Due Date: November 17, 2015

Problem 1: Write the dual of the following linear program.

$$\begin{aligned} \max x_1 + x_2 \\ 2x_1 + x_2 \leq 3 \\ x_1 + 3x_2 \leq 5 \\ x_1, x_2 \geq 0 \end{aligned}$$

Find the optimal solutions to both primal and dual LPs using the simplex method. For the primal you can use $x_1 = 0$ and $x_2 = 0$ as the initial feasible solution (BFS), and for the dual, you can use $y_1 = 1$ and $y_2 = 0$ as the initial BFS, where y_1 and y_2 are the dual variables associated with the first and second constraint in the primal, respectively.

Problem 2: In the following simple two-player game, the players (call them R and C) each choose an outcome, *heads* or *tails*. If both outcomes are equal, C gives a dollar to R; if the outcomes are different, R gives a dollar to C.

(i) Represent the payoffs by a 2×2 matrix.

(ii) What is the value of this game, and what are the optimal strategies for the two players?

Problem 3: Integer linear programming (ILP) is the same as linear programming except that one considers only integer solutions, i.e.,

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0, x \in \mathbb{Z} \end{aligned}$$

The vertex cover of a graph $G = (V, E)$ is a subset $C \subseteq V$ of vertices so that each edge in E is incident to at least one of the vertices in C.

(i) Show that the problem of computing the minimum-size vertex cover can be formulated as an instance of ILP.

(ii) Suppose we relax the ILP in (i) to an LP by removing the integer constraint $x \in \mathbb{Z}$. Show that there is graph for which the optimal LP solution is not integral (i.e., values of some of the variables are not integers).

Problem 4: Polynomial Multiplication of FFT: Suppose that you want to multiply the two polynomials $1 + x + 2x^2$ and $2 + 3x$ using the FFT. Choose an appropriate power of two, find the FFT of the two sequences, multiply the results componentwise, and compute the inverse FFT to get the final result.

Problem 5: Let X and Y be two sets of natural numbers such that the largest number in $X \cup Y$ is M. Let $Z = \{x + y \mid x \in X, y \in Y\}$ be the Minkowski sum of X and Y. Describe an

$O(M \log M)$ algorithm to compute the set Z. (**Hint:** Write each X and Y as coefficients of a polynomial.)

Problem 6: (Extra Credit)
Give an efficient algorithm to compute K_j for each j from the following formula,

$$K_j = \sum_{i<j} \frac{A p_i p_j}{(j-i)^2} - \sum_{i>j} \frac{A p_i p_j}{(j-i)^2}$$

where A, $p_i \in \mathbb{R}$ are known, $i, j \in [1, \ldots, n]$, and $n \in \mathbb{Z}^+$.