Michael Lin
HW 11
Stat 360

2) In Gibbs sampler, the proposal distribution for U is the full conditional distribution of U given $V = v$. Thus in Metropolis - Hastings sampler,

$$J_u(u^* \mid u^{(s)}, v^{(s)}) = p_0(u^* \mid v^{(s)})$$

where J_u is proposal distribution and p_0 is the full conditional.

The Metropolis - Hastings acceptance ratio for U,

$$r = \frac{p_0(u^*, v^{(s)})}{p_0(u^{(s)}, v^{(s)})} \times \frac{J_u(u^{(s)} \mid u^*, v^{(s)})}{J_u(u^* \mid u^{(s)}, v^{(s)})}$$

$$= \frac{p_0(u^*, v^{(s)})}{p_0(u^{(s)}, v^{(s)})} \times \frac{p_0(u^{(s)} \mid v^{(s)})}{p_0(u^* \mid v^{(s)})}$$

$$= \frac{\cancel{p_0(u^* \mid v^{(s)})}\,\cancel{p_0(v^{(s)})}\,\cancel{p_0(u^{(s)} \mid v^{(s)})}}{\cancel{p_0(u^{(s)} \mid v^{(s)})}\,\cancel{p_0(v^{(s)})}\,\cancel{p_0(u^* \mid v^{(s)})}}$$

$$= 1$$

If we propose a value from the full conditional, the acceptance probability is 1, which is equivalent to Gibbs sampler.

3) Let u and v be 2 possible states

a) $$T = \begin{array}{c|cc} & u & v \\ \hline u & 0.5 & 0.5 \\ v & 0 & 1 \end{array}$$

T is not irreducible because $\forall t$,

$$\mathbb{P}(X_t = u \mid X_0 = v) = 0$$

b) $$T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

The identity matrix has that given <u>any</u> row vector π, $\pi I = \pi$, i.e., there are infinitely many possibilities for π.

c) $$T = \begin{array}{c|ccc} & u & v & w \\ \hline u & 0.7 & 0.2 & 0.1 \\ v & 0.1 & 0.7 & 0.2 \\ w & 0.2 & 0.1 & 0.7 \end{array}$$

$$\pi = \begin{array}{ccc} u & v & w \\ (1/3 & 1/3 & 1/3) \end{array}$$

$\pi T = \pi$, so π is stationary distribution of T, but does not have detailed balance, e.g.,

$$\pi(u)\, T(uv) = 1/3 \cdot 0.2 = 1/15$$
$$\pi(v)\, T(vu) = 1/3 \cdot 0.1 = 1/30$$

Since they don't equal, π and T do not have detailed balance.

Scanned by CamScanner

$\forall u, v$ finite

d) Suppose transition matrix T is irreducible.
Thus, there is a "t" such that $\mathbb{P}(X_t = u \mid X_0 = v) > 0$
In addition by hypothesis that there is a state "b" such that $T_{bb} > 0$, at least one diagonal entry is positive. Therefore, for all states besides "b" (say "a"), $t_a \geq 1$ such that

$$\mathbb{P}(X_{t_a} = a \mid X_0 = a) > 0$$

and for state "b", $t_b = 1$ such that

$$P(X_{t_b} = b \mid X_0 = b) > 0$$

Clearly, $GCD\{t_a, t_b\} = 1$ for all "a".
Thus the MC is aperiodic.

4)

$$\pi^{(1)} = \pi^{(0)} T = (0.7 \quad 0.2 \quad 0.1)$$
$$\pi^{(2)} = \pi^{(1)} T = (0.53 \quad 0.29 \quad 0.18)$$
$$\pi^{(3)} = \pi^{(2)} T = (0.436 \quad 0.327 \quad 0.237)$$
$$\pi^{(4)} = \pi^{(3)} T = (0.3853 \quad 0.3398 \quad 0.2749)$$
$$\pi^{(5)} = \pi^{(4)} T = (0.35867 \quad 0.34241 \quad 0.29892)$$
$$\pi^{(6)} = \pi^{(5)} T = (0.34509 \quad 0.34131 \quad 0.31359)$$
$$\pi^{(7)} = \pi^{(6)} T = (0.33842 \quad 0.3393 \quad 0.32229)$$
$$\pi^{(8)} = \pi^{(7)} T = (0.33528 \quad 0.33742 \quad 0.3273)$$

$$\pi = (1/3 \quad 1/3 \quad 1/3) \; ?$$

$$\pi T = (1/3 \quad 1/3 \quad 1/3) \begin{pmatrix} 7/10 & 2/10 & 1/10 \\ 1/10 & 7/10 & 2/10 \\ 2/10 & 1/10 & 7/10 \end{pmatrix} = (1/3 \quad 1/3 \quad 1/3)$$

so $\pi = (1/3 \quad 1/3 \quad 1/3)$

5) • First, we need to find proposal distribution:

$$a^* = a_i e^{x_i} \implies x_i = \log(a^*) - \log(a_i)$$

• Given $x_i \sim No(0, \sigma^2)$, we know

$$p_{x_i}(x_i) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{x_i^2}{2\sigma^2}\right\}$$

