COURSE MATERIALS (STUDENT COPY)

PRELIMINARIES FOR PHYSICS HONOURS I

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PRELIM 1: VECTOR MULTIPLICATION

1.1. Revision on Vectors

1.1.1. Definition

Vector is a quantity that has magnitude and direction. It is represented by an arrow on top of a letter. The notation is very crucial as we can't equate vector to scalar. For example, the following are ill representations of vectors:

$$\vec{\mathbf{A}} = 5$$
$$\vec{\mathbf{v}} = -2$$
$$\vec{\mathbf{A}} + B = 3 \text{ to the right}$$
$$a = \frac{\Delta\vec{\mathbf{v}}}{\Delta t}$$

To represent directions, various notations can be used:

- words, e.g., "to the right", "upwards", "into the page", "30° east of north", "S49W"etc.
- signs (only for one-dimensional vectors), e.g., "+" means to the right, "$-$" means to the left.
- unit vectors, e.g., "$\hat{\mathbf{i}}$" means towards positive x-axis, "$\hat{\mathbf{j}}$" means towards positive y-axis, "$\hat{\mathbf{k}}$" means towards positive z-axis.

The magnitude of vector $\vec{\mathbf{A}}$ is written as $|\vec{\mathbf{A}}|$ or A. When dealing with calculus-based physics, it is important to stick to this convention, e.g., F represents the magnitude of force, v represents the magnitude of velocity, etc., and therefore they are always positive. We will write $F = kx$ for elastic force, for instance, instead of $F = -kx$.

Is sign important then? Sign is more important when we deal with scalars, rather than vectors. For example, the work done by friction is commonly negative, and hence, positive result will be marked as a wrong answer. There is no such thing as "magnitude of work done" or "absolute value of work done" and they are physically meaningless.

1.1.2. Unit Vector

Unit vector is a vector with magnitude one. A unit vector in the direction of vector $\vec{\mathbf{A}}$ is written as $\hat{\mathbf{A}}$.

$$\hat{\mathbf{A}} = \frac{\vec{\mathbf{A}}}{A} \tag{1.1}$$

Common unit vectors used in Cartesian coordinates are $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. For example, if vector $\vec{\mathbf{A}} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$, its magnitude is $A = 5$. Hence,

$$\hat{\mathbf{A}} = \frac{3}{5}\hat{\mathbf{i}} + \frac{4}{5}\hat{\mathbf{j}}$$

Vector $\vec{\mathbf{A}}$ can also be expressed as a column vector:

$$\vec{\mathbf{A}} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$$

1.1.3. Vector Addition and Subtraction

If two vectors are expressed in terms of unit vectors in Cartesian coordinates, the addition or subtraction of these two vectors can be done via the addition or subtraction of each of their components. Otherwise, if the magnitude of each vector and the angle between the two vectors are known, Cosine rule can be applied.

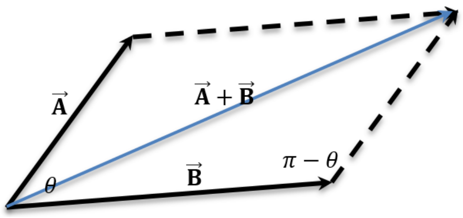

Fig. 1.1

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$$\left|\vec{\mathbf{A}}+\vec{\mathbf{B}}\right| = \sqrt{A^2+B^2-2AB\cos(\pi-\theta)}$$
$$\left|\vec{\mathbf{A}}+\vec{\mathbf{B}}\right| = \sqrt{A^2+B^2+2AB\cos\theta}$$

(1.2)

where θ denotes the angle between vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.

By noticing that $\vec{\mathbf{A}}-\vec{\mathbf{B}} = \vec{\mathbf{A}}+\left(-\vec{\mathbf{B}}\right)$,

$$\left|\vec{\mathbf{A}}-\vec{\mathbf{B}}\right| = \sqrt{A^2+B^2-2AB\cos\theta}$$

(1.3)

In physics, **displacement vector** is defined as the change in the position vector, $\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1$.

1.2. Dot Product (a.k.a. Scalar Product) of Vectors

The **dot product** between vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is defined as

$$\vec{\mathbf{A}}\cdot\vec{\mathbf{B}} = AB\cos\theta$$

(1.4)

where θ denotes the angle between vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. <u>The result of a dot product is a scalar</u>.

The following are important corollaries:

- $\vec{\mathbf{A}}\cdot\vec{\mathbf{B}} = \vec{\mathbf{B}}\cdot\vec{\mathbf{A}}$
- $\vec{\mathbf{A}}\cdot\vec{\mathbf{A}} = A^2$
- $\vec{\mathbf{A}}\cdot\left(\vec{\mathbf{B}}\pm\vec{\mathbf{C}}\right) = \vec{\mathbf{A}}\cdot\vec{\mathbf{B}}\pm\vec{\mathbf{A}}\cdot\vec{\mathbf{C}}$
- $\vec{\mathbf{A}}\cdot\vec{\mathbf{B}} = 0 \leftrightarrow \vec{\mathbf{A}}\perp\vec{\mathbf{B}}$
- $\vec{\mathbf{A}}\cdot\vec{\mathbf{B}} = AB \leftrightarrow \vec{\mathbf{A}}\parallel\vec{\mathbf{B}}$
- $\vec{\mathbf{A}}\cdot\vec{\mathbf{B}} = -AB \leftrightarrow \vec{\mathbf{A}}\parallel-\vec{\mathbf{B}}$

The dot products of unit vectors are listed below:

$$\begin{array}{ccc} \hat{\mathbf{i}}\cdot\hat{\mathbf{i}}=1 & \hat{\mathbf{j}}\cdot\hat{\mathbf{i}}=0 & \hat{\mathbf{k}}\cdot\hat{\mathbf{i}}=0 \\ \hat{\mathbf{i}}\cdot\hat{\mathbf{j}}=0 & \hat{\mathbf{j}}\cdot\hat{\mathbf{j}}=1 & \hat{\mathbf{k}}\cdot\hat{\mathbf{j}}=0 \\ \hat{\mathbf{i}}\cdot\hat{\mathbf{k}}=0 & \hat{\mathbf{j}}\cdot\hat{\mathbf{k}}=0 & \hat{\mathbf{k}}\cdot\hat{\mathbf{k}}=1 \end{array}$$

