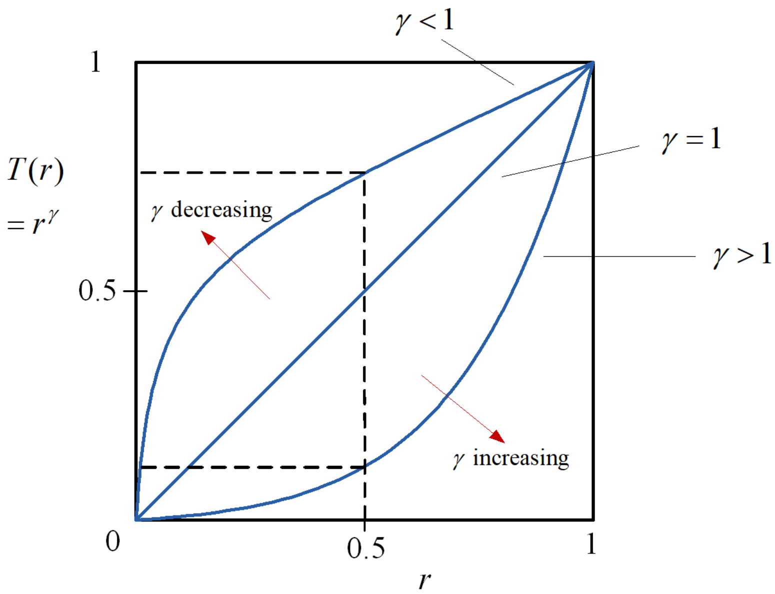
EE4704 Image Processing and Analysis

Tutorial Set D – Solutions

Question 1



- $0 < \gamma < 1$: The image tends to get brighter as the gray levels get transformed to higher levels.
- $\gamma = 1$: No change.
- $\gamma > 1$: The image tends to get darker as the gray levels get transformed to lower levels.

We obviously cannot just replace r by r_k to give

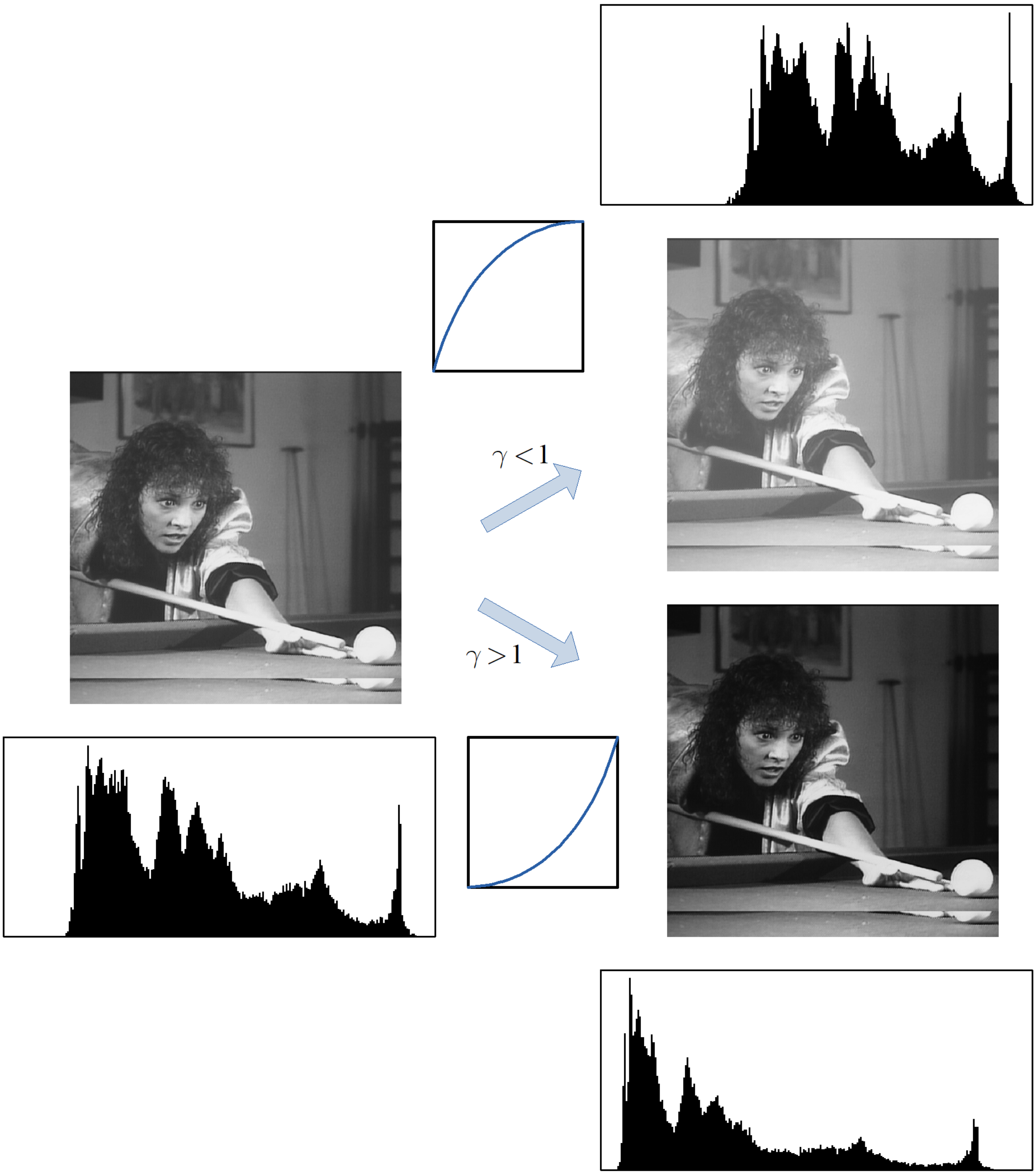
$$T_\gamma(r_k) = (r_k)^\gamma \tag{1}$$

