Collins

Cambridge International AS & A Level Mathematics

Pure Mathematics 1

STUDENT'S BOOK: Worked solutions

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Worked solutions

1 Quadratics

Please note: Full worked solutions are provided as an aid to learning, and represent one approach to answering the question. In some cases, alternative methods are shown for contrast.

All sample answers have been written by the authors. Cambridge Assessment International Education bears no responsibility for the example answers to questions taken from its past question papers, which are contained in this publication.

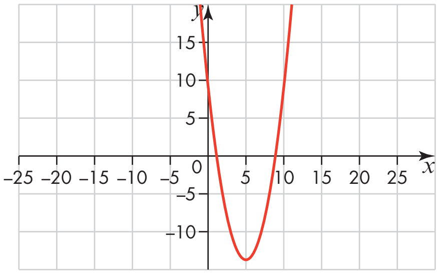
Non-exact numerical answers should be given correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

Prerequisite knowledge

1 $2x^2 - 5x - 3 = 0$

$(2x + 1)(x - 3) = 0$

$x = -\frac{1}{2}$ or $x = 3$



2 $3x^2 + x - 7 = 0$

$a = 3,\ b = 1,\ c = -7$

$$x = \frac{-1 \pm \sqrt{1^2 - (4)(3)(-7)}}{6}$$

$$x = \frac{-1 \pm \sqrt{1 + 84}}{6}$$

$$x = \frac{-1 \pm \sqrt{85}}{6}$$

$x = 1.37$ or -1.70

3 $5x - y = 13$ ①

$2x + y = 1$ ②

Add ① and ②.

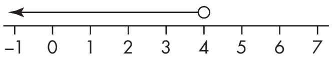
$7x = 14$

$x = 2,\ y = -3$

4 $3x - 5 < 7$

$3x < 12$

$x < 4$ ④



Exercise 1.1A

1 a $x^2 + 4x = (x + 2)^2 - 4$

b $2x^2 - 8x = 2(x^2 - 4x)$

$= 2[(x - 2)^2 - 4]$

$= 2(x - 2)^2 - 8$

c $x^2 + 8x + 7 = (x + 4)^2 - 16 + 7$

$= (x + 4)^2 - 9$

2 $x^2 - 10x + 11 = (x - 5)^2 - 25 + 11$

$= (x - 5)^2 - 14$

3 a $x^2 - 8x - 5 = (x - 4)^2 - 16 - 5$

$= (x - 4)^2 - 21$

b $x^2 + 3x - 7 = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} - 7$

$= \left(x + \frac{3}{2}\right)^2 - \frac{37}{4}$

c $2x^2 + 3x + 9 = 2\left(x^2 + \frac{3x}{2} + \frac{9}{2}\right)$

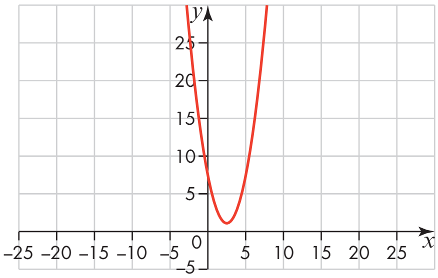
$= 2\left[\left(x + \frac{3}{4}\right)^2 - \frac{9}{16} + \frac{9}{2}\right]$

$= 2\left[\left(x + \frac{3}{4}\right)^2 + \frac{63}{16}\right]$

4 $x^2 - 5x + 7 = \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + 7$

$y = \left(x - \frac{5}{2}\right)^2 + \frac{3}{4}$

At the turning point y has a minimum value i.e. when $x = \frac{5}{2}$. When $x = \frac{5}{2}, y = \frac{3}{4}$, the coordinates of the turning point.



5 $2\left[(x+2)^2-\frac{13}{2}\right]=2\left(x^2+4x+4-\frac{13}{2}\right)$

$=2x^2+8x+8-13$

$=2x^2+8x-5$

6 **a** $5x^2+\frac{1}{2}x-3=5\left(x^2+\frac{x}{10}-\frac{3}{5}\right)$

$=5\left[\left(x+\frac{1}{20}\right)^2-\frac{1}{400}-\frac{3}{5}\right]$

$=5\left[\left(x+\frac{1}{20}\right)^2-\frac{241}{400}\right]$

$=5\left(x+\frac{1}{20}\right)^2-\frac{241}{80}$

b $5x^2-18=5\left(x^2-\frac{18}{5}\right)$ but can go no further.

c $5-7x-3x^2=-3\left(x^2+\frac{7x}{3}-\frac{5}{3}\right)$

$=-3\left[\left(x+\frac{7}{6}\right)^2-\frac{49}{36}-\frac{5}{3}\right]$

$=-3\left[\left(x+\frac{7}{6}\right)^2-\frac{109}{36}\right]$

$=-3\left(x+\frac{7}{6}\right)^2+\frac{109}{12}$

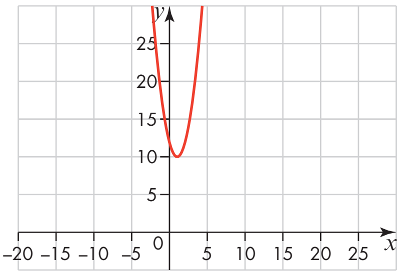
7 $2x^2-3x+11=2\left(x^2-\frac{3x}{2}+\frac{11}{2}\right)$

$=2\left[\left(x-\frac{3}{4}\right)^2-\frac{9}{16}+\frac{11}{2}\right]$

$=2\left[\left(x-\frac{3}{4}\right)^2+\frac{79}{16}\right]$

$2\left[\left(x-\frac{3}{4}\right)^2+\frac{79}{16}\right]>0$

Consequently $b^2-4ac<0$ because the curve of this equation will not intersect the x-axis and so there are no real roots.



8 $(1-2x)\ [(x+4)^2-1]=(1-2x)(x^2+8x+16-1)$

$=(1-2x)\ (x^2+8x+15)$

$=x^2+8x+15-2x^3-16x^2-30x$

$=15-22x-15x^2-2x^3$

9 $4x^2+5x-3=4\left[x^2+\frac{5}{4}x-\frac{3}{4}\right]$

$=4\left[\left(x+\frac{5}{8}\right)^2-\left(\frac{5}{8}\right)^2-\frac{3}{4}\right]$

$=4\left[\left(x+\frac{5}{8}\right)^2-\frac{73}{64}\right]$

$=4\left(x+\frac{5}{8}\right)^2-\frac{73}{16}$

When $4x^2+5x-3=0$

$4\left(x+\frac{5}{8}\right)^2-\frac{73}{16}=0$

$4\left(x+\frac{5}{8}\right)^2=\frac{73}{16}$

$\left(x+\frac{5}{8}\right)^2=\frac{73}{64}$

$x+\frac{5}{8}=\pm\sqrt{\frac{73}{64}}$

$x=\frac{-5}{8}\pm\sqrt{\frac{73}{64}}$

$x=\frac{-5\pm\sqrt{73}}{8}$

10 $ax^2+bx+c=\left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right)^2-\frac{b^2}{4a}+c$

$ax^2+bx+c=0$

$\left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right)^2-\frac{b^2}{4a}+c=0$

$\left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right)^2=\frac{b^2}{4a}-c=\frac{b^2-4ac}{4a}$

$\sqrt{a}x+\frac{b}{2\sqrt{a}}=\pm\sqrt{\frac{b^2-4ac}{4a}}$

$\sqrt{a}x+\frac{b}{2\sqrt{a}}=\frac{\pm\sqrt{b^2-4ac}}{2\sqrt{a}}$

$\sqrt{a}x=-\frac{b}{2\sqrt{a}}\pm\frac{\sqrt{b^2-4ac}}{2\sqrt{a}}$

$\sqrt{a}x=\frac{-b\pm\sqrt{b^2-4ac}}{2\sqrt{a}}$

$x=\frac{-b\pm\sqrt{b^2-4ac}}{\sqrt{a}\times2\sqrt{a}}$

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Exercise 1.2A

1 **a** $x^2 + 2x + 13 = 0$

$b^2 - 4ac = (2)^2 - (4)(1)(13)$

$= -48$

$b^2 - 4ac < 0$ so no real roots.

b $2x^2 + 3x + 1 = 0$

$b^2 - 4ac = (3)^2 - (4)(2)(1)$

$= 1$

$b^2 - 4ac > 0$ so two distinct real roots.

c $x^2 + 4x + 4 = 0$

$b^2 - 4ac = (4)^2 - (4)(1)(4)$

$= 0$

$b^2 - 4ac = 0$ so two equal real roots.

2 $2x^2 - 5x + 3 = 0$

$b^2 - 4ac = (-\mathbf{5})^2 - (4)(\mathbf{2})(3)$

$= 25 - 24$

$= \mathbf{1}$

$b^2 - 4ac > 0$ so has two **distinct** real roots.

3 **a** $x^2 - 3x - 5 = 0$

$b^2 - 4ac = (-3)^2 - (4)(1)(-5)$

$= 29$

$b^2 - 4ac > 0$ so two distinct real roots.

b $3x^2 - 2x + 7 = 0$

$b^2 - 4ac = (-2)^2 - (4)(3)(7)$

$= -80$

$b^2 - 4ac < 0$ so no real roots.

c $x^2 - 12x + 36 = 0$

$b^2 - 4ac = (-12)^2 - (4)(1)(36)$

$= 0$

$b^2 - 4ac = 0$ so two equal real roots.

4 $x^2 + (2k + 2)x + (5k - 1) = 0$

$b^2 - 4ac = 0$ for one repeated real roots.

$(2k + 2)^2 - (4)(1)(5k - 1) = 0$

Expanding and simplifying.

$k^2 - 3k + 2 = 0$ *or* $-k^2 + 3k - 2 = 0$

Both quadratic equations have the solutions $k = 1$ or $k = 2$.

When $k = 1$, $x^2 + 4x + 4 = 0$ and $x = -2$;

when $k = 2$, $x^2 + 6x + 9 = 0$ and $x = -3$

5 **a** $4 + 2x - 3x^2 = 0$

$b^2 - 4ac = (2)^2 - (4)(-3)(4)$

$= 52$

$b^2 - 4ac > 0$ so two distinct real roots.

b $49 + x^2 - 14x = 0$

$b^2 - 4ac = (-14)^2 - (4)(1)(49)$

$= 0$

$b^2 - 4ac = 0$ so two equal real roots.

c $3x^2 + 12 - 5x = 0$

$b^2 - 4ac = (-5)^2 - (4)(3)(12)$

$= -119$

$b^2 - 4ac < 0$ so no real roots.

6 $2x^2 + 5x - c = 0$

a No real roots:

$b^2 - 4ac < 0$

$(5)^2 - (4)(2)(-c) < 0$

$25 + 8c < 0$

$c < -\dfrac{25}{8}$

b One repeated real root:

$b^2 - 4ac = 0$

$(5)^2 - (4)(2)(-c) = 0$

$25 + 8c = 0$

$c = -\dfrac{25}{8}$

c Two distinct real roots:

$b^2 - 4ac > 0$

$(5)^2 - (4)(2)(-c) > 0$

$25 + 8c > 0$

$c > -\dfrac{25}{8}$

7 $x = \sqrt{\dfrac{13 - 3x}{3}}$

$x^2 = \dfrac{13 - 3x}{3}$

$3x^2 = 13 - 3x$

$3x^2 + 3x - 13 = 0$

$b^2 - 4ac = 9 + 4 \times 3 \times 13 = 165 > 0$

As $b^2 - 4ac > 0$ there are 2 distinct solutions.

8 $x^2 - 3x + a^2 = 0$

No real roots when $b^2 - 4ac < 0$

$b^2 - 4ac = 9 - 4a^2$

$9 - 4a^2 < 0$

$9 < 4a^2$

$a^2 > \dfrac{9}{4}$

$a < \dfrac{-3}{2}$ or $a > \dfrac{3}{2}$

9 $y = 3 - x$ meets $y = x^2 - 5x + 7$ when $3 - x = x^2 - 5x + 7$

ie $0 = x^2 - 4x + 4$

$b^2 - 4ac = 16 - 16 = 0$

One repeated root, so the line touches, but does not cross the curve.

10 $3x - xy + 1 = 0$

$y = \frac{3x+1}{x}$

substitute into $3y - xy + 1 = 0$

$3\left(\frac{3x+1}{x}\right) - x\left(\frac{3x+1}{x}\right) + 1 = 0$

$3\left(\frac{3x+1}{x}\right) - 3x = 0$

$3x^2 - 9x - 3 = 0$

$b^2 - 4ac = 81 + 4 \times 3 \times 3 = 117 > 0$

As $b^2 - 4ac > 0$ there are two distinct real roots

$\therefore$ the curves intersect in two places.

Exercise 1.3A

1 a i $x^2 - 9 = 0$

$(x + 3)(x - 3) = 0$

$x = 3$ or -3

ii $x = \frac{0 \pm \sqrt{0-(4)(1)(-9)}}{2}$

$x = 3$ or -3

iii $x^2 = 9$

$x = 3$ or -3

b i $x^2 + 3x + 2 = 0$

$(x + 1)(x + 2) = 0$

$x = -1$ or -2

ii $x = \frac{-3 \pm \sqrt{9-(4)(1)(2)}}{2}$

$x = -1$ or -2

iii $\left(x + \frac{3}{2}\right)^2 - \frac{1}{4} = 0$

$x = -1$ or -2

c i $x^2 - 5x - 7 = 0$

Cannot be factorised.

ii $x = \frac{5 \pm \sqrt{25-(4)(1)(-7)}}{2}$

$x = \frac{5 \pm \sqrt{53}}{2}$

iii $\left(x - \frac{5}{2}\right)^2 - \frac{53}{4} = 0$

$x = \frac{5 \pm \sqrt{53}}{2}$

d i $x(x - 6) = 0$

$x = 0$ or 6

ii $x = \frac{6 \pm \sqrt{36-(4)(1)(0)}}{2}$

$x = 0$ or 6

iii $(x - 3)^2 - 9 = 0$

$x = 0$ or 6

2 **Factorisation:**

- can only be used on equations that factorise
- sometimes spotting factors can be difficult
- can solve: $x^2 + 3x + 2 = 0$
- cannot solve: $x^2 - 5x - 7 = 0$.

Quadratic formula:

- can be used to solve any equation with real roots including ones that don't factorise
- cumbersome and consequently easy to make a mistake
- can solve: $x^2 - 5x - 7 = 0$
- cannot solve: $x^2 - 5x + 7 = 0$.

Completing the square:

- can be used to solve any equation with real roots including ones that don't factorise
- can be cumbersome manipulations if b is odd and $a > 1$, and consequently easy to make a mistake
- can solve: $x^2 - 5x - 7 = 0$
- cannot solve: $x^2 - 5x + 7 = 0$.

3 a $(x - 4)^2 = 13$

$x = 4 \pm \sqrt{13}$

Completing the square. The equation was already in form of a completed square.

b $2x^2 - 5x - 11 = 0$

$x = \frac{5 \pm \sqrt{25-(4)(2)(-11)}}{4}$

$x = \frac{5 \pm \sqrt{113}}{4}$

Does not factorise so used the quadratic formula. Alternatively, could have completed the square but chose not to as $a > 1$.

c $3x^2 = 5 - 14x$

$3x^2 + 14x - 5 = 0$

$(3x - 1)(x + 5) = 0$

$x = \frac{1}{3}$ or -5

Equation factorises so easy to do this. Alternatively, could have completed the square but chose not to as $a > 1$.

d $4x^2 - 16x + 7 = 0$

$(2x - 1)(2x - 7) = 0$

$x = \frac{1}{2}$ or $\frac{7}{2}$

Equation factorises so easy to do this. Alternatively, could have completed the square but chose not to as $a > 1$.

4 $x^2 + bx + c = 0$

Subtract any extra values that completing the square has produced, in this case $-\frac{b^2}{4}$.

$$\left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c = 0$$

Manipulate the equation to make x the subject.

$$\left(x + \frac{b}{2}\right)^2 = \frac{b^2 - 4c}{4}$$

$$x + \frac{b}{2} = \frac{\pm\sqrt{b^2 - 4c}}{2}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

5 **a** Cannot be factorised if $b^2 - 4ac$ is not a square number.

$b^2 - 4ac = 16 - (4)(3)(-11) = 148$

148 not a square number so does not factorise.

b $3x^2 + 4x - 11 = 0$

$$x = \frac{-4 \pm \sqrt{148}}{6}$$

$$x = \frac{-2 \pm \sqrt{37}}{3}$$

6 $(x - 3)(x - 5) = 8$

$x^2 - 8x + 15 - 8 = 0$

$x^2 - 8x + 7 = 0$

$(x - 7)(x - 1) = 0$

$x = 1$ or 7

7 $ax^2 + bx + c = 0$

$$a\left(x^2 + \frac{bx}{a} + \frac{c}{a}\right) = 0$$

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

8 $x = \sqrt{\frac{74x + 33}{7}}$

$x^2 = \frac{74x + 33}{7}$

$7x^2 - 74x - 33 = 0$

$(7x + 3)(x - 11) = 0$

The equation can be solved by factorising a quadratic expression.

9 **a** $9x^2 + 55x - 56 = 0$

$b^2 - 4ac = 55^2 + 4 \times 9 \times 56 = 5041$

$\sqrt{b^2 - 4ac} = 71$ so the quadratic can be factorised.

b $(9x - 8)(x + 7) = 0$

$x = \frac{8}{9}$ or $x = -7$

Exercise 1.4A

1 **a** $x^2 + 3x - 4 > 0$

$x^2 + 3x - 4 = 0$

$(x + 4)(x - 1) = 0$

$x < -4$ or $x > 1$

b $x^2 - 6x + 8 \leqslant 0$

$(x - 4)(x - 2) = 0$

$2 \leqslant x \leqslant 4$

c $x^2 - 9 \geqslant 0$

$(x + 3)(x - 3) \geqslant 0$

$x \leqslant -3$ or $x \geqslant 3$

d $x^2 - 6x < 0$

$x(x - 6) = 0$

$0 < x < 6$

2 **a** $2x^2 + 7x < -3$

$2x^2 + 7x + 3 < 0$

$(2x + 1)(x + 3) = 0$

$-3 < x < -\frac{1}{2}$

b $-x^2 - 3x + 4 > 0$

$(-x + 1)(x + 4) = 0$

$-4 < x < 1$

c $x^2 > 4$

$x^2 - 4 > 0$

$(x - 2)(x + 2) > 0$

$x < -2$ or $x > 2$

d $3x^2 \leqslant 5x$

$3x^2 - 5x \leqslant 0$

$x(3x - 5) \leqslant 0$

$0 \leqslant x \leqslant \frac{5}{3}$

3 a $6 - 5x - x^2 > 0$

$(6 + x)(1 - x) = 0$

$-6 < x < 1$

and $-x + 6 > 0$

$6 > x$

$x < 6$

$\{-5, -4, -3, -2, -1, 0\}$

b $2x^2 + 9x - 5 \leqslant 0$

$(2x - 1)(x + 5) = 0$

$-5 \leqslant x \leqslant \frac{1}{2}$

or $2x < 5$

$x < \frac{5}{2}$

So $x < \frac{5}{2}$

{all integers less than 2.5}

c $(2x + 1)^2 - 9 \leqslant 0$

$2x + 1 = \pm 3$

$-2 \leqslant x \leqslant 1$ and $2x < 6$

$x < 3$

$-2 \leqslant x \leqslant 1$

$\{-2, -1, 0, 1\}$

d $4x^2 < 3x$

$4x^2 - 3x < 0$

$x(4x - 3) < 0$

$0 < x < \frac{3}{4}$ or $\frac{1 - 4x}{4} > 0$

$1 > 4x$

$x < \frac{1}{4}$

So $x < \frac{3}{4}$

{all integers less than 0.75}

4 $r < 10$

$\pi r^2 > 250$

$r > 8.9$

So $8.9 < r < 10$

5 a $x^2 + 4 < 7x - 2$

$x^2 - 7x + 6 < 0$

$(x - 1)(x - 6) = 0$

$1 < x < 6$

$\{2, 3, 4, 5\}$

b $2x^2 - 3x - 15 < 2x - 3$

$2x^2 - 5x - 12 < 0$

$(2x + 3)(x - 4) = 0$

$-\frac{3}{2} < x < 4$

$\{-1, 0, 1, 2, 3\}$

c $12 + 9x + x^2 > 3(x + 1)$

$12 + 9x + x^2 > 3x + 3$

$9 + 6x + x^2 > 0$

$(3 + x)^2 > 0$

Inequality true for all values of x.

d $x + 1 > 3 - 4x - 3x^2$

$0 > 2 - 5x - 3x^2$

$3x^2 + 5x - 2 > 0$

$(3x - 1)(x + 2) > 0$

$x < -2$ or $x \geqslant \frac{1}{3}$ [all integers excluding $-2, -1$ and 0]

6 a $x^2 + 5x - 6 < 0$

$(x + 6)(x - 1) = 0$

$-6 < x < 1$ and $x^2 + 3x - 4 < 0$

$(x + 4)(x - 1) = 0$

$-4 < x < 1$

So $-4 < x < 1$

$\{-3, -2, -1, 0\}$

b $x^2 + 5x < 6$ or $x^2 + 3x < 4$

So $-6 < x < 1$ or $-4 < x < 1$

$\{-5, -4, -3, -2, -1, 0\}$

c $x^2 - 6 < -5x$ and $x^2 - 4 > -3x$

$-6 < x < 1$ and $x < -4$ or $x > 1$

So $-6 < x < -4$

$\{-5\}$

d $x^2 + 5x - 6 > 0$ or $x^2 + 3x - 4 < 0$

$x < -6$ or $x > 1$ or $-4 < x < 1$

{all integers excluding $-6, -5, -4$ and 1}

7 $x^2 - 4x - 3 \leqslant 0,\ 1 - 2x^2 \geqslant 0$

First consider $x^2 - 4x - 3 \leqslant 0$

$x^2 - 4x - 3 = 0$ when $x = \frac{4 \pm \sqrt{16 + 12}}{2} = 2 \pm \sqrt{7}$

when $x = 0$ $x^2 - 4x - 3 = -3$

$2 - \sqrt{7} \leqslant x \leqslant 2 + \sqrt{7}$

Now consider $1 - 2x^2 \geqslant 0$

$1 \geqslant 2x^2,\ x^2 \leqslant \frac{1}{2}$, so $\frac{-1}{\sqrt{2}} \leqslant x \leqslant \frac{1}{\sqrt{2}}$

The range of x values for which $x^2 - 4x - 3 \leqslant 0$, and $1 - 2x^2 \geqslant 0$

is $2 - \sqrt{7} \leqslant x \leqslant \frac{1}{\sqrt{2}}$

which can be written as $2 - \sqrt{7} \leqslant x \leqslant \frac{\sqrt{2}}{2}$

8 $x \geqslant \sqrt{\dfrac{1-3x}{5}}$

$x^2 \geqslant \dfrac{1-3x}{5}$

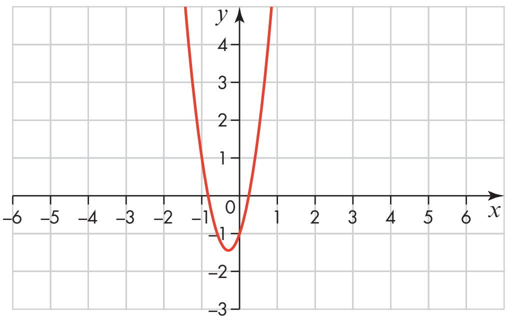
$5x^2 + 3x - 1 \geqslant 0$

$x = \dfrac{-3 \pm \sqrt{3^2 - (4)(5)(-1)}}{10}$

$x = \dfrac{-3+\sqrt{29}}{10}$

Or

$x = \dfrac{-3-\sqrt{29}}{10}$



So for $5x^2 + 3x - 1 \geqslant 0$ either $x \leqslant \dfrac{-3-\sqrt{29}}{10}$ or

$x \geqslant \dfrac{-3+\sqrt{29}}{10}$

9 $4x - 7x^2 - 8 > 0$

Consider $4x - 7x^2 - 8 = 0$

$7x^2 - 4x + 8 = 0$

$b^2 - 4ac = 16 - 224 = -208 < 0$

There are no solutions, $\therefore$ the function does not cross the x-axis, ie it is all above or below the x-axis.

When $x = 0$, $4x - 7x^2 - 8 = -8$

There are no x values for which $4x - 7x^2 - 8 > 0$

10 $-3x^2 \leqslant x - 5$

$3x^2 + x - 5 \geqslant 0$

$3x^2 + x - 5 = 0$ when $x = \dfrac{-1 \pm \sqrt{1+60}}{6} = \dfrac{-1 \pm \sqrt{61}}{6}$

The higher value is $\dfrac{-1+\sqrt{61}}{6}$

$\sqrt{49} < \sqrt{61} < \sqrt{64}$

Consider $\dfrac{-1+\sqrt{49}}{6} = \dfrac{-1+7}{6} = \dfrac{6}{6} = 1$

$\therefore \dfrac{-1+\sqrt{61}}{6} > 1$

No, the solution set for $-3x^2 \leqslant x - 5$ is not a subset of $x \leqslant 1$.

Exercise 1.5A

1 **a** $2x + y = -4$ ①

$5x + y = -1$ ②

Subtract ① from ②.

$3x = 3$

$x = 1$ and $y = -6$

b $3x - 2y = 2$ ①

$5x + y = 25$ ②

Multiply ② by 2.

$10x + 2y = 50$ ③

Add ① and ③.

$13x = 52$

$x = 4$ and $y = 5$

c $x - 2y = 13$ ①

$-x + 3y = -15$ ②

Add ① and ②.

$y = -2$ and $x = 9$

2 $5x + 6y = 6$ ①

$3x - y = 22$ ②

Multiply ② by 6.

$5x + 6y = 6$ ①

$18x - 6y = 132$ ③

Add ① and ③.

$23x = 138$

$x = 6$

Substitute x into ②.

$18 - y = 22$

$y = -4$

3 If there are n unknowns then you need n distinct equations involving the n unknowns.

4 $3x - 4y = 3$ ①

$6x + 4y = 3$ ②

Add ① and ②.

$9x = 6$

$x = \dfrac{2}{3}$ and $y = -\dfrac{1}{4}$

5 $x + y = 50$ ①

$x - 1 = 15(y - 1)$

$x - 15y = -14$ ②

Subtract ② from ①.

$16y = 64$

$y = 4$ and $x = 46$. Helen was 42 when her daughter was born.

6 $2x - y = -14$ ①

$3y - 2z = 16$ ②

$z - x = 3$ ③

Multiply ① by 3.

$6x - 3y = -42$ ④

$3y - 2z = 16$ ②

Add ④ and ②.

$6x - 2z = -26$ ⑤

Multiply ③ by 2.

$2z - 2x = 6$ ⑥

Add ⑤ and ⑥.

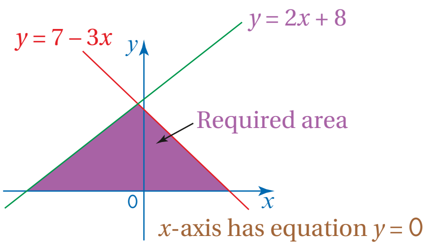
$4x = -20$

$x = -5$

Substitute x into 3: $z = -2$

Substitute x into 1: $y = 4$

7 A sketch diagram helps see what is required.



Find the coordinates of the point of intersection of the lines $y = 2x + 8$ and $y = 7 - 3x$

$7 - 3x = 2x + 8$, so $x = -0.2$, $y = 7.6$

The intersections of the lines with the line $y = 0$ are when $0 = 2x + 8$, $x = -4$ and $0 = 7 - 3x$, $x = \frac{7}{3}$

Area of the triangle $= \frac{1}{2} \times \text{base} \times \text{height}$

$= \frac{1}{2} \times \left(4 + \frac{7}{3}\right) \times 7.6 = 24\frac{1}{15}$ square units

8 $y = 4 - 5x$ meets $y = 7 - 3x$ when $4 - 5x = 7 - 3x$, so, $x = -1.5$, $y = 11.5$ $(-1.5, 11.5)$

$y = 4 - 5x$ meets $y = x + 2$ when $4 - 5x = x + 2$, so, $x = \frac{1}{3}$, $y = \frac{7}{3}$ $\left(\frac{1}{3}, \frac{7}{3}\right)$

$y = 7 - 3x$ meets $y = x + 2$ when $7 - 3x = x + 2$, so $x = 1.25$, $y = 3.25$

The coordinates of the vertices are $(-1.5, 11.5)$, $\left(\frac{1}{3}, \frac{7}{3}\right)$, and $(1.25, 3.25)$.

9 Use the information to make 3 equations.

Let m be the cost of a maths textbook, h the cost of a history textbook, p the cost of a pen.

$2m + 1p = 33.6$ A

$2h + 2p = 42.6$ B

$1h + 1m = 33.9$ C

Multiply A by 2

$4m + 2p = 67.2$ 2A

$2h + 2p = 42.6$ B

Work out 2A – B

$4m - 2h = 24.6$ D

Multiply C by 2

$2h + 2m = 67.8$ 2C

Work out D + 2C

$6m = 92.4$

$m = \frac{92.4}{6} = 15.4$

1 maths textbook costs \$15.40 (remember to add the '0' to make the money notation correct).

Exercise 1.5B

1 a $x + y = 3$ ①

$2x^2 - y = 25$ ②

Add ① and ②.

$2x^2 + x - 28 = 0$

$(2x - 7)(x + 4) = 0$

$x = \frac{7}{2}$ or -4

$y = -\frac{1}{2}$ or 7

b $2x - y = 20$ ①

$x^2 + xy = -12$ ②

From ①.

$2x - 20 = y$

Substitute into ②.

$x^2 + x(2x - 20) = -12$

$3x^2 - 20x + 12 = 0$

$(3x - 2)(x - 6) = 0$

$x = \frac{2}{3}$ or 6

$y = \frac{-56}{3}$ or -8

c $y = 4x$ ①

$5 - x^2 = y$ ②

① = ②

$5 - x^2 = 4x$

$x^2 + 4x - 5 = 0$

$(x + 5)(x - 1) = 0$

$x = -5$ or 1

$y = -20$ or 4

2 **a** Substitution or elimination: for elimination the first equation would need to be multiplied by 2 so that y can subsequently be eliminated.

b Substitution only: neither addition or subtraction of the equations will eliminate a variable.

c Substitution only: neither addition or subtraction of the equations will eliminate a variable.

3 **a** $2x^2 + y = 14$ (1)

$x - 2y = 11$ (2)

Multiply (1) by 2.

$4x^2 + 2y = 28$ (3)

Add (2) and (3).

$4x^2 + x - 39 = 0$

$(4x + 13)(x - 3) = 0$

$x = \frac{-13}{4}$ or 3

$y = \frac{-57}{8}$ or -4

b $xy - x = -4$ (1)

$x + y = 1$ (2)

From (2).

$y = 1 - x$

Substitute into (1).

$x(1 - x) - x = -4$

$x - x^2 - x = -4$

$x = \pm 2$, $y = -1$ or 3

c $x - y = 10$ (1)

$xy = 140$ (2)

From (1).

$x = y + 10$

Substitute into (2).

$y(y + 10) = 140$

$y^2 + 10y - 140 = 0$

$$y = \frac{-10 \pm \sqrt{100 - (4)(1)(-140)}}{2}$$

$$y = \frac{-10 \pm \sqrt{660}}{2}$$

$y = -5 \pm \sqrt{165}$ $x = 5 \pm \sqrt{165}$

4 $2x - 3y = 13$ (1)

$x^2 - y = 7$ (2)

From (2).

$x^2 - 7 = y$

Substitute into (1).

$2x - 3(x^2 - 7) = 13$

$3x^2 - 2x - 8 = 0$

$(3x + 4)(x - 2) = 0$

$x = \frac{-4}{3}$ or 2

$y = \frac{-47}{9}$ or -3

So the coordinates of the points of intersection of this line and this curve are $\left(-\frac{4}{3}, -\frac{47}{9}\right)$ and $(2, -3)$

5 $y - x = 1$ (1)

$x^2 + y^2 = 64$ (2)

From (1).

$y = x + 1$

Substitute into (2).

$x^2 + (x + 1)^2 = 64$

$2x^2 + 2x - 63 = 0$

$$x = \frac{-2 \pm \sqrt{4 - (4)(2)(-63)}}{4}$$

$x = 5.135$ or -6.135

$y = 6.135$ or -5.135

So the coordinates of the points of intersection of this line and this circle are $(5.135, 6.135)$ and $(-6.135, -5.135)$.

6 $y - x = 10$ (1)

$x^2 + y^2 = 50$ (2)

From (1).

$y = x + 10$

Substitute into (2).

$x^2 + (x + 10)^2 = 50$

$2x^2 + 20x + 50 = 0$

$x^2 + 10x + 25 = 0$

$(x + 5)^2 = 0$

$x = -5$, $y = 5$

The line is a tangent to the circle at $(-5, 5)$.

7 Let the numbers be x and y.

Then $x + y = 129$, $x^2 + y^2 = 8433$

The two numbers must satisfy both equations.

$x + y = 129$ so $y = 129 - x$

Substitute into $x^2 + y^2 = 8433$

$x^2 + (129 - x)^2 = 8433$

$x^2 + 16\,641 - 258x + x^2 = 8433$

$2x^2 - 258x + 8208 = 0$

$x^2 - 129x + 4104 = 0$

$(x - 57)(x - 72) = 0$

$x = 57$, $x = 72$

The two numbers are 57 and 72.

8 The line $y = 9 - 2x$ meets the circle $(x-3)^2 + (y+2)^2 = 25$

when $(x-3)^2 + (9 - 2x + 2)^2 = 25$

$$(x-3)^2 + (11-2x)^2 = 25$$

$$x^2 - 6x + 9 + 121 - 44x + 4x^2 = 25$$

$$5x^2 - 50x + 105 = 0$$

$$x^2 - 10x + 21 = 0$$

$$(x-3)(x-7) = 0$$

$x = 3, x = 7$

$y = 3, y = -5$

There are two intersections, and their coordinates are (3, 3) and (7, –5).

9 $3x^2 - 7x + 2 > 5x - 6$

$3x^2 - 12x + 8 > 0$

$3x^2 - 12x + 8 = 0$ when

$$x = \frac{12 \pm \sqrt{144-96}}{6} = \frac{12 \pm \sqrt{48}}{6} = 2 \pm \frac{2\sqrt{3}}{3}$$

When $x = 2$, $3x^2 - 12x + 8 = -4$,

$\therefore$ the solution set of $3x^2 - 7x + 2 > 5x - 6$ is

$$x < 2 - \frac{2\sqrt{3}}{3},\ 2 + \frac{2\sqrt{3}}{3} < x$$

10 $2x + 3y = 7$, so $y = \dfrac{7-2x}{3}$

Substitute into $x^2 + y^2 = 8$

$$x^2 + \left(\frac{7-2x}{3}\right)^2 = 8$$

$$9x^2 + (7-2x)^2 = 72$$

$$9x^2 + 49 - 28x + 4x^2 = 72$$

$$13x^2 - 28x - 23 = 0$$

$$x = \frac{28 \pm \sqrt{28^2 + 4 \times 13 \times 23}}{26} = \frac{28 \pm \sqrt{1980}}{26}$$

$x = 2.788$, or $x = -0.6345$

$y = 0.474$, $y = 2.756$

The points of intersection are (2.79, 0.474) and (–0.635, 2.76).

Exercise 1.6A

1 $\sqrt[3]{x^2 - 1} = 0$

$$x^2 - 1 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

2 $x + 10\sqrt{x} = -25$

$$x + 10\sqrt{x} + 25 = 0$$

$\left(\sqrt{x} + 5\right)\left(\sqrt{x} + 5\right) = 0$ square root sign missed

$\left(\sqrt{x} + 5\right)^2 = 0$ square root sign missed

$\sqrt{x} = -5$ square root sign missed

No real solutions.

3 $x^4 - 7x^2 + 1 = 0$

Let $y = x^2$

$y^2 - 7y + 1 = 0$

$$y = \frac{7 \pm \sqrt{49-4}}{2}$$

$$y = \frac{7 \pm \sqrt{45}}{2} = \frac{7 \pm 3\sqrt{5}}{2}$$

$y = x^2$

$$\therefore x = -\sqrt{\frac{7+3\sqrt{5}}{2}}, \sqrt{\frac{7+3\sqrt{5}}{2}}, -\sqrt{\frac{7-3\sqrt{5}}{2}}, \sqrt{\frac{7-3\sqrt{5}}{2}}$$

4 $10^{2x} + 10^x - 2 = 0$

Let $y = 10^x$

$y^2 + y - 2 = 0$

$(y+2)(y-1) = 0$

So $y = -2$ or 1

So $10^x = -2$ or $10^x = 1$

So $10^x = 1$

5 $2x^8 = 5x^4 - 1$

$2x^8 - 5x^4 + 1 = 0$

Let $y = x^4$

$2y^2 - 5y + 1 = 0$

$$y = \frac{5 \pm \sqrt{25-8}}{4} = \frac{5 \pm \sqrt{17}}{4}$$

$y = x^4 = 2.280776406$, $x = \pm 1.22891\ldots = \pm 1.23$ (3 s.f.)

$y = x^4 = 0.21922359\ldots$, $x = \pm 0.684261\ldots = \pm 0.684$ (3 s.f.)

6 $p + \dfrac{\sqrt{p}}{2} = \dfrac{3}{2}$

$2p + \sqrt{p} - 3 = 0$

Let $q = \sqrt{p}$

$2q^2 + q - 3 = 0$

$(2q+3)(q-1) = 0$

So $q = -\dfrac{3}{2}$ or 1

So $\sqrt{p} = -\dfrac{3}{2}$ or 1

$\sqrt{p} = 1$ so $p = 1$

7 $(5^{2x} + 5^x + 4)^3 = 343$

$5^{2x} + 5^x + 4 = 7$

$5^{2x} + 5^x - 3 = 0$

Let $y = 5^x$

$y^2 + y - 3 = 0$

$a = 1, b = 1, c = -3$

$$y = \frac{-1 \pm \sqrt{1 - 4.1.-3}}{2}$$

$$= \frac{-1 \pm \sqrt{13}}{2}$$

$$5^x = \frac{-1+\sqrt{13}}{2} \text{ or } 5^x = \frac{-1-\sqrt{13}}{2}$$

So $5^x = \dfrac{-1+\sqrt{13}}{2}$

8 $x^4 - 19x^2 + 16 = 0$

Let $y = x^2$

$y^2 - 19y + 16 = 0$

$a = 1, b = -19, c = 16$

$$y = \frac{19 \pm \sqrt{361 - 4.1.16}}{2}$$

$$= \frac{19 \pm \sqrt{297}}{2}$$

$$x^2 = \frac{19+\sqrt{297}}{2} \text{ or } x^2 = \frac{19-\sqrt{297}}{2}$$

$x = \pm 4.26$ or ± 0.940

9 $2x^6 + x^3 - 1 = 0$

Let $y = x^3$

$2y^2 + y - 1 = 0$

$$y = \frac{-1 \pm \sqrt{1^2 + 8}}{4} = \frac{-1 \pm 3}{4}$$

$y = \frac{1}{2}, y = -1$

So, $x = \dfrac{1}{\sqrt[3]{2}}$ or $x = -1$

Exam-style questions

1 a $y = 2x^2 - 3x - 7$

$b^2 - 4ac = 9 - (4)(2)(-7) = 65 > 0$ two distinct real roots.

b $2x^2 - 3x - 7 = 0$

$a = 2, b = -3, c = -7$

$$x = \frac{3 \pm \sqrt{9 - (4)(2)(-7)}}{4}$$

$$= \frac{3 \pm \sqrt{65}}{4}$$

$x = 2.77$ or $x = -1.27$

2 a $$x = \frac{3 \pm \sqrt{9 - (4)(3)(-11)}}{6}$$

$$= \frac{3 \pm \sqrt{141}}{6}$$

$x = -1.48$ or $x = 2.48$

b $3x^2 - 3x - 11 > 0$

$x < -1.48$ or $x > 2.48$

3 $x^2 - 9x - 10 \geqslant 0$

$(x - 10)(x + 1) = 0$

$x = 10$ or -1

$x \leqslant -1$ or $x \geqslant 10$

4 $$f(x) = 2x^2 + 3x + 13 = 2\left[x^2 + \frac{3}{2}x + \frac{13}{2}\right]$$

$$= 2\left[\left(x + \frac{3}{4}\right)^2 - \frac{9}{16} + \frac{13}{2}\right]$$

$$= 2\left(x + \frac{3}{4}\right)^2 - \frac{9}{8} + 13$$

$$= 2\left(x + \frac{3}{4}\right)^2 + 11\frac{7}{8}$$

Minimum value is $11\frac{7}{8}$ and always greater than 0, i.e. f(x) has no real roots.

So, if f(x) = p has no real roots, then

$$2\left(x + \frac{3}{4}\right)^2 + 11\frac{7}{8} = p$$

and $2\left(x + \frac{3}{4}\right)^2 + 11\frac{7}{8} - p = 0$ has no roots.

So $p < 11\frac{7}{8}$

5 a $y = 4x^2 - 7x - 2$

$$= 4\left(x^2 - \frac{7x}{4} - \frac{1}{2}\right)$$

$$= 4\left[\left(x - \frac{7}{8}\right)^2 - \frac{49}{64} - \frac{1}{2}\right]$$

$$= 4\left(x - \frac{7}{8}\right)^2 - \frac{81}{16}$$

b At the stationary point y will have a minimum value i.e. when $x = \frac{7}{8}$.

When $x = \frac{7}{8}$.

$$y = 4\left(x - \frac{7}{8}\right)^2 - \frac{81}{16}$$

$$= -\frac{81}{16}$$

So coordinates of stationary point are $\left(\frac{7}{8}, -\frac{81}{16}\right)$.

6 a $3x - 2y = 7$ (1)

$x^2 - y^2 = 8$ (2)

Rearrange (1).

$$x = \frac{7 + 2y}{3}$$

Substitute into (2).

$$\left(\frac{7+2y}{3}\right)^2 - y^2 = 8$$

$49 + 28y + 4y^2 - 9y^2 = 72$

$5y^2 - 28y + 23 = 0$

$(5y - 23)(y - 1) = 0$

$\left(\frac{27}{5}, \frac{23}{5}\right)$ and $(3, 1)$

b $3x - 2y = 7$, so $y = \frac{3x-7}{2}$

Substitute into $x^2 - y^2 = c$

$$x^2 - \left(\frac{3x-7}{2}\right)^2 = c$$

$4x^2 - (3x - 7)^2 = 4c$

$4x^2 - (9x^2 - 42x + 49) = 4c$

$-5x^2 + 42x - 49 - 4c = 0$

$5x^2 - 42x + 49 + 4c = 0$

$b^2 - 4ac$

$= 1764 - 4 \times 5 \times (49 + 4c)$

$= 1764 - 980 - 80c$

$= 784 - 80c$

$= 8(98 - 10c)$

For no intersection $b^2 - 4ac < 0$

$8(98 - 10c) < 0$

$98 - 10c < 0$

$98 < 10c$

$c > 9.8$

7 $6x^2 > 3 - 7x$

$6x^2 + 7x - 3 > 0$

$(2x + 3)(3x - 1) > 0$

$x < -\frac{3}{2}$ or $x > \frac{1}{3}$

8 a $x^2 + 10x + 21 > 0$

$(x + 3)(x + 7) > 0$

$x < -7$ or $x > -3$ and $3x - 5 > 0$

$x > \frac{5}{3}$

So $x > \frac{5}{3}$

b $y = \text{gf}(x) = 3(x^2 + 10x + 21) - a$

$= 3x^2 + 30x + 63 - a$

$= 3(x^2 + 10x) + 63 - a$

$= 3(x + 5)^2 - 75 + 63 - a$

$= 3(x + 5)^2 - (12 + a)$

$y = \text{gf}(x)$ has 2 real roots when $12 + a > 0$,
i.e. $a > -12$

9 $4^x = 1 + 2^{1-2x} = 1 + 2^1 \times 2^{-2x} = 1 + \frac{2}{2^{2x}}$

$4^x = 2^{2x}$

$2^{2x} = 1 + \frac{2}{2^{2x}}$

$2^{2x} \times 2^{2x} = 2^{2x} + 2$

$2^{4x} - 2^{2x} - 2 = 0$

$\therefore\ 4^{2x} - 4^x - 2 = 0$

Let $y = 2^{2x}$

$y^2 - y - 2 = 0$

$(y + 1)(y - 2) = 0$

$y = -1, y = 2$

$2^{2x} = -1$ not possible, so $2^{2x} = 2$

$2x = 1$, so $x = \frac{1}{2}$

10 $3x^2 + 5x - 2 > 7x + 15$

$3x^2 - 2x - 17 > 0$

$$x = \frac{2 \pm \sqrt{4 - (4)(3)(-17)}}{6}$$

$$= \frac{2 \pm \sqrt{208}}{6}$$

$x = -2.070$ or $x = 2.737$

$x < -2.07$ or $x > 2.74$ (3 s.f.)

11 a $px^2 + 5x - 6 = 0$

$b^2 - 4ac < 0$

$25 - 4 \times p \times (-6) < 0$

$25 + 24p < 0$

$p < -\frac{25}{24}$

b $\text{f}(x + p) = p^3$

$p(x + p)^2 + 5(x + p) - q = p^3$

$px^2 + (2p^2 + 5)x + 5p - q = 0$

One repeated root when $b^2 - 4ac = 0$

$b^2 - 4ac = (2p^2 + 5)^2 - 4p(5p - q)$

$4p^4 + 20p^2 + 25 - 20p^2 + 4pq = 0$

$4p^4 + 4pq + 25 = 0$ as required.

12 a $a^2x^2 + 3bx + \sqrt{c} = 0$

$$a^2\left(x^2 + \frac{3bx}{a^2} + \frac{\sqrt{c}}{a^2}\right) = 0$$

$$\left(x + \frac{3b}{2a^2}\right)^2 - \frac{9b^2}{4a^4} + \frac{\sqrt{c}}{a^2} = 0$$

$$\left(x+\frac{3b}{2a^2}\right)^2=\frac{9b^2}{4a^4}-\frac{\sqrt{c}}{a^2}$$

$$=\frac{9b^2}{4a^4}-\frac{4a^2\sqrt{c}}{4a^4}$$

$$=\frac{9b^2-4a^2\sqrt{c}}{4a^4}$$

$$x+\frac{3b}{2a^2}=\frac{\pm\sqrt{9b^2-4a^2\sqrt{c}}}{2a^2}$$

$$x=\frac{-3b\pm\sqrt{9b^2-4a^2\sqrt{c}}}{2a^2}$$

b $y=a^2x^2+3bx+\sqrt{c}$ has a repeated root when $c=1$

i.e. $b^2-4ac=0$

$9b^2-4a^2=0$

i.e. $a^2=\frac{9b^2}{4}$

$$x=\frac{-3b\pm\sqrt{b^2-4ac}}{2a^2}=\frac{-3b}{2a^2}=-\frac{3b}{2}\times\frac{4}{9b^2}=-\frac{2}{3b}$$

13 a Pythagoras' theorem $AB^2=BD^2-AD^2$

$AB^2=(5x^2+14xy+10y^2)-(2x+3y)^2$

$AB^2=5x^2+14xy+10y^2-(4x^2+12xy+9y^2)$

$AB^2=x^2+2xy+y^2$

$AB^2=(x+y)^2$

$AB=x+y$

b So length $=19+-12=7$ or $-1+3=2$.

14 $13\pi r\leqslant 260$

$r\leqslant 6.4$

$\pi r^2>60$

$r>4.4$

$4.4<r\leqslant 6.4$

15 a $\left(\sqrt{p}+p-1\right)^3=125$

$\sqrt{p}+p-1=5$

$\sqrt{p}+p-6=0$

$p+\sqrt{p}-6=0$

b Let $y=\sqrt{p}$

$y^2+y-6=0$

$(y+3)(y-2)=0$

So $y=-3$ or $y=2$.

$\sqrt{p}=-3$ or $\sqrt{p}=2$

So $p=4$. (No solution from $\sqrt{p}=-3$.)

16 $h_A=t+4,\ t\geqslant 0$

$h_B=8+3t-t^2$

$h_B>h_A$

$8+3t-t^2>t+4$

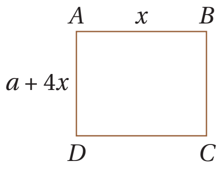
$4+2t-t^2>0$

$0>t^2-2t-4$

$$t=\frac{2\pm\sqrt{4+16}}{2}=\frac{2\pm\sqrt{20}}{2}=1\pm\sqrt{5}$$

When $t=0$ $h_B>h_A$ $\therefore$ $0\leqslant t\leqslant 1+\sqrt{5}$ seconds.