(1.5)

This tells us that if two vectors are expressed in terms of unit vectors in Cartesian coordinates, their dot product can be computed by multiplying each corresponding component and adding them up.

$$\vec{\mathbf{A}} = \begin{pmatrix}A_x\\A_y\\A_z\end{pmatrix}, \vec{\mathbf{B}} = \begin{pmatrix}B_x\\B_y\\B_z\end{pmatrix} \rightarrow \vec{\mathbf{A}}\cdot\vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$$

(1.6)

For example,

$$\vec{\mathbf{A}} = \begin{pmatrix}1\\2\\3\end{pmatrix} \quad \vec{\mathbf{B}} = \begin{pmatrix}-2\\1\\4\end{pmatrix}$$

the angle between them is

Some applications of dot product in physics:

- work, $W = \vec{\mathbf{F}}\cdot\vec{\mathbf{s}}$
- power, $P = \vec{\mathbf{F}}\cdot\vec{\mathbf{v}}$

1.3. Cross Product (a.k.a. Vector Product) of Vectors

The **cross product** between vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is defined as

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = AB \sin\theta\, \hat{\mathbf{n}} \tag{1.7}$$

where θ denotes the angle between vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, and $\hat{\mathbf{n}}$ is the unit vector in the perpendicular direction to the plane formed by $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. The result of a cross product is a vector.

Right-hand rule is used to determine the direction of the cross product. As long as your four fingers can turn from the direction of the first vector to the second one, the cross product of the two vectors points to the direction of the thumb, as shown in Fig. 1.2. This also implies that cross product is anticommutative.

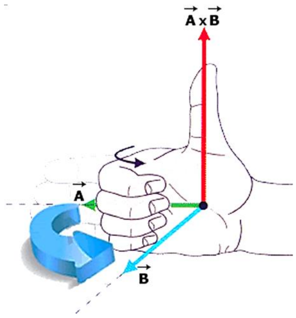

Fig. 1.2

The following are important corollaries:

- $\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}}$
- $\vec{\mathbf{A}} \times \vec{\mathbf{A}} = \vec{\mathbf{0}}$
- $\vec{\mathbf{A}} \times \left(\vec{\mathbf{B}} \pm \vec{\mathbf{C}}\right) = \vec{\mathbf{A}} \times \vec{\mathbf{B}} \pm \vec{\mathbf{A}} \times \vec{\mathbf{C}}$
- $\left(\vec{\mathbf{A}} \pm \vec{\mathbf{B}}\right) \times \vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{C}} \pm \vec{\mathbf{B}} \times \vec{\mathbf{C}}$
- $\vec{\mathbf{A}} \times \vec{\mathbf{B}} = 0 \leftrightarrow \vec{\mathbf{A}} \parallel \vec{\mathbf{B}}$ or $\vec{\mathbf{A}} \parallel -\vec{\mathbf{B}}$
- $\left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right| = AB \leftrightarrow \vec{\mathbf{A}} \perp \vec{\mathbf{B}}$

The cross products of unit vectors are listed below:

$$\begin{array}{lll}
\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \vec{\mathbf{0}} & \hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}} & \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \\
\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} & \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \vec{\mathbf{0}} & \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}} \\
\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}} & \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}} & \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \vec{\mathbf{0}}
\end{array} \tag{1.8}$$

The easiest way to do cross product between two vectors, when they are expressed in terms of unit vectors in Cartesian coordinates, is by computing determinant of a 3×3 matrix:

$$\vec{\mathbf{A}} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}, \vec{\mathbf{B}} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} \rightarrow \vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{pmatrix} A_yB_z - A_zB_y \\ A_zB_x - A_xB_z \\ A_xB_y - A_yB_x \end{pmatrix} \tag{1.9}$$

For example,

$$\vec{\mathbf{A}} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \vec{\mathbf{B}} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$$

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} =$$

Some applications of cross product in physics:

- torque, $\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$
- angular momentum, $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$
- magnetic force, $\vec{\mathbf{F}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$

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1.4. Physical Interpretation and Vector Identities

Notice that the area of a parallelogram can be expressed by the vector cross product of any two adjacent sides.

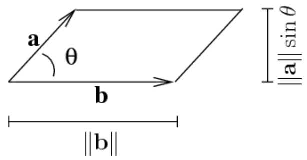

Fig. 1.3

$$A = ab\sin\theta$$

and that justifies the definition of <u>area as a vector physical quantity</u>,

$$\vec{\mathbf{A}} \coloneqq \vec{\mathbf{a}} \times \vec{\mathbf{b}} \tag{1.10}$$

and that the magnitude of $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ should equal the magnitude of $\vec{\mathbf{b}} \times \vec{\mathbf{a}}$, i.e.,

$$\left|\vec{\mathbf{a}} \times \vec{\mathbf{b}}\right| = \left|\vec{\mathbf{b}} \times \vec{\mathbf{a}}\right|$$

In physics, area is commonly regarded as a vector physical quantity. The direction of the area vector is perpendicular to the plane formed by $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. It is also easy to see that the area of a triangle can be expressed by <u>half</u> of the vector cross product of any two adjacent sides.

