EE4704 Image Processing and Analysis

Tutorial Set B – Solution

Question 1

Part (a)

(i) The FT of $\delta(x, y)$ is

$$\begin{aligned}
\mathcal{F}\{\delta(x,y)\} &= \iint_{-\infty}^{+\infty} \delta(x,y)\exp[-j2\pi(ux+vy)]\,dx\,dy \quad \text{(by definition)} \\
&= \exp[-j2\pi(ux+vy)]\big|_{x,y=0} \\
&= 1
\end{aligned}$$

i.e.,

$$\delta(x,y) \leftrightarrow 1$$

(ii) The inverse FT of $\delta(u, v)$ is

$$\begin{aligned}
\mathcal{F}^{-1}\{\delta(u,v)\} &= \iint_{-\infty}^{+\infty} \delta(u,v)\exp[j2\pi(ux+vy)]\,du\,dv \quad \text{(by definition)} \\
&= \exp[j2\pi(ux+vy)]\big|_{u,v=0} \\
&= 1
\end{aligned}$$

Taking the FT of both sides,

$$\delta(u,v) = \mathcal{F}\{1\}$$

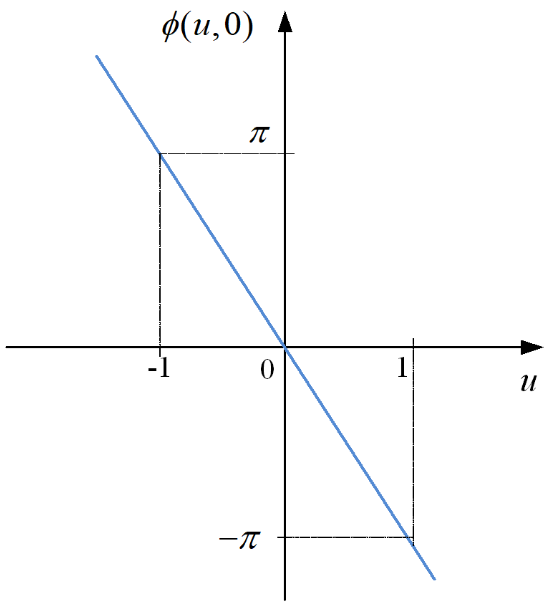
i.e.,

$$1 \leftrightarrow \delta(u,v)$$

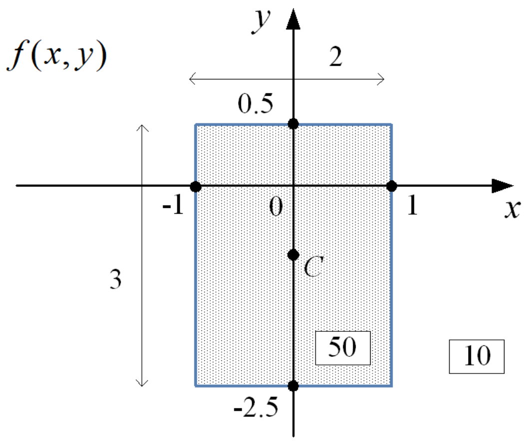
Part (b)

$$
\begin{aligned}
\mathcal{F}\{\delta(x-0.5, y+0.2)\} &= (1)\exp[-j2\pi(au+bv)]|_{a=0.5,\ b=-0.2} \\
&= \exp[-j2\pi(0.5u-0.2v)] \\
&= e^{-j\pi u}e^{j0.4\pi v}
\end{aligned}
$$

$$
\begin{aligned}
\phi(u,v) &= -\pi u + 0.4\pi v \\
\phi(u,0) &= -\pi u
\end{aligned}
$$



Question 2



The image function $f(x, y)$ can be regarded as the sum of two component functions $f_1(x, y)$ and $f_2(x, y)$:

$$f(x, y) = f_1(x, y) + f_2(x, y)$$

where

$$\begin{aligned} f_1(x,y) &= 10 \quad \text{for all} \quad x, y \\ f_2(x,y) &= \begin{cases} 40 & -1 \le x \le +1, \; -2.5 \le y \le +0.5 \\ 0 & \text{otherwise} \end{cases} \\ &= 40\,\mathrm{rect}(x/2, (y+1)/3) \end{aligned}$$

Method I - (not recommended)

$$\begin{aligned} F(u,v) &= \mathcal{F}\{f(x,y)\} \\ &= \int\!\!\int_{-\infty}^{+\infty} 10 \exp[-j2\pi(ux+vy)]dxdy \\ &\quad + \int_{-2.5}^{0.5}\int_{-1}^{1} 40 \exp[-j2\pi(ux+vy)]dxdy \\ &= \ldots \\ \text{Fourier spectrum} &= |F(u,v)| \end{aligned}$$

