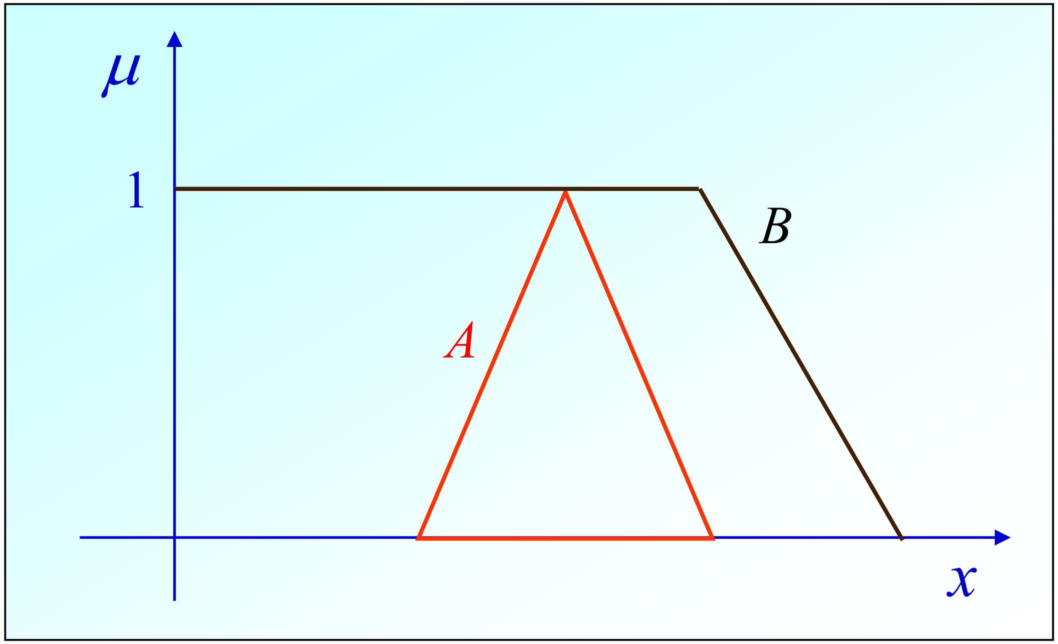
Extension from crisp set

Involution	$\overline{\overline{A}} = A$
Commutativity	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
Associativity	$A \cup (B \cup C) = (A \cup B) \cup C$
	$A \cap (B \cap C) = (A \cap B) \cap C$
Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Idempotency	$A \cup A = A$
	$A \cap A = A$
Identity	$A \cup \varnothing = A \qquad A \cap \varnothing = \varnothing$
	$A \cap X = A \qquad A \cup X = X$
Transitivity	If $A \subseteq B \subseteq C$, then $A \subseteq C$

♦ Fuzzy Inclusion and equality

$A \subseteq B$ (A is a *subset* of B) if $\mu_A(x) \le \mu_B(x)$, for every $x \in X$.
$A \subset B$ (A is a *proper subset* of B) if $\mu_A(x) \le \mu_B(x)$ for every $x \in X$ and $\mu_A(x) < \mu_B(x)$ for at least one $x \in X$.
$A = B$ (A is *equal* to B) if $\mu_A(x) = \mu_B(x)$, for every $x \in X$.
$A \ne B$ (A and B are *not equal*) if $\mu_A(x) \ne \mu_B(x)$, for at least one $x \in X$.

Example:



Is $A = B$?

Is $A \subseteq B$?

♦ Fuzzy Set Operations

To find out the standard operations of fuzzy sets, we just follow the SOP:

Step one: find out the operations of crisp sets in terms of characteristic functions.

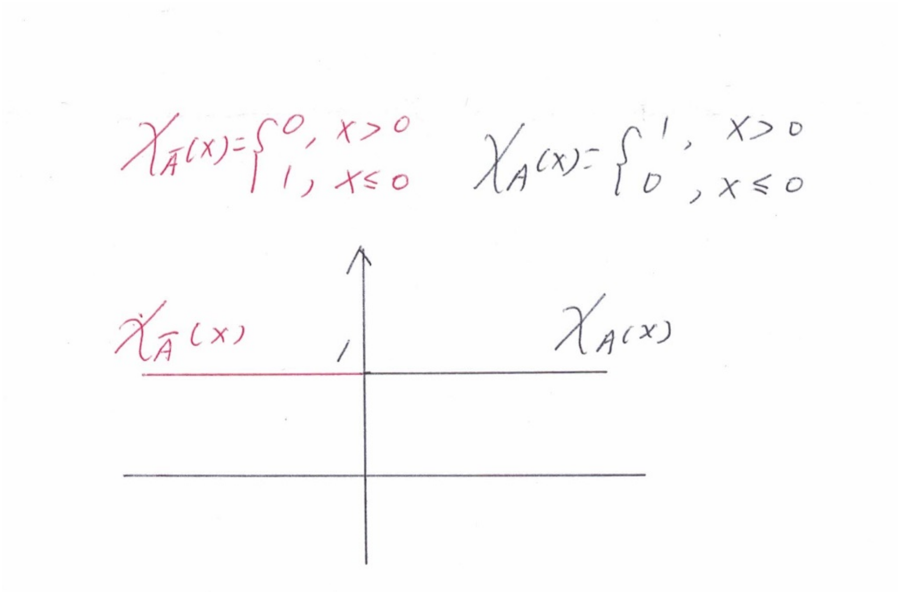
Step two: replace the characteristic functions by membership functions.

We are going to do that for complement, intersection and union.