17 A sketch diagram often helps, it doesn't need to be drawn accurately.



Area $=x(a+4x)=4x^2+ax$

When $a=3$, Area $\geqslant 45\text{ m}^2$

$4x^2+3x\geqslant 45$

$4x^2+3x-45\geqslant 0$

Consider $4x^2+3x-45=0$

$$x=\frac{-3\pm\sqrt{3^2-4\times4\times(-45)}}{2\times4}$$

$$x=\frac{-3\pm27}{8}=\frac{-30}{8}\text{ or }\frac{24}{8}$$

As $x>0$, $x=\frac{24}{8}=3$ is the only solution.

The values of x for which the area is at least 45 m^2 are $x\geqslant 3\text{ m}$

18 $6574t^2+776t+100>1\,000\,000$

$6574t^2+776t-999\,900>0$

Consider $6574t^2+776t-999\,900=0$

$$t=\frac{-776\pm\sqrt{776^2+4\times6574\times999\,900}}{2\times6574}$$

$$t=\frac{-776\pm162\,154.163}{13\,148}$$

Reject the negative value, as $t\geqslant 0$.

$t=12.27$°C

The values of t for which the number of bacteria is great than $1\,000\,000$ is $t>12.3$° C (3 s.f.).

Mathematics in life and work

1 $2x^2 - 3x - \frac{3}{2} = 0$

$a = 2, b = -3, c = -\frac{3}{2}$

$$x = \frac{3 \pm \sqrt{9 + \frac{4 \times 2 \times 3}{2}}}{4}$$

$$x = \frac{3 \pm \sqrt{21}}{4}$$

So $x = \frac{3 + \sqrt{21}}{4}$ or $\frac{3 - \sqrt{21}}{4}$.

2 $\frac{3 - \sqrt{21}}{4} < x < \frac{3 + \sqrt{21}}{4}$

3 $2x^2 - 3x - \frac{3}{2} = 2\left[x^2 - \frac{3}{2}x - \frac{3}{4}\right]$

$$= 2\left[\left(x - \frac{3}{4}\right)^2 - \frac{9}{16} - \frac{3}{4}\right]$$

$$= 2\left[\left(x - \frac{3}{4}\right)^2 - \frac{21}{16}\right]$$

So coordinates of turning point are $x = \frac{3}{4}, y = \frac{-21}{8}$.

Line of symmetry $x = \frac{3}{4}$.

4 $\frac{21}{8}$ units

2 Functions

Please note: Full worked solutions are provided as an aid to learning, and represent one approach to answering the question. In some cases, alternative methods are shown for contrast.

All sample answers have been written by the authors. Cambridge Assessment International Education bears no responsibility for the example answers to questions taken from its past question papers, which are contained in this publication.

Non-exact numerical answers should be given correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

Prerequisite knowledge

1 **a** $f(0) = 2 - 0 = 2$

b $f(1) = 2 - 1 = 1$

c $f(-3) = 2 + 3 = 5$

d $f\left(\frac{1}{2}\right) = 2 - \frac{1}{2} = \frac{3}{2}$

2 **a** $3x - 5 = 1$

$x = 2$

b $3x - 5 = -17$

$x = -4$

c $3x - 5 = 0$

$x = \frac{5}{3}$

d $3x - 5 = 6$

$x = \frac{11}{3}$

3 $2 - x = 3x - 5$

$4x - 7 = 0$

$x = \frac{7}{4}$

4 $g(2) = 1$

$fg(2) = 2 - 1 = 1$

$f(2) = 0$

$gf(2) = 0 - 5 = -5$

Exercise 2.1A

1 **a** $f(x) = 5 - 3x$ is a linear relationship whereby every input value is related to only one output value and every output value is related to only one input value. This is a one–one function.

b $g(x) = x^2 + x$ is a quadratic relationship whereby every input value is related to only one output value. As an output value may be related to more than one input value, this is a many–one function.

c $h(x) = \sqrt{(x + 1)}$ is not a function because it is not defined for $x < -1$.

d This is not a function because an input value of $x = 0$ does not map to any output value because there is a discontinuity at $x = 0$.

e This is an ellipse where an input value can map to two output values. Consequently, this is not a function.

f This is not a function because some input values do not map to any output value because there is a discontinuity at these values.

2 **a** $f(-3) = -17$

$f(-2) = -12$

$f(-1) = -7$

$f(0) = -2$

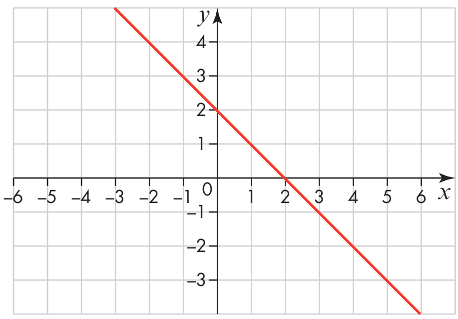
$f(1) = 3$

$f(2) = 8$

So the range of $f(x)$ is $\{-17, -12, -7, -2, 3, 8\}$.

b This is a linear relationship whereby every input value is related to only one output value and every output value is related to only one input value. This is a one–one function.

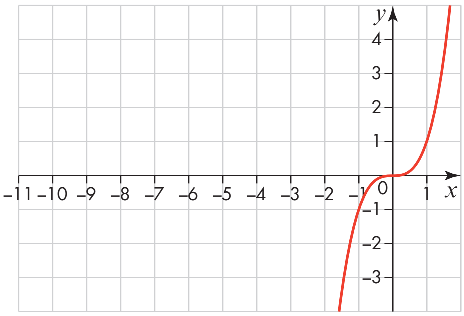
3 **a** $f(x) = 2 - x$ for $x \in \mathbb{R}$



$f(x) \in \mathbb{R}$

$f(x)$ can have any real number value.

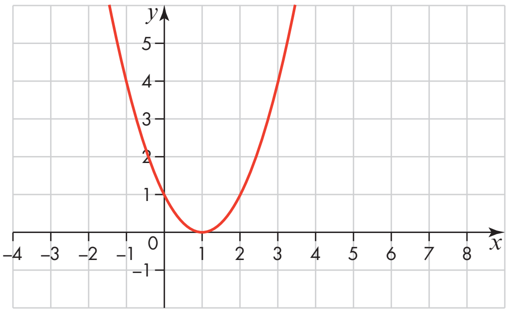
b $g(x) = x^3$ for $\{x: x \leqslant 1, x \in \mathbb{R}\}$



g(x) $\{g(x): g(x) \leqslant 1, g(x) \in \mathbb{R}\}$

All the values of g(x) are less than or equal to 1

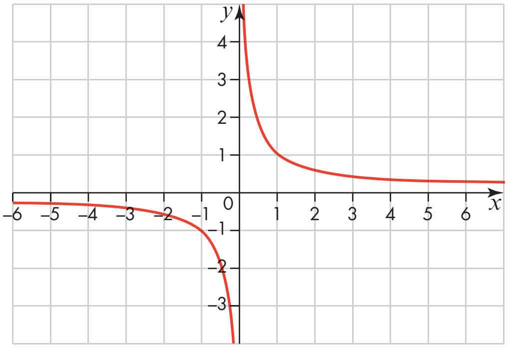
c $h(x) = (x-1)^2$ for $x \in \mathbb{R}$



h(x) $\{h(x): h(x) \geqslant 0, h(x) \in \mathbb{R}\}$

All the values of h(x) are greater than or equal to zero (when you square a number it is either zero or a positive number).

d $j(x) = \frac{1}{x}$ for $\{x: x \neq 0, x \in \mathbb{R}\}$

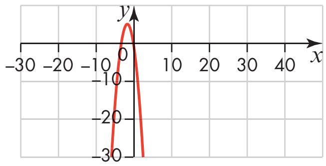


j(x)$\{j(x): j(x) \neq 0, j(x) \in \mathbb{R}\}$

j(x) can have any value except 0.

4 **a** $g(-2) = 3 + 10 - 8 = 5$

b



$g(-5) = 3 + 25 - 50 = -22$

Turning point:

$$-2\left[\left(x+\frac{5}{4}\right)^2 - \frac{49}{16}\right] = 0$$

$$x = -\frac{5}{4},\ g\left(-\frac{5}{4}\right) = \frac{49}{8}$$

Range of g(x) is $-22 \leqslant g(x) \leqslant \frac{49}{8}$.

c $3 - 5b - 2b^2 = 6$

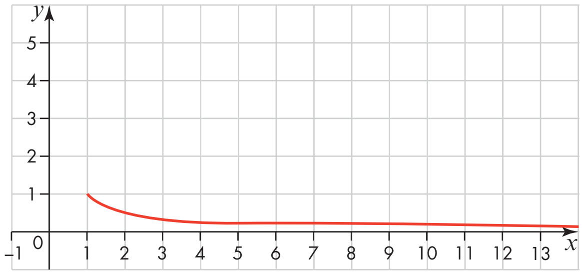
$2b^2 + 5b + 3 = 0$

$(2b + 3)(b + 1) = 0$

$b = -\frac{3}{2}$ or $b = -1$

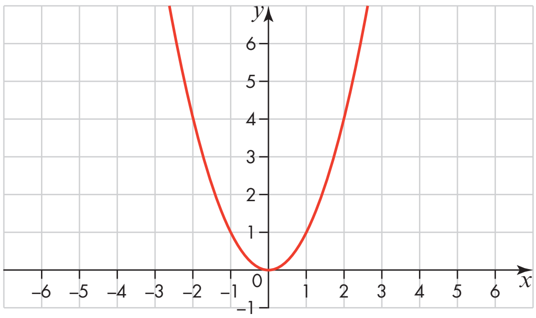
d This is a quadratic relationship whereby every input value is related to only one output value. As an output value may be related to more than one input value, this is a many–one function.

5 **a** $f(x) = \frac{1}{x}$ for $\{x: x \geqslant 1, x \in \mathbb{R}\}$



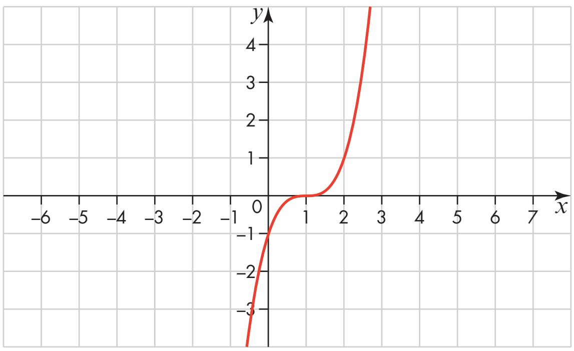
Range $\{f(x): 0 < f(x) \leqslant 1, f(x) \in \mathbb{R}\}$

b $g(x) = x^2$ for $x \in \mathbb{R}$



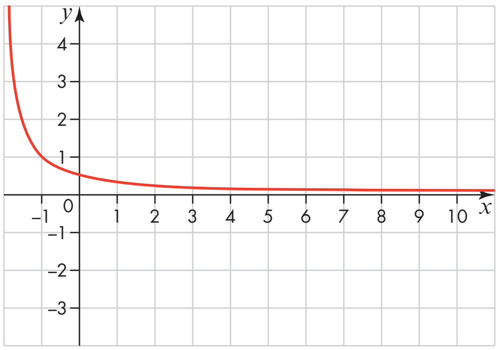
Range $\{g(x): g(x) \geqslant 0, g(x) \in \mathbb{R}\}$

c $h(x) = (x-1)^3$ for $x \in \mathbb{R}$



Range $\{h(x): h(x) \in \mathbb{R}\}$

d $\mathrm{j}(x) = \dfrac{1}{x+2}$ for $\{x: x > -2, x \in \mathbb{R}\}$



Range $\{\mathrm{f}(x): \mathrm{f}(x) \geqslant 0, \mathrm{f}(x) \in \mathbb{R}\}$

6 $\mathrm{h}(2) = d - 2$

$\mathrm{j}(2) = \dfrac{2+2}{4} = 1$

$d - 2 = 1$

$d = 3$

7 The relationship $y = \dfrac{1}{(x+a)^2}$ isn't currently a function because there is a discontinuity at $x = -a$.

The relationship could become a function if the domain was limited to either $\{x < -a, x \in -\mathbb{R}\}$ or $\{x > -a, x \in +\mathbb{R}\}$

8 **a** $\{x: x > 3, x \in \mathbb{R}\}$ As the curve continuous for $x > 3$.

b $\{x: x < 0, x \in \mathbb{R}\}$ Although the curve is continuous for $x < 3$, the question specifies $\{x \in -\mathbb{R}\}$

9 $4x^2 + 6x - 15 = \left(2x + \dfrac{3}{2}\right)^2 - \dfrac{9}{4} - 15 = \left(2x + \dfrac{3}{2}\right)^2 - \dfrac{69}{4}$

This has a minimum value of $-\dfrac{69}{4}$ when $x = -\dfrac{3}{4}$.

$\mathrm{f}(x)$ is one-one for $x \geqslant -\dfrac{3}{4}$.

Hence $\mathrm{f}(x)$ is one-one for the set of values $\left\{x: x \in \mathbb{R}, x \geqslant -\dfrac{3}{4}\right\}$.

10 $\mathrm{f}(x) = px^5 + qx^3 + 2$

$\mathrm{f}(-1) = -p - q + 2 = 7$

$-p - q - 5 = 0$ A

$\mathrm{f}(2) = 32p + 8q + 2 = 10$ $32p + 8q - 8 = 0$

$4p + q - 1 = 0$ B

Work out A + B

$3p - 6 = 0$, so $p = 2$, $q = -7$

11 $2x^2 - 4x + 13 = 2(x^2 - 2x) + 13 = 2(x-1)^2 - 2 + 13 = 2(x-1)^2 + 11$

This is a one-one function for values $\{x: x \in \mathbb{R}, x \geqslant 1\}$, so $p = 1$.

Exercise 2.2A

1 **a** $\mathrm{g}(1) = 1^2 = 1$

$\mathrm{f}(1) = (4 \times 1) + 3 = 7$

b $\mathrm{f}(1) = (4 \times 1) + 3 = 7$

$\mathrm{g}(7) = 7^2 = 49$

c $\mathrm{g}(-2) = (-2)^2 = 4$

$\mathrm{g}(4) = 4^2 = 16$

d $\mathrm{f}(-2) = (4 \times -2) + 3 = -5$

$\mathrm{f}(-5) = (4 \times -5) + 3 = -17$

2 **a** Cannot be found as the range of $\mathrm{g}(x)\{x : x \geqslant -1\}$ is not fully included in the domain of $\mathrm{f}(x)\{x : x \neq 0\}$.

b $\mathrm{gf}(x) = \left(\dfrac{1}{x}\right)^2 - 1 = \dfrac{1 - x^2}{x^2}$

c $\mathrm{gg}(x) = (x^2 - 1)^2 - 1 = x^4 - 2x^2$

d $\mathrm{ff}(x) = \dfrac{1}{\frac{1}{x}} = x$

3 $\mathrm{fg}(a) = 4$

$(a - 5)^2 = 4$

$a^2 - 10a + 25 = 4$

$a^2 - 10a + 21 = 0$

$(a - 3)(a - 7) = 0$

So $a = 3$ or $a = 7$.

4 $\mathrm{fg}(x) = 5(x^2 + 3) - 2 = 5x^2 + 13$

$\mathrm{gf}(x) = (5x - 2)^2 + 3 = 25x^2 - 20x + 7$

So $\mathrm{fg} \neq \mathrm{gf}$.

5 **a** $x + 2 = \mathrm{ff}(x)$

b $x^2 + 2x - 6 = \mathrm{gf}(x)$

6 $\mathrm{f}(x) = 2x - 3$, $\mathrm{g}(x) = x^3$ and $\mathrm{h}(x) = \dfrac{1}{x+1}$

$\mathrm{hg}(x) = \dfrac{1}{x^3 + 1}$

$\mathrm{fhg}(x) = \dfrac{2}{x^3 + 1} - 3$

$\mathrm{ffhg}(x) = \dfrac{4}{x^3 + 1} - 6 - 3$

$\mathrm{f}^2\mathrm{hg} = \dfrac{4}{x^3 + 1} - 9$

7 $\mathrm{fg}(x) = 2^{x-3}$

$\mathrm{gf}(x) = 2^x - 3$

$\mathrm{fg} = \mathrm{gf}$

$2^{x-3} = 2^x - 3$

$\dfrac{2^x}{2^3} = 2^x - 3$

$2^x = (8)(2^x) - 24$

$24 = (7)(2^x)$

$$x = \frac{\ln\left(\frac{24}{7}\right)}{\ln 2}$$

$= 1.78$ (3 s.f.)

fg(x) can be found as the range of g is included in the domain of f, but gf(x) cannot be found as f(x) > 0 for all x and the domain of g is x: $x < 0$.

8 $f(x) = \dfrac{2}{x-1}$

$$ff(x) = f\left(f(x)\right) = \frac{2}{\frac{2}{x-1} - 1} = \frac{2}{\frac{2}{x-1} - \frac{x-1}{x-1}}$$

$$= \frac{2}{\frac{3-x}{x-1}} = \frac{2(x-1)}{(3-x)}$$

The domain is $\{x: x \in \mathbb{R}, x > 3\}$.

9 $f(x) = 4x^3 - 2x^2 + 7$, $g(x) = x^2 - 1$

$fg(x) = f(g(x)) = 4(x^2 - 1)^3 - 2(x^2 - 1)^2 + 7$

$= 4(x^6 - 3x^4 + 3x^2 - 1) - 2(x^4 - 2x^2 + 1) + 7$

$= 4x^6 - 14x^4 + 16x^2 + 1$

10 $f(x) = 4x - 5$, for $fg(x) = x$; $g(x)$ must involve $\frac{1}{4}x$ in order for $4 \times \frac{1}{4}x = x$.

It must also involve $\frac{5}{4}$ in order for $4 \times \frac{5}{4} = +5$.

Check: If $g(x) = \frac{x}{4} + \frac{5}{4}$:

$$fg(x) = 4 \times \left(\frac{x}{4} + \frac{5}{4}\right) - 5 = x + 5 - 5 = x$$

$$g(x) = \frac{x}{4} + \frac{5}{4}, \text{ or } g(x) = \frac{x+5}{4}$$

Exercise 2.3A

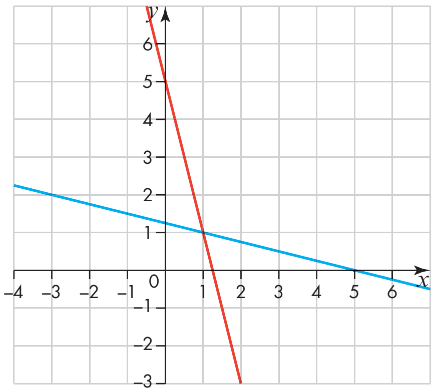
1 **a** This is a linear function so it is one-one.

$f(x) = 5 - 4x$ $\{x \in \mathbb{R}\}$

$y = 5 - 4x$

$x = \dfrac{5-y}{4}$

$f^{-1}(x) = \dfrac{5-x}{4}$ $\{x \in \mathbb{R}\}$



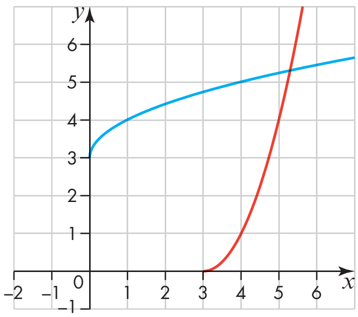
b This is one-one for the given domain.

$f(x) = (x-3)^2$ $\{x \in \mathbb{R}, x \geqslant 3\}$

$\boldsymbol{y = (x-3)^2}$

$\boldsymbol{x = 3 + \sqrt{y}}$

$f^{-1}(x) = 3 + \sqrt{x}$ $\{x \geqslant 0, x \in \mathbb{R}\}$



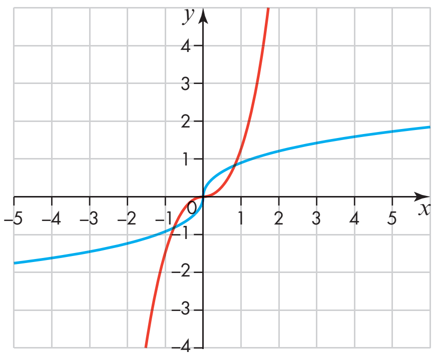
c The cube of any number is unique so the function is one-one.

$f(x) = x^3$ $\{x \in \mathbb{R}\}$

$y = x^3$

$x = \sqrt[3]{y}$

$f^{-1}(x) = \sqrt[3]{x}$ $\{x \in \mathbb{R}\}$



2 $f(x) = 7x - 2$

Let $y = 7x - 2$.

$$\frac{y+2}{7} = x$$

$$f^{-1}(x) = \frac{x+2}{7}$$

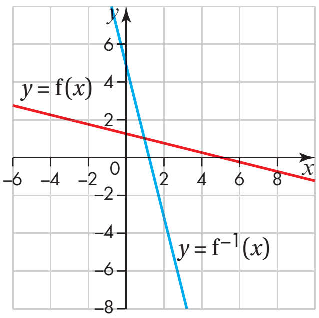
$$f^{-1}(2) = \frac{2+2}{7} = \frac{4}{7}$$

3 **a** $f^{-1}(x) = 5 - 4x \ \{x \in \mathbb{R}\}$

Let $y = 5 - 4x$.

$$x = \frac{5-y}{4}$$

$$f(x) = \frac{5-x}{4} \ \{x \in \mathbb{R}\}$$

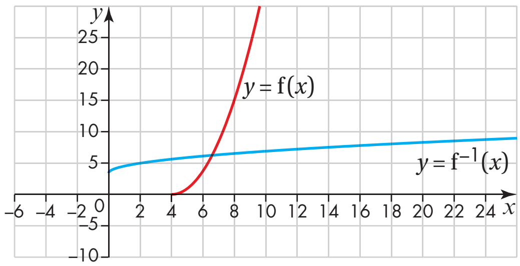


b $f^{-1}(x) = \sqrt{x} + 4 \ \{x \in \mathbb{R}, x \geqslant 0\}$

Let $y = \sqrt{x} + 4$.

$$x = (y-4)^2$$

$$f(x) = (x-4)^2 \ \{x \in \mathbb{R}, x \geqslant 4\}$$



4 $f(x) = x^2 + 7x - 11$

Let $y = x^2 + 7x - 11$.

$$y = \left[\left(x+\frac{7}{2}\right)^2 - \frac{49}{4} - 11\right]$$

$$y = \left[\left(x+\frac{7}{2}\right)^2 - \frac{93}{4}\right]$$

$$x = \sqrt{y + \frac{93}{4}} - \frac{7}{2}$$

$$f^{-1}(x) = \sqrt{x + \frac{93}{4}} - \frac{7}{2}$$

The domain is $\left\{x : x \in \mathbb{R}, x > \frac{-93}{4}\right\}$.

5 **a** $f(x) = \frac{2}{x-1} + 5$, where $\{x \in \mathbb{R}, x > 1\}$.

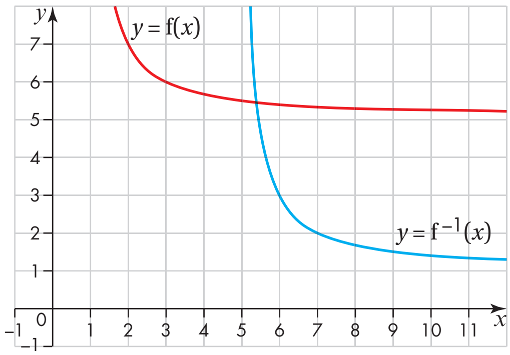
Let $y = \frac{2}{x-1} + 5$.

$$y - 5 = \frac{2}{x-1}$$

$$x = \frac{2}{y-5} + 1$$

$f^{-1}(x) = \frac{2}{x-5} + 1$, where $\{x \in \mathbb{R}, x > 5\}$.

b



c $f(x) = f^{-1}(x)$

$$\frac{2}{x-1} + 5 = \frac{2}{x-5} + 1$$

$$2(x-5) + 5(x-1)(x-5) = 2(x-1) + (x-1)(x-5)$$

$$4x^2 - 24x + 12 = 0$$

$$x^2 - 6x + 3 = 0$$

$$x = 3 \pm \sqrt{6}$$

(Only $3 + \sqrt{6}$ is valid as $3 - \sqrt{6}$ is not included in the domain of either function.)

6 $f(x) = 2x^2 + 5x - 13$

Let $y = 2x^2 + 5x - 13$.

$$y = 2\left[\left(x+\frac{5}{4}\right)^2 - \frac{129}{16}\right]$$

$$x = \sqrt{\frac{8y+129}{16}} - \frac{5}{4}$$

$$f^{-1}(x) = \sqrt{\frac{8x+129}{16}} - \frac{5}{4}$$

$$ff^{-1}(x) = 2\left(\sqrt{\frac{8x+129}{16}} - \frac{5}{4}\right)^2 + 5\left(\sqrt{\frac{8x+129}{16}} - \frac{5}{4}\right) - 13$$

$$ff^{-1}(x) = \frac{8x+129}{8} - 5\sqrt{\frac{8x+129}{16}} + \frac{25}{8} + 5\sqrt{\frac{8x+129}{16}} - \frac{25}{4} - \frac{52}{4}$$

$$ff^{-1}(x) = x + \frac{129}{8} + \frac{25}{8} - \frac{50}{8} - \frac{104}{8}$$

$$ff^{-1}(x) = x$$

7 $\mathrm{f}(x) = \dfrac{3x-1}{x-2}$ Let $y = \dfrac{3x-1}{x-2}$

Then $(x-2)\,y = 3x-1$

$xy - 2y = 3x - 1$

$xy - 3x = 2y - 1$

$x(y-3) = 2y - 1$

$x = \dfrac{2y-1}{y-3}$

$\mathrm{f}^{-1}(x) = \dfrac{2x-1}{x-3}$

The domain is $\{x : x \in \mathbb{R}, x > 3\}$.

8 $\mathrm{f}(x) = 2x^2 - 4x + 13$

Let $y = 2x^2 - 4x + 13$

Then $y = 2(x^2 - 2x) + 13$

$y = 2[(x-1)^2 - 1] + 13$

$y = 2(x-1)^2 + 11$

$\dfrac{y-11}{2} = (x-1)^2$

$x = \sqrt{\dfrac{y-11}{2}} + 1$

$\mathrm{f}^{-1}(x) = \sqrt{\dfrac{x-11}{2}} + 1$

$\mathrm{ff}^{-1}(x) = 2 \times \left(\sqrt{\dfrac{x-11}{2}} + 1\right)^2 - 4\left(\sqrt{\dfrac{x-11}{2}} + 1\right) + 13$

$= 2\left(\dfrac{x-11}{2} + 2\sqrt{\dfrac{x-11}{2}} + 1\right) - 4\left(\sqrt{\dfrac{x-11}{2}} + 1\right) + 13$

$= x - 11 + 4\sqrt{\dfrac{x-11}{2}} + 2 - 4\sqrt{\dfrac{x-11}{2}} + 9$

$= x$

$\mathrm{f}^{-1}\mathrm{f}(x) = \sqrt{\dfrac{2x^2 - 4x + 13 - 11}{2}} + 1$

$= \sqrt{\dfrac{2x^2 - 4x + 2}{2}} + 1$

$= \sqrt{x^2 - 2x + 1} + 1$

$= \sqrt{(x-1)^2} + 1$

$= x - 1 + 1 = x$

$\mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x) = x$ as required.

9 $\mathrm{f}(x) = 4x^3 - 2x^2 + 7$, $\mathrm{g}(x) = x^2 - 1$

Let $y = x^2 - 1$; then $x = \sqrt{y+1}$, i.e. $\mathrm{g}^{-1}(x) = \sqrt{x+1}$,

$\mathrm{fg}^{-1}(x) = 4\sqrt{x+1}^{\,3} - 2\left(\sqrt{x+1}\right)^2 + 7$

$= 4\sqrt{x+1}^{\,3} - 2x - 2 + 7$

$= 4\sqrt{x+1}^{\,3} - 2x + 5$

The domain is $\{x : x \in \mathbb{R}, x > -1\}$.

10 $\mathrm{f}(x) = x^6 - 2x^3 - 7$

Let $y = x^6 - 2x^3 - 7$

Then $y = (x^3 - 1)^2 - 1 - 7$

$y = (x^3 - 1)^2 - 8$

$y + 8 = (x^3 - 1)^2$

$x^3 - 1 = \sqrt{y+8}$

$x^3 = \sqrt{y+8} + 1$

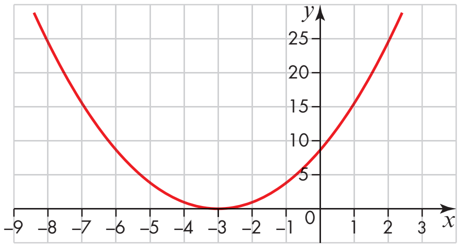
$x = \sqrt[3]{\sqrt{y+8} + 1}$

$\mathrm{f}^{-1}(x) = \sqrt[3]{\sqrt{x+8} + 1}$

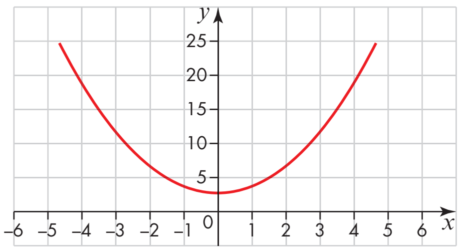
The domain is $\{x : x \in \mathbb{R}, x \geqslant -8\}$.

Exercise 2.4A

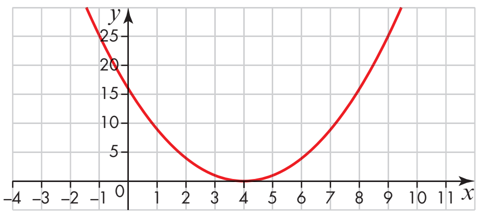
1 **a** $\mathrm{f}(x+3)$ one solution



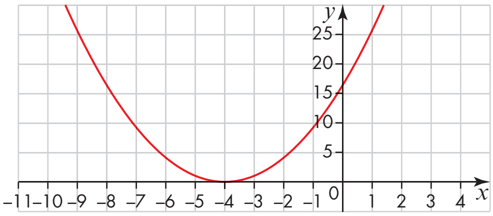
b $\mathrm{f}(x) + 3$ no solutions



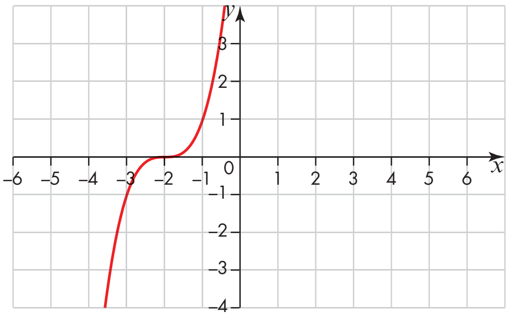
c $\mathrm{f}(x-4)$ one solution



d $f(x + 4)$ one solution

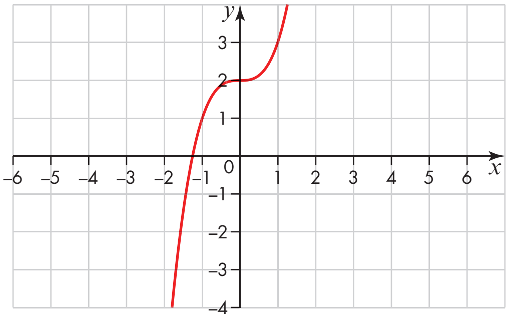


2 a



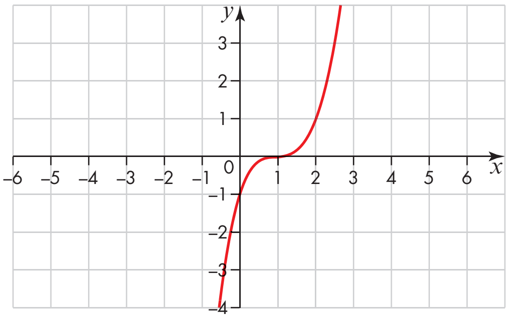
One solution

b



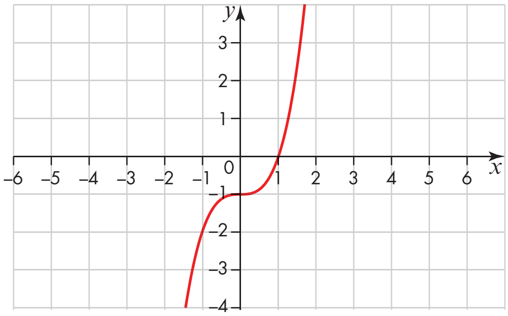
One solution

c



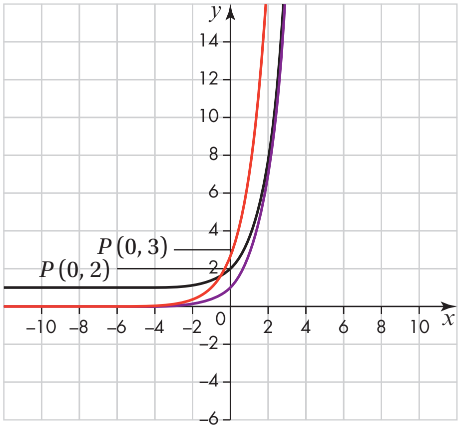
One solution

d



One solution

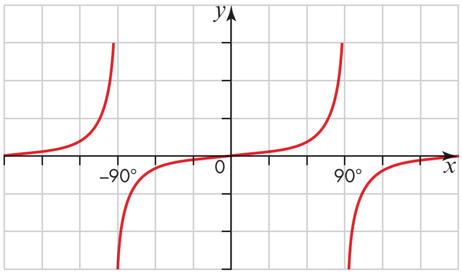
3



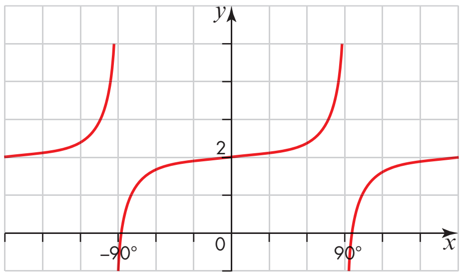
$f(x + 1) = f(x) + 1$ has one solution

4 a, b $f(x – 180)$, where $–180° \leqslant x \leqslant 180°$; asymptotes at $x = –90$ and 90.

Axis intercepts at (–180, 0), (0, 0) and (180, 0).

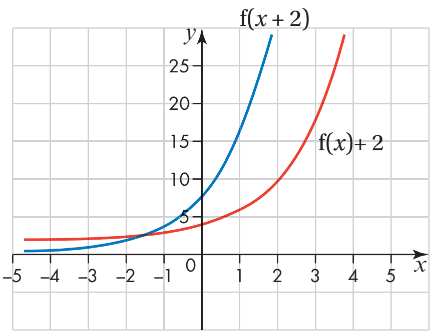


c, d $f(x) + 2$; asymptotes at $x = –90$ and 90



5 **a, b** $f(x + 2) = 2^{x+3}$

$f(x) + 2 = 2^{x+1} + 2$



c **i** If $g(x) = 2^x$, then $g(x + 3) = 2^{x+3} = f(x + 2)$.

ii If $g(x) = 2^x$, then $g(x + 1) + 2 = 2^{x+1} + 2$

$= f(x) + 2.$

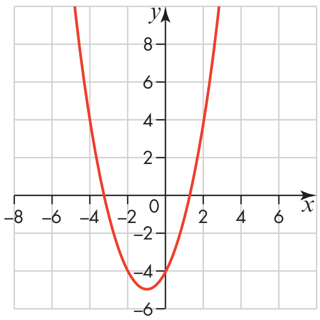
6 $f(x) = x^2 + 6x - 15 = (x + 3)^2 - 24$

$f(x)$ is a translation of $y = x^2$ with vector $\begin{pmatrix} -3 \\ -24 \end{pmatrix}$

So, the transformation which maps $f(x)$ to x^2 is a translation with vector $\begin{pmatrix} 3 \\ 24 \end{pmatrix}$.

7 $f(x) = x^2 - 2,\quad g(x) = x - 1,\ \therefore\ g^{-1}(x) = x + 1$

$fg^{-1}(x) - 3 = (x + 1)^2 - 2 - 3 = (x + 1)^2 - 5$



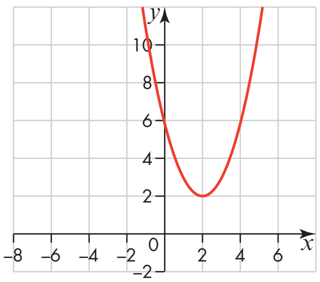
8 $f(x) = x^2 - 2x + 1$, $g(x) = x - 1$, $h(x) = x + 2$

$hfg(x) = hf(x - 1) = (x - 1)^2 - 2(x - 1) + 1 + 2$

$= x^2 - 2x + 1 - 2x + 2 + 3$

$= x^2 - 4x + 6 = (x - 2)^2 - 4 + 6$

$= (x - 2)^2 + 2$

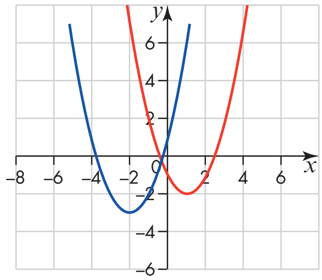


$f(x) = x^2 - 2x + 1 = (x - 1)^2$

Translation, vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

9 $f(x) = (x + 2)^2 - 3$ drawn in blue,

$g(x) = (x - 1)^2 - 2$ in red



a Translation vector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

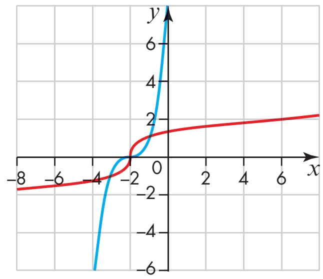
b Translation vector $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$

c Translation vector $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$

d Translation vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

10 $f(x) = x^3,\ \therefore\ f^{-1}(x) = \sqrt[3]{x}$

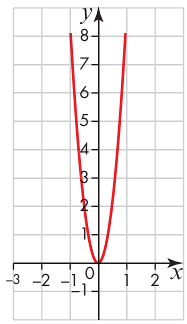
$f(x + 2) = (x + 2)^3$, drawn in blue $\therefore\ f^{-1}(x) = \sqrt[3]{(x + 2)}$ drawn in red



Exercise 2.4B

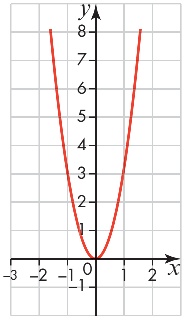
1 **a** $f(3x)$

one solution



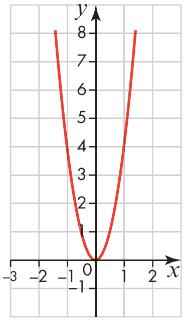
b 3f(x)

one solution



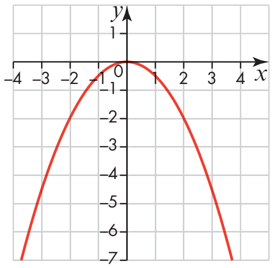
c f(2x)

one solution

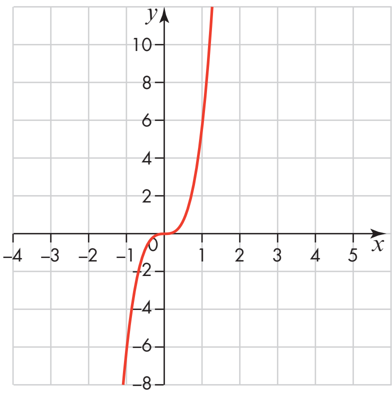


d $-\frac{1}{2}$f(x)

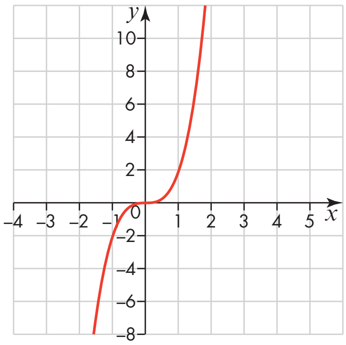
one solution



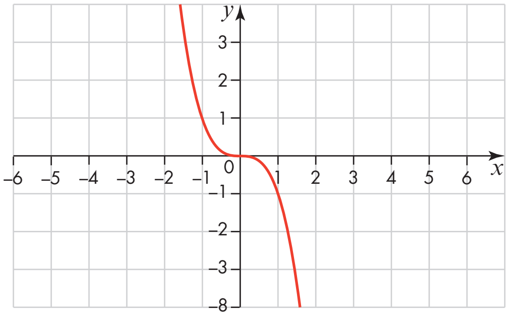
2 a f(2x)



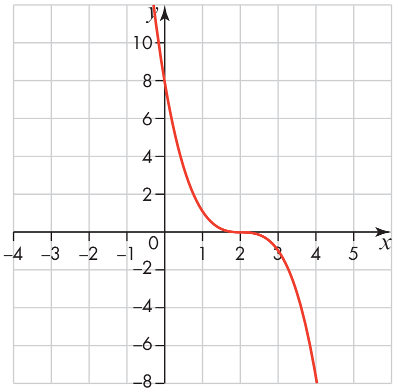
b 2f(x)



c f($-x$)

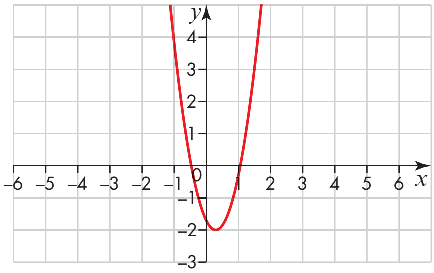


d $-$f($x - 2$)



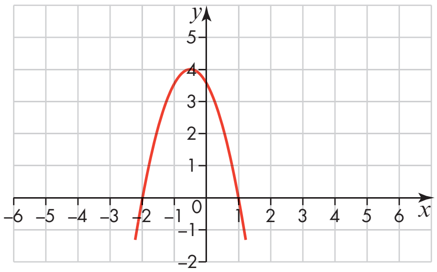
3 f($-2x$); P (1, 0), $R\left(-\frac{1}{2}, 0\right)$, Q (0, -2)

f($-2x$) = 0 has two solutions.



-2f(x); P (-2, 0), R (1, 0), Q (0, 4)

-2f(x) = 0 has two solutions.

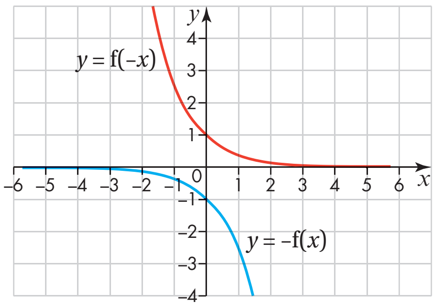


4 $y = f(-x)$ and $y = -f(x)$ in both cases asymptote at $y = 0$.

For $y = f(-x)$ P is at (0, 1).

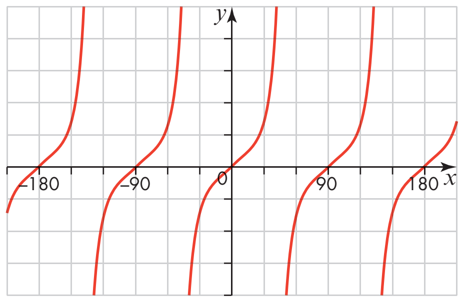
For $y = -f(x)$ P is at (0, –1).

$f(-x) = -f(x)$ has no solutions (they do not intersect).



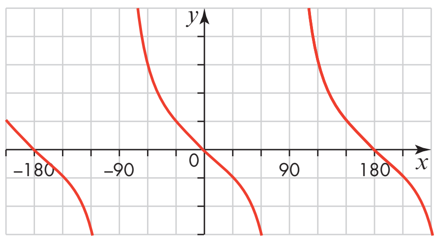
5 **a, b** $y = f(2x)$, where $-180° \leqslant x \leqslant 180°$; asymptotes at $x = -135°, -45°, 45°$ and $135°$.

Axes intercepts at (–180°, 0), (–90°, 0), (0, 0), (90°, 0) and (180°, 0).

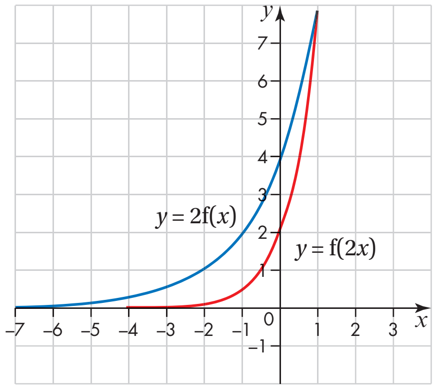


c, d $y = -2f(x)$, where $-180° \leqslant x \leqslant 180°$; asymptotes at $x = -90°$ and $90°$.

Axes intercepts at (–180°, 0), (0, 0) and (180°, 0).



6 **a, b, d** $f(2x) = 2^{2x+1}$ no solutions and $2f(x) = (2)(2^{x+1})$ no solutions



c **i** If $g(x) = 2^x$, $g(2x + 1) = 2^{2x+1}$ and

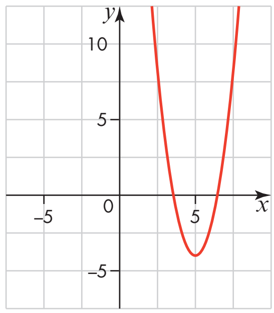
ii $2g(x + 1) = (2)(2^{x+1})$

7 $f(x) = x^2 - 2$, $\quad g(x) = x - 5$, $\quad \therefore g^{-1}(x) = x + 5$

$g^{-1}(-x) = -x + 5$

$2fg^{-1}(-x) = 2[(5 - x)^2 - 2] = 2(5 - x)^2 - 4$

This is the most useful form to use to produce a sketch graph.



8 $f(x) = x^2 - 2x + 1$, or $f(x) = (x - 1)^2$

$g(x) = 2x$,

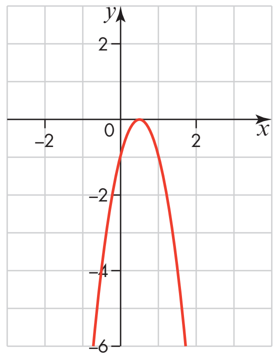
$h(x) = -x$

$hfg(x) = -[(2x)^2 - 2(2x) + 1] = -4x^2 + 4x - 1$

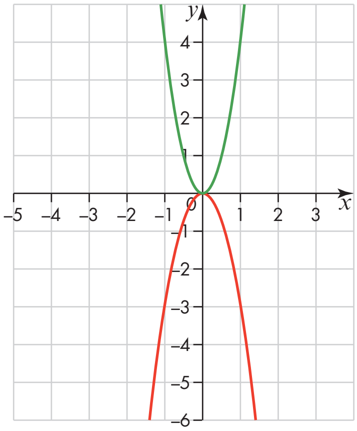
Or, using $f(x) = (x - 1)^2$

$hfg(x) = -(2x - 1)^2$

Horizontal stretch with scale factor $\frac{1}{2}$ and a reflection in the x-axis



9 $y = \mathrm{f}(x)$ drawn in red, $y = \mathrm{g}(x)$ drawn in green.



a One-way stretch vertically with scale factor $-\frac{4}{3}$

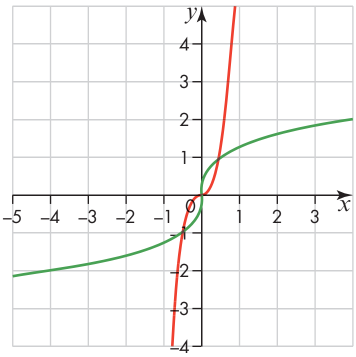
b One-way stretch vertically with scale factor $-\frac{3}{4}$

c One-way stretch vertically with scale factor -3

d One-way stretch vertically with scale factor 4

10 $\mathrm{f}(x) = x^3$, so $\mathrm{f}^{-1}(x) = \sqrt[3]{x}$

$\mathrm{f}(2x) = (2x)^3$ (red), $\mathrm{f}^{-1}(2x) = \sqrt[3]{(2x)}$ (green)



Exam-style questions

1 **a** $\{\mathrm{f}(x) : \mathrm{f}(x) \geqslant -11.75, \mathrm{f}(x) \in \mathbb{R}\}$

b $y = 3(x^2 - x) - 11$

$$y = 3\left(x - \frac{1}{2}\right)^2 - \frac{47}{4}$$

$$y + \frac{47}{4} = 3\left(x - \frac{1}{2}\right)^2$$

$$x = \frac{1}{2} + \sqrt{\frac{y}{3} + \frac{47}{12}}$$

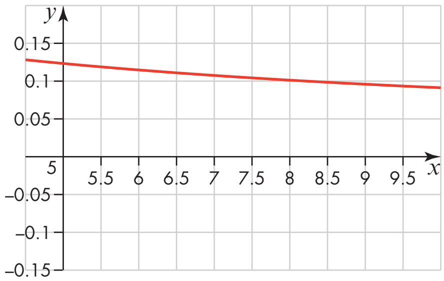
$$\mathrm{f}^{-1}(x) = \frac{1}{2} + \sqrt{\frac{x}{3} + \frac{47}{12}}$$

2 **a** Translation with vector $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

b Reflection in the y-axis

c Translation with vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

3 **a** $\left\{\mathrm{f}(x) \in \mathbb{R}, \mathrm{f}(x) < \frac{1}{7}\right\}$



b Let $y = \dfrac{1}{x+2}$.

$$x + 2 = \frac{1}{y}$$

$$x = \frac{1}{y} - 2$$

$$= \frac{1 - 2y}{y}$$

$$\mathrm{f}^{(-1)}(x) = \frac{1 - 2x}{x}$$

4 **a** $\mathrm{g}^2(x) = \dfrac{1}{\frac{1}{x}} = x$

$$\mathrm{g}^{-1}(x) = \frac{1}{x}$$

$$\mathrm{gg}^{-1}(x) = \frac{1}{\frac{1}{x}} = x$$

$\mathrm{g}^2(x) = \mathrm{gg}^{-1}(x)$

The inverse of $\mathrm{g}(x)$ is the same as $\mathrm{g}(x)$.
Consequently $\mathrm{g}^2(x) = \mathrm{gg}^{-1}(x)$ become the same composition of functions.

b $\mathrm{fg}(x) = \left(\dfrac{1}{x}\right)^2 + \dfrac{5}{x} - 13$ range of $\mathrm{g}(x)$ included within domain of $\mathrm{f}(x)$

$\mathrm{gf}(x) = \dfrac{1}{x^2 + 5x - 13}$ range of $\mathrm{f}(x)$ not all included within domain of $\mathrm{g}(x)$ so not an appropriate composition of functions.