Method II

Use $|F(u,v)| = |\mathcal{F}\{f(x,y)\}| = |\mathcal{F}\{f(x,y-1)\}|$

We have

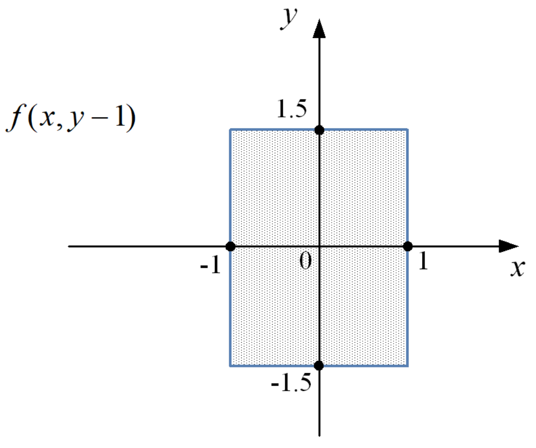
$$f(x,y) = f_1(x,y) + f_2(x,y)$$

Hence,

$$\begin{aligned}
f(x,y-1) &= f_1(x,y) + f_2(x,y-1) \\
\mathcal{F}\{f(x,y-1)\} &= \iint\limits_{-\infty}^{+\infty} 10\exp[-j2\pi(ux+vy)]\,dxdy \\
&\qquad +40\int_{-1.5}^{+1.5}\int_{-1}^{1}\exp[-j2\pi(ux+vy)]\,dxdy \\
&= 10\delta(u,v) + 40\left[\frac{\exp(-j2\pi ux)}{-j2\pi u}\right]_{-1}^{+1}\left[\frac{\exp(-j2\pi vy)}{-j2\pi v}\right]_{-1.5}^{+1.5} \\
&= 10\delta(u,v) + 40[2\text{sinc}(2u)]\,[3\text{sinc}(3v)] \\
&= 10\delta(u,v) + 240\text{sinc}(2u)\text{sinc}(3v)
\end{aligned}$$

The Fourier spectrum of $f(x,y)$ is

$$\begin{aligned}
|F(u,v)| &= |\mathcal{F}\{f(x,y)\}| \\
&= |\mathcal{F}\{f(x,y-1)\}| \\
&= 10\delta(u,v) + 240|\text{sinc}(2u)||\text{sinc}(3v)|
\end{aligned}$$



Method III

We have

$$f(x,y) = f_1(x,y) + f_2(x,y)$$

where

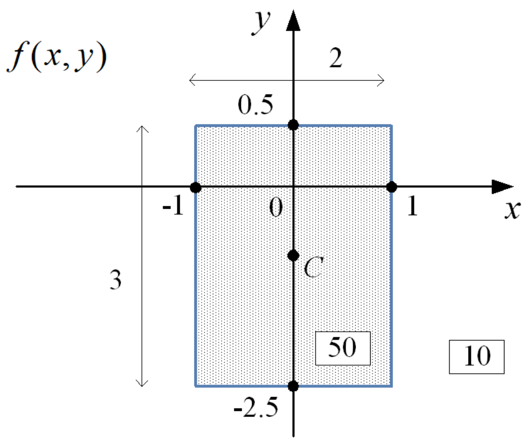
$$f_1(x,y) = 10, \quad f_2(x,y) = 40\,\mathrm{rect}\left(\frac{x}{2}, \frac{y+1}{3}\right)$$

$$\begin{aligned} F(u,v) &= \mathcal{F}\{f_1(x,y)\} + \mathcal{F}\{f_2(x,y)\} \\ &= F_1(u,v) + F_2(u,v) \end{aligned}$$

$$F_1(u,v) = \mathcal{F}\{10\} = 10\delta(u,v)$$

$$\begin{aligned} F_2(u,v) &= \mathcal{F}\left\{40\,\mathrm{rect}\left(\frac{x}{2}, \frac{y+1}{3}\right)\right\} = 40(2)(3)\mathrm{sinc}(2u,3v)e^{-j2\pi(-1)v} \\ |F_2(u,v)| &= 240\,|\mathrm{sinc}(2u,3v)| \end{aligned}$$

$$\begin{aligned} |F(u,v)| &= |F_1(u,v) + F_2(u,v)| \\ &= |10\delta(u,v) + 240\,\mathrm{sinc}(2u,3v)| \\ &= 10\delta(u,v) + 240\,|\mathrm{sinc}(2u,3v)| \end{aligned}$$



Question 3

$$\begin{aligned}
\mathcal{F}^{-1}\{\delta(u-a,v-b)\} &= \int\int \delta(u-a,v-b)\exp[j2\pi(ux+vy)]dudv \\
&= \exp[j2\pi(ux+vy)]|_{u=a,v=b} \\
&= \exp[j2\pi(ax+by)]
\end{aligned}$$

Similarly,

$$\mathcal{F}^{-1}\{\delta(u+a,v+b)\} = \exp[-j2\pi(ax+by)]$$

Hence,

$$\begin{aligned}
\mathcal{F}^{-1}\{\delta(u-a,v-b)+\delta(u+a,v+b)\} &= \exp[j2\pi(ax+by)]+\exp[-j2\pi(ax+by)] \\
&= 2\cos 2\pi(ax+by)
\end{aligned}$$

Thus,

$$\cos 2\pi(ax+by) \leftrightarrow \frac{1}{2}\delta(u-a,v-b)+\frac{1}{2}\delta(u+a,v+b) \tag{1}$$