since we have the requirement

$$r_k = 255 \;\rightarrow\; s_k = 255 \tag{2}$$

Hence, we normalise r_k by 255, then multiply by 255 after taking the exponential, i.e.,

$$s_k = T_\gamma(r_k) = 255\left(\frac{r_k}{255}\right)^\gamma \tag{3}$$

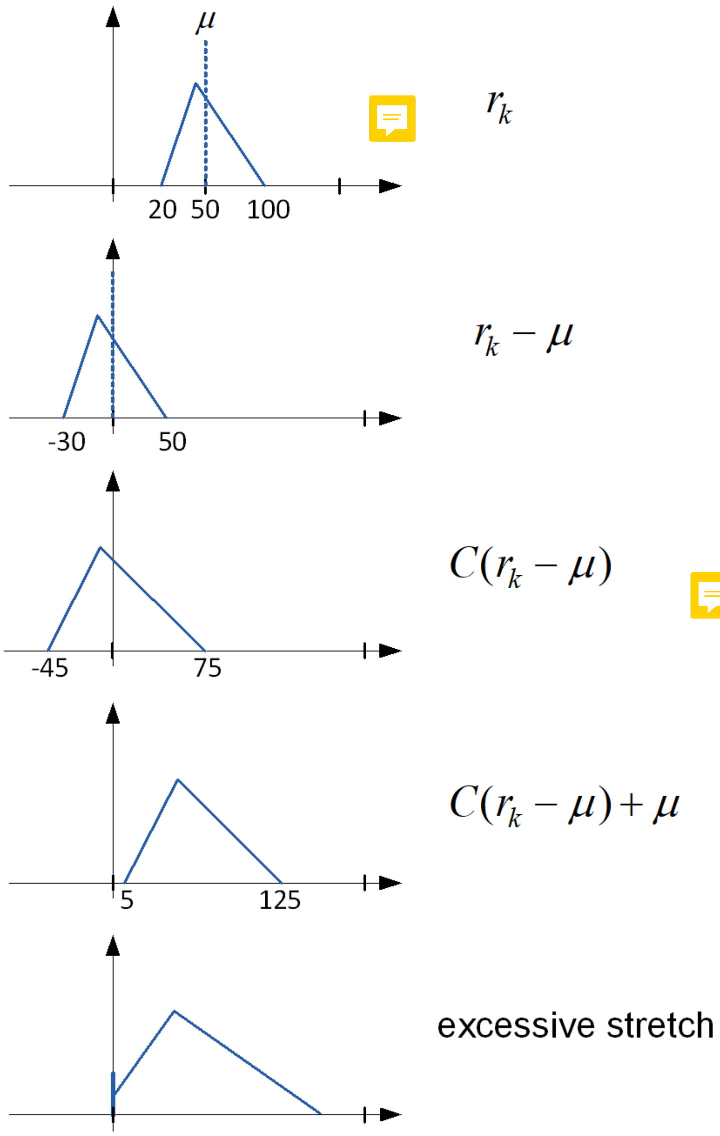

$\gamma < 1$
$\gamma > 1$

Question 2

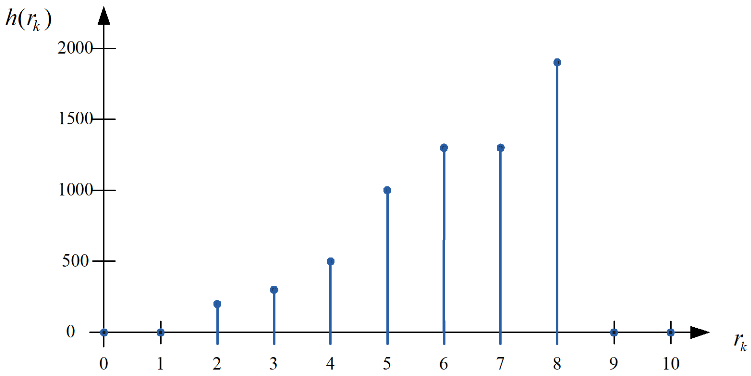
The transformation function is

$$s_k = C(r_k - \mu) + \mu \tag{4}$$

The histogram is shifted such that its mean is at the origin. It is then stretched and shifted back by the same amount. C is chosen such that the stretched histogram occupies the full range of gray levels (without significant clipping occurring).



