STA 360: Lab 9

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1. See below for priors used and the resulting posterior distributions for μ and σ^2.

Given data has Lognormal distribution:

$$f_X(x|\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\ln x-\mu)^2}{2\sigma^2}\right\}, \quad x>0$$

Suppose priors of μ and σ^2 are:

$$p(\mu) = \text{Normal}(\mu \mid \mu_0, \sigma_0^2)$$

$$p(\sigma^2) = \text{InvGamma}(\sigma^2 \mid \alpha, \beta)$$

Therefore,

$$p(\mu|X_{1:n}) = p(\mu)\, p(X_{1:n}|\mu,\sigma^2)$$

$$\propto \exp\left\{-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right\} \prod_{i=1}^{n} \exp\left\{-\frac{(\ln x_i-\mu)^2}{2\sigma^2}\right\}$$

$$= \exp\left\{-\frac{(\mu-\mu_0)^2}{2\sigma_0^2} - \frac{\sum_{i=1}^{n}(\ln x_i-\mu)^2}{2\sigma^2}\right\}$$

$$= \exp\left\{-\frac{\sigma^2(\mu-\mu_0)^2 + \sigma_0^2\sum_{i=1}^{n}(\ln x_i-\mu)^2}{2\sigma_0^2\sigma^2}\right\}$$

$$\propto \exp\left\{-\frac{\sigma^2(\mu^2-2\mu\mu_0) + \sigma_0^2\sum_{i=1}^{n}(\mu^2-2\mu\ln x_i)}{2\sigma_0^2\sigma^2}\right\}$$

$$= \exp\left\{-\frac{\mu^2\sigma^2 - 2\mu\mu_0\sigma^2 + \sigma_0^2(n\mu^2 - 2\mu\sum_{i=1}^{n}\ln x_i)}{2\sigma_0^2\sigma^2}\right\}$$

$$= \exp\left\{-\frac{\mu^2\sigma^2 - 2\mu\mu_0\sigma^2 + \mu^2 n\sigma_0^2 - 2\mu\sigma_0^2\sum_{i=1}^{n}\ln x_i}{2\sigma_0^2\sigma^2}\right\}$$

$$= \exp\left\{-\frac{(\sigma^2+n\sigma_0^2)\mu^2 - 2\mu(\mu_0\sigma^2 + \sigma_0^2\sum_{i=1}^{n}\ln x_i)}{2\sigma_0^2\sigma^2}\right\}$$

$$= \exp\left\{-\frac{\mu^2 - 2\mu\,\frac{\mu_0\sigma^2 + \sigma_0^2\sum_{i=1}^{n}\ln x_i}{\sigma^2+n\sigma_0^2}}{2\sigma_0^2\sigma^2/(\sigma^2+n\sigma_0^2)}\right\}$$

$$\propto \text{Normal}\left(\mu \,\middle|\, \frac{\mu_0\sigma^2 + \sigma_0^2\sum_{i=1}^{n}\ln x_i}{\sigma^2+n\sigma_0^2},\ \frac{\sigma_0^2\sigma^2}{\sigma^2+n\sigma_0^2}\right)$$

$(\sigma^2)^{-1/2}$

$$p(\sigma^2 \mid x_{1:n}) = p(\sigma^2)\, p(x_{1:n} \mid \mu, \sigma^2)$$

$$\propto \frac{\beta^\alpha}{\Gamma(\alpha)} (\sigma^2)^{-\alpha-1} \exp\left(\frac{-\beta}{\sigma^2}\right) \prod_{i=1}^{n} (2\pi\sigma^2)^{-1/2} \exp\left\{\frac{(\ln x_i - \mu)^2}{2\sigma^2}\right\}$$

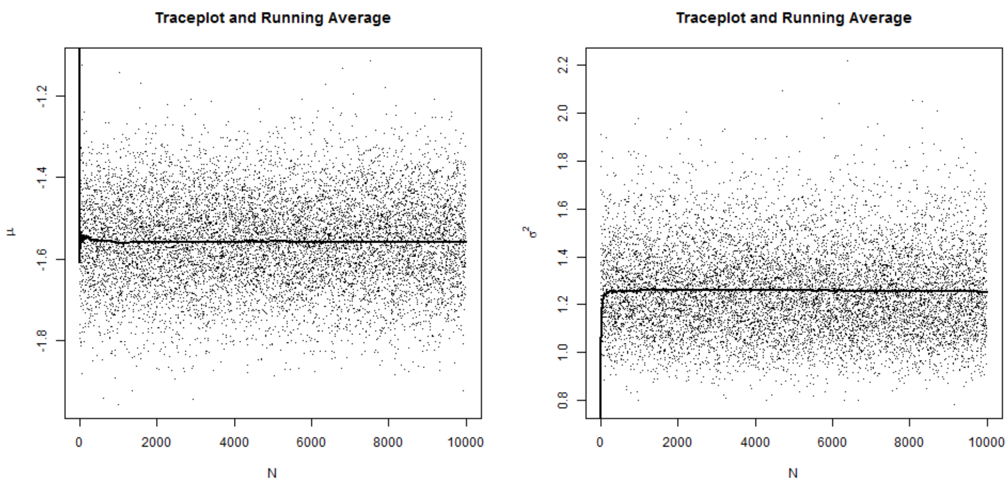
$$\propto (\sigma^2)^{-\alpha-1} \cdot (\sigma^2)^{-n/2} \exp\left\{- \frac{2\beta + \sum_{i=1}^{n} (\ln(x_i) - \mu)^2}{2\sigma^2}\right\}$$