♦ Fuzzy Complement

Step one: find out the complement of crisp sets in terms of characteristic functions.

Let X be the real line, and A={x|x>0}, what is the complement of A, $\bar{A}$?

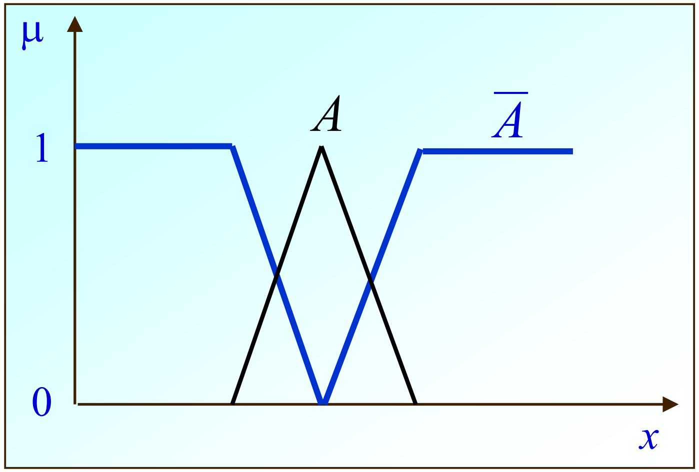


How do we relate the two characteristic functions?

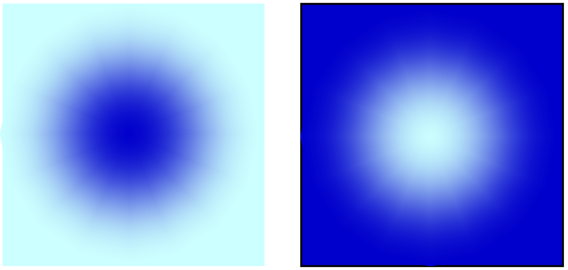
$$\chi_{\bar{A}}(x) = 1 - \chi_A(x)$$

Step two: replace the characteristic functions by membership functions.

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$$



Or



A $\overline{A}$

- old = 0/5 + 0/10 + 0.1/20 + 0.2/30 + 0.4/40 +0.6/50 +0.8/60 + 1/70 + 1/80

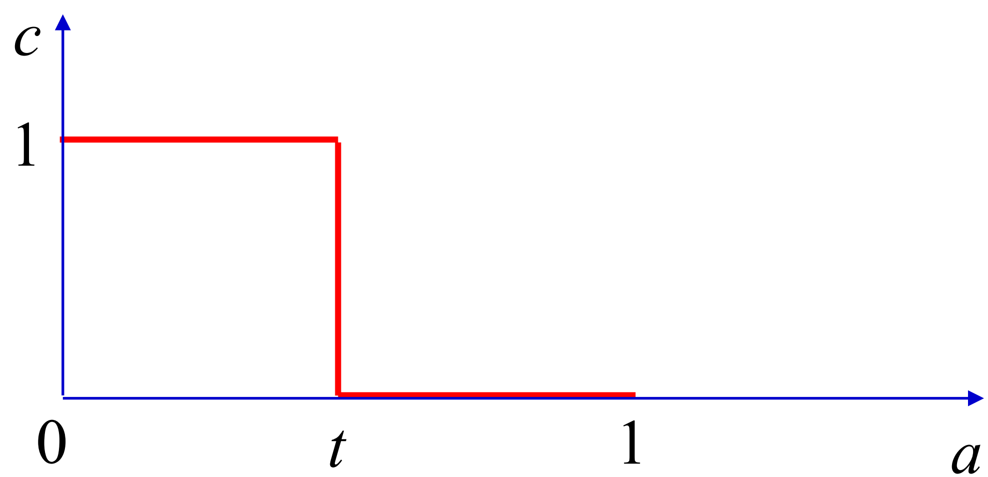
  $NOT\ old$ = 1/5 + 1/10 + 0.9/20 +0.8/30 + 0.6/40 +0.4/50 +0.2/60

- We have mentioned that $c(a) = 1 - a$, but must it always be $1 - a$? NO
- A fuzzy complement function must at least satisfy:

1. $c(0) = 1$ and $c(1) = 0$
2. For all $\mu_A(x), \mu_A(y) \in [0, 1]$, if $\mu_A(x) \leq \mu_A(y)$, then $c(\mu_A(x)) \geq c(\mu_A(y))$

E.g. A threshold-type complement function: $c(a)\begin{cases} 1 & \text{for } a \leq t, \\ 0 & \text{for } a > t, \end{cases}$

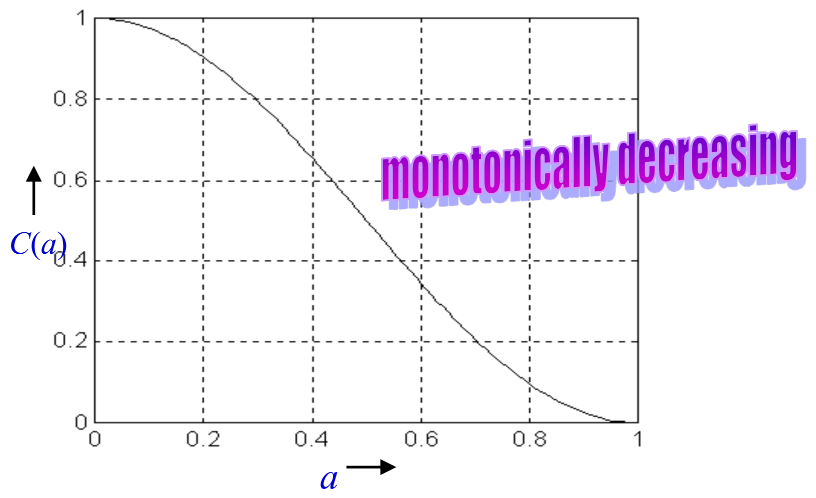
where $a \in [0, 1]$ and $t \in [0,1)$; t is called the threshold of c.