5 Let $y = \dfrac{1}{3-x}$; then $3 - x = \dfrac{1}{y}$ and $x = 3 - \dfrac{1}{y}$

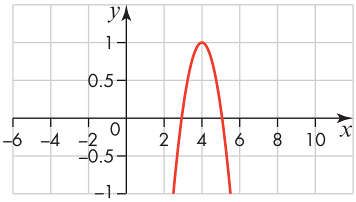
$$\mathrm{f}^{-1}(x) = 3 - \frac{1}{x}$$

6 **a** $\mathrm{fg}(x) = 2(x-3) - (x-3)^2$

$$= 2x - 6 - (x^2 - 6x + 9)$$

$$= 2x - 6 - x^2 + 6x - 9$$

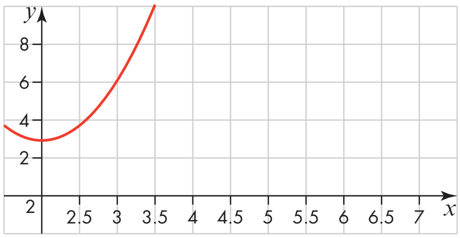
$$= -x^2 + 8x - 15$$

b 

7 a $f(x) = x^2 - 4x + 7$

$x^2 - 4x + 7 = (x - 2)^2 + 3$

When $x > 2$, $(x - 2)^2 + 3 > 0$.

b 

$\{f(x) \in \mathbb{R}, f(x) > 3\}$

c Let $y = (x - 2)^2 + 3$.

$x = 2 + \sqrt{y - 3}$

$f^{-1}(x) = 2 + \sqrt{x - 3}$

8 a $f(x) = 2x^2 + 8x - 14 = 2(x^2 + 4x) - 14$

$= 2[(x + 2)^2 - 4] - 14 = 2(x + 2)^2 - 22$

$f(x) = 2(x + 2)^2 - 22$

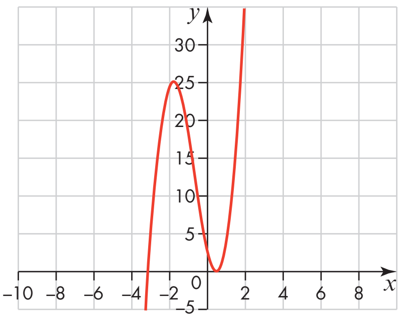
b $f(x) = 2(x + 2)^2 - 22$

Let $y = 2(x + 2)^2 - 22$. Then $(x + 2)^2 = \dfrac{y + 22}{2}$

$x = \sqrt{\dfrac{y + 22}{2}} - 2$

$f^{-1}(x) = \sqrt{\dfrac{x + 22}{2}} - 2$

c One-way stretch scale factor 2, translation with vector $\begin{pmatrix} -2 \\ -22 \end{pmatrix}$.

9 a 

b $f(-x)$ reflection in y-axis

10 a $\sqrt{x + 7} = h^2fg(x)$

b $x^{-\frac{1}{2}} + 7 = ghf(x)$

c $x^{\frac{1}{2}} + 7 = gh^2f(x)$

11 a Let $y = \dfrac{2x^2 + 7x + 6}{x^2 + x - 2}$

$= \dfrac{(2x + 3)(x + 2)}{(x + 2)(x - 1)}$

$= \dfrac{2x + 3}{x - 1}$

$y(x - 1) = 2x + 3$

$yx - 2x = y + 3$

$x(y - 2) = y + 3$

$x = \dfrac{y + 3}{y - 2}$

$f^{-1}(x) = \dfrac{x + 3}{x - 2}$

b $f^{-1}(x) = 7$

$\dfrac{x + 3}{x - 2} = 7$

$x + 3 = 7x - 14$

$6x - 17 = 0$

$x = \dfrac{17}{6}$

12 a $f(x) = 3x^2 + 15x - 10 = 3(x^2 + 5x) - 10$

$= 3\left(x + \dfrac{5}{2}\right)^2 - \dfrac{75}{4} - 10$

Minimum value of $-\dfrac{115}{4}$.

b $f(5) = 75 + 75 - 10 = 140$, so range is $f(x) \geqslant 140$.

13 a $\dfrac{2^{2x^2}}{2^x} = 2^{2x^2 - x} = gf(x)$

b $2^x(2^{x+1} - 1) = 2^x(2 \times 2^x - 1) = 2 \times 2^x \times 2^x - 2^x$

Let $y = 2^x$.

$2 \times 2^x \times 2^x - 2^x = 2y^2 - y = fg(x)$

c $8x^4 - 8x^3 + x = 8x^4 - 8x^3 + 2x^2 - 2x^2 + x$

$= 8x^4 - 8x^3 + 2x^2 - (2x^2 - x)$

$= 2(4x^4 - 4x^3 + x^2) - (2x^2 - x)$

$= 2(2x^2 - x)^2 - (2x^2 - x)$

$= ff(x)$

14 a $\mathrm{gf}(x) = \dfrac{1}{1-3(2x-5)} = \dfrac{1}{1-6x+15} = \dfrac{1}{16-6x}$

b $2\mathrm{g}(x) = \mathrm{f}(x)$

$\dfrac{2}{1-3x} = 2x-5$

$2 = (2x-5)(1-3x)$

$= 2x - 5 - 6x^2 + 15x$

$0 = -6x^2 + 17x - 7$

$x = \dfrac{-17 \pm \sqrt{121}}{-12} = \dfrac{1}{2}$ or $2\dfrac{1}{3}$

15 a $\mathrm{f}(x) = x^3 + x^2 + x + 1$

$\mathrm{g}(x) = (x-1)^3 + (x-1)^2 + (x-1) + 1 - 3$

$\mathrm{g}(x) = (x-1)^3 + (x-1)^2 + x - 3$

b $\mathrm{h}(x) = -2x^3 + 2x^2 - 2x + 1$

$= 2(-x)^3 + 2(-x)^2 + 2(-x) + 1$

$= 2[(-x)^3 + (-x)^2 + (-x) + 1] - 1$

One-way stretch vertically scale factor 2,

reflection in y-axis, translation with vector $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

16 a Let $y = \dfrac{x}{5-x}$

$y(5-x) = x$

$5y = x + xy = x(1+y)$

$x = \dfrac{5y}{1+y}$

$\mathrm{f}^{-1}(x) = \dfrac{5x}{1+x}$

b $\mathrm{ff}(x) = \mathrm{f}(x)$,

$\dfrac{\frac{x}{5-x}}{5-\frac{x}{5-x}} = \dfrac{\frac{x}{5-x}}{\frac{5(5-x)-x}{5-x}} = \dfrac{x}{5-x} \times \dfrac{5-x}{25-6x}$

$\dfrac{x}{25-6x} = \dfrac{x}{5-x}$

$x(5-x) = x(25-6x)$

$5x - x^2 = 25x - 6x^2$

$5x^2 - 20x = 0$

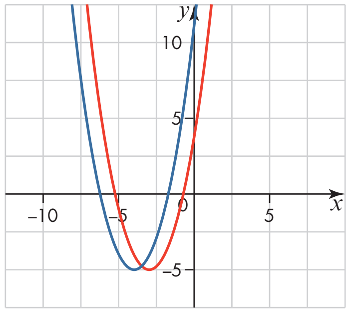
$5x(x-4) = 0$

$x = 0$ or $x = 4$

17 a $x^2 + 6x + 4 = (x+3)^2 - 9 + 4 = (x+3)^2 - 5$

b Translation vector $\begin{pmatrix} -3 \\ -5 \end{pmatrix}$

c $y = \mathrm{g}(x)$ in red, $y = \mathrm{h}(x)$ in blue



d Translation vector $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

18 a $\mathrm{f}(x) = ax^2 + bx + c = a\left(x^2 + \dfrac{b}{a}x\right) + c$

$= a\left[\left(x + \dfrac{b}{2a}\right)^2 - \dfrac{b^2}{4a^2}\right] + c$

$= a\left(x + \dfrac{b}{2a}\right)^2 - \dfrac{b^2}{4a} + c$

$q \geqslant c - \dfrac{b^2}{4a}$

b $\mathrm{f}(-1) = -9$, $\mathrm{f}(1) = 1$, $\mathrm{f}(2) = 15$

$a - b + c = -9$ A

$a + b + c = 1$ B

$4a + 2b + c = 15$ C

A + B

$2a + 2c = -8$, ie $a + c = -4$ D

2A + C

$6a + 3c = -3$, ie $2a + c = -1$ E

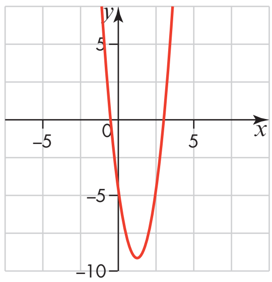
E – D

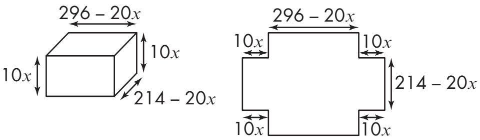
$a = 3$

$c = -7$

$b = 5$

c



19 

a $v(x) = (296 - 20x)(214 - 20x)10x$
$= 40(148 - 10x)(107 - 10x)x$

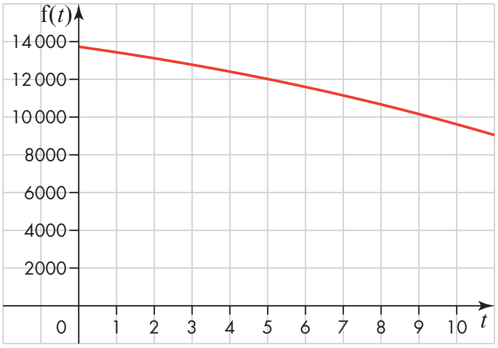
b Domain is $\{x: x \in \mathbb{R}, 0 < x < 10.7\}$. Range is $\{v(x): v(x) \in \mathbb{R}, 0 \leqslant v(x) < 1158\}$.

c $x = 3\text{ cm}$

$v = 236 \times 154 \times 30 = 1\,090\,320\text{ mm}^3$, or 1090 cm^3 (3 s.f.)

Mathematics in life and work

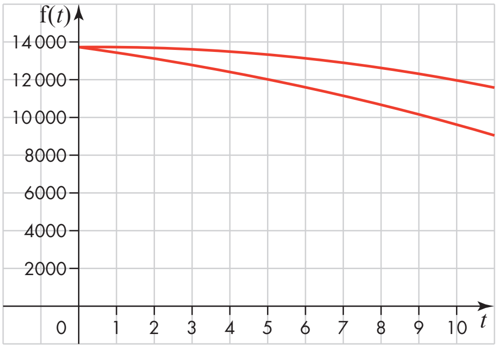
1 $\{t \geqslant 0, t \in \mathbb{R}\}$

2 

3 As the car ages its value depreciates.

4 $\text{fg}(t) = 15\,000 - 1500\left(2^{\frac{t}{10}}\right)$

5 Car Model A has the greater rate of depreciation, as illustrated on the following graph.



3 Coordinate geometry

Please note: Full worked solutions are provided as an aid to learning, and represent one approach to answering the question. In some cases, alternative methods are shown for contrast.

All sample answers have been written by the authors. Cambridge Assessment International Education bears no responsibility for the example answers to questions taken from its past question papers, which are contained in this publication.

Non-exact numerical answers should be given correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

Prerequisite knowledge

1 Gradient $= \frac{9-5}{(-2)-8} = \frac{4}{-10} = \frac{-2}{5}$.

2 Gradient of first line $= -3$.

Perpendicular gradient $= \frac{1}{3}$.

Equation of perpendicular line through (5, 8) is given by $y - 8 = \frac{1}{3}(x-5)$.

$3y - 24 = x - 5$

$x - 3y + 19 = 0$

3 Substitute $y = \frac{2}{3}x - 2$ into $x + 2y = 17$.

$x + 2\left(\frac{2}{3}x - 2\right) = 17$

$x + \frac{4}{3}x - 4 = 17$

$\frac{7}{3}x = 21$

$7x = 63$

$x = 9$

When $x = 9$, $y = \frac{2}{3} \times 9 - 2 = 4$.

Hence point of intersection = (9, 4).

4 Complete square on $x^2 - 14x + 23$.

$x^2 - 14x + 23 = (x-7)^2 - 49 + 23$

$(x-7)^2 - 26$

Exercise 3.1A

1 The lines not in the form $ax + by + c = 0$, where a, b and c are integers are:

c, because it does not equal zero

d, because it does not equal zero

e, because c is not an integer

f, because a, b and c are not integers.

2 **a** Rearrange $y = 4 + 5x$.

$5x - y + 4 = 0$, where $a = 5$, $b = -1$ and $c = 4$

b Rearrange $y = 3 - 2x$.

$2x + y - 3 = 0$, where $a = 2$, $b = 1$ and $c = -3$

3 **a** Multiply by 3: $3y = x - 21$, and rearrange.

$x - 3y - 21 = 0$

b Multiply by 5: $5y = -2x + 30$, and rearrange.

$2x + 5y - 30 = 0$

c Multiply by 3: $3y = 4x + \frac{21}{2}$.

Multiply by 2: $6y = 8x + 21$, and rearrange.

$8x - 6y + 21 = 0$

4 Rearrange: $8x + 3 = 2y$

Rearrange and divide by 2: $y = 4x + \frac{3}{2}$

gradient = 4 and the y-intercept has coordinates $\left(0, \frac{3}{2}\right)$

5 The line in the form $ax + by + c = 0$, where a, b and c are integers is:

$\frac{5y}{3} = x - 4$

$5y = 3(x-4)$

$= 3x - 12$

$3x - 5y - 12 = 0$

6 When the line intersects the x-axis, $y = 0$. Substituting $y = 0$

$-3x + 2 = 0$

$2 = 3x$

$x = \frac{2}{3}$

So the coordinates of the x-intercept are $\left(\frac{2}{3}, 0\right)$.

When the line intersects the y-axis, $x = 0$.

Substituting $x = 0$

$-5y + 2 = 0$

$2 = 5y$

$y = \frac{2}{5}$

So the coordinates of the y-intercept are $\left(0, \frac{2}{5}\right)$.

7 $x + \frac{y}{3} - \frac{1}{2} = 0$.

Multiply by 6: $6x + 2y - 3 = 0$, $a = 6$, $b = 2$, $c = -3$

8 $\frac{4x}{5} + \frac{7}{3} = \frac{5y}{4}$

Multiply by 60: $48x + 140 = 75y$,

so $48x - 75y + 140 = 0$

9 $\frac{5}{7x} = -\frac{1}{4y}$, $5 \times 4y = -1 \times 7x$, $20y = -7x$,

$20y + 7x = 0$, $a = 7$, $b = 20$, $c = 0$

10 To write $\frac{1}{2x} - 3n = \frac{5}{3y}$ in the required form multiply by the denominators, $2x$, $3y$.

This gives a term $2x \times 3y \times (-3n) = -18nxy$, which is non-zero if $n \neq 0$.

$n = 0$ is the only value.

Exercise 3.1B

1 a $m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (2, 3)$

$(x_2, y_2) = (7, 8)$

$m = \frac{8-3}{7-2} = \frac{5}{5} = 1$

b $m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (3, 3)$

$(x_2, y_2) = (5, 9)$

$m = \frac{9-3}{5-3} = \frac{6}{2} = 3$

c $m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (1, -3)$

$(x_2, y_2) = (3, -9)$

$m = \frac{-9 - -3}{3-1} = \frac{-6}{2} = -3$

2 a $m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (3a, 3a)$

$(x_2, y_2) = (8a, 5a)$

$m = \frac{5a - 3a}{8a - 3a} = \frac{2a}{5a} = \frac{2}{5}$

b $m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (a, a)$

$(x_2, y_2) = (3a, -5a)$

$m = \frac{-5a - a}{3a - a} = \frac{-6a}{2a} = -3$

3 Should be $m = \frac{y_2 - y_1}{x_2 - x_1}$.

y-coordinate incorrect in (x_2, y_2). It should be $(5, -8)$.

Numerator and denominator confused. Should be:

$m = \frac{-8-2}{5-1} = \frac{-10}{4} = \frac{-5}{2}$

4 a $m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (1, 1)$

$(x_2, y_2) = (4, -8)$

$m = \frac{-8-1}{4-1} = \frac{-9}{3} = -3$

Lie on a straight line with a gradient of –3.

b $m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (1, 3)$

$(x_2, y_2) = (4, 9)$

$m = \frac{9-3}{4-1} = \frac{6}{3} = 2$

Do not lie on a straight line with a gradient of –3.

c $m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (1, -7)$

$(x_2, y_2) = (4, -16)$

$m = \frac{-16 - -7}{4-1} = \frac{-9}{3} = -3$

Lie on a straight line with a gradient of –3.

d $m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (1, 7)$

$(x_2, y_2) = (3, 1)$

$m = \frac{1-7}{3-1} = \frac{-6}{2} = -3$

Lie on a straight line with a gradient of –3.

5 Ascent:

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (-10, 0)$

$(x_2, y_2) = \left(0, \frac{1}{2}\right)$

$m = \frac{\frac{1}{2} - 0}{0 + 10} = \frac{\frac{1}{2}}{10} = \frac{1}{20}$

Descent:

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = \left(0, \frac{1}{2}\right)$

$(x_2, y_2) = \left(2, \frac{3}{8}\right)$

$$m = \frac{\frac{3}{8}-\frac{1}{2}}{2-0} = \frac{-\frac{1}{8}}{2} = -\frac{1}{16}$$

The descent is the steepest because $\frac{1}{16} > \frac{1}{20}$.

6 $m = \frac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = \left(\frac{1}{2}, \frac{1}{3}\right)$

$(x_2, y_2) = \left(\frac{3}{4}, -\frac{2}{3}\right)$

$$m = \frac{-\frac{2}{3}-\frac{1}{3}}{\frac{3}{4}-\frac{1}{2}} = \frac{-1}{\frac{1}{4}} = -4$$

7 $\frac{1}{2x} - 3n = \frac{5}{3y}$

Multiply by $3y \times 2x$.

$3y - 3n \times 2x \times 3y = 5 \times 2x$

For this to be a straight line $n = 0$.

$3y = 10x$. Gradient $= 3\frac{1}{3}$.

8 $$\frac{y_1 - y_2}{x_1 - x_2} = \frac{a+\frac{1}{5}}{\frac{1}{3}-\frac{1}{2}} = \frac{a+\frac{1}{5}}{-\frac{1}{6}} = -6\left(a+\frac{1}{5}\right) = -6a - 1.2$$

Gradient $= -2 \therefore -6a - 1.2 = -2$, $6a = 0.8$,

$a = \frac{0.8}{6} = \frac{8}{60} = \frac{2}{15}$

9 $\frac{4x}{5} + \frac{7}{3} = \frac{5y}{4}$. Multiply by 60 : $48x + 140 = 75y$,

so $y = \frac{48x}{75} + \frac{140}{75}$

Gradient $= \frac{48}{75}$, and $0 < \frac{48}{75} < 1$ as required.

10 For AB:

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{\frac{1}{6}+\frac{1}{9}}{-\frac{1}{5}-\frac{7}{8}} = \frac{\frac{5}{18}}{-\frac{43}{40}} = -\frac{5}{18} \times \frac{40}{43} = -\frac{200}{774}$$

$-1 < -\frac{200}{774} < 0$

For CD:

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{\frac{7}{8}-\frac{1}{5}}{\frac{1}{9}-\frac{1}{6}} = \frac{\frac{27}{40}}{-\frac{1}{18}} = -\frac{27}{40} \times \frac{18}{1} = -\frac{486}{40}$$

$-\frac{486}{40} < -12$

AB is shallowest, as it has a gradient closer to 0.

Exercise 3.1C

1 **a** $m = 2$ and $(x_1, y_1) = (3, 0)$

$y - y_1 = m(x - x_1)$

$y - 0 = 2(x - 3)$

$y = 2x - 6$

b $m = 3$ and $(x_1, y_1) = (0, 3)$

$y - y_1 = m(x - x_1)$

$y - 3 = 3(x - 0)$

$= 3x$

$y = 3x + 3$

c $m = 2$ and $(x_1, y_1) = (3, 4)$

$y - y_1 = m(x - x_1)$

$y - 4 = 2(x - 3)$

$= 2x - 6$

$y = 2x - 2$

d $m = -5$ and $(x_1, y_1) = (2, 3)$

$y - y_1 = m(x - x_1)$

$y - 3 = -5(x - 2)$

$= -5x + 10$

$y = -5x + 13$

2 **a** $m = -4$ and $(x_1, y_1) = (-2, -5)$

$y - y_1 = m(x - x_1)$

$y - -5 = -4(x + 2)$

$y + 5 = -4x - 8$

$y = -4x - 13$

b $m = -1$ and $(x_1, y_1) = (2, -2)$

$y - y_1 = m(x - x_1)$

$y - -2 = -1(x - 2)$

$y + 2 = -x + 2$

$y = -x$

3 $m = -2$ and $(x_1, y_1) = (0, 3)$

$y - y_1 = m(x - x_1)$

$y - 3 = -2(x - 0)$

$= -2x$

$2x + y - 3 = 0$

4 First find the point of intersection.

$2x + 4 = 7 - x$

$3x = 3$

$x = 1$

$y = 2 + 4 = 6$

$m = 3$ and $(x_1, y_1) = (1, 6)$

$y - y_1 = m(x - x_1)$

$y - 6 = 3(x - 1)$

$= 3x - 3$

$y = 3x + 3$

5 Find the equation of the first line.

$m = 3$ and $(x_1, y_1) = (1, 1)$

$$y - y_1 = m(x - x_1)$$
$$y - 1 = 3(x - 1)$$
$$= 3x - 3$$
$$y = 3x - 2$$

Find the equation of the second line.

$m = -1$ and $(x_1, y_1) = (4, 6)$

$$y - y_1 = m(x - x_1)$$
$$y - 6 = -1(x - 4)$$
$$= -x + 4$$
$$y = -x + 10$$

Find the point of intersection.

$$3x - 2 = -x + 10$$
$$4x = 12$$
$$x = 3$$
$$y = -3 + 10 = 7$$

Point of intersection is (3, 7).

6 $m = \frac{5}{2}$ and $(x_1, y_1) = (2, 3)$

$$y - y_1 = m(x - x_1)$$
$$y - 3 = \frac{5}{2}(x - 2)$$

When you substitute in $x = 0$, y will equal 0 if the line goes through the origin.

$$y - 3 = \frac{5}{2}(0 - 2)$$
$$= \frac{5}{2}(-2)$$
$$= -5$$
$$y = -2$$

The line does not go through the origin.

7 Line with gradient –2 passing through $\left(-\frac{3}{2}, \frac{1}{3}\right)$ has equation $y - \frac{1}{3} = -2\left(x + \frac{3}{2}\right)$.

Intersects the y-axis when $x = 0$.

$$y - \frac{1}{3} = -2 \times \frac{3}{2}$$
$$y = \frac{1}{3} - 3 = -\frac{8}{3}$$

8 Line with gradient $\frac{1}{2}$ passing through $\left(\frac{4}{5}, \frac{1}{10}\right)$ has equation $y - \frac{1}{10} = \frac{1}{2}\left(x - \frac{4}{5}\right)$.

Passes through point with x-coordinate 2 when

$$y - \frac{1}{10} = \frac{1}{2}\left(2 - \frac{4}{5}\right).$$

$y = 0.7 \neq 3$, $\therefore$ the line does not pass through (2, 3).

9 Line A: $y + 7 = \frac{1}{3}(x - 1)$, $3y + 21 = x - 1$, $3y = x - 22$

Line B: $y - \frac{2}{5} = -2\left(x + \frac{1}{4}\right)$, $5y - 2 = -10x - \frac{10}{4}$,

$$y = -2x - \frac{1}{10}$$

A and B meet when $3\left(-2x - \frac{1}{10}\right) = x - 22$,

$-6x - \frac{3}{10} = x - 22$, $22 - \frac{3}{10} = 7x$, $x = 3.1$, $y = -6.3$

Line with gradient 3 passing through (3.1, –6.3) has equation $y + 6.3 = 3(x - 3.1)$.

$$y + 6.3 = 3x - 9.3$$
$$y = 3x - 15.6$$

10 $\dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{\frac{1}{6} + \frac{1}{9}}{-\frac{1}{5} - \frac{7}{8}} = \dfrac{\frac{5}{18}}{-\frac{43}{40}} = \frac{5}{18} \times -\frac{40}{43} = -\frac{100}{387}$

Exercise 3.2A

1 **a** $m = 2$ in both equations so the lines are parallel.

b $m = -3$ in both equations so the lines are parallel.

2 Line A:

$(x_1, y_1) = (1, 7)$

$(x_2, y_2) = (3, 11)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{11 - 7}{3 - 1} = 2$$

Line B:

$(x_1, y_1) = (2, -3)$

$(x_2, y_2) = (5, 3)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - -3}{5 - 2} = 2$$

$m = 2$ for both lines so the lines are parallel.

3 **a** Arrange the equations in the form $y = mx + c$:

$y = 4x + 2$, $y = -4x - 3$

m is not equal so the lines are not parallel.

b Arrange the equations in the form $y = mx + c$:

$y = 3x + 1$, $y = 3x - 3$

$m = 3$ in both equations so the lines are parallel.

4 From $-4x + y + 7 = 0$, $y = 4x - 7$, so $m = 4$.

The y-intercept of $2x - y + 3 = 0$ is when $x = 0$, so is at (0, 3).

$(x_1, y_1) = (0, 3)$

$$y - y_1 = m(x - x_1)$$
$$y - 3 = 4(x - 0)$$
$$y = 4x + 3$$

5 **a** Arrange the equations in the form $y = mx + c$:

$y = \frac{5}{3}x - \frac{7}{3}, y = \frac{5}{3}x + \frac{1}{2}$

$m = \frac{5}{3}$ in both equations so the lines are parallel.

b Arrange the equations in the form $y = mx + c$:

$y = \frac{8}{3}x - \frac{7}{3},\ y = \frac{4}{3}x - \frac{1}{3}.$

m is not equal so the lines are not parallel.

6 Arrange the equations in the form $y = mx + c$:

$2x - y + 6 = 0,\ y = 2x + 6,\ m = 2$

$2x + 4y - 44 = 0,\ 4y = 44 - 2x,\ y = 11 - \frac{1}{2}x,\ m = -\frac{1}{2}$

$2x - y - 4 = 0,\ y = 2x - 4,\ m = 2$

$2x + 4y - 24 = 0,\ 4y = 24 - 2x,\ y = 6 - \frac{1}{2}x,\ m = -\frac{1}{2}$

So $2x - y + 6 = 0$ and $2x - y - 4 = 0$ are parallel lines as $m = 2$ for both lines.

And $2x + 4y - 44 = 0$ and $2x + 4y - 24 = 0$ are parallel lines as $m = -\frac{1}{2}$ for both lines.

7 $\frac{x}{4} - \frac{5y}{6} + 7 = 0,\ \frac{5y}{6} = \frac{x}{4} + 7,\ y = \frac{3x}{10} + \frac{42}{5}$

Gradient $= \frac{3}{10}$

Line with gradient 0.3 passing through (1, 3) has equation

$y - 3 = 0.3(x - 1)$

$y - 3 = 0.3x - 0.3$

$y = 0.3x + 2.7$

$10y = 3x + 27$

8 $\frac{4x}{5} + \frac{7}{3} = \frac{5y}{4},\ y = \frac{16x}{25} + \frac{28}{15}$, gradient $= \frac{16}{25}$

$8x - \frac{25y}{2} - 7 = 0,\ \frac{25y}{2} = 8x - 7,$

$y = \frac{2}{25} \times 8x - 7 \times \frac{2}{25},\ y = \frac{16x}{25} - \frac{14}{25}$, gradient $= \frac{16}{25}$

As the two lines have the same gradient they are parallel.

9 $\frac{4}{11x} = -\frac{1}{4y},\ 16y = -11x,\ y = -\frac{11x}{16}$

Line with gradient $-\frac{11}{16}$ passing through (0, 1) has equation

$y - 1 = -\frac{11}{16}(x - 0)$

$16y + 11x - 16 = 0$

10 $\frac{2x}{3} + \frac{8}{11} = \frac{7y}{2},\ y = \frac{2}{7} \times \left(\frac{2x}{3} + \frac{8}{11}\right)$, gradient $= \frac{4}{21}$

$2x - \frac{15y}{2} - 9 = 0,\ \frac{15y}{2} = 2x - 9,\ y = \frac{2}{15} \times (2x - 9),$

gradient $= \frac{4}{15}$

As $\frac{4}{21} \neq \frac{4}{15}$ the lines are not parallel.

Exercise 3.2B

1 **a** $m_1 = 2$ and $m_2 = \frac{1}{2}$ so $m_1 m_2 \neq -1$, so the lines are not perpendicular.

b $m_1 = -3$ and $m_2 = \frac{1}{3}$ so $m_1 m_2 = -1$, so the lines are perpendicular.

2 $m_1 = -\frac{1}{2}$ for the given line but the gradient of a line perpendicular to this will be $m_2 = 2$.

Let $(x_1, y_1) = (1, 0)$.

$y - y_1 = m(x - x_1)$

(x_1, y_1) incorrectly substituted. Correcting this and using correct value of m_2 gives:

$y - 0 = 2(x - 1)$

So $y = 2x - 2$

3 Arrange the equations in the form $y = mx + c$:

$2x - y + 6 = 0,\ y = 2x + 6,\ m = 2$

$2x + 4y - 44 = 0,\ 4y = 44 - 2x,\ y = 11 - \frac{1}{2}x,\ m = -\frac{1}{2}$

$2x - y - 4 = 0,\ y = 2x - 4,\ m = 2$

$2x + 4y - 24 = 0,\ 4y = 24 - 2x,\ y = 6 - \frac{1}{2}x,\ m = -\frac{1}{2}$

So $2x - y + 6 = 0$ and $2x + 4y - 44 = 0$ are perpendicular lines as $m_1 m_2 = -1$.

And $2x - y + 6 = 0$ and $2x + 4y - 24 = 0$ are perpendicular lines as $m_1 m_2 = -1$.

And $2x - y - 4 = 0$ and $2x + 4y - 44 = 0$ are perpendicular lines as $m_1 m_2 = -1$.

And $2x - y - 4 = 0$ and $2x + 4y - 24 = 0$ are perpendicular lines as $m_1 m_2 = -1$.

4 From $8x - 2y - 7 = 0,\ 8x - 7 = 2y,\ 4x - \frac{7}{2} = y,\ m_1 = 4.$

$m_1 m_2 = -1$

So $m_2 = -\frac{1}{4}$.

The y-intercept of $4x - 2y + 5 = 0$ is when $x = 0$ so is at $\left(0, \frac{5}{2}\right)$.

$(x_1, y_1) = \left(0, \frac{5}{2}\right)$

$y - y_1 = m(x - x_1)$

$y - \frac{5}{2} = -\frac{1}{4}(x - 0)$

$y = \frac{5}{2} - \frac{1}{4}x$

$x + 4y = 10$

5 **a** $m_1 = \frac{5}{3}$ and $m_2 = -\frac{3}{5}$ so $m_1 m_2 = -1$ and the lines are perpendicular.

b $m_1 = \frac{9}{10}$ and $m_2 = \frac{10}{9}$ so $m_1 m_2 \neq -1$ and the lines are not perpendicular.

6 Describe as line A

$(x_1, y_1) = (0, 6)$

$(x_2, y_2) = (4, 4)$

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-6}{4-0} = -\frac{2}{4} = -\frac{1}{2}$

Describe as line B

$(x_1, y_1) = (0, 6)$

$(x_2, y_2) = (2, 10)$

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10-6}{2-0} = \frac{4}{2} = 2$

Describe as line C

$(x_1, y_1) = (6, 8)$

$(x_2, y_2) = (2, 10)$

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10-8}{2-6} = \frac{2}{-4} = -\frac{1}{2}$

Describe as line D

$(x_1, y_1) = (4, 4)$

$(x_2, y_2) = (6, 8)$

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-4}{6-4} = \frac{4}{2} = 2$

So A is parallel to C, B is parallel to D, A is perpendicular to B and D and C is perpendicular to B and D.

So the quadrilateral can only be a square or a rectangle.

7 $\frac{x}{4} - \frac{5y}{6} + 7 = 0$, $\frac{5y}{6} = \frac{x}{4} + 7$, $y = \frac{3x}{10} + \frac{42}{5}$

Gradient $= \frac{3}{10}$.

Perpendicular line has gradient $-\frac{10}{3}$.

Line with gradient $-\frac{10}{3}$ passing through (1, 3) has equation

$y - 3 = -\frac{10}{3}(x - 1)$

$3y - 9 = -10x + 10$

$3y + 10x - 19 = 0$

8 $\frac{4x}{5} + \frac{7}{3} = \frac{5y}{4}$, $y = \frac{16x}{25} + \frac{28}{15}$, gradient $= \frac{16}{25}$

$8x - \frac{25y}{2} - 7 = 0$, $\frac{25y}{2} = 8x - 7$,

$y = \frac{2}{25} \times 8x - 7 \times \frac{2}{25}$, $y = \frac{16x}{25} - \frac{14}{25}$, gradient $= \frac{16}{25}$

As the two lines have the same gradient they are parallel, so cannot be perpendicular.

9 $\frac{4}{11x} = -\frac{1}{4y}$, $16y = -11x$, $y = -\frac{11y}{16}$

Perpendicular line has gradient $\frac{16}{11}$.

Line with gradient $\frac{16}{11}$ passing through (0, 1) has equation

$y - 1 = \frac{16}{11}(x - 0)$

$11y - 11 = 16x$

10 $\frac{2x}{3} + \frac{8}{11} = \frac{7y}{2}$, $y = \frac{2}{7} \times \left(\frac{2x}{3} + \frac{8}{11}\right)$, gradient $= \frac{4}{21}$

$2x + \frac{2y}{15} - 9 = 0$, $\frac{2y}{15} = -2x + 9$, $y = \frac{15}{2} \times (-2x + 9)$,

gradient $= -15$

As $\frac{4}{21} \times -15 = -\frac{20}{7} \neq -1$ the lines are not perpendicular.

Exercise 3.2C

1 **a** mid-point $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$(x_1, y_1) = (2, 3)$

$(x_2, y_2) = (7, 8)$

$\left(\frac{2+7}{2}, \frac{3+8}{2}\right)$

$= (4.5, 5.5)$

length $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(7-2)^2 + (8-3)^2}$

$= \sqrt{25 + 25} = 5\sqrt{2}$

b mid-point $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$(x_1, y_1) = (5, 9)$

$(x_2, y_2) = (3, 3)$

$\left(\frac{5+3}{2}, \frac{9+3}{2}\right)$

$= (4, 6)$

length $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(3-5)^2 + (3-9)^2}$

$= \sqrt{4 + 36}$

$= 2\sqrt{10}$

2 Gradient of line through (2, 2), $\left(5, \frac{1}{2}\right)$

$m = \frac{2 - \frac{1}{2}}{2 - 5} = -\frac{1}{2}$

Equation of line through (2, 2), $\left(5, \frac{1}{2}\right)$

$y - 2 = -\frac{1}{2}(x - 2)$

$y = -\frac{1}{2}x + 3$

$\left(11, \frac{-5}{2}\right)$ on line? Substitute into equation.

$x = 11$

$y = -\frac{11}{2} + \frac{6}{2} = -\frac{5}{2}$

So all points on a straight line.

length $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$(x_1, y_1) = (2, 2)$

$(x_2, y_2) = \left(11, -\frac{5}{2}\right)$

$\sqrt{(11-2)^2 + \left(-\frac{5}{2} - 2\right)^2}$

$= \sqrt{9^2 + \left(-\frac{9}{2}\right)^2}$

$= \sqrt{81 + \frac{81}{4}}$

$= \sqrt{101.25}$

≈ 10.1

3 Mid-point of A:

$\left(\frac{-1+3}{2}, \frac{3+-1}{2}\right)$

$= (1, 1)$

Gradient of A:

$m = \frac{3 - -1}{-1-3} = \frac{4}{-4} = -1$

Mid-point of B:

$\left(\frac{-2-3}{2}, \frac{2+3}{2}\right)$

$\left(-\frac{5}{2}, \frac{5}{2}\right)$

Gradient of B:

$m = \frac{3-2}{-3 - -2} = \frac{1}{-1} = -1$

Line through the mid-points:

$y - 1 = \left(\frac{\frac{5}{2} - 1}{-\frac{5}{2} - 1}\right)(x - 1)$

$y = -\frac{3}{7}x + \frac{10}{7}$

Gradient of line through mid-points is $\frac{-3}{7}$. This is not perpendicular to A or B ($m = -1$).

4 Line A length:

$d = \sqrt{[(5-2)^2 + (16-7)^2]}$

$= \sqrt{[9 + 81]}$

$= \sqrt{90}$

Line B length:

$d = \sqrt{[(7-5)^2 + (-6 - -4)^2]}$

$= \sqrt{[4+4]}$

$= \sqrt{8}$

Line A is the longest.

5 Line A:

$y - 5 = \frac{9-5}{4-3}(x - 3)$

$y = 4x - 7$

Mid-point of A:

$\left(\frac{3+4}{2}, \frac{5+9}{2}\right)$

$= \left(\frac{7}{2}, 7\right)$

Line B:

$y - -3 = \frac{-31 - -3}{5-1}(x - 1)$

$y = -7x + 4$

Point of intersection:

$4x - 7 = -7x + 4$

$x = 1, y = -3$

The point of intersection is not the mid-point of A.

6 a $\left(\frac{\frac{2}{7} + p}{2}, \frac{-\frac{1}{5} + q}{2}\right) = \left(-\frac{1}{3}, -\frac{1}{2}\right)$

$\frac{\frac{2}{7} + p}{2} = -\frac{1}{3}$

$p = -\frac{20}{21}$

$\frac{-\frac{1}{5} + q}{2} = -\frac{1}{2}$

$q = -\frac{4}{5}$

So $B = \left(-\frac{20}{21}, -\frac{4}{5}\right)$.

Note: The coords of B are the values of p and q,

So $B = \left(-\frac{20}{21}, -\frac{4}{5}\right)$.

Length of AB:

$$d = \sqrt{\left[\left(-\frac{20}{21} - \frac{2}{7}\right)^2 + \left(-\frac{4}{5} + \frac{1}{5}\right)^2\right]}$$

$$= \sqrt{\left[\left(-\frac{26}{21}\right)^2 + \left(-\frac{3}{5}\right)^2\right]}$$

$$= \sqrt{\frac{20869}{11025}}$$

b Gradient of $AB = \frac{63}{130}$, so the gradient of the perpendicular is $-\frac{130}{63}$. The equation of the perpendicular through M is

$y + \frac{1}{2} = -\frac{130}{63}\left(x + \frac{1}{3}\right)$ from which

$378y + 780x + 449 = 0.$

7 Gradient -3, through $\left(-\frac{3}{4}, -\frac{1}{3}\right)$, x-intercept has coordinates $(x, 0)$.

$$\text{Gradient } = \frac{y_1 - y_2}{x_1 - x_2} = \frac{0 + \frac{1}{3}}{x + \frac{3}{4}} = -3$$

$$-\frac{1}{3} = -3\left(x + \frac{3}{4}\right)$$

$$x = -\frac{1}{9} - \frac{3}{4} = -\frac{31}{36}$$

$$\text{Length} = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{31-27}{36}\right)^2} = \sqrt{\frac{1}{9} + \left(\frac{1}{9}\right)^2}$$

$$= \sqrt{\frac{10}{81}} = 0.351 \text{ (3 s.f.)}$$

8 **a** Consider the y-coordinates of the points.

$\frac{1}{2} > \frac{1}{10}$ as $\frac{5}{10} > \frac{1}{10}$

$\frac{1}{2} > \frac{1}{3}$ as $\frac{3}{6} > \frac{2}{6}$

Therefore, the point $\left(\frac{3}{5}, \frac{1}{2}\right)$ is not the mid-point of AB.

b Gradient of AB is $-\frac{7}{9}$

Equation is $y - \frac{1}{3} = -\frac{7}{9}\left(x - \frac{1}{2}\right)$

$18y - 6 = -14x + 7$

$14x + 18y - 13 = 0.$

c Gradient of perpendicular is $\frac{9}{7}$

Midpoint of AB is $\left(\frac{13}{20}, \frac{13}{60}\right)$

Equation of perpendicular bisector is

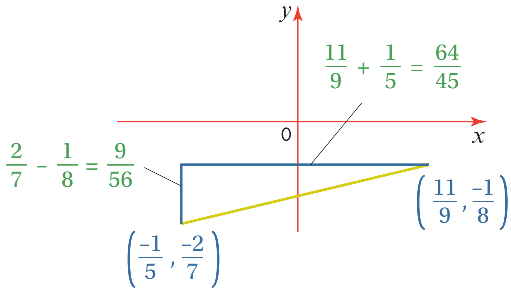
$y - \frac{13}{60} = \frac{9}{7}\left(x - \frac{13}{20}\right)$

$420y - 91 = 540x - 351$

$540x - 420y = 260.$

9 $\left(\frac{11}{9}, -\frac{1}{8}\right)$ and $\left(-\frac{1}{5}, -\frac{2}{7}\right)$ have mid-point.

$$\left(\frac{\frac{11}{9} - \frac{1}{5}}{2}, \frac{-\frac{1}{8} - \frac{2}{7}}{2}\right) \text{ ie } \left(\frac{23}{45}, -\frac{23}{112}\right)$$



$$\text{Length } = \sqrt{\left(\frac{64}{45}\right)^2 + \left(\frac{9}{56}\right)^2} = 1.43 \text{ 3 s.f.}$$

Exercise 3.3A

1 **a** centre $= (-5, 8)$, radius $= 6$

b centre $= (19, 33)$, radius $= 20$

c centre $= (0, -4)$, radius $= 3\sqrt{5}$

d centre $= (-3, -10)$, radius $= 2\sqrt{7}$

2 **a** $(x + 5)^2 + (y - 9)^2 = 49$ and
$x^2 + y^2 + 10x - 18y + 57 = 0$

b $(x + 11)^2 + (y + 1)^2 = 169$ and
$x^2 + y^2 + 22x + 2y - 47 = 0$

c $(x - 3)^2 + y^2 = 48$ and $x^2 + y^2 - 6x - 39 = 0$

d $(x - 14)^2 + (y - 6)^2 = 44$ and
$x^2 + y^2 - 28x - 12y + 188 = 0$

3 **a** $(x - 5)^2 + (y + 3)^2 = 16$

b $(x + 4)^2 + (y - 2)^2 = 9$

c $(x + 1)^2 + (y - 4)^2 = 20$

4 **a** $(x - 9)^2 - 81 + (y + 7)^2 - 49 - 14 = 0$

$(x - 9)^2 + (y + 7)^2 = 144$

centre $= (9, -7)$, radius $= 12$

b $(x + 4)^2 - 16 + y^2 = 9$

$(x + 4)^2 + y^2 = 25$

centre $= (-4, 0)$, radius $= 5$

c $x^2 + y^2 + 10x + 18y + 79 = 0$

$(x + 5)^2 - 25 + (y + 9)^2 - 81 + 79 = 0$

$(x + 5)^2 + (y + 9)^2 = 27$

centre $= (-5, -9)$, radius $= 3\sqrt{3}$

d $x^2 + y^2 + 30x - 6y + \frac{855}{4} = 0$

$(x + 15)^2 - 225 + (y - 3)^2 - 9 + \frac{855}{4} = 0$

$(x + 15)^2 + (y - 3)^2 = \frac{81}{4}$

centre = (–15, 3), radius = $\frac{9}{2}$

5 $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (-4 - 1)^2 + (7 - 3)^2$

$= (-5)^2 + (4)^2 = 41$

The circle has the equation $(x + 4)^2 + (y - 7)^2 = 41$.

6 $(x - 9)^2 + (y - 2)^2 = 81$

7 $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$= (4 - 1)^2 + (7 - 3)^2$

$= (3)^2 + (4)^2 = 25$

The circle has the equation $(x - 1)^2 + (y - 3)^2 = 25$.

The locations on the x-axis will lie on the circumference of the circle.

When $y = 0$, $(x - 1)^2 + (-3)^2 = 25$

$(x - 1)^2 + 9 = 25$

$(x - 1)^2 = 16$

$x - 1 = \pm 4$

$x = 1 \pm 4 = 5$ or -3

The required coordinates are (5, 0) and (–3, 0).

8 Rewrite the equation of the circle by completing the square.

$\left(x + \frac{1}{2}p\right)^2 - \frac{1}{4}p^2 + (y + 3)^2 - 9 = 96$

$\left(x + \frac{1}{2}p\right)^2 + (y + 3)^2 = \frac{1}{4}p^2 + 9 + 96$

Since $r^2 = 121$, $\frac{1}{4}p^2 + 9 + 96 = 121$

$\frac{1}{4}p^2 + 9 + 96 = 121$

$\frac{1}{4}p^2 = 16$

$p^2 = 64$

Since p is a positive constant, $p = \sqrt{64} = 8$.

The centre of the circle has the coordinates $\left(-\frac{1}{2}p, -3\right)$.

Since $p = 8$, the centre has the coordinates (–4, –3).

This is a distance of 5 units from the origin.

9 $3x^2 + 3y^2 + 7x - 5y = \frac{1}{2}$

$x^2 + y^2 + \frac{7}{3}x - \frac{5}{3}y = \frac{1}{6}$

$\left(x + \frac{7}{6}\right)^2 - \left(\frac{7}{6}\right)^2 + \left(y - \frac{5}{6}\right)^2 - \left(\frac{5}{6}\right)^2 = \frac{6}{36}$

$\left(x + \frac{7}{6}\right)^2 + \left(y - \frac{5}{6}\right)^2 = \frac{6}{36} + \left(\frac{7}{6}\right)^2 + \left(\frac{5}{6}\right)^2$

$= \frac{6}{36} + \frac{49}{36} + \frac{25}{36} = \frac{80}{36}$

Area $= \pi r^2 = \frac{80}{36}\pi$

10 Smaller circle radius 3 units centre (7, 3) has equation $(x - 7)^2 + (y - 3)^2 = 9$

Larger circle radius 7 units centre (7, 3) has equation $(x - 7)^2 + (y - 3)^2 = 49$

Area of path = area of larger circle – area smaller circle

$= 49\pi - 9\pi = 40\pi$

Exercise 3.3B

1 $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$= (-17 - 27)^2 + (25 - (-8))^2$

$= (-44)^2 + (33)^2 = 3025$

$d = \sqrt{3025} = 55$

2 a The centre is the midpoint between D and E.

Centre $= \left(\frac{2 + 14}{2}, \frac{9 - 7}{2}\right) = (8, 1)$

b The radius is the distance between the centre and a point on the circumference.

Let the centre $(8, 1) = (x_1, y_1)$ and $D(2, 9) = (x_2, y_2)$.

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (8 - 2)^2 + (1 - 9)^2$

$= (6)^2 + (-8)^2 = 100$

$r = \sqrt{100} = 10$

c The circle has the equation $(x - 8)^2 + (y - 1)^2 = 100$.

