10 — MORPHOLOGICAL PROCESSING

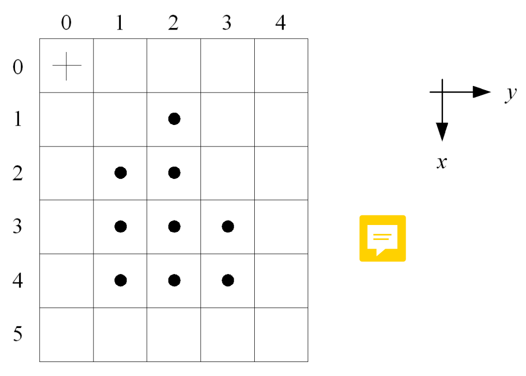
Morphology is a branch of biology that deals with the form and structure of animals and plants. In the image processing context, *mathematical morphology* is a tool

- for extracting image components that are useful in representation and description of region shapes, such as boundaries, skeletons and the convex hull
- for processing techniques such as morphological filtering, thinning, thickening, region filling and pruning.

The language of mathematical morphology is *set theory*. In this context, the coordinates of each pixel are a pair of elements from the 2D integer space $Z \times Z$ (or Z^2), (a, b), $a, b \in Z$.

Sets in mathematical morphology represents the shapes of objects in an image. In binary images, the sets in question are members of Z^2, where each element of a set is a 2-tuple (or 2-D vector) whose coordinates are the x, y coordinates of a black (by convention) pixel in the image.

Example



A

(+ denotes the origin, • represents logical "1")
$A = \{(1,2),(2,1),(2,2),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3)\}$

PRELIMINARIES

Let A be a set in Z^2, with elements

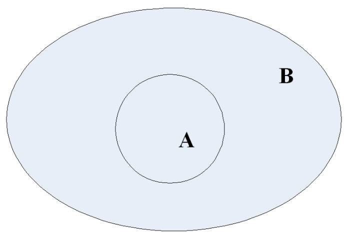
$$a_i = (a_{i1}, a_{i2})$$

The empty set is denoted by $\emptyset$. If a_i is an element of set A, we write

$$a_i \in A \tag{1}$$

If every element of set A is also an element of another set B, then A is said to be a subset of B (or A is contained in B), denoted by

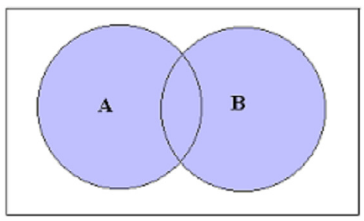
$$A \subseteq B \tag{2}$$



The *union* of two sets A and B, denoted by

$$C = A \cup B \tag{3}$$

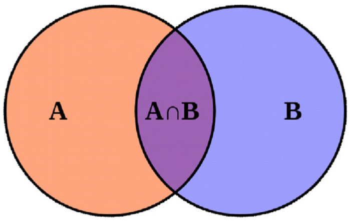
is the set of all elements belonging to either A, B, or both.



The *intersection* of two sets A and B, denoted by

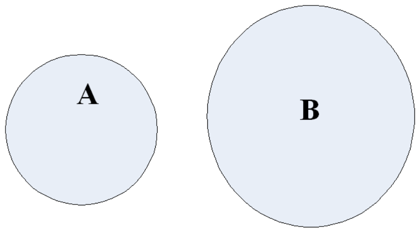
$$D = A \cap B \tag{4}$$

is the set of all elements belonging to both A and B.



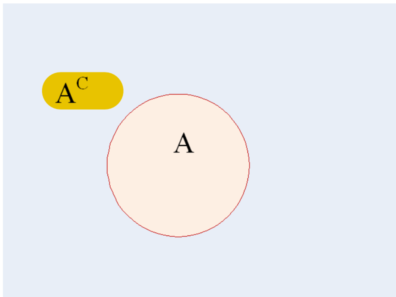
Two sets A and B are said to be *disjoint* or mutually exclusive if they have no common elements. In this case,

$$A \cap B = \emptyset \tag{5}$$



The *complement* of set A is the set of elements not contained in A:

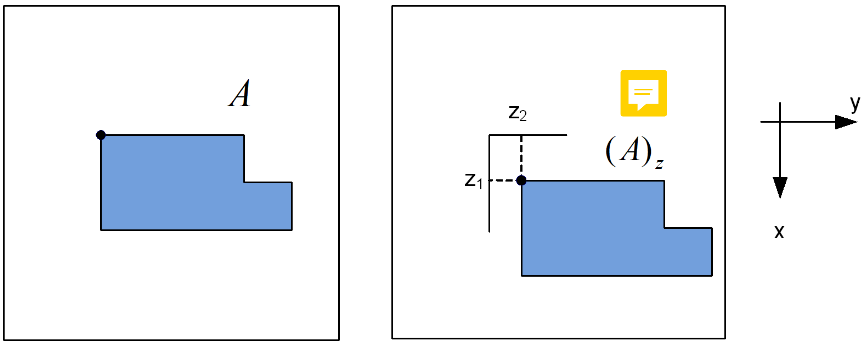
$$A^c \equiv \{x | x \notin A\} \tag{6}$$



The *translation* of A by $z = (z_1, z_2)$, denoted $(A)_z$, is

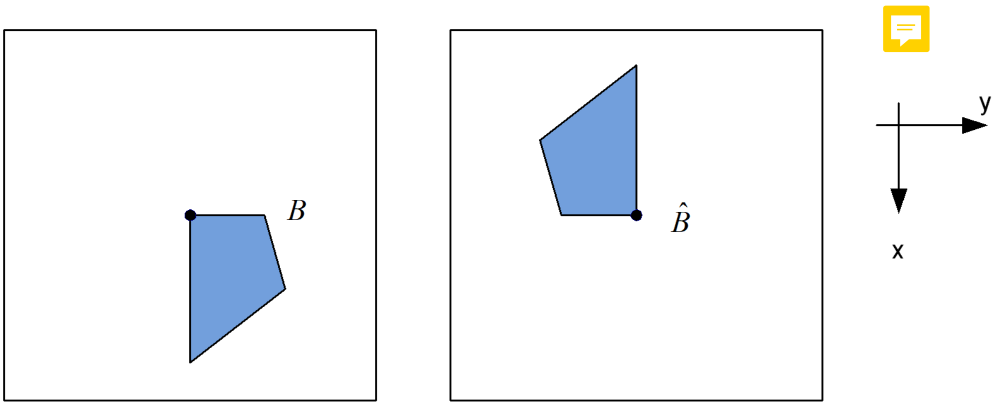
$$(A)_z \equiv \{x | x = a + z, \text{ for } a \in A\} \tag{7}$$

$(A)_z$ is called a translate of A.



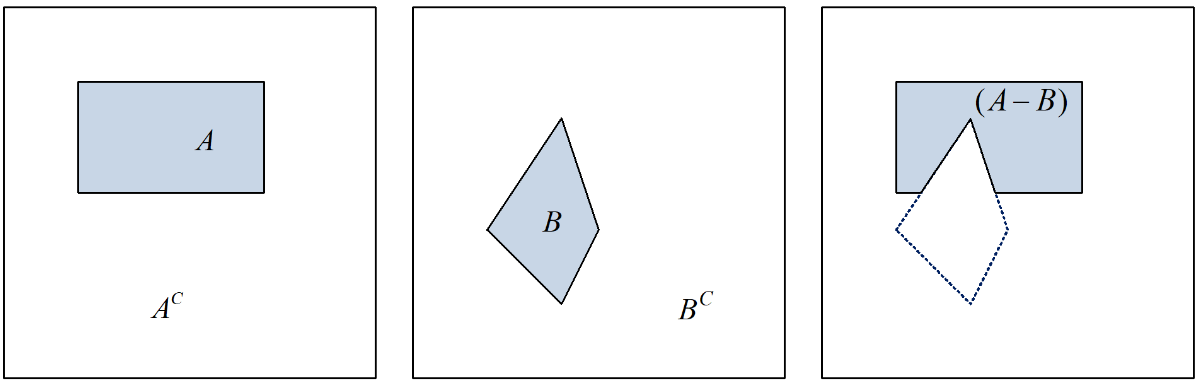
The *reflection* of B, denoted $\hat{B}$, is

$$\hat{B} \equiv \{x | x = -b, \text{ for } b \in B\} \tag{8}$$



The *difference* of two sets A and B is

$$A - B = \{x | x \in A, x \notin B\} = A \cap B^c \tag{9}$$



DILATION AND EROSION

Dilation and *erosion* are the bases for many morphological operations. Dilation expands an image and erosion shrinks it.