- In addition, the following axioms should be satisfied (as noted in some textbooks):

3. c is a continuous function

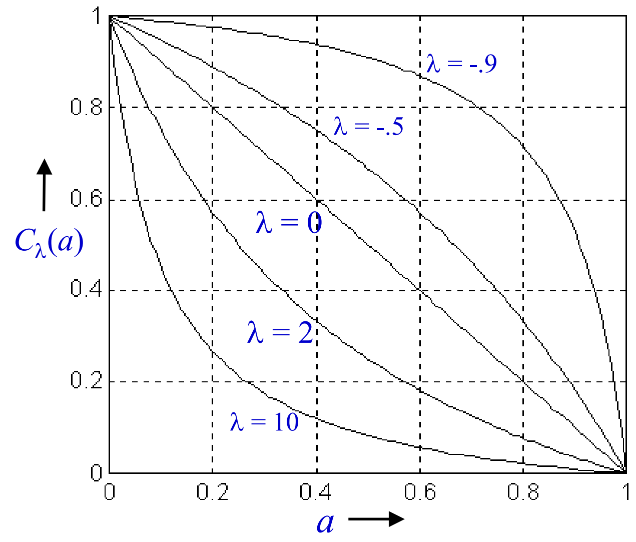
$$e.g., \quad c(a) = \tfrac{1}{2}(1 + \cos \pi a)$$



4. c is *involutive*, i.e., $c(c(a)) = a$ for all $a \in [0, 1]$

e.g., Sugeno class: $c_\lambda(a) = \dfrac{1-a}{1+\lambda a}$

where $\lambda \in (-1, \infty)$



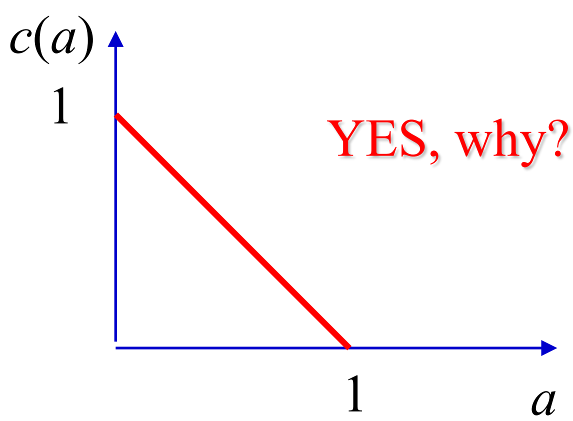
- In this course, a function must satisfy the above 4 axioms in order to be regarded as a fuzzy complement function.

- Example: Sugeno class with λ = 1 *(e.g. for involutive)*

$$c_{\lambda=1}(a)=\frac{1-a}{1+a}$$

$$c[c(a)]=\frac{1-\frac{1-a}{1+a}}{1+\frac{1-a}{1+a}}=\frac{2a}{2}=a \quad \Rightarrow \quad c\,[c(a)]=a \;\checkmark$$

- Example: Is $c(a) = 1 - a$ (e.g., $\lambda = 0$) satisfying the 4 axioms?



YES, why?

The function satisfies all the four axioms.

For example, (*for involutive*)

$$c\,[c(a)] = 1-(1-a) = a \;\checkmark$$

♦ (Def) An *equilibrium* of a fuzzy complement c is defined as any value a for which $c(a) = a$.

E.g., For the fuzzy complement $c(a) = 1 - a$, *equilibrium* = ?

$$c(a) = 1 - a = a \quad \Rightarrow \quad a = 0.5$$

Although there are many types of complement operations for fuzzy set, the most commonly used one is the standard fuzzy complement:

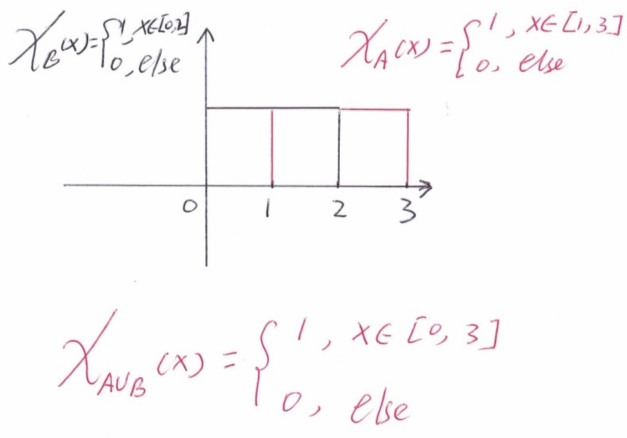
$$c(a) = 1 - a$$

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$$

♦ **Fuzzy Union**

Step one: find out the union of crisp sets in terms of characteristic functions.

Let X be the real line, and A=[1,3], and B=[0,2], what is the union of A and B?