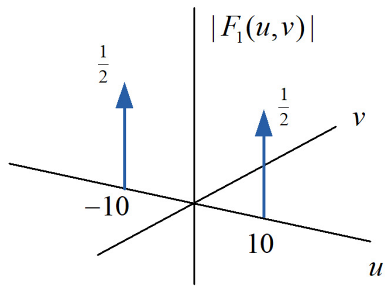
Part (a)

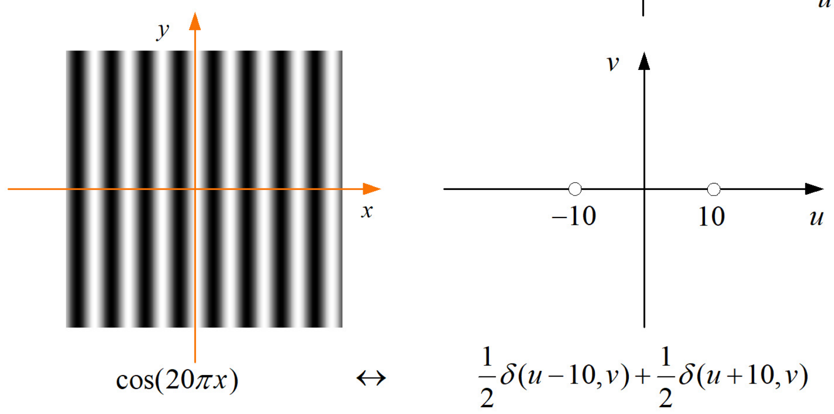
From Eq. (1), we let $a=10$ and $b=0$:

$$f_1(x,y)=\cos(20\pi x) \leftrightarrow \frac{1}{2}\delta(u-10,v)+\frac{1}{2}\delta(u+10,v)$$

Hence, the Fourier spectrum of $f_1(x,y)$ is

$$|F_1(u,v)| = \frac{1}{2}\delta(u-10,v)+\frac{1}{2}\delta(u+10,v)$$





Part (b)

We first obtain the Fourier transform of $\sin 2\pi(ax+by)$. Similar to Eq. (1),

$$\begin{aligned}\mathcal{F}^{-1}\{\delta(u-a,v-b)-\delta(u+a,v+b)\} &= \exp[j2\pi(ax+by)]-\exp[-j2\pi(ax+by)]\\ &= 2j\sin 2\pi(ax+by)\end{aligned}$$

Hence,

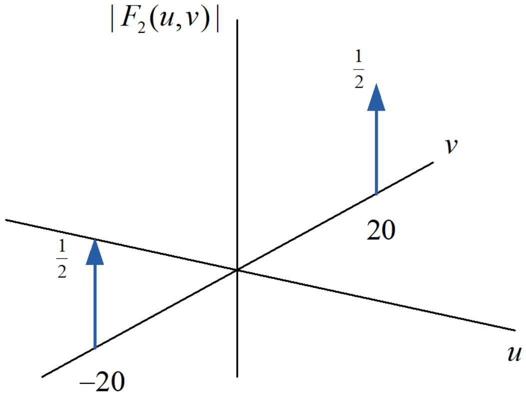
$$\begin{aligned}\sin 2\pi(ax+by) &\leftrightarrow \frac{1}{2j}\delta(u-a,v-b)-\frac{1}{2j}\delta(u+a,v+b) &(2)\\ &= -\frac{j}{2}\delta(u-a,v-b)+\frac{j}{2}\delta(u+a,v+b) &(3)\end{aligned}$$

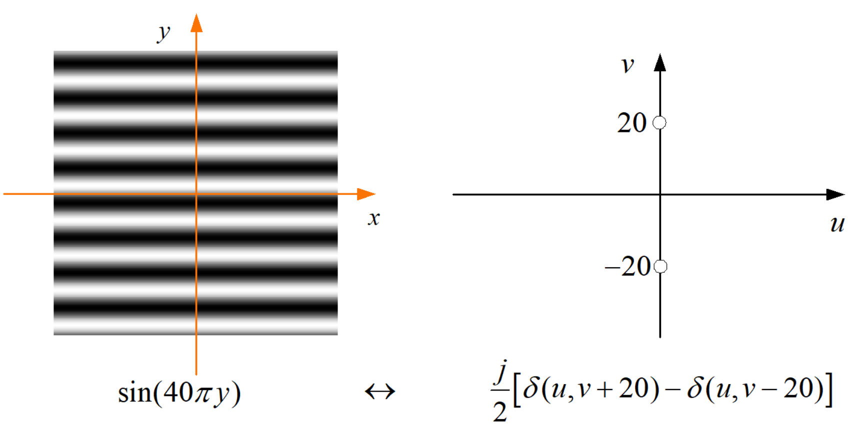
From Eq. (3), we let $a=0$ and $b=20$:

$$f_2(x,y)=\sin(40\pi y) \leftrightarrow \frac{j}{2}\delta(u,v+20)-\frac{j}{2}\delta(u,v-20)$$