3 a The student has correctly worked out that $AB^2 = 1369$, but the equation of the circle needs the square of the radius, not the square of the diameter. The diameter is $\sqrt{1369} = 37$, so the radius is $\frac{37}{2}$.

b $(x + 0.5)^2 + (y - 17)^2 = \left(\frac{37}{2}\right)^2$

4 a i $\left(\frac{x + 9}{2}, \frac{y + 11}{2}\right) = (5, -3)$

$x + 9 = 2 \times 5 = 10$

$x = 1$

$y + 11 = 2 \times -3 = -6$

$y = -17$

$G = (1, -17)$

ii $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (9-5)^2 + (5-(-3))^2$

$= (4)^2 + (8)^2 = 80$

$r = \sqrt{80} = 4\sqrt{5}$

iii $(x-5)^2 + (y+3)^2 = 80$

b Substitute the coordinates of H into the equation of the circle.

$(-3-5)^2 + (1+3)^2 = (-8)^2 + (4)^2 = 80$

Since both sides of the equation are satisfied, H lies on C.

5 **a** The centre of the circle is the mid-point of $ST = \left(\frac{-9+5}{2}, \frac{4+10}{2}\right) = (-2, 7)$.

The midpoint of UV will also be the centre of the circle, $(-2, 7)$.

Hence $\left(\frac{1+p}{2}, \frac{14+q}{2}\right) = (-2, 7)$.

$1 + p = 2 \times -2 = -4$

$p = -5$

$14 + q = 2 \times 7 = 14$

$q = 0$

b $r^2 = (-9-(-2))^2 + (4-7)^2$

$= (-7)^2 + (-3)^2 = 58$

The circle has the equation $(x+2)^2 + (y-7)^2 = 58$.

6 The equation of the circle can be rewritten as $(x+3)^2 + (y-8)^2 = 125$, so $(-3, 8)$ is the centre of the circle.

Substitute $y = 2x + 14$ into $x^2 + y^2 + 6x - 16y = 52$.

$x^2 + (2x+14)^2 + 6x - 16(2x+14) = 52$

Expand and simplify.

$x^2 + 4x^2 + 56x + 196 + 6x - 32x - 224 = 52$

$5x^2 + 30x - 80 = 0$

$x^2 + 6x - 16 = 0$

Solve.

$(x+8)(x-2) = 0$

$x = -8$ or 2

M and N are the two intersection points, it doesn't matter which is which.

When $x = -8$, $y = 2 \times -8 + 14 = -2$, so $M = (-8, -2)$.

When $x = 2$, $y = 2 \times 2 + 14 = 18$, so $N = (2, 18)$.

The midpoint of $MN = \left(\frac{-8+2}{2}, \frac{-2+18}{2}\right) = (-3, 8)$.

Since the midpoint of M and N is the centre of the circle, MN is a diameter of the circle.

7 Centre of circle is the mid-point of AB and has coordinates $\left(\frac{\frac{1}{2}+\frac{5}{2}}{2}, \frac{\frac{1}{2}-\frac{3}{2}}{2}\right)$ i.e. $\left(\frac{3}{2}, \frac{-1}{2}\right)$

Radius is distance from centre to one of the points

$= \sqrt{\left(\frac{5}{2}-\frac{3}{2}\right)^2 + \left(\frac{3}{2}-\frac{1}{2}\right)^2} = \sqrt{1+1} = \sqrt{2}$

Equation of circle is $\left(x - \frac{3}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 2$

Substitute $x = 2,\ y = -\frac{2}{5}$ into the LHS of the equation

$\left(x-\frac{3}{2}\right)^2 + \left(y+\frac{1}{2}\right)^2 = \left(2-\frac{3}{2}\right)^2 + \left(-\frac{2}{5}+\frac{1}{2}\right)^2$

$= \frac{1}{4} + \frac{1}{100} = 0.26 \neq 2$

The point with coordinates $x = 2,\ y = -\frac{2}{5}$ does not lie on the circle.

8 $x^2 - 10x + y^2 - 14y + 73 = 0$ meets $y = x + 2$ when

$x^2 - 10x + (x+2)^2 - 14(x+2) + 73 = 0$

$x^2 - 10x + x^2 + 4x + 4 - 14x - 28 + 73 = 0$

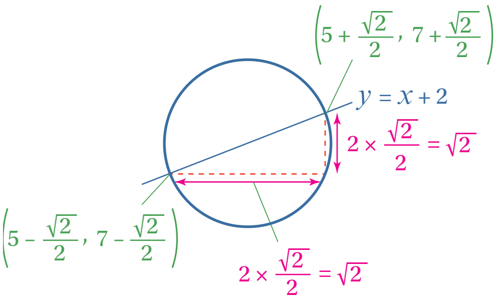
$2x^2 - 20x + 49 = 0$

$x = \frac{20 \pm \sqrt{400-392}}{4} = \frac{20 \pm \sqrt{8}}{4} = 5 \pm \frac{\sqrt{2}}{2}$

$x = 5 - \frac{\sqrt{2}}{2},\ y = 7 - \frac{\sqrt{2}}{2}$

$x = 5 + \frac{\sqrt{2}}{2},\ y = 7 + \frac{\sqrt{2}}{2}$

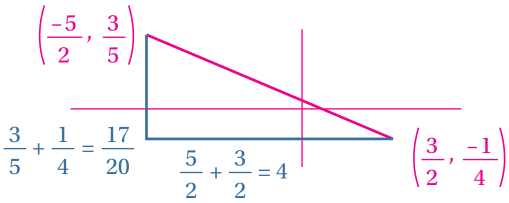
A sketch diagram helps:



Use Pythagoras:

Diameter $= \sqrt{\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2} = 2$ units.

9 A sketch diagram helps:



Diameter $\sqrt{\left(\frac{17}{20}\right)^2 + 4^2} = 4.0893$,

radius = 2.04466.

Area = πr^2 = 13.1 square units (3 s. f.)

10 Centre of the circle has coordinates $\left(\frac{\frac{1}{3}+2}{2}, \frac{-1}{2}\right)$

i.e. $\left(\frac{7}{6}, -\frac{1}{2}\right)$

Distance of centre from the origin is $\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{7}{6}\right)^2}$

= 1.27 units (3 s.f.)

Exercise 3.4A

1 **a** $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$DE^2 = (-1 - 1)^2 + (11 - 5)^2$

$= (-2)^2 + (6)^2 = 40$

$EF^2 = (1 - 13)^2 + (5 - 9)^2$

$= (-12)^2 + (-4)^2 = 160$

$DF^2 = (-1 - 13)^2 + (11 - 9)^2$

$= (-14)^2 + (2)^2 = 200$

Since 40 + 160 = 200, $DE^2 + EF^2 = DF^2$.

b Gradient of $DE = \frac{11-5}{-1-1} = -3$.

Gradient of $EF = \frac{9-5}{13-1} = \frac{1}{3}$.

Since $m_1 \times m_2 = -1$, DE and EF are perpendicular.

c DF is a diameter of the circle.

Centre = $\left(\frac{-1+13}{2}, \frac{11+9}{2}\right)$ = (6, 10).

For the radius.

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (6 - 1)^2 + (10 - 5)^2$

$= (5)^2 + (5)^2 = 50$

The equation of the circle is $(x - 6)^2 + (y - 10)^2 = 50$.

2 **a** Proof using Pythagoras' theorem (could compare gradients).

$PQ^2 = (-5 - 11)^2 + (-4 - 8)^2 = (-16)^2 + (-12)^2 = 400$

$PR^2 = (-5 - 13)^2 + (-4 - 2)^2 = (-18)^2 + (-6)^2 = 360$

$QR^2 = (11 - 13)^2 + (8 - 2)^2 = (-2)^2 + (6)^2 = 40$

Since $PR^2 + QR^2 = PQ^2$, Pythagoras' theorem is satisfied and the triangle PQR is right-angled.

b Centre = $\left(\frac{-5+11}{2}, \frac{-4+8}{2}\right)$ = (3, 2).

For the radius.

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (3 - 11)^2 + (2 - 8)^2$

$= (-8)^2 + (-6)^2 = 100$

Area of triangle $= \frac{1}{2}bh = \frac{1}{2} \times \sqrt{360} \times \sqrt{40} = 60$.

Area of circle $= \pi r^2 = \pi \times 10^2 = 100\pi$.

Green area = $(100\pi - 60)$ m^2

c $100\pi - 60 = 254.16$ m^2

254.16 ÷ 5 = 50.8, so 51 tins required.

51 × 4 = \$204

3 Since AB is a diameter, ACB = 90° and the lengths of the sides satisfy Pythagoras' theorem.

$$AC^2 + BC^2 = AB^2$$

$$(4x - 9)^2 + (14 - x)^2 = (2x + 5)^2$$

$$16x^2 - 72x + 81 + 196 - 28x + x^2 = 4x^2 + 20x + 25$$

$$13x^2 - 120x + 252 = 0$$

$$(13x - 42)(x - 6) = 0$$

$$x = \frac{42}{13} \text{ or } 6$$

When $x = \frac{42}{13}$, $d = 2 \times \frac{42}{13} + 5 = \frac{149}{13}$

Radius $= \frac{1}{2} \times \frac{149}{13} = \frac{149}{26}$

When $x = 6$, $d = 2 \times 6 + 5 = 17$

Radius $= \frac{1}{2} \times 17 = \frac{17}{2}$

4 **a** Substitute $x = 1$ and $y = 0$ into $(x + 2)^2 + (y - 1)^2 = r^2$

$(1 + 2)^2 + (0 - 1)^2 = r^2$

$(3)^2 + (-1)^2 = r^2$

$r^2 = 10$

b Substitute $x = -3$ and $y = q$ into $(x + 2)^2 + (y - 1)^2 = 10$

$(-3 + 2)^2 + (q - 1)^2 = 10$

$(-1)^2 + (q - 1)^2 = 10$

$(q - 1)^2 = 9$

$q - 1 = \pm 3$

$q = 1 \pm 3$

$q = -2$ or 4

Since $q > 0$, $q = 4$

c Gradient of $AB = \frac{2-0}{-5-1} = -\frac{1}{3}$.

Gradient of $AC = \frac{2-4}{-5-(-3)} = 1$.

Gradient of $BC = \frac{0-4}{1-(-3)} = -1$.

d $m_1 \times m_2 = -1$ so AC and BC are perpendicular.

Since the angle in a semicircle is a right angle, AB is a diameter of the circle.

5 **a** Gradient of $L_1 = \frac{6-1}{5-(-10)} = \frac{1}{3}$.

Gradient of $L_2 = \frac{4-(-2)}{9-11} = -3$.

$m_1 \times m_2 = -1$ so L_1 and L_2 are perpendicular.

b Equation of L_1 is given by $y - 6 = \frac{1}{3}(x-5)$,

from which $y = \frac{1}{3}(x-5) + 6$.

Equation of L_2 is given by $y - 4 = -3(x-9)$, from which $y = -3(x-9) + 4$.

At W, $\frac{1}{3}(x-5) + 6 = -3(x-9) + 4$.

$$x - 5 + 18 = -9(x-9) + 12$$
$$x + 13 = -9x + 81 + 12$$
$$10x = 80$$
$$x = 8$$

When $x = 8$, $y = -3(8-9) + 4 = 7$.

Because L_1 and L_2 are perpendicular and intersect at point W on the circumference, SV is a diameter.

Centre $= \left(\frac{5+11}{2}, \frac{6-2}{2}\right) = (8, 2)$.

For the radius.

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (5-8)^2 + (6-2)^2$

$= (-3)^2 + (4)^2 = 25$

The equation of C is given by $(x-8)^2 + (y-2)^2 = 25$.

Expanding and simplifying: $x^2 + y^2 - 16x - 4y + 43 = 0$.

6 **a** $PR^2 = PQ^2 + QR^2$

$PR^2 = (1-(t+17))^2 + ((t+7)-3)^2$

$= (-16-t)^2 + (t+4)^2 = 2t^2 + 40t + 272$

$PQ^2 = (1-3)^2 + ((t+7)-(t+11))^2$

$= (-2)^2 + (-4)^2 = 20$

$QR^2 = (3-(t+17))^2 + ((t+11)-3)^2$

$= (-14-t)^2 + (t+8)^2 = 2t^2 + 44t + 260$

Hence $2t^2 + 40t + 272 = 2t^2 + 44t + 260 + 20$.

$-8 = 4t$

$t = -2$

Hence the coordinates of P and R are $(1, 5)$ and $(15, 3)$.

Centre of circle $= \left(\frac{1+15}{2}, \frac{5+3}{2}\right) = (8, 4)$.

For the radius.

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (8-1)^2 + (4-5)^2$

$= (7)^2 + (-1)^2 = 50$

The equation of C is given by $(x-8)^2 + (y-4)^2 = 50$.

b $JL^2 = JK^2 + KL^2$

$JL^2 = (-11-7)^2 + (6-22)^2 = (-18)^2 + (-16)^2 = 580$

$JK^2 = (-11-(-1))^2 + (6-(p-8))^2 = (-10)^2 + (14-p)^2 = p^2 - 28p + 296$

$KL^2 = (-1-7)^2 + ((p-8)-22)^2 = (-8)^2 + (p-30)^2 = p^2 - 60p + 964$

Hence $580 = p^2 - 28p + 296 + p^2 - 60p + 964$.

$0 = 2p^2 - 88p + 680$

$= p^2 - 44p + 340$

$= (p-10)(p-34)$

$p = 10$ or 34

7 Gradient of $AB = \frac{y_1 - y_2}{x_1 - x_2} = \frac{2-1}{6+1} = \frac{1}{7}$

Gradient $AC = \frac{y_1 - y_2}{x_1 - x_2} = \frac{2 - \frac{23}{5}}{6 - \frac{4}{5}} = \frac{-\frac{13}{5}}{\frac{26}{5}}$

$= -\frac{13}{5} \times \frac{5}{26} = -\frac{1}{2}$

Gradient $BC = \frac{y_1 - y_2}{x_1 - x_2} = \frac{\frac{23}{5} - 1}{\frac{4}{5} + 1} = \frac{\frac{18}{5}}{\frac{9}{5}} = \frac{18}{5} \times \frac{5}{9} = 2$

AC is perpendicular to BC (product of gradients is -1).

$\therefore$ AB is the diameter.

Mid-point of AB has coordinates $\left(\frac{6-1}{2}, \frac{2+1}{2}\right)$ i.e. $(2.5, 1.5)$

Length $AB = \sqrt{(6+1)^2 + (2-1)^2} = \sqrt{49+1} = \sqrt{50}$

Radius of circle $= \frac{\sqrt{50}}{2}$

Equation of circle is

$(x-2.5)^2 + (y-1.5)^2 = \frac{50}{4}$

$x^2 - 5x + 6.25 + y^2 - 3y + 2.25 - 12.5 = 0$

$x^2 + y^2 - 5x - 3y - 4 = 0$

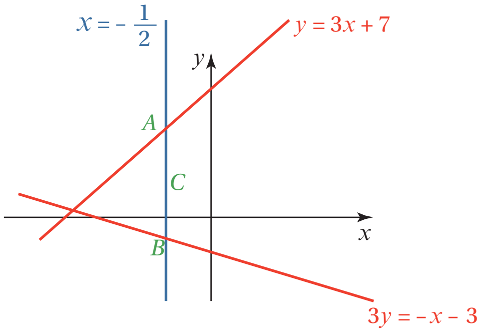
8 If PR is diameter and Q lies on the circle then PQ and QR must be perpendicular to each other.

Gradient $PQ = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{6 - \frac{17}{3}}{-1 + \frac{4}{3}} = \dfrac{\frac{1}{3}}{\frac{1}{3}} = 1$

Gradient $QR = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{\frac{17}{3} + 3}{-\frac{4}{3} - 0} = \dfrac{\frac{26}{3}}{-\frac{4}{3}} = -\dfrac{26}{3} \times \dfrac{3}{4} = -\dfrac{13}{2}$

As $-\dfrac{13}{2} \times 1 \neq -1$ PQ is not perpendicular to QR

9 A sketch diagram helps.



The diagram shows the intersection of the line $x = -\frac{1}{2}$ with the two lines, these are labelled as A and B. AB is the diameter of the circle. C, the centre of the circle, is the mid-point of AB.

Coordinates of A are $x = -\frac{1}{2}$, $y = 3 \times \left(-\frac{1}{2}\right) + 7 = \frac{11}{2}$

Coordinates of B are $x = -\frac{1}{2}$, $y = -\frac{1}{3} \times \left(-\frac{1}{2}\right) - 1 = -\frac{5}{6}$

Distance $AB = \frac{11}{2} + \frac{5}{6} = \frac{38}{6} = 6\frac{1}{3}$

Radius $= \frac{1}{2} AB = 3\frac{1}{6}$

y-coordinate of C is $\frac{11}{2} - 3\frac{1}{6} = \frac{7}{3}$

Equation of the circle with centre $\left(-\frac{1}{2}, \frac{7}{3}\right)$, $r = 3\frac{1}{6}$ is

$$\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{7}{3}\right)^2 = \left(\frac{19}{6}\right)^2$$

10 If one of the line segments is a diameter then the other two line segments must be perpendicular to each other.

Gradient $AB = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{-23 + 19}{-5 + 17} = -\dfrac{4}{12} = -\dfrac{1}{3}$

Gradient $BC = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{-3 + 19}{-1 + 17} = \dfrac{16}{16} = 1$

Gradient $AC = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{-3 + 23}{-1 + 5} = \dfrac{20}{4} = 5$

As none of the gradients multiply to give -1 none of the lines are perpendicular to each other.

Exercise 3.4B

1 a $\left(\dfrac{9+7}{2}, \dfrac{5-3}{2}\right) = (8, 1)$

b Gradient of $DE = \dfrac{5-(-3)}{9-7} = 4$.

c Gradient of $FG = -\dfrac{1}{4}$.

d Equation is given by $y - 1 = -\dfrac{1}{4}(x - 8)$

from which $4y + x = 12$

e When $x = 4$, $y - 1 = -\dfrac{1}{4}(4 - 8)$.

$y = 1 - 1 + 2 = 2$ as required.

f For the radius.

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (9 - 4)^2 + (5 - 2)^2$

$= (5)^2 + (3)^2 = 34$

The equation is given by $(x - 4)^2 + (y - 2)^2 = 34$.

2 Gradient of $JK = \dfrac{7-2}{3-10} = -\dfrac{5}{7}$.

Gradient of perpendicular bisector $= \dfrac{7}{5}$.

Mid-point $= \left(\dfrac{3+10}{2}, \dfrac{7+2}{2}\right) = \left(\dfrac{13}{2}, \dfrac{9}{2}\right)$.

Equation of bisector is given by

$y - \dfrac{9}{2} = \dfrac{7}{5}\left(x - \dfrac{13}{2}\right)$.

When $y = 8$, $8 - \dfrac{9}{2} = \dfrac{7}{5}\left(x - \dfrac{13}{2}\right)$.

$\dfrac{7}{2} = \dfrac{7}{5}x - \dfrac{91}{10}$

$\dfrac{63}{5} = \dfrac{7}{5}x$

$x = 9$

Centre of circle $= (9, 8)$.

For the radius.

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (9 - 3)^2 + (8 - 7)^2$

$= (6)^2 + (1)^2 = 37$

The equation is given by $(x - 9)^2 + (y - 8)^2 = 37$.

3 a Mid-point $= \left(\dfrac{10+x}{2}, \dfrac{11+y}{2}\right) = (6, 3)$.

$10 + x = 2 \times 6$

$x = 2$

$11 + y = 2 \times 3$

$y = -5$

Coordinates of V are $(2, -5)$.

b Gradient of $UV = \frac{11-3}{10-6} = 2$.

Gradient of perpendicular bisector $= -\frac{1}{2}$.

Equation of bisector is given by $y - 3 = -\frac{1}{2}(x-6)$.

When $x = 8$, $y - 3 = -\frac{1}{2}(8-6)$.

$y = 2$

Centre of circle = (8, 2).

For the radius.

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (8-10)^2 + (2-11)^2$

$= (-2)^2 + (-9)^2 = 85$

The equation is given by $(x-8)^2 + (y-2)^2 = 85$.

4 a Equation of circle can be rewritten as $(x+3)^2 + (y+5)^2 = 185$.

Hence centre of circle = (–3, –5).

b When $y = 6$, $(x+3)^2 + (6+5)^2 = 185$.

$(x+3)^2 + 121 = 185$

$(x+3)^2 = 64$

$x + 3 = \pm 8$

$x = 5$ or -11

Length of chord = 5 – (–11) = 16.

c Height of triangle = 6 – (–5) = 11.

Base of triangle = PQ = 16.

Area of triangle $= \frac{1}{2}bh = \frac{1}{2} \times 16 \times 11 = 88$.

5 a Mid-point of $AB = \left(\frac{-4+2}{2}, \frac{3+7}{2}\right) = (-1, 5)$.

Gradient of $AB = \frac{7-3}{2-(-4)} = \frac{2}{3}$.

Gradient of perpendicular bisector $= -\frac{3}{2}$.

Equation of bisector is given by $y - 5 = -\frac{3}{2}(x+1)$.

$2y - 10 = -3x - 3$

$3x + 2y = 7$

b Mid-point of $BC = \left(\frac{2+10}{2}, \frac{7-5}{2}\right) = (6, 1)$.

Gradient of $BC = \frac{7-(-5)}{2-10} = -\frac{3}{2}$.

Gradient of perpendicular bisector $= \frac{2}{3}$.

Equation of bisector is given by $y - 1 = \frac{2}{3}(x-6)$.

$3y - 3 = 2x - 12$

$2x = 3y + 9$

c 90° because gradients are perpendicular.

d The quadrilateral $MCNX$ has $\angle XMC = \angle XNC = 90°$ because MX and NX are perpendicular bisectors. $\angle MXN = 90°$ from **c**. Hence $\angle ABC = \angle MBN = 360° - (3 \times 90°) = 90°$.

e AC is a diameter.

6 $A(12, 8)$, $B(11, 1)$ and $C(20, 4)$

Mid-point of $AB = \left(\frac{12+11}{2}, \frac{8+1}{2}\right) = \left(\frac{23}{2}, \frac{9}{2}\right)$.

Gradient of $AB = \frac{8-1}{12-11} = 7$.

Gradient of perpendicular bisector $= -\frac{1}{7}$.

Equation of bisector is given by $y - \frac{9}{2} = -\frac{1}{7}\left(x - \frac{23}{2}\right)$.

$y = -\frac{1}{7}x + \frac{43}{7}$

Mid-point of $AC = \left(\frac{12+20}{2}, \frac{8+4}{2}\right) = (16, 6)$.

Gradient of $AC = \frac{8-4}{12-20} = -\frac{1}{2}$.

Gradient of perpendicular bisector = 2.

Equation of bisector is given by $y - 6 = 2(x - 16)$.

$y = 2x - 26$

For point of intersection, $-\frac{1}{7}x + \frac{43}{7} = 2x - 26$.

$\frac{225}{7} = \frac{15}{7}x$

$x = 15$

When $x = 15$, $y = 2 \times 15 - 26 = 4$.

Centre of circle = (15, 4).

For the radius:

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$= (15-12)^2 + (4-8)^2$

$r^2 = (3)^2 + (-4)^2 = 25$

Final answer is $x^2 - 30x + 225 + y^2 - 8y + 16 = 25$

$x^2 + y^2 - 30x - 8y + 216 = 0$

7 a $(x+6)^2 + (y-6)^2 = 145 \quad (x-8)^2 + (y+1)^2 = 40$

Multiply out the brackets and collect terms.

$x^2 + 12x + 36 + y^2 - 12y + 36 = 145$

$x^2 + 12x + y^2 - 12y = 73$ A

$x^2 - 16x + 64 + y^2 + 2y + 1 = 40$

$x^2 - 16x + y^2 + 2y = -25$ B

Find $A - B$

$28x - 14y = 98$

$2x - y = 7$

$y = 2x - 7$

Substitute this into equation B.

$x^2 - 16x + (2x-7)^2 + 2(2x-7) = -25$

$x^2 - 16x + 4x^2 - 28x + 49 + 4x - 14 = -25$

$5x^2 - 40x + 60 = 0$

$x^2 - 8x + 12 = 0$

$(x - 6)(x - 2) = 0$

$x = 2, y = -3$ i.e. $(2, -3)$

$x = 6, y = 5$ i.e. $(6, 5)$

b Length $= \sqrt{4^2 + 8^2} = \sqrt{16 + 64} = \sqrt{80} = 4\sqrt{5}$

8 $A(3, 8)$, $B(1, 4)$ and $C(-6, 5)$

Mid-point of $AB = \left(\frac{3+1}{2}, \frac{8+4}{2}\right) = (2, 6)$.

Gradient of $AB = \frac{8-4}{3-1} = 2$.

Gradient of perpendicular bisector $= -\frac{1}{2}$.

Equation of bisector is given by $y - 6 = -\frac{1}{2}(x - 2)$.

$y = -\frac{1}{2}x + 7$

Mid-point of $AC = \left(\frac{3-6}{2}, \frac{8+5}{2}\right) = \left(-\frac{3}{2}, \frac{13}{2}\right)$.

Gradient of $AC = \frac{8-5}{3-(-6)} = \frac{1}{3}$.

Gradient of perpendicular bisector $= -3$.

Equation of bisector is given by

$y - \frac{13}{2} = -3\left(x - \left(-\frac{3}{2}\right)\right)$.

$y = -3x + 2$

For point of intersection, $-\frac{1}{2}x + 7 = -3x + 2$.

$\frac{5}{2}x = -5$

$x = -2$

When $x = -2$, $y = -3 \times -2 + 2 = 8$.

Centre of circle $= (-2, 8)$.

For the radius:

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (-2 - 1)^2 + (8 - 4)^2$

$= (-3)^2 + (4)^2 = 25$

As all three points lie on a circle with centre $(-2, 8)$ with radius 5 none of the line segments could be a radius, but all three line segments are chords.

9 $A(19, 4)$, $B(17, 0)$ and $C(4, -1)$

Mid-point of $AB = \left(\frac{19+17}{2}, \frac{4+0}{2}\right) = (18, 2)$.

Gradient of $AB = \frac{4-0}{19-17} = 2$.

Gradient of perpendicular bisector $= -\frac{1}{2}$.

Equation of bisector is given by

$y - 2 = -\frac{1}{2}(x - 18)$.

$y = -\frac{1}{2}x + 11$

Mid-point of $AC = \left(\frac{19+4}{2}, \frac{4-1}{2}\right) = \left(\frac{23}{2}, \frac{3}{2}\right)$.

Gradient of $AC = \frac{4-(-1)}{19-4} = \frac{1}{3}$.

Gradient of perpendicular bisector $= -3$.

Equation of bisector is given by

$y - \frac{3}{2} = -3\left(x - \frac{23}{2}\right)$.

$y = -3x + 36$

For point of intersection, $-\frac{1}{2}x + 11 = -3x + 36$.

$\frac{5}{2}x = 25$

$x = 10$

When $x = 10$, $y = -3 \times 10 + 36 = 6$.

Centre of circle $= (10, 6)$.

For the radius:

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (10 - 17)^2 + (6 - 0)^2$

$= (-7)^2 + (6)^2 = 85$

The equation is given by $(x - 10)^2 + (y - 6)^2 = 85$.

Exercise 3.4C

1 Gradient of radius $= \frac{6-8}{4-7} = \frac{2}{3}$.

Gradient of the tangent $= -\frac{3}{2}$.

The equation of the tangent is given by

$y - 8 = -\frac{3}{2}(x - 7)$.

$2y - 16 = -3x + 21$

$3x + 2y = 37$

2 **a** $2^2 + (-2 - 1)^2 = 4 + 9 = 13$

Since both sides of the equation agree, T lies on the circle.

b Centre of circle $= (0, 1)$.

Gradient of radius $= \frac{1-(-2)}{0-2} = -\frac{3}{2}$.

Gradient of the tangent $= \frac{2}{3}$.

The equation of the tangent is given by

$y + 2 = \frac{2}{3}(x - 2)$.

$3y + 6 = 2x - 4$

Hence $2x = 3y + 10$.

3 **a** Centre $= (3, 6)$.

b Radius $= 8$.

Let centre $= X$, point $P = (15, 7)$ and point where tangent meets circle $= T$.

c The length of PT is required, where $PT^2 + XT^2 = PX^2$.

For PX:

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$PX^2 = (3 - 15)^2 + (6 - 7)^2$$
$$PX^2 = (-12)^2 + (-1)^2 = 145$$
$$PT^2 + XT^2 = PX^2$$
$$PT^2 + 8^2 = 145$$
$$PT^2 = 81$$
$$PT = \sqrt{81} = 9\,\text{m}$$

4 **a** For A:

Gradient of radius $= \frac{11-7}{7-4} = \frac{4}{3}$.

Gradient of the tangent $= -\frac{3}{4}$.

The equation of the tangent is given by

$y - 7 = -\frac{3}{4}(x - 4)$.

Hence $y = -\frac{3}{4}(x - 4) + 7$.

For B:

Gradient of radius $= \frac{11-7}{7-10} = -\frac{4}{3}$.

Gradient of the tangent $= \frac{3}{4}$.

The equation of the tangent is given by

$y - 7 = \frac{3}{4}(x - 10)$.

Hence $y = \frac{3}{4}(x - 10) + 7$.

Put tangents equal.

$$-\frac{3}{4}(x - 4) + 7 = \frac{3}{4}(x - 10) + 7$$
$$-\frac{3}{4}(x - 4) = \frac{3}{4}(x - 10)$$
$$-(x - 4) = x - 10$$
$$-x + 4 = x - 10$$
$$14 = 2x$$

$x = 7$ (this could also be deduced by symmetry since AB is horizontal)

When $x = 7$, $y = \frac{3}{4}(7 - 10) + 7$.

$$y = \frac{3}{4}(-3) + 7$$
$$= \frac{19}{4}$$

Hence the coordinates of Q are $\left(7, \frac{19}{4}\right)$.

b $AXBQ$ is a kite. Split into congruent triangles AXQ and BXQ.

Base of triangle $AXQ = 11 - \frac{19}{4} = \frac{25}{4}$.

Perpendicular height of triangle $AXQ = 3$.

Area of triangle $AXQ = \frac{1}{2} \times \frac{25}{4} \times 3 = \frac{75}{8}$.

Area of $AXBQ = 2 \times \frac{75}{8} = \frac{75}{4}$.

5 Rewrite the equation of the circle as $(x + 4)^2 + (y - 1)^2 = 81$.

Centre = $(-4, 1)$; radius = 9.

Let centre = X and point where tangent meets circle = T.

$PT^2 + XT^2 = PX^2$.

For PX:

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$PX^2 = (7 - (-4))^2 + (4 - 1)^2$$
$$= (11)^2 + (3)^2 = 130$$
$$PT^2 + XT^2 = PX^2$$
$$PT^2 + 9^2 = 130$$
$$PT^2 = 49$$
$$PT = \sqrt{49} = 7$$

6 **a** Rewrite the equation of the circle as $(x - 15)^2 + (y - 23)^2 = 400$.

Centre = $(15, 23)$, radius = 20.

Gradient of radius $= \frac{35-23}{31-15} = \frac{3}{4}$.

Gradient of the tangent $= -\frac{4}{3}$.

The equation of the tangent is given by

$y - 35 = -\frac{4}{3}(x - 31)$.

$$3y - 105 = -4x + 124$$
$$4x + 3y = 229$$

When $x = 10$, $y = q$.

$$40 + 3q = 229$$
$$3q = 189$$
$$q = 63$$

b Use $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

For PT:

$$PT^2 = (10 - 31)^2 + (63 - 35)^2$$
$$= (-21)^2 + (28)^2 = 1225$$

For PX:

$$PX^2 = (10 - 15)^2 + (63 - 23)^2$$
$$= (-5)^2 + (40)^2 = 1625$$

TX is the radius, so $TX^2 = 400$.

Hence $PT^2 + TX^2 = PX^2$ because $1225 + 400 = 1625$.

c Since triangle is right-angled, area $= \frac{1}{2} \times TX \times PT$.

$TX = 20$

$PT = \sqrt{1225} = 35$

Area $= \frac{1}{2} \times 20 \times 35 = 350$.

7 Gradient of tangent = 1.

Gradient of radius = –1.

Equation of line with gradient –1 through (5, 11) is given by $y - 11 = -(x - 5)$.

$y = 16 - x$

Find point of intersection of $y = 16 - x$ and $y = x + 2$.

$16 - x = x + 2$

$14 = 2x$

$x = 7$

$y = 7 + 2 = 9$

Hence the point (7, 9) lies on the circumference.

For the radius:

$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (7 - 5)^2 + (9 - 11)^2$

$= (2)^2 + (-2)^2 = 8$

The equation of the circle is given by $(x - 5)^2 + (y - 11)^2 = 8$.

8 **a** Gradient of $CT = \frac{y_1 - y_2}{x_1 - x_2} = \frac{4 + 6}{-3 - 1} = \frac{10}{-4} = -\frac{5}{2}$

Gradient of the tangent at $T = \frac{2}{5}$

Equation of tangent is $y + 6 = \frac{2}{5}(x - 1)$

$5y + 30 = 2x - 2$

$-2x + 5y + 32 = 0$

b Equation of diameter

Gradient $-\frac{5}{2}$ through the point (1, – 6)

$y + 6 = -\frac{5}{2}(x - 1)$

$2y + 12 = -5x + 5$

$2y + 5x + 7 = 0$

9 Gradient of tangent is 3, so gradient of radius to point of contact is $-\frac{1}{3}$.

Equation of this radius is

$y - 23 = -\frac{1}{3}(x + 8)$

$3y - 69 = -x - 8$

$3y + x - 61 = 0$

This line meets the tangent when

$3(3x + 5) + x - 61 = 0$

$9x + 15 + x - 61 = 0$

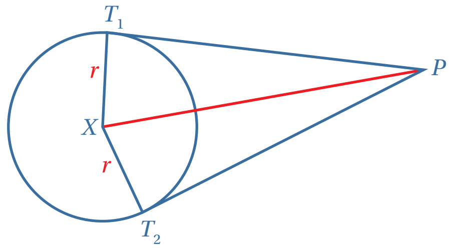
$10x - 46 = 0$

$x = 4.6$, $y = 18.8$

Radius, $r = \sqrt{12.6^2 + 4.2^2} = \sqrt{176.4}$

$(x + 8)^2 + (y - 23)^2 = 176.4$

10 Draw a sketch



Let the tangents from P meet the circle at T_1 and T_2 and the centre of the circle be X.

PXT_1 is a right angled triangle (tangent meets radius at 90°)

PXT_2 is a right angled triangle (tangent meets radius at 90°)

PX is common to both triangles and is the hypotenuse in each triangle.

$T_1X = T_2X =$ radius of the circle.

In triangle PXT_1, by Pythagoras' theorem

$PT_1 = \sqrt{PX^2 - r^2}$

In triangle PXT_2, by Pythagoras' theorem

$PT_2 = \sqrt{PX^2 - r^2}$

Therefore $PT_1 = PT_2$

In this question, $r = 4$, $PX = \sqrt{145}$,

so $PT = \sqrt{145 - 16} = \sqrt{129}$ for both T_1 and T_2.

Exercise 3.4D

1 **a** Substitute $y = x - 10$ into $x^2 + y^2 = 50$.

$x^2 + (x - 10)^2 = 50$

$x^2 + x^2 - 20x + 100 = 50$

$2x^2 - 20x + 50 = 0$

$x^2 - 10x + 25 = 0$

$(x - 5)^2 = 0$

The equation has repeated roots so $y = x - 10$ is a tangent of the circle.

b Substitute $x = 7y - 50$ into $x^2 + y^2 = 50$.

$$(7y-50)^2 + y^2 = 50$$
$$49y^2 - 700y + 2500 + y^2 = 50$$
$$50y^2 - 700y + 2450 = 0$$
$$y^2 - 14y + 49 = 0$$
$$(y-7)^2 = 0$$

The equation has repeated roots so $7y = x + 50$ is a tangent of the circle.

2 Substitute $y = 4x - 5$ into $x^2 + y^2 + 10x - 18y + 38 = 0$.

$$x^2 + (4x-5)^2 + 10x - 18(4x-5) + 38 = 0$$
$$x^2 + 16x^2 - 40x + 25 + 10x - 72x + 90 + 38 = 0$$
$$17x^2 - 102x + 153 = 0$$
$$x^2 - 6x + 9 = 0$$
$$(x-3)^2 = 0$$

The equation has repeated roots so $y = 4x - 5$ is a tangent of the circle.

3 Substitute $y = 3x + c$ into $x^2 + y^2 - 10x - 6y - 6 = 0$.

$$x^2 + (3x+c)^2 - 10x - 6(3x+c) - 6 = 0$$
$$x^2 + 9x^2 + 6cx + c^2 - 10x - 18x - 6c - 6 = 0$$
$$10x^2 + (6c-28)x + (c^2 - 6c - 6) = 0$$
$$b^2 - 4ac = 0$$
$$(6c-28)^2 - 4 \times 10 \times (c^2 - 6c - 6) = 0$$
$$36c^2 - 336c + 784 - 40c^2 + 240c + 240 = 0$$
$$-4c^2 - 96c + 1024 = 0$$
$$c^2 + 24c - 256 = 0$$
$$(c-8)(c+32) = 0$$

$c = 8$ or -32

4 Substitute $y = -\frac{1}{2}x + c$ into

$x^2 + y^2 - 12x - 26y + 125 = 0$.

$$x^2 + (-\tfrac{1}{2}x + c)^2 - 12x - 26(-\tfrac{1}{2}x + c) + 125 = 0$$
$$x^2 + \frac{1}{4}x^2 - cx + c^2 - 12x + 13x - 26c + 125 = 0$$
$$\frac{5}{4}x^2 + (1-c)x + (c^2 - 26c + 125) = 0$$
$$5x^2 + (4-4c)x + (4c^2 - 104c + 500) = 0$$
$$b^2 - 4ac = 0$$
$$(4-4c)^2 - 4 \times 5 \times (4c^2 - 104c + 500) = 0$$
$$16 - 32c + 16c^2 - 80c^2 + 2080c - 10\,000 = 0$$
$$-64c^2 + 2048c - 9984 = 0$$
$$c^2 - 32c + 156 = 0$$
$$(c-6)(c-26) = 0$$

$c = 6$ or 26

The tangents are $y = -\frac{1}{2}x + 6$ and $y = -\frac{1}{2}x + 26$.

5 Substitute $y = mx + 24$ into $(x-18)^2 + (y-10)^2 = 52$.

$$(x-18)^2 + (mx + 24 - 10)^2 = 52$$
$$(x-18)^2 + (mx+14)^2 = 52$$
$$x^2 - 36x + 324 + m^2x^2 + 28mx + 196 = 52$$
$$(1+m^2)x^2 + (28m - 36)x + 468 = 0$$
$$b^2 - 4ac = 0$$
$$(28m-36)^2 - 4 \times (1+m^2) \times 468 = 0$$
$$784m^2 - 2016m + 1296 - 1872 - 1872m^2 = 0$$
$$-1088m^2 - 2016m - 576 = 0$$
$$34m^2 + 63m + 18 = 0$$
$$(17m+6)(2m+3) = 0$$
$$m = -\frac{6}{17} \text{ or } -\frac{3}{2}$$

For the tangent $y = -\frac{6}{17}x + 24$.

$17y = -6x + 408$

$6x + 17y - 408 = 0$

For the tangent $y = -\frac{3}{2}x + 24$.

$$2y = -3x + 48$$
$$3x + 2y - 48 = 0$$

6 Substitute $y = 3x - 2$ into $x^2 + y^2 + 10x - 18y + 38 = 0$.

$x^2 + (3x-2)^2 + 10x - 18(3x-2) + 38 = 0$

$x^2 + 9x^2 - 12x + 4 + 10x - 54x + 36 + 38 = 0$

$10x^2 - 56x + 78 = 0$

$5x^2 - 28x + 39 = 0$

$b^2 - 4ac = 784 - 780 = 4$

As $b^2 - 4ac > 0$ there are two intersections of the line with the circle, $\therefore$ the line is not a tangent to the circle.

7 $x^2 + y^2 - 8x - 4y - 6 = 0$ has 2 tangents $y = -\frac{1}{2}x + c$

Substitute the equation of the tangent into the circle.

$$x^2 + \left(-\frac{1}{2}x + c\right)^2 - 8x - 4\left(-\frac{1}{2}x + c\right) - 6 = 0$$

$$x^2 + \frac{1}{4}x^2 - cx + c^2 - 8x + 2x - 4c - 6 = 0$$

$$\frac{5}{4}x^2 - (c+6)x + c^2 - 4c - 6 = 0$$

For the line to be a tangent $b^2 - 4ac = 0$

$$b^2 - 4ac = (c+6)^2 - 4 \times \frac{5}{4} \times (c^2 - 4c - 6) = c^2 + 12c + 36 - 5c^2 + 20c + 30$$

$= -4c^2 + 32c + 66$

$-4c^2 + 32c + 66 = 0$

$$c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-32 \pm \sqrt{1024 + 1056}}{-8}$$

$$= \frac{-32 \pm 45.607}{-8}$$

The values of c are -1.70 and 9.70.

8 $x^2 + y^2 - 16x - 22y + 115 = 0$ or $(x-8)^2 + (y-11)^2 = 70$

Circle has centre $(8, 11)$.

Gradient of tangent(s) is $\frac{1}{3}$ so gradient of the diameter between the points of contact is -3.

Equation of diameter:

$y - 11 = -3(x - 8)$, so $y = -3x + 35$

Substitute this in the equation of the circle:

$(x - 8)^2 + (-3x + 35 - 11)^2 = 70$

Multiply out the brackets and simplify:

$10x^2 - 160x + 570 = 0$

$x^2 - 16x + 57 = 0$

$x = 5.35$, $y = 18.9$

$x = 10.6$, $y = 3.06$

For the tangent through $x = 5.35$, $y = 18.9$

$y = \frac{1}{3}x + 17.2$

For the tangent through $x = 10.6$, $y = 3.06$

$y = \frac{1}{3}x - 0.0486$

9 $(x - 7)^2 + (y + 4)^2 = 57$; tangents have gradient -2, and circle has centre $(7, -4)$.

Gradient of the diameter joining the points of intersection is $\frac{1}{2}$.

Equation of diameter

$y + 4 = \frac{1}{2}(x - 7)$

$y = 0.5x - 7.5$

Diameter meets the circle when

$(x - 7)^2 + (0.5x - 7.5 + 4)^2 = 57$

$x^2 - 14x + 49 + 0.25x^2 - 3.5x + 12.25 - 57 = 0$

$1.25x^2 - 17.5x + 4.25 = 0$

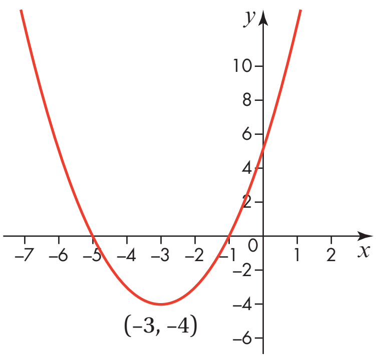
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{17.5 \pm \sqrt{306.25 - 21.25}}{2.5}$$
$$= \frac{17.5 \pm 16.882}{2.5}$$

$x = 13.8$, $y = -0.624$, the point is $(13.8, -0.624)$

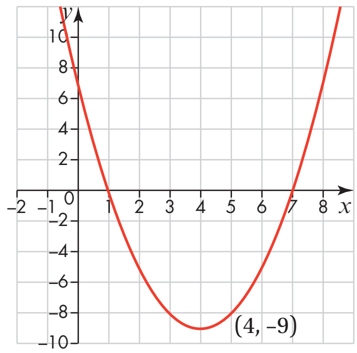
$x = 0.246$, $y = -7.38$, the point is $(0.246, -7.38)$

Exercise 3.5A

1 **a**

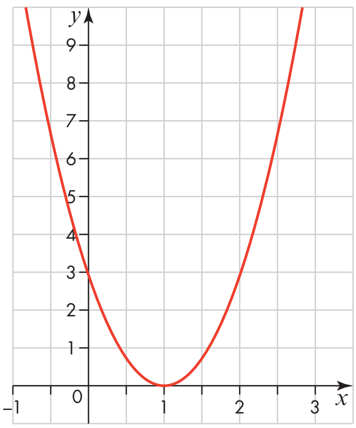


b

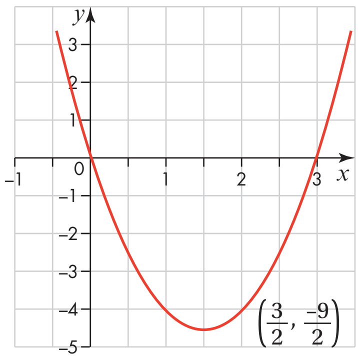


2 Roots at $x = 1$ and $x = 3$ so factors are $(x - 1)$ and $(x - 3)$.
$(x - 1)(x - 3) = x^2 - 4x + 3$, so the equation is $y = x^2 - 4x + 3$

3 **a**

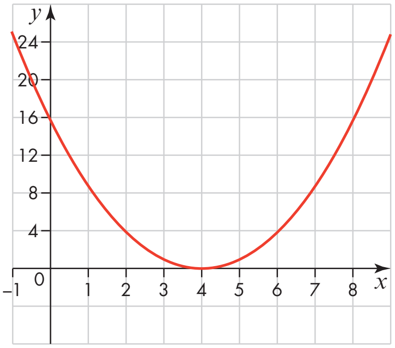


b

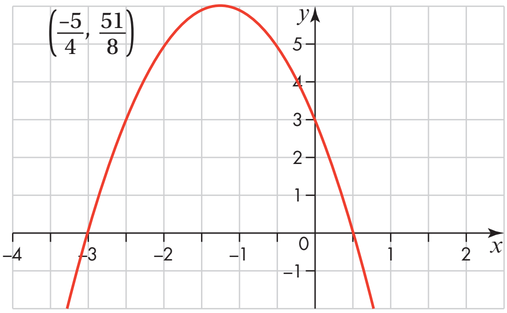


4 The sketch is *not* correct.
$x^2 - 4x - 21 = (x - 7)(x + 3)$
So should cut x-axis at $(-3, 0)$ and $(7, 0)$.
$a > 0$ so graph should be U-shaped.
When $x = 0$, $y = -21$ so y-intercept at $(0, -21)$.
$x^2 - 4x - 21 = (x - 2)^2 - 25$
So turning point at $(2, -25)$.

5 **a**



b



6 **a** 4 m

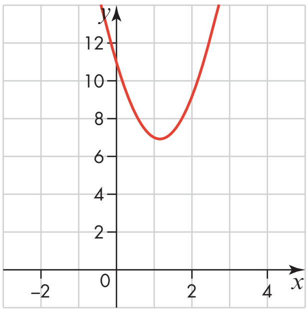
b 3 s

c Roots at $t = -1$ and $t = 3$ so factors are $(t + 1)$ and $(t - 3)$. $\cap - t^2$

$-(t + 1)(t - 3) = -t^2 + 2t + 3$, the equation is $h = -t^2 + 2t + 3$

7 $y = 3x^2 - 7x + 11 = 3\left(x - \frac{7}{6}\right)^2 - \frac{49}{12} + 11$

$= 3\left(x - \frac{7}{6}\right)^2 + \frac{83}{12}$

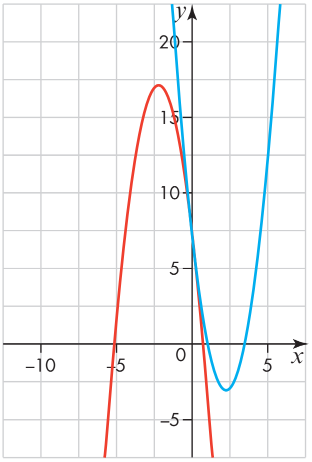


Intersects the y-axis at $y = 11$.

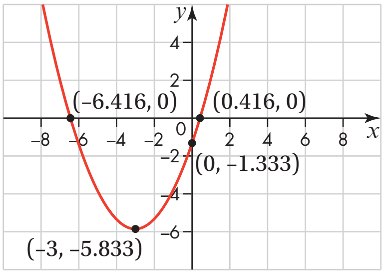
Turning point is $\left(\frac{7}{6}, \frac{83}{12}\right)$.

8 Red graph is of $y = -2x^2 - 9x + 7$.

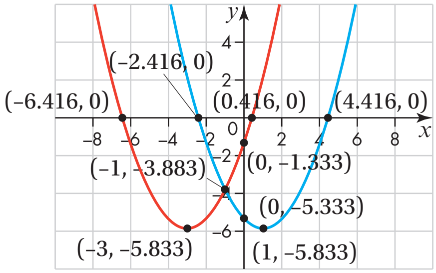
Blue graph is of $y = 2x^2 - 9x + 7$.