Gray level:	0	1	2	3	4	5	6	7	8	9	10
Number of pixels:	0	0	200	300	500	1000	1300	1300	1800	0	0



We first compute the image mean

$$
\begin{aligned}
\mu &= \frac{1}{N}\sum_k n_k r_k \\
&= \frac{1}{6400}(0(0) + 1(0) + 2(200) + 3(300) + \ldots) \\
&= 6.19
\end{aligned}
$$

C has to satisfy two constraints:

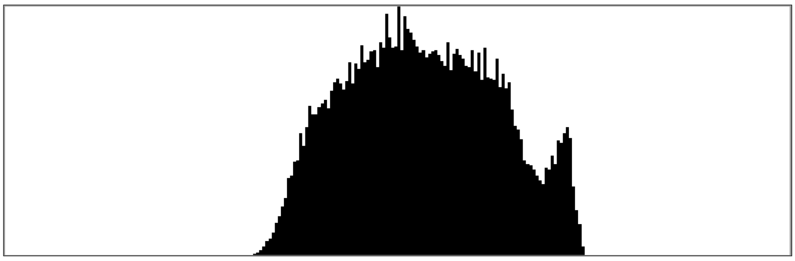
1. For $r_k = 2$, $s_k \geq 0$

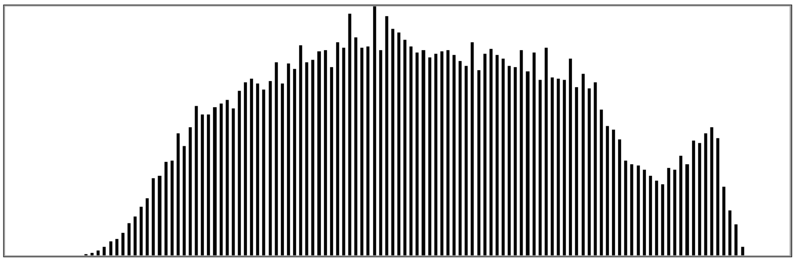
$$
\begin{aligned}
\text{i.e.} \quad C(2 - 6.19) + 6.19 &\geq 0 \\
\text{or} \quad C &\leq 1.5
\end{aligned}
$$

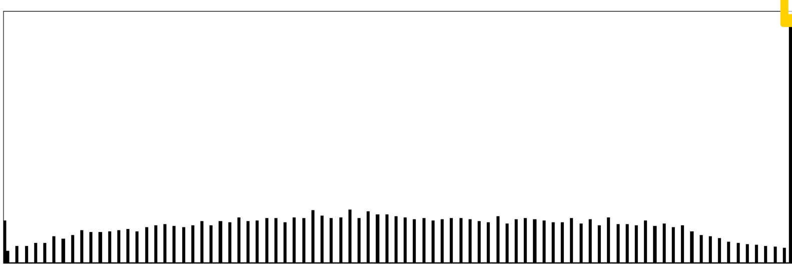
2. For $r_k = 8$, $s_k \leq 10$

$$
\begin{aligned}
\text{i.e.} \quad C(8 - 6.19) + 6.19 &\leq 10 \\
\text{or} \quad C &\leq 2.1
\end{aligned}
$$