Hence, the Fourier spectrum of $f_2(x,y)$ is

$$|F_2(u,v)|=\frac{1}{2}\delta(u,v+20)+\frac{1}{2}\delta(u,v-20)$$





Part (c)

$$\begin{aligned} f_3(x,y) &= \sin(30x+40y) \\ &= \sin 2\pi\left(\frac{15}{\pi}x+\frac{20}{\pi}y\right) \\ F_3(u,v) &= \frac{j}{2}\delta\left(u+\frac{15}{\pi},v+\frac{20}{\pi}\right)-\frac{j}{2}\delta\left(u-\frac{15}{\pi},v-\frac{20}{\pi}\right) \end{aligned}$$

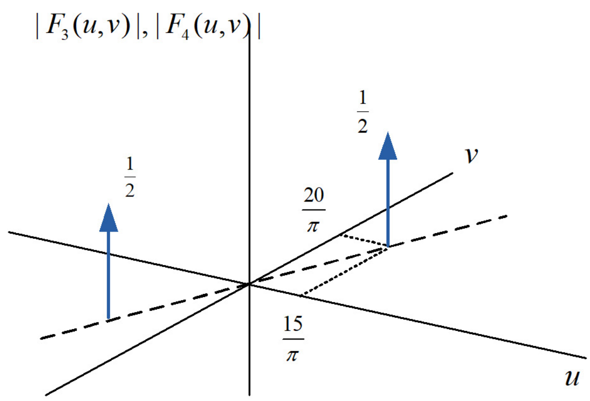
Hence, the Fourier spectrum of $f_3(x,y)$ is

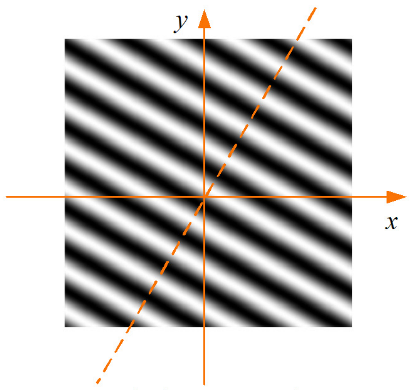
$$|F_3(u,v)| = \frac{1}{2}\delta\left(u+\frac{15}{\pi},v+\frac{20}{\pi}\right)+\frac{1}{2}\delta\left(u-\frac{15}{\pi},v-\frac{20}{\pi}\right) \tag{4}$$

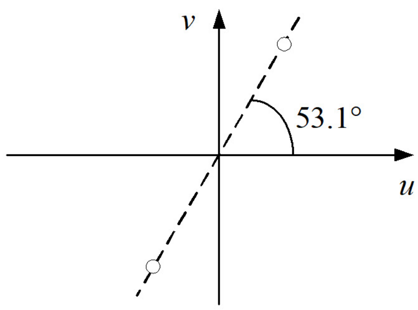
Part (d)

$f_4(x,y) = \sin(30x+40y+30)$ is a translated version of $f_3(x,y)$. Hence

$$|F_4(u,v)| = |F_3(u,v)|$$







$$\sin(30x+40y) \quad \leftrightarrow \quad \frac{j}{2}\left[\delta\left(u+\frac{15}{\pi},v+\frac{20}{\pi}\right)-\delta\left(u-\frac{15}{\pi},v-\frac{20}{\pi}\right)\right]$$

Question 4

Consider the transform $F(u)$ shifted by u_o, i.e.,

$$F_t(u) = F(u - u_0)$$

The inverse DFT of $F_t(u)$ is

$$\begin{aligned} f_t(x) &= \mathcal{F}^{-1}\{F(u-u_0)\} \\ &= f(x)\exp(j2\pi u_0 x/N) \end{aligned}$$

from the translation property of the DFT (see below).

We wish to shift the transform $F(u)$ to the right by $N/2$:

$$F(u) \rightarrow F(u - N/2)$$

i.e., $u_0 = N/2$. Shifting $F(u)$ by u_0 is obtained by multiplying $f(x)$ by $\exp(j2\pi u_0 x/N)$. In this case, the exponential term is

$$\begin{aligned} \exp(j2\pi u_0 x/N) &= \exp\left(j2\pi \frac{N}{2}\frac{x}{N}\right) \\ &= \exp(j\pi x) \\ &= (e^{j\pi})^x \\ &= (-1)^x \end{aligned}$$

Thus,

$$f_t(x) = f(x)(-1)^x$$

and

$$\mathcal{F}\{f(x)(-1)^x\} = F(u - N/2)$$

Proof of translation property (1D)

$$\begin{aligned} F(u) &= \frac{1}{N}\sum_{x=0}^{N-1} f(x)\, e^{-j2\pi ux/N} \\ F(u-u_0) &= \frac{1}{N}\sum_{x=0}^{N-1} f(x)\, e^{-j2\pi(u-u_0)x/N} \\ &= \frac{1}{N}\sum_{x=0}^{N-1} \{f(x)\, e^{j2\pi u_0 x/N}\}\, e^{-j2\pi ux/N} \\ &= \mathcal{F}\{f(x)\, e^{j2\pi u_0 x/N}\} \end{aligned}$$

Question 5

Given

$$f(x) = 1, 1, 1, 1, 0, 0, 0, 0$$

Part (a)