$$\propto (\sigma^2)^{-\alpha-n/2-1} \exp\left\{- \frac{2\beta + \sum_{i=1}^{n} (\ln x_i - \mu)^2}{2\sigma^2}\right\}$$

$$= (\sigma^2)^{-(\alpha+n/2)-1} \exp\left\{- \frac{\beta + \frac{1}{2}\sum_{i=1}^{n} (\ln x_i - \mu)^2}{\sigma^2}\right\}$$

$$= IG\left(\sigma^2 \,\middle|\, \alpha + \frac{n}{2},\ \beta + \frac{1}{2}\sum_{i=1}^{n} (\ln x_i - \mu)^2\right)$$

2. See below for trace plots. We used $\alpha, \beta = 0$ for σ^2 because it's a Jeffery's prior. We also set $\mu_0 = 0$ and $\sigma_0^2 = 0$ to allow large prior variance (i.e. uninformative prior). The Gibbs sampler appears to have converged and explored the space reasonably well.



3. See below for respective 95% confidence interval.

$$\text{CI(mean)} = [0.7066048, 0.1072423]$$

$$\text{CI(variance)} = [0.1586194, 1.0671797]$$

See below for R code:

```
# load data
x = read.table("data.txt", header = F)[,1]
library(MCMCpack)

## Define prior parameters ##
a = 0; b = 0;
mu.0 = 0; sig2.0 = 10;
n = length(x)

## Define prior ##
mu.temp = rnorm(1, mean = (mu.0*sig2.0 + sig2.0*sum(log(x)))/(sig2.0 + n*sig2.0),
                sd = sqrt((sig2.0*sig2.0)/(sig2.0 + n*sig2.0)))
sig2.temp = rinvgamma(1, shape = a + n/2,
                      scale = b + 1/2*sum( (log(x) - mu.temp)^2 ))

N = 10000
MU = rep(0, N)
SIG2 = rep(0, N)

## Gibbs Sampler ##
for(i in 1:N){
  mu.temp = rnorm(1, mean = (mu.0*sig2.temp + sig2.0*sum(log(x)))/(sig2.temp + n*sig2.0),
             sd = sqrt((sig2.0*sig2.temp)/(sig2.temp + n*sig2.0)))
  sig2.temp = rinvgamma(1, shape = a + n/2,
                        scale = b + 1/2*sum( (log(x) - mu.temp)^2 ))
  MU[i] = mu.temp
  SIG2[i] = sig2.temp
}

## Running Avg ##
MU.sum = rep(0, N)
MU.avg = rep(0, N)
MU.sum[1] = MU[1]
for(i in 2:N){
  MU.sum[i] = MU[i] + MU.sum[i-1]
  MU.avg[i] = MU.sum[i]/i
}

SIG2.sum = rep(0, N)
SIG2.avg = rep(0, N)
SIG2.sum[1] = MU[1]
for(i in 2:N){
  SIG2.sum[i] = SIG2[i] + SIG2.sum[i-1]
  SIG2.avg[i] = SIG2.sum[i]/i
}

## Plot ##
png("MUplot.png")
plot(MU, pch = '.', xlab = "N", ylab = expression(mu),
     main = "Traceplot and Running Average")
lines(MU.avg, lwd = 2)
dev.off()
```

```
png("SIGplot.png")
plot(SIG2, pch = '.', xlab = "N", ylab = expression(sigma^{2}),
     main = "Traceplot and Running Average")
lines(SIG2.avg, lwd = 2)
dev.off()

## Confidence Interval ##
mean.vec = exp(MU + SIG2/2)
var.vec = (exp(SIG2)-1)*exp(2*MU + SIG2)

mean.ci = quantile(mean.vec, c(0.025, 0.975))
var.ci = quantile(var.vec, c(0.025, 0.975))
```