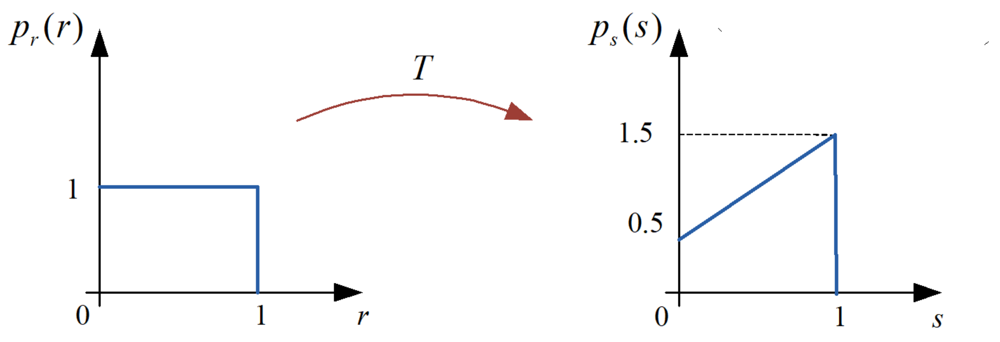
Therefore, we choose $C = 1.5$.





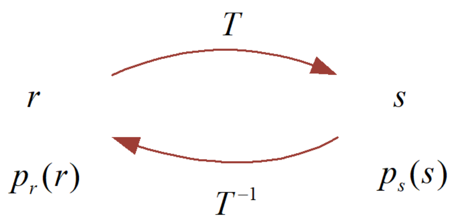


Question 3



$p_s(s) = s + 0.5$

We first find the transformation $T^{-1}(s)$ that equalises $p_s(s)$:

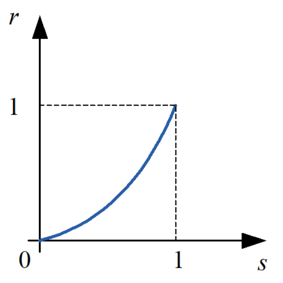


$$
\begin{aligned}
r &= T^{-1}(s) \\
&= \int_0^s p_s(w)dw \\
&= \int_0^s (w + 0.5)dw \\
&= \left[\frac{1}{2}w^2 + \frac{1}{2}w\right]_0^s \\
&= \frac{1}{2}s^2 + \frac{1}{2}s
\end{aligned}
$$

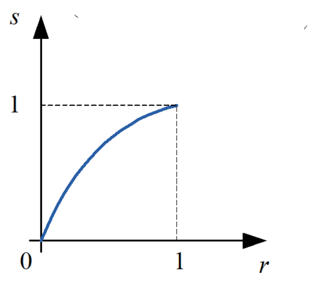
The transformation that is applied to $p_r(r)$ to give $p_s(s)$ is

$$
\begin{aligned}
s &= T(r) \\
&= -\frac{1}{2} \pm \frac{\sqrt{1+8r}}{2} \\
&= -\frac{1}{2} + \frac{\sqrt{1+8r}}{2}
\end{aligned}
$$

We take the positive square root so that $0 \leq s \leq 1$.