Scalar Triple Product: Volume of a Parallelepiped

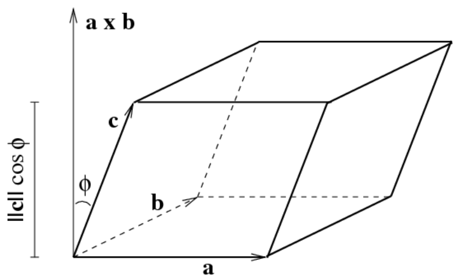

Fig. 1.4

$$V = \text{base area} \cdot \text{height} = (ab\sin\theta)\cdot(c\cos\phi)$$

$$V = \left(\vec{\mathbf{a}} \times \vec{\mathbf{b}}\right)\cdot\vec{\mathbf{c}} \tag{1.11}$$

Eq. (1.11) tells us that volume is a scalar physical quantity. As $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ are interchangeable,

$$\vec{\mathbf{a}}\cdot\left(\vec{\mathbf{b}} \times \vec{\mathbf{c}}\right) = \vec{\mathbf{b}}\cdot\left(\vec{\mathbf{c}} \times \vec{\mathbf{a}}\right) = \vec{\mathbf{c}}\cdot\left(\vec{\mathbf{a}} \times \vec{\mathbf{b}}\right) = \begin{vmatrix} a_x & b_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \tag{1.12}$$

The identity in Eq. (1.12) is referred to as the **scalar triple product**.

Stop to Think 1.1

(a) What will be the scalar triple product if any two vectors are equal?

(b) Hence, what can you deduce about any of the two vectors if the scalar triple product equals zero?

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Vector Triple Product (a.k.a. Lagrange's Formula)

The **vector triple product** is given by

$$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{b}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) - \vec{\mathbf{c}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}) \tag{1.13}$$

As $\vec{\mathbf{b}} \times \vec{\mathbf{a}}$ points perpendicularly with respect to the plane of $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$, thus $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$ must be in the plane of $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$. Vector triple product is also known as the **BAC-CAB rule**.

The relation between position vector $\vec{\mathbf{r}}$, angular velocity vector $\vec{\boldsymbol{\omega}}$ and linear velocity vector $\vec{\mathbf{v}}$ is given by $\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}$, and all the three vectors are known to be mutually perpendicular. Now, use vector identities to simplify the following expressions:

$\vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}} =$

$\vec{\mathbf{r}} \times \vec{\mathbf{v}} =$

Power is defined as the rate of change of energy, and it can be expressed as $P = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}$, where $\vec{\mathbf{F}}$ is force and $\vec{\mathbf{v}}$ is velocity. Consider a charge q of mass m, moving with velocity $\vec{\mathbf{v}}$ in a region of magnetic field $\vec{\mathbf{B}}$. Show that the speed of the charge remains constant.

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1.5. Tutorials

(1) The magnitude of the resultant of two vectors is four times the smaller one and makes an angle of 60° with the smaller one. What is the ratio of the magnitude of the vectors? What is the angle between them?

(2) If $|\vec{\mathbf{A}}+\vec{\mathbf{B}}| = |\vec{\mathbf{A}}-\vec{\mathbf{B}}|$, prove that $\vec{\mathbf{A}} \perp \vec{\mathbf{B}}$.

(3) If $|\vec{\mathbf{A}}| = 8$ and $|\vec{\mathbf{B}}| = 15$, and the angle between them is 45°, what is the angle between $\vec{\mathbf{A}}+\vec{\mathbf{B}}$ and $\vec{\mathbf{A}}$?

(4) If $\vec{\mathbf{A}}+\vec{\mathbf{B}} \perp \vec{\mathbf{A}}-\vec{\mathbf{B}}$, prove that $|\vec{\mathbf{A}}| = |\vec{\mathbf{B}}|$.

(5) Work done is defined as the dot product between force vector and displacement vector. If the work done by a force is negative, what can you conclude? Does it make sense to take the absolute value of this work done and call it 'the magnitude of work done'?

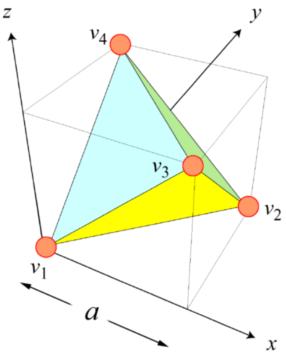
(6) If vectors $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$ and $\vec{\mathbf{C}}$ satisfy $\vec{\mathbf{A}}+\vec{\mathbf{B}}+\vec{\mathbf{C}}=\vec{\mathbf{0}}$, then they could form a triangle. Show that the area of triangle is given by

$$\overrightarrow{\mathbf{area}} = \frac{1}{2}\vec{\mathbf{A}}\times\vec{\mathbf{B}} = \frac{1}{2}\vec{\mathbf{B}}\times\vec{\mathbf{C}} = \frac{1}{2}\vec{\mathbf{C}}\times\vec{\mathbf{A}}$$

This is the reason why area is defined as a vector physical quantity.

(7) Find a unit vector perpendicular to $\vec{\mathbf{A}} = \hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = -2\hat{\mathbf{i}}-\hat{\mathbf{j}}+3\hat{\mathbf{k}}$.

(8) Consider a regular tetrahedron whose vertices v_1, v_2, v_3, v_4 are given by the position vectors $\vec{\mathbf{r}}_1 = \vec{\mathbf{0}}$, $\vec{\mathbf{r}}_2 = a(\hat{\mathbf{i}}+\hat{\mathbf{j}})$, $\vec{\mathbf{r}}_3 = a(\hat{\mathbf{i}}+\hat{\mathbf{k}})$, $\vec{\mathbf{r}}_4 = a(\hat{\mathbf{j}}+\hat{\mathbf{k}})$, respectively, where a is the length of the side of the cube formed by the vertices of the tetrahedron. Compute the total surface area of the tetrahedron.



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PRELIM 2: CALCULUS

2.1. Differentiation

2.1.1. The First Principle

Let $y = f(x)$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{2.1}$$

For example, let us try to determine the change in the following polynomial and trigonometry functions as x changes by an infinitesimal change Δx.

$$y = 2x^3$$

$$y = \sin x$$

Hence, for $y = \sin x$, $\frac{dy}{dx} = y' = \cos x$.

In physics, prime ($'$) usually indicates derivative with position, and dot ($\dot{}$) indicates derivative with time.