Dilation

With A and B as sets in Z^2 and $\emptyset$ denoting the empty set, a definition of the *dilation* of A by B is

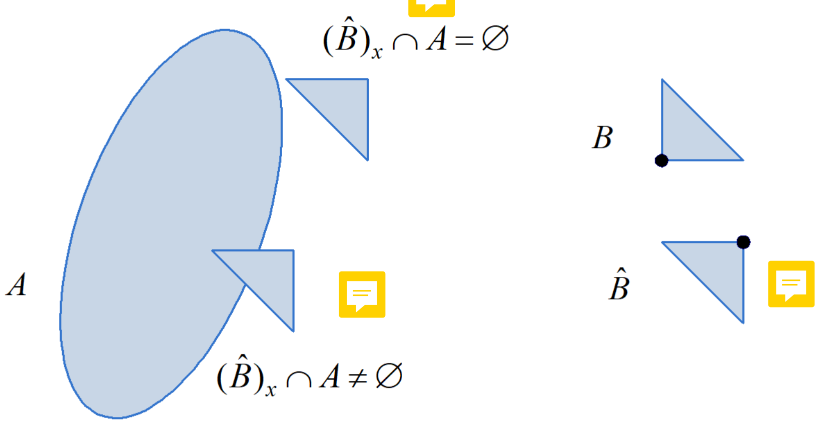
$$A \oplus B = \{x | (\hat{B})_x \cap A \neq \emptyset\} \tag{10}$$

i.e., we first obtain the reflection of B about its origin and then shift this reflection by x. The dilation of A by B is then the set of all x displacements such that $(\hat{B})_x$ and A overlap by at least one non-zero element (i.e., they have a non-empty intersection).

This equation may be rewritten as

$$A \oplus B = \{x | [(\hat{B})_x \cap A] \subseteq A\} \tag{11}$$

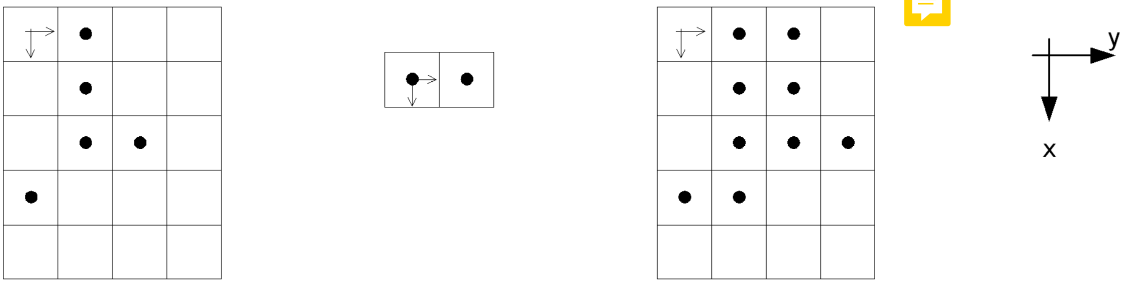
Set B is commonly referred to as the *structuring element*.



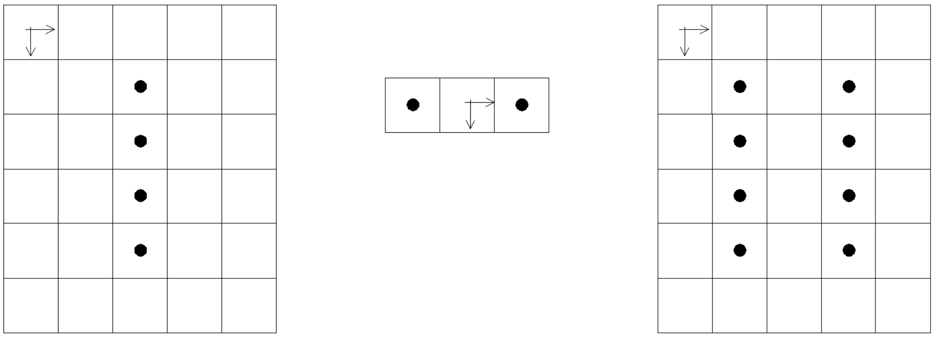
Example

$A = \{(0,1),(1,1),(2,1),(2,2),(3,0)\}$ $\quad B = \{(0,0),(0,1)\}$

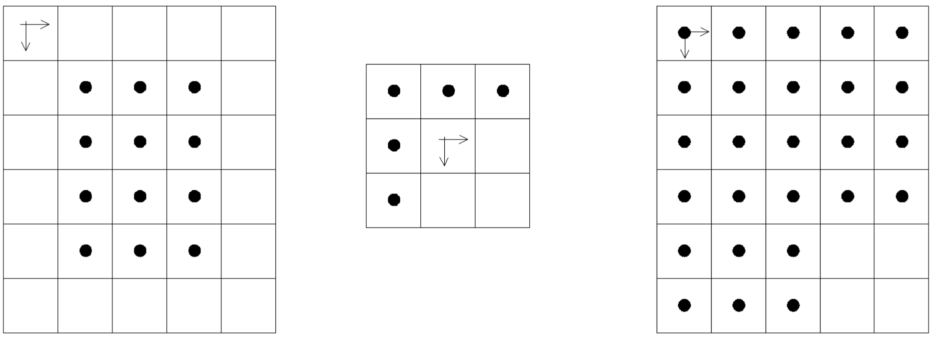
$$A \oplus B = \{(0,1),(1,1),(2,1),(2,2),(3,0)$$
$$(0,2),(1,2),(2,2),(2,3),(3,1)\}$$



A $\qquad B$ $\qquad A \oplus B$

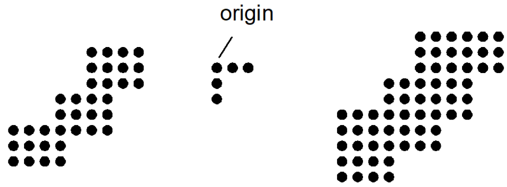


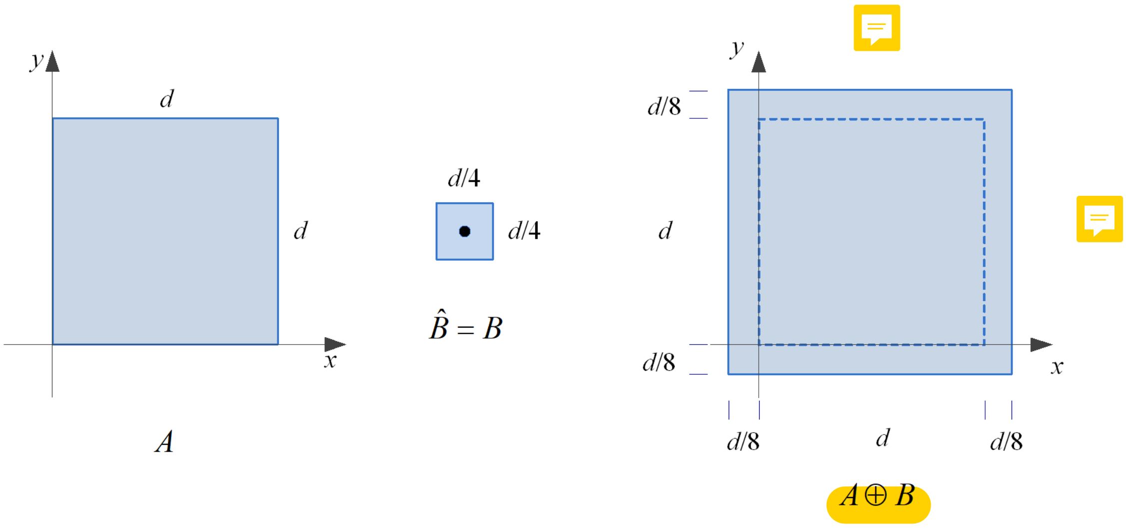
A $\qquad B$ $\qquad A \oplus B$

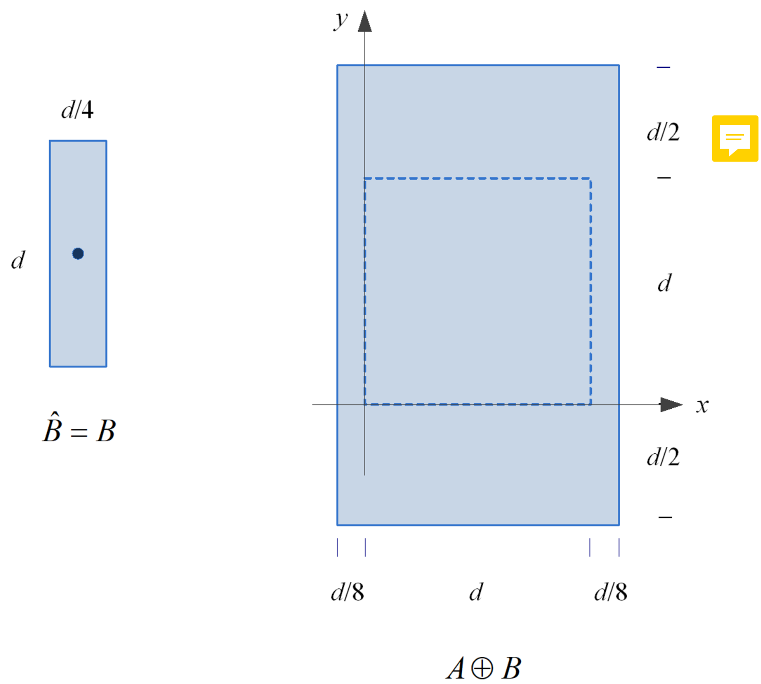


A $\qquad B$ $\qquad A \oplus B$

Note: background may be extended as necessary.