How do we express $\chi_{A\cup B}(x)$ in terms of $\chi_A(x)$ and $\chi_B(x)$?

$$\chi_{A\cup B}(x) = \max[\chi_A(x), \chi_B(x)]$$

Step two: replace the characteristic functions by membership functions.

$$\mu_{A\cup B}(x) = \max[\mu_A(x), \mu_B(x)]$$

♦ Fuzzy Union

(Def) The union of two fuzzy sets A and B is specified by a function of the form $u : [0, 1] \times [0, 1] \rightarrow [0, 1]$. For every $x \in X$,

$\mu(x) = \max\{\mu_A(x), \mu_B(x)\} = A \cup B = A \vee B$ (*unless otherwise specified*). The fuzzy union operator is often referred to as *S-norm*.

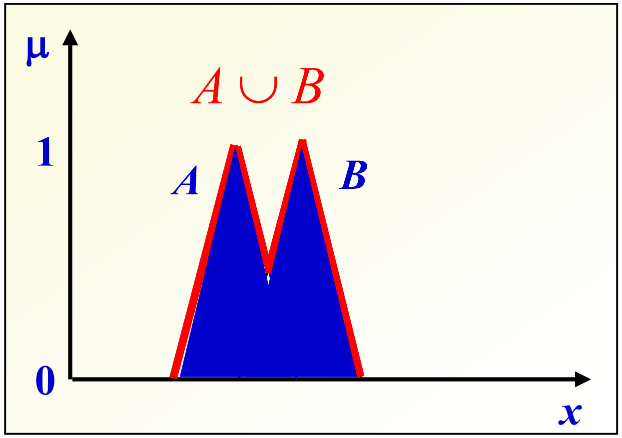
♦ E.g., $young = 1/5 + 1/10 + 0.8/20 + 0.5/30 + 0.2/40 + 0.1/50$

$old = 0.1/20 + 0.2/30 + 0.4/40 + 0.6/50 + 0.8/60 + 1/70 + 1/80$

$$young \cup old = 1/5 + 1/10 + 0.8/20 + 0.5/30 + 0.4/40 + 0.6/50 + 0.8/60 + 1/70 + 1/80$$

♦ Fuzzy union must satisfy:

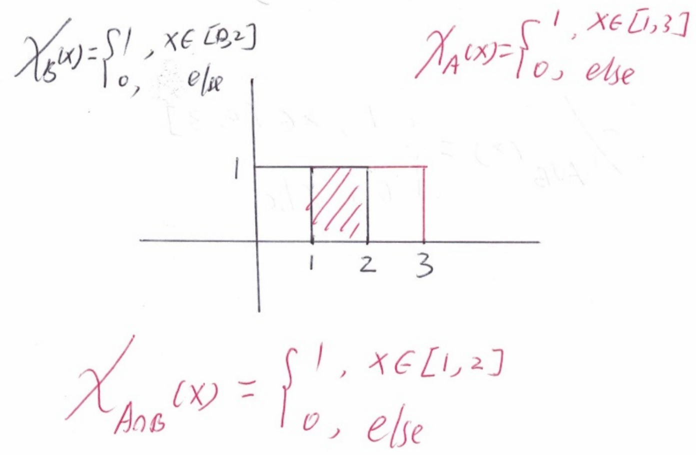
1. $u(0, 0) = 0$; $u(0, 1) = u(1, 0) = u(1, 1) = 1$
2. $u(a, b) = u(b, a)$
3. If $a \leq a'$ and $b \leq b'$, then $u(a, b) \leq u(a', b')$
4. $u(u(a, b), c) = u(a, u(b, c))$



♦ Fuzzy Intersection

Step one: find out the intersection of crisp sets in terms of characteristic functions.

Let X be the real line, and A=[1,3], and B=[0,2], what is the intersection of A and B?



How do we express $\chi_{A\cap B}(x)$ in terms of $\chi_A(x)$ and $\chi_B(x)$?

$$\chi_{A\cap B}(x) = \min[\chi_A(x), \chi_B(x)]$$

Step two: replace the characteristic functions by membership functions.

$$\mu_{A\cap B}(x) = \min[\mu_A(x), \mu_B(x)]$$

♦ Fuzzy Intersection

(Def) The fuzzy intersection of two fuzzy sets A and B is specified by a function $i : [0, 1] \times [0, 1] \rightarrow [0, 1]$.

For every $x \in X$, $\mu(x) = \min\{\mu_A(x), \mu_B(x)\} = A \cap B = A \wedge B$ (*unless otherwise specified*).

Fuzzy intersection operator is also referred to as *T-norm*.

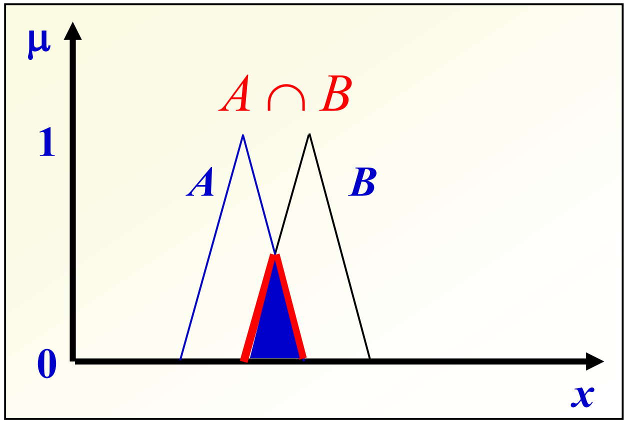
♦ E.g. $young = 1/5 + 1/10 + 0.8/20 + 0.5/30 + 0.2/40 + 0.1/50$