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2.1.2. Derivative of Known Functions

$$\frac{d}{dx}(ax^n) = nax^{n-1}$$
$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(e^x) = e^x$$
$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

2.1.3. Product Rule and Quotient Rule

Let $y = uv$, $u = f(x)$ and $v = g(x)$.

Using the product rule,

$$y' = u'v + uv' = f'(x)g(x) + f(x)g'(x) \tag{2.2}$$

Let $y = \frac{u}{v}$, $u = f(x)$ and $v = g(x)$.

Using the quotient rule,

$$y' = \frac{u'v - uv'}{v^2} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \tag{2.3}$$

For example, the derivative of the following functions can be obtained by using product rule or quotient rule.

$$y = e^x \sin x$$

$$y = \cot x$$

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2.1.4. Chain Rule

Let $y = f(a)$ and $a = g(x)$.

Using the chain rule,

$$y' = \frac{df(a)}{da}\frac{dg(x)}{dx} \tag{2.4}$$

For example, the derivative of the following functions can be obtained by using chain rule.

$$y = \sec x$$

$$y = (7\sin(5x^2))^3$$

2.1.5. Partial Derivatives

Let z be a function of x and y, $z = f(x, y)$.

$$\begin{cases} \dfrac{\partial z}{\partial x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \\ \dfrac{\partial z}{\partial y} = \lim\limits_{\Delta y \to 0} \dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \end{cases} \tag{2.5}$$

The symbol ∂ is read as "partial" or "tho" (Miller, 2006). The infinitesimal change in z can be obtained from

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$\frac{\partial f}{\partial x}dx$ is the change in z when y is kept constant, and $\frac{\partial f}{\partial y}dy$ is the change in z when x is kept constant.

For example,

$$z = f(x, y) = 5x^2y^3 + 4y\sin(3x)$$

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What if we have a function $f(x, y, z)$?

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \tag{2.6}$$

Last but not least, let $z = f(u, v)$. We know that in general,

$$dz = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv$$

If $z = uv$, we have

and if $z = \frac{u}{v}$, we have

and they verify both the product rule and quotient rule.

2.2. Integration

2.2.1. Integration as Summation

Let $y = 2x$.

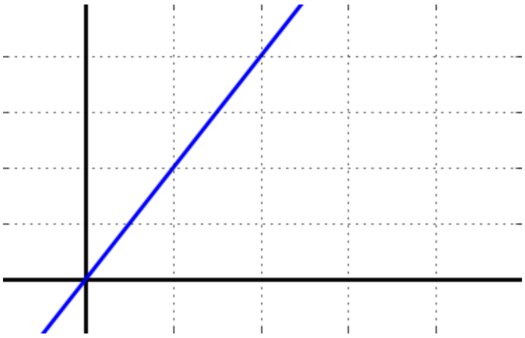

Fig. 3.1

Suppose we want to determine the area under the graph, i.e., a triangle in this case. We know that the result should be $\frac{1}{2}(x)(2x) = x^2$. But what if we split the triangle into infinitesimally thin rectangles?

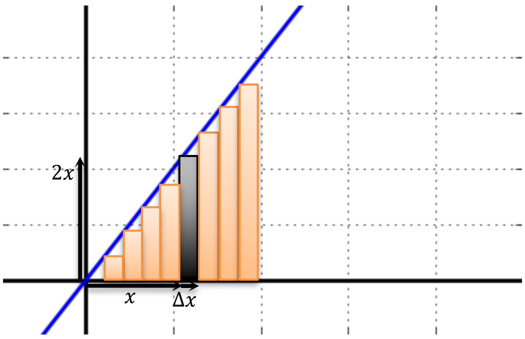

Fig. 3.2

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The area of a rectangle is given by $2x\Delta x$. As Δx becomes smaller and smaller, the sum of the areas of the thin rectangles should become equal to the area of the triangle.

Mathematically,

And this is how integration is defined: **a continuous summation** of a physical quantity, in this case, area, instead of a discrete one. It can then be inferred that integration is the inverse process of differentiation.

2.2.2. Definite vs. Indefinite Integrals

The type of integral in the previous section is called an **indefinite integral**. The easiest way to verify it is by differentiating the result and comparing it with the integrand. Thus, any additional constant in the result is meaningless. Below are the integrations of some known functions.

$$\int ax^n\,dx = \frac{a}{n+1}x^{n+1} + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \frac{1}{x}dx = \ln x + C$$

$$\int e^x dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

Let's go back to the $y = 2x$ graph. What if we want to perform the summation within a certain range of x, say between $x = a$ and $x = b$? This type of integral is called a **definite integral**.

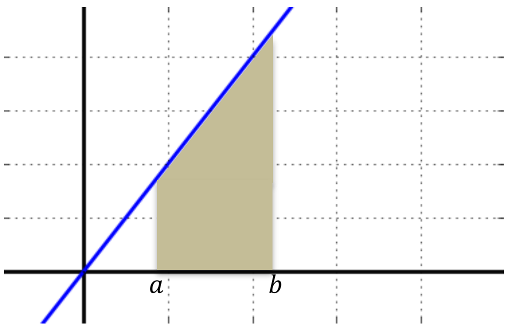

Fig. 3.3

It is obvious that the area under the graph is the difference between the area of a triangle with $x = b$ (the upper bound) and the area of a triangle with $x = a$ (the lower bound). Mathematically, *2.2.3. Integration Techniques*

(1) u-Substitution

Rule of thumb: Let u be the expression in the denominator without the exponent. Below are some examples.

$$\int_0^2 \frac{6x}{(x^2+16)^{3/2}}\,dx$$

$$\int \tan x\,dx$$

(2) Trigonometry Substitution

The following identities can be used to solve an integral provided it simplifies the integrand.

$$\begin{gathered} 1-\sin^2 x=\cos^2 x \\ 1-\cos^2 x=\sin^2 x \\ 1+\tan^2 x=\sec^2 x \\ 1+\cot^2 x=\csc^2 x \end{gathered} \tag{2.7}$$

Let's solve the same integral,

$$\int_0^2 \frac{6x}{(x^2+16)^{3/2}}\,dx$$

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(3) Integration by Parts

Following the product rule, $d(uv) = u\,dv + v\,du$, we have

$$\int u\,dv = uv - \int v\,du \tag{2.8}$$

Below is an example.

$$\int 4xe^x dx$$

2.3. Ordinary Differential Equation (ODE)

A **differential equation** is an equation involving derivatives of a function or functions. It is categorised into two: ordinary differential equation (ODE) and partial differential equation (PDE). In simple term, an **ordinary differential equation** (ODE) is an equation consisting of the dependent variable and its full derivatives only, i.e., y, y', y'', and so on, as functions of the independent variable x. The **standard form** of an ODE is achieved by putting all terms consisting the dependent variable and its derivatives on the left hand side and putting the remaining terms on the right hand side. A **solution** to an ODE is a function $y(x)$ that satisfies the ODE. An ODE can have more than one solution.