origin


y
d
d
x
A
d/4
d/4
$\hat{B} = B$
d/8
d
d/8
d/8
d
d/8
$A \oplus B$

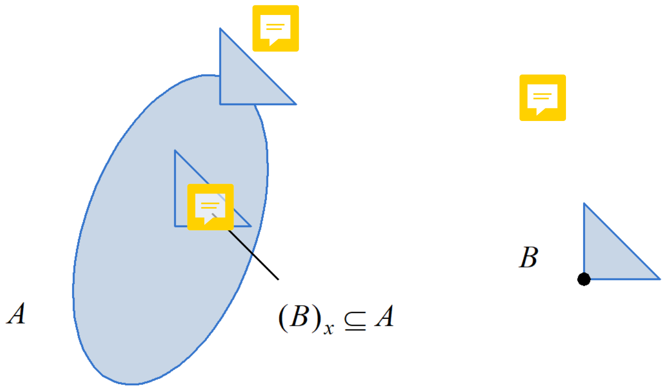

d/4
d
$\hat{B} = B$
y
d/2
d
x
d/2
d/8
d
d/8
$A \oplus B$

Erosion

For sets A and B in Z^2, the erosion of A by B may be defined as

$$A \ominus B = \{x | (B)_x \subseteq A\} \tag{12}$$

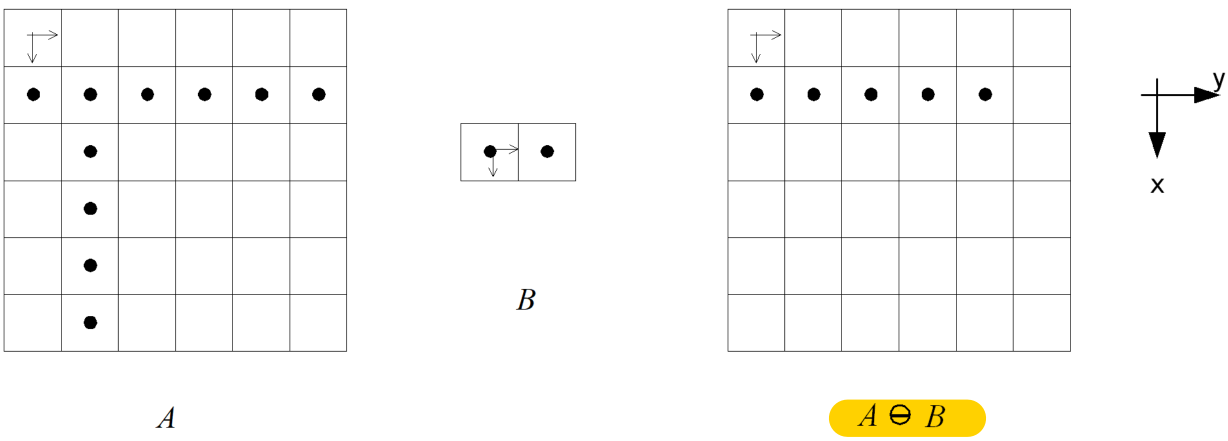
which says that the erosion of A by B is the set of all points x such that B, translated by x, is contained in A.

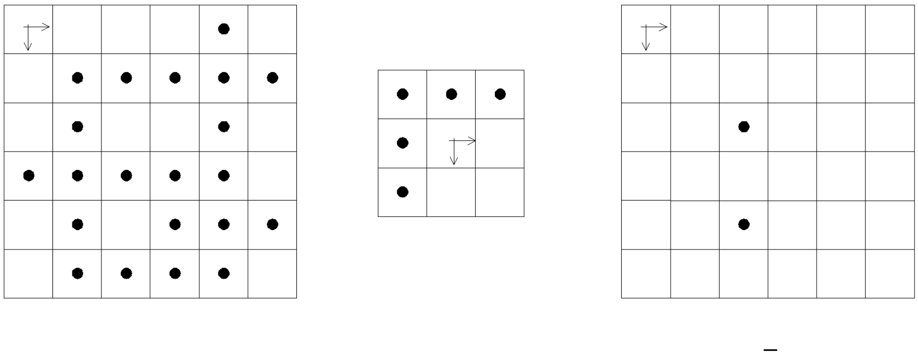


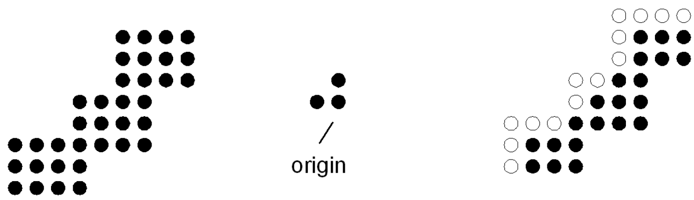
$A = \{(1,0), (1,1), (1,2), (1,3), (1,4)$
$(1,5), (2,1), (3,1), (4,1), (5,1)\}$

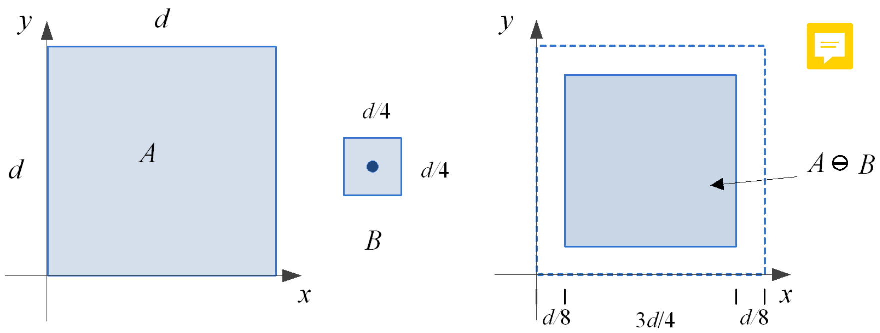
$B = \{(0,0), (0,1)\}$

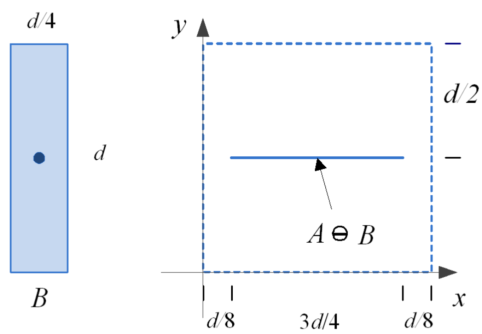
$A \ominus B = \{1,0), (1,1), (1,2), (1,3), (1,4)\}$





