Due Date: September 15, 2015

Problem 1:
(a) Rank the functions by order of their growth. ($\log n$ means $\log_2 n$)

$\left(\frac{5}{4}\right)^n$, n^4, $n!$, $(2)^{\sqrt{2\log n}}$, $\log n$, $(\log n)^{\log n}$, $\left(\sqrt{2}\right)^{\log n}$, 2^{2^n}

(b) Show that $\log(n!) = \Theta(n \log n)$.

Problem 2: Solve the following recurrences using induction and give a Θ bound for each of them. In all cases, assume $T(k) = O(1)$ if k is a constant.

- $T(n) = 5T(n/4) + n$
- $T(n) = \sqrt{n}T(\sqrt{n}) + n \log n$

Problem 3: Let A and B be two sorted lists of size m and n, respectively, and a natural number $k \geq 1$. Describe an $O(\log m + \log n)$-time algorithm for computing the kth smallest element in $A \cup B$. For example, if $A = \langle 2, 10, 15, 32, 45, 93 \rangle$, $B = \langle 8, 12, 18, 22, 28, 56, 85, 92 \rangle$, and $k = 10$, then the algorithm should return 45.

Problem 4: For a sequence of natural numbers $S = \langle a_1, \ldots, a_n \rangle$, an *inversion* is a pair of elements (a_i, a_j) such that $i < j$ but $a_i > a_j$. For example, the inversions in sequence $\langle 1, 4, 2, 3 \rangle$ are $(4, 2)$ and $(4, 3)$, 4 appears before both 2 and 3. Given a sequence S of n natural numbers, describe an $O(n \log n)$-time algorithm to count the number of inversions in S. Prove its correctness and running time.

Problem 5: An oil company is planning to build a large pipeline running East to West through an oil field of n wells. The company wants to connect a spur-pipeline from each well directly to the main pipeline along a shortest (North-South) route, as shown in Figure 1. Given x and y coordinates of the wells, how should the company pick the optimal location of the main pipeline, that minimizes the total length of the spur-pipelines? Show how to determine the optimal location in linear time.

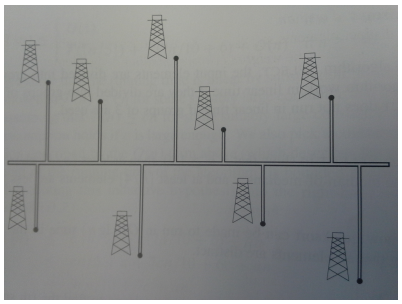

Figure 1: Main pipeline and spur pipelines. Figure is taken from T. H. Cormen, C. E. Leiserson, R. L. Rivest, C. Stein, Introduction to Algorithms, The MIT Press, 3rd ed., 2009.