If the right-hand side of the ODE, written in its standard form, is zero, the equation is said to be **homogeneous**; and if the right hand side of the ODE, written in its standard form, is non-zero, the equation is said to be **non-homogeneous (inhomogeneous)**.

What about the following ODEs?

$$3y''^3x^2 - 4\sin y' + 7x^4 + 10 = 4\cos 2x - 9\frac{y'^2}{x}$$

$$3\ln x - 4(\ln y)^2 = 5y' - 3$$

$$5y''^3 + 4\sin^2 y' = 8\ln y$$

An ODE is said to be **linear** if the terms containing the dependent variable and its full derivatives are in their linear form. That is, ODEs containing $\sin y'$, $\ln y$, $y''y$, y'^2, etc., are non-linear.

What about the following ODEs?

$$5y'\sin^2 x - 4y'' = 3yx^3 + 10x + 1$$

$$2y''y - 4y' = 9$$

$$y'' - 5y' + 6y = 18$$

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Stop to Think 2.1
Consider a linear homogeneous ODE. What can you say about the function $y = 0$?

Consider a linear homogeneous ODE, $a(x)y'' + b(x)y' + c(x)y = 0$, where $a(x)$, $b(x)$, and $c(x)$ are functions of x. Let y_1 and y_2 be the solutions to this ODE. Check whether $y_3 = k_1y_1 + k_2y_2$ is also a solution, where k_1 and k_2 are constants.

We will limit our discussion to linear, first order, homogeneous and non-homogeneous, ODEs.

2.3.1. Variable Separation (Only for Separable Function)

This method only works if the dependent and independent variables can be separated in a single equation. Suppose a differential equation can be expressed as

$$\frac{dy}{dx} = f(x)g(y)$$

Thus

$$\frac{dy}{g(y)} = f(x)\,dx$$

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

After solving the integration, the solution to the ODE, $y(x)$, can be obtained.

Consider a radioactive molecule. It is known that the decay rate of the molecule at any time is proportional to the number of molecules that are present at that time. Thus, we can write $\frac{dN}{dt} = -\lambda N$, where N is the number of molecules at any time t, and λ is the decay constant. If the initial number of molecules is given by N_0, how do you find the number of molecules at any time t?

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2.3.2. Integrating Factor

This is used to solve first-order linear ODEs in the form of

$$y' + P(x)y = Q(x)$$

We are looking for a new variable z where $z = ye^{f(x)}$. The factor $e^{f(x)}$ is called the **integrating factor** (I.F.).

$$z' = y'e^{f(x)} + yf'(x)e^{f(x)} = e^{f(x)}(y' + f'(x)y)$$
$$y' + f'(x)y = z'e^{-f(x)}$$

Hence, we need to look for a function such that $f'(x) = P(x)$, or

$$f(x) = \int P(x)\,dx \rightarrow \text{I.F.} = e^{\int P(x)\,dx}$$

and ignore the integration constant.

$$z' = Q(x)e^{f(x)} = Q(x)e^{\int P(x)dx}$$
$$(y \times \text{I.F.})' = Q(x) \times \text{I.F.}$$

Consider a charged particle moving in one-dimension under an influence of a one-dimensional electric potential $V(x)$. The electric field in this region is given by

$$E = 2xV - 4xe^{-x^2}$$

The point $x = 1$ is taken to be the reference point for the electric potential function. How do you determine the electric potential as a function of x?

2.3.3. Euler's Method (Only for Linear Homogeneous ODEs)

This method is used to solve a linear homogeneous ODE with constant coefficients. This method was found by Leonhard Euler who realised that the solution to this ODE is always in the form of e^{sx} where s is a constant to be determined by the **characteristic equation**.

Consider the following linear first-order homogeneous ODE.

$$y' - 3y = 0$$

Let $y = Ae^{sx}$, where A is a constant, $y' = sAe^{sx}$. Thus,

$$Ae^{sx}(s - 3) = 0$$

If we let $A = 0$ or $e^{sx} = 0$, then what we have is a trivial solution, $y = 0$. The only accepted solution is thus

$$s - 3 = 0$$

This equation is called the <u>characteristic equation</u> of the ODE. For first order ODE, the characteristic equation has only one root – in this case, $s = 3$. Therefore,

$$y = Ae^{3x}$$

The constant A can be determined if a point (x, y) on the graph is given.

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Consider a particle moving in one dimension. The particle moves under the influence of a resistive force bv with an initial velocity v_0. If there is no other force acting on the particle, how do you find the velocity of the particle as a function of time?

Now, consider the following linear second order homogeneous ODE.

$$y'' - 8y' + 15y = 0$$

Let $y = Ae^{sx}$, $y' = sAe^{sx}$, $y'' = s^2Ae^{sx}$. Thus,

$$Ae^{sx}(s^2 - 8s + 15) = 0$$

If we let $A = 0$ or $e^{sx} = 0$, then what we have is a trivial solution, $y = 0$. The only accepted solution is thus

$$s^2 - 8s + 15 = 0$$

It is obvious that this characteristic equation has two roots, $s = 3$ and $s = 5$; implying two solutions, e^{3x} and e^{5x}.