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\exp(-j2\pi ux/N) = \frac{1}{8}\sum_{x=0}^{3}\exp(-j2\pi ux/8)$$

$$\begin{aligned}
F(0) &= \frac{1}{8}\sum_{x=0}^{3}\exp(0) = 0.5 \\
F(1) &= \frac{1}{8}\sum_{x=0}^{3}\exp(-j2\pi(1)x/8) \\
&= \frac{1}{8}\exp(0) + \frac{1}{8}\exp(-j2\pi/8) + \frac{1}{8}\exp(-j2\pi\times 2/8) \\
&\qquad + \frac{1}{8}\exp(-j2\pi\times 3/8) \\
&= 0.327\angle -1.18
\end{aligned}$$

$$\begin{aligned}
F(2) &= \frac{1}{8}\sum_{x=0}^{3}\exp(-j2\pi(2)x/8) \\
&= \frac{1}{8}\exp(0) + \frac{1}{8}\exp(-j2\pi\times 2/8) + \frac{1}{8}\exp(-j2\pi\times 4/8) \\
&\qquad + \frac{1}{8}\exp(-j2\pi\times 6/8) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
F(3) &= \frac{1}{8}\sum_{x=0}^{3}\exp(-j2\pi(3)x/8) \\
&= \frac{1}{8}\exp(0) + \frac{1}{8}\exp(-j2\pi\times 3/8) + \frac{1}{8}\exp(-j2\pi\times 6/8) \\
&\qquad + \frac{1}{8}\exp(-j2\pi\times 9/8) \\
&= 0.135\angle -0.393
\end{aligned}$$

$$
\begin{aligned}
F(4) &= \frac{1}{8}\sum_{x=0}^{3}\exp(-j2\pi(4)x/8) \\
&= \frac{1}{8}\exp(0) + \frac{1}{8}\exp(-j2\pi\times 4/8) + \frac{1}{8}\exp(-j2\pi\times 8/8) \\
&\qquad + \frac{1}{8}\exp(-j2\pi\times 12/8) \\
&= 0
\end{aligned}
$$

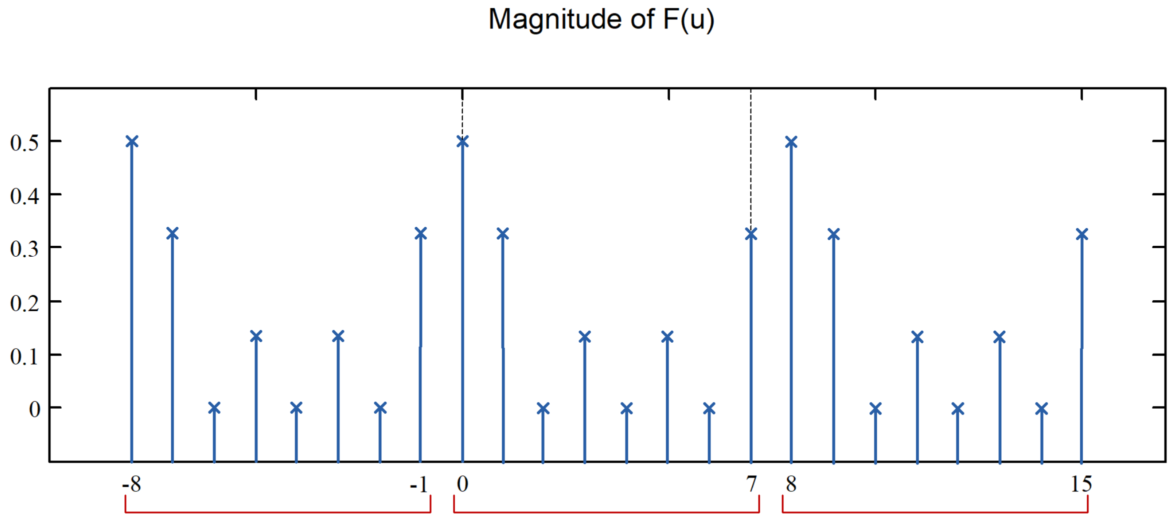
$$
\begin{aligned}
F(5) &= F^*(-5) \quad \text{(conjugate symmetry)} \\
&= F^*(8-5) \quad \text{(periodicity)} \\
&= F^*(3) \\
&= 0.135\angle + 0.393
\end{aligned}
$$