$old = 0.1/20 + 0.2/30 + 0.4/40 + 0.6/50 + 0.8/60 + 1/70 + 1/80$

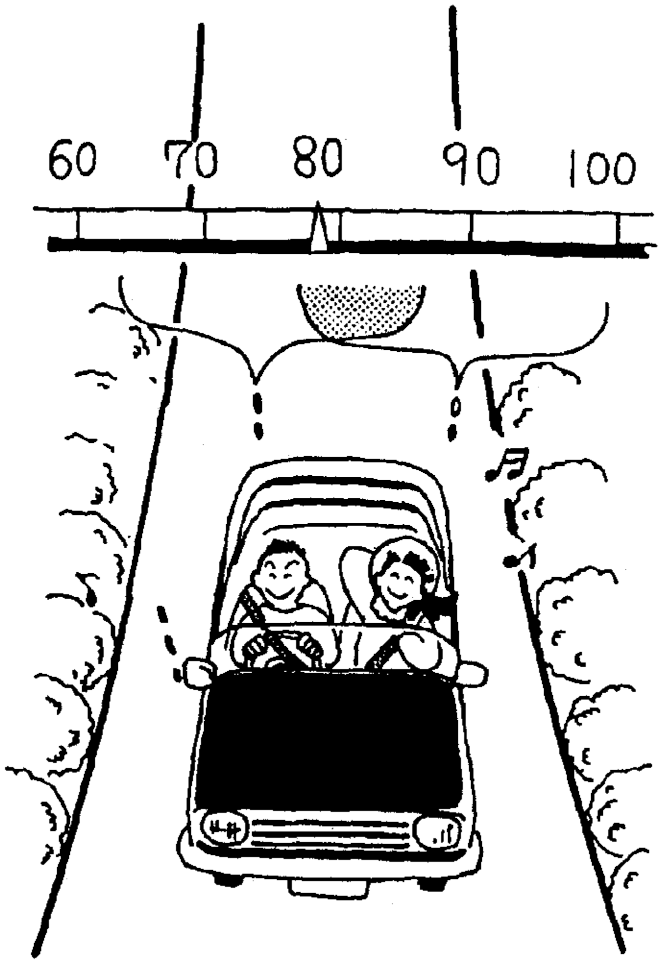
$young \cap old = 0.1/20 + 0.2/30 + 0.2/40 + 0.1/50$

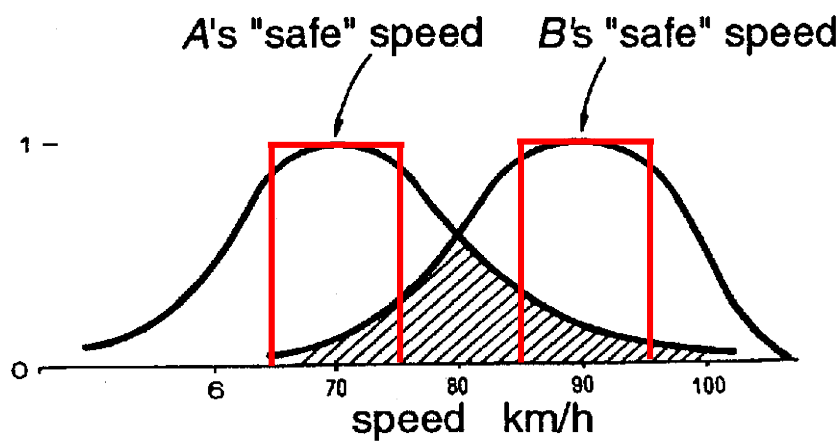
♦ Fuzzy intersection must satisfy:

1. $i(1, 1) = 1$; $i(0, 1) = i(1, 0) = 0$
2. $i(a, b) = i(b, a)$
3. If $a \leq a'$ and $b \leq b'$, then $i(a, b) \leq i(a', b')$
4. $i(i(a, b), c) = i(a, i(b, c))$



Fuzzy Intersection Example: Safe Speed





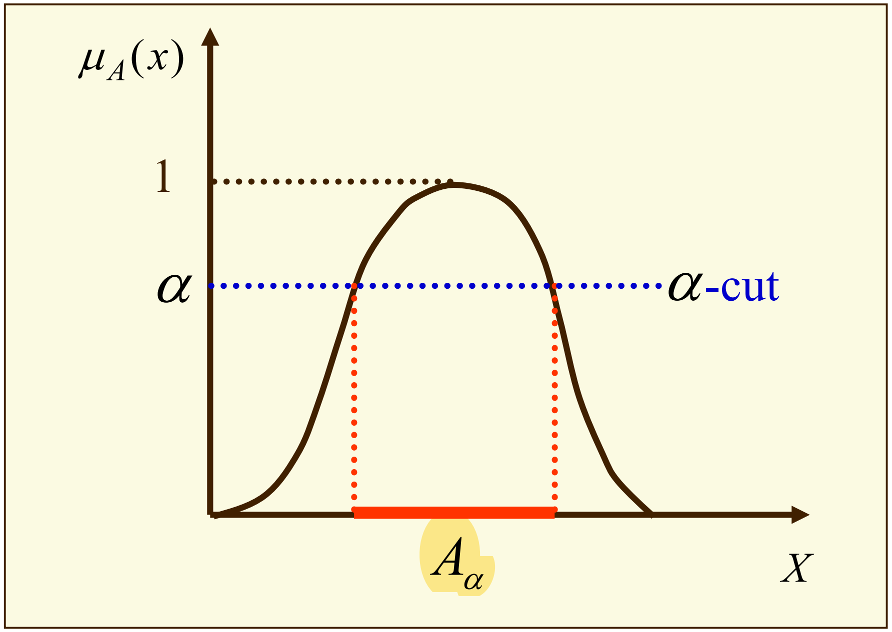
Fuzzy Sets "Safe Speed"

Next, we want to introduce an important tool to represent fuzzy set by the crisp set such that we can define various operations of fuzzy sets through our knowledge on operations of crisp sets.

