STA 360: Lab 10

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1. From last week:

$$(\sigma^2)^{-1/2}$$

$$p(\sigma^2 \mid x_{1:n}) = p(\sigma^2)\, p(x_{1:n} \mid \mu, \sigma^2)$$

$$\propto \frac{\beta^\alpha}{\Gamma(\alpha)} (\sigma^2)^{-\alpha-1} \exp\left(\frac{-\beta}{\sigma^2}\right) \prod_{i=1}^{n} (2\pi\sigma^2)^{-1/2} \exp\left\{\frac{(\ln x_i - \mu)^2}{2\sigma^2}\right\}$$

$$\propto (\sigma^2)^{-\alpha-1} \cdot (\sigma^2)^{-n/2} \exp\left\{-\frac{2\beta + \sum_{i=1}^{n} (\ln(x_i) - \mu)^2}{2\sigma^2}\right\}$$

$$\propto (\sigma^2)^{-\alpha-n/2-1} \exp\left\{-\frac{2\beta + \sum_{i=1}^{n} (\ln x_i - \mu)^2}{2\sigma^2}\right\}$$

$$= (\sigma^2)^{-(\alpha+n/2)-1} \exp\left\{-\frac{\beta + \frac{1}{2}\sum_{i=1}^{n} (\ln x_i - \mu)^2}{\sigma^2}\right\}$$

$$= IG\left(\sigma^2 \,\middle|\, \alpha + \frac{n}{2},\ \beta + \frac{1}{2}\sum_{i=1}^{n} (\ln x_i - \mu)^2\right)$$

The model for this week:

From last time, we saw that given priors:

$$p(\mu) = \text{Normal}(\mu \mid \mu_0, \sigma_0^2)$$
$$p(\sigma^2) = IG(\sigma^2 \mid \alpha, \beta)$$

The posteriors are:

$$p(\mu \mid x_{1:n}) = \text{Normal}\left(\mu \,\middle|\, \frac{\mu_0\sigma^2 + \sigma_0^2 \Sigma \ln x_i}{\sigma^2 + n\sigma_0^2}, \frac{\sigma_0^2\sigma^2}{\sigma^2 + n\sigma_0^2}\right)$$

$$p(\sigma^2 \mid x_{1:n}) = IG\left(\sigma^2 \,\middle|\, \alpha + \frac{n}{2}, \beta + \frac{1}{2}\Sigma_i(\ln x_i - \mu)^2\right)$$

1) Since we want $\mu_1 > \mu_2$, we will draw from these priors

$$\mu_2 \sim No(\mu_2 \mid m_2, S_2^2)$$

$$\mu_1 = \mu_2 + \delta \quad \text{where } \delta \sim No(\delta \mid m_1, s_1^2)\mathbb{1}_{\delta > 0}$$

Thus $\delta = \mu_1 - \mu_2 \sim No(\delta \mid m_1, s_1^2)\mathbb{1}_{\delta > 0}$

For σ_1 and σ_2, we have

$$\sigma_1^2 \sim IG(\sigma_1^2 \mid a_1, b_1)$$
$$\sigma_2^2 \sim IG(\sigma_2^2 \mid a_2, b_2)$$

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$$p(\mu_2 | y_2) \propto p(y_2 | \mu_2, \sigma_2^2) \cdot p(\mu_2)\, p(\sigma_2^2)$$

$$\propto \prod_{i=1}^{n_2} \exp\left\{-\frac{(\ln(y_{2i})-\mu_2)^2}{2\sigma_2^2}\right\} \cdot \exp\left\{-\frac{(\mu_2-m_2)^2}{2s_2^2}\right\}$$

$$= \exp\left\{-\frac{\sum_{i=1}^{n_2}(\ln(y_{2i})-\mu_2)^2}{2\sigma_2^2}\right\} \cdot \exp\left\{-\frac{(\mu_2-m_2)^2}{2s_2^2}\right\}$$