9 **a**



b



10 Let the equation of the curve be $y = a(x - b)^2 + c$

Minimum point is at $\left(\frac{5}{2}, \frac{7}{2}\right)$ so $b = \frac{5}{2}$, $c = \frac{7}{2}$

$$y = a\left(x - \frac{5}{2}\right)^2 + \frac{7}{2}$$

When $x = 0$, $y = 16$

$$16 = a\left(0 - \frac{5}{2}\right)^2 + \frac{7}{2}$$

$$16 = a \times \frac{25}{4} + \frac{7}{2}$$

$$a = 12.5 \times \frac{4}{25} = 2$$

$$\therefore\ y = 2\left(x - \frac{5}{2}\right)^2 + \frac{7}{2}$$

Exercise 3.5B

1 a $x^2 + x - 2 = x + 2$

$x^2 - 4 = 0$

$x = \pm 2$ so two points of intersection.

b $x^2 + x - 2 = x + k$

$x^2 - 2 - k = 0$

$x^2 - (2 + k) = 0$

$b^2 - 4ac = 0 - 4.1. - (2 + k) = 8 + 4k$

i intersects: $b^2 - 4ac > 0$

$8 + 4k > 0$

$k > -2$

ii is a tangent to: $b^2 - 4ac = 0$

$8 + 4k = 0$

$k = -2$

iii does not touch: $b^2 - 4ac < 0$

$8 + 4k < 0$

$k < -2$

2 $x^2 + x - 12 = x - 13$

$x^2 = -1$

No real solutions to this equation consequently the line and the curve do not intersect.

3 a $x^2 - 2x + 5 = 5 - x$

$x^2 - x = 0$

$b^2 - 4ac = 1 - 0 = 1 > 0$ so the curve and the line intersect

b $x^2 - 2x + 5 = x + \frac{5}{2}$

$x^2 - 3x + \frac{5}{2} = 0$

$b^2 - 4ac = 9 - \frac{4.1.5}{2} = -1$ so the curve and the line do not intersect

c $x^2 - 2x + 5 = x + 3$

$x^2 - 3x + 2 = 0$

$b^2 - 4ac = 9 - 4.1.2 = 1$ so the curve and the line intersect

4 a $-x^2 + 2x + 3 = x + k$

$-x^2 + x + 3 - k = 0$

$b^2 - 4ac = 1 - 4. - 1(3 - k) = 0$

$k = \frac{13}{4}$

b $-x^2 + 2x + 3 = -x + k$

$-x^2 + 3x + 3 - k = 0$

$b^2 - 4ac = 9 - 4. - 1(3 - k) = 0$

$k = \frac{21}{4}$

c $x + \frac{13}{4} = -x + \frac{21}{4}$

$2x = 2$

$x = 1, y = \frac{17}{4}$, the point is $\left(1, \frac{17}{4}\right)$

5 $2x^2 + 5x - 3 = x + k$

$2x^2 + 4x - (3 + k) = 0$

$b^2 - 4ac = 16 - 4.2. - (3 + k) = 0$

$k = -5$

Tangent: $y = x - 5$

Point where curve meets tangent:

$2x^2 + 5x - 3 = x - 5$

$x^2 + 2x + 1 = 0$

$(x + 1)^2 = 0$

$x = -1, y = -6$

Equation of perpendicular:

$y + 6 = -1(x + 1)$

$y = -x - 7$

Points where curve meets perpendicular to tangent:

$2x^2 + 5x - 3 = -x - 7$

$2x^2 + 6x + 4 = 0$

$x^2 + 3x + 2 = 0$

$(x + 2)(x + 1) = 0$

Intersect at $x = -1, y = -6$

and $x = -2, y = -5$

6 a For $y = x + k$: $3x^2 - 2x + 5 = x + k$

$3x^2 - 3x + (5 - k) = 0$

$b^2 - 4ac = 9 - 4.3.(5 - k) = -51 + 12k$

i intersects: $b^2 - 4ac > 0$

$-51 + 12k > 0$

$k > \frac{51}{12}$

ii is a tangent to: $b^2 - 4ac = 0$

$-51 + 12k = 0$

$k = \frac{51}{12}$

iii does not touch: $b^2 - 4ac < 0$

$-51 + 12k < 0$

$k < \frac{51}{12}$

For $y = -x + k$: $3x^2 - 2x + 5 = -x + k$

$3x^2 - x + (5 - k) = 0$

$b^2 - 4ac = 1 - 4.3.(5 - k) = -59 + 12k$

i intersects: $b^2 - 4ac > 0$

$-59 + 12k > 0$

$k > \frac{59}{12}$

ii is a tangent to: $b^2 - 4ac = 0$

$-59 + 12k = 0$

$k = \frac{59}{12}$

iii does not touch: $b^2 - 4ac < 0$

$-59 + 12k < 0$

$k < \frac{59}{12}$

b For line of symmetry:

$x + \frac{51}{12} = -x + \frac{59}{12}$

$2x = \frac{8}{12}$

$x = \frac{1}{3}$

7 $y = 3x^2 - \frac{1}{2}x - 7$ meets the line $y = x + m$ when

$3x^2 - \frac{1}{2}x - 7 = x + m$

$3x^2 - \frac{3}{2}x - (7 + m) = 0$

For the line to be a tangent $b^2 - 4ac = 0$

$b^2 - 4ac = \left(-\frac{3}{2}\right)^2 + 4 \times 3 \times (7 + m) = \frac{9}{4} + 84 + 12m$

$= 86.25 + 12m = 0$

$m = -\frac{115}{16}$

$y = x - \frac{115}{16}$

$y = 3x^2 - \frac{1}{2}x - 7$ meets the line $y = -x + n$ when

$3x^2 - \frac{1}{2}x - 7 = -x + n$

$3x^2 + \frac{1}{2}x - (7 + n) = 0$

For the line to be a tangent $b^2 - 4ac = 0$

$b^2 - 4ac = \left(\frac{1}{2}\right)^2 + 4 \times 3 \times (7 + n) = \frac{1}{4} + 84 + 12n$

$= 84.25 + 12n = 0$

$n = -\frac{337}{48}$

$y = -x - \frac{337}{48}$

The tangents meet when $x - \frac{115}{16} = -x - \frac{337}{48}$

$-2x = \frac{337}{48} - \frac{115}{16} = -\frac{1}{6}$

$x = \frac{1}{12}$, $y = -\frac{341}{48}$. Point of intersection is $\left(\frac{1}{12}, -\frac{341}{48}\right)$

8 The line $y = 2x + k$ meets the curve

$y = \left(2x - \frac{7}{3}\right)^2 + 5$ when

$\left(2x - \frac{7}{3}\right)^2 + 5 = 2x + k$

$4x^2 - \frac{28}{3}x + \frac{49}{9} + 5 = 2x + k$

$0 = 4x^2 - \frac{34}{3}x + \frac{94}{9} - k$

Tangent when $b^2 - 4ac = 0$

$b^2 - 4ac = \left(-\frac{34}{3}\right)^2 - 4 \times 4 \times \left(\frac{94}{9} - k\right)$

$= \frac{1156}{9} - \frac{1504}{9} + 16k$

$= -\frac{348}{9} + 16k = 0$

$16k = \frac{348}{9}$

$k = \frac{29}{12}$

Line is not a tangent when $k \neq \frac{29}{12}$

9 $2y - 6x - 31 = 0$, so $y = 3x + 15.5$

Meets the curve $y = -2x^2 - 7x + 3$ when

$-2x^2 - 7x + 3 = 3x + 15.5$

$-2x^2 - 10x - 12.5 = 0$

$2x^2 + 10x + 12.5 = 0$

A tangent when $b^2 - 4ac = 0$

$b^2 - 4ac = 100 - 4 \times 2 \times 12.5 = 100 - 100 = 0$

The line is a tangent to the curve.

10 $y = 2x^2 - 7x + 11$ meets the line $y = -2x + k$

when $2x^2 - 7x + 11 = -2x + k$

$2x^2 - 5x + 11 - k = 0$

Tangent when $b^2 - 4ac = 0$

$b^2 - 4ac = 25 - 4 \times 2 \times (11 - k) = -63 + 8k = 0$

$k = \frac{63}{8}$

To find the coordinates:

$2x^2 - 7x + 11 = -2x + \frac{63}{8}$

$2x^2 - 5x + \frac{25}{8} = 0$

$16x^2 - 40x + 25 = 0$

$(4x - 5)^2 = 0$

$x = \frac{5}{4}$, $y = -2 \times \frac{5}{4} + \frac{63}{8} = \frac{43}{8}$. The point is $\left(\frac{5}{4}, \frac{43}{8}\right)$

Exam-style questions

1 a $m = 3$, $(x_1, y_1) = (1, 6)$

$y - y_1 = m(x - x_1)$

$y - 6 = 3(x - 1)$

$y - 6 = 3x - 3$

$y = 3x + 3$

so $3x - y + 3 = 0$

$a = 3$, $b = -1$

b $m = -\frac{1}{3}$, $(x_1, y_1) = (1, 6)$

$y - y_1 = m(x - x_1)$

$y - 6 = -\frac{1}{3}(x - 1)$

$y - 6 = -\frac{1}{3}x + \frac{1}{3}$

$y = -\frac{1}{3}x + \frac{19}{3}$

so $m = -\frac{1}{3}, c = \frac{19}{3}$

2 a Gradient of $AB = \frac{y_1 - y_2}{x_1 - x_2} = \frac{7-3}{0-2} = -2$

Gradient $BC = \frac{1}{2} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{k-3}{5-2}$

$k - 3 = \frac{3}{2}$, $k = 4.5$

b Equation of BC:

Gradient $\frac{1}{2}$ through (2, 3)

$y - 3 = \frac{1}{2}(x - 2)$

$2y = x + 4$

c Length $AB = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$

Length $BC = \sqrt{3^2 + 1.5^2} = \sqrt{11.25} = \frac{3}{2}\sqrt{5}$

Ratio $AB : BC$ is 2 : 1.5 i.e. 4 : 3

3 a $m = -1$, $(x_1, y_1) = (0, 5)$

$y - y_1 = m(x - x_1)$

$y - 5 = -1(x - 0)$

$y = -x + 5$, or $f(x) = -x + 5$

b $-x + 5 = x - 8$

$x = \frac{13}{2}, y = -\frac{3}{2}$

Point of intersection at $\left(\frac{13}{2}, -\frac{3}{2}\right)$.

c $\left(\frac{13}{2}, -\frac{3}{2}\right)$ and (0, 5)

$d = \sqrt{\left(\frac{13}{2}\right)^2 + \left(\frac{-13}{2}\right)^2}$

$= \sqrt{\frac{338}{4}}$

$= \sqrt{\frac{169}{2}}$

$= \frac{13\sqrt{2}}{2}$

4 a Centre of the circle is (9, –8), A(14, 4).

AB is a diameter, i.e. from A, through the centre to B

14 – 9 = 5, and 4 – (–8) = 12

B has coordinates (9 – 5, –8 – 12), i.e. (4, –20)

b Although the coordinates of B have been found it is better (safer) to use those given in the question.

Gradient AB is $= \frac{y_1 - y_2}{x_1 - x_2} = \frac{4-(-8)}{14-9} = \frac{12}{5}$.

Perpendicular bisector has gradient $-\frac{5}{12}$ and passes through (9, –8).

$y + 8 = -\frac{5}{12}(x - 9)$

$12y + 96 = -5x + 45$

$12y + 5x + 51 = 0$

5 $x^2 + 6x + y^2 - 14y + 22 = 0$

$(x + 3)^2 - 9 + (y - 7)^2 - 49 + 22 = 0$

$(x + 3)^2 + (y - 7)^2 = 36$

The circle has centre (–3, 7) and radius 6.

6 a On the x-axis, $y = 0$.

$(x + 5)^2 + (-7)^2 = 50$

$(x + 5)^2 = 1$

$x + 5 = \pm 1$

$x = -5 \pm 1 = -6$ or -4

Coordinates for A and B are (–6, 0) and (–4, 0).

On the y-axis, $x = 0$

$(5)^2 + (y - 7)^2 = 50$

$(y - 7)^2 = 25$

$y - 7 = \pm 5$

$y = 7 \pm 5 = 2$ or 12

Coordinates for D and E are (0, 2) and (0, 12).

b D(0, 2) and E(0, 12)

Mid-point (0, 7)

Perpendicular bisector of AB has equation $y = 7$.

7 $x^2 - 5x + 4 = x + k$

$x^2 - 6x + (4 - k) = 0$

$b^2 - 4ac = 36 - 4.1.(4 - k) = 20 + 4k$

does not touch: $b^2 - 4ac < 0$

$20 + 4k < 0$

$k < -5$

8 **a** Circle has centre (1, 3) and radius 5.

It meets the line $x = 4$ when

$3^2 + (y - 3)^2 = 25$

$(y - 3)^2 = 16$

$y = 3 \pm 4$

The points of intersection are (4, 7) and (4, –1).

b Gradient $AC = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{7 - 3}{4 - 1} = \dfrac{4}{3}$.

Gradient of tangent $= -\dfrac{3}{4}$.

Equation of tangent:

$y - 7 = -\dfrac{3}{4}(x - 4)$

$4y - 28 = -3x + 12$

$4y = -3x + 40$

Gradient $BC = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{3 - (-1)}{1 - 4} = \dfrac{4}{-3}$

Gradient of tangent $= \dfrac{3}{4}$.

Equation of tangent:

$y + 1 = \dfrac{3}{4}(x - 4)$

$4y + 4 = 3x - 12$

$4y = 3x - 16$

c The tangents meet at D.

$-3x + 40 = 3x - 16$

$6x = 56$

$3x = 28$

$x = \dfrac{28}{3}, y = 3$

$CD = \dfrac{28}{3} - 1 = \dfrac{25}{3}$

d Area triangle $= \dfrac{1}{2}$ base × height

$CD = \dfrac{25}{3}$ units

Area $= \dfrac{1}{2} \times 4 \times \dfrac{25}{3} = \dfrac{50}{3}$ square units.

9 **a** $m = -\dfrac{1}{2}, (x_1, y_1) = (12, 4)$

$y - y_1 = m(x - x_1)$

$y - 4 = -\dfrac{1}{2}(x - 12)$

$= -\dfrac{1}{2}x + 6$

$y = -\dfrac{1}{2}x + 10$, or $f(x) = -\dfrac{1}{2}x + 10$

b When $x = 0$, $y = 10$ (0, 10).

When $y = 0$, $x = 20$ (20, 0).

Area of triangle $= \dfrac{1}{2}bh = \dfrac{1}{2} \times 20 \times 10 = 100$ units^2.

10 **a** $y = f(x)$ has gradient 1 and passes through (8, 5)

$y - 5 = 1(x - 8)$

$y = x - 3$

b Gradient of $y = f(x)$ $\dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{a - 5}{-12 - 8} = \dfrac{5 - a}{20} = 1$,

$\therefore a = -15$

$y = g(x)$ has gradient $= -1$,

$y + 15 = -1(x - 0)$

$y + x + 15 = 0$, $y = -x - 15$

11 **a** $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (180 - 60)^2 + (40 - 130)^2$

$= (120)^2 + (-90)^2 = 22\,500$

Radius $= \sqrt{22500} = 150$ cm

b The circle has the equation $(x - 60)^2 + (y - 130)^2 = 22\,500$.

c Gradient of radius $= \dfrac{130 - 40}{60 - 180} = -\dfrac{3}{4}$.

Gradient of the tangent $= \dfrac{4}{3}$.

The equation of the tangent is given by

$y - 40 = \dfrac{4}{3}(x - 180)$.

$3y - 120 = 4x - 720$

$4x - 3y - 600 = 0$

d $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$r^2 = (5940 - 180)^2 + (4360 - 40)^2$

$= (5760)^2 + (4320)^2 = 51\,840\,000$

Radius $= \sqrt{51840000} = 7200$ cm $= 72$ m

12 Mid-point of $AB = \left(\dfrac{9 + 13}{2}, \dfrac{10 - 2}{2}\right) = (11, 4)$.

For the radius, $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$.

$r^2 = (11 - 9)^2 + (4 - 10)^2$

$= (2)^2 + (-6)^2 = 40$

The circle has the equation

$(x - 11)^2 + (y - 4)^2 = 40$.

Gradient of radius $= \dfrac{10 - 4}{9 - 11} = -3$.

Gradient of the tangent $= \dfrac{1}{3}$.

The equation of the tangent is given by

$y - 10 = \dfrac{1}{3}(x - 9)$.

$3y - 30 = x - 9$

$x - 3y + 21 = 0$

13 Centre of circle, C is (–9, 4)

Radius $= \sqrt{49} = 7$

$AC^2 = (8 - (-9))^2 + (21 - 4)^2 = 17^2 + 17^2 = 578$

$AB^2 = AD^2 = AC^2 - r^2 = 578 - 49 = 529$

$AB = AD = 23$

Area $= r \times AB = 7 \times 23 = 161$ square units

14 a $-x^2 + 6x + 5 = x + k$

$-x^2 + 5x + 5 - k = 0$

$b^2 - 4ac = 25 - 4 - 1(5 - k) = 0$

$k = \frac{45}{4}$

$-x^2 + 6x + 5 = -x + l$

$-x^2 + 7x + 5 - l = 0$

$b^2 - 4ac = 49 - 4 - 1(5 - l) = 0$

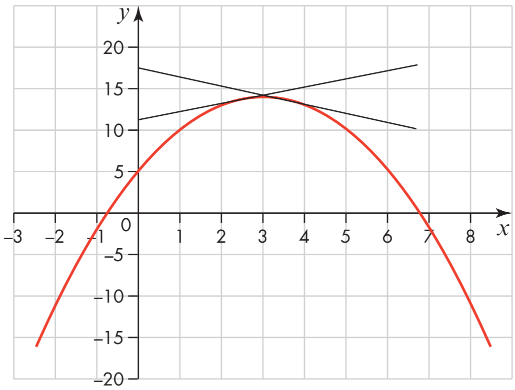
$l = \frac{69}{4}$

$x + \frac{45}{4} = -x + \frac{69}{4}$

$2x = 6$

$x = 3,\ y = \frac{57}{4}$, the point is $\left(3, \frac{57}{4}\right)$.

b



15 $(x_1, y_1) = (2\sqrt{2}, \sqrt{2})$

$(x_2, y_2) = (\sqrt{2}, 2\sqrt{3})$

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{2\sqrt{3} - \sqrt{2}}{\sqrt{2} - 2\sqrt{2}}$

$= \frac{2\sqrt{3} - \sqrt{2}}{-\sqrt{2}}$

Gradient perpendicular to this:

$m = \frac{\sqrt{2}}{2\sqrt{3} - \sqrt{2}}$

$(x_1, y_1) = (0, 3)$

$y - y_1 = m(x - x_1)$

$y - 3 = \frac{\sqrt{2}}{2\sqrt{3} - \sqrt{2}}(x - 0)$

$y = \frac{\sqrt{6} + 1}{5}x + 3$

16 a Line AB

$(x_1, y_1) = (-2, 8)$

$(x_2, y_2) = (4, 5)$

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{5 - 8}{4 + 2}$

$m_1 = -\frac{1}{2}$

Line BC

$(x_1, y_1) = (-2, 8)$

$(x_2, y_2) = (-10, -8)$

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{-8 - 8}{-10 + 2}$

$m_2 = 2$

$m_1\, m_2 = \left(-\frac{1}{2}\right)(2) = -1$ so AB and BC are perpendicular to each other and ABC is a right-angled triangle.

OR

Length of AB:

$d^2 = (4 - -2)^2 + (5 - 8)^2$

$= 45$

Length of BC:

$d^2 = (-2 - -10)^2 + (8 - -8)^2$

$= 320$

Length of AC:

$d^2 = (4 - -10)^2 + (5 - -8)^2$

$= 365$

$AB^2 + BC^2 = AC^2$

b Centre of circle has coordinates $\left(\frac{4 - 10}{2}, \frac{5 - 8}{2}\right)$

ie $\left(-3, -\frac{3}{2}\right)$

Radius $= \frac{AC}{2} = \frac{\sqrt{365}}{2}$

Circle has equation $(x + 3)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{365}{4}$

17 $x + 4y - 8 = 0$

$x = 8 - 4y$

$3(8 - 4y) + 5y + 15 = 0$

Solving gives $y = \frac{39}{7}$.

Substituting this value back into the original equation gives $x = -\frac{100}{7}$.

Point of intersection $(x_1, y_1) = \left(-\frac{100}{7}, \frac{39}{7}\right)$.

$3x + 5y + 15 = 0$

Rearranging gives $m_1 = -\frac{3}{5}$.

So $m_2 = \frac{5}{3}$.

$y - y_1 = m(x - x_1)$

$y - \frac{39}{7} = \frac{5}{3}\left(x - -\frac{100}{7}\right)$

Rearranging gives

$y = \frac{5}{3}x + \frac{617}{21}$

18 $x^2 + y^2 - 8x + 10y + 5 = 0,$

$(x - 4)^2 + (y + 5)^2 - 16 - 25 + 5 = 0$

$(x - 4)^2 + (y + 5)^2 = 36$

Centre $(4, -5)$, radius 6.

After translation centre has coordinates $(1, 2)$ and radius 6 units.

19 a $r^2 = (31 - 7)^2 + (12 - 2)^2$

$= (24)^2 + (10)^2 = 676$

Equation is given by $(x - 7)^2 + (y - 2)^2 = 676$.

b Mid-point of $XA = \left(\frac{7+31}{2}, \frac{2+12}{2}\right) = (19, 7)$.

Gradient of $XA = \frac{12-2}{31-7} = \frac{5}{12}$.

Gradient of the perpendicular $= -\frac{12}{5}$.

The equation of the perpendicular bisector is given by $y - 7 = -\frac{12}{5}(x - 19)$, i.e. $12x + 5y = 263$.

c Let P equal the midpoint of XA.

$XM = XA = AM$ because XM and XA are radii and since MN is the bisector of XA it is a line of symmetry for triangle MAX (hence $XM = AM$).

$MP^2 = MX^2 - XP^2$

$= 26^2 - 13^2 = 507$

$MP = \sqrt{507} = 13\sqrt{3}$

$MN = 2 \times 13\sqrt{3} = 26\sqrt{3}$

20 a $(x + 4)^2 + (y - 11)^2 = 50$

b $y = 7 - x$

$(x + 4)^2 + (7 - x - 11)^2 = 50$

$(x + 4)^2 + (-4 - x)^2 = 50$

$x^2 + 8x + 16 + x^2 + 8x + 16 = 50$

$2x^2 + 16x - 18 = 0$

$x^2 + 8x - 9 = 0$

$(x + 9)(x - 1) = 0$

$x = -9$ or 1

When $x = -9$, $y = 7 - (-9) = 16$.

When $x = 1$, $y = 7 - 1 = 6$.

Since the x-coordinate of A is less than the x-coordinate of B, A is $(-9, 16)$ and B is $(1, 6)$.

c Tangent at A:

Gradient of radius $= \frac{16-11}{-9-(-4)} = -1$.

Gradient of the tangent $= 1$.

(Note that since the centre of the circle lies on $x + y = 7$, AB is a diameter, so the tangent is perpendicular to $x + y = 7$.)

The equation of the tangent is given by $y - 16 = 1(x + 9)$.

Hence $y = x + 25$.

Tangent at P:

Gradient of radius $= \frac{18-11}{-3-(-4)} = 7$.

Gradient of the tangent $= -\frac{1}{7}$.

The equation of the tangent is given by $y - 18 = -\frac{1}{7}(x + 3)$.

Hence $y = -\frac{1}{7}(x + 3) + 18$.

From which, $x + 25 = -\frac{1}{7}(x + 3) + 18$.

$x + 25 = -\frac{1}{7}x - \frac{3}{7} + 18$

$7x + 175 = -x - 3 + 126$

$8x = -52$

$x = -\frac{13}{2}$

When $x = -\frac{13}{2}$, $y = -\frac{13}{2} + 25 = \frac{37}{2}$.

Coordinates of $T = \left(-\frac{13}{2}, \frac{37}{2}\right)$.

21 a Equation is given by $(x - 6)^2 + (y - 3)^2 = 9$.

The centre is $(6, 3)$. The radius is 3.

Method one:

Similar triangles: $\frac{XP}{OP} = \frac{TX}{OA}$.

Let length of $XP = a$.

$$\frac{a}{\sqrt{6^2+(3+a)^2}} = \frac{3}{6}$$

$$2a = \sqrt{36+(3+a)^2}$$

$4a^2 = 36 + (3 + a)^2$

$= 36 + 9 + 6a + a^2$

$3a^2 - 6a - 45 = 0$

$a^2 - 2a - 15 = 0$

$(a - 5)(a + 3) = 0$

$a = 5$ (can't be -3)

Method two:

Since $y = mx$ at T, $(x - 6)^2 + (mx - 3)^2 = 9$.

$x^2 - 12x + 36 + m^2x^2 - 6mx + 9 = 9$

$(1 + m^2)x^2 + (-12 - 6m)x + 36 = 0$

Since $y = mx$ is a tangent, $b^2 - 4ac = 0$.

$(-12 - 6m)^2 - 4 \times (1 + m^2) \times 36 = 0$

$144 + 144m + 36m^2 - 144 - 144m^2 = 0$

$4 + 4m + m^2 - 4 - 4m^2 = 0$

$3m^2 - 4m = 0$

$m(3m - 4) = 0$

$m = 0$ or $\frac{4}{3}$

Since $m = \frac{4}{3}$, then $\frac{AP}{OA} = \frac{4}{3}$.

$$\frac{3+XP}{6} = \frac{4}{3}$$

$3(3 + XP) = 24$

$3 + XP = 8$

$XP = 5$

b Quadrilateral $OAXT$ is made from two congruent right-angled triangles each with a base of 6 and a height of 3.

Area of triangle $OAX = \frac{1}{2} \times 6 \times 3 = 9$.

Area of $OAXT = 2 \times 9 = 18$.

22 a Substitute $y = mx$ into the equation of the circle C.

$(x - 5)^2 + (mx - 3)^2 = 2$

$x^2 - 10x + 25 + m^2x^2 - 6mx + 9 = 2$

$(1 + m^2)x^2 + (-10 - 6m)x + 32 = 0$

Given that $y = mx$ is a tangent to the circle C,

$b^2 - 4ac = 0$:

$(-10 - 6m)^2 - 4 \times (1 + m^2) \times 32 = 0$

$100 + 120m + 36m^2 - 128 - 128m^2 = 0$

$0 = 92m^2 - 120m + 28$

$0 = 23m^2 - 30m + 7$

b $(23m - 7)(m - 1) = 0$

$m = 1$ or $\frac{7}{23}$

When $m = 1$, the tangent equation is given by $y = x$.

Substitute $y = x$ into the circle equation.

$(x - 5)^2 + (x - 3)^2 = 2$

$2x^2 - 16x + 32 = 0$

$x^2 - 8x + 16 = 0$

$(x - 4)^2 = 0$

$x = 4$

$y = 4$

so coordinates of point where tangent meets the circle are (4, 4)

Mathematics in life and work

1 The equation for tunnel A is:

$y - 12 = \frac{3}{2}(x - 8)$

$2y - 24 = 3x - 24$

$y = \frac{3}{2}x$

Tunnel B is 10 vertically below, so the y-intercept will be -10. Therefore, an equation for tunnel B is:

$y = \frac{3}{2}x - 10$

2 The entrance to tunnels A and B are at $y = 12$.

$12 = \frac{3}{2}x - 10$

$22 = \frac{3}{2}x$

$x = \frac{44}{3} = 14\frac{2}{3}$

The coordinates of the entrance to tunnel B are $\left(14\frac{2}{3}, 12\right)$.

3 The gradient of tunnel C is $-\frac{2}{3}$.

The mid-point of OP is $\left(\frac{8+0}{2}, \frac{12+0}{2}\right) = (4, 6)$.

Therefore, an equation for tunnel C is:

$y - 6 = -\frac{2}{3}(x - 4)$

$3y - 18 = -2x + 8$

$2x + 3y - 26 = 0$

4 Rearranging the equations for tunnels B and C, we can get:

Tunnel B: $9x - 6y - 60 = 0$ (1)

Tunnel C: $4x + 6y - 52 = 0$ (2)

(1) + (2)

$13x - 112 = 0$

$13x = 112$

$x = \frac{112}{13}$

Substituting back to get y:

$y = \frac{3}{2} \times \frac{112}{13} - 10$

$y = \frac{38}{13}$

The point of intersection would be $\left(\frac{112}{13}, \frac{38}{13}\right)$.

4 Circular measure and trigonometry

Please note: Full worked solutions are provided as an aid to learning, and represent one approach to answering the question. In some cases, alternative methods are shown for contrast.

All sample answers have been written by the authors. Cambridge Assessment International Education bears no responsibility for the example answers to questions taken from its past question papers, which are contained in this publication.

Non-exact numerical answers should be given correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

Prerequisite knowledge

1 $30\sin 35°$ (or $30\cos 55°$) = 17.2 cm

2 $\tan C = \frac{8}{6}$

$C = \tan^{-1}\frac{8}{6} = 53.1°$

Alternatively, $\sin A = 0.8$ or $\cos C = 0.6$

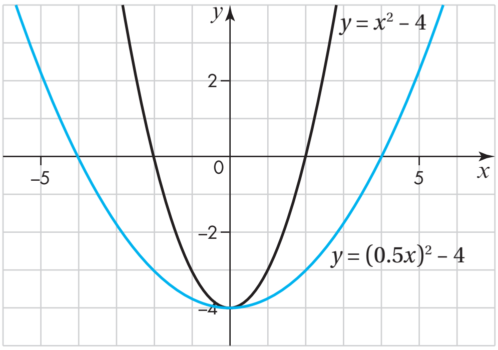
3 Factorise: $(2x - 1)(x + 1) = 0$

Either $2x - 1 = 0 \rightarrow x = \frac{1}{2}$

or $x + 1 = 0 \rightarrow x = -1$

$x = \frac{1}{2}$ or -1

4 **a** and **b**



b This is a stretch of $y = x^2 - 4$ from the y-axis with a factor of 2.

Exercise 4.1A

1 **a** $0.5 \times \frac{180}{\pi} = 28.6°$

b $1.2 \times \frac{180}{\pi} = 68.8°$

c $0.1 \times \frac{180}{\pi} = 5.7°$

d $4 \times \frac{180}{\pi} = 229.2°$

e $5.5 \times \frac{180}{\pi} = 315.1°$

2 **a** $30 \times \frac{\pi}{180} = \frac{\pi}{6}$

b $150 \times \frac{\pi}{180} = \frac{5\pi}{6}$

c $270 \times \frac{\pi}{180} = \frac{3\pi}{2}$

d $315 \times \frac{\pi}{180} = \frac{7\pi}{4}$

e $720 \times \frac{\pi}{180} = 4\pi$

3 **a** $180 \div 5 = 36°$

b $180 + 30 = 210°$

c $\frac{13}{8} \times 180 = 292.5°$

d $3.5 \times 180 = 630°$

e $0.3 \times 180 = 54°$

4 **a** 0.644 **b** 0.454 **c** 0.0500 **d** 0.809 **e** 2.41

5 **a** $\frac{CB}{42} = \sin 0.85$

$CB = 42 \times \sin 0.85$

$= 31.6$ cm

b $\frac{3.7}{XZ} = \tan 1.05$

$XZ = \frac{3.7}{\tan 1.05}$

$= 2.12$ cm

6 **a** **i** $15 \times 1.8 = 27$ cm

ii $24 \times 2.7 = 64.8$ cm

iii $6.5 \times 4 = 26$ cm

b **i** $27 + 2 \times 15 = 57$ cm

ii $64.8 + 2 \times 24 = 112.8$ cm

iii $26 + 2 \times 6.5 = 39$ cm

c **i** $\frac{1}{2} \times 15^2 \times 1.8 = 202.5\text{ cm}^2$

ii $\frac{1}{2} \times 24^2 \times 2.7 = 777.6\text{ cm}^2$

iii $\frac{1}{2} \times 6.5^2 \times 4 = 84.5\text{ cm}^2$

7 $\pi \times 20^2 - \frac{1}{2} \times 20^2 \times 0.6 = 1140\text{ cm}^2$ to 3 s.f.

8 **a** $\frac{\pi}{2} \times 15^2 \times 1.6 - \frac{3\pi}{4} \times 10^2 \times 1.6 = 100\text{ cm}^2$

b $10 \times 1.6 + 15 \times 1.6 + 2 \times 5 = 50$ cm

9 $\frac{1}{2}r^2\theta = 500$ and $r\theta = 20$

$\frac{1}{2}r \times 20 = 500$

$r = 50$

$50\theta = 20$

$\theta = 0.4$

10 **a** Let r be the radius in metres.

Then $100r\theta = \frac{1}{2}r^2\theta$

$200r\theta = r^2\theta$

$r = 200$ m

b Substituting back into the first equation gives $\theta = \theta$, so θ can be any angle from 0 to 2π

11 The area of sector OCD is $\frac{1}{2} \times 6^2 \times 2 = 36$ cm²

The area of sector OAB is $\frac{1}{2} \times (6+x)^2 \times 2$

$= 36 + 12x + x^2$ cm²

Hence area of $ABDC = OAB - OCD = 12x + x^2 = 64$

Rearrange: $x^2 + 12x - 64 = 0$; $(x-4)(x+16) = 0$

The positive solution is $x = 4$

12 Shaded area = area of sector – area of triangle

$= \frac{1}{2} \times 6.4^2 \times 1.2 - \frac{1}{2} \times 6.4^2 \sin 1.2 = 5.49$ cm² to 3 s.f.

13 First find the angle θ.

Area of sector $= \frac{1}{2} \times 20^2 \times \theta = 200\theta$; $200\theta = 164$;

$\theta = \frac{164}{200} = 0.82$

Area of triangle $= \frac{1}{2} \times 20^2 \times \sin 0.82 = 146$ cm² to 3 s.f.

14 **a** The angle of the sector is 2 radians.

The area of the segment is

$\frac{1}{2} \times r^2 \times 2 - \frac{1}{2} \times r^2 \times \sin 2 = 24$

Hence $r^2(2 - \sin 2) = 48$ and

$r^2 = \frac{48}{2 - \sin 2} = 44.01$ and $r = 6.63$ cm to 3 s.f.

b From triangle OAB, $AB = 2r\sin 1 = 11.16$ cm

The perimeter $= 11.16 + 2 \times 6.63 = 24.4$ cm to 3 s.f.

15 **a** $CB = 12 \times 0.8 = 9.6$ cm

b Area of sector $= \frac{1}{2} \times 0.8 \times 12^2 = 57.6$;

$DB = 12 \tan 0.8 = 12.36$

area of triangle $= \frac{1}{2} \times 12 \times 12.36 = 74.1$

area of $BCD = 74.1 - 57.6 = 16.5$ cm²

c $CD = \frac{12}{\cos 0.8} - 12 = 5.22$;

perimeter $= 5.22 + 9.6 + 12.36$

$= 27.2$ cm to 3 s.f.

16 **a** $CD = CE$; EBC is an isosceles right-angled triangle so $EB = r$;

$CE = \sqrt{CB^2 + EB^2} = \sqrt{2r^2} = \sqrt{2}r = CD$

b Angle $ECD = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

c $AE = \frac{\pi}{2} \times 10 = 15.71$; $DE = \frac{3\pi}{4} \times \sqrt{2} \times 10 = 33.32$;

$AD = 20 + \sqrt{2} \times 10 = 34.14$; the perimeter is $15.71 + 33.32 + 34.14 = 83.2$ cm to 3 s.f.

d The area of triangle $EBC = \frac{1}{2} \times 10 \times 10 = 50$

Area of shape $= \frac{1}{2} \times \frac{\pi}{4} \times 10^2 + 50 + \frac{1}{2} \times \frac{3\pi}{4} \times 200 =$

$39.27 + 50 + 235.62 = 325$ cm² to 3 s.f.

Exercise 4.2A

1 **a** 0.5 **b** –0.5 **c** –0.5
d –0.5 **e** 0.5

2 **a** –0.966 **b** 0.966 **c** 0.966
d –0.966 **e** 0.966

3 **a** 0.940 **b** 0.940 **c** 0.940
d 0.940 **e** –0.940

4 **a** **i** $\sin^{-1} 0.95 = 71.8°$ and $180° - 71.8° = 108.2°$

ii $\sin^{-1}(-0.35) = -20.5°$ which is out of the range $0° \leqslant x \leqslant 360°$

The solutions are $180° + 20.5° = 200.5°$ and $360° - 20.5° = 339.5°$.

iii $\sin^{-1}(-0.812) = -54.3°$

The solutions are $180° + 54.3° = 234.3°$ and $360° - 54.3° = 305.7°$.

iv No solution because $-1 \leqslant \sin x \leqslant 1$.

b **i** $\cos^{-1}(-0.25) = 1.82$ and the other solution is $2\pi - 1.82 = 4.46$

ii $\cos^{-1} 0.1 = 1.47$ and $2\pi - 1.47 = 4.81$

iii $\frac{\pi}{2}$ and $\frac{3\pi}{2}$

iv π is the only solution.

5 **a** **i** 77.3°, 360° – 77.3° = 282.7°, 360° + 77.3° = 437.3°, 720° – 77.3° = 642.7°

ii 180° + 12.7° = 192.7°, 360° – 12.7° = 347.3°, 540° + 12.7° = 552.7°, 720° – 12.7° = 707.3°

iii 0°, 180°, 360°, 540°, 720°

iv 90°, 450°

b **i** $-2\pi, 0, 2\pi$

ii $-\frac{\pi}{6}, -\frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

iii –5.49, –0.795, 0.795, –5.49

iv $\frac{3\pi}{2}, \frac{5\pi}{4}$

6 $x = -315°, -135°, 45°, 225°$

7 **a** For example $\begin{pmatrix} 90 \\ 0 \end{pmatrix}$

b Rotation of 180° about (–45, 0)

8 **a** $\frac{\pi}{3} = 60°$ so one solution is 30° or $\frac{\pi}{6}$. The other is $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$

b $\sin\theta = -\cos\frac{7\pi}{4}$; $\cos\frac{7\pi}{4} = \cos\left(2\pi - \frac{7\pi}{4}\right)$

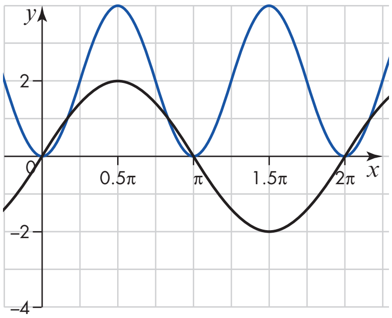
$= \cos\frac{\pi}{4} = \sin\frac{\pi}{4}$

The equation is $\sin\theta = -\sin\frac{\pi}{4}$ so $\theta = \frac{5\pi}{4}$ or $\frac{7\pi}{4}$

9 $4\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ or $\frac{7\pi}{3}$ or $\frac{11\pi}{3}$ or …

Hence $\theta = \frac{\pi}{12}$ or $\frac{5\pi}{12}$ or $\frac{7\pi}{12}$ or $\frac{11\pi}{12}$. There are four solutions in the interval.

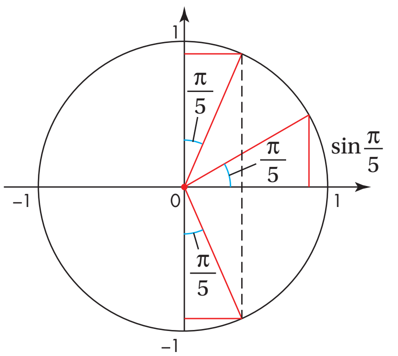
10 **a** and **b**



c $4(\sin x)^2 = 1$; $(\sin x)^2 = \frac{1}{4}$; $\sin x = \frac{1}{2}$ or $-\frac{1}{2}$;

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

11



The diagram shows that one possible solution is

$\theta = \frac{\pi}{2} - \frac{\pi}{5} = \frac{3\pi}{10}$.

Another solution is $\theta = 2\pi - \frac{3\pi}{10} = \frac{17\pi}{10}$.

12 **a** If $\theta = \sin^{-1}x = \cos^{-1}0.5$ then $\theta = 60°$. This is the principal value.

Hence $x = \sin 60° = 0.866$ to 3 d.p.

b If $\theta = \sin^{-1}x = \cos^{-1}x$ then $\sin\theta = \cos\theta = x$

If $\sin\theta = \cos\theta$, then $\theta = 45°$ and $x = \sin 45° = 0.707$ to 3 d.p.

Exercise 4.3A

1 **a** $\tan\theta = \frac{-0.5}{0.866} = -0.577$

b $\tan\theta = \frac{\sin\theta}{\cos\theta} \rightarrow -4.121 = \frac{\sin\theta}{-0.236} \rightarrow \sin\theta =$

$-4.121 \times -0.236 = 0.973$

2 **a** $\cos^2 x = 1 - \sin^2 x$

$= 1 - 0.36$

$= 0.64$

$\cos x = \pm 0.8$

b $\tan x = \frac{\sin x}{\cos x}$

$= \frac{0.6}{\pm 0.8}$

$= \pm 0.75$

3 $305° - 180° = 125°$ and $305° - 360° = -55°$

4 **a** $\tan^{-1}0.05 = 0.050$ and $\pi + 0.050 = 3.19$

b $\tan^{-1}(-0.5) = -0.464$ which is outside the interval.

The solutions are $\pi - 0.464 = 2.68$ and $2\pi - 0.464 = 5.82$.

c $\tan^{-1}5 = 1.373$ and $\pi + 1.373 = 4.52$

d $\tan^{-1}(-50) = -1.551$ which is outside the interval.

The solutions are $\pi - 1.551 = 1.59$ and $2\pi - 1.551 = 4.73$.

5 **a** $\frac{\sin x}{\cos x} = \tan x = 4 \rightarrow x = 76.0°$ and

$180° + 76.0° = 256.0°$

b $5\sin x = -3\cos x \rightarrow \frac{\sin x}{\cos x} = \frac{-3}{5} \rightarrow \tan x = -0.6$

$\tan^{-1}(-0.6) = -0.540$ so the solutions are $\pi - 0.540 = 2.60$ and $2\pi - 0.540 = 5.74$

6 **a** For example, $\begin{pmatrix} 180 \\ 0 \end{pmatrix}$.

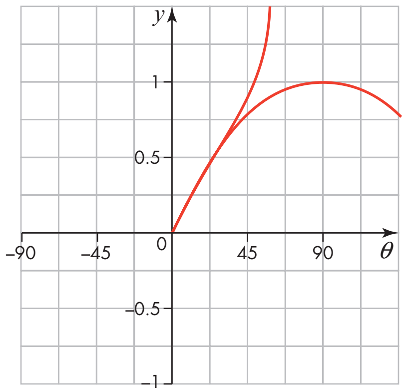
b For example, rotation of 180° about the origin or rotation of 180° about (180, 0)

7 **a** $\tan\theta = \frac{\sin\theta}{\cos\theta}$

For $0° < \theta < 90°$, $\cos\theta < 1$

so $\frac{\sin\theta}{\cos\theta} = \tan\theta > \sin\theta$

b



8 **a** $\tan 3\theta = \frac{1}{3}$; $3\theta = 18.4°$ or $198.4°$ or $378.4°$ or…

Hence $\theta = 6.1°$ or $66.1°$ or $126.1°$ or $186.1°$ …

The only obtuse angle is $126.1°$

b $4\theta + 30° = 78.7°$ or $258.7°$ or $438.7°$ or $618.7°$ or …

Hence $\theta = 12.2°$ or $57.2°$ or $102.2°$ or $147.2°$ or …

The two obtuse solutions are $102.2°$ and $147.2°$.

9 $\left(\frac{\sin x}{\cos x}\right)^2 = 4$; $\tan x = \pm 2$

If $\tan x = 2$, then $x = 1.11$ or $1.11 - \pi = -2.03$

If $\tan x = -2$, then $x = -1.11$ or $-1.11 + \pi = 2.03$.

10 **a** $2\sin x = \tan x \Rightarrow 2\sin x = \frac{\sin x}{\cos x}$

$\Rightarrow 2\sin x\cos x - \sin x \Rightarrow \sin x(2\cos x - 1) = 0$

Either $\sin x = 0$ or $\cos x = \frac{1}{2}$. Hence $x = 0°$, $180°$ $360°$, $60°$ or $300°$

b Either $\sin(2y + 35°) = 0$ or $\cos(2y + 35°) = \frac{1}{2}$.

Either $2y + 35° = 0°$ or $180°$ or $360°$ or $540°$ or $720°$… so $y = 72.5°$ or $162.5°$ or $252.5°$ or $342.5°$ are the values in the interval given.

Either $2y + 35° = 60°$ or $300°$ or $420°$ or $660°$ or $780°$ or …

Possible values are $y = 12.5°$ or $132.5°$ or $192.5°$ or $312.5°$ or $372.5°$.

c In this case $\sin z(\cos z - 2) = 0$ and hence either $\sin z = 0$ or $\cos z = 2$.

The second equation has no solution. Hence $z = 0°$ or $180°$ or $360°$ are the only solutions.

11 **a** $x^2 - 2x + 1 = 0$; $(x - 1)(x - 1) = 0$; $x = 1$ is the only solution.

b This factorises to $(\tan\theta - 1)^2 = 0$ so $\tan\theta = 1$ and so $\theta = 45°$ or $225°$.

c Now $((\tan\theta)^2 - 1)^2 = 0$ so $(\tan\theta)^2 = 1$.

$\tan\theta = 1$ or -1 and so $\theta = 45°, 135°, 225°$ or $315°$.

12 Write t for $\tan\theta$ to get $t(t - 2) = 3$; rearrange as $t^2 - 2t - 3 = 0$.

Factorise: $(t - 3)(t + 1) = 0$; $\tan\theta = 3$ or -1; hence $\theta = 1.25$ or 2.36.

Exercise 4.4 A

1 **a** $\frac{1}{2}$

b $\sin 120° = \sin 60° = \frac{\sqrt{3}}{2}$

c $\sin 225° = -\sin 45° = -\frac{1}{\sqrt{2}}$

d $\frac{\sqrt{3}}{2}$

e $\cos 135° = -\cos 45° = -\frac{1}{\sqrt{2}}$

f $\cos 300° = \cos 60° = \frac{1}{2}$

2 **a** $\tan 30° = \frac{1}{\sqrt{3}}$

b $\tan 120° = -\tan 60° = -\sqrt{3}$

c $\tan 225° = \tan 45° = 1$

3 **a** $\sin\frac{3\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

b $\cos\frac{3\pi}{4} = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

c $\tan\frac{3\pi}{4} = -\tan\frac{\pi}{4} = -1$

4 **a** $\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{2}$ **c** $-\sqrt{3}$

5 **a** $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$ **b** $\theta = \frac{\pi}{6}, \frac{7\pi}{6}$

6 **a** $\left(\sin\frac{3\pi}{4}\right)^2 + \left(\sin\frac{4\pi}{3}\right)^2 + \left(\cos\frac{5\pi}{4}\right)^2$

$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{3}{4} + \frac{1}{2} = 1\frac{3}{4}$

b $\sin\frac{\pi}{3} - \sin\frac{2\pi}{3} + \sin\pi - \sin\frac{4\pi}{3} + \sin\frac{5\pi}{3} - \sin 2\pi$

$= \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} + (0) - \left(-\frac{\sqrt{3}}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right) - 0 = 0$

7 **a** $\tan 210° + \tan 240° = \tan 30° + \tan 60°$

$= \frac{1}{\sqrt{3}} + \sqrt{3} = \frac{\sqrt{3}}{3} + \sqrt{3} = \frac{4\sqrt{3}}{3}$

b $\cos 30° + \cos 45° + \cos 60° = \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} + \frac{1}{2}$

$= \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} + \frac{1}{2} = \frac{1}{2}\left(1 + \sqrt{2} + \sqrt{3}\right)$

c $\dfrac{\sin 30° + \sin 60°}{\sin 45°} = \dfrac{\frac{1}{2} + \frac{\sqrt{3}}{2}}{\frac{1}{\sqrt{2}}} = \sqrt{2}\left(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\right)$

$= \dfrac{1}{2}\left(\sqrt{2} + \sqrt{2} \times \sqrt{3}\right) = \dfrac{1}{2}\left(\sqrt{2} + \sqrt{6}\right)$

8 $2\theta + \dfrac{3\pi}{8} = \dfrac{\pi}{4}$ or $\dfrac{5\pi}{4}$ or $\dfrac{9\pi}{4}$ or $\ldots \Rightarrow 2\theta = -\dfrac{\pi}{8}$ or $\dfrac{7\pi}{8}$

or $\dfrac{15\pi}{8}$ or $\ldots \Rightarrow \theta = -\dfrac{\pi}{16}$ or $\dfrac{7\pi}{16}$ or $\dfrac{15\pi}{16}$ or …

The two solutions in the interval are $\dfrac{7\pi}{16}$ or $\dfrac{15\pi}{16}$.