Considering the linearity of the ODE,

$$y = Ae^{3x} + Be^{5x}$$

The constants A and B can be determined if two points (x_1, y_1) and (x_2, y_2), or one point and the gradient at that point (x, y, y'), on the graph are given. In physics, they are usually boundary conditions and/or initial conditions.

Thus, we have observed that for second-order ODEs the characteristic equation is quadratic. Hence, there are three classifications based on the roots of the characteristic equation.

- Real distinct roots

$$s = s_1\,, s_2 \quad ; \quad s_1 \neq s_2 \quad ; \quad s_1, s_2 \in \mathbb{R}$$

- Complex pair roots

$$s = \sigma \pm i\omega \quad ; \quad \sigma, \omega \in \mathbb{R}$$

It can be derived (proof omitted) that the real part and the imaginary part of the roots give rise to exponential and sinusoidal functions, respectively, in the solution.

- Twin roots

It can be derived (proof omitted) that the general solution is obtained by raising the polynomial degree of the coefficient.

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2.3.4. *Method of Undetermined Coefficients (Only for Particular Solutions)*

Solution for a linear non-homogeneous ODE consists of two terms, **homogeneous solution** and **particular solution** – it can be expressed it as $y = y_h + y_p$. This method is to find the particular solution to certain non-homogeneous ODEs. In order to find the best possible form of particular solution, a "guess" is made as to the appropriate form, which is then tested by differentiating the resulting equation.

Consider the following linear first-order non-homogeneous ODE.

$$y' - 3y = 12$$
$$\left(y_h + y_p\right)' - 3\left(y_h + y_p\right) = 12$$
$$(y_h' - 3y_h) + \left(y_p' - 3y_p\right) = 12$$

The homogeneous solution y_h to the above equation is one that fulfils

$$y_h' - 3y_h = 0$$

and the solution can easily be determined using Euler's method, $y_h = Ae^{3x}$.

The first guess of a possible particular solution will be a constant; let $y_p = B$, hence

$$0 - 3B = 12$$
$$B = -4$$
$$y_p = -4$$

Thus, the more general solution for this ODE is

$$y = y_h + y_p = Ae^{3x} - 4$$

The constant A can be determined if a point (x, y) on the graph is given.

The above steps form the standard procedure for solving a linear non-homogeneous ODE. The homogeneous solution is obtained by making the right hand side of the equation to be zero, and the particular solution is obtained by making the right guess.

The following are some examples of particular solutions, depending on the terms on the right hand side of the equation. In this table, A, B, and C are arbitrary constants. These constants can be determined by substituting the particular solution into the ODE and comparing the coefficients.

Right Hand Side	Particular Solution
5	A
$2x - 1$	$Ax + B$
$5x^2 + 7$	$Ax^2 + Bx + C$
$3\sin(2x)$	$A\sin(2x) + B\cos(2x)$
$7\cos(5x + 1)$	$A\sin(5x) + B\cos(5x)$
$4e^{7x}$	Ae^{7x}

Suppose you drop a rock of mass m from rest at the surface of a pond and let it fall to the bottom under fluid resistance kv. How do you find the velocity of the rock as a function of time?

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Consider a horizontal force $F_0 \cos(\alpha t)$ acting on a block of mass m that is put on a smooth horizontal table. Throughout the motion, an air resistance kv is acting on the block. Find the velocity of the block as a function of time if the block starts from rest.

2.4. References

Miller, J. (2006). Earliest uses of various mathematical symbols. *Retrieved July*, *2*, 2007.

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2.5. Tutorials

(1) Consider a cylinder with base radius 1 cm and height 5 cm. How much will the volume of the cylinder change (in cm^3 and in %) if its radius is increased by 5% and its height is decreased by 1%?

(2) Solve $\int \frac{3x^2 dx}{\sqrt{16-9x^2}}$ using an appropriate trigonometry substitution.
(Ans: $\mathbf{\frac{8}{9}\left(\sin^{-1}\left(\frac{3x}{4}\right) - \frac{3x}{4}\sqrt{1-\left(\frac{3x}{4}\right)^2}\right)}$**)**

(3) Solve the integral $\int \ln x \, dx$ using integration by parts.
(Ans: $\boldsymbol{x \ln x - x + C}$**)**

(4) Solve the following indefinite integrals:

$$\int \sin^2 x \, dx$$
$$\int \sin^3 x \, dx$$
$$\int \cos^2 x \, dx$$
$$\int \cos^3 x \, dx$$

(5) Angular momentum of a point mass is defined as the cross product between its position and linear momentum vectors, $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$. By definition, linear momentum can be expressed as $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$ and linear velocity can be expressed as $\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt}$, where t denotes time. Newton's second law states that $\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt}$, and equivalently, $\vec{\boldsymbol{\tau}} = \frac{d\vec{\mathbf{L}}}{dt}$, where $\vec{\mathbf{F}}$ and $\vec{\boldsymbol{\tau}}$ denotes the total external force and total external torque acting on the mass, respectively. Show that $\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$.

(6) Suppose you have just poured a cup of freshly brewed coffee with temperature 95°C in a room where the temperature is a constant 20°C.

(a) When do you think the coffee cools most quickly? What happens to the rate of cooling as time goes by? Explain.

(b) **Newton's Law of Cooling** states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. Write down a differential equation that expresses Newton's Law of Cooling for this particular situation. What is the initial condition? In view of your answer to part (a), do you think this differential equation is an appropriate model for cooling?

(c) What is the expression of the temperature of the coffee as a function of time?

(d) Make a rough sketch of the graph of the solution of the initial-value problem in part (b).

