STA 360: Assignment 3

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2. Since Z has distribution Binomial(n, θ), the p.d.f. of z is:

$$\begin{aligned}
p(z|\theta) &= \binom{n}{z}\theta^z(1-\theta)^{n-z}\mathbb{1}(z \in S) \\
&= \binom{n}{z}\exp\{\log[\theta^z(1-\theta)^{n-z}]\}\mathbb{1}(z \in S) \\
&= \binom{n}{z}\exp\{\log\theta^z + \log(1-\theta)^{n-z}\}\mathbb{1}(z \in S) \\
&= \binom{n}{z}\exp\{z\log\theta + (n-z)\log(1-\theta)\}\mathbb{1}(z \in S) \\
&= \binom{n}{z}\exp\{z\log\theta + n\log(1-\theta) - z\log(1-\theta)\}\mathbb{1}(z \in S) \\
&= \binom{n}{z}\exp\{z\log\frac{\theta}{1-\theta} + n\log(1-\theta)\}\mathbb{1}(z \in S) \\
&= \binom{n}{z}\exp\{z\log\frac{\theta}{1-\theta} - n\log\frac{1}{1-\theta}\}\mathbb{1}(z \in S)
\end{aligned}$$

where $S = \{0, 1, 2, ..., n\}$. From here, we see that:

$$\begin{aligned}
t(z) &= z \\
\phi(\theta) &= \log\frac{\theta}{1-\theta} \\
\kappa(\theta) &= n\log\frac{1}{1-\theta} \\
h(z) &= \binom{n}{z}\mathbb{1}(z \in S)
\end{aligned}$$

Thus, for a fixed n and given parameter θ, Binomial(n, θ) distributions form a one-parameter exponential family.

3. Given that the generating distribution is $p(x|a, b) = \text{Gamma}(x|a, b)$ where a is fixed, the form of this distribution is:

$$\begin{aligned}
p(x_{1:n}|b) &= \prod_{i=1}^{n}\frac{b^a}{\Gamma(a)}x_i^{a-1}e^{-bx_i} \\
&\propto b^{an}e^{-b\sum x_i}
\end{aligned}$$

We will show that a conjugate prior for b is $p(b) = \text{Gamma}(b|\alpha, \beta)$ by considering the corresponding posterior distribution.

$$\begin{aligned}
p(b|x_{1:n}) &= p(b)p(x_{1:n}|b) \\
&\propto \frac{\beta^\alpha}{\Gamma(\alpha)} b^{\alpha-1} e^{-\beta b} b^{an} e^{-b\sum x_i} \\
&\propto b^{(\alpha+an)-1} e^{-b(\beta+\sum x_i)} \\
&\propto \mathrm{Gamma}(b|\alpha+an, \beta+\sum x_i)
\end{aligned}$$

Since the posterior, like the prior, is also a Gamma distribution, thus $p(b) = \mathrm{Gamma}(b|\alpha,\beta)$ is a conjugate prior for b.

5. Given conjugate prior $p_{n_0,t_0}(\theta)$ and i.i.d. probability distributions $p(x|\theta)$, the posterior distribution $p(\theta|x_{1:n})$ is the following:

$$\begin{aligned}
p(\theta|x_{1:n}) &= p_{n_0,t_0}(\theta) \prod_{i=1}^{n} p(x_i|\theta) \\
&\propto \exp\{n_0 t_0 \phi(\theta) - n_0\kappa(\theta)\} \prod_{i=1}^{n} \{\exp(\phi(\theta)t(x_i) - \kappa(\theta))h(x_i)\} \\
&= \exp\{n_0 t_0 \phi(\theta) - n_0\kappa(\theta)\}[\prod_{i=1}^{n} h(x_i)] \exp\{\sum_{i=1}^{n}(\phi(\theta)t(x_i) - \kappa(\theta))\} \\
&\propto \exp\{n_0 t_0 \phi(\theta) - n_0\kappa(\theta)\} \exp\{\sum_{i=1}^{n}(\phi(\theta)t(x_i) - \kappa(\theta))\} \\
&= \exp\{n_0 t_0 \phi(\theta) - n_0\kappa(\theta) + \phi(\theta)\sum_{i=1}^{n} t(x_i) - n\kappa(\theta)\} \\
&= \exp\{(n_0 t_0 + \sum_{i=1}^{n} t(x_i))\phi(\theta) - (n_0+n)\kappa(\theta)\} \\
&= \exp\{(n_0+0)\frac{n_0 t_0 + \sum_{i=1}^{n} t(x_i)}{n_0+n}\phi(\theta) - (n_0+n)\kappa(\theta)\} \\
&= \exp\{n' t' \phi(\theta) - n'\kappa(\theta)\} \\
&\propto p_{n',t'}(\theta)
\end{aligned}$$

where $n' = n_0 + n$ and

$$t' = \frac{n_0 t_0 + \sum_{i=1}^{n} t(x_i)}{n_0+n}$$

7. See image below:

7. Suppose $\{p_\alpha(\theta) : \alpha \in H\}$ is a conjugate family for some generator family, say a set G.

By definition of conjugate family, the resulting posterior, where $p(x_{1:n}|\theta) \in G$, is.