- (Def) The α-cut of a fuzzy set A is a crisp set A_α that contains all the elements of the universal set X that have a membership grade in A greater than or equal to the specified value of α:

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$$

$$\text{Strong } \alpha\text{-cut}: \ A_{\alpha^+} = \{x \in X \mid \mu_A(x) > \alpha\}$$



$\mu_A \geq 0$ $\quad A_{\alpha=0} = X$

$\mu_A \geq 1$ $\quad A_{\alpha=1} = Core(A)$

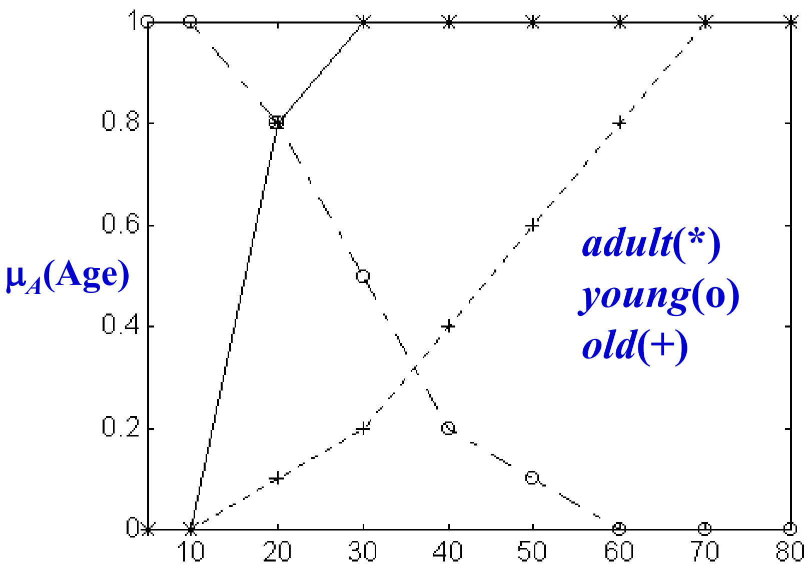
$\mu_A > 0$ $\quad A_{\alpha^+=0} = Supp(A)$

$\mu_A > 1$ $\quad A_{\alpha^+=1} = \varnothing$

Elements (ages)	*adult*	*young*	*old*
5	0	1	0
10	0	1	0
20	0.8	0.8	0.1
30	1	0.5	0.2
40	1	0.2	0.4
50	1	0.1	0.6
60	1	0	0.8
70	1	0	1
80	1	0	1

- *Supp*(*adult*) = {20, 30, 40, 50, 60, 70, 80}
- *Supp*(*young*) = {5, 10, 20, 30, 40, 50}
- *Supp*(*old*) = {20, 30, 40, 50, 60, 70, 80}

$\mu_A(x) > 0$



- $\alpha = 0.2$

  $young_{0.2} = \{5, 10, 20, 30, 40\}$
- $\alpha = 0.8$

  $young_{0.8} = \{5, 10, 20\}$
- $\alpha = 1.0$

  $young_{1.0} = \{5, 10\}$

$\mu_A(x) \geq \alpha$

♦ (Def) Level set: The set of all levels $\alpha \in [0, 1]$ that represent distinct α-cut of a fuzzy set A is $\Lambda_A = \{\alpha \mid \mu_A(x) = \alpha \text{ for some } x \in X\}$

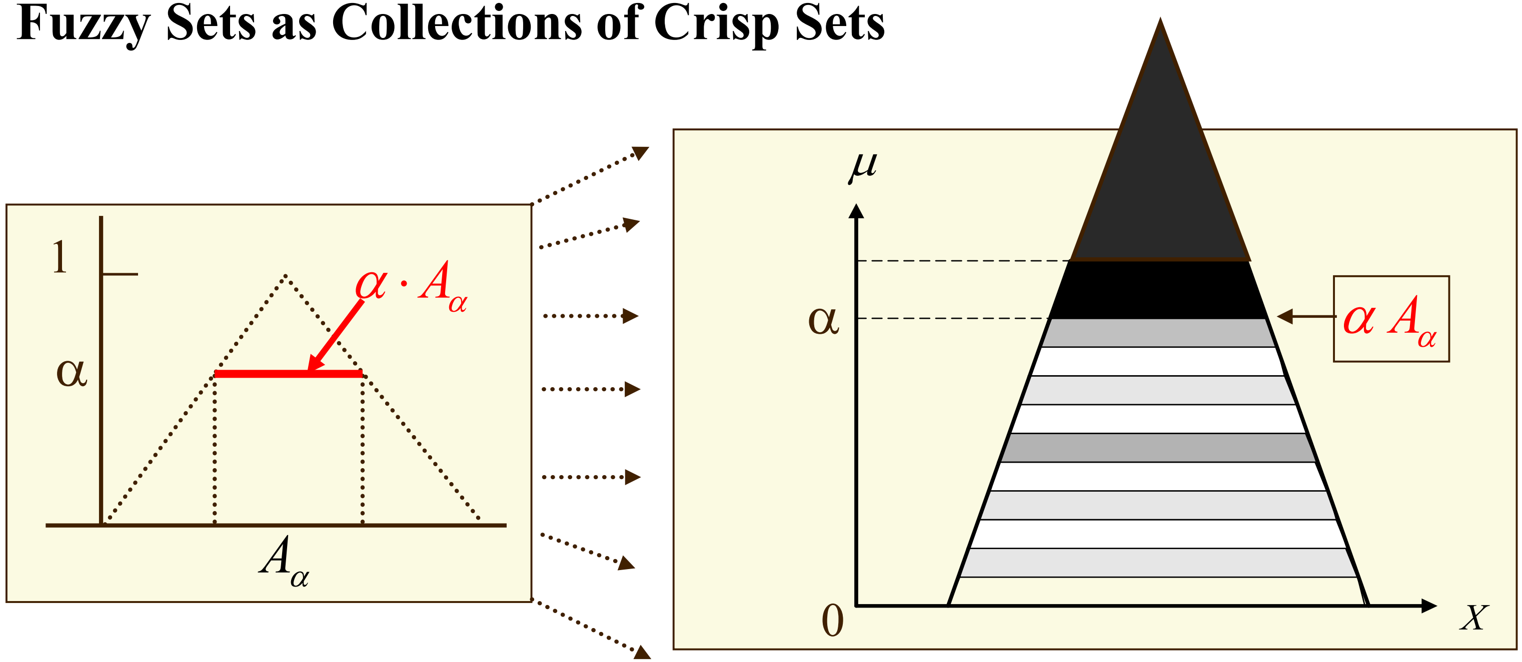
Elements (ages)	*adult*	*young*	*old*
5	0	1	0
10	0	1	0
20	0.8	0.8	0.1
30	1	0.5	0.2
40	1	0.2	0.4
50	1	0.1	0.6
60	1	0	0.8
70	1	0	1
80	1	0	1