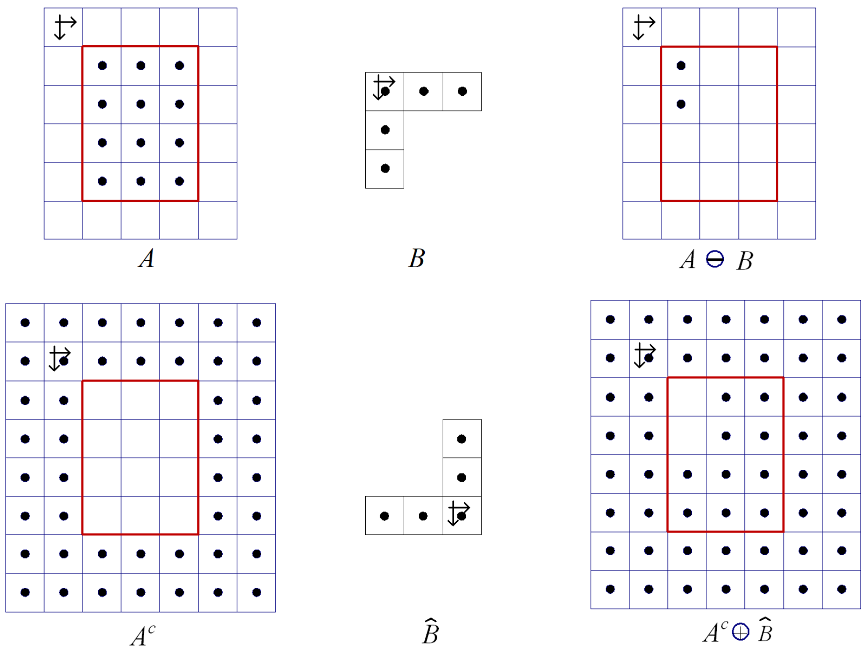

origin

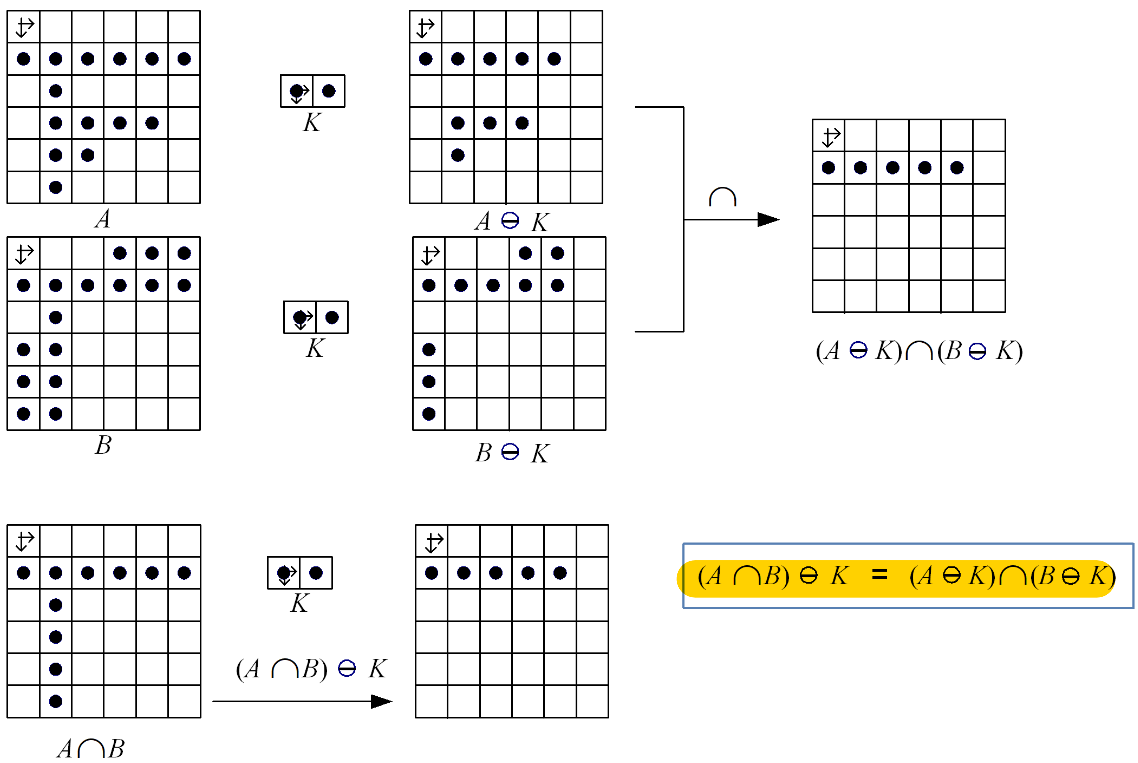

y
d
d
A
x
d/4
d/4
B
y
A ⊖ B
x
d/8
3d/4
d/8


d/4
d
B
y
d/2
A ⊖ B
x
d/8
3d/4
d/8

Dilation and erosion are duals of each other with respect to set complementation and reflection. That is,

$$(A \ominus B)^c = A^c \oplus \hat{B} \tag{13}$$





OPENING AND CLOSING

Opening basically opens up strips; it generally smooths the contours of an image, breaks narrow isthmuses, and eliminates thin protrusions.

Closing basically closes gaps; it tends to smooth contours, but as opposed to closing, it generally fuses narrow breaks and long thin gulfs, eliminates small holes, and fill gaps in the contour.

The opening of set A by structuring element B, is

$$A \circ B = (A \ominus B) \oplus B \tag{14}$$

which says that the opening of A by B is simply the erosion of A by B, followed by a dilation of the result by B.

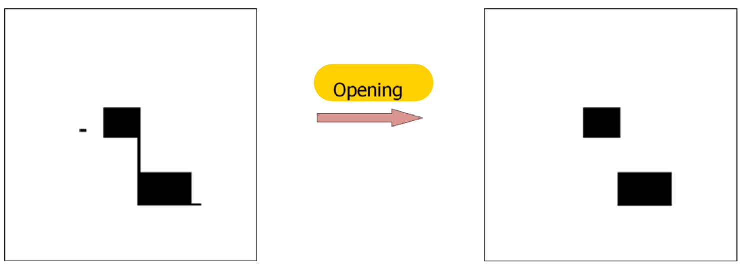
The closing of set A by structuring element B is

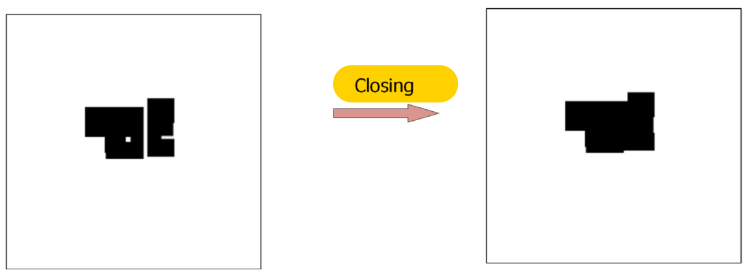
$$A \bullet B = (A \oplus B) \ominus B \tag{15}$$

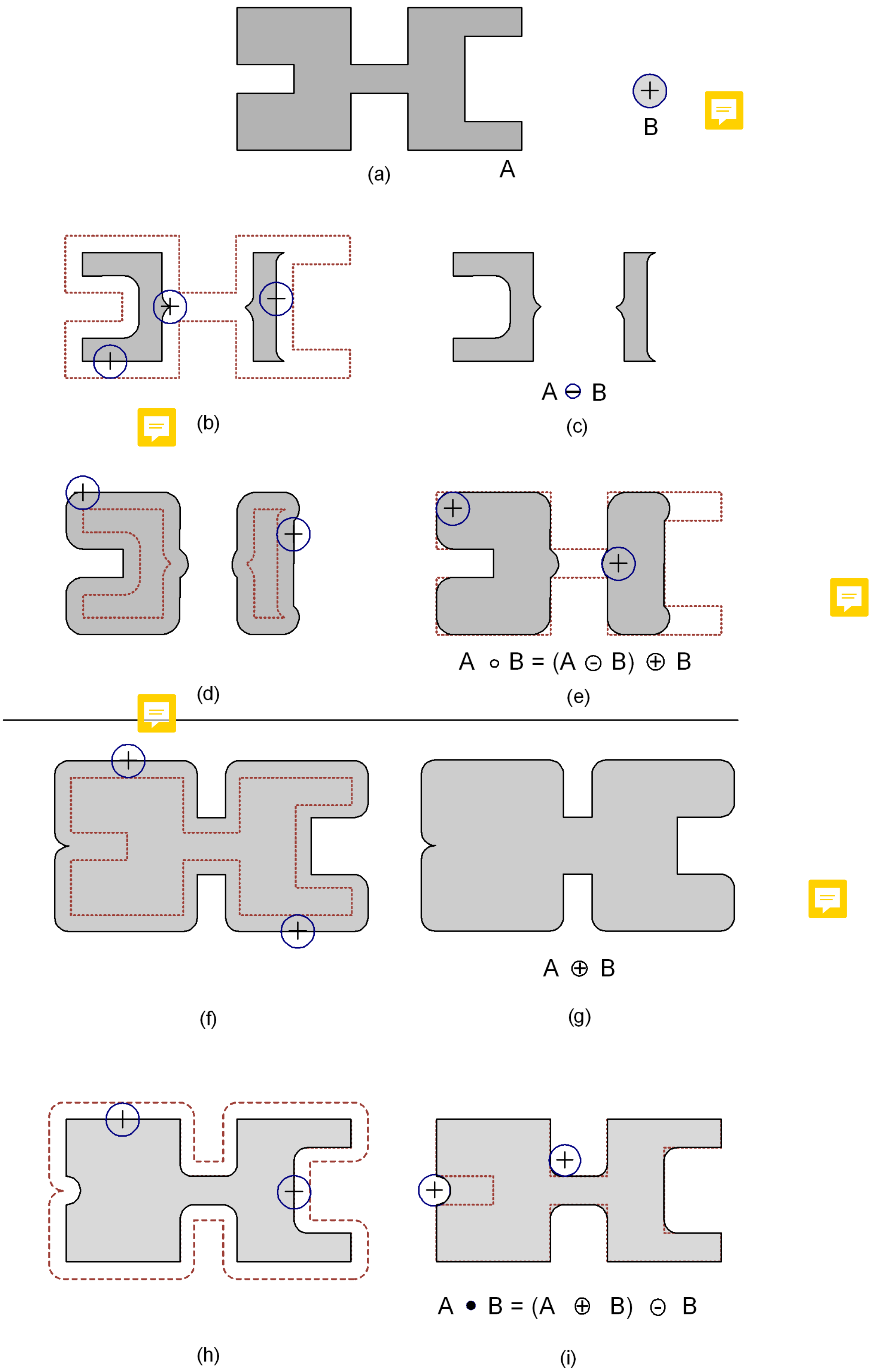
i.e., the dilation of A by B, followed by the erosion of the result by B.