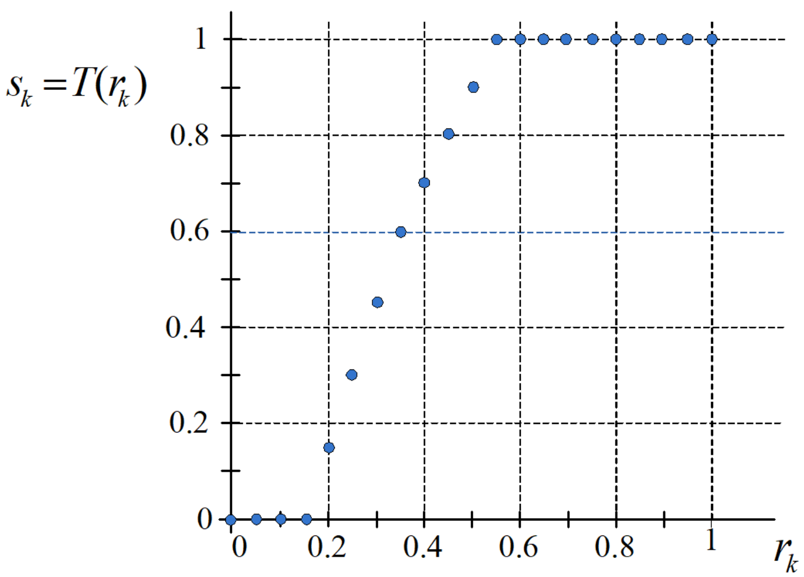
$r = T^{-1}(s)$

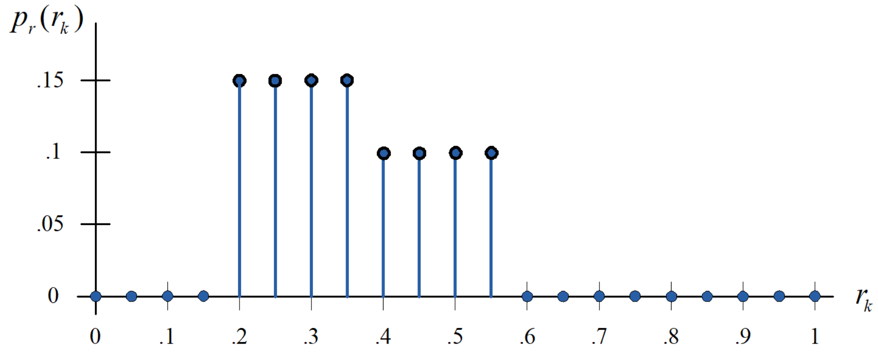


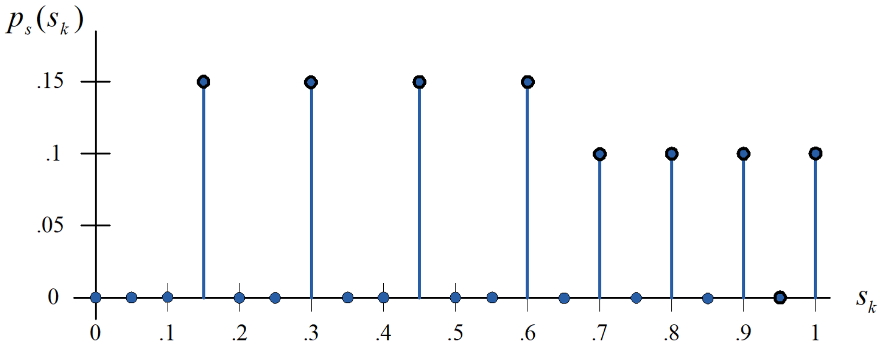
$s = T(r)$

Question 4

k	r_k	$p_r(r_k)$	s_k	$p_s(s_k)$
0	0	0	0	
1	0.05	0	0	
2	0.1	0	0	0
3	0.15	0	0	
4	0.2	0.15	0.15	0.15
5	0.25	0.15	0.3	0.15
6	0.3	0.15	0.45	0.15
7	0.35	0.15	0.6	0.15
8	0.4	0.1	0.7	0.1
9	0.45	0.1	0.8	0.1
10	0.5	0.1	0.9	0.1
11	0.55	0.1	1	
12	0.6	0	1	
13	0.65	0	1	
14	0.7	0	1	
15	0.75	0	1	
16	0.8	0	1	0.1
17	0.85	0	1	
18	0.9	0	1	
19	0.95	0	1	
20	1	0	1	




$p_r(r_k)$
.15
.1
.05
0
r_k
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1


$p_s(s_k)$
.15
.1
.05
0
s_k
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1

Question 5

In the frequency domain, the output image after one application of the filter is

$$
\begin{aligned}
G_1(u,v) &= H(u,v) \times F(u,v) && (5)\\
&= e^{-\omega^2/2\sigma^2} F(u,v) && (6)
\end{aligned}
$$

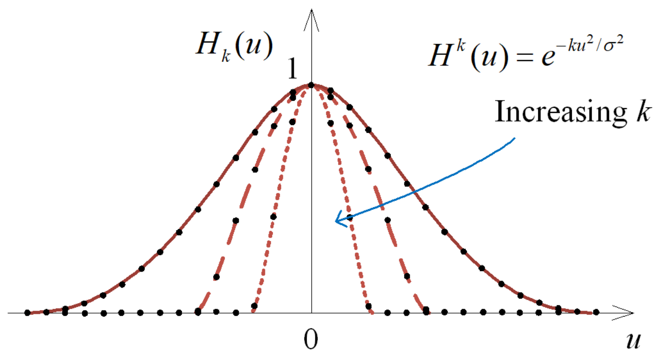
Applying the filter two times:

