STA 360: Lab 8

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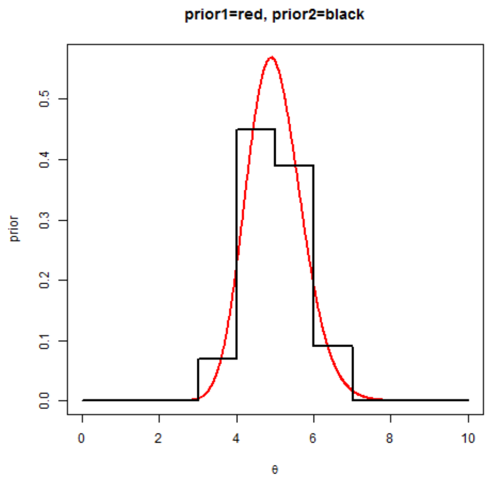
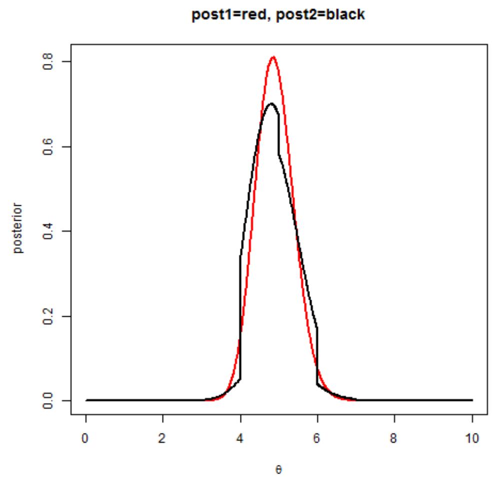
1. Given the following prior:
$$p_1(\theta) = \text{Gamma}(\theta|a, b)$$
the posterior is:
$$\begin{aligned} p_1(\theta|y_{1:n}) &\propto p_1(\theta)p(y_{1:n}|\theta) \\ &\propto \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}e^{-n\theta}\theta^{\sum y_i} \\ &\propto \theta^{a+\sum y_i - 1}e^{-(b+n)\theta} \\ &\propto \text{Gamma}(\theta|a + \sum y_i, b+n) \end{aligned}$$

2. Given the following prior:
$$p_2(\theta) = \begin{cases} 0.07 & : \theta \in (3,4] \\ 0.45 & : \theta \in (4,5] \\ 0.39 & : \theta \in (5,6] \\ 0.09 & : \theta \in (6,7] \end{cases}$$
the posterior is:
$$\begin{aligned} p_2(\theta|y_{1:n}) &= \frac{p_2(\theta)p(y_{1:n}|\theta)}{\int_{-\infty}^{\infty} p_2(\theta)p(y_{1:n}|\theta)d\theta} \\ &= \frac{p_2(\theta)\frac{e^{-n\theta}\theta^{\sum y_i}}{\prod y_i!}}{\int_{-\infty}^{\infty} p_2(\theta)\frac{e^{-n\theta}\theta^{\sum y_i}}{\prod y_i!}d\theta} \\ &= \frac{p_2(\theta)e^{-n\theta}\theta^{\sum y_i}}{\int_{-\infty}^{\infty} p_2(\theta)e^{-n\theta}\theta^{\sum y_i}d\theta} \\ &= \frac{p_2(\theta)e^{-n\theta}\theta^{\sum y_i}}{\int_3^4 0.07e^{-n\theta}\theta^{\sum y_i}d\theta + \cdots + \int_6^7 0.09e^{-n\theta}\theta^{\sum y_i}d\theta} \\ &= \frac{p_2(\theta)e^{-n\theta}\theta^{\sum y_i}}{\frac{\Gamma(\sum y_i+1)}{n^{\sum y_i+1}}\{0.07[F(4)-F(3)] + \cdots + 0.09[F(7)-F(6)]\}} \end{aligned}$$

3. See below for plots of prior and posterior distributions:





4. The 95% posterior credible interval for $p_1(\theta|y_{1:n})$ is [3.978057, 5.916589]. The 95% posterior credible interval for $p_2(\theta|y_{1:n})$ is $[x_1, x_2]$ where:

$$\int_{-\infty}^{x_1} p_2(\theta|y_{1:n}) = 0.025$$

$$\int_{-\infty}^{x_2} p_2(\theta|y_{1:n}) = 0.975$$

These integrals can be calculated by noting that

$$p_2(\theta|y_{1:n}) \propto \text{Gamma}(\theta|\sum y_i + 1, n)$$

and simply evaluating the appropriate c.d.f.

R code:

```
y = c(2,1,9,4,3,3,7,7,5,7)
x = seq(0,10,0.01)

## Define priors ##
n = 10
prior1 = dgamma(x, shape = 50, rate = 10)
prior2 = rep(0,length(x))
prior2[x<=3 | x>7] = 0
prior2[x>3 & x<=4] = 0.07
prior2[x>4 & x<=5] = 0.45
prior2[x>5 & x<=6] = 0.39
prior2[x>6 & x<=7] = 0.09

## Graph priors ##
png("prior.png")
plot(x, prior1,type="l", col='red', lwd = 2, main = "prior1=red, prior2=black",
     xlab = expression(theta), ylab="prior")
lines(x, prior2, lwd = 2)
dev.off()

## Posteriors ##
post1 = dgamma(x, shape = 50+sum(y), rate = 10+n)
c = gamma(sum(y)+1)/(n^(sum(y)+1))*(0.07*(pgamma(4, sum(y)+1,n) - pgamma(3, sum(y)+1,n))
                                    +0.45*(pgamma(5, sum(y)+1,n) - pgamma(4, sum(y)+1,n))
                                    +0.39*(pgamma(6, sum(y)+1,n) - pgamma(5, sum(y)+1,n))
                                    +0.09*(pgamma(7, sum(y)+1,n) - pgamma(6, sum(y)+1,n)))
post2.func = function(t) prior2*t^sum(y)*exp(-n*t)/c
post2 = post2.func(x)

## Graph posteriors ##
png("post.png")
plot(x,post1,type="l", col='red', lwd = 2, main = "post1=red, post2=black",
     xlab = expression(theta), ylab="posterior")
lines(x,post2, lwd = 2)
dev.off()

## 95% CI ##
qpost1 = qgamma(c(0.025, 0.975), shape = 50+sum(y), rate = 10+n)
```