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STA 360
HW 9

1) We want to show
$$\int \cdots \int p(\vec{x})\, dx_1 \cdots dx_K = \int \cdots \int \prod_{k=1}^{K} p(x_k \mid pa(k))\, dx_1 \cdots dx_K = 1$$

Assume that x_K is independent of all lower numbered nodes, and each node is only dependent on higher numbered nodes, i.e., we would write

$p(x_K)$
$p(x_{K-1} \mid x_K)$
$p(x_{K-2} \mid x_K, x_{K-1})$
$\vdots$
$p(x_1 \mid x_K, x_{K-1}, \ldots, x_2)$

Thus
$$\int \cdots \int \prod_{k=1}^{K} p(x_k \mid pa(k))\, dx_1 \cdots dx_K$$
$$= \int \cdots \int p(x_1 \mid pa(1)) \prod_{k=2}^{K} p(x_k \mid pa(k))\, dx_1 \cdots dx_K$$
$$= \int \cdots \int \prod_{k=2}^{K} p(x_k \mid pa(k))\, dx_2 \cdots dx_K$$
$\vdots$ (repeat)
$$= \int p(x_K \mid pa(K))\, dx_K = \int p(x_K)\, dx_K = 1$$

Since each conditional distribution is correctly normalized.

2) (1) $a \perp\!\!\!\perp b, c \mid d$

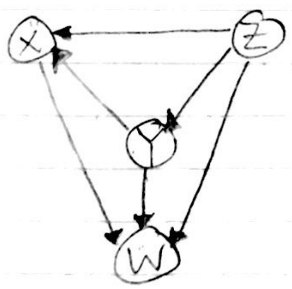
(2) $p(a,b,c \mid d) = p(a \mid d)\, p(b,c \mid d)$

(3) $\int p(a,b,c \mid d)\, dc = \int p(a \mid d)\, p(b,c \mid d)\, dc$

(4) $p(a,b \mid d) = p(a \mid d)\, p(b \mid d)$

(5) $a \perp\!\!\!\perp b \mid d$

3) $p(w,x,y,z) = p(w \mid x,y,z)\, p(x \mid y,z)\, p(y \mid z)\, p(z)$

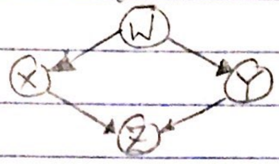


The decomposition in the equation is true for any $p(w,x,y,z)$. The decomposition respects the graph because in the graph, w depends on x, y, z; x depends on y, z; y depends on z; and z is independent.

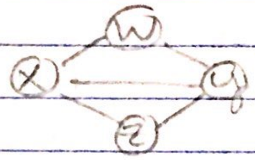
4) We need

$$p(x, y \mid w) = p(x \mid w)\, p(y \mid w)$$

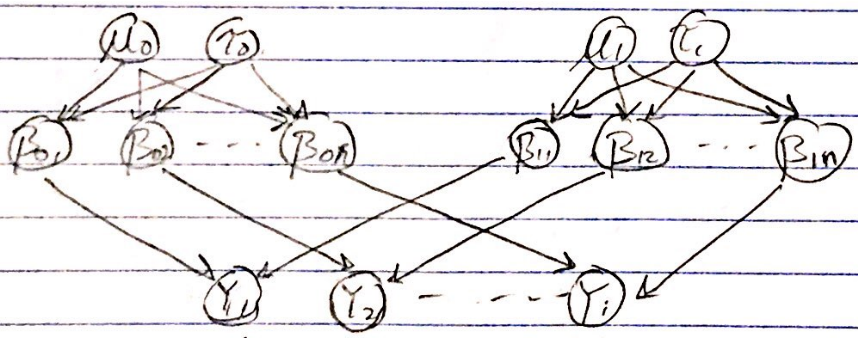
Clearly, the PGM below is respected by P.



However, the corresponding moral graph does not show $x \perp\!\!\!\perp y \mid w$ because not every path from x to y goes through w.

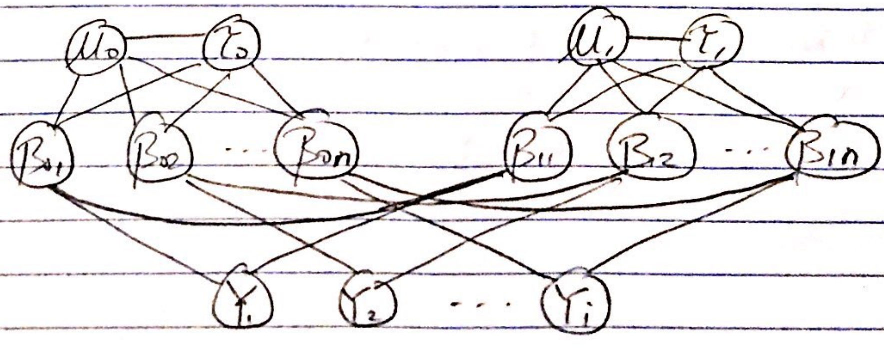


5) a)



x_j's are not random variables, so not shown.

b)



x_j's are not random variables, so not shown.

c) Show (μ_0, τ_0) is conditionally independent of everything else given $\beta_{01}, \ldots, \beta_{0n}$.

To get from (μ_0, τ_0) to anything else, we must pass through one of $\beta_{01}, \beta_{02}, \ldots, \beta_{0n}$. Thus (μ_0, τ_0) is conditionally independent of everything else given $\beta_{01}, \ldots, \beta_{0n}$.

d) $\varepsilon_{ij} \sim N(0, \tau)$ are i.i.d, so they do not connect to any node in the directed or undirected graphs. Thus, we look at the subset $\{\mu_0, \tau_0, \mu_1, \tau_1\}$. To get from Y_1 to Y_2 in the moral graph, we need to traverse through at least one of $\mu_0, \tau_0, \mu_1, \tau_1$. For example,

$$Y_1 \rightarrow \beta_{01} \rightarrow \mu_0 \rightarrow \beta_{12} \rightarrow Y_2 .$$

Thus $Y_1 \perp\!\!\!\perp Y_2 \mid \{\mu_0, \tau_0, \mu_1, \tau_1\}$. Since ε_{ij} is i.i.d,

$$Y_1 \perp\!\!\!\perp Y_2 \mid \{\mu_0, \tau_0, \mu_1, \tau_1, \tau\}$$