9 $BD = a\sin 60° = \dfrac{\sqrt{3}a}{2}$; $AB = a\cos 60° = \dfrac{a}{2}$;

$BC = AB\tan 30° = \dfrac{a}{2} \times \dfrac{1}{\sqrt{3}} = \dfrac{a}{2\sqrt{3}}$

Hence $CD = BD - BC = \dfrac{\sqrt{3}a}{2} - \dfrac{a}{2\sqrt{3}}$

$= \dfrac{a}{2}\left(\sqrt{3} - \dfrac{1}{\sqrt{3}}\right) = \dfrac{a}{2}\left(\dfrac{3-1}{\sqrt{3}}\right) = \dfrac{a}{2} \times \dfrac{2}{\sqrt{3}} = \dfrac{a}{\sqrt{3}}$

10 a In triangle ABC, angle BAC is 30°.

The length of $AC = 2 \times AB\cos 30°$

$= 2a \times \dfrac{\sqrt{3}}{2} = \sqrt{3}a.$

b The area of triangle ABC is

$\dfrac{1}{2} \times AB \times AC\sin 30° = \dfrac{1}{2}a \times \sqrt{3}a \times \dfrac{1}{2} = \dfrac{\sqrt{3}}{4}a^2.$

The area of triangle ACE is

$\dfrac{1}{2} \times AC \times AE\sin 60°$

$= \dfrac{1}{2} \times \sqrt{3}a \times \sqrt{3}a \times \dfrac{\sqrt{3}}{2} = \dfrac{3\sqrt{3}}{4}a^2.$

The area of the kite is $\dfrac{\sqrt{3}}{4}a^2 + \dfrac{3\sqrt{3}}{4}a^2 = \sqrt{3}a^2.$

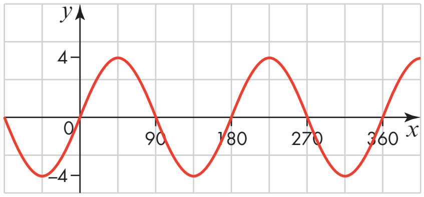
Exercise 4.5A

1 a $y = 15\sin x$ **b** $y = \tan 2x$

2 a $y = -2\cos x$ **b** $y = 10 + 5\sin x$

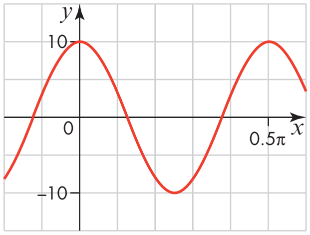
3 a period $= \dfrac{360}{2} = 180$, amplitude $= 4$

b



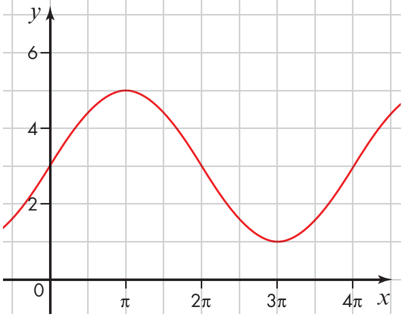
4 a period $= \dfrac{2\pi}{4} = \dfrac{\pi}{2}$, amplitude $= 10$

b

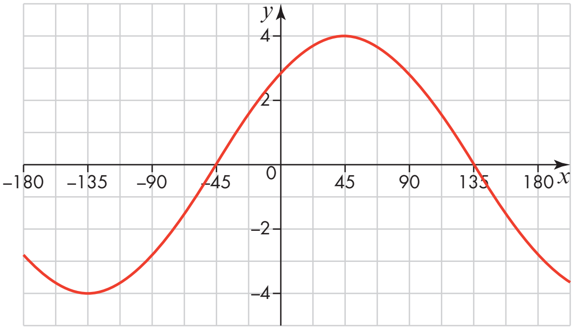


c period $= \dfrac{2\pi}{0.5} = 4\pi$; amplitude $= 2$

d

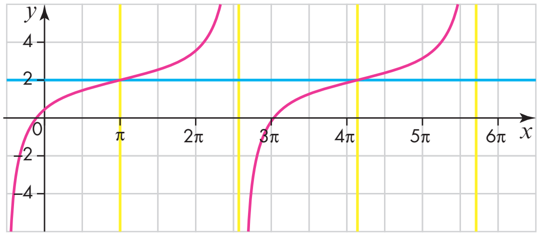


5 It is a sine wave of amplitude 4 and period 360 translated by $\begin{pmatrix} -45 \\ 0 \end{pmatrix}$.

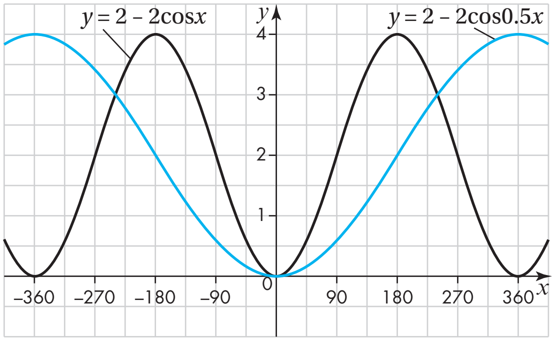


6 a $y = 2 + \tan(x - 1)$

b



7 **a** and **b**



8 **a** Start with the graph of $y = \cos x$.

Reflect it in the y-axis. The equation is $y = \cos(-x)$ but the graph is symmetrical and unchanged.

Translate by $\begin{pmatrix} 90 \\ 0 \end{pmatrix}$ so that it is now identical to the graph of $y = \sin x$.

The equation is $y = \cos(-(x - 90°))$ or $y = \cos(90° - x)$.

b $\cos(90° - x) = 2\cos x \Rightarrow \sin x = 2\cos x \Rightarrow \tan x = 2$. Hence $x = 63.4°$ or $243.4°$.

9 The amplitude is 30 so start with $y = 30\sin x$ Stretch by 4 parallel to the x-axis to get the correct period. The equation is $y = 30\sin\frac{1}{4}x$.

Finally translate by $\begin{pmatrix} \pi \\ 0 \end{pmatrix}$.

The equation becomes $y = 30\sin\frac{1}{4}(x - \pi)$ or $y = 30\sin\left(\frac{1}{4}x - \frac{\pi}{4}\right)$.

Hence $a = 30$, $b = \frac{1}{4}$ and $c = -\frac{\pi}{4}$.

Another possible value for c is $\frac{7\pi}{4}$ corresponding to a final translation of $\begin{pmatrix} -7\pi \\ 0 \end{pmatrix}$.

10 **a** The period is $\frac{\pi}{2}$ and so $b = 2$.

The graph crosses the y-axis at 0.4 and so $c = 0.4$.

The equation is $y = a\tan 2x + 0.4$.

Substitute the coordinates $\left(\frac{\pi}{8}, 0.5\right)$;

$0.5 = a\tan\frac{\pi}{4} + 0.4$.

Hence $0.5 = a + 0.4$ and so $a = 0.1$.

b The curve will look the same but cross the y-axis at $(0, -0.4)$.

The equation is $y = 0.1\tan 2x - 0.4$.

Exercise 4.6A

1 Calculator operation

2 **a** $\cos\theta = \pm\sqrt{1 - 0.68^2} = \pm 0.733$ two possible values

b $\tan\theta = \frac{\sin\theta}{\cos\theta} = \pm\frac{0.68}{0.733} = \pm 0.927$

3 **a** $\sin\theta = \pm\sqrt{1 - 0.44^2} = \pm 0.898$

b $\tan\theta = \frac{\sin\theta}{\cos\theta} = \pm\frac{0.898}{0.44}$ ± 2.041

4 **a** $\sin^2 x = \frac{5}{9}$ so $\cos^2 x = 1 - \sin^2 x = 1 - \frac{5}{9} = \frac{4}{9}$;

$\cos x = \frac{2}{3}$

The value is positive because x is acute.

b $\tan x = \frac{\sin x}{\cos x} = \frac{\sqrt{5}}{3} \div \frac{2}{3} = \frac{\sqrt{5}}{2}$

5 One is a reflection of the other in the line $y = \frac{1}{2}$.

6 $(\cos x + 1)(\cos x - 1) = \cos^2 x - 1 = -(1 - \cos^2 x) = -\sin^2 x$

7 $1 - \sin\theta\cos\theta\tan\theta = 1 - \sin\theta \times \cos\theta \times \frac{\sin\theta}{\cos\theta}$
$= 1 - \sin\theta \times \sin\theta$
$= 1 - \sin^2\theta = \cos^2\theta$

8 **a** $\tan x + \frac{1}{\tan x} = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\cos x \sin x}$
$= \frac{1}{\cos x \sin x}$

b $\tan x + \frac{1}{\tan x} = \frac{1}{\sin x\cos x} = \frac{2}{2\sin x\cos x} = \frac{2}{\sin 2x}$.

The range of values for $\sin 2x$ is $-1 \leqslant \sin 2x \leqslant 1$.

Hence $\frac{1}{\sin 2x} \geqslant 1$ or $\leqslant -1$ and hence

$\tan x + \frac{1}{\tan x} \geqslant 2$ or $\leqslant -2$.

9 $(\cos x + \sin x)^2 + (\cos x - \sin x)^2$
$= \cos^2 x + 2\cos x\sin x + \sin^2 x + \cos^2 x - 2\cos x\sin x + \sin^2 x = 2(\cos^2 x + \sin^2 x) = 2$

10 $\sin^2 x + \cos^2 x = 1$ so $(\sin^2 x + \cos^2 x)^2 = 1$

Therefore $\sin^4 x + 2\sin^2 x\cos^2 x + \cos^4 x = 1$ and so $\sin^4 x + \cos^4 x = 1 - 2\sin^2 x\cos^2 x$.

11 $\tan^2 x + 2 + \frac{1}{\tan^2 x} \equiv \frac{\sin^2 x}{\cos^2 x} + 2 + \frac{\cos^2 x}{\sin^2 x}$

$\equiv \frac{\sin^4 x + 2\sin^2 x\cos^2 x + \cos^4 x}{\sin^2 x\cos^2 x}$

$\equiv \frac{\left(\sin^2 x + \cos^2 x\right)^2}{\sin^2 x\cos^2 x} \equiv \frac{1}{\sin^2 x\cos^2 x}$

12 $\dfrac{\sin^4\theta-\cos^4\theta}{\sin\theta-\cos\theta}\equiv\dfrac{\left(\sin^2\theta+\cos^2\theta\right)\left(\sin^2\theta-\cos^2\theta\right)}{\sin\theta-\cos\theta}$

$\equiv\dfrac{1\times(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}{\sin\theta-\cos\theta}\equiv\sin\theta+\cos\theta$

13 $\cos^2 A=1-\sin^2 A=1-\dfrac{9}{25}=\dfrac{16}{25}$; hence

$\cos A=\pm\dfrac{4}{5}$

$\cos^2 B=1-\sin^2 B=1-\dfrac{25}{169}=\dfrac{144}{169}$; hence

$\cos A=\pm\dfrac{12}{13}$

Hence $\cos A\cos B+\sin A\sin B=\pm\dfrac{4}{5}\times\dfrac{12}{13}+\dfrac{3}{5}\times\dfrac{5}{13}$

$=\pm\dfrac{48}{65}+\dfrac{15}{65}=\dfrac{63}{65}$ or $-\dfrac{33}{65}$

14 $\sin\theta-\dfrac{1}{\sin\theta}=\dfrac{\sin^2\theta-1}{\sin\theta}=-\dfrac{\cos^2\theta}{\sin\theta}$

$\cos\theta-\dfrac{1}{\cos\theta}=\dfrac{\cos^2\theta-1}{\cos\theta}=-\dfrac{\sin^2\theta}{\cos\theta}$

$\tan\theta+\dfrac{1}{\tan\theta}=\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}=\dfrac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}$

$=\dfrac{1}{\cos\theta\sin\theta}$

The product is $-\dfrac{\cos^2\theta}{\sin\theta}\times\left(-\dfrac{\sin^2\theta}{\cos\theta}\right)\times\dfrac{1}{\cos\theta\sin\theta}$

$=\dfrac{\cos\theta\sin\theta}{\cos\theta\sin\theta}=1$

Exercise 4.7A

1 a $\sin 2x=\dfrac{\sqrt{3}}{2}$. One solution is $2x=\dfrac{\pi}{3}$. Other

solutions are $\pi-\dfrac{\pi}{3}, 2\pi+\dfrac{\pi}{3}, 3\pi-\dfrac{\pi}{3}, 4\pi+\dfrac{\pi}{3}$, etc.

That is $2x=\dfrac{\pi}{3},\dfrac{2\pi}{3},\dfrac{7\pi}{3},\dfrac{8\pi}{3},\dfrac{13\pi}{3}$ etc.

So $x=\dfrac{\pi}{6},\dfrac{\pi}{3},\dfrac{7\pi}{6}$ and $\dfrac{4\pi}{3}$. There are four solutions.

The next value, $\dfrac{13\pi}{6}$ is outside the interval.

b $4\tan 0.5x=-3$ and so $\tan 0.5x=-0.75$
$\tan^{-1}(-0.75)=-36.9°$.

Therefore $0.5x=-36.9°, -216.9°, 143.1°$, etc.

Therefore $x=-73.7°$ or $286.3°$. These are the only values in the interval $-360°\leqslant x\leqslant 360°$.

2 a $\cos(x+75)°=0.5$ and $\cos^{-1}0.5=60°$

So $x+75=60, 300, 420, 660, \ldots$ and $x=-15, 225, 345, 585, \ldots$.

The solutions in the interval $0°\leqslant x\leqslant 360°$ are $x=225°$ or $345°$.

b Rearrange as $50\sin 2x=-35$ and $\sin 2x=-0.7$;
$\sin^{-1}(-0.7)=-0.775$.

$2x=-0.775, -\pi+0.775, \pi+0.775, 2\pi-0.775\ldots$

$x=-0.388, -\dfrac{\pi}{2}+0.388, \dfrac{\pi}{2}+0.388, \pi-0.388\ldots$

The four solutions in the interval $-\pi\leqslant x\leqslant\pi$ are $x=-0.388$, $x=-1.18$, 1.96 and 2.75.

3 $3\cos^2 x-2\cos x=0\to\cos x(3\cos x-2)=0$

Either $\cos x=0\to x=90°$ or $270°$

or $3\cos x-2=0\to\cos x=\dfrac{2}{3}\to x=48.2°$ or $311.8°$.

There are four solutions.

4 $\cos^2 x=0.25\to\cos x=\pm0.5$

If $\cos x=0.5$, $x=\dfrac{\pi}{3}$ or $\dfrac{5\pi}{3}$

If $\cos x=-0.5$, $x=\dfrac{2\pi}{3}$ or $\dfrac{4\pi}{3}$. These are the four solutions.

5 $(3\sin x-1)(2\sin x-1)=0$

Either $3\sin x-1=0\to\sin x=\dfrac{1}{3}\to x=19.5°, 160.5°$

or $2\sin x-1=0\to\sin x=\dfrac{1}{2}\to x=30°, 150°$.

6 $4\sin^2 x-7\sin x-2=0\to(4\sin x+1)(\sin x-2)=0$

Either $4\sin x+1=0\to\sin x=-\dfrac{1}{4}\to x=-0.253$

which is not in the range so $x=\pi+0.253$ and $2\pi-0.253=3.39$ and 6.03.

or $\sin x-2=0\to\sin x=2$.

This has no solution.

7 $\sin x+1=1-\sin^2 x\to\sin x+\sin^2 x=0\to$
$\sin x(1+\sin x)=0$

Either $\sin x=0\to x=0°, 180°, 360°$

or $1+\sin x=0\to\sin x=-1\to x=270°$.

There are four solutions.

8 a Diameter $=67.5-67.5\cos180$

$=67.5-67.5(-1)$

$=135$ m

b First solve $67.5-67.5\cos\theta=100$

$67.5\cos\theta=-32.5$

$\cos\theta=-0.481$

$\theta=180-61.2, 180+61.2$

$=118.8°, 241.2°$

The wheel is above the ground for $122.4°$ out of $360°$.

Length of time $=\dfrac{122.4}{360}\times30$ mins

$=10.2$ mins

9 **a** The graphs cross twice between 0° and 360° so there are 2 solutions.

b $\cos x = \tan x \rightarrow \cos x = \dfrac{\sin x}{\cos x}$

Multiply by $\cos x$: $\cos^2 x = \sin x \rightarrow 1 - \sin^2 x = \sin x$.

Rearrange: $\sin^2 x + \sin x - 1 = 0$

Use the quadratic formula.

$\sin x = \dfrac{-1 \pm \sqrt{1+4}}{2} = 0.618$ or -1.618

$\sin x = -1.618$ has no solution.

The smallest solution of $\sin x = 0.618$ is 38.2°.

10 $20(1 - \sin^2 x) + 27 \sin x = 29$

Rearrange as $20 \sin^2 x - 27 \sin x + 9 = 0$.

This can be factorised as $(5 \sin x - 3)(4 \sin x - 3) = 0$

Hence $\sin x = \dfrac{3}{5}$ or $\sin x = \dfrac{3}{4}$

solutions to 3 s.f. are $x = 0.644, 2.50, 0.848$ or 2.29.

11 $8(1 - \cos^2 x) = 2\cos x + 5$

$8\cos^2 x + 2\cos x - 3 = 0$

$(4\cos x + 3)(2\cos x - 1) = 0$

$\cos x = -\dfrac{3}{4}$ or $\cos x = \dfrac{1}{2}$

$x = -138.6°, 138.6°, -60°, 60°$

12 $\cos x = 1 - \cos^2 x + 0.3$

$\cos^2 x + \cos x - 1.3 = 0$

$\cos x = \dfrac{-1 \pm \sqrt{1+5.2}}{2} = 0.745$ or -1.745

$\cos x = -1.745$ has no solution.

$x = 0.730, 5.55$

13 **a** $6(1 - \sin^2 x) + \sin x = 5 \Rightarrow 6\sin^2 x - \sin x - 1 = 0$

$\Rightarrow (2\sin x - 1)(3\sin x + 1) = 0 \Rightarrow \sin x = \dfrac{1}{2}$ or $-\dfrac{1}{3}$

Hence $x = 30°, 150°, 199.5°$ or $340.5°$.

b If you write $\sin^2 y = \sin x$ then you find

$\sin^2 y = \dfrac{1}{2}$ or $-\dfrac{1}{3}$

$\sin^2 y = -\dfrac{1}{3}$ has no solution.

If $\sin^2 y = \dfrac{1}{2}$ then $\sin y = \dfrac{1}{\sqrt{2}}$ or $-\dfrac{1}{\sqrt{2}}$

Hence $y = 45°, 135°, 225°$ or $315°$

14 **a** $\tan^2 x + k = 2\tan x$; $\tan^2 x - 2\tan x + k = 0$

This is a quadratic in $\tan x$ and the discriminant is $(-2)^2 - 4 \times 1 \times k = 4 - 4k$.

There is no solution if $4 - 4k < 0 \Rightarrow k > 1$.

b $\tan^2 x - 2\tan x - 8 = 0$; $(\tan x - 4)(\tan x + 2) = 0$

$\tan x = 4$ or -2; $x = 76.0°, -104.0°, -63.4$ or $116.6°$

Exam-style questions

1 If θ is the angle of the sector, $16\theta = 36$ so $\theta = 2.25$

Area $= \dfrac{1}{2} \times 2.25 \times 16^2 = 288\,\text{cm}^2$.

2 **a** The angle of the sector is $2\pi - \theta$ so the area is $\dfrac{1}{2}a^2(2\pi - \theta)$ or $\pi a^2 - \dfrac{1}{2}a^2\theta$.

b The area of the sector containing the triangle is $\dfrac{1}{2}a^2\theta$ and the area of the triangle is $\dfrac{1}{2}a^2\sin\theta$ and so $\dfrac{1}{2}a^2\theta = 2 \times \dfrac{1}{2}a^2\sin\theta$ and hence $\theta = 2\sin\theta$.

3 **a** $\sin^2 x + \cos^2 x = 1$ therefore

$\sin^2 x = 1 - \left(-\dfrac{\sqrt{3}}{4}\right)^2 = 1 - \dfrac{3}{16} = \dfrac{13}{16}$.

b $\sin x = \pm\dfrac{\sqrt{13}}{4}$

Then $\tan x = \dfrac{\sin x}{\cos x} = \pm\dfrac{\sqrt{13}}{4} \div -\dfrac{\sqrt{3}}{4} = \pm\dfrac{\sqrt{13}}{\sqrt{3}}$.

This could be written as $\pm\dfrac{\sqrt{39}}{3}$.

4 **a** $2\sin^2 x = 3\cos x$; $2(1 - \cos^2 x) = 3\cos x$; $2\cos^2 x + 3\cos x - 2 = 0$

$(2\cos x - 1)(\cos x + 2) = 0$; $\cos x = \dfrac{1}{2}$ or -2

$\cos x = -2$ has no solution. If $\cos x = \dfrac{1}{2}$ then $x = \dfrac{\pi}{3}$.

b Use the solution to part a

$\cos 2\theta = \dfrac{1}{2}$ so $2\theta = \dfrac{\pi}{3}$ or $\dfrac{5\pi}{3}$; $\theta = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$

are the two solutions in the domain $0 \leqslant \theta \leqslant \pi$.

5 **a** The amplitude is $a = 5$; $b = \dfrac{7\pi}{4}$ as only the positive value is required.

b Again the amplitude is $c = 5$; $d = \dfrac{5\pi}{4}$ as only the positive value is required.

6 **a** $2\sin^2 x - \sin x = 0$

Factorise: $\sin x(2\sin x - 1) = 0$.

Either $\sin x = 0$ or $2\sin x - 1 = 0$.

If $\sin x = 0$ then $x = 0$ or π.

If $2\sin x - 1 = 0$ then $\sin x = \dfrac{1}{2}$ so $x = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$.

There are 4 values for x.

b Using the solution to part a, either $\sin\dfrac{\theta}{2} = 0$ or $\sin\dfrac{\theta}{2} = \dfrac{1}{2}$

If $\sin\frac{\theta}{2} = 0$, then $\frac{\theta}{2} = 0$ and $\theta = 0$ is the only solution in the domain $0 \leqslant \theta \leqslant \pi$.

If $\sin\frac{\theta}{2} = \frac{1}{2}$, then $\frac{\theta}{2} = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ or … and the only solution is $\theta = \frac{\pi}{3}$.

7 Area of triangle $= \frac{1}{2}a^2\sin 60°$.

Area of sector $= \frac{1}{2}a^2\theta$ so $\theta = \sin 60° = 0.866$.

The perimeter of the triangle is $3a$ and of the sector is $0.866a + 2a = 2.866a$.

The sector perimeter is $\frac{2.866}{3} \times 100 = 95.53\%$ of the triangle's, about 4.5% less.

8 a $2y + 40 = 30°$ or $180 - 30 = 150°$ or $360 + 30 = 390°$, etc

$2y + 40 = 30 \rightarrow y = -5°$ which is outside the interval.

$2y + 40 = 150 \rightarrow y = 55°$; $2y + 40 = 390 \rightarrow y = 175°$;

$2y + 40 = 510 \rightarrow y = 235°$; $2y + 40 = 750 \rightarrow y = 355°$. There are four solutions.

b $\sin(2y + 40°) = \frac{1}{2}$; $\cos^2(2y + 40°)$

$= 1 - \sin^2(2y+40°) = 1 - \frac{1}{4} = \frac{3}{4}$

So $\cos(2y + 40°) = \pm\frac{\sqrt{3}}{2}$

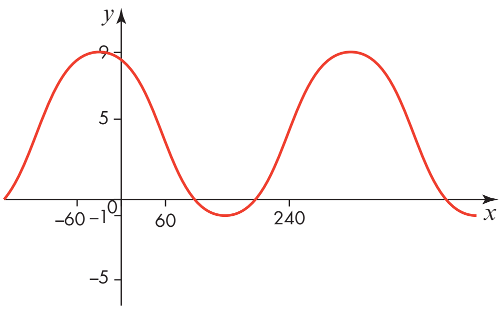
c $\tan(2y + 40°) = \frac{\sin(2y + 40°)}{\cos(2y + 40°)} = \frac{1}{2} \div \pm\frac{\sqrt{3}}{2} = \pm\frac{1}{\sqrt{3}}$

9 a $\cos(x + 30°) = -0.8$

$x + 30° = 143.1°$ or $360 - 143.1 = 216.9°$

$x = 113.1°$ or $186.9°$

b



10 a $\frac{1}{\cos\theta} - \cos\theta = \frac{1 - \cos^2\theta}{\cos\theta}$

$= \sin\theta \times \frac{\sin\theta}{\cos\theta} = \sin\theta\tan\theta$

b $\sin\theta\tan\theta = 4\sin\theta$ In the domain, $\sin\theta \neq 0$ so $\tan\theta = 4$ and $\theta = 1.33$ to 3 s.f.

11 a Divide by $\cos\theta$: $\tan\theta = 4 \rightarrow \theta = 76.0°$ or $76.0 - 180 = -104.0°$.

b This is the equation in part a with $2x + 30° = \theta$.

So $\tan(2x+30°) = 4$; $2x+30° = 76°$ or $256°$ or $-104°$ or $-284°$.

The solutions are $x = 23°$ or $113°$ or $-67°$ or $-157°$.

12 $\cos A = \frac{6}{10} = 0.6$; $A = 0.9273$ and the reflex angle at A is $2\pi - 0.9273$.

The length of the major arc on the left is $6(2\pi - 0.9273) = 32.135$.

$\cos B = \frac{8}{10} = 0.8$; $A = 0.6435$ and the reflex angle at B is $2\pi - 0.6435$.

The length of the major arc on the right is $8(2\pi - 0.6435) = 45.117$

The total length is 77.3 cm.

13 a $\frac{\sin\theta}{\cos\theta} = 2\sin\theta \rightarrow \sin\theta = 2\sin\theta\cos\theta$

Rearrange: $2\sin\theta\cos\theta - \sin\theta = 0$.

$\sin\theta(2\cos\theta - 1) = 0$

Either $\sin\theta = 0$ so $\theta = 0, \pi$ or 2π

or $\cos\theta = \frac{1}{2}$ so $\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$.

The solutions are $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ and 2π.

b Using the answer to part a, $\sin(4x - \pi) = 0$ or $\cos(4x - \pi) = \frac{1}{2}$

If $\sin(4x - \pi) = 0$ then $4x - \pi = 0, \pm\pi, \pm 2\pi, \pm 3\pi \ldots$

So $4x = \pi, 0, 2\pi, -\pi, 3\pi, -2\pi, 4\pi, \ldots$

Hence $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$ or 2π.

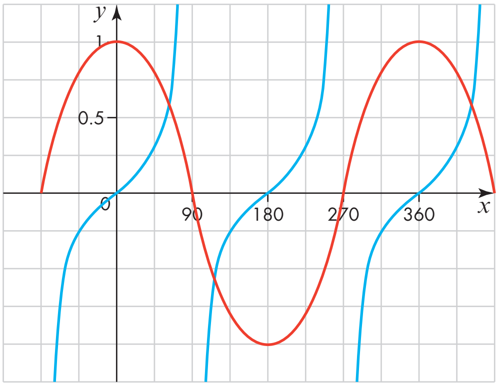
These are the values in the given domain.

If $\cos(4x - \pi) = \frac{1}{2}$ then $4x - \pi = -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \ldots$

So $4x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}, \ldots$

and $x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}$ or $\frac{11\pi}{6}$.

14 a



The graphs of $y = \cos x$ and $y = \tan x$ cross twice between 0° and 360°.

b $\tan x = \cos x$ therefore $\dfrac{\sin x}{\cos x} = \cos x$

Therefore $\sin x = \cos^2 x$ therefore $\sin x = 1 - \sin^2 x$.

Rearrange: $\sin^2 x + \sin x - 1 = 0$

Use the quadratic formula:

$$\sin x = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$
$$= 0.618 \text{ or } -1.618$$

$\sin x = -1.618$ has no solutions

$x = 38.2°$ or $141.8°$

15 a $\tan x + \dfrac{1}{\tan x} = 5.$

Multiply by $\tan x$: $\tan^2 x + 1 = 5\tan x$.

Rearrange: $\tan^2 x - 5\tan x + 1 = 0$.

b Use the quadratic formula.

$\tan x = \dfrac{5 \pm \sqrt{25-4}}{2} = 4.791$ or 0.2087. Therefore

$x = 1.37$ or 0.206.

c $\tan x + \dfrac{1}{\tan x} = k$; $\tan^2 x - k\tan x + 1 = 0$. This only has a solution if $k^2 - 4 \geqslant 0$.

Hence $k^2 \geqslant 4$ so that $k \geqslant 2$ or $k \leqslant -2$.

16 a $6(1 - \cos^2 x) + \cos x = 5$; $6 - 6\cos^2 x + \cos x = 5$;
$6\cos^2 x - \cos x - 1 = 0$

b Factorise: $(2\cos x - 1)(3\cos x + 1) = 0$ so $\cos x = \dfrac{1}{2}$ or $-\dfrac{1}{3}$.

If $\cos x = \dfrac{1}{2}$ then $x = \dfrac{\pi}{3}$ or $\dfrac{5\pi}{3}$; if $\cos x = -\dfrac{1}{3}$ then $x = 1.91$ or 4.37.

17 a The perimeter is $2r + r\theta$ cm and hence $2r + r\theta = 50$.

Rearrange: $r\theta = 50 - 2r$; $\theta = \dfrac{50-2r}{r}$

b The area is $\dfrac{1}{2}r^2\theta = 150$; $\dfrac{1}{2}r^2\left(\dfrac{50-2r}{r}\right) = 150$;

$25r - r^2 = 150$; $r^2 - 25r + 150 = 0$.

c $(r-10)(r-15) = 0$; Hence $r = 10$ or 15 and

$\theta = \dfrac{50-2r}{r} = \dfrac{50-20}{10} = 3$ or $\dfrac{50-30}{15} = 1\dfrac{1}{3}$.

18 a If angle $AOB = \theta$, then $\tan\dfrac{\theta}{2} = \dfrac{2r}{r} = 2$; $\dfrac{\theta}{2} = 1.107$ and $\theta = 2.21$ to 3 s.f.

b Area of sector $AOB = \dfrac{1}{2}r^2 \times \theta = 1.107r^2$; area of $ACBO = r \times 2r = 2r^2$.

Area of $ACBO$ outside the circle is $2r^2 - 1.107r^2 = 0.893r^2$.

c Percentage $= \dfrac{0.893r^2}{2r^2} \times 100 = 44.6\%$.

19 a $6\cos^2 x + \sin x = 5$; $6(1 - \sin^2 x) + \sin x = 5$;
$6 - 6\sin^2 x = -\sin x + 5$;

$6\sin^2 x - \sin x - 1 = 0$

b Factorise: $(3\sin x + 1)(2\sin x - 1) = 0$;

$\sin x = -\dfrac{1}{3}$ or $\dfrac{1}{2}$.

$x = 30°$, $150°$, $199.5°$ or $340.5°$

c $6\sin^4 y + 5(1 - \sin^2 x) = 6$

$6\sin^4 y + 5 - 5\sin^2 y = 6$

$6\sin^4 y - 5\sin^2 y - 1 = 0$

d This is a quadratic in $\sin^2 y$:

$(6\sin^2 y + 1)(\sin^2 y - 1) = 0$

$\sin^2 y = -\dfrac{1}{6}$ or 1

$\sin^2 y = -\dfrac{1}{6}$ has no solution.

If $\sin^2 y = 1$ then $\sin y = 1$ or -1; $y = 90°$ or $270°$.

20 $\cos AXY = 0.75$; Angle $AXB = 2 \times 0.7227 = 1.4455$

The area of sector $AXB = \dfrac{1}{2} \times 1.4455 \times r^2 = 0.7227r^2$.

The area of triangle $AXB = \dfrac{1}{2} \times \sin AXB \times r^2 = 0.4961r^2$.

The difference is $0.2267\,r^2$. The area in common is $2 \times 0.2267\,r^2 = 0.4533\,r^2$.

The percentage of one circle is $\dfrac{0.4533}{\pi} \times 100 = 14.4\%$.

Mathematics in life and work

1 The first oscillation is completed when $512\pi t = 2\pi$

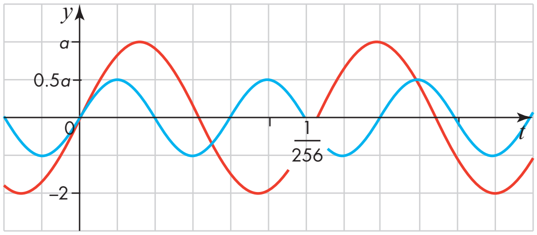
Hence $t = \frac{2}{512} = \frac{1}{256}$. There are 256 cycles in one second so the frequency is 256 Hz.

2 The frequency is $256 \times \frac{3}{2} = 384$.

If the equation is $y = \frac{1}{2} a \sin kt$, then $kt = 2\pi$ when $t = \frac{1}{384}$ so $k = 2\pi \times 384 = 768\pi$.

The equation is $= \frac{1}{2} a \sin 768\pi t$.

3



5 Series

Please note: Full worked solutions are provided as an aid to learning, and represent one approach to answering the question. In some cases, alternative methods are shown for contrast.

All sample answers have been written by the authors. Cambridge Assessment International Education bears no responsibility for the example answers to questions taken from its past question papers, which are contained in this publication.

Non-exact numerical answers should be given correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

Prerequisite knowledge

1 **a** $\frac{1}{2} \times 5 \times 6 = 15$

b If $\frac{1}{2} n(n+1) = 54$, then $n^2 + n - 108 = 0$.

This cannot be factorised with whole numbers so 54 is not a triangular number.

2 $(x+1)^3 - (x-1)^3 = x^3 + 3x^2 + 3x + 1 - (x^3 - 3x^2 + 3x - 1)$
$= x^3 + 3x^2 + 3x + 1 - x^3 + 3x^2 - 3x + 1 = 6x^2 + 2$ or $2(3x^2 + 1)$

3 The multiplier for a 5% increase is 1.05.

$\$24\,000 \times 1.05^5 = \$30\,630.76$.

Exercise 5.1A

1 **a** $\binom{6}{1} = \frac{6!}{5!1!} = \frac{6 \times 5!}{5! \times 1} = 6$

b $\binom{10}{2} = \frac{10!}{2!8!} = \frac{10 \times 9}{2 \times 1} = 5 \times 9 = 45$

c $\binom{12}{10} = \frac{12!}{10!2!} = \frac{12 \times 11}{2} = 6 \times 11 = 66$

d $\binom{12}{3} = \frac{12!}{3!9!} = \frac{12 \times 11 \times 10}{3 \times 2 \times 1} = 2 \times 11 \times 10 = 220$

e $\binom{8}{4} = \frac{8!}{4!4!} = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = 7 \times 2 \times 5 = 70$

2 The coefficient is $\binom{10}{4}$ or $\binom{10}{6} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1}$

$= 210$. This can also be found directly with a calculator.

3 **a** $(2+x)^4 = 2^4 + 4 \times 2^3 x + 6 \times 2^2 x^2 + 4 \times 2x^3 + x^4$
$= 16 + 32x + 24x^2 + 8x^3 + x^4$

b $(3+5x)^3 = 3^3 + 3 \times 3^2 \times 5x + 3 \times 3 \times (5x)^2 + (5x)^3$
$= 27 + 135x + 225x^2 + 125x^3$

c $(x+4)^5 = x^5 + 5 \times x^4 \times 4 + \binom{5}{2} \times x^3 \times 4^2 + \binom{5}{3} \times x^2 \times 4^3 + 5x \times 4^4 + 4^5$

Now $\binom{5}{2} = \binom{5}{3} = \frac{5 \times 4}{2 \times 1} = 10$ so $(x+4)^5 = x^5 + 20x^4 + 160x^3 + 640x^2 + 1280x + 1024$

4 **a** $(x-y)^4 = x^4 + 4x^3(-y) + 6x^2(-y)^2 + 4x(-y)^3 + (-y)^4$
$= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

b $(1-2y)^4 = 1 + 4(-2y) + 6(-2y)^2 + 4(-2y)^3 + (-2y)^4$
$= 1 - 8y + 24y^2 - 32y^3 + 16y^4$

c $(2-y)^5 = 2^5 + 5 \times 2^4(-y) + 10 \times 2^3(-y)^2 + 10 \times 2^2(-y)^3 + 5 \times 2(-y)^4 + (-y)^5$
$= 32 - 80y + 80y^2 - 40y^3 + 10y^4 - y^5$

5 **a** The x^3 term is $\binom{5}{3}(3x)^3 = \frac{5 \times 4}{2 \times 1} \times 27x^3 = 10 \times 27x^3 = 270x^3$. The value of $\binom{5}{3} = \binom{5}{2}$ could also be found directly with a calculator.

b The x^3 term is $\binom{6}{3} 2^3(4x)^3 = 20 \times 8 \times 64x^3 = 10\,240x^3$. Here $\binom{6}{3} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1}$.

6 **a** $\left(2 + \frac{x}{2}\right)^3 = 2^3 + 3 \times 2^2 \times \frac{x}{2} + 3 \times 2\left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3$
$= 8 + 6x + \frac{3}{2}x^2 + \frac{1}{8}x^3$

b $(5x-4y)^4 = (5x)^4 + 4(5x)^3(-4y) + 6(5x)^2(-4y)^2 + 4(5x)(-4y)^3 + (-4y)^4$
$= 625x^4 - 2000x^3y + 2400x^2y^2 - 1280xy^3 + 256y^4$

7 **a** The term involving x^3 is $\binom{10}{3} \times 2^7 x^3$, where $\binom{10}{3} = \frac{10!}{7!3!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120$, so the coefficient is $120 \times 2^7 = 15\,360$.

b The term involving x^3 is $\binom{9}{3} \times 2^6(3x)^3$, where $\binom{9}{3} = \frac{9!}{6!3!} = \frac{9\times8\times7}{3\times2\times1} = 84$, so the coefficient is $84 \times 64 \times 27 = 145\,152$.

8 The x^3 term is $\binom{7}{3} \times 2^4(bx)^3$ and $\binom{7}{3} = \frac{7\times6\times5}{3\times2\times1} = 35$ so the term is $35 \times 16 \times b^3x^3 = 560b^3x^3$. Hence $560b^3 = 70\,000$ and $b^3 = 125$ so $b = \sqrt[3]{125} = 5$.

9 **a** $(x^2+2)^4 = (x^2)^4 + 4 \times (x^2)^3 \times 2 + 6(x^2)^2 \times 2^2 + 4x^2 \times 2^3 + 2^4$
$= x^8 + 8x^6 + 24x^4 + 32x^2 + 16$

b $(1-y^3)^5 = 1 + 5(-y^3) + 10(-y^3)^2 + 10(-y^3)^3 + 5(-y^3)^4 + (-y^3)^5$
$= 1 - 5y^3 + 10y^6 - 10y^9 + 5y^{12} - y^{15}$

10 $(1+2x)^8 = 1 + 8(2x) + \binom{8}{2}(2x)^2 + \binom{8}{3}(2x)^3 + \cdots$, where $\binom{8}{2} = \frac{8\times7}{2} = 28$ and $\binom{8}{3} = \frac{8\times7\times6}{3\times2\times1} = 56$.

So $(1+2x)^8 = 1 + 16x + 28 \times 4x^2 + 56 \times 8x^3 + \cdots$
$= 1 + 16x + 112x^2 + 448x^3 + \ldots$

So $(1+x)(1+2x)^8 = (1+x)(1 + 16x + 112x^2 + 448x^3 + \ldots)$.

The x^3 term is $1 \times 448x^3 + x \times 112x^2 = (448+112)\,x^3 = 560x^3$. The coefficient is 560.

11 $(x+1)(x-1) = x^2 - 1$ so $(x+1)^8(x-1)^8 = (x^2-1)^8$

The x^{10} term is $\binom{8}{5}(x^2)^5(-1)^3$ and $\binom{8}{5} = \frac{8\times7\times6}{3\times2\times1} = 56$.

The x^{10} term is $56x^{10} \times (-1) = -56x^{10}$.

12 **a** $(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

b If $x = \sqrt{2}$, then $(1+\sqrt{2})^4 = 1 + 4\sqrt{2} + 6\left(\sqrt{2}\right)^2 + 4\left(\sqrt{2}\right)^3 + \left(\sqrt{2}\right)^4$
$= 1 + 4\sqrt{2} + 12 + 8\sqrt{2} + 4 = 17 + 12\sqrt{2}$

Hence $a = 17$ and $b = 12$.

13 **a** The constant term is
$\binom{6}{3}(2x)^3\left(\frac{1}{x}\right)^3 = 20 \times 8 \times 1 = 160$

b $\binom{4}{2}\left(\frac{x}{2}\right)^2\left(\frac{a}{x}\right)^2 = 6 \times \frac{1}{4}a^2$ hence $\frac{3}{2}a^2 = 96$;
$a^2 = 64$; $a = \pm 8$

14 $(1+x^2)^3(1-x^2)^3 = \{(1+x^2)(1-x^2)\}^3 = (1-x^4)^3$
$= 1 - 3x^4 + 3x^8 - x^{12}$

15 **a** $(1+x)^3 + (1-x)^3 = 1 + 3x + 3x^2 + x^3 + 1 - 3x + 3x^2 - x^3 = 2 + 6x^2$

b $(1+3y)^3 + (1-3y)^3 = 2 + 6(3y)^2 = 2 + 54y^2$

c $2 + 54y^2 = 11 \Rightarrow 54y^2 = 9 \Rightarrow y^2 = \frac{1}{6}$ and so $y = \pm\frac{1}{\sqrt{6}}$

16 The terms in x^2 and x^3 are $\binom{8}{2}a^6(bx)^2$ and $\binom{8}{3}a^5(bx)^3$.

Hence $28a^6b^2 = 56a^5b^3$; hence $28a = 56b$ and $\frac{a}{b} = \frac{56}{28} = \frac{2}{1}$ and so $a : b = 2 : 1$.

Exercise 5.2A

1 **a** $58 - 25 = 33$; $91 - 58 = 33$; $124 - 91 = 33$; an arithmetic progression with $d = 33$.

b The 15th term is $25 + 14 \times 33 = 487$.

c The 30th term is $25 + 29 \times 33 = 982$.

2 **a** $d = 57 - 45 = 12$; $u_{12} = 45 + 11 \times 12 = 177$

b $d = 191 - 200 = -9$; $u_{14} = 200 - 9 \times 13 = 83$

c $d = 55.1 - 48.7 = 6.4$; $u_{20} = 48.7 + 19 \times 6.4 = 170.3$

d $d = 192 - 215 = -23$; $u_{15} = 215 - 14 \times 23 = -107$

3 **a** $S_{10} = \frac{10}{2}(2 \times 20 + 9 \times 5) = 425$

b $S_{12} = \frac{12}{2}(2 \times 120 + 11 \times (-9)) = 846$

c $S_{15} = \frac{15}{2}(2 \times 9 + 14 \times 2.5) = 397.5$

4 **a** $a = 20$ and $d = 7$; 10th term $= 20 + 9 \times 7 = 83$

b $S_{10} = \frac{10}{2}\{2 \times 20 + 9 \times 7\} = 515$.

c If $20 + 7(n-1) = 200$ then $n - 1 = \frac{180}{7}$ so $n = 26.7$.

So the term required is the 26th.

5 The series is $2 + 6 + 10 + \ldots$ (20 terms). $S_{20} = \frac{20}{2}(2 \times 2 + 19 \times 4) = 800$.

6 **a** $a = 15$, and $d = 6 \quad \Rightarrow \quad u_n = 6n + 9$

On Saturday, $n = 7 \quad \Rightarrow \quad u_7 = 6 \times 7 + 9 = 51$

Alex read 51 pages on Saturday.

b $S_n = \frac{n}{2}(30 + 6(n-1))$

$S_7 = \frac{7}{2}(30 + 6 \times 6) = \frac{7}{2} \times 66 = 231$ pages

7 $a = 50$, $d = 5$ and $n = 24$; $S_{24} = \frac{24}{2}(2 \times 50 + 23 \times 5) = 2580$. So she will save \$2580.

8 $u_5 = a + 4d = 27.6$ and $u_{10} = a + 9d = 24.1$.
Subtract to get $5d = -3.5$ so $d = -0.7$.
Substitute into the first equation: $a - 2.8 = 27.6$ so $a = 30.4$.
Therefore $u_{20} = 30.4 - 19 \times 0.7 = 17.1$.

9 **a** The multiples form an arithmetic progression with $a = 6$ and $d = 6$.
$S_{10} = \frac{10}{2}(2 \times 6 + 9 \times 6) = 330$

b $S_n = \frac{n}{2}(12 + (n-1) \times 6) = \frac{n}{2}(12 + 6n - 6)$
$= \frac{n}{2}(6n + 6) = 3n(n+1)$

10 **a** The odd numbers form an arithmetic progression with $a = 1$ and $d = 2$ so $S_n = \frac{n}{2}(2 + (n-1) \times 2)$
$= \frac{n}{2}(2 + 2n - 2) = \frac{n}{2} \times 2n = n^2$.

b Each even number is 1 more than the corresponding odd number so $S_n = n^2 + n$.

11 $S_n = \frac{n}{2}(20 + 4(n-1)) = 792$
Therefore $\frac{n}{2}(16 + 4n) = 792$.
$8n + 2n^2 = 792$
$n^2 + 4n - 396 = 0$
$(n + 22)(n - 18) = 0$
$n = -22$ or 18
You want the positive root; there are 18 terms.

12 **a** The differences between terms are 1, 3, 1, 3, 1, 3, … They are not constant so it is not an arithmetic sequence.

b Here are two possible methods:
Method 1: Add pairs of terms to get this sequence: 21, 29, 37, …
This is an arithmetic progression with $a = 21$ and $d = 8$. The sum required is 50 terms of this sequence. $S_{50} = 25\{42 + 49 \times 8\} = 10\,850$.
Method 2: Split the sequence into two: 10, 14, 18, … and 11, 15, 19, …
Each is an arithmetic progression with $d = 4$. Add 50 terms of each one.
Sum $= 25\{20 + 49 \times 4\} + 25\{22 + 49 \times 4\}$
$= 5400 + 5450 = 10\,850$.

13 **a** The difference is $y - x$ and the next term is $y + (y - x) = 2y - x$.

b $x = a + 7d$ and hence $x = a + 7(y - x)$;
$x = a + 7y - 7x$; $a = 8x - 7y$.

14 The 10th term is twice the 4th term so $a + 9d = 2(a + 3d)$; $a + 9d = 2a + 6d$; $a = 3d$.
The 18th term is 50 so $a + 17d = 50$; $3d + 17d = 50$; $20d = 50$; $d = 2.5$.
Then the first term $a = 3d = 7.5$.

15 If $n = 1$, then $1 + 4 = 5 = a$.
If $n = 2$, then $4 + 8 = 12 = a + a + d = 2a + d = 10 + d$ and so $d = 2$.
The nth term is $5 + 2(n-1) = 2n + 3$.
Hence the 100th term is 203

16 If $n = 1$, then the first term is $6 + 8 = 14$.
If $n = 2$, then the first + the second term is $24 + 16 = 40$ so the second term is $40 - 14 = 26$.
If an arithmetic sequence starts 14, 26 …, then $a = 14$ and $d = 12$ and the sum of n terms is
$\frac{1}{2}n\{28 + 12(n-1)\} = \frac{1}{2}n(28 + 12n - 12)$
$= \frac{1}{2}n(12n + 16) = 6n^2 + 8n$, so the sequence is arithmetic.

Exercise 5.3A

1 **a** $a = 0.04$ and $= \frac{0.2}{0.04} = 5$; $u_8 = 0.04 \times 5^7 = 3125$

b $a = 20$ and $r = \frac{24}{20} = 1.2$; $u_6 = 20 \times 1.2^5 = 49.7664$

c $a = 0.005$ and $r = \frac{0.01}{0.005} = 2$;
$u_{15} = 0.005 \times 2^{14} = 81.92$

d $a = 3$ and $r = \frac{-6}{3} = -2$; $u_{10} = 3 \times (-2)^9 = -1536$

2 **a** $a = 5$ and $r = 10 \div 5 = 2$; $u_{15} = 5 \times 2^{14} = 81\,920$

b $S_{15} = \frac{5(2^{15} - 1)}{2 - 1} = 163\,835$

3 **a** $S_{10} = \frac{2(3^{10} - 1)}{3 - 1} = 59\,048$

b $S_8 = \frac{100(1 - 0.9^8)}{1 - 0.9} = 570$ (3 s.f.)

c $a = 4$ and $r = -2$ so $S_{12} = \frac{4((-2)^{12} - 1)}{-2 - 1} = -5460$

d $a = 20$ and $r = 1.1$ so $S_{20} = \frac{20(1.1^{20} - 1)}{1.1 - 1} = 1150$ (3 s.f.)

4 The total amount is a geometric series with $a = 12\,000$, $r = 1.03$ and $n = 10$.
$S_{10} = \frac{12\,000(1.03^{10} - 1)}{0.03} = \$137\,566.55$ (3 s.f.)