$$\propto \exp\left\{-\frac{\sum_{i=1}^{n_2}(\mu_2^2-2\mu_2\ln(y_{2i}))}{2\sigma_2^2}\right\} \exp\left\{-\frac{\mu_2^2-2\mu_2 m_2}{2s_2^2}\right\}$$

$$= \exp\left\{-\frac{n_2\mu_2^2-2\mu_2\sum_{i=1}^{n_2}\ln(y_{2i})}{2\sigma_2^2} - \frac{\mu_2^2-2\mu_2 m_2}{2s_2^2}\right\}$$

$$= \exp\left\{-\left[\frac{s_2^2(n_2\mu_2^2-2\mu_2\sum\ln(y_{2i})) + \sigma_2^2(\mu_2^2-2\mu_2 m_2)}{2\sigma_2^2 s_2^2}\right]\right\}$$

$$= \exp\left\{-\frac{s_2^2 n_2\mu_2^2 - 2s_2^2\mu_2\sum_i\ln(y_{2i}) + \sigma_2^2\mu_2^2 - 2\sigma_2^2\mu_2 m_2}{2\sigma_2^2 s_2^2}\right\}$$

$$= \exp\left\{-\frac{(s_2^2 n_2+\sigma_2^2)\mu_2^2 - 2[s_2^2\sum\ln(y_{2i}) + \sigma_2^2 m_2]\mu_2}{2\sigma_2^2 s_2^2}\right\}$$

$$= \exp\left\{-\frac{\mu_2^2 - 2\left(\frac{s_2^2\sum\ln(y_{2i})+\sigma_2^2 m_2}{s_2^2 n_2+\sigma_2^2}\right)\mu_2}{2\sigma_2^2 s_2^2 / (s_2^2 n_2+\sigma_2^2)}\right\}$$

$$\propto \text{No}\left(\mu_2 \,\middle|\, \frac{s_2^2\sum\ln(y_{2i})+\sigma_2^2 m_2}{s_2^2 n_2+\sigma_2^2}, \frac{\sigma_2^2 s_2^2}{s_2^2 n_2+\sigma_2^2}\right)$$

$p(\sigma_2^2 | y_2) = \cdots$ (See last week's derivation) $\cdot$

$$= IG\left(\sigma_2^2 \,\middle|\, \alpha_2 + \frac{n_2}{2}, \beta_2 + \frac{1}{2}\sum_{i=1}^{n_2}(\ln y_{2i}-\mu_2)^2\right)$$

Similarly,

$$p(\sigma_1^2 | y_1) = IG\left(\sigma_1^2 \,\middle|\, \alpha_1 + \frac{n_1}{2}, \beta_1 + \frac{1}{2}\sum_{i=1}^{n_1}(\ln y_{1i}-\mu_1)^2\right)$$

$$p(\delta \mid y_1) \propto p(y_1 \mid \mu_1, \sigma_1^2)\, p(\delta) \cdot p(\sigma_1^2)$$

$$\propto \prod_{i=1}^{n_1} \exp\left\{-\frac{(\ln(y_{1i}) - \mu_1)^2}{2\sigma_1^2}\right\} \cdot \exp\left\{-\frac{(\delta - m_1)^2}{2s_1^2}\right\} \mathbb{1}_{\delta>0}$$

$$= \prod_{i=1}^{n_1} \exp\left\{-\frac{(\ln y_{1i} - (\delta+\mu_2))^2}{2\sigma_1^2}\right\} \exp\left\{-\frac{(\delta - m_1)^2}{2s_1^2}\right\} \mathbb{1}_{\delta>0}$$

$$\propto \exp\left\{-\frac{n(\delta+\mu_2)^2 - 2(\delta+\mu_2)\sum \ln y_{1i}}{2\sigma_1^2}\right\} \exp\left\{-\frac{\delta^2 - 2m_1\delta}{2s_1^2}\right\} \mathbb{1}_{\delta>0}$$

$$\propto \exp\left\{-\frac{n(\delta^2 + 2\delta\mu_2) - 2\delta \sum_i \ln y_{1i}}{2\sigma_1^2}\right\} \exp\left\{-\frac{\delta^2 - 2m_1\delta}{2s_1^2}\right\} \mathbb{1}_{\delta>0}$$