(7) The velocity of a particle moving in one dimension is given by $\frac{dx}{dt} = t^3 x^2$, where x is its position and t is time. Obtain the position of the particle as a function of time.
(Ans: $\boldsymbol{x = \frac{4}{C - t^4}}$**)**

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PRELIM 3: BASIC PYTHON

3.1. Use of Python Programming in Physics Honours

When we study physics, we delve into the behaviour of physical systems and the reason behind it. The fundamental principles and laws governing such behaviour are often expressed mathematically. Although solving these mathematical expressions can involve techniques like handling simultaneous equations, finding polynomial roots, or tackling differential equations, our primary focus remains on understanding and applying the physics laws. At times, the mathematics that describe physical systems are too complex to solve analytically by hand, necessitating **numerical solutions**. Programming languages like **Python** are tools that assist us in this regard. Remember, the goal is not to master programming for its own sake, but to effectively use it to model physical laws and thus deepen our understanding of the systems governed by these laws.

Google Colab (https://colab.google/) is a free, cloud-based service provided by Google that allows anyone to write and execute Python code through the browser. It is designed to simplify the process of coding in Python and to make computational resources more accessible to a broader audience. You will mainly use Google Colab whenever you need to perform numerical computations and submit the **.ipynb** file for grading. You should name your notebook according to the instructions given to avoid ambiguity. Notebooks that are not appropriately named will not be accepted, as their authorship and purpose are unclear.

There are many tutorials on the internet. You may check out one of them:
https://colab.research.google.com/drive/16pBJQePbqkz3QFV54L4NIkOn1kwpuRrj

3.2. Key Python Programming Skills

NumPy, commonly imported as np, is a Python library which is specially designed for scientific computing. NumPy arrays supports advanced mathematical operations through its many inbuilt functions and array operations are efficiently executed on large arrays and with less code. For further study, check out https://numpy.org.

SciPy, commonly imported as sp, is a scientific computation library that uses NumPy underneath. It provides more utility functions for optimisation, stats, and signal processing. For further study, check out https://scipy.org.

3.2.1. Basic Python Data Types

Python has basic native data types such as “int”, “float”, “string”, “bool”, “list” which represent an integer, floating point variable, string of text, boolean, and a group of variables.

3.2.2. Basic Python Operators

The main types of operators are **Arithmetic Operators**, **Comparison Operators**, and **Logical Operators**.

- Examples of Arithmetic Operators are Addition (+), Subtraction (–), Multiplication (*), Division (/), Modulus (%), and Exponent (**).
- Comparison Operators compare the values on both sides and decide the relation among them. They are also called Relational Operators. Examples include Equal (==), Not Equal (!=), Greater than (>), Less than (<), Greater than or Equal to (>=), and Less than or Equal to (<=).
- Logical Operators do not deal with numbers but with Boolean True/False states. Examples include “and”, “or” and “not”. “and” can be alternatively coded as “&”, and “or” can be alternatively coded as “|”.

3.2.3. Control Flow

Control flow refers to codes that direct the computing flow of the program. The essentials are the **if-else** statements, **loops**, and **functions**.

- Writing **loops** is essential for any programming. Loops are a way to instruct the computer to repeatedly do something while changing some conditions. The **break** statement allows us to break out of a current loop. The **continue** statement allows us to skip the rest of the code in the loop and continue the execution in the next run of the loop.
- **Functions** are groups of codes that serve a particular purpose. Your codes are expected to be modular and segregated into functions as much as possible.

The key steps are:

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(1) Start by listing the separate tasks we want the program to achieve. Seek to implement a functional module for each task. Determine what are the inputs and outputs for each functional module.
(2) Implement glue logic to use these functions. The glue logic can be consolidated in the **main** function, which is the entry point of the program. The glue logic passes inputs to one function, gets the outputs and then passes them to the next function and so on.

3.2.4. Reading from and Writing to Files

Python provides file reading and writing functionality. In this way, data collected from a data-logger can be read from file for processing, and the outputs can also be saved to file if desired.

3.2.5. Debugging

Debugging is a process of fixing the codes when it does not work. Googling error messages often help greatly as many people would have encountered the same error message and are kind to help. Writing highly readable codes with ample commenting and the use of functions will certainly reduce the number of bugs in your codes. The time and discipline spent doing these will reap dividends in time saved in debugging many times over.

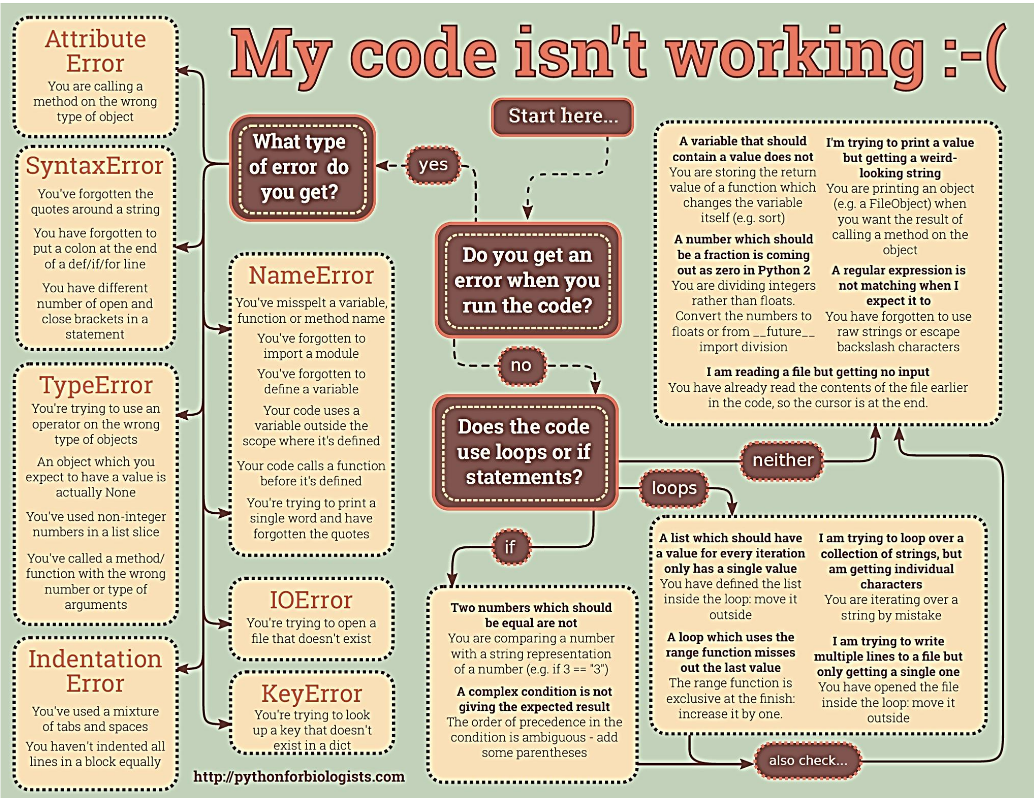

Fig. 3.1

3.3. Writing Readable Codes

We cannot overemphasise the importance of writing codes that are readable, i.e., the codes are easily understood by others, analogous to good handwriting. Writing readable codes will help you to spend less time in debugging.