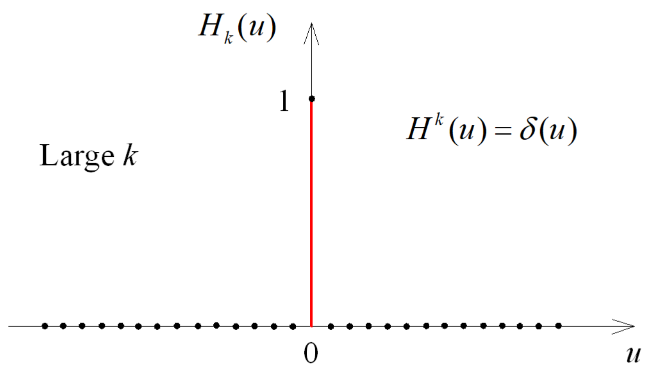
$$
\begin{aligned}
G_2(u,v) &= H(u,v) \times H(u,v) \times F(u,v) && (7)\\
&= H^2(u,v)F(u,v) && (8)\\
&= \left[e^{-2\omega^2/2\sigma^2}\right] F(u,v) && (9)
\end{aligned}
$$

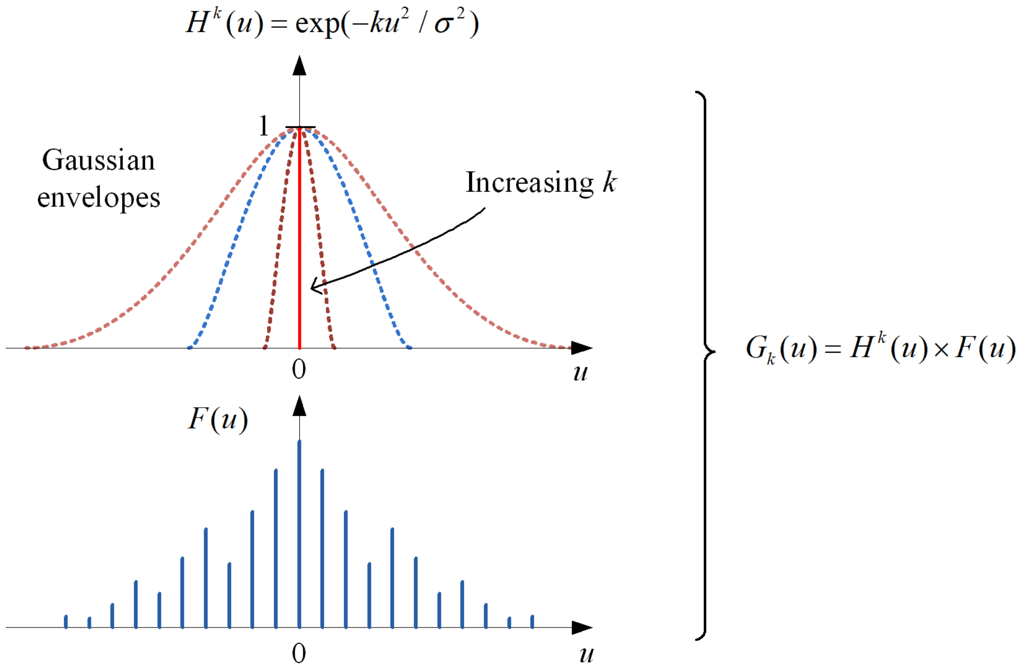
Applying the filter k times:

$$
\begin{aligned}
G_k(u,v) &= H^k(u,v)F(u,v) && (10)\\
&= [e^{-k\omega^2/2\sigma^2}]F(u,v) && (11)
\end{aligned}
$$

As k increases, $H^k(u,v) = \exp(-k\omega^2/2\sigma^2)$ tends towards an impulse, i.e., it is equal to 1 at $(0,0)$ and 0 elsewhere.