5 **a** $a = 1$ and $r = \frac{1}{2}$; $S_\infty = \frac{1}{1-\frac{1}{2}} = 2$

b $a = 2$ and $r = \frac{1}{3}$; $S_\infty = \frac{2}{1-\frac{1}{3}} = 3$

c $a = 5$ and $r = -\frac{1}{3}$; $S_\infty = \frac{5}{1+\frac{1}{3}} = \frac{15}{4}$

d $a = 1$ and $r = \frac{3}{4}$; $S_\infty = \frac{1}{1-\frac{3}{4}} = 4$

e $a = 6$ and $r = \frac{-4}{6} = -\frac{2}{3}$; $S_\infty = \frac{6}{1+\frac{2}{3}} = \frac{18}{5}$

6 **a** The amounts are a geometric progression with $a = 0.01$ and $r = 2$.

The 21st term is $0.01 \times 2^{20} = \$10\,485.76$.

b $S_{21} = \frac{0.01(2^{21}-1)}{2-1} = \$20\,971.51$

c On his 65th birthday he would receive $\$0.01 \times 2^{64}$ which is more than $\$1.8 \times 10^{17}$. This is an absurdly large amount.

7 **a** The amounts each year are a geometric progression with $a = 2000$ and $r = 1.25$.

The amount in the third year is $\$2000 \times 1.25^2 = \$3125$.

b After 8 years she has $S_8 = \frac{2000(1.25^8 - 1)}{1.25-1} =$ \$39 684 which is less than \$50 000.

After 9 years she has $S_9 = \frac{2000(1.25^9 - 1)}{1.25-1} =$ \$51 605 so it does take 9 years.

8 **a** $\frac{1}{4} + \frac{1}{4}$ of $\frac{1}{4} = \frac{1}{4} + \left(\frac{1}{4}\right)^2 = \frac{5}{16}$

b $\frac{5}{16} + \left(\frac{1}{4}\right)^3 = \frac{21}{64}$

c This is a geometric series with $a = r = \frac{1}{4}$ so

$$S_\infty = \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \ldots = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

One-third of the square is shaded.

9 **a** The distance north is $800 - 400 + 200 - 100 + \ldots$ m.

This a geometric series with $a = 800$ and $r = -\frac{1}{2}$.

So $S_\infty = \frac{800}{1+\frac{1}{2}} = 800 \times \frac{2}{3} = \frac{1600}{3} = 533\frac{1}{3}$ m north of the starting point.

b The distance walked is $800 + 400 + 200 + \ldots$ m.

This is a geometric series with $a = 800$ and $r = \frac{1}{2}$.

$S_\infty = \frac{800}{1-\frac{1}{2}} = 1600$ so she walks 1600 m.

c Distance north $= 800 + 200 + 50 + \ldots$

$= \frac{800}{1-\frac{1}{4}} = \frac{800}{\frac{3}{4}} = \frac{3200}{3}$

Distance east $= 400 + 100 + 25 + \ldots$

$= \frac{400}{1-\frac{1}{4}} = \frac{400}{\frac{3}{4}} = \frac{1600}{3}$

By Pythagoras, distance from start

$= \sqrt{\left(\frac{3200}{3}\right)^2 + \left(\frac{1600}{3}\right)^2} = 1193$ m

10 This is a geometric series with $a = 0.45$ and $r = 0.01$.

$$S_\infty = \frac{0.45}{1-0.01} = \frac{0.45}{0.99} = \frac{45}{99} = \frac{5}{11}$$

11 $a = 80$ and $\frac{a}{1-r} = 200$ so $\frac{80}{1-r} = 200$.

$1 - r = \frac{80}{200} = 0.4$ and therefore $r = 0.6$.

$S_{11} = \frac{80(1-0.6^{11})}{1-0.6} = 199.27$ but

$S_{10} = \frac{80(1-0.6^{10})}{1-0.6} = 198.79$ so 11 terms are needed before the total is greater than 199.

12 Split the sum into two separate series: $1 + 2 + 3 + 4 + \ldots$ and $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

The first is an arithmetic series, $a = d = 1$ and $S_n = \frac{n}{2}\{2 + (n-1)\} = \frac{1}{2}n(n+1)$.

The second is a geometric series with $a = r = \frac{1}{2}$ and $S_n = \frac{0.5(1-0.5^n)}{1-0.5} = 1 - 0.5^n$.

So for the series given $S_n = \frac{1}{2}n(n+1) + 1 - 0.5^n$.

13 $a = 20$ and $\frac{a}{1-r} = 200$; $1 - r = \frac{20}{200} = 0.1$; $r = 0.9$

1% of 200 = 2 so the total must be at least 198.

$S_{43} = \frac{20(1-0.9^{43})}{1-0.9} = 197.84$;

$S_{44} = \frac{20(1-0.9^{44})}{1-0.9} = 198.06$

This shows that the n must be at least 44

14 The second term is ar and the sum to infinity is $\frac{a}{1-r}$.

Hence $\frac{a}{1-r} = 4ar$; $1 = 4r(1-r)$; $4r^2 - 4r + 1 = 0$;
$(2r-1)(2r-1) = 0$.

The only solution is $r = \frac{1}{2}$.

15 **a** From her aunt she receives amounts in an arithmetic progression, $a = 30$ and $d = 10$.

On her 10th birthday, $n = 6$ and she receives $30 + 5 \times 10 = \$80$.

From her uncle she receives amounts in a geometric progression, $a = 0.1$ and $r = 2$.

On her 10th birthday, $n = 6$ and she receives $0.1 \times 2^5 = \$3.20$.

The total is \$83.20.

b Anna has had 17 birthday presents so $n = 17$.

From her aunt $\frac{17}{2}\{2 \times 30 + 16 \times 10\} = \$1870$.

From her uncle $\frac{0.1\left(2^{17}-1\right)}{2-1} = \$13107.10$.

The total is \$14977.10.

16 **a** $S_\infty = \frac{a}{1-r} = \frac{x}{1-3x} = 8$; $x = 8(1-3x)$; $x = 8 - 24x$;

$25x = 8$; $x = \frac{8}{25}$ or 0.32

b $ar^2 = 4$; $r^2 = \frac{4}{a}$; $r = \sqrt{\frac{4}{a}}$; $r = \pm\frac{2}{\sqrt{a}}$

$$S_\infty = \frac{a}{1-r} = \frac{a}{1 \pm \frac{2}{\sqrt{a}}} = \frac{a\sqrt{a}}{\sqrt{a} \pm 2}$$

Exam-style questions

1 **a** $(1+2x)^{10} = 1^{10} + 10 \times 2x + \binom{10}{2} \times (2x)^2 + \ldots$

$= 1 + 20x + 45 \times 4x^2 + \ldots$

$= 1 + 20x + 180x^2 + \ldots$

b $(3x-4)(1+2x)^{10} = (3x-4)(1 + 20x + 180x^2 + \ldots) =$

The term x^2 is $3x \times 20x - 4 \times 180x^2 = 60x^2 - 720x^2$

$= -660x^2$.

The coefficient is -660.

2 **a** $(1-3x)^4 = 1 + 4(-3x) + \binom{4}{2}(-3x)^2 + 4(-3x)^3 + (-3x)^4$

$= 1 - 12x + 6 \times 9x^2 - 108x^3 + 81x^4$

$= 1 - 12x + 54x^2 - 108x^3 + 81x^4$

b $(2 + ax^2)(1-3x)^4 = (2 + ax^2)(1 - 12x + 54x^2 - 108x^3 + 81x^4)$

The coefficient of x^3 is $2 \times -108 + a \times -12 = -216 - 12a$.

If $-216 - 12a = -132$ then $-12a = 84$ and $a = -7$.

3 The term independent of x, where the x terms cancel out is $\binom{6}{2}(x^2)^2 \times \left(\frac{1}{2x}\right)^4 = \frac{6 \times 5}{2 \times 1}x^4 \times \frac{1}{16x^4} = \frac{15}{16}$

4 **a** $S_{20} = 10(2a + 19d)$ and $S_{30} = 15(2a + 29d)$

Hence $15(2a + 29d) = 20(2a + 19d)$;

$3(2a + 29d) = 4(2a + 19d)$;

$6a + 87d = 8a + 76d$; $2a = 11d$; $a = \frac{11d}{2}$

b $15(2a + 29d) = 400$; substitute to get

$15(11d + 29d) = 400$; $15 \times 40d = 400$;

$d = \frac{10}{15} = \frac{2}{3}$

Then $a = \frac{11}{2} \times \frac{2}{3} = \frac{11}{3}$

5 **a** An arithmetic sequence with $a = 1200$ and $d = 70$.

$S_{12} = \frac{12}{2}(2 \times 1200 + 11 \times 70) = 19\,020$. So he earns \$19 020.

b In the second month he earns $1200 \times 1.05 = \$1260$ which is less than $1200 + 70 = \$1270$.

In the year he earns $\frac{1200(1.05^{12}-1)}{1.05-1} = \$19\,100.55$, which is more than \$19 020.

6 **a** $ar^2 = \frac{8}{3}$ and $ar^4 = \frac{32}{27}$

Hence $r^2 = \frac{32}{27} \div \frac{8}{3} = \frac{32}{27} \times \frac{3}{8} = \frac{4}{9}$

and $a = \frac{8}{3} \div \frac{4}{9} = 6$.

b If $r^2 = \frac{4}{9}$, then $r = \pm\frac{2}{3}$.

If $r = \frac{2}{3}$, then $S_\infty = \frac{6}{1-\frac{2}{3}} = 18$.

If $r = -\frac{2}{3}$, then $S_\infty = \frac{6}{1+\frac{2}{3}} = 6 \div \frac{5}{3} = \frac{18}{5}$ or $3\frac{3}{5}$.

7 **a** The term in x^3 is $\binom{6}{3}(2x)^3(-k)^3$

$= -20 \times 8x^3 k^3 = -160k^3 x^3$.

The term in x^5 is $\binom{6}{5}(2x)^5(-k) = -6 \times 32x^5 k$

$= -192k\,x^5$.

The coefficients are equal so $-160k^3 = -192k$;

hence $k^2 = \frac{192}{160} = 1.2$; $k = \sqrt{1.2}$.

b The term in x^2 is $\binom{6}{2}(2x)^2(-k)^4$

$= 15 \times 4x^2 \times 1.2^2 = 86.4\,x^2$.

The coefficient is 86.4.

8 a $a = 24$ and $a + d = 18$ so $d = -6$

$$S_n = \frac{n}{2}\{2a + (n-1)d\} = \frac{n}{2}\{48 - 6(n-1)\}$$

$$= \frac{n}{2}(48 - 6n + 6) = \mathbf{27}n - \mathbf{3}n^2$$

If $S_n < -30$ then $27n - 3n^2 < -30$;
$3n^2 - 27n - 30 > 0$; $n^2 - 9n - 10 > 0$

$(n-10)(n+1) > 0$; hence $n > 10$ or $n < -1$.

Since n is positive, $n > 10$.

b $a = 24$ and $ar = 18$; $r = \frac{18}{24} = \frac{3}{4}$;

$$S_\infty = \frac{a}{1-r} = \frac{24}{1 - \frac{3}{4}} = 96$$

9 a $a + 2d = 51$ and $a + 10d = 187$; hence
$8d = 187 - 51 = 136$; $d = 136 \div 8 = 17$.

Hence $a + 34 = 51$ and $a = 17$; the terms of the sequence are 17, 34, 51, … the multiples of 17.

b $\frac{1000}{17} = 58.8$, so there are 58 multiples of 17 between 0 and 1000.

With $a = 17$ and $d = 17$ then

$S_{58} = 26(34 + 57 \times 17) = 26\,078$

10 a The term in x^3 is $3x \times \begin{pmatrix} 7 \\ 1 \end{pmatrix} \times 2x^2 \times (-b)^6$

$= 3 \times 7 \times 2b^6 x^3 = 42b^6 x^3$.

Hence $42b^6 = 2688$; $b^6 = 64$; $b = \pm 2$.

b The term in x^2 is $a \times \begin{pmatrix} 7 \\ 1 \end{pmatrix} \times 2x^2 \times (-b)^6$
$= 14ab^6 x^2$.

Hence $14ab^6 = 2688$; $a = \frac{2688}{14 \times 64} = 3$.

11 a \$400 has 1 month's interest, another \$400 has 2 month's interest and so on.

Total $= 400 \times 1.005 + 400 \times 1.005^2 + \ldots + 400 \times 1.005^{12}$.

This is a geometric series with $a = 400 \times 1.005 = 402$ and $r = 1.005$.

b $S_{12} = \frac{402(1.005^{12} - 1)}{1.005 - 1} = 4958.90$, she has \$4960 (3 s.f.).

12 a $(a + 5d) + (a + 6d) + (a + 7d) = 12$; hence
$3a + 18d = 12$; $a + 6d = 4$; $a = 4 - 6d$

b $a + 5d = 12$; hence $(4 - 6d) + 5d = 12$;
$4 - d = 12$; $d = -8$

Hence $a = 4 - 6d = 4 + 48 = 52$.

13 $a = 20$, $a + d = 24$, $d = 4$

$$S_k = \frac{k}{2}\{2a + (k-1)d\} = \frac{k}{2}\{40 + 4(k-1)\}$$

$$= \frac{k}{2}\{4k + 36\} = 2k^2 + 18k$$

If $S_k = 504$, then $2k^2 + 18k = 504$; $k^2 + 9k - 252 = 0$;
$(k + 21)(k - 12) = 0$.

Hence $k = -21$ or 12. The value of k is positive.
Hence $k = 12$.

14 a The total length is 20 + 10 + 5 + ... which is a geometric series with $a = 20$ and $r = 0.5$ so the sum $S_\infty = \frac{20}{1 - 0.5} = 40$.

b The x-coordinate is 20 – 5 + 1.25 – ... which is a geometric series with $a = 20$ and $r = -0.25$ so the x-coordinate is $S_\infty = \frac{20}{1 + 0.25} = 16$.

The y-coordinate is 10 – 2.5 + 0.625 – ... which is a geometric series with $a = 10$ and $r = -0.25$ so the y-coordinate is $S_\infty = \frac{10}{1 + 0.25} = 8$.

The coordinates are (16, 8).

15 a The ratios of successive terms are equal so

$$\frac{6p + 2}{4p + 4} = \frac{4p + 4}{3p + 3}.$$

Hence $\frac{6p + 2}{4p + 4} = \frac{4}{3}$ and $3(6p + 2) = 4(4p + 4)$;

$18p + 6 = 16p + 16$;

$2p = 10$ and so $p = 5$.

b The first term is $6 \times 5 + 2 = 32$, the second is $4 \times 5 + 4 = 24$ and $r = \frac{24}{32} = \frac{3}{4}$.

$$S_\infty = \frac{32}{1 - \frac{3}{4}} = \frac{32}{\frac{1}{4}} = 128$$

16 a If the common difference is d then $b = a + 2d$ and $d = \frac{1}{2}(b - a)$.

The sixth term is $a + 5d = a + \frac{5}{2}(b - a)$

$= a + \frac{5}{2}b - \frac{5}{2}a = \frac{5}{2}b - \frac{3}{2}a$ or $\frac{1}{2}(5b - 3a)$.

b If the common ratio is r, then $ar^2 = b$; $r^2 = \frac{b}{a}$;

$r = \pm\sqrt{\frac{b}{a}}$.

The fifth term is $ar^4 = a \times \left(\frac{b}{a}\right)^2 = \frac{ab^2}{a^2} = \frac{b^2}{a}$.

17 a $\frac{63}{64}$

b The nth term is $\frac{2^n - 1}{2^n}$.

c $\frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$.

Hence $S_n = \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{4}\right) + \left(1 - \frac{1}{8}\right) + \ldots$

$\left(1 - \frac{1}{2^n}\right) = n - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \frac{1}{2^n}\right)$.

The terms in the bracket form a geometric sequence with $a = r = \frac{1}{2}$ and n terms.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\frac{1}{2^n}=\frac{\frac{1}{2}\left\{1-\left(\frac{1}{2}\right)^n\right\}}{1-\frac{1}{2}}=1-\left(\frac{1}{2}\right)^n$$

So $S_n = n-\left\{1-\left(\frac{1}{2}\right)^n\right\} = n-1+\left(\frac{1}{2}\right)^n$

18 If the common ratio is r, then $S_\infty = \frac{a}{1-r} = ka$.

Therefore $\frac{1}{1-r} = k; \frac{1}{k} = 1-r; r = 1-\frac{1}{k}$.

The second term is $ar = a\left(1-\frac{1}{k}\right) = a-\frac{a}{k}$.

19 a The first term is $\frac{12}{r}$;

$$S_\infty = \frac{a}{1-r} = \frac{\frac{12}{r}}{1-r} = \frac{12}{r(1-r)} = 50$$

Hence $12 = 50r(1-r)$; $6 = 25r-25r^2$;

$25r^2-25r+6=0$

b $(5r-2)(5r-3)=0$; $r = \frac{2}{5}$ or $\frac{3}{5} = 0.4$ or 0.6

c $a = \frac{12}{r}$ so $a = \frac{12}{0.4} = 30$ or $\frac{12}{0.6} = 20$

The 4th term is $ar^3 = 30\times 0.4^3 = 1.92$ or $20\times 0.6^3 = 4.32$.

20 a If the shortest piece is a then the longest is $a+9d = 4a$.

Hence $3a = 9d$ and $d = \frac{1}{3}a$

The sum is 100 so $\frac{10}{2}(2a+9d) = 100$;

$5\left(2a+9\times\frac{1}{3}a\right) = 100$; $25a = 100$ so $a = 4$.

The shortest piece is 4 cm.

b If the shortest piece is a, then the longest is $ar^9 = 4a$; $r^9 = 4$; $r = \sqrt[9]{4} = 1.1665$

The sum is $\frac{a\left(r^{10}-1\right)}{r-1} = 100$; $a = \frac{100}{22.015} =$ 4.54 cm to 3 s.f.

21 The fifth term is r^4. The sum of the subsequent terms is $ar^5+ar^6+ar^7+\ldots = \frac{ar^5}{1-r}$.

Hence $\frac{ar^5}{1-r} = 9ar^4$; $\frac{r}{1-r} = 9$; $r = 9(1-r)$; $10r = 9$; $r = 0.9$.

Then $S_\infty = \frac{a}{1-r} = \frac{a}{0.1} = 10a$.

Mathematics in life and work

1 The value of the first payment after 1 year is \$ $\frac{5000}{1.02}$, the value of the second after 2 years is \$ $\frac{5000}{1.02^2}$ and so on.

Total value $= \frac{5000}{1.02}+\frac{5000}{1.02^2}+\ldots+\frac{5000}{1.02^{15}}$.

This is a geometric progression with $a = \frac{5000}{1.02}$, $r = \frac{1}{1.02}$ and $n = 15$.

$$S_{15} = \frac{a(r^n-1)}{r-1} = \frac{\frac{5000}{1.02}\left(\left(\frac{1}{1.02}\right)^{15}-1\right)}{\frac{1}{1.02}-1} = \frac{5000\left(\left(\frac{1}{1.02}\right)^{15}-1\right)}{-0.02}$$

$= 64\,246$. The value is \$64 200 (3 s.f.)

2 If you write $c = 1+\frac{R}{100}$, then in this case it is a geometric progression with $a = \frac{5000}{c}$ and $r = \frac{1}{c}$.

Then

$$S_{15} = \frac{5000}{c}\left(1-c^{-15}\right)\Big/\left(1-\frac{1}{c}\right) = \frac{5000}{c}(1-c^{-15})\times\frac{c}{c-1}$$

$$= \frac{5000}{c-1}\times\left(1-c^{-15}\right) = \frac{500000}{R}\times(1-c^{-15}).$$

3 If the annuity is for n years, then

$S_n = \frac{500\,000}{R}\times(1-c^{-n})$.

Now $\lim_{n\to\infty} c^{-n} = 0$ so the value of a perpetuity,

$S_\infty = \frac{500\,000}{R}$.

6 Differentiation

Please note: Full worked solutions are provided as an aid to learning, and represent one approach to answering the question. In some cases, alternative methods are shown for contrast.

All sample answers have been written by the authors. Cambridge Assessment International Education bears no responsibility for the example answers to questions taken from its past question papers, which are contained in this publication.

Non-exact numerical answers should be given correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

Prerequisite knowledge

1 **a** 7 **b** −0.3
c $\frac{1}{2}$ **d** −2

2 **a** $2x^2 - 8x$
b $x^3 + 3x^2$
c $x^2 + 4x - 5x - 20 = x^2 - x - 20$
d $9x^2 + 3x + 3x + 1 = 9x^2 + 6x + 1$

3 **a** x^{-3} **b** $x^{\frac{1}{2}}$ **c** $x^{-\frac{1}{2}}$
d $x^{\frac{1}{3}}$ **e** $x^{\frac{3}{2}}$

Exercise 6.1A

1 **a** 3 **b** $6x$ **c** $6x + 4$
2 **a** $2x + 4$ **b** $2x - 4$ **c** $4 - 2x$

3 **a** $y = x^2 + 6x, \frac{dy}{dx} = 2x + 6$

b $y = x^2 - 2x - 3, \frac{dy}{dx} = 2x - 2$

c $y = 4x^2 + 12x + 9, \frac{dy}{dx} = 8x + 12$

4 **a** $\frac{dy}{dx} = 2x + 1$

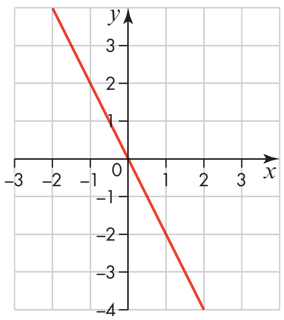
b **i** at (0, −6) **ii** If $x = 0$, $\frac{dy}{dx} = 1$.

c At (2, 0) and (−3, 0), the gradients are 5 and −5.

5 **a** $f'(x) = -2x$

b $f'(-1) = 2$ and $f'(3) = -6$

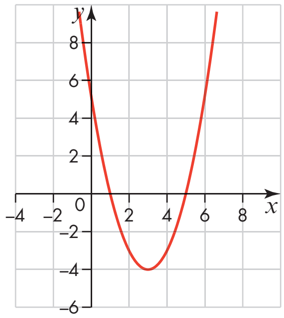
c

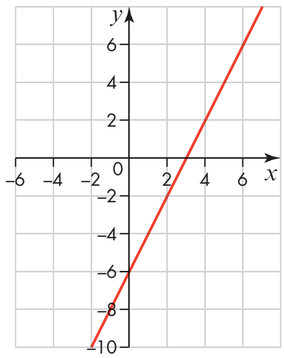


6 **a** $f(x) = x^2 - 6x + 5$ so $f'(x) = 2x - 6$

b $f'(1) = -4$ and $f'(11) = 16$

c





7 $\frac{dy}{dx} = 30x - 30$

a If $x = 1$, then $\frac{dy}{dx} = 0$ **b** If $x = 5$, then $\frac{dy}{dx} = 120$

8 $y = kx(x - 4) = kx^2 - 4kx$

$\frac{dy}{dx} = 2kx - 4k$

When $x = 4$, $\frac{dy}{dx} = 2$.

$2k \times 4 - 4k = 2$

$8k - 4k = 2$

$4k = 2$

$k = \frac{1}{2}$ or 0.5

9 a $\frac{dy}{dx} = 2x - 6$; where the curve crosses the y-axis $x = 0$ and then the gradient is -6.

b The curve crosses the x-axis where $x^2 - 6x - 16 = 0$; $(x - 8)(x + 2) = 0$; $x = 8$ or -2.

If $x = 8$, $\frac{dy}{dx} = 16 - 6 = 10$;

if $x = -2$, $\frac{dy}{dx} = -4 - 6 = -10$.

10 a P is (1, 1) and A is $(1.1, 1.1^3)$ so the gradient of PA is $\frac{1.1^3 - 1}{1.1 - 1} = 3.31$.

b $(1 + h)^3 = 1 + 3h + 3h^2 + h^3$ so B is $(1 + h, 1 + 3h + 3h^2 + h^3)$

The gradient of PB is

$$\frac{1 + 3h + 3h^2 + h^3 - 1}{1 + h - 1} = \frac{3h + 3h^2 + h^3}{h} = 3 + 3h + h^2.$$

c As h approaches $0, 3 + 3h + h^2$ approaches 3 so the gradient = 3.

11 a A is (1, 2) and B is (3, 12). The gradient of $AB = \frac{12 - 2}{3 - 1} = 5$

b The y-coordinate of Q is $(2 - h)^2 + (2 - h) = 4 - 4h + h^2 + 2 - h = 6 - 5h + h^2$.

The y-coordinate of P is $(2 + h)^2 + (2 + h) = 4 + 4h + h^2 + 2 + h$
$= 6 + 5h + h^2$.

The gradient of PQ =

$$\frac{(6 + 5h + h^2) - (6 - 5h + h^2)}{(2 + h) - (2 - h)} = \frac{10h}{2h} = 5.$$

$\frac{dy}{dx} = 2x + 1$ so the gradient at (2, 6) is $2 \times 2 + 1 = 5 =$ the gradient of PQ.

12 a If $y = 0$, then $(x + 3)(x - 4) = 0$ and so $x = -3$ or 4

$y = x^2 - x - 12$ and so $\frac{dy}{dx} = 2x - 1$.

At $(-3, 0)$, $\frac{dy}{dx} = (2 \times -3) - 1 = -7$; at (4, 0),

$\frac{dy}{dx} = (2 \times 4) - 1 = 7$.

b At (5, 8) $\frac{dy}{dx} = 2 \times 5 - 1 = 9$ so the equation of the tangent is $y - 8 = 9(x - 5)$ or $y = 9x - 37$.

c If $\frac{dy}{dx} = 2$, then $2x - 1 = 2$ and $x = \frac{3}{2} = 1.5$;

$y = (1.5 + 3)(1.5 - 4) = -11.25$.

The point is $(1.5, -11.25)$.

13 a $\sqrt{4 + h} = \sqrt{4\left(1 + \frac{h}{4}\right)} = \sqrt{4}\sqrt{1 + \frac{h}{4}} = 2\sqrt{1 + \frac{h}{4}}$

Using the result given with $a = \frac{h}{4}$, if h is small then $\sqrt{4 + h} \approx 2\left(1 + \frac{h}{8}\right) = 2 + \frac{h}{4}$.

b P and Q are (4, 2) and $\left(4 + h, \sqrt{4 + h}\right)$.

The gradient of

$PQ = \frac{\sqrt{4 + h} - 2}{4 + h - 4} \approx \frac{2 + \frac{h}{4} - 2}{h} = \frac{1}{4}$; the gradient of the curve at P is $\frac{1}{4}$.

14 a $\frac{dy}{dx} = 18 - 4x$; at A $18 - 4x = -2$; $4x = 20$; $x = 5$ and $y = 45$; A is (5, 45)

b At A the equation of the tangent is $y - 45 = -2(x - 5)$ or $y + 2x = 55$.

This crosses the y-axis at (0, 55) and the x-axis at (27.5, 0).

The area of the triangle is $55 \times 27.5 \div 2 = 756.25$.

Exercise 6.2A

1 a $2 \times 3x^2 = 6x^2$ **b** $0.5 \times 4x^3 = 2x^3$

c $0.1 \times 5x^4 = 0.5x^4$ **d** $50 \times 3x^2 = 150x^2$

2 a $3x^2 + 8x - 8$ **b** $6x^2 - 10x + 6$

3 a $4x^3 + 16x$ **b** $5x^4 - 30x^2 + 2$

4 a $3x^2 - 4x$

b i When $x = 2$, $3x^2 - 4x = 12 - 8 = 4$.

ii When $x = -1$, $3x^2 - 4x = 3 + 4 = 7$.

5 a $f'(x) = 0.5 \times 4x^3 - 2 \times 2x + 1 = 2x^3 - 4x + 1$

b i When $x = -1$, $2x^3 - 4x + 1 = -2 + 4 + 1 = 3$.

ii When $x = 1$, $2x^3 - 4x + 1 = 2 - 4 + 1 = -1$.

iii When $x = 2$, $2x^3 - 4x + 1 = 16 - 8 + 1 = 9$.

6 a $y = 2x^3 + 5x^2$

$\frac{dy}{dx} = 6x^2 + 10x$

b $y = x(x^2 - 8x + 16) = x^3 - 8x^2 + 16x$

$\frac{dy}{dx} = 3x^2 - 16x + 16$

c $y = x^3 + x^2 + x + 1$

$\frac{dy}{dx} = 3x^2 + 2x + 1$

7 **a** $f(x) = x^3 + 3x^2$

$f'(x) = 3x^2 + 6x$

b **i** $f'(-3) = 27 - 18 = 9$

ii $f'(-2) = 12 - 12 = 0$

iii $f'(1) = 3 + 6 = 9$

8 $y = x(x^2 - 6x + 8) = x^3 - 6x^2 + 8x$

$\frac{dy}{dx} = 3x^2 - 12x + 8$

If $x = 0$, $\frac{dy}{dx} = 8$.

If $x = 2$, $\frac{dy}{dx} = 12 - 24 + 8 = -4$.

If $x = 4$, $\frac{dy}{dx} = 48 - 48 + 8 = 8$.

9 **a** $\frac{dy}{dx} = 3x^2 - 4x + 1$; at A $x = 0$, $y = 3$ and $\frac{dy}{dx} = 1$; the equation of the tangent is $y - 3 = x$ or $y = x + 3$.

b If $x = 2$, then $x^3 - 2x^2 + x + 3 = 5$ and so $(2, 5)$ is on the curve.

Also if $x = 2$, then $x + 3 = 5$ so the point $(2, 5)$ is also on the tangent.

10 **a** $y = 8x^2 - x^4 \Rightarrow \frac{dy}{dx} = 16x - 4x^3$; if $x = -3$, then

$\frac{dy}{dx} = -48 + 108 = 60$

b If $16x - 4x^3 = 0$, then $4x - x^3 = 0$; $x(4 - x^2) = 0$; $x(2 + x)(2 - x) = 0$

So $x = 0$, -2 or 2; The three points are $(0, 0)$, $(-2, 16)$ and $(2, 16)$

11 **a** $f(x) = x^4 - 2cx^2 + c^2$ so $f'(x) = 4x^3 - 4cx$

b If $f'(x) = 0$, then $4x^3 - 4cx = 0$; $x(x^2 - c) = 0$; $x = 0$ or $x^2 = c$

If $c \leqslant 0$ the only solution is $x = 0$ and there is one point where the gradient is 0.

If $c > 0$, there are three solutions, 0, $\sqrt{c}$ and $-\sqrt{c}$ and hence there are three points.

12 $y = x(x^2 - 6x + 9) - 8 = x^3 - 6x^2 + 9x - 8$

Hence $\frac{dy}{dx} = 3x^2 - 12x + 9$

If the gradient is -3, then $3x^2 - 12x + 9 = -3$; $3x^2 - 12x + 12 = 0$

$x^2 - 4x + 4 = 0$; $(x - 2)^2 = 0$; $x = 2$

If $x = 2$, then $y = 2(2 - 3)^2 - 8 = -6$ so P is $(2, -6)$

The equation of the tangent is $y + 6 = -3(x - 2)$ or $y = -3x$ and this passes through the origin.

13 $\frac{dy}{dx} = 3x^2 - 4x - 5$. The gradient of the straight line is 10.

If $\frac{dy}{dx} = 10$, then $3x^2 - 4x - 5 = 10$;

$3x^2 - 4x - 15 = 0$; $(3x + 5)(x - 3) = 0$

So $x = -\frac{5}{3}$ or 3 and the points on the curve are $(-1.67, 4.15)$ and $(3, 0)$

The second of these is on the straight line and so the line is the tangent at that point.

14 **a** $(x + a)(x - a) = x^2 - a^2$ so $f(x) = (x^2 - a^2)^2$

$(x^2 - a^2)^2 = x^4 - 2a^2x^2 + a^4$ hence $f'(x) = 4x^3 - 4a^2x$

b If $f'(x) = 0$, then $4x^3 - 4a^2x = 0$; $x^3 - a^2x = 0$; $x(x^2 - a^2) = 0$;

$x(x + a)(x - a) = 0$; $x = 0$, a or $-a$; then $y = a^4$, 0 or 0

The points are $(0, a^4)$, $(a, 0)$ and $(-a, 0)$

Exercise 6.3A

1 **a** $y = x^{-2}$ and so $\frac{dy}{dx} = -2x^{-3} = -\frac{2}{x^3}$

b $y = 2x^{-3}$ and so $\frac{dy}{dx}$ $2 \times (-3)x^{-4} = -\frac{6}{x^4}$

c $y = x^{\frac{1}{3}}$; $\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$

d $\frac{dy}{dx} = 4 \times \frac{5}{2}x^{\frac{5}{2}-1} = 10x^{\frac{3}{2}}$

2 **a** $s = 10t^{\frac{1}{2}}$ so $\frac{ds}{dt} = 10 \times \frac{1}{2}t^{-\frac{1}{2}} = \frac{5}{\sqrt{t}}$

b $s = 50t^{-1} + 10$ so $\frac{ds}{dt} = 50 \times -1 \times t^{-2} = -\frac{50}{t^2}$

c $s = 10t^2 - 10t^{-2}$ so $\frac{ds}{dt} = 20t + 20t^{-3} = 20t + \frac{20}{t^3}$

3 **a** $f(x) = 24x^{-1}$

$f'(x) = -24x^{-2} = -\frac{24}{x^2}$

b **i** $f'(6) = -\frac{24}{6^2} = -\frac{2}{3}$

ii $f'(4) = -\frac{24}{4^2} = -\frac{3}{2}$

iii $f'(-2) = -\frac{24}{(-2)^2} = -6$

iv $f'(24) = -\frac{24}{24^2} = -\frac{1}{24}$

c If $x \neq 0$, then x^2 is positive and the gradient $-\frac{24}{x^2}$ is negative.

4 a $y = x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

i If $x = 4$, $\frac{dy}{dx} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$.

ii If $x = 9$, $\frac{dy}{dx} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$.

iii If $x = 100$, $\frac{dy}{dx} = \frac{1}{2\sqrt{100}} = \frac{1}{20}$.

b $\frac{1}{2\sqrt{x}} = \frac{1}{2}$

$x = 1$ and the coordinates are $(1, 1)$.

c $\frac{1}{2\sqrt{x}} = 1$

$\sqrt{x} = \frac{1}{2}$

$x = \frac{1}{4}$ and the coordinates are $\left(\frac{1}{4}, \frac{1}{2}\right)$.

5 a $y = \frac{1}{2}x + 2x^{-1}$

$\frac{dy}{dx} = \frac{1}{2} - 2x^{-2} = \frac{1}{2} - \frac{2}{x^2}$

If $x = 2$, $\frac{dy}{dx} = \frac{1}{2} - \frac{2}{2^2} = \frac{1}{2} - \frac{1}{2} = 0$.

If $x = -2$, $\frac{dy}{dx} = \frac{1}{2} - \frac{2}{(-2)^2} = \frac{1}{2} - \frac{1}{2} = 0$.

b If $x = 0.5$, $\frac{dy}{dx} = \frac{1}{2} - \frac{2}{0.5^2} = \frac{1}{2} - 8 = -7\frac{1}{2}$.

If $x = 4$, $\frac{dy}{dx} = \frac{1}{2} - \frac{2}{4^2} = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$.

c $\frac{dy}{dx} = \frac{1}{2} - \frac{2}{x^2}$

If x is large, $\frac{2}{x^2}$ is a small positive number and the gradient is close to $\frac{1}{2}$.

The larger x is, the closer the gradient is to $\frac{1}{2}$.

6 a $f(x) = 2 + \frac{5}{x} = 2 + 5x^{-1}$

$f'(x) = -5x^{-2} = -\frac{5}{x^2}$

b i $f'(2) = -\frac{5}{2^2} = -\frac{5}{4}$

ii $f'(10) = -\frac{5}{10^2} = -\frac{1}{20} = -0.05$

c $f'(x) = -\frac{5}{x^2} = -5$

$x^2 = 1$

$x = 1$ or -1

Points are $(1, 7)$ and $(-1, -3)$.

7 a $y = 2x^{\frac{1}{3}} \Rightarrow \frac{dy}{dx} = \frac{2}{3}x^{-\frac{2}{3}}$; if $x = 125$ then

$\frac{dy}{dx} = \frac{2}{3}125^{-\frac{2}{3}} = \frac{2}{3} \times \frac{1}{5^2} = \frac{2}{75}$

b The equation is $x = \sqrt[3]{8y}$; $8y = x^3$; $y = \frac{1}{8}x^3$

c For the reflection $\frac{dy}{dx} = \frac{1}{8} \times 3x^2 = \frac{3}{8}x^2$; Q is the point $(10, 125)$.

The gradient is $\frac{3}{8} \times 100 = \frac{75}{2}$ = reciprocal of $\frac{2}{75}$.

8 a $f(x) = x^4 + 6x^2 + 9$ so $f'(x) = 4x^3 + 12x$

b $h(x) = x^3(x^4 + 6x^2 + 9) = x^7 + 6x^5 + 9x^3$

So $h'(x) = 7x^6 + 30x^4 + 27x^2$

9 $y = \frac{x^2 + 2ax + a^2}{x} = x + 2a + a^2x^{-1}$; hence

$\frac{dy}{dx} = 1 - a^2x^{-2} = 1 - \frac{a^2}{x^2}$.

If $1 - \frac{a^2}{x^2} = 0$, then $x^2 = a^2$ and $x = \pm a$

If $x = a$, then $y = \frac{(2a)^2}{a} = 4a$. If $x = -a$, then $y = 0$. The points are $(a, 4a)$ and $(-a, 0)$.

10 a $f(4) = \left(4 + \frac{4}{4}\right)^2 = 5^2 = 25$ so $(4, 25)$ is on the curve.

b $f(x) = x^2 + 8 + 16x^{-2}$ so

$f'(x) = 2x - 32x^{-3} = 2x - \frac{32}{x^3}$

c $f'(4) = 8 - \frac{32}{64} = 8 - \frac{1}{2} = \frac{15}{2}$

The equation of the tangent is

$y - 25 = \frac{15}{2}(x - 4)$; $2y - 50 = 15x - 60$;

$2y = 15x - 10$.

Exercise 6.4A

1 a $\frac{dy}{dx} = 2(4x+2) \times 4 = 8(4x+2)$ or $16(2x+1)$

b $\frac{dy}{dx} = 3(4x+2)^2 \times 4 = 12(4x+2)^2$

c $\frac{dy}{dx} = 5(4x+2)^4 \times 4 = 20(4x+2)^4$

2 a $\frac{dy}{dx} = 10(8-x)^9 \times -1 = -10(8-x)^9$

b $\frac{dy}{dx} = 10(1+3x^2)^9 \times 6x = 60x(1+3x^2)^9$

c $\frac{dy}{dx} = 10(6x-3x^2)^9 \times (6-6x) = 60(6x-3x^2)^9(1-x)$

3 a $f(x) = (x-3)^{\frac{1}{2}}$ so $f'(x) = \frac{1}{2}(x-3)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x-3}}$.

b $f(x) = (2x-3)^{\frac{1}{2}}$ so $f'(x) = \frac{1}{2}(2x-3)^{-\frac{1}{2}} \times 2 = \frac{1}{\sqrt{2x-3}}$.

c $f(x) = \left(x^2-3\right)^{\frac{1}{2}}$ so $f'(x) = \frac{1}{2}\left(x^2-3\right)^{-\frac{1}{2}} \times 2x$

$= \frac{x}{\sqrt{x^2-3}}$.

4 a $y = 20(2+x)^{-1}$

$\frac{dy}{dx} = 20 \times -1 \times (2+x)^{-2} \times 1 = -\frac{20}{(2+x)^2}$

If $x = 2$, the gradient $\frac{dy}{dx} = -\frac{20}{16} = -1.25$

b Where the gradient is -0.2, $-\frac{20}{(2+x)^2} = -0.2$.

Therefore $(2+x)^2 = 100$.

Take the positive root: $2 + x = 10$.

Therefore $x = 8$ and $y = 2$. The point is $(8, 2)$.

5 a $f(x) = 20(x-2)^{-1}$ so

$f'(x) = 20 \times (-1)(x-2)^{-2} \times 1 = -\frac{20}{(x-2)^2}$.

$f'(4) = -\frac{20}{2^2} = -5$

b $g(x) = 20(x+3)^{-1}$ so

$g'(x) = 20 \times (-1)(x+3)^{-2} \times 1 = -\frac{20}{(x+3)^2}$

$g'(-1) = -\frac{20}{2^2} = -5$

c A translation of $y = f(x)$ by $\begin{pmatrix} -5 \\ 0 \end{pmatrix}$ is

$\frac{20}{(x+5)-2} = \frac{20}{x-3} = g(x)$.

The gradient at $x = 4$ on the first is the gradient at $x = 4 - 5 = -1$ on the second.

6 a $y = (2x+5)^{\frac{1}{2}}$ so

$\frac{dy}{dx} = \frac{1}{2} \times (2x+5)^{-\frac{1}{2}} \times 2 = \frac{1}{\sqrt{2x+5}} = \frac{1}{y}$

b If $x = 2$, then $y = \sqrt{4+5} = 3$.

c At $(2, 3)$ $\frac{dy}{dx} = \frac{1}{3}$ and the equation of the tangent is $y - 3 = \frac{1}{3}(x-2)$.

On the x-axis $y = 0$ and $-3 = \frac{1}{3}(x-2)$;

$x - 2 = -9$; $x = -7$. The point is $(-7, 0)$.

7 a When $x = 10$, $y = \frac{600}{10^2+50} = \frac{600}{150} = 4$

so $(10, 4)$ is on the curve.

b $y = 600(x^2+50)^{-1}$

$\frac{dy}{dx} = -600(x^2+50)^{-2} \times 2x$

$= -\frac{1200x}{(x^2+50)^2}$

c At $x = 10$, $\frac{dy}{dx} = -\frac{12000}{150^2} = -\frac{12000}{22500} = -\frac{24}{45} = -\frac{8}{15}$

So $y - 4 = -\frac{8}{15}(x-10)$, $8x + 15y = 140$

8 a $f(-6) = \sqrt{100-(-6)^2} = \sqrt{64} = 8$ so $(-6, 8)$ is on the curve.

b $f(x) = \left(100-x^2\right)^{\frac{1}{2}}$;

$f'(x) = \frac{1}{2}\left(100-x^2\right)^{-\frac{1}{2}} \times (-2x) = \frac{-x}{\sqrt{100-x^2}}$

So the gradient of the tangent $= f'(-6) = \frac{6}{8} = \frac{3}{4}$.

The gradient of $OP = \frac{8}{-6} = -\frac{4}{3}$

$-\frac{4}{3} \times \frac{3}{4} = -1$ so the lines are perpendicular.

9 $\frac{dy}{dx} = 3(2x-3)^2 \times 2 = 6(2x-3)^2$; if $\frac{dy}{dx} = 24$, then

$6(2x-3)^2 = 24$; $(2x-3)^2 = 4$;

$2x - 3 = 2$ or -2; $x = 2.5$ or 0.5; then $y = 8$ or -8;

points $(2.5, 8)$ and $(0.5, -8)$

10 a $1 - \frac{1}{x+1} = \frac{x+1}{x+1} - \frac{1}{x+1} = \frac{x+1-1}{x+1} = \frac{x}{x+1}$

b $f'(x)$ is the derivative of

$1 - (x+1)^{-1} = (x+1)^{-2} = \frac{1}{(x+1)^2}$.

c $g(x) = \dfrac{x}{x-1} = \dfrac{1}{x-1} + 1$ and the derivative

is $-1 \times (x-1)^{-2} = -\dfrac{1}{(x-1)^2}$.

11 a $f(x) = (x^2+4)^{-1}$; $f'(x) = -1 \times (x^2+4)^{-2} \times 2x = \dfrac{-2x}{\left(x^2+4\right)^2}$

b $g(x) = \dfrac{x^2}{x^2+4} = \dfrac{x^2+4-4}{x^2+4} = 1 - \dfrac{4}{x^2+4} = 1 - 4f(x)$

So $g'(x) = -4f'(x) = \dfrac{8x}{\left(x^2+4\right)^2}$.

12 a If $x = 2$, then $y = \dfrac{80}{4^2} = 5$.

b $y = 80(x+2)^{-2}$; then

$\dfrac{dy}{dx} = 80 \times (-2) \times (x+2)^{-3} = -\dfrac{160}{(x+2)^3}$.

If $x = 2$, then $\dfrac{dy}{dx} = -\dfrac{160}{4^3} = -2.5$.

The equation of the tangent is $y - 5 = -2.5(x-2)$ or $y + 2.5x = 10$.

If $x = 0$, $y = 10$; if $y = 0$, $x = \dfrac{10}{2.5} = 4$; the area of the triangle is $\dfrac{1}{2} \times 10 \times 4 = 20$.

Exercise 6.5A

1 a $\dfrac{dy}{dx} = 2x - 12$

At a stationary point, $\dfrac{dy}{dx} = 0$.

$2x - 12 = 0$

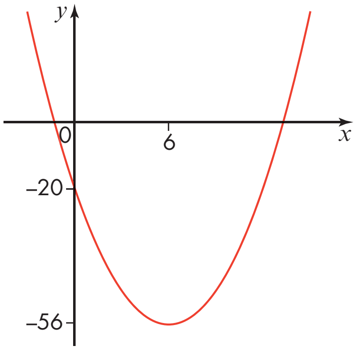
$x = 6$ and $y = 6^2 - 12 \times 6 - 20 = -56$

Coordinates are $(6, -56)$.

b $\dfrac{d^2y}{dx^2} = 2$ which is positive when $x = 6$

(in fact it is always positive) so the point is a minimum point.

c



2 a $\dfrac{dy}{dx} = 6 - 4x$

At a stationary point, $\dfrac{dy}{dx} = 0$.

$6 - 4x = 0$

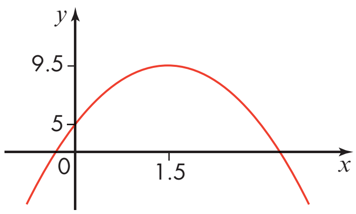
$x = 1\frac{1}{2}$ and $y = 5 + 6 \times 1\frac{1}{2} - 2 \times \left(1\frac{1}{2}\right)^2 = 9.5$.

Coordinates are $(1.5, 9.5)$.

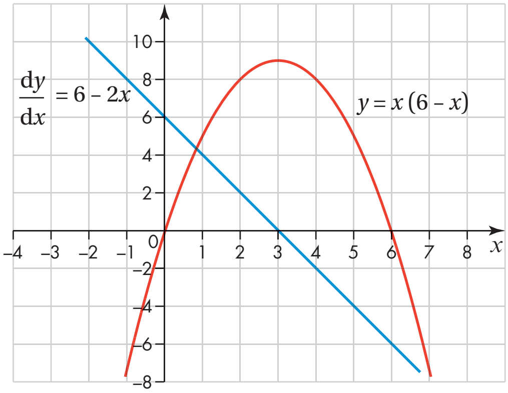
b $\dfrac{d^2y}{dx^2} = -4$ which is negative when $x = 1.5$

(in fact it is always negative) so the point is a maximum point.

c



3 a and **b**

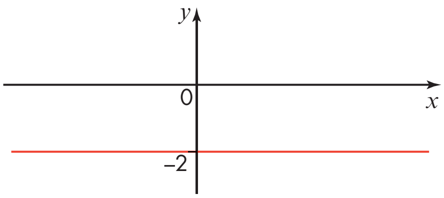


$y = 6x - x^2$

$\dfrac{dy}{dx} = 6 - 2x$

c Where the graph of $\dfrac{dy}{dx}$ crosses the x-axis gives the x-coordinate of a turning point.

d $\dfrac{d^2y}{dx^2} = -2$, a constant value



4 **a** $f'(x) = 6x^2 - 18x + 12$

At a stationary point: $f'(x) = 0$

$6x^2 - 18x + 12 = 0$

Divide by 6 and factorise: $x^2 - 3x + 2 = 0$

$(x - 1)(x - 2) = 0$;

$x = 1$ or 2.

$f(1) = 2 - 9 + 12 + 8 = 13$ and
$f(2) = 16 - 36 + 24 + 8 = 12$.

The stationary points are (1, 13) and (2, 12).

b $f''(x) = 12x - 18$

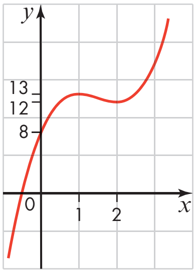
$f''(1) = 12 - 18 = -6$

(1, 13) is a maximum point.

$f''(2) = 24 - 18 = 6$

(2, 12) is a minimum point.

c



5 **a** $\dfrac{dy}{dx} = 3x^2 - 12x - 180$

At a stationary point, $\dfrac{dy}{dx} = 0$.

$3x^2 - 12x - 180 = 0$

Divide by 3 and factorise.