$$p_1(\theta | x_{1:n}) \propto p(x_{1:n}|\theta)\, p_\alpha(\theta)$$
$$\propto p_{\alpha'}(\theta) \quad \text{such that } \alpha' \in H$$

Now, given the set

$$A = \left\{ \sum_{i=1}^{k} \pi_i\, p_{\alpha_i}(\theta) : \alpha_1, \ldots, \alpha_k \in H,\ \pi \in \Delta_k \right\}$$

where $\Delta_k = \left\{ \pi \in \mathbb{R}^k : \pi_1, \ldots, \pi_k \geq 0,\ \sum_{i=1}^{k} \pi_i = 1 \right\}$,

consider the following:

$$p_2(\theta | x_{1:n}) \propto p(x_{1:n}|\theta) \sum_{i=1}^{k} \pi_i\, p_{\alpha_i}(\theta)$$
$$= \sum_{i=1}^{k} \pi_i\, p(x_{1:n}|\theta)\, p_{\alpha_i}(\theta)$$
$$\propto \sum_{i=1}^{k} \pi_i\, p_{\alpha_i'}(\theta) \quad \text{(by hypothesis)}$$
$$\propto p_{\alpha_i'}(\theta)$$

since all π_i's are constant. Since $\alpha_i' \in H$, $p_{\alpha_i'}(\theta)$ is also in A. Thus A is also a conjugate family of the same generator family G.

Additional exercise: Given prior distribution $p(\theta)$ and generated distribution $p(x|\theta)$, the posterior distribution is as follows:

$$
\begin{aligned}
p(\theta|x_{1:n}) &= p(x_{1:n}|\theta)p(\theta) \\
&= \Big\{ \prod_{i=1}^{n} \sqrt{\frac{\lambda}{2\pi}} \exp(-\frac{\lambda}{2}(x_i-\theta)^2) \Big\} \sqrt{\frac{\lambda_0}{2\pi}} \exp(-\frac{\lambda_0}{2}(\theta-\mu_0)^2) \\
&\propto \exp(-\frac{\lambda}{2}\sum_{i=1}^{n}(x_i-\theta)^2) \exp(\frac{\lambda_0}{2}(\theta-\mu_0)^2) \\
&= \exp(-\frac{1}{2}[\lambda\sum_{i=1}^{n}(x_i-\theta)^2 + \lambda_0(\theta-\mu_0)^2]) \\
&= \exp(-\frac{1}{2}[\lambda\sum_{i=1}^{n}(x_i^2-2x_i\theta+\theta^2) + \lambda_0(\theta^2-2\theta\mu_0+\mu_0^2)]) \\
&= \exp(-\frac{1}{2}[\lambda\sum_{i=1}^{n}x_i^2 - 2\lambda\theta\sum_{i=1}^{n}x_i + n\lambda\theta^2 + \lambda_0\theta^2 - 2\lambda_0\theta\mu_0 + \lambda_0\mu_0^2]) \\
&\propto \exp(-\frac{1}{2}[(n\lambda+\lambda_0)\theta^2 - 2(\lambda\sum_{i=1}^{n}x_i + 2\lambda_0\mu_0)\theta]) \\
&= \exp(-\frac{n\lambda+\lambda_0}{2}[\theta^2 - 2(\frac{\lambda\sum\limits_{i=1}^{n}x_i + 2\lambda_0\mu_0}{n\lambda+\lambda_0})\theta]) \\
&\propto \exp(-\frac{n\lambda+\lambda_0}{2}(\theta - \frac{\lambda\sum\limits_{i=1}^{n}x_i + 2\lambda_0\mu_0}{n\lambda+\lambda_0})^2) \\
&= \exp(-\frac{L}{2}(\theta-M)^2) \\
&\propto \text{Normal}(\theta|M, L^{-1})
\end{aligned}
$$

where $L = n\lambda + \lambda_0$ and

$$M = \frac{\lambda\sum\limits_{i=1}^{n}x_i + 2\lambda_0\mu_0}{n\lambda+\lambda_0}$$