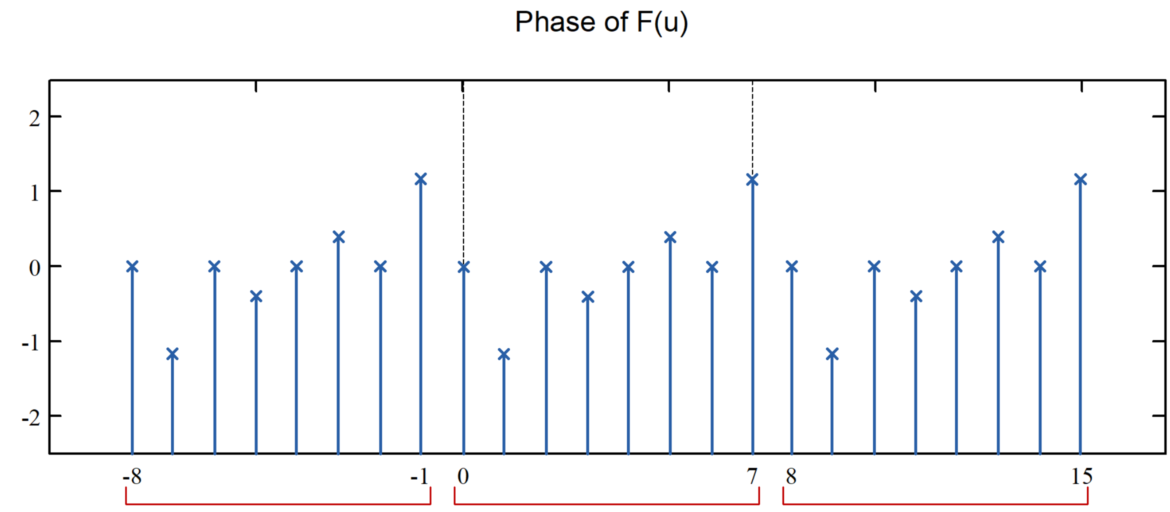
$$
\begin{aligned}
F(6) &= F^*(2) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
F(7) &= F^*(1) \\
&= 0.327\angle 1.18
\end{aligned}
$$

Part (b)

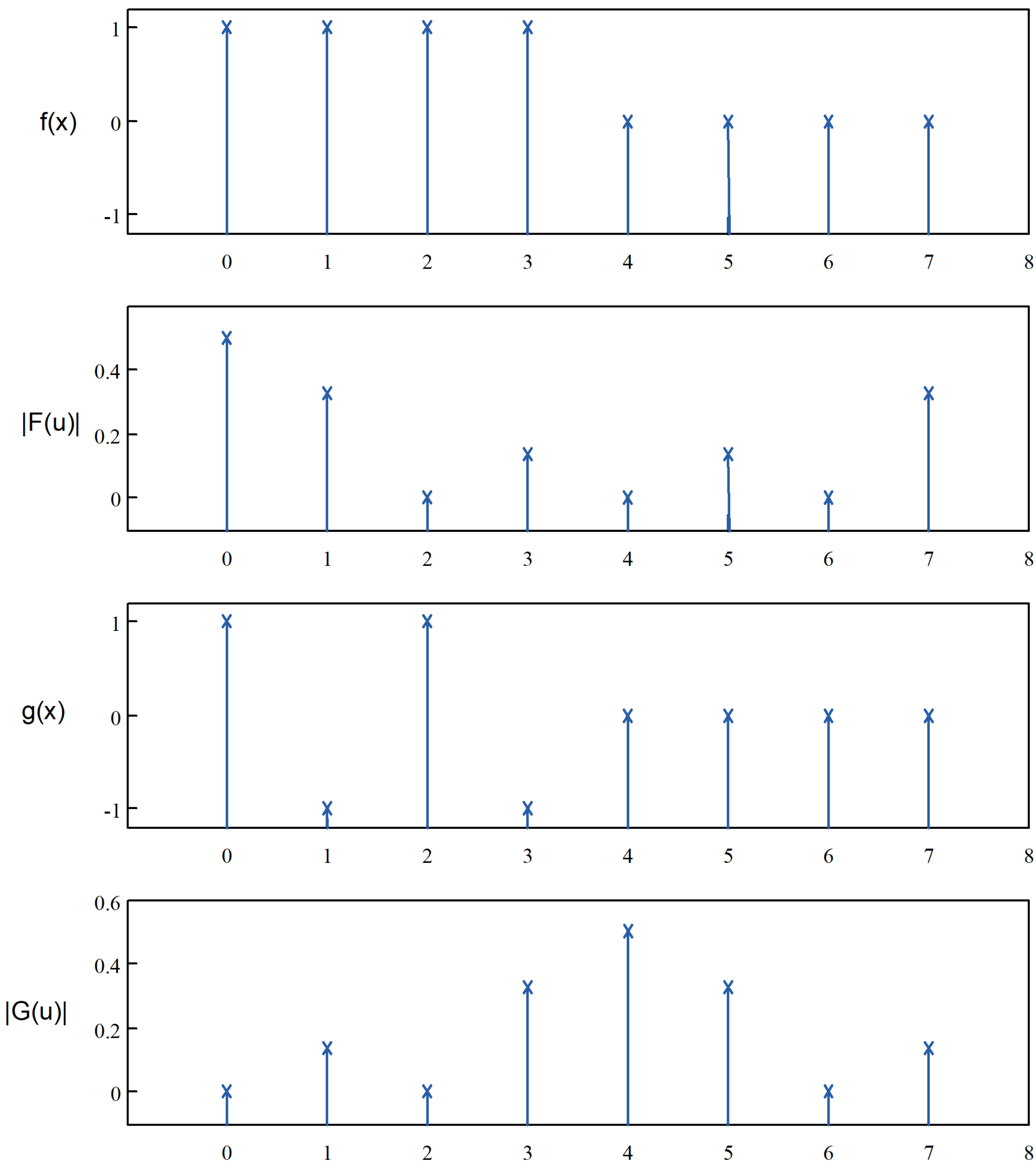
$$
\begin{aligned}
f(x) &= 1, \quad 1, \quad 1, \quad 1, \quad 0, \quad 0, \quad 0, \quad 0 \\
F(u) &= 0.5 \quad 0.327\angle-1.18, \quad 0, \quad 0.135\angle-0.393 \\
&\quad\; 0, \quad 0.135\angle+0.393, \quad 0, \quad 0.327\angle 1.18
\end{aligned}
$$