These are some tips on making your codes readable:
(1) Write ample comments.
(2) Use functions as much as possible.
(3) Use descriptive variable and function names.

3.4. Riemann Sum

Riemann sum (from Bernhard Riemann) is an approximation of a region's area, obtained by adding up the areas of multiple simplified slices of the region. It is one of the numerical integration methods. Let

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$$\frac{dy}{dx} = f(x)$$
$$\int dy = \int f(x) dx$$

Numerically, y can be approximated by slicing the area under the graph into N rectangles of width Δx. This width is called the **step-size**.

$$y \approx \sum_{i=0}^{N-1} f(x_i)\, \Delta x$$
$$y \approx \Delta x\big(f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{N-1})\big)$$

(3.1)

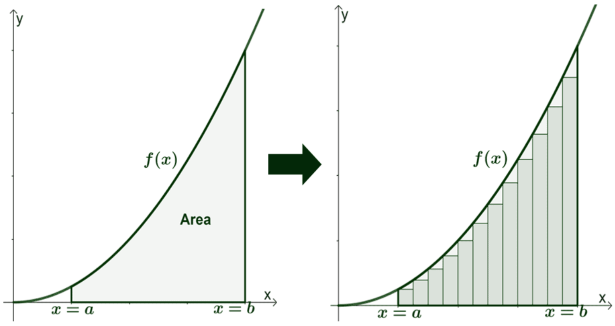

Fig. 3.2

The total area calculated using rectangles is a numerical estimate, differing from the exact analytical area. Reducing the step-size by using more rectangles decreases the error but requires more computational effort and time.

Suppose we wish to determine the area under the graph described by the function $y = 3x^3 - 2x^2 + 10$ from $x = 1$ to $x = 3$. Determine the answer analytically using integration and then write a program to determine the minimum number of rectangles needed to numerically compute this area such that the computational error is less than (i) 0.1, (ii) 0.01 and (iii) 0.001.

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3.5. Python Arrays and NumPy Arrays

Arrays are a group of variables. In Python's native data objects, arrays are called **Lists** or **Tuples**. Lists are bracketed by [], while tuples by (). Lists is mutable (their values can be changed after they are created) while Tuples are not. However, you are encouraged **NumPy arrays** (np.array) which is more powerful than Python's native List and Tuples but works in similar way. For example, NumPy arrays support "vectorised" operations such as element-by-element addition and multiplication. If we have an array of angles $[0,1,2,\ldots,365]/(365*2\pi)$, and we wish to compute the sine of these angles. It is both more computationally efficient and code efficient to use the NumPy function np.sin() than to use the core Python math package function math.sin().

To help you learn how to import and use the NumPy library and to study examples of their common usage, a Notebook has been created for this purpose. This Notebook replaces the texts that would have been shown in this set of notes.

Suppose you want to create a NumPy array containing angles from 0 radian to 2π radian such that
(a) the array includes both angles and that the step between adjacent array elements is exactly $1°$.
(b) the array includes 0 radian but not 2π radian and that the step between adjacent array elements is exactly $1°$.
Use the function `np.linspace` and also alternatively the function `np.arange`.

3.6. Visualisation with Python

In Physics Honours courses, we will use the **Matplotlib** python library to plot graphs. Matplotlib is a comprehensive library for creating static, animated, and interactive visualisations in Python. It is quite common to import it the following way:

```
import matplotlib.pyplot as plt
```

and to import mplot3d toolkits:

```
from mpl_toolkits import mplot3d
```

Here is an example of a program, with the output given in Fig. 3.3.

```
fig = plt.figure()
ax = plt.axes(projection ='3d')
z = np.linspace(0, 1, 100)
x = z * np.sin(25 * z)
y = z * np.cos(25 * z)
ax.plot3D(x, y, z, 'green')
plt.show()
```

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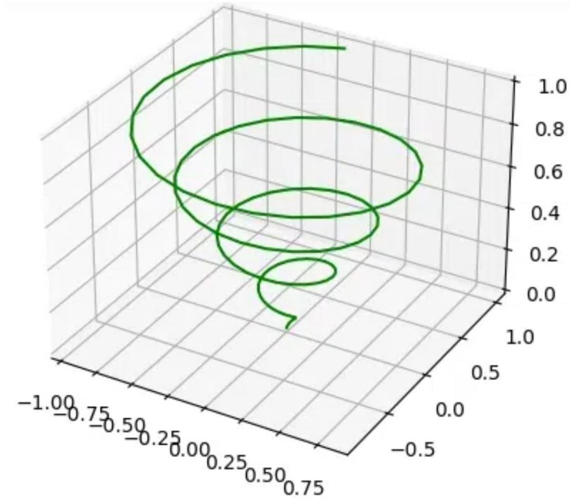

Fig. 3.3

Try to generate the following subplots in the domain 0° to 360° inclusive and thus show on the plots that $\sin^2\theta + \cos^2\theta = 1$. Line them up vertically.

(a) $\sin\theta$
(b) $\cos\theta$
(c) $\sin^2\theta$
(d) $\cos^2\theta$
(e) $\sin^2\theta + \cos^2\theta$

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