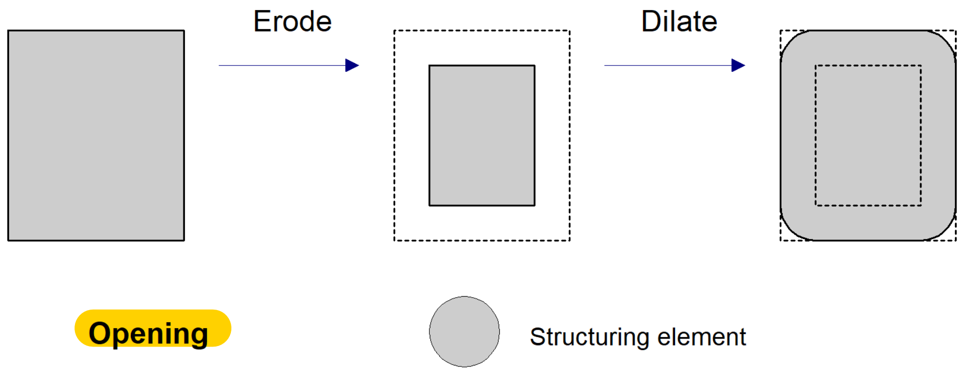
Opening and closing are duals with respect to set complementation and reflection:

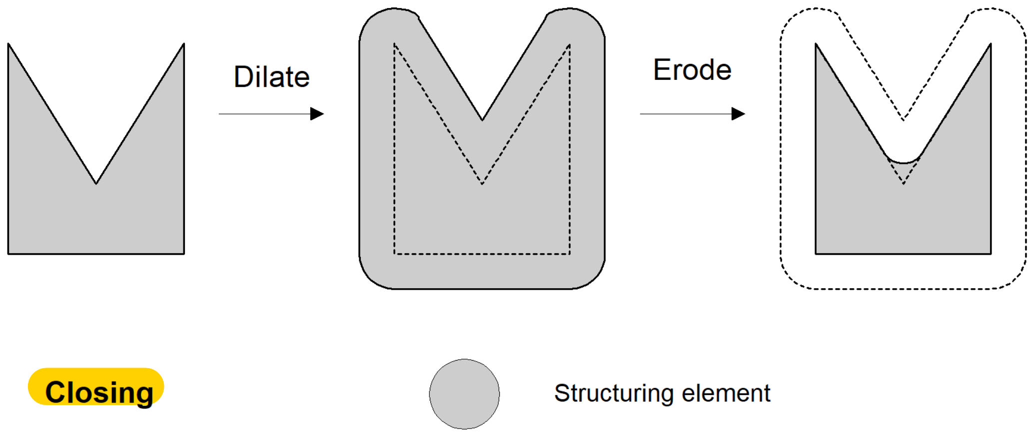
$$(A \bullet B)^c = (A^c \circ \hat{B}) \tag{16}$$

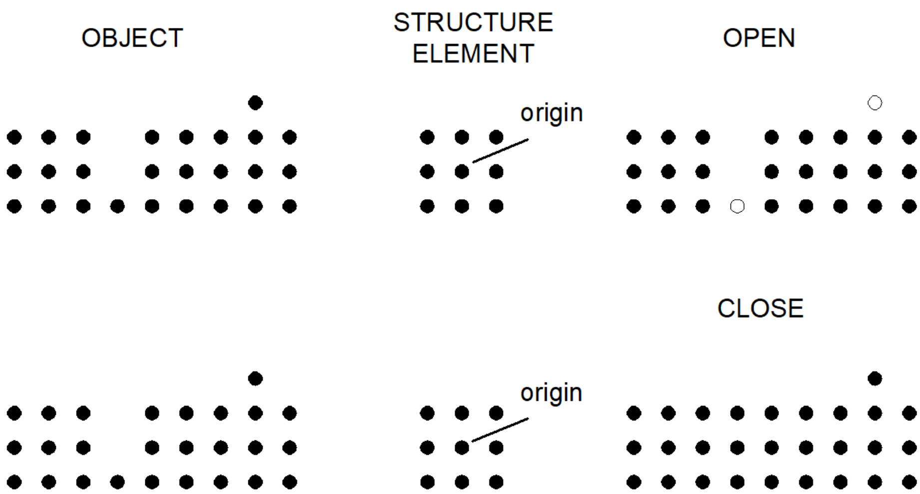


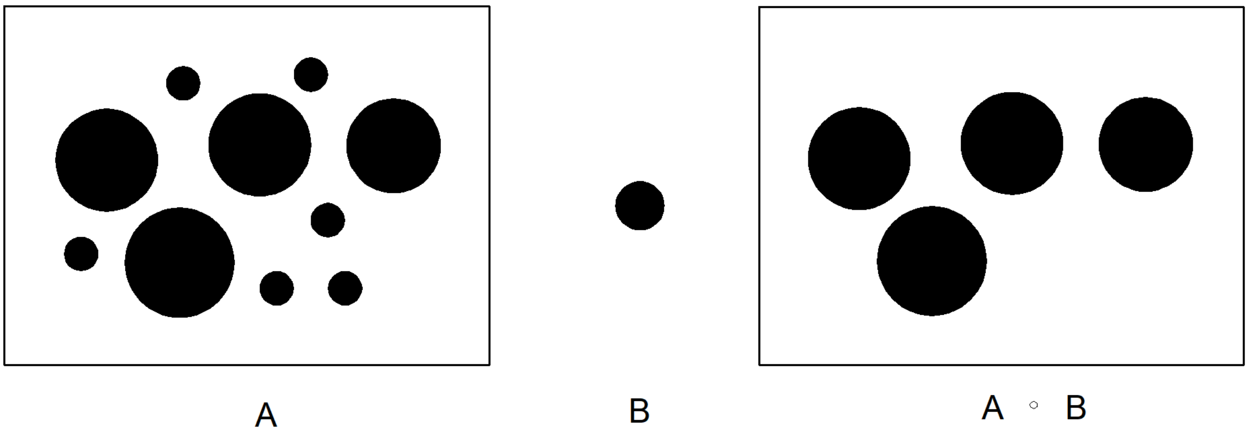



B
(a)
A
(b)
A ⊖ B
(c)
(d)
A ∘ B = (A ⊖ B) ⊕ B
(e)
A ⊕ B
(f)
(g)
A • B = (A ⊕ B) ⊖ B
(h)
(i)

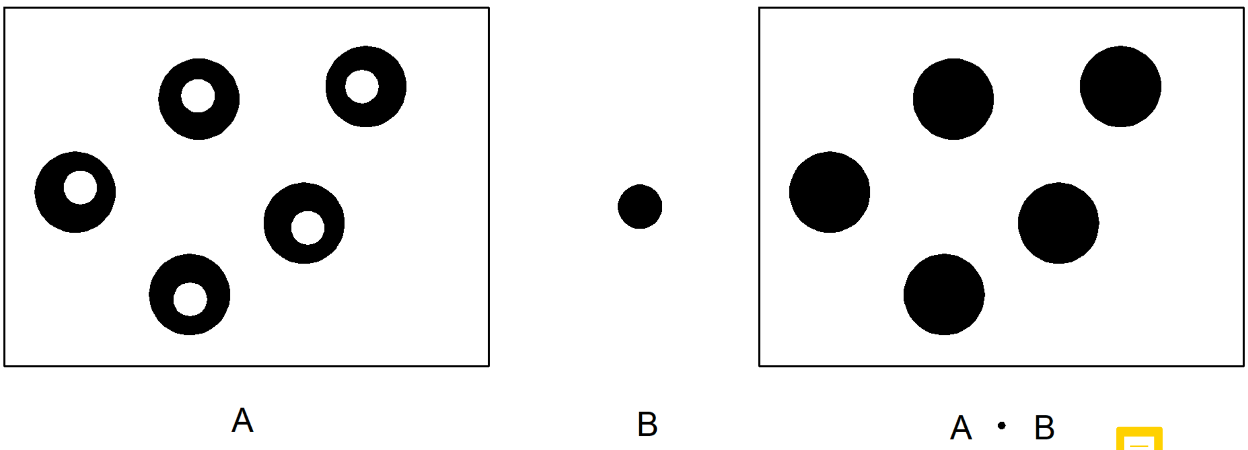

Erode
Dilate
Opening
Structuring element


Dilate
Erode
Closing
Structuring element


OBJECT
STRUCTURE ELEMENT
OPEN
origin
CLOSE
origin



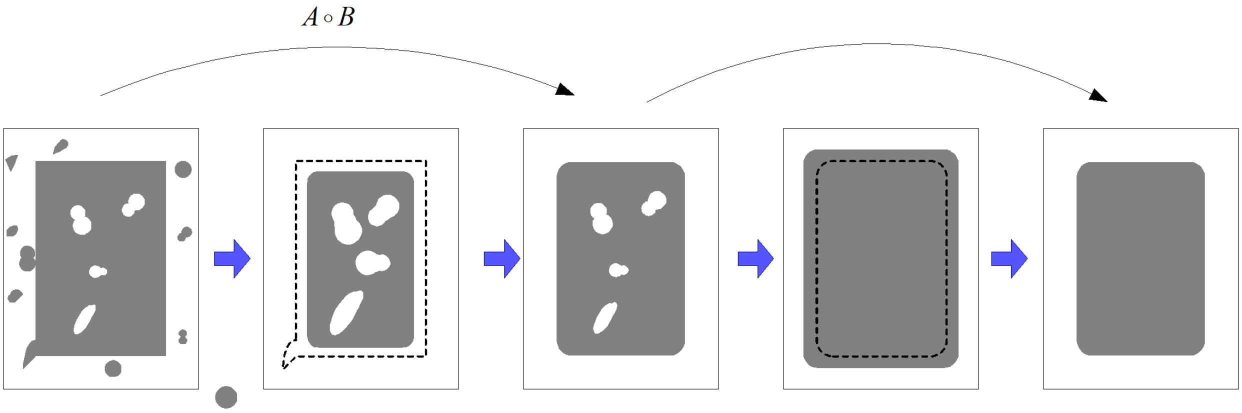
Using opening to separate objects of different sizes.