We have

$$H^k(u,v) = \exp(-k\omega^2/2\sigma^2) \tag{12}$$

For large k:

$$\begin{aligned} H^k(u,v) &\rightarrow \delta(u,v) && (13) \\ G_k(u,v)|_{k \text{ large}} &= F(u,v) \times \delta(u,v) && (14) \\ &= F(0,0)\delta(u,v) && (15) \\ &= \bar{f}(x,y)\delta(u,v) && (16) \end{aligned}$$

$G_k(u,v)$ is zero everywhere except at the origin where it is equal to $F(0,0) = \bar{f}(x,y)$. Hence, the output image is

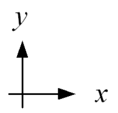
$$\begin{aligned} g_k(x,y)|_{k \text{ large}} &= \mathcal{F}^{-1}\left\{\bar{f}(x,y)\delta(u,v)\right\} && (17) \\ &= \bar{f}(x,y) && (18) \end{aligned}$$

which is a constant equal to the average value of the input image.

Question 6

Part(a)

0	0	0	0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0	0	0	0
0	0	1	1	1	0	0	0	0	0	0
0	0	1	1	1	1	0	0	0	0	0
0	0	1	1	1	1	1	0	0	0	0
0	0	1	1	1	1	1	1	0	0	0
0	0	1	1	1	1	1	1	1	0	0
0	0	0	0	0	0	0	0	0	0	0



$f(x,y)$

0	0	0	0	0	0	0	0	0	0	0
0	1	1	-1	-1	0	0	0	0	0	0
0	3	3	-2	-3	-1	0	0	0	0	0
0	4	4	-1	-3	-3	-1	0	0	0	0
0	4	4	0	-1	-3	-3	-1	0	0	0
0	4	4	0	0	-1	-3	-3	-1	0	0
0	4	4	0	0	0	-1	-3	-3	-1	0
0	3	3	0	0	0	0	-1	-3	-2	0
0	1	1	0	0	0	0	0	-1	-1	0

$G_x(x,y)$

0	0	0	0	0	0	0	0	0	0	0
0	-1	-3	-3	-1	0	0	0	0	0	0
0	-1	-3	-4	-3	-1	0	0	0	0	0
0	0	0	-1	-3	-3	-1	0	0	0	0
0	0	0	0	-1	-3	-3	-1	0	0	0
0	0	0	0	0	-1	-3	-3	-1	0	0
0	0	0	0	0	0	-1	-3	-3	-1	0
0	1	3	4	4	4	4	3	1	0	0
0	1	3	4	4	4	4	4	3	1	0
0	0	0	0	0	0	0	0	0	0	0

$G_y(x,y)$

0	0	0	0	0	0	0	0	0	0	0
0	1.4	3.2	3.2	1.4	0	0	0	0	0	0
0	3.2	4.2	4.5	4.2	1.4	0	0	0	0	0
0	4.0	4.2	1.4	4.2	4.2	1.4	0	0	0	0
0	4.0	4.0	0	1.4	4.2	4.2	1.4	0	0	0
0	4.0	4.0	0	0	1.4	4.2	4.2	1.4	0	0
0	4.0	4.0	0	0	0	1.4	4.2	4.2	1.4	0
0	3.2	4.2	4.0	4.0	4.0	4.0	3.2	3.2	2.0	0
0	1.4	3.2	4.0	4.0	4.0	4.0	4.0	3.2	1.4	0
0	0	0	0	0	0	0	0	0	0	0

Gradient magnitude

$$= \sqrt{(G_x)^2 + (G_y)^2}$$

x	x	x	x	x	x	x	x	x	x	x
x	-45	-72	-108	-135	x	x	x	x	x	x
x	-18	-45	-117	-135	-135	x	x	x	x	x
x	0	0	-135	-135	-135	-135	x	x	x	x
x	0	0	x	-135	-135	-135	-135	x	x	x
x	0	0	x	x	-135	-135	-135	-135	x	x
x	0	0	x	x	x	-135	-135	-135	-135	x
x	18	45	90	90	90	90	108	162	180	x
x	45	72	90	90	90	90	90	108	135	x
x	x	x	x	x	x	x	x	x	x	x

Gradient angle

$$= \operatorname{atan}\left(G_y / G_x\right)$$

(in degrees)