Part (c)

$$
\begin{aligned}
g(x) &= (-1)^x f(x) = +1, \quad -1, \quad +1, \quad -1 \quad 0, \quad 0, \quad ,0, \quad 0 \\
G(u) &= 0, \quad 0.135\angle 0.393, \quad 0, \quad 0.327\angle 1.18, \\
&\quad 0.5, \quad 0.327\angle -1.18, \quad 0, \quad 0.135\angle -0.393
\end{aligned}
$$



Question 6

Part (a)

$$\begin{aligned}
f_1(x) &= 1, 1, 1, 1 \\
F_1(u) &= \frac{1}{N}\sum_{x=0}^{N-1} f_1(x)\exp(-j2\pi ux/N) = \frac{1}{4}\sum_{x=0}^{3}\exp(-j2\pi ux/4) \\
F_1(0) &= \frac{1}{4}\sum_{x=0}^{3}\exp(0) = \frac{1}{4}\times 4 = 1 \\
F_1(1) &= \frac{1}{4}\sum_{x=0}^{3}\exp(-j2\pi(1)x/4) = \frac{1}{4}\times 0 = 0 \\
F_1(2) &= \frac{1}{4}\sum_{x=0}^{3}\exp(-j2\pi(2)x/4) = \frac{1}{4}\times 0 = 0 \\
F_1(3) &= \frac{1}{4}\sum_{x=0}^{3}\exp(-j2\pi(3)x/4) = \frac{1}{4}\times 0 = 0
\end{aligned}$$

Hence,

$$\begin{aligned}
F_1(u) &= 1, 0, 0, 0 \\
&= \delta(u)
\end{aligned}$$

Part(b)

$$\begin{aligned}
f_2(x) &= 1, 0, 0, 0 \\
F_2(u) &= \frac{1}{N}\sum_{x=0}^{N-1} f_2(x)\exp(-j2\pi ux/N) \\
&= \frac{1}{4}\sum_{x=0}^{0} 1 \\
&= \frac{1}{4}
\end{aligned}$$

$$F_2(0) = \frac{1}{4}, \qquad F_2(1) = \frac{1}{4}, \qquad F_2(2) = \frac{1}{4}, \qquad F_2(3) = \frac{1}{4}$$

Hence,

$$F(u) = \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$$

Part (c)

From the periodicity property, we note that

$$f_3(x) = f_2(x-1)$$

Hence,

$$\begin{aligned} F_3(u) &= F_2(u)\exp(-j2\pi ua/4) \quad \text{(translation property)} \\ &= F_2(u)\exp(-j\pi u/2) \end{aligned}$$

$$\begin{aligned} F_3(0) &= F_2(0)\exp(0) = F_2(0) \\ F_3(1) &= F_2(1)\exp(-j\pi/2) = -jF_2(1) \\ F_3(2) &= F_2(2)\exp(-j\pi) = -F_2(2) \\ F_3(3) &= F_2(3)\exp(-j3\pi/2) = jF_2(3) \end{aligned}$$

$$F_3(u) = \frac{1}{4}, -\frac{j}{4}, -\frac{1}{4}, \frac{j}{4}$$

Part (d)

$$f_4(x,y) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$F_4(x,v) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$F_4(u,v) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that $F_4(u,v) = \delta(u,v)$.

Part (e)

$$f_5(x,y) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F_5(x,v) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F_5(u,v) = \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Part (f)

$f_5(x,y)$

0	0	0	0	0	0	0
0	0	0	0	0	0	0
1	1	1	1	1	1	1
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
1	1	1	1	1	1	1
0	0	0	0	0	0	0

y x

$f_6(x,y) = f_5(x-1, y-2)$

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	**0**	0	0	0	0	0	0
1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1

y x

$$f_6(x,y) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F_6(x,v) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F_6(u,v) = \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ -j & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ +j & 0 & 0 & 0 \end{bmatrix}$$

Question 7

Step 1 – Compute the DFT $\rightarrow F(u,v) = \mathcal{F}\{f(x,y)\}$
Step 2 – Take the complex conjugate of the transform $\rightarrow G(u,v) = F^*(u,v)$
Step 3 – Compute the inverse DFT $\rightarrow g(x,y) = \mathcal{F}^{-1}\{G(u,v)\}$

$$\begin{aligned} F(u,v) &= \frac{1}{200^2}\sum_{x=0}^{199}\sum_{y=0}^{199} f(x,y)\exp[-j2\pi(ux/200 + vy/200)] && (5)\\ G(u,v) &= F^*(u,v) && (6)\\ g(x,y) &= \sum_{u=0}^{199}\sum_{v=0}^{199} F^*(u,v)\exp[j2\pi(ux/200 + vy/200)] && (7) \end{aligned}$$