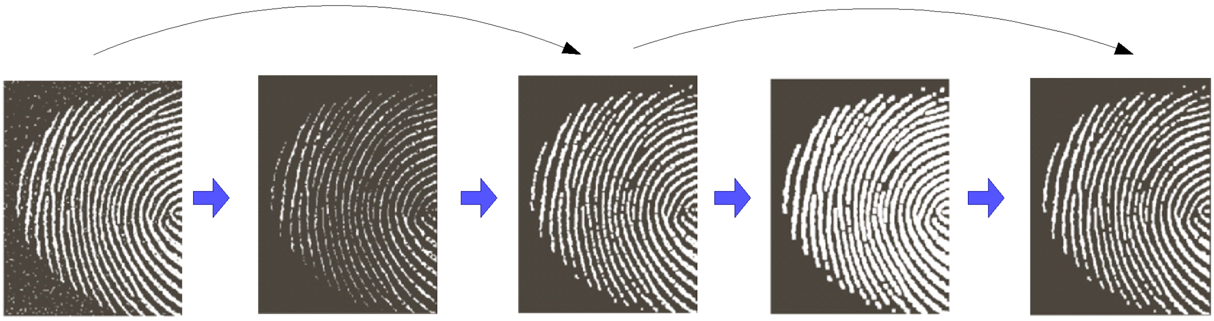
Using closing to fill in holes within objects.

Examples

Using the morphological filter $(A \circ B) \bullet B$ to remove noise.



(black pixel = logical 1)



$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$

(white pixel = logical 1)

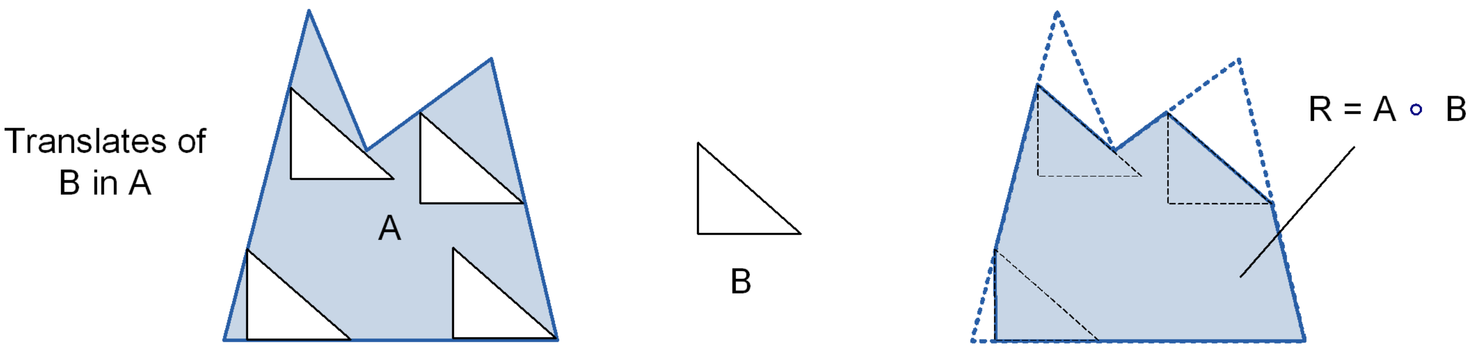
Opening and closing have a simple geometric interpretation.

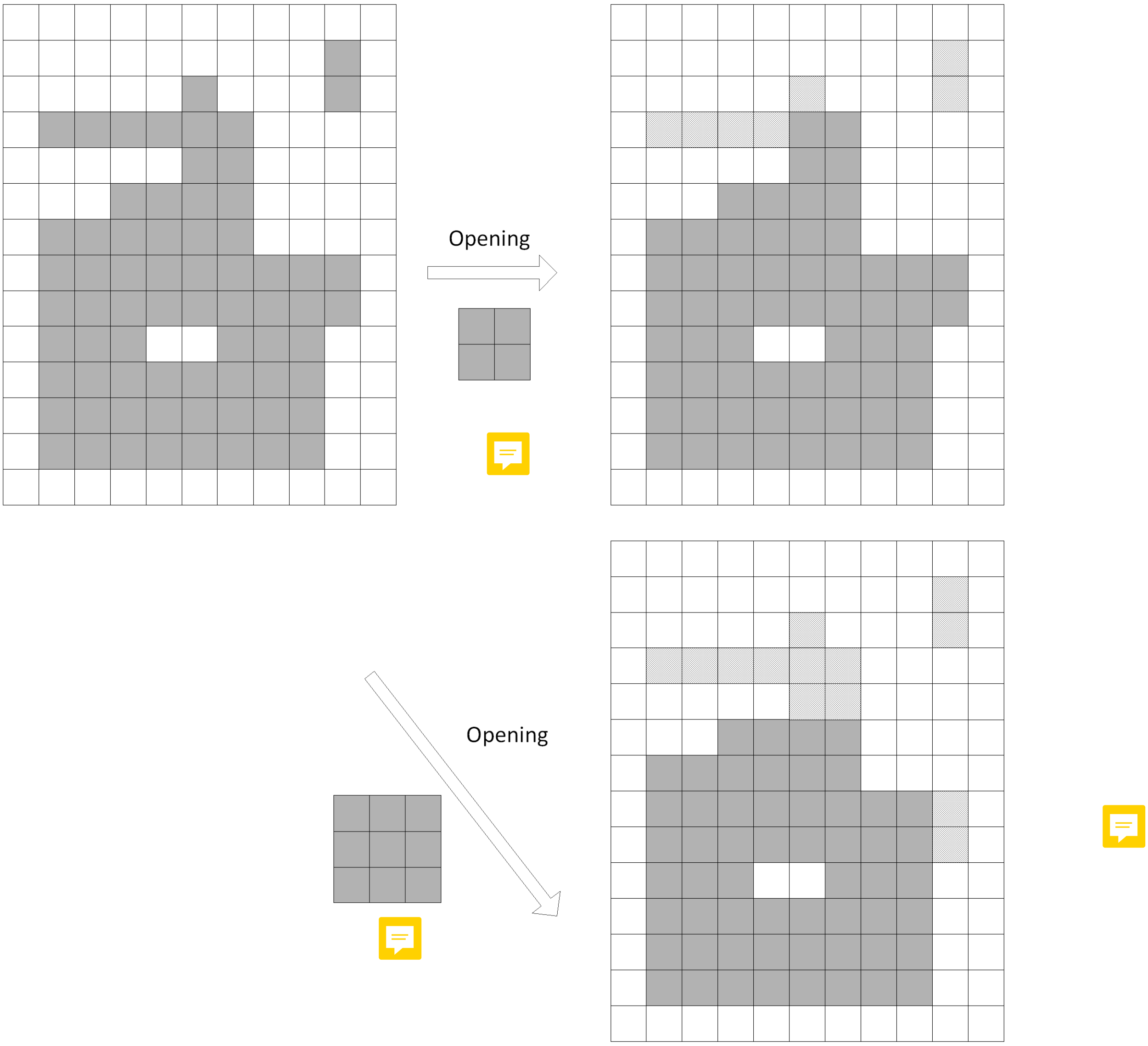
The boundary of $R = A \circ B$ is given by the points on the boundary of B that reach the farthest towards the boundary of A as B is translated around the inside of this boundary.

Outward pointing corners become rounded whereas inward pointing corners are not affected. Protruding elements where B does not fit are eliminated.

In other words, the opening of A by B can be obtained by taking the union of all translates of B that fit into A, i.e.,