$$= \exp\left\{-\frac{\delta^2 + 2\delta\mu_2 - 2\delta \frac{\sum \ln y_{1i}}{n_1}}{2\sigma_1^2/n_1} - \frac{\delta^2 - 2m_1\delta}{2s_1^2}\right\} \mathbb{1}_{\delta>0}$$

$$= \exp\left\{-\frac{s_1^2\delta^2 + 2\delta\mu_2 s_1^2 - 2\delta s_1^2 \frac{\sum \ln y_{1i}}{n_1} + \left(\delta^2\frac{\sigma_1^2}{n_1} - 2m_1\delta\frac{\sigma_1^2}{n_1}\right)}{2\sigma_1^2 s_1^2/n_1}\right\} \mathbb{1}_{\delta>0}$$

$$= \exp\left\{-\frac{\left(s_1^2 + \frac{\sigma_1^2}{n_1}\right)\delta^2 + 2\left(\mu_2 s_1^2 - s_1^2\frac{\sum \ln y_{1i}}{n_1} - m_1\frac{\sigma_1^2}{n_1}\right)\delta}{2\sigma_1^2 s_1^2/n_1}\right\} \mathbb{1}_{\delta>0}$$

$$= \exp\left\{-\frac{\left(s_1^2 + \frac{\sigma_1^2}{n_1}\right)\delta^2 - 2\left(\frac{s_1^2\sum \ln y_{1i} + m_1\sigma_1^2 - n_1\mu_2 s_1^2}{n_1}\right)\delta}{2\sigma_1^2 s_1^2/n_1}\right\} \mathbb{1}_{\delta>0}$$

$$= \exp\left\{-\frac{\delta^2 - 2\left(\frac{s_1^2\sum_i \ln y_{1i} + m_1\sigma_1^2 - n_1\mu_2 s_1^2}{n_1\left(s_1^2 + \frac{\sigma_1^2}{n_1}\right)}\right)\delta}{2\,\frac{\sigma_1^2 s_1^2}{n_1\left(s_1^2 + \frac{\sigma_1^2}{n_1}\right)}}\right\} \mathbb{1}_{\delta>0}$$

$$\propto \mathrm{No}\left(\delta \,\middle|\, \frac{s_1^2\sum_i \ln y_{1i} + m_1\sigma_1^2 - n_1\mu_2 s_1^2}{n_1 s_1^2 + \sigma_1^2},\ \frac{\sigma_1^2 s_1^2}{n_1 s_1^2 + \sigma_1^2}\right)$$