• Using transformation formula, we have

$$p_{a^*}(a^* | a_i) = p_{x_i}(\log(a^*) - \log(a_i)) \left| \frac{\partial}{\partial a^*}(\log(a^*) - \log(a_i)) \right|$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(\log a^* - \log a_i)^2}{2\sigma^2}\right\} \cdot \left|\frac{1}{a^*}\right|$$

$$= LN(a^* | \log a_i, \sigma^2)$$

• The acceptance ratio is

$$r = \frac{p(y, a^*, b_i)}{p(y, a_i, b_i)} \times \frac{p_{a^*}(a^* | a_i)}{p_{a^*}(a_i | a^*)}$$

$$= \frac{p(y | a^*, b_i)\, p(a^*)\, \cancel{p(b_i)}}{p(y | a_i, b_i)\, p(a_i)\, \cancel{p(b_i)}} \times \frac{p_{a^*}(a^* | a_i)}{p_{a^*}(a_i | a^*)}$$

$$= \frac{\prod_{j=1}^{n} Gamma(y_j | a^*, b_i)}{\prod_{j=1}^{n} Gamma(y_j | a_i, b_i)} \times \frac{Gamma(a^* | r, s)}{Gamma(a_i | r, s)} \times \frac{LN(a^* | \log a_i, \sigma^2)}{LN(a_i | \log a^*, \sigma^2)}$$

$$= \frac{\left(\prod_{j=1}^{n} \frac{b_i^{a^*}}{\Gamma(a^*)} y_j^{a^*-1} \cancel{e^{-b_i y_j}}\right)}{\left(\prod_{j=1}^{n} \frac{b_i^{a_i}}{\Gamma(a_i)} y_j^{a_i-1} \cancel{e^{-b_i y_j}}\right)} \times \frac{\cancel{\frac{s^r}{\Gamma(r)}} a^{*\,r-1} e^{-sa^*}}{\cancel{\frac{s^r}{\Gamma(r)}} a_i^{r-1} e^{-sa_i}} \times \frac{\frac{1}{a^*\sqrt{2\pi\sigma^2}} \cancel{\exp\left\{-\frac{(\log a^* - \log a_i)^2}{2\sigma^2}\right\}}}{\frac{1}{a_i\sqrt{2\pi\sigma^2}} \cancel{\exp\left\{-\frac{(\log a_i - \log a^*)^2}{2\sigma^2}\right\}}}$$

$$= \frac{\frac{b_i^{na^*}}{\Gamma^n(a^*)} \left(\prod_{j=1}^{n} y_j\right)^{a^*-1}}{\frac{b_i^{na_i}}{\Gamma^n(a_i)} \left(\prod_{j=1}^{n} y_j\right)^{a_i-1}} \cdot \left(\frac{a^*}{a_i}\right)^{r-1} e^{-s(a^*-a_i)} \cdot \frac{a_i}{a^*}$$

$$= \frac{b_i^{n(a^*-a_i)}}{\frac{\Gamma^n(a^*)}{\Gamma^n(a_i)}} \left(\prod_{j=1}^{n} y_j\right)^{a^*-a_i} \left(\frac{a^*}{a_i}\right)^{r-2} e^{-s(a^*-a_i)}$$

- also, we need $p(b|y,a) = p(b|y)$.

$$p(b|y) \underset{b}{\propto} p(y|b)\,p(b) = \left[\prod_{j=1}^{n} Gamma(y_j|a,b)\right] Gamma(u,v)$$

$$= \left(\prod_{j=1}^{n} \frac{b^a}{\Gamma(a)} y_j^{a-1} e^{-by_j}\right)\left(\frac{v^u}{\Gamma(u)} b^{u-1} e^{-vb}\right)$$

$$\propto \left(b^{na} e^{-b\sum y_j}\right) b^{u-1} e^{-vb} = b^{(u+na)-1} e^{-b(v+\sum y_j)}$$

$$\propto Gamma\left(b \,\middle|\, u+na,\ v+\sum_{j=1}^{n} y_j\right)$$

The MH-within-Gibbs sample is as follows:

1) Set initial starting point
   - sample $a_r \sim Gamma(r,s)$

2) Update b_i
   - sample $b_i \sim Gamma(u+na_i,\ v+\sum_{j=1}^{n} y_j)$

3) Update a_i
   - sample $a^* \sim LN(\log a_{i-1}, \sigma^2)$
   - compute acceptance ratio r as derived above
   - set $a_i = a^*$ with probability $\min(1,r)$ or set $a_i = a_{i-1}$ with probability $\max(0, 1-r)$.

4) Repeat (2), (3) until sampler converges.