STA 360: Lab 5

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1. Given $\beta \sim$Uniform(0,1), $N \sim$Poisson(λ), and $y \sim$Binomial(N, β), they have the following prior distributions:

$$\begin{aligned} p(\beta) &= \mathbb{1}(0 \leq \beta \leq 1) \\ p(N) &= \frac{\lambda^N}{N!} e^{-\lambda} \\ p(y|N, \beta) &= \binom{N}{y} \beta^y (1-\beta)^{N-y} \end{aligned}$$

   Therefore, the joint posterior distribution is:

$$\begin{aligned} p(N, \beta|y) &= p(y|N, \beta)p(\beta)p(N) \\ &= \binom{N}{y} \beta^y (1-\beta)^{N-y} \cdot \mathbb{1}(0 \leq \beta \leq 1) \cdot \frac{\lambda^N}{N!} e^{-\lambda} \\ &= \frac{N!}{y!(N-y)!} \beta^y (1-\beta)^{N-y} \frac{\lambda^N}{N!} e^{-\lambda} \\ &= \frac{\lambda^N e^{-\lambda}}{y!(N-y)!} \beta^y (1-\beta)^{N-y} \\ &= \left(\frac{\beta}{1-\beta}\right)^y \frac{(1-\beta)^N \lambda^N e^{-\lambda}}{y!(N-y)!} \end{aligned}$$

2. Marginal distribution of N. First, we find the marginal distribution of $N - y$:

$$p(N-y|y, \beta) \propto \frac{[(1-\beta)\lambda]^{(N-y)+y}}{(N-y)!} \propto \text{Poisson}((1-\beta)\lambda)$$

   Thus, the marginal distribution of N is:

$$p(N|y, \beta) = y + \text{Poisson}((1-\beta)\lambda)$$

   Marginal distribution of β.

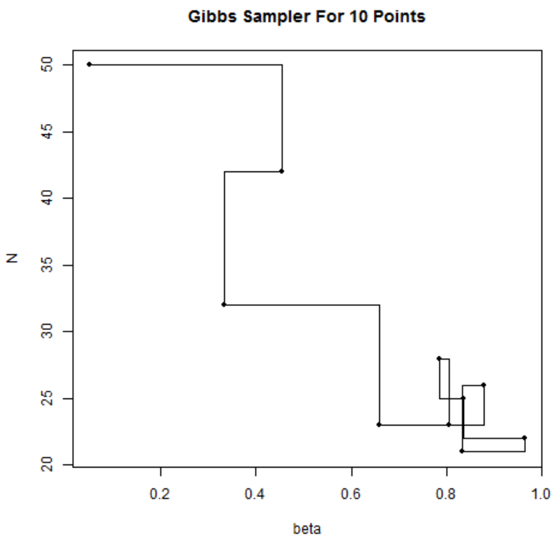
$$p(\beta|y, N) \propto \left(\frac{\beta}{1-\beta}\right)^y (1-\beta)^N = \beta^y (1-\beta)^{N-y} \propto \text{Beta}(y+1, N-y+1)$$

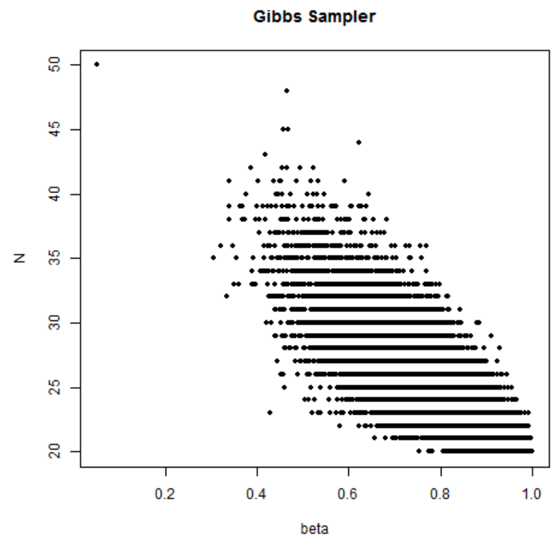
   Thus, the marginal distribution of β is:

$$p(\beta|y, N) = \text{Beta}(y+1, N-y+1)$$

3. See R code

4. Graphs for Gibbs sampling are shown below.





5. Using the `quantile` function, we found the 90 % posterior credible interval to be:

$$(0.55, 0.97)$$

6. The probability that exactly 20 people were polled is 0.073.

```r
## Distribution parameters ##
b = 0.05
n = 50
y = 20
lam = 25

## Gibbs Sampling ##
N = 11000
b.samp = rep(NA,N)
b.samp[1] = b
n.samp = rep(NA,N)
n.samp[1] = n

for(i in 2:N){
  b.samp[i] = rbeta(1, y+1, n.samp[i-1]-y+1)
  n.samp[i] = y + rpois(1, (1-b.samp[i])*lam)
}

## Plot for first 10 ##
png("gibb10.png")
plot(b.samp[1:10], n.samp[1:10], pch = 20, xlab = "beta",
     ylab = "N", main = "Gibbs Sampler For 10 Points")

for(i in 1:(9)){
  lines(c(b.samp[i],b.samp[i+1]), c(n.samp[i],n.samp[i]))
  lines(c(b.samp[i+1],b.samp[i+1]), c(n.samp[i],n.samp[i+1]))
}
dev.off()

## Plot for all points ##
png("gibb.png")
plot(b.samp, n.samp, pch = 20, xlab = "beta",
     ylab = "N", main = "Gibbs Sampler")
dev.off()

## Credible Interval ##
print(quantile(b.samp, prob = c(0.05,0.95)))

## Burn-in discard ##
n.burn = n.samp[seq(1001,11000,1)]

## Probability N=20 ##
N.20 = length(n.burn[n.burn==20])
prob = N.20/length(n.burn)
print(prob)
```