$$R = A \circ B = \cup\{(B)_x \mid (B)_x \subseteq A\} \tag{17}$$

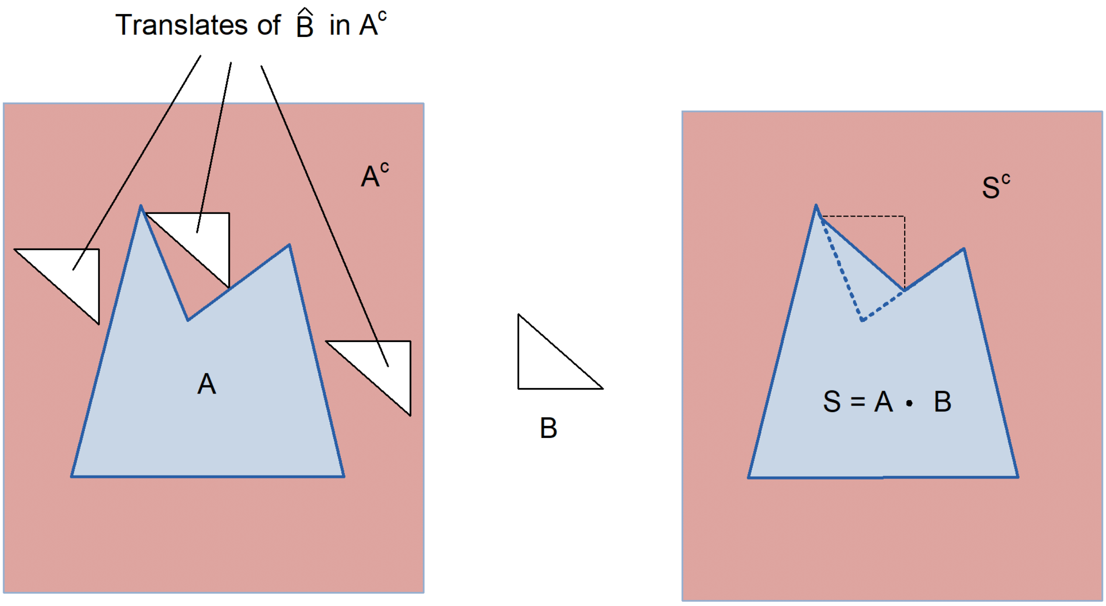


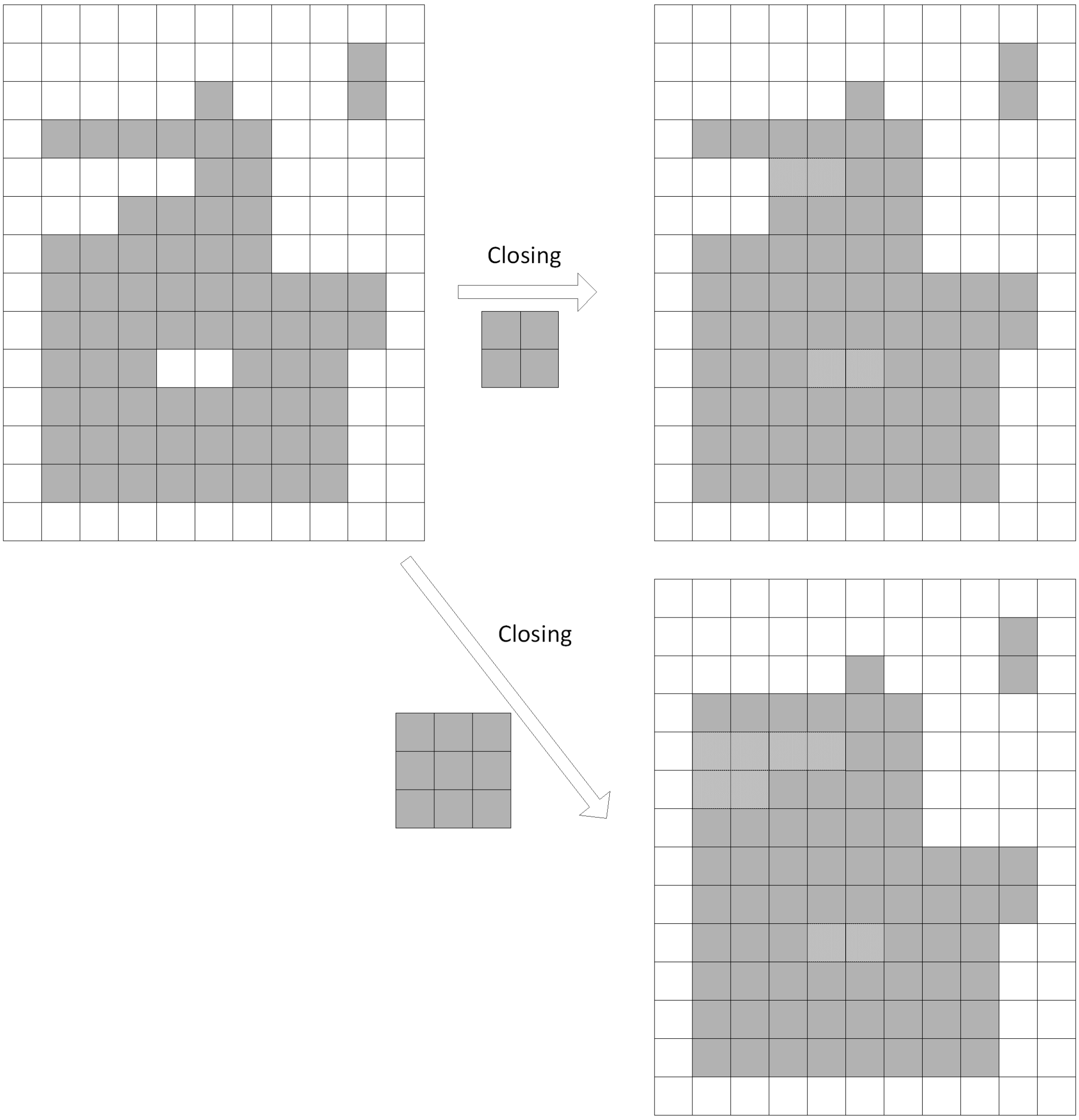

Opening
Opening

Closing has a similar geometric interpretation, except that now we translate $\hat{B}$ on the outside of the boundary. The boundary of $S = A \bullet B$ is then given by the points on the boundary of $\hat{B}$ that reach the farthest towards the boundary of A. Mathematically, we have

$$S^C = (A \bullet B)^c = \cup\{(\hat{B})_x \mid (\hat{B})_x \subseteq A^c\} \tag{18}$$
$$S = (A \bullet B) \tag{19}$$

Inward pointing corners become rounded whereas outward pointing corners remain unchanged. Intrusions may be reduced in size if the structuring element does not fit there.




Closing
Closing

MORPHOLOGICAL ALGORITHMS

One application of morphology is extracting image components that are useful in the representation and description of shape, e.g.,

- boundaries
- connected components
- convex hull
- skeleton of a region

Another application is in the implementation of processing techniques such as

- region filling
- thinning
- thickening
- pruning

In the following binary images, 1's are shown shaded and 0's shown in white.

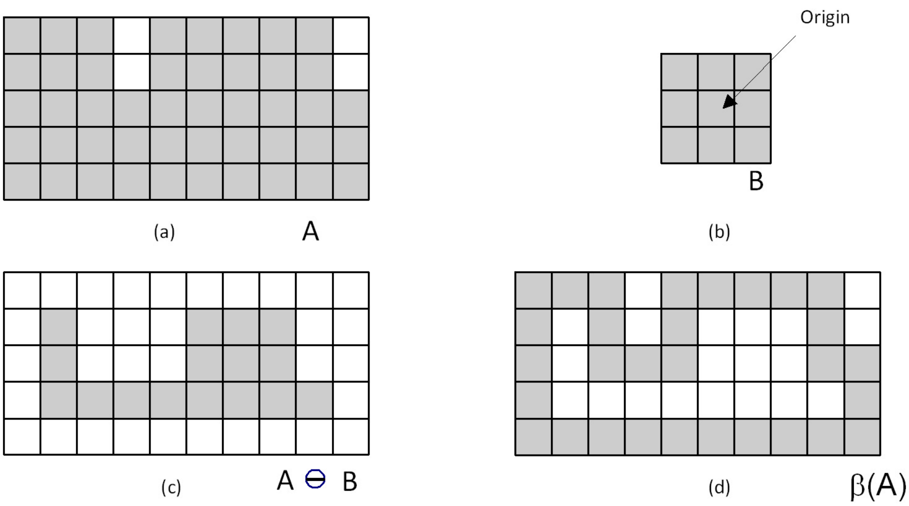
Boundary Extraction

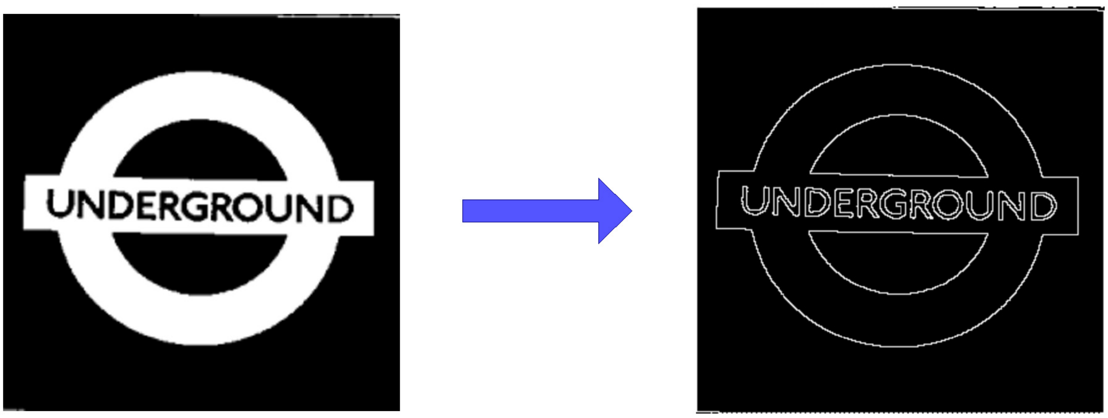
The boundary of a set A, denoted by $\beta(A)$, can be obtained by

$$\beta(A) = A - (A \ominus B) \tag{20}$$

i.e., first eroding A by a suitable structuring element B, and then performing the set difference between A and its erosion.

The 3×3 structuring element is commonly used. Other structuring elements may also be employed; e.g., with a 5×5 structuring element, the boundary would be between 2 and 3 pixels thick.