$$\Lambda_{adult} = \{0,\ 0.8,\ 1\}$$

$$\Lambda_{young} = \{0,\ 0.1,\ 0.2,\ 0.5,\ 0.8,\ 1\}$$

$$\Lambda_{old} = \{0,\ 0.1,\ 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1\}$$

♦ Fuzzy Sets as Collections of Crisp Sets



♦ Decomposition Theorem:

A fuzzy set *A* can be represented by $A = \bigcup_{\alpha \in [0,1]} \alpha \cdot A_\alpha$

Fuzzy Set   Crisp Sets

$\alpha \cdot A_\alpha$ denotes the algebraic product of a scalar α with the α-cut A_α; $\cup$ denotes the union operation.

Fuzzy set: $A = 0.2/10 + 0.5/20 + 1/30 + 0.5/40$

α-cuts:

$$A_{0.2} = 1/10 + 1/20 + 1/30 + 1/40$$

$$A_{0.5} = 0/10 + 1/20 + 1/30 + 1/40$$

$$A_1 = 0/10 + 0/20 + 1/30 + 0/40$$

We now convert each of the α-cuts to special fuzzy set: $\alpha \cdot A_\alpha$

$$0.2 \cdot A_{0.2} = 0.2/10 + 0.2/20 + 0.2/30 + 0.2/40$$

$$0.5 \cdot A_{0.5} = 0/10 + 0.5/20 + 0.5/30 + 0.5/40$$

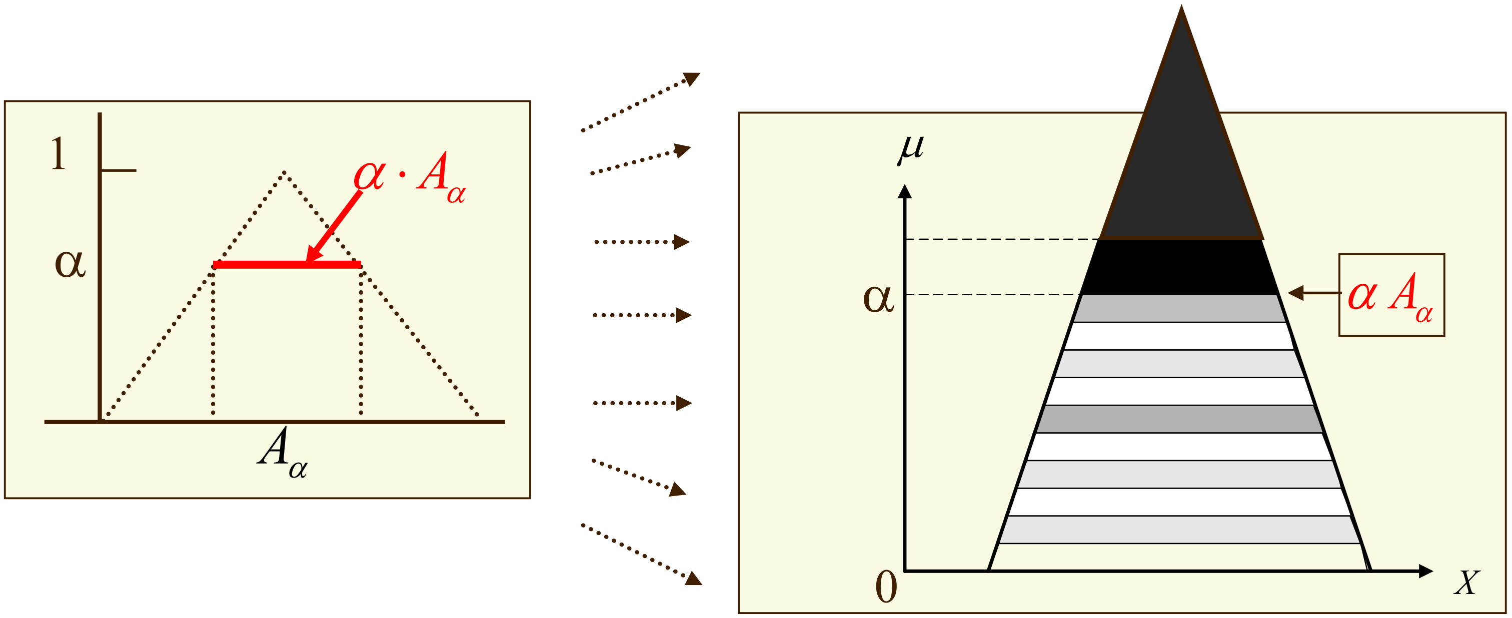
$$1 \cdot A_1 = 0/10 + 0/20 + 1/30 + 0/40$$