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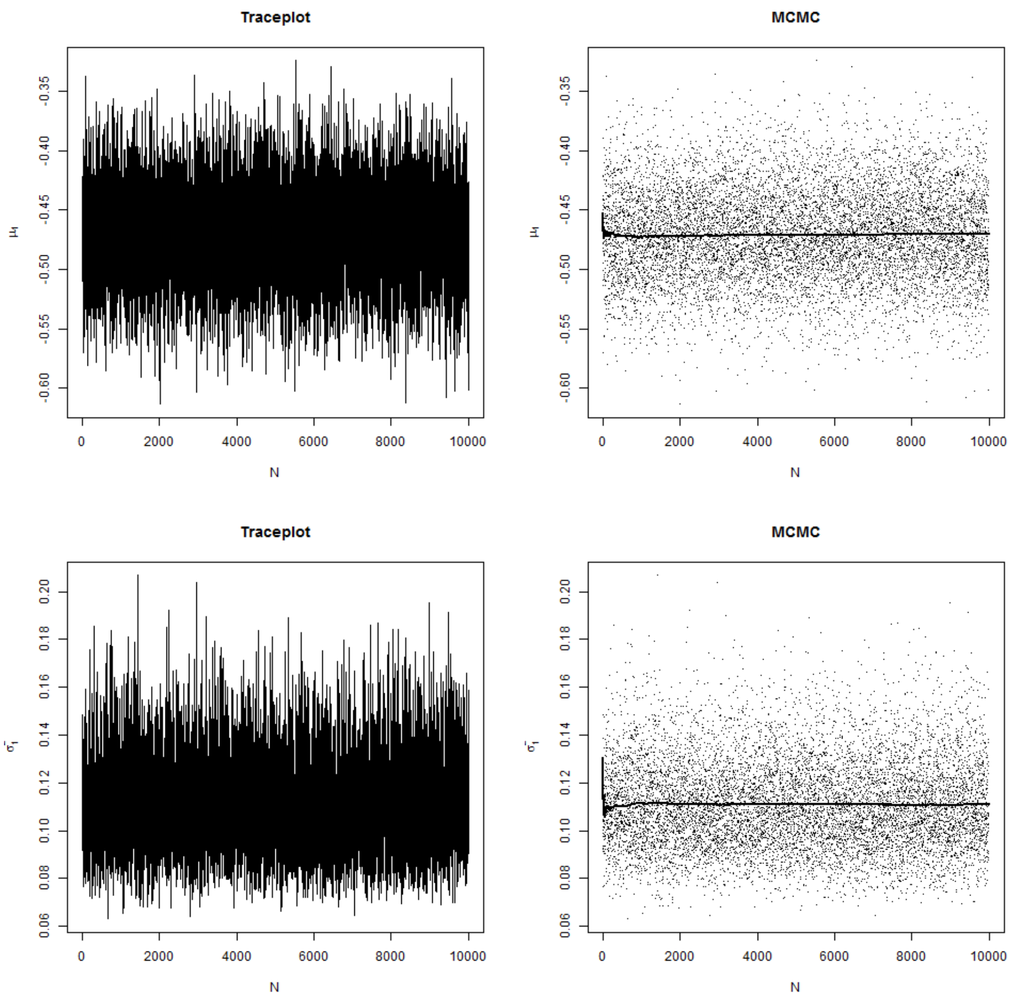
2. Below are traceplots of μ_1 and σ_1^2:



3. Below are point estimates and 95% confidence intervals:

	Mean	Confidence Interval
μ_1	-0.4708718	[-0.5466545, -0.3949325]
μ_2	-1.377839	[-1.5240060, -1.2294750]
σ_1^2	0.1104881	[0.07978934, 0.15334020]
σ_2^2	0.1597617	[0.09405123, 0.26961285]

4. Posterior probability that $\mu_1 > \mu_2$ is 1. Posterior probability that $\sigma_1^2 > \sigma_2^2$ is 0.136.

5. The posterior probability that the pollution level on a randomly chosen future Tuesday is higher than the pollution level on a randomly chosen future Saturday is 0.9579.

See below for R code:

```
## load data ##
data = read.table("data.txt", header = T)
library(MCMCpack)
data$weekend = rep(0,nrow(data))
data$weekend[data$day==’Saturday’ | data$day==’Sunday’] = 1
X = data[,-2]
Y1 = subset(X, weekend==0)
row.names(Y1) = NULL
Y1 = Y1[,1]
Y2 = subset(X, weekend==1)
row.names(Y2) = NULL
Y2 = Y2[,1]

## set priors ##
m1 = m2 = 0
s21 = s22 = 1
a1 = a2 = 1
b1 = b2 = 1
n1 = length(Y1)
n2 = length(Y2)
B = 2000
N = 12000

## set seed ##
sig1.temp = 1 #for delta variance
sig2.temp = 1

## sample priors ##
mu2.temp = rnorm(1, mean = (s22*sum(log(Y2))+sig2.temp*m2)/(s22*n2 + sig2.temp),
                 sd = sqrt((sig2.temp*s22)/(s22*n2+sig2.temp)))
del.temp = -1
while(del.temp < 0){
del.temp = rnorm(1, mean = (s21*sum(log(Y1))+m1*sig1.temp-n1*mu2.temp*s21)/(n1*s21+sig1.temp),
                 sd = sqrt(sig1.temp*s21/(n1*s21+sig1.temp)) )
}
mu1.temp = mu2.temp + del.temp


DEL = rep(0, N)
MU1 = rep(0, N)
MU2 = rep(0, N)
SIG1 = rep(0, N)
SIG2 = rep(0, N)


for(i in 1:N){
  sig1.temp = rinvgamma(1, shape = a1+n1/2, scale = b1 + 0.5*sum((log(Y1)-mu1.temp)^2))

  sig2.temp = rinvgamma(1, shape = a2+n2/2, scale = b2 + 0.5*sum((log(Y2)-mu2.temp)^2))

  mu2.temp = rnorm(1, mean = (s22*sum(log(Y2))+sig2.temp*m2)/(s22*n2 + sig2.temp),
                   sd = sqrt((sig2.temp*s22)/(s22*n2+sig2.temp)))