White pixel = logical 1

Region Filling

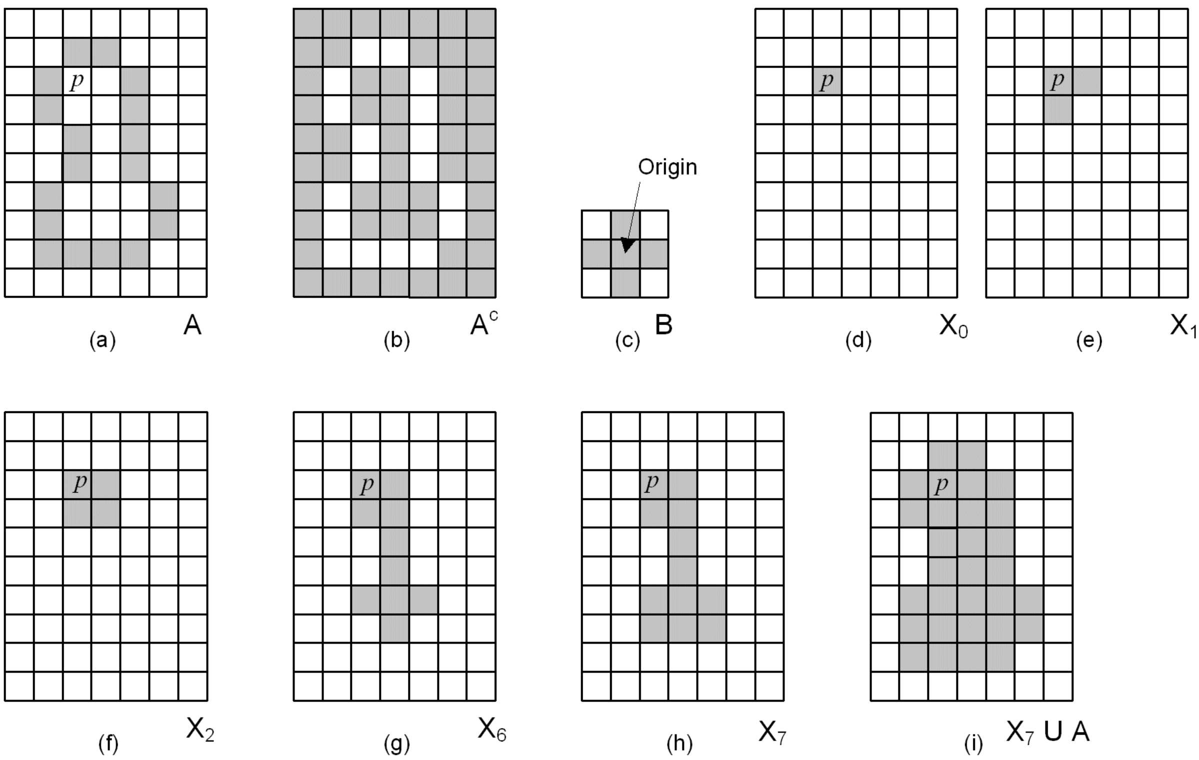
In (a), A denotes a set containing a subset whose elements are 8-connected boundary points of a region. Beginning with a point p inside the boundary, the objective is to fill the entire region with 1's.

Since all non-boundary points are labelled 0, we first assign a value of 1 to p. The following procedure then fills the region with 1's:

$$X_k = (X_{k-1} \oplus B) \cap A^c \qquad k = 1, 2, 3, \ldots \tag{21}$$

where $X_0 = p$, and B is the symmetric structuring element shown in (c).

The algorithm terminates at iteration step k if $X_k = X_{k-1}$. The set union of X_k and A contains the filled set and its boundary. The intersection at each step with A^c limits the result to the inside of the region of interest.



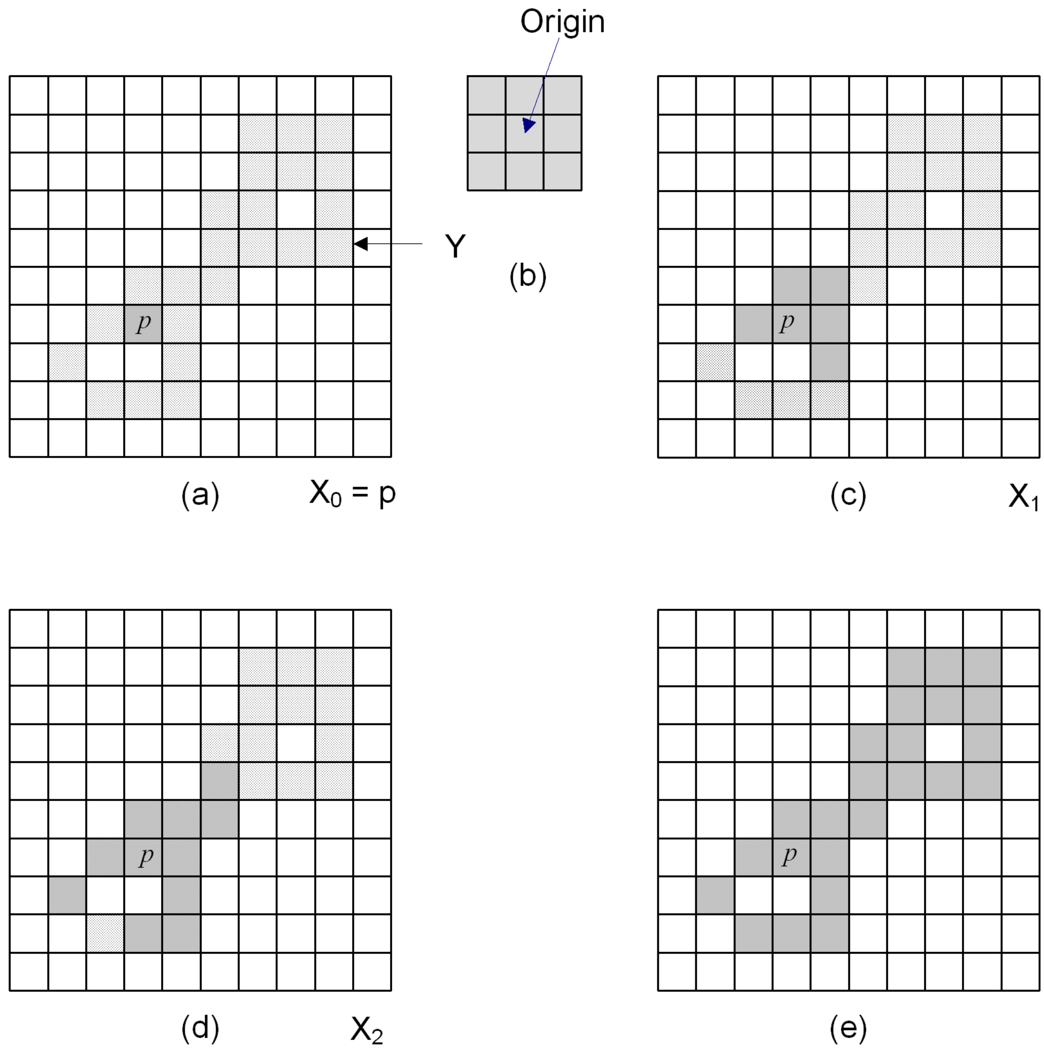
Extraction of Connected Components

Let Y represent a connected component. Assume that a point p of a connected component Y is known. All the elements of Y are obtained by

$$X_k = (X_{k-1} \oplus B) \cap Y \qquad k = 1, 2, 3, \ldots \tag{22}$$

where $X_0 = p$, and B is a suitable structuring element. If $X_k = X_{k-1}$, the algorithm has converged and we let $Y = X_k$.

The intersection with Y at each iterative step eliminates dilations centred on elements labelled 0.



Skeletons

The skeleton of a set (region) A can be expressed in terms of erosions and openings. With $S(A)$ denoting the skeleton of A, it can be shown that

$$S(A) = \bigcup_{k=0}^{K} S_k(A) \tag{23}$$

with

$$S_k(A) = (A \ominus kB) - [(A \ominus kB) \circ B] \tag{24}$$

where

B is a structuring element,

$(A \ominus kB)$ indicates k successive erosions of A; i.e.,

$$(A \ominus kB) = ((\ldots(A \ominus B) \ominus B) \ominus \ldots)B$$

k times, and

K is the last iterative step before A erodes to an empty set, i.e.,

$$K = \max\{k | (A \ominus kB) \neq \emptyset\}$$

$S(A)$ can be obtained as the union of the skeleton subsets $S_k(A)$. It can also be shown that A can be reconstructed from these subsets by using the equation

$$A = \bigcup_{k=0}^{K} (S_k(A) \oplus kB) \tag{25}$$

where

$(S_k(A) \oplus kB)$ denotes k successive dilations of $S_k(A)$; that is,

$$(S_k(A) \oplus kB) = ((\ldots(S_k(A) \oplus B) \oplus B) \oplus \ldots) \oplus B$$

k times, and

K is the limit of the summation as defined before.

$$
\begin{aligned}
& S_k(A) &=& \ (A \ominus kB) - [(A \ominus kB) \circ B] \\
k = 0: \quad & S_0(A) &=& \ A - (A \circ B) \\
k = 1: \quad & S_1(A) &=& \ (A \ominus B) - [(A \ominus B) \circ B] \\
k = 2: \quad & S_2(A) &=& \ (A \ominus 2B) - [(A \ominus 2B) \circ B]
\end{aligned}
$$

$$
\begin{aligned}
A &= \bigcup_{k=0}^{K} (S_k(A) \oplus kB) \\
&= S_0(A) \cup [S_1(A) \oplus B] \cup [S_2(A) \oplus 2B]
\end{aligned}
$$