It is now easy to see that the standard fuzzy union

$$(A \cup B)(x) = \max[A(x), B(x)]$$

of these three fuzzy sets will lead to A:

$$A = 0.2 \cdot A_{0.2} \bigcup 0.5 \cdot A_{0.5} \bigcup 1 \cdot A_1$$



- Decomposition Theorem: $$A = \bigcup_{\alpha \in [0,1]} \alpha \cdot A_\alpha$$

Let $A(x)=m$, we need to show that the membership of x in the union of $\alpha \cdot A_\alpha$ is also m.

Does x belong to the α-cut A_α where $\alpha>m$?

No. By the definition of α-cut, x only belongs to the set A_α where $\alpha \le m$.

For the fuzzy set $\alpha \cdot A_\alpha$, the membership of all the elements is the same, α.

What is the membership of x in the fuzzy set $m \cdot A_m$? *m.*

The membership of x in the fuzzy set $\alpha \cdot A_\alpha$, where $\alpha<m$, is smaller than m, right?

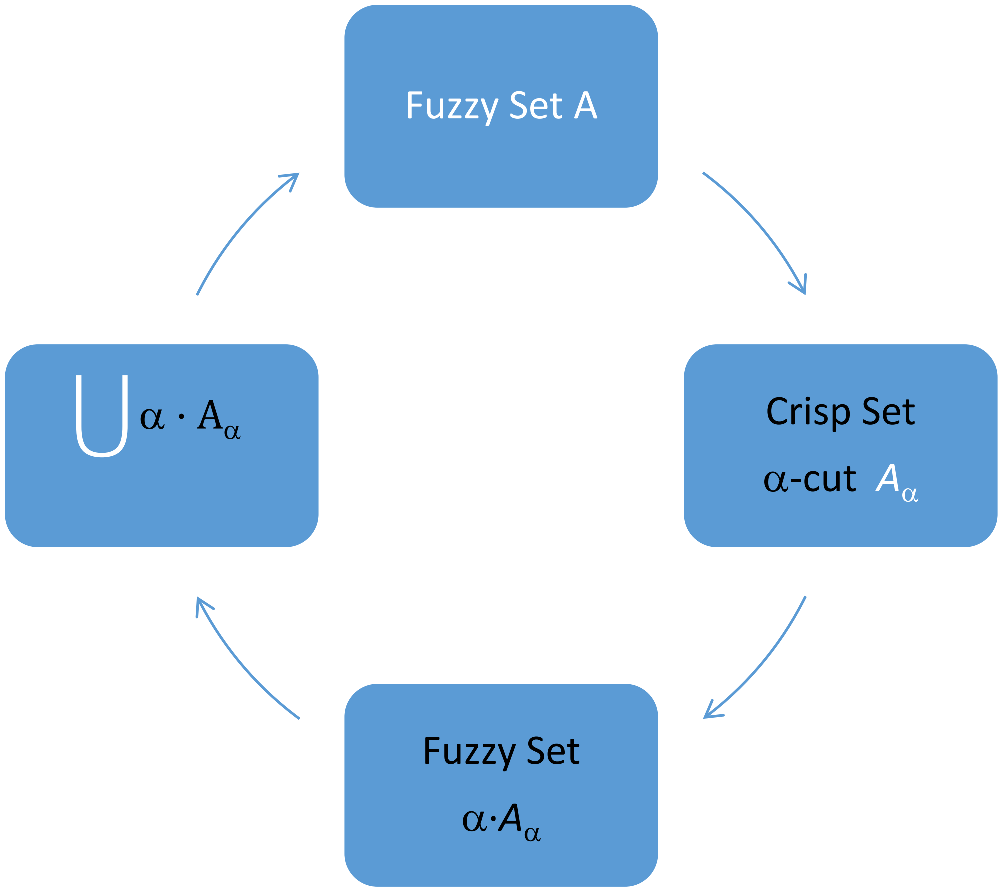
Since the union operation takes the maximum of the memberships, we can conclude that the membership of x in the union of $\alpha \cdot A_\alpha$ must be m.

The principal role of α-cuts in fuzzy set theory is their capability to represent fuzzy sets.

With α-cuts, we can easily extend various properties of crisp sets and operations on crisp sets to their fuzzy counterparts.

In each extension, a given classical (crisp) property or operation is required to be valid for all the α-cuts of the fuzzy set.

If the operation can be done for all the α-cuts of the fuzzy set, then we just put together the results from all the α-cuts, and use the decomposition theorem to construct the Fuzzy set.


Fuzzy Set A
Crisp Set
α-cut A_α
Fuzzy Set
$\alpha \cdot A_\alpha$
$\bigcup \alpha \cdot A_\alpha$