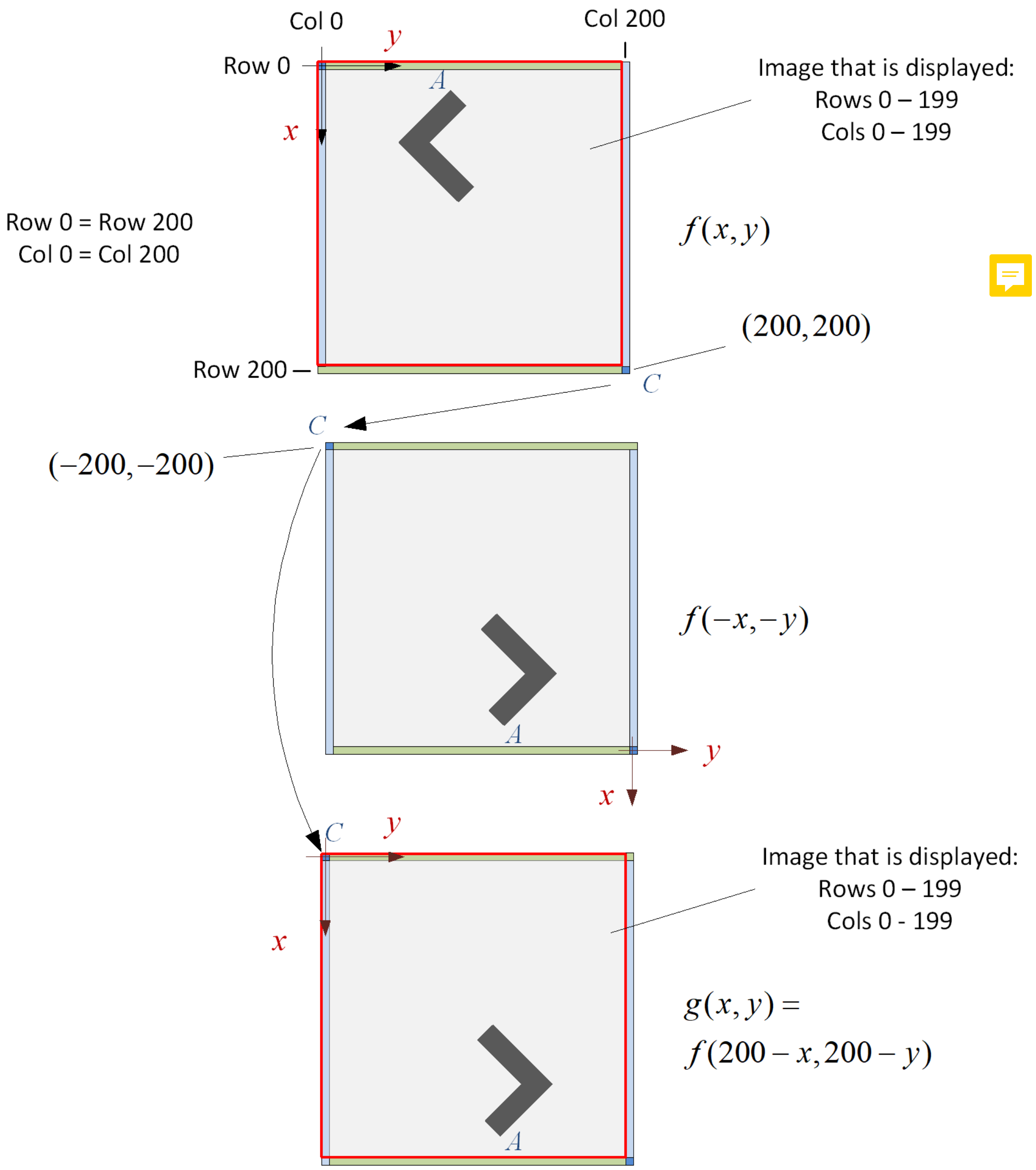
For convenience, we use the 1-D equations for the time being:

$$\begin{aligned} g(x) &= \sum_{u=0}^{199} F^*(u)\exp[j2\pi ux/200] \\ g^*(x) &= \sum_{u=0}^{199} F(u)\exp[-j2\pi ux/200] \quad \text{(by conjugating both sides)} \\ g^*(-x) &= \sum_{u=0}^{199} F(u)\exp[j2\pi ux/200] \\ &= f(x) \\ g(-x) &= f^*(x) \quad \text{(by conjugating both sides)} \\ &= f(x) \quad \text{(since } f(x) \text{ is real)} \\ g(x) &= f(-x) \\ &= f(200-x) \quad \text{since } f(x) \text{ is periodic } (N = 200) \end{aligned}$$

In 2-D, we have

$$g(x,y) = f(200-x, 200-y)$$

$$g(x, y) = f(200 - x, 200 - y)$$



$$\begin{array}{rcccccc}
 & f(x,y) & & f(-x,-y) & & g(x,y) \\
\text{Origin} \; : & (0,0) & \to & (0,0) & \to & (200,200) \\
\text{Corner } C: & (200,200) & \to & (-200,-200) & \to & (0,0) \\
A: & (20,90) & \to & (-20,-90) & \to & (180,110)
\end{array}$$