```

```
  del.temp = -1
  while(del.temp <= 0){
    del.temp = rnorm(1, mean = (s21*sum(log(Y1))+m1*sig1.temp-n1*mu2.temp*s21)/(n1*s21+sig1.temp),
                     sd = sqrt(sig1.temp*s21/(n1*s21+sig1.temp)) )
  }

  mu1.temp = mu2.temp + del.temp

  DEL[i] = del.temp
  MU1[i] = mu1.temp
  MU2[i] = mu2.temp
  SIG1[i] = sig1.temp
  SIG2[i] = sig2.temp
}


MU1 = MU1[(B+1):N]
MU2 = MU2[(B+1):N]
SIG1 = SIG1[(B+1):N]
SIG2 = SIG2[(B+1):N]

N = 10000
## Running Avg ##
MU1.sum = rep(0, N)
MU1.avg = rep(0, N)
MU1.avg[1] = MU1[1]
MU1.sum[1] = MU1[1]
for(i in 2:N){
  MU1.sum[i] = MU1[i] + MU1.sum[i-1]
  MU1.avg[i] = MU1.sum[i]/i
}

SIG1.sum = rep(0, N)
SIG1.avg = rep(0, N)
SIG1.avg[1] = SIG1[1]
SIG1.sum[1] = SIG1[1]
for(i in 2:N){
  SIG1.sum[i] = SIG1[i] + SIG1.sum[i-1]
  SIG1.avg[i] = SIG1.sum[i]/i
}


## Plot MU1 and SIG1 ##
png("MU1plot.png")
traceplot(mcmc(MU1), xlab = "N", ylab = expression(mu[1]),
     main = "Traceplot")
dev.off()

png("SIG1plot.png")
traceplot(mcmc(SIG1), xlab = "N", ylab = expression(sigma[1]^{2}),
     main = "Traceplot")
dev.off()

png("MU1MCMC.png")
```

```
plot(MU1, pch = '.', xlab = "N", ylab = expression(mu[1]),
          main = "MCMC")
lines(MU1.avg, lwd = 2)
dev.off()

png("SIG1MCMC.png")
plot(SIG1, pch = '.', xlab = "N", ylab = expression(sigma[1]^{2}),
          main = "MCMC")
lines(SIG1.avg, lwd = 2)
dev.off()

## Mean ##
MU1.mean = mean(MU1)
MU2.mean = mean(MU2)
SIG1.mean = mean(SIG1)
SIG2.mean = mean(SIG2)

## CI ##
MU1.ci = quantile(MU1, c(0.025, 0.975))
MU2.ci = quantile(MU2, c(0.025, 0.975))
SIG1.ci = quantile(SIG1, c(0.025, 0.975))
SIG2.ci = quantile(SIG2, c(0.025, 0.975))

## Posterior Probability ##
mean(MU1>MU2)
mean(SIG1>SIG2)

## Posterior Predictive ##
wk = we = rep(0, 10000)
for(i in 1:10000){
  wk[i] = rlnorm(1, meanlog = MU1[i], sdlog = sqrt(SIG1[i]))
  we[i] = rlnorm(1, meanlog = MU2[i], sdlog = sqrt(SIG2[i]))
}
mean(wk>we)
```