x: undefined

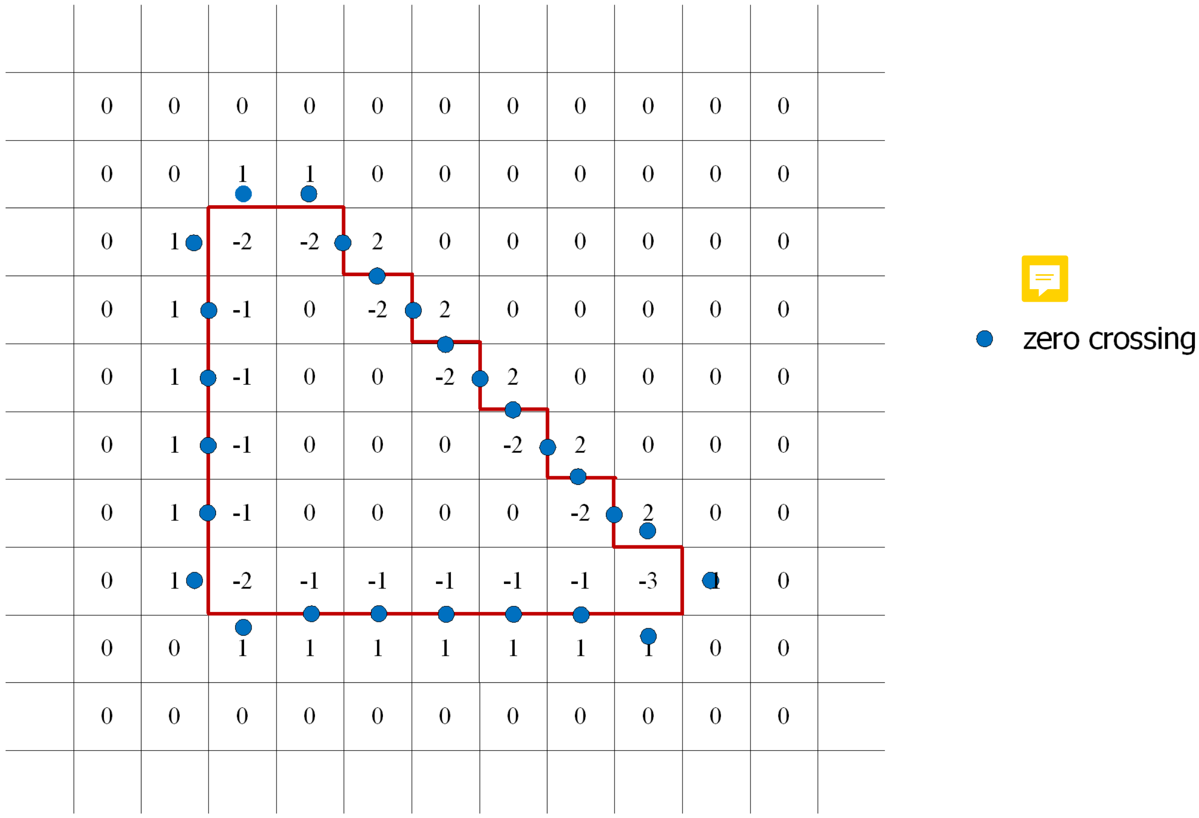
Note:

- edge strength depends on edge orientation
- diagonal edges give a stronger response compared to vertical and horizontal edges

Part (b)

0	0	0	0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0	0	0	0
0	1	-2	-2	2	0	0	0	0	0	0
0	1	-1	0	-2	2	0	0	0	0	0
0	1	-1	0	0	-2	2	0	0	0	0
0	1	-1	0	0	0	-2	2	0	0	0
0	1	-1	0	0	0	0	-2	2	0	0
0	1	-2	-1	-1	-1	-1	-1	-3	1	0
0	0	1	1	1	1	1	1	1	0	0
0	0	0	0	0	0	0	0	0	0	0

$g_L(x,y)$



Question 7

$$
\begin{aligned}
f(x,y) &= \exp(-ax^2 - by^2) && (19) \\
G_x(x,y) &= \frac{\partial f}{\partial x} && (20) \\
&= -2ax\exp(-ax^2 - by^2) && (21) \\
G_y(x,y) &= \frac{\partial f}{\partial y} && (22) \\
&= -2by\exp(-ax^2 - by^2) && (23)
\end{aligned}
$$

Hence

$$
\begin{aligned}
|\mathbf{G}(x,y)| &= [G_x^2 + G_y^2]^{1/2} && (24) \\
&= 2\sqrt{a^2x^2 + b^2y^2}\exp(-ax^2 - by^2) && (25) \\
\theta(x,y) &= \tan^{-1}(G_y/G_x) && (26) \\
&= \tan^{-1}(by/ax) && (27) \\
\mathbf{G}(x,y) &= 2\sqrt{a^2x^2 + b^2y^2}\exp(-ax^2 - by^2)\ \angle\tan^{-1}(by/ax) && (28)
\end{aligned}
$$