$x^2 - 4x - 60 = 0$

$(x - 10)(x + 6) = 0$

$x = 10$ or -6

If $x = 10$, $y = 10^3 - 6 \times 10^2 - 180 \times 10 = -1400$.

(10, −1400)

$x = -6$, $y = (-6)^3 - 6 \times (-6)^2 - 180 \times (-6) = 648$

The stationary points are (10, −1400) and (− 6, 648).

b $\dfrac{d^2y}{dx^2} = 6x - 12$

If $x = 10$, $\dfrac{d^2y}{dx^2} = 48 > 0$, so (10, −1400) is a minimum point.

If $x = -6$, $\dfrac{d^2y}{dx^2} = -48 < 0$, so (− 6, 648) is a maximum point.

6 $\dfrac{dy}{dx} = 4x^3 - 4x$

At a stationary point:

$4x^3 - 4x = 0$

$4x(x - 1)(x + 1) = 0$

$x = 0$, 1 or −1

The stationary points are (0, 0), (1, −1) and (−1, −1).

$\dfrac{d^2y}{dx^2} = 12x^2 - 4$

If $x = 0$, $\dfrac{d^2y}{dx^2} = -4$ and is negative so (0, 0) is a maximum point.

If $x = 1$, $\dfrac{d^2y}{dx^2} = 8$ and is positive so (1, −1) is a minimum point.

If $x = -1$, $\dfrac{d^2y}{dx^2} = 8$ and is positive so (−1, −1) is a minimum point.

7 **a** $f'(x) = 3x^2 + 6x + 3 = 3(x^2 + 2x + 1) = 3(x + 1)^2$ which, because of the square, is $\geqslant 0$ for all values of x. The function is increasing.

b $f'(x) = 3(x^2 + 4)^2 \times 2x = 6x(x^2 + 4)^2$. This is negative if x is negative and positive if x is positive. The function is neither increasing nor decreasing.

c $f(x) = 100x^{-3}$ so $f'(x) = 100 \times -3x^{-4} = -\dfrac{300}{x^4}$. Now $x \neq 0$ so x^4 is always positive and so $-\dfrac{300}{x^4}$ is always negative. The function is decreasing.

8 $y = 10x^{-1} + \dfrac{1}{4}x$

$\dfrac{dy}{dx} = -10x^{-2} + \dfrac{1}{4}$

At a stationary point:

$-10x^{-2} + \dfrac{1}{4} = 0$

Rearrange.

$10x^{-2} = \dfrac{1}{4}$

$40 = x^2$

$x = \sqrt{40} = 2\sqrt{10}$

9 $\dfrac{dv}{dt} = 6 - t$

At a maximum point: $\dfrac{dv}{dt} = 0$.

$6 - t = 0$

$t = 6$

$\frac{d^2v}{dt^2} = -1$ which is negative for any value of t so this is a maximum point.

If $t = 6$, $v = 10 + 6 \times 6 - 0.5 \times 6^2 = 10 + 36 - 18 = 28$.

The maximum speed is $28\,\text{m s}^{-1}$.

10 a $\frac{dy}{dt} = 23 - 10t$

When the height is a maximum:

$\frac{dy}{dt} = 0$

$23 - 10t = 0$

$t = 2.3$

$\frac{d^2y}{dt^2} = -10 < 0$ so this is a maximum point.

When $x = 2.3$, $y = 23 \times 2.3 - 5 \times 2.3^2$

$= 26.45$.

The maximum height is 26.45 m.

b Air resistance will reduce the maximum height of the ball.

11 a $\frac{dy}{dx} = 3x^2 + 12x - 15$. At a stationary point

$\frac{dy}{dx} = 0$ so $3x^2 + 12x - 15 = 0$;

$x^2 + 4x - 5 = 0$; $(x + 5)(x - 1) = 0$; $x = -5$ or 1.

If $x = -5$, then $y = -125 + 150 + 75 - 90 = 10$ so $(-5, 10)$ is a stationary point.

$\frac{d^2y}{dx^2} = 6x + 12$, and if $x = -5$ then $\frac{d^2y}{dx^2} = -18$ and so $(-5, 10)$ is a maximum point.

b If $y = x^3 + ax^2 + bx + c$, then $\frac{dy}{dx} = 3x^2 + 2ax + b$.

There are two stationary points if the quadratic equation has two distinct roots.

That is when the determinant '$b^2 - 4ac$' > 0; $(2a)^2 - 4 \times 3b > 0$; $4a^2 - 12b > 0$; $4a^2 > 12b$; $a^2 > 3b$.

12 a $f'(x) = 3x^2 + 2ax$, and if $f'(x) = 0$ then $3x^2 + 2ax = 0$; $x(3x + 2a) = 0$;

$x = 0$ or $-\frac{2a}{3}$. There is a stationary point if $x = 0$.

b $f''(x) = 6x + 2a$; if $x = 0$, then $f''(x) = 2a$; if $a > 0$ then $f''(x) > 0$ and the point is a minimum point.

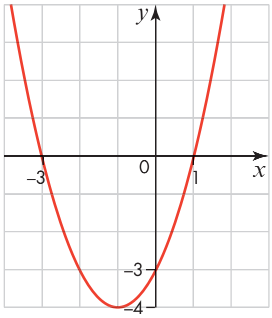
c There is another stationary point where

$x = -\frac{2a}{3} = -\frac{24}{3} = -8$.

Then $f(-8) = (-8)^3 + 12 \times (-8)^2 + 6 = 262$; the point is $(-8, 262)$.

Exercise 6.6A

1 a This parabola crosses the x-axis at 1 and -3.



b $y = x^2 + 2x - 3$

$\frac{dy}{dx} = 2x + 2$

c If $x = 3$, $\frac{dy}{dx} = 2 \times 3 + 2 = 8$.

The equation of the tangent is $y - 12 = 8(x - 3)$

$y - 12 = 8x - 24$

$y = 8x - 12$

d The gradient of the normal is $-\frac{1}{8}$

The equation of the normal is

$y - 12 = -\frac{1}{8}(x - 3)$.

$8y - 96 = -x + 3$

$x + 8y = 99$

e If $x = -3$, $\frac{dy}{dx} = 2 \times -3 + 2 = -4$, the tangent is

$y - 0 = -4(x + 3)$

$y + 4x + 12 = 0$

The gradient of the normal is $\frac{1}{4}$ and the equation is $y = \frac{1}{4}(x + 3)$ or $4y = x + 3$.

2 a If $x = -1$, $y = (-1)^3 - 4 \times (-1)^2 = -5$ so $(-1, -5)$ is on the curve.

b $\frac{dy}{dx} = 3x^2 - 8x$

c If $x = -1$, $\frac{dy}{dx} = 3 \times (-1)^2 - 8 \times (-1) = 3 + 8 = 11$.

The equation of the tangent is $y + 5 = 11(x + 1)$.

$y + 5 = 11x + 11$

$y = 11x + 6$

d The equation of the normal is $y + 5 = -\frac{1}{11}(x + 1)$.

$11y + 55 = -x - 1$

$11y + x = -56$

3 $y = 12x^{-2}$

$\frac{dy}{dx} = -24x^{-3}$

If $x = 1$, $\frac{dy}{dx} = -24$.

The gradient of the normal is $\frac{1}{24}$.

The equation of the normal is $y - 12 = \frac{1}{24}(x - 1)$.

$24y - 288 = x - 1$

$24y = x + 287$

4 **a** $y = 12x^{\frac{1}{2}}$

$\frac{dy}{dx} = 6x^{-\frac{1}{2}} = \frac{6}{\sqrt{x}}$

If $x = 4$, $\frac{dy}{dx} = \frac{6}{\sqrt{4}} = 3$.

Gradient of normal $= -\frac{1}{3}$.

Equation of normal is $y - 24 = -\frac{1}{3}(x - 4)$.

$3y - 72 = -x + 4$

$x + 3y = 76$

b If $x = 100$, $\frac{dy}{dx} = \frac{6}{\sqrt{100}} = \frac{3}{5}$.

Gradient of tangent $= \frac{3}{5}$.

Equation of tangent is $y - 120 = \frac{3}{5}(x - 100)$.

$5y - 600 = 3x - 300$

$5y = 3x + 300$

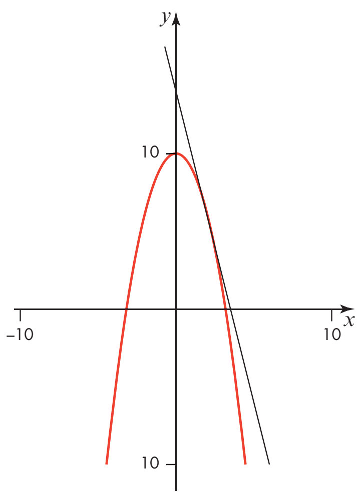
5 **a** $\frac{dy}{dx} = -2x$

If $x = 2$, $\frac{dy}{dx} = -4$ and the equation of the tangent is $y - 6 = -4(x - 2)$.

$y - 6 = -4x + 8$

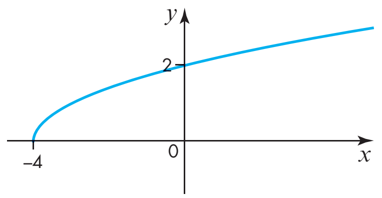
$y + 4x = 14$

b Draw a diagram. Only the tangent is required.



The tangent crosses the x-axis at 3.5 and the area of the triangle is $\frac{1}{2} \times 14 \times 3.5 = 24.5$.

6 **a**



b $y = (x + 4)^{\frac{1}{2}}$ so $\frac{dy}{dx} = \frac{1}{2}(x + 4)^{-\frac{1}{2}}$.

At P, $x = 0$ and so $y = 2$ and $\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

The equation of the tangent at P is $y - 2 = \frac{1}{4}(x - 0)$ or just $y - 2 = \frac{1}{4}x$.

At Q, $y = 0$ and so $x = -8$ and the coordinates of Q are $(-8, 0)$.

7 **a** $\frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}}$

If $x = 8$, $\frac{dy}{dx} = \frac{2}{3} \times 8^{-\frac{1}{3}} = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$.

The equation of the tangent is $y - 4 = \frac{1}{3}(x - 8)$.

$3y - 12 = x - 8$

$3y = x + 4$

b You need to find the coordinates of the point.

If $\frac{dy}{dx} = \frac{1}{6}$, $\frac{2}{3}x^{-\frac{1}{3}} = \frac{1}{6}$.

$4x^{-\frac{1}{3}} = 1$

$4 = x^{\frac{1}{3}}$

$x = 4^3 = 64$

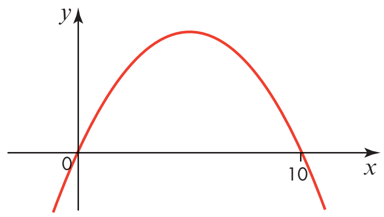
$y = 64^{\frac{2}{3}} = 4^2 = 16$

Equation of tangent is $y - 16 = \frac{1}{6}(x - 64)$.

$6y - 96 = x - 64$

$6y = x + 32$

8 **a** $y = x(10 - x)$ so it crosses the x-axis at 0 and 10.



b $\frac{dy}{dx} = 10 - 2x$

If $x = 3$, $\frac{dy}{dx} = 10 - 2 \times 3 = 4$.

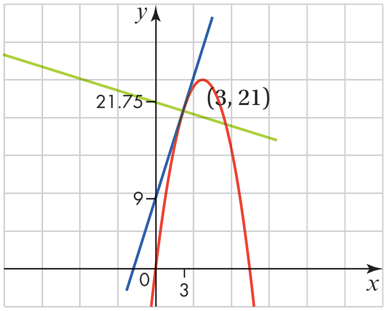
Equation of tangent is $y - 21 = 4(x - 3)$.

$$y - 21 = 4x - 12$$
$$y = 4x + 9$$

Equation of normal is $y - 21 = -\frac{1}{4}(x - 3)$.

$$4y - 84 = -x + 3$$
$$4y + x = 87$$

Sketch the tangent and the normal (the curve is not necessary).



The tangent meets the y-axis at 9. The normal meets the y-axis at $\frac{87}{4} = 21\frac{3}{4}$.

The area of the triangle is

$\frac{1}{2}\left(21\frac{3}{4} - 9\right) \times 3 = 19\frac{1}{8}$ or 19.125.

9 $y = a(1 + x^2)^{-1}$

Using the chain rule, $\frac{dy}{dx} = a \times -1(1 + x^2)^{-2} \times 2x$

$$= \frac{-2ax}{(1 + x^2)^2}.$$

If $x = 1$ then $y = \frac{a}{2}$ and $\frac{dy}{dx} = \frac{-2a}{2^2} = \frac{-a}{2}$.

The equation of the tangent is $y - \frac{a}{2} = -\frac{a}{2}(x - 1)$.

Where this meets the y-axis, $x = 0$ and then $y - \frac{a}{2} = \frac{a}{2}$ so $y = a$ and this is the point where the curve crosses the y-axis.

10 $f(x) = 100x^{-1}$ so $f'(x) = -100x^{-2} = -\frac{100}{x^2}$;

$f'(a) = -\frac{100}{a^2}$

The equation of the tangent is

$y - \frac{100}{a} = -\frac{100}{a^2}(x - a)$

Where it meets the y-axis, $x = 0$ and

$y - \frac{100}{a} = -\frac{100}{a^2} \times (-a) = \frac{100}{a}$ so $y = \frac{200}{a}$.

Where it meets the x-axis $y = 0$

and $-\frac{100}{a} = -\frac{100}{a^2} \times (x - a)$ so $a = x - a$ and $x = 2a$

The area of the triangle is $\frac{1}{2} \times \frac{200}{a} \times 2a = 200$.

11 **a** $\frac{dy}{dx} = 2x$; at P $\frac{dy}{dx} = 2a$ and the gradient of the tangent is $2a$.

The equation of the tangent is $y - a^2 = 2a(x - a)$; at the point T, $x = 0$ and $y - a^2 = -2a^2$; $y = -a^2$; so $OT = a^2$

A is the point $(0, a^2)$ and so $OA = a^2 = OT$.

b The gradient of the normal is $-\frac{1}{2a}$ and the equation is $y - a^2 = -\frac{1}{2a}(x - a)$

At N, $x = 0$ and $y - a^2 = -\frac{1}{2a} \times (-a)$; $y = a^2 + \frac{1}{2}$;

hence $ON = a^2 + \frac{1}{2}$ and $AN = ON - OA = \frac{1}{2}$.

12 The normal is $y = \frac{1}{2}x - \frac{3}{2}$ which has a gradient of $\frac{1}{2}$ so the gradient of the tangent at P is -2

If $y = x^2 + 4x$, then $\frac{dy}{dx} = 2x + 4$; if $2x + 4 = -2$, then $2x = -6$ and $x = -3$.

Then $y = 9 - 12 = -3$; P is $(-3, -3)$.

13 **a** If $x = 16$ then $y = \sqrt{16 + 9} = \sqrt{25} = 5$ so P is on the curve.

b $y = (x + 9)^{\frac{1}{2}}$ so $\frac{dy}{dx} = \frac{1}{2}(x + 9)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x + 9}}$; if $x = 16$, then $\frac{dy}{dx} = \frac{1}{10}$.

The equation of the tangent is $y - 5 = \frac{1}{10}(x - 16)$;

at A $y = 0$ and $-5 = \frac{1}{10}(x - 16)$; $-50 = x - 16$; $x = -34$ and A is $(-34, 0)$.

The gradient of the normal is -10; The equation of the normal is $y - 5 = -10(x - 16)$; at B $y = 0$ and $-5 = -10(x - 16)$; $\frac{1}{2} = x - 16$; $x = 16.5$ and A is $(16.5, 0)$.

The length of AB is $34 + 16.6 = 50.5$.

Exercise 6.7A

1 When the depth is h cm, the volume $V = \pi \times 30^2 \times h = 900\pi h \text{ cm}^3$.

$\frac{dV}{dt} = -400 \text{ cm}^3 \text{ s}^{-1}$ and $\frac{dV}{dh} = 900\pi$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ so $-400 = 900\pi \times \frac{dh}{dt}$

Therefore $\frac{dh}{dt} = -\frac{400}{900\pi} = -0.141$.

The depth is decreasing at a rate of 0.141 cm s^{-1}.

2 The area is $A = \pi r^2$ and $\frac{dA}{dr} = 2\pi r$.

$\frac{dr}{dt} = 0.3$ and $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 2\pi r \times 0.3 = 0.6\pi r$

a When $r = 5$, $\frac{dA}{dt} = 0.6\pi \times 5 = 9.42$ cm^2 s^{-1}.

b When $r = 10$, $\frac{dA}{dt} = 0.6\pi \times 10 = 18.8$ cm^2 s^{-1}.

c When $r = 15$, $\frac{dA}{dt} = 0.6\pi \times 15 = 28.3$ cm^2 s^{-1}.

3 If the radius is r then the surface area $A = 4\pi r^2$.

$\frac{dr}{dt} = 0.5$ and $\frac{dA}{dr} = 8\pi r$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 8\pi r \times 0.5 = 4\pi r$

When $r = 2.8$, $\frac{dA}{dt} = 4\pi \times 2.8 = 35.2$ cm^2 s^{-1}.

4 The volume $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dt} = 30$ and $\frac{dV}{dr} = 4\pi r^2$.

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ so $30 = 4\pi r^2 \times \frac{dr}{dt}$ and

$\frac{dr}{dt} = \frac{30}{4\pi r^2} = \frac{7.5}{\pi r^2}$.

a When $r = 4$, $\frac{dr}{dt} = \frac{7.5}{2\pi \times 16} = 0.0746$ cm s^{-1}.

b When $r = 8$, $\frac{dr}{dt} = \frac{7.5}{2\pi \times 64} = 0.0187$ cm s^{-1}.

5 When the depth is h the radius of the circle is $\frac{8}{20}h = 0.4h$.

The volume of water $V = \frac{1}{3}\pi \times (0.4h)^2 \times h = 0.1676h^3$ and $\frac{dV}{dh} = 0.5027h^2$.

$\frac{dV}{dt} = 25$ and $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ so $25 = 0.5027h^2 \times \frac{dh}{dt}$

and $\frac{dh}{dt} = \frac{49.74}{h^2}$.

When $h = 12$, $\frac{dh}{dt} = \frac{49.74}{12^2} = 0.345$ cm s^{-1}.

6 If the side of the cube is x cm then the volume $V = x^3$ and the surface area $A = 6x^2$.

$\frac{dV}{dt} = 50$, $\frac{dV}{dx} = 3x^2$ and $\frac{dA}{dx} = 12x$

$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$ so $50 = 3x^2 \times \frac{dx}{dt}$ and $\frac{dx}{dt} = \frac{50}{3x^2}$

The rate of increase of the surface area is

$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt} = 12x \times \frac{50}{3x^2} = \frac{200}{x}$.

When $x = 5$, $\frac{dA}{dt} = \frac{200}{5} = 40$ cm^2 s^{-1}.

7 $\frac{dx}{dt} = 0.2$ and the volume is $x^2h = 1000$ and

$\frac{dh}{dt} = \frac{dh}{dx} \times \frac{dx}{dt}$.

$h = \frac{1000}{x^2} = 1000x^{-2}$ so $\frac{dh}{dx} = -2000x^{-3} = -\frac{2000}{x^3}$.

Therefore $\frac{dh}{dt} = -\frac{2000}{x^3} \times 0.2 = -\frac{400}{x^3}$.

When the shape is a cube, $x = h$ and $x^3 = 1000$ so

$\frac{dh}{dt} = -\frac{400}{1000} = -0.4$.

The height is decreasing at a rate of 0.4 cm s^{-1}

8 **a** $u = 150w^{\frac{2}{3}}$ so $\frac{du}{dw} = 150 \times \frac{2}{3}w^{-\frac{1}{3}} = \frac{100}{\sqrt[3]{w}}$; if $w = 8$,

$\frac{du}{dw} = \frac{100}{\sqrt[3]{8}} = 50$

b $\frac{dw}{dt} = 0.05$

c $\frac{du}{dt} = \frac{du}{dw} \times \frac{dw}{dt} = \frac{100}{\sqrt[3]{w}} \times \frac{dw}{dt} = \frac{5}{\sqrt[3]{w}}$; when $w = 64$,

$\frac{du}{dt} = \frac{5}{\sqrt[3]{64}} = \frac{5}{4} = 1.25$

9 **a** $\frac{dV}{dA} \times \frac{dA}{dr} = \frac{dV}{dr}$; $\frac{dV}{dr} = 4\pi r^2$ and $\frac{dA}{dr} = 2\pi r$ so

$\frac{dV}{dA} \times 2\pi r = 4\pi r^2$; $\frac{dV}{dA} = 2r$

b $V = \frac{1000}{5+5} = 100$; the volume is 100 cm³

c $V = 1000(t+5)^{-1}$; $\frac{dV}{dt} = -1000(t+5)^{-2}$; when $t = 5$,

$\frac{dV}{dt} = -\frac{1000}{10^2} = 10$ cm³s⁻¹

10 **a** $10 + 0.4t^{\frac{1}{2}}$; $\frac{dp}{dt} = 0.2t^{-\frac{1}{2}} = \frac{0.2}{\sqrt{t}}$; when $t = 100$,

$\frac{dp}{dt} = \frac{0.2}{10} = 0.02$

b $\frac{dv}{dt} = \frac{dv}{dp} \times \frac{dp}{dt}$; $v = \frac{1200}{p}$ so $\frac{dv}{dp} = -\frac{1200}{p^2}$

When $t = 100$ $p = 10 + 0.4\sqrt{100} = 14$;

$\frac{dv}{dt} = \frac{dv}{dp} \times \frac{dp}{dt} = -\frac{1200}{14^2} \times 0.02 = -0.122$ to 3d.p.

Exam-style questions

1 **a** $y = 2x + 8x^{-1}$

$\frac{dy}{dx} = 2 - 8x^{-2} = 2 - \frac{8}{x^2}$

b If $y = 10$, then $2x + \frac{8}{x} = 10$; $2x^2 + 8 = 10x$;

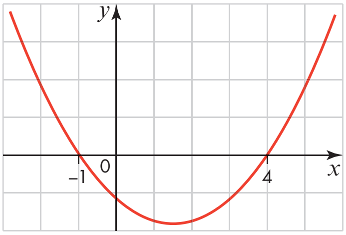
$2x^2 - 10x + 8 = 0$.

Divide by 2: $x^2 - 5x + 4 = 0$;
factorise: $(x - 1)(x - 4) = 0$.

So $x = 1$ or 4

If $x = 1$, then $\frac{dy}{dx} = 2 - 8 = -6$; if $x = 4$, then $\frac{dy}{dx} = 2 - 0.5 = 1.5$.

2 The gradient is 0 when $x = -1$ or 4.



3 **a** $y = 3x^2 - 3x + ax - a$

$\frac{dy}{dx} = 6x - 3 + a$

b $6x - 3 + a = a$

$6x = 3$

$x = \frac{1}{2}$

$y = \left(\frac{3}{2} + a\right)\left(-\frac{1}{2}\right)$

$= -\frac{3}{4} - \frac{a}{2}$

$\left(\frac{1}{2}, -\frac{3}{4} - \frac{a}{2}\right)$

4 **a** $y = x^{\frac{1}{2}}(x + 4) = x^{\frac{3}{2}} + 4x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}$

b At a stationary point $\frac{3}{2}x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} = 0$.

Multiply by $x^{\frac{1}{2}}$: $\frac{3}{2}x + 2 = 0$; $\frac{3}{2}x = -2$;

$x = -\frac{4}{3}$ or $-1\frac{1}{3}$.

5 **a** $y = (a + bx^2)^{\frac{1}{2}}$ and

$\frac{dy}{dx} = \frac{1}{2}(a + bx^2)^{-\frac{1}{2}} \times 2bx = \frac{bx}{(a + bx^2)^{\frac{1}{2}}}$

b Since $y = (a + bx^2)^{\frac{1}{2}}$, $\frac{x}{y} = \frac{x}{(a + bx^2)^{\frac{1}{2}}} = \frac{1}{b} \times \frac{dy}{dx}$.

6 **a** $y = x^4 - px^2$; $\frac{dy}{dx} = 4x^3 - 2px$

b If $p = 8$, then $\frac{dy}{dx} = 4x^3 - 16x$

At a stationary point $4x^3 - 16x = 0$.

Divide by 4 and factorise.

$x(x^2 - 4) = 0$

$x(x - 2)(x + 2) = 0$

$x = 0$, 2 or -2

If $x = 0$, $y = 0$.

If $x = 2$, $y = 2^4 - 8 \times 2^2 = -16$.

If $x = -2$, $y = (-2^4) - 8 \times (-2^2) = -16$.

Stationary points are (0, 0), (2, –16) and (–2, –16).

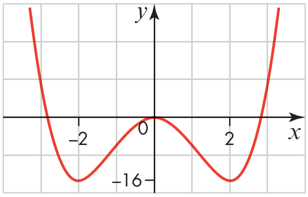
c $\frac{d^2y}{dx^2} = 12x^2 - 16$

If $x = 2$, then $\frac{d^2y}{dx^2} = 48 - 16 = 32 > 0$ so (2, –16) is a minimum point.

If $x = -2$, then $\frac{d^2y}{dx^2} = 32 > 0$ so (–2, –16) is a minimum point.

If $x = 0$, then $\frac{d^2y}{dx^2} = 0$ and this does not determine the type of stationary point at (0, 0).
However the other two stationary points are below (0, 0) so it is a maximum point.

d



7 $\frac{dy}{dx} = 6x^2 - 12x - 12$

If the gradient is 6, then $6x^2 - 12x - 12 = 6$.

$x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x = 3$ or -1

If $x = 3$, $y = 2 \times 27 - 6 \times 9 - 12 \times 3 + 4 = -32$.

If $x = -1$, $y = -2 - 6 + 12 + 4 = 8$.

The points are $(3, -32)$ and $(-1, 8)$.

8 a $\frac{dh}{dt} = 0.4t - 0.03t^2$

When $t = 10$, $\frac{dh}{dt} = 4 - 3 = 1$.

Speed is $1\,\mathrm{m\,s^{-1}}$ upwards.

b At the highest point, $\frac{dh}{dt} = 0$.

$0.4t - 0.03t^2 = 0$

$t(0.4 - 0.03t) = 0$

Either $t = 0$, which is when the drone starts, or $0.4 - 0.03t = 0$.

$t = \frac{0.4}{0.03} = 13\frac{1}{3}$

This is $\frac{2}{3}$ of 20 seconds, showing the ascent is twice as long as the descent.

9 a $y = x^2 + 250x^{-1}$

$\frac{dy}{dx} = 2x - 250x^{-2}$

b At a stationary point, $2x - 250x^{-2} = 0$.

$2x^3 = 250$

$x^3 = 125$

$x = 5$

$\frac{d^2y}{dx^2} = 2 + 500x^{-3}$

If $x = 5$, $\frac{d^2y}{dx^2} = 2 + 500 \times 5^{-3} = 6 > 0$ so this is a minimum point.

If $x = 5$, $y = 5^2 + \frac{250}{5} = 75$.

The minimum is at $(5, 75)$.

10 a $y = px^3 - 4px^2 + 5x - 11$ so $\frac{dy}{dx} = 3px^2 - 8px + 5$

At a stationary point $3px^2 - 8px + 5 = 0$.
This quadratic in x has two distinct solutions if $b^2 - 4ac > 0$.
That is $(-8p)^2 - 4 \times 3p \times 5 > 0$; $64p^2 - 60p > 0$;
$16p^2 - 15p > 0$; $p(16p - 15) > 0$

The roots of the equation $p(16p - 15) = 0$ are
$p = 0$ or $\frac{15}{16}$
There are two distinct stationary points if
$p < 0$ or $p > \frac{15}{16}$

b If $p = 1$ the stationary points are given by
$3x^2 - 8x + 5 = 0$.
$x = 1$ or $\frac{5}{3}$.
If $x = 1$, $y = -9$ and
if $x = \frac{5}{3}$, $y = -\frac{247}{27}$ or -9.148.
The coordinates of the stationary points are
$(1, -9)$ and $\left(\frac{5}{3}, -\frac{247}{27}\right)$

11 a $f(x) = (5x + 4)^{\frac{1}{2}}$ and so using the chain rule

$f'(x) = \frac{1}{2} \times (5x + 4)^{-\frac{1}{2}} \times 5 = \frac{5}{2\sqrt{5x + 4}}$.

If $x \geqslant 0$, then $\sqrt{5x + 4}$ is always positive. So is $f'(x)$ and this means the function is increasing.

b When $x = 1$, $f'(x) = \frac{5}{2 \times 3} = \frac{5}{6}$ The gradient of the tangent is $\frac{5}{6}$.
Hence the gradient of the normal is $-\frac{6}{5}$ and the equation is $y - 3 = -\frac{6}{5}(x - 1)$ which can be rearranged as $5y - 15 = -6x + 6$ or $5y + 6x = 21$.

c $f'(x) = \frac{5}{2\sqrt{5x + 4}}$ so $f'(1) = \frac{5}{6}$ and the tangent at $(1, 3)$ is $y - 3 = \frac{5}{6}(x - 1)$.

Where it meets the y-axis, $x = 0$ and $y - 3 = -\frac{5}{6}$,

$y = \frac{13}{6}$.

Where it meets the x-axis, $y = 0$ and
$-3 = \frac{5}{6}(x - 1)$, $x - 1 = -\frac{18}{5}$; $x = -\frac{13}{5}$.

The area of the triangle is $\frac{1}{2} \times \frac{13}{6} \times \frac{13}{5} = \frac{169}{60}$ or 2.82 to 3 s.f.

12 a The other side is $= 240 - 2x$.
The area $A = x(240 - 2x)$.

b $A = 240x - 2x^2$

$\frac{dA}{dx} = 240 - 4x$

For maximum area, $\frac{dA}{dx} = 240 - 4x = 0$.
$x = 60$

$\frac{d^2A}{dx^2} = -4 < 0$ so this will be a maximum.

Maximum area $= 60 \times 120 = 7200\,\mathrm{m}^2$.

13 a $\frac{dy}{dx} = 2x + 4$

If $x = 2$, $\frac{dy}{dx} = 2 \times 2 + 4 = 8$.

The equation of the tangent is $y - 7 = 8(x - 2)$; $y = 8x - 9$.

b $\frac{dy}{dt} = 2.4$; $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$ so at A,

$\frac{dx}{dt} = \frac{dy}{dt} \div \frac{dy}{dx} = 2.4 \div 8 = 0.3$ units s^{-1}.

c The gradient is $\frac{dy}{dx}$ so call this z.

Then the rate of change of the gradient is

$\frac{dz}{dt} = \frac{dz}{dx} \times \frac{dx}{dt}$.

But $\frac{dz}{dx} = \frac{d^2y}{dx^2} = 2$ and so $\frac{dz}{dt} = 2 \times 0.3 = 0.6$ units s^{-1}.

14 a Volume of a cone, $V = \frac{1}{3}\pi r^2 h$.

$h = 60 - r$ so $V = \frac{1}{3}\pi r^2(60 - r)$.

b $V = \frac{1}{3}\pi(60r^2 - r^3)$

$\frac{dV}{dr} = \frac{1}{3}\pi(120r - 3r^2)$

When the volume is a maximum, $\frac{dV}{dr} = 0$.

$\frac{1}{3}\pi(120r - 3r^2) = 0$

$120r - 3r^2 = 0$

Divide by 3 and factorise.

$r(40 - r) = 0$

$r = 0$ or 40

$\frac{d^2V}{dr^2} = \frac{1}{3}\pi(120 - 6r)$

When $r = 40$, $\frac{d^2V}{dr^2} = \frac{1}{3}\pi(120 - 240) < 0$ so this is a maximum point.

c The maximum volume is $\frac{1}{3}\pi 40^2(60 - 40)$

$= \frac{32\,000\pi}{3}$ cm^3.

15 $y = (x + 1)(x - p) = x^2 + x - px - p$ and hence

$\frac{dy}{dx} = 2x + 1 - p$.

At $(p, 0)$, $x = p$ and $\frac{dy}{dx} = 2p + 1 - p = p + 1$.

The gradient of the normal is $-\frac{1}{p+1}$ and the equation is $y = -\frac{1}{p+1}(x - p)$.

If $x = 0$, $y = \frac{p}{p+1}$ and this is the intercept on the y-axis.

If $y = 0$, $x = p$ and this is the intercept on the x-axis.

The area of the triangle is $\frac{1}{2} \times \frac{p}{p+1} \times p = \frac{p^2}{2(p+1)}$.

16 When the radius is r, the volume $V = \frac{4}{3}\pi r^3$ and the surface area $A = 4\pi r^2$.

$\frac{dV}{dr} = 4\pi r^2$, $\frac{dA}{dr} = 8\pi r$ and $\frac{dV}{dt} = 36$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ so $36 = 4\pi r^2 \times \frac{dr}{dt}$ and $\frac{dr}{dt} = \frac{9}{\pi r^2}$

Then $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 8\pi r \times \frac{9}{\pi r^2} = \frac{72}{r}$.

When $V = 2000$, $\frac{4}{3}\pi r^3 = 2000$ and $r^3 = \frac{1500}{\pi}$ and $r = 7.816$.

Then $\frac{dA}{dt} = \frac{72}{7.816} = 9.21$.

The area is increasing at a rate of 9.21 cm^2 s^{-1}.

17 a $f(x) = 10x - x^2$; $f(3) = 21$; $f(3.1) = 21.39$; $f(3.5) = 22.75$; $f(4) = 24$

The gradient of AD is $\frac{24 - 21}{1} = 3$.

The gradient of AC is $\frac{22.75 - 21}{0.5} = 3.5$.

The gradient of AB is $\frac{21.39 - 21}{0.1} = 3.9$.

b The answers suggest that $f'(3)$ is greater than 3.9 and could be 4.

18 a Where the curve crosses the x-axis, $x(x + 2)(x - 5) = 0$ so $x = 0, -2$ or 5; at P, $x = -2$.

$y = x^3 - 3x^2 - 10x$ and so $\frac{dy}{dx} = 3x^2 - 6x - 10$; if $x = -2$, then $\frac{dy}{dx} = 12 + 12 - 10 = 14$.

The gradient at P is 14.

b Where the gradient is 14, $3x^2 - 6x - 10 = 14$; $3x^2 - 6x - 24 = 0$; $x^2 - 2x - 8 = 0$;

$(x - 4)(x + 2) = 0$; $x = 4$ or -2. Q is the point where $x = 4$ and then $y = 4 \times 6 \times -1 = -24$

Q is $(4, -24)$.

19 a The height of the box is x cm, the length is $30 - 2x$ cm and the width is $20 - 2x$ cm.

Multiply these three to get $v = x(30 - 2x)(20 - 2x)$.

b $v = x(600 - 60x - 40x + 4x^2) = 600x - 100x^2 + 4x^3$

$\frac{dv}{dx} = 600 - 200x + 12x^2$; where the volume has a maximum value, $\frac{dv}{dx} = 0$;

$600 - 200x + 12x^2 = 0$; divide by 4: $3x^2 - 50x + 150 = 0$

$$x = \frac{50 \pm \sqrt{2500 - 1800}}{6} = \frac{50 \pm \sqrt{700}}{6};$$

$x = 12.74$ or 3.924. The first value is impossible; it would give a negative volume. The volume is a maximum when $x = 3.92$.

$$\frac{d^2v}{dx^2} = -200 + 24x = -105.8 < 0 \text{ when } x = 3.92$$

which confirms that the volume is a maximum.

20 $f(x) = \frac{a}{x}$; $f'(x) = -\frac{a}{x^2}$; At P, $f'(p) = -\frac{a}{p^2}$ and the tangent is $y - \frac{a}{p} = -\frac{a}{p^2}(x - p)$.

At Q, $y = 0$ and $-\frac{a}{p} = -\frac{a}{p^2}(x - p)$; $p = x - p$; $x = 2p$.

At R, $x = 0$ and $y - \frac{a}{p} = -\frac{a}{p^2} \times (-p)$; $y - \frac{a}{p} = \frac{a}{p}$; $y = \frac{2a}{p}$.

The area of triangle $OQR = \frac{1}{2} \times 2p \times \frac{2a}{p} = 2a$.

A is $(p, 0)$ so $OA = p$; B is $\left(0, \frac{a}{p}\right)$ so $OB = \frac{a}{p}$; the area of $OAPB$ is $p \times \frac{a}{p} = a$. This proves the result.

Mathematics in life and work

1 **a** Volume $= x^2h = 1000$

Rearrange: $h = \frac{1000}{x^2}$

b The total surface area is $2x^2 + 4xh$

$= 2x^2 + 4x \times \frac{1000}{x^2} = 2x^2 + \frac{4000}{x}$.

c If $a = 2x^2 + \frac{4000}{x} = 2x^2 + 4000x^{-1}$, then

$$\frac{da}{dx} = 4x - 4000x^{-2} = 4x - \frac{4000}{x^2}.$$

When the surface area has a minimum value, $\frac{da}{dx} = 0$.

$$4x - \frac{4000}{x^2} = 0$$

$$x^3 = 1000$$

$$x = 10$$

$$\frac{d^2a}{dx^2} = 4 - 8000x^{-3}$$

When $x = 10$, $\frac{d^2a}{dx^2} = 4 + \frac{8000}{10^3} = 12$ which is positive, so the surface area has a minimum value. In fact, in this case the cuboid is a cube.

2 If the side of the triangle is x cm the area is $\frac{1}{2}x^2 \sin 60° = 0.433x^2$.

Then the volume is $0.433x^2l = 1000$ and so $l = \frac{2309}{x^2}$.

The surface area is $A = 2 \times 0.433x^2 + 3xl$

$= 0.866x^2 + \frac{6928}{x}$.

Then $\frac{dA}{dx} = 1.732x - \frac{6928}{x^2}$, and if $\frac{dA}{dx} = 0$, then

$$x^3 = \frac{6928}{1.732} = 4000.$$

Hence $x = \sqrt[3]{4000} = 15.9$.

The length is $l = \frac{2309}{15.9 \times 15.9} = 9.2$ cm.

7 Integration

Please note: Full worked solutions are provided as an aid to learning, and represent one approach to answering the question. In some cases, alternative methods are shown for contrast.

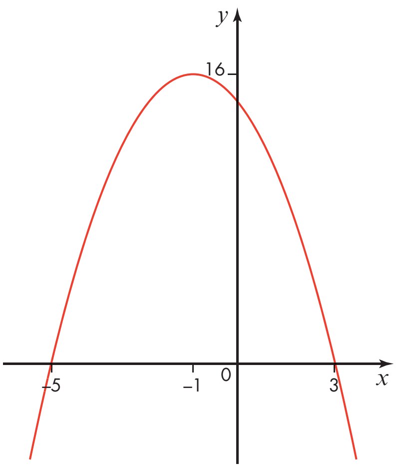
All sample answers have been written by the authors. Cambridge Assessment International Education bears no responsibility for the example answers to questions taken from its past question papers, which are contained in this publication.

Non-exact numerical answers should be given correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

Prerequisite knowledge

1 $4x^{-\frac{1}{2}}$

2 The graph crosses the x-axis at 3 and –5.



3 **a** $12x^2 - 6$

b $y = \frac{1}{2}x + 2x^{-1}$

$\frac{dy}{dx} = \frac{1}{2} - 2x^{-2}$ or $\frac{1}{2} - \frac{2}{x^2}$

c $y = 6x^{\frac{1}{3}}$

$\frac{dy}{dx} = 2x^{-\frac{2}{3}}$

Exercise 7.1A

1 **a** $\frac{6}{2}x^2 + c = 3x^2 + c$ **b** $2x^2 + 2x + c$

c $\frac{7}{2}x^2 - 5x + c$ **d** $3x - 2x^2 + c$

2 **a** $\frac{2}{5}x^5 + c$ **b** $\frac{2}{-3}x^{-3} = -\frac{2}{3}x^{-3} + c$

c $\frac{5}{4}x^4 + c$ **d** $\frac{5}{-2}x^{-2} + c = -\frac{5}{2}x^{-2} + c$

3 **a** $2x^3 - 2x^2 + c$ **b** $\frac{2}{5}x^5 - \frac{5}{2}x^2 + 10x + c$

c $4x^2 - \frac{10}{3}x^3 + c$

4 **a** $\int x^{\frac{1}{2}} dx = \frac{2}{3}x^{\frac{3}{2}} + c$ **b** $2 \times \frac{2}{5}x^{\frac{5}{2}} + c = \frac{4}{5}x^{\frac{5}{2}} + c$

c $3 \times 2x^{\frac{1}{2}} + c = 6x^{\frac{1}{2}} + c$

d $\int 10x^{-\frac{1}{2}} dx = 20x^{\frac{1}{2}} + c$ or $20\sqrt{x} + c$

5 **a** $\int (x^2 - 4x)\ dx = \frac{1}{3}x^3 - 2x^2 + c$

b $\int (x^3 - 8x^2)\ dx = \frac{1}{4}x^4 - \frac{8}{3}x^3 + c$

c $\int 10x^{-3}\ dx = \frac{10}{-2}x^{-2} + c = -5x^{-2} + c$ or $-\frac{5}{x^2} + c$

6 **a** $\int 4x^{\frac{1}{2}}\ dx = 4 \times \frac{2}{3}x^{\frac{3}{2}} + c = \frac{8}{3}x^{\frac{3}{2}} + c$

b $\int \sqrt{4x}\ dx = \int 2\sqrt{x}\ dx = 2 \times \frac{2}{3}x^{\frac{3}{2}} + c = \frac{4}{3}x^{\frac{3}{2}} + c$

c $\int 4x^{-\frac{1}{2}}\ dx = 4 \times 2x^{\frac{1}{2}} + c = 8x^{\frac{1}{2}} + c$ or $8\sqrt{x} + c$

7 $f(x) = \int (2x^3 - 2x)\, dx = \frac{2}{4}x^4 - x^2 + c = \frac{1}{2}x^4 - x^2 + c$

8 **a** You cannot integrate a product by integrating each term separately and multiplying the results.

b First multiply out the brackets:

$\int (2x^2 - x - 1)\, dx = \frac{2}{3}x^3 - \frac{1}{2}x^2 - x + c$

9 **a** $\int (4 - 2x^{-3})\, dx = 4x + x^{-2} + c$

b $\int (2x^{\frac{1}{2}} + 8x^{-\frac{1}{2}})\, dx = 2 \times \frac{2}{3}x^{\frac{3}{2}} + 16x^{\frac{1}{2}} + c$

$= \frac{4}{3}x^{\frac{3}{2}} + 16x^{\frac{1}{2}} + c$

c $\int (4x^{-2} + 2x^{-\frac{3}{2}})\ dx = -4x^{-1} - 4x^{-\frac{1}{2}} + c$

10 The derivatives of f(x) and g(x) are the same so the derivative of f(x) – g(x) is 0.
This means that f(x) – g(x) is $\int 0\ dx$ = a constant.

11 a $\int (x-1)^2\,dx = \int (x^2-2x+1) = \frac{1}{3}x^3 - x^2 + x + c$

b $\frac{1}{3}(x-1)^3 + c = \frac{1}{3}\left(x^3 - 3x^2 + 3x - 1\right) + c$

$= \frac{1}{3}x^3 - x^2 + x - \frac{1}{3} + c$

12 $\int (x+1)(x-3)(x+5)\,dx = \int \left(x^2-2x-3\right)(x+5)\,dx$

$= \int \left(x^3+3x^2-13x-15\right)dx$

$= \frac{1}{4}x^4 + x^3 - \frac{13}{2}x^2 - 15x + c$

13 a $\int \frac{x^2-1}{x-1}\,dx = \int \frac{(x+1)(x-1)}{x-1}\,dx$

$= \int (x+1)\,dx = \frac{1}{2}x^2 + x + c$

b $\int \frac{4-x^2}{2+x}\,dx = \int \frac{(2-x)(2+x)}{2+x}\,dx$

$= \int (2-x)\,dx \;=\; 2x - \frac{1}{2}x^2 + \mathrm{c}$

14 a $f'(x) = 3(x^2+1)^2 \times 2x = 6x(x^2+1)^2$

b $\int 6x\left(x^2+1\right)^2 dx = \left(x^2+1\right)^3 + c$ and hence

$\int x\left(x^2+1\right)^2 dx = \frac{1}{6}\left(x^2+1\right)^3 + \frac{1}{6}c$ or just

$\frac{1}{6}\left(x^2+1\right)^3 + c$, including the $\frac{1}{6}$ in the arbitrary constant.

c $\frac{1}{12}(x^2+1)^6 + c$

15 a $\int \frac{x^2+1}{x^2}\,dx = \int \left(1 + \frac{1}{x^2}\right)dx = \int \left(1 + x^{-2}\right)dx$

$= x + \frac{1}{-1}x^{-1} + c$

$= x - \frac{1}{x} + c$

b $\int \left(\frac{x^2+1}{x^2}\right)^2 dx = \int \left(1 + \frac{1}{x^2}\right)^2 dx$

$= \int \left(1 + 2x^{-2} + x^{-4}\right)dx = x + \frac{2}{-1}x^{-1} + \frac{1}{-3}x^{-3} + c$

$= x - \frac{2}{x} - \frac{1}{3x^3} + c$

16 a $y = \left(x^2+a^2\right)^{\frac{1}{2}}$; using the chain rule for differentiation, $\frac{dy}{dx} = \frac{1}{2}\left(x^2+a^2\right)^{-\frac{1}{2}} \times 2x$

$= \frac{x}{\sqrt{x^2+a^2}}$.

b $\int \frac{x}{\sqrt{x^2+a^2}}\,dx = \sqrt{x^2+a^2} + c$

17 $\int (x+2)^2\,dx = \int (x^2+4x+4)\,dx = \frac{1}{3}x^3 + 2x^2 + 4x + c$

so Ari is correct.

However, $(x+2)^3 = x^3 + 3x^2 \times 2 + 3x \times 2^2 + 2^3$

$= x^3 + 6x^2 + 12x + 8$

Hence $\frac{1}{3}(x+2)^3 = \frac{1}{3}x^3 + 2x^2 + 4x + \frac{8}{3}$ and this is the same as Ari's answer except for the constant. This means that $\frac{1}{3}(x+2)^3 + c$ is the same solution written in a different way. Both students are correct.

18 a $\int x \times x^{\frac{1}{2}}dx = \int x^{\frac{3}{2}}dx = \frac{2}{5}x^{\frac{5}{2}} + c$

b $\int \frac{x^2-1}{x^{\frac{1}{3}}}dx = \int \left(x^{\frac{5}{3}} - x^{-\frac{1}{3}}\right)dx = \frac{3}{8}x^{\frac{8}{3}} - \frac{3}{2}x^{\frac{2}{3}} + c$

Exercise 7.2A

1 a If $y = (x+7)^3$, then $\frac{dy}{dx} = 3 \times (x+7)^2 \times 1 = 3(x+7)^2$.

Hence $\int (x+7)^2\,dx = \frac{1}{3}(x+7)^3 + c$.

b If $y = (x+7)^4$, then $\frac{dy}{dx} = 4 \times (x+7)^3 \times 1 = 4(x+7)^3$.

Hence $\int (x+7)^3\,dx = \frac{1}{4}(x+7)^4 + c$.

c If $y = (x+7)^6$, then $\frac{dy}{dx} = 6 \times (x+7)^5 \times 1 = 6(x+7)^5$.

Hence $\int (x+7)^5\,dx = \frac{1}{6}(x+7)^6 + c$.

2 a If $y = (2x-3)^3$, then $\frac{dy}{dx} = 3(2x-3)^2 \times 2 = 6(2x-3)^2$.

Hence $\int (2x-3)^2\,dx = \frac{1}{6}(2x-3)^3 + c$.

b If $y = (6x+1)^4$, then $\frac{dy}{dx} = 4(6x+1)^3 \times 6 = 24(6x+1)^3$.

Hence $\int (6x+1)^3\,dx = \frac{1}{24}(6x+1)^4 + c$.

c If $y = (0.5x-4)^5$, then $\frac{dy}{dx} = 5(0.5x-4)^4 \times 0.5$

$= 2.5(0.5x-4)^4$.

Hence $\int (0.5x-4)^4\,dx = \frac{1}{2.5}(0.5x-4)^5 + c$

$= 0.4(0.5x-4)^5 + c$.

3 a $y = (10x+1)^{-1}$ thus $\frac{dy}{dx} = -1 \times (10x+1)^{-2} \times 10$

$= \frac{-10}{(10x+1)^2}$.

b $\int \frac{3}{(10x+1)^2}\,dx = \frac{3}{-10}(10x+1)^{-1} + c = -\frac{0.3}{10x+1} + c$

4 a $\int \sqrt{x+1}\,dx = \int (x+1)^{\frac{1}{2}}\,dx$

If $y = (x+1)^{\frac{3}{2}}$, then

$$\frac{dy}{dx} = \frac{3}{2} \times (x+1)^{\frac{1}{2}} \times 1 = \frac{3}{2}(x+1)^{\frac{1}{2}}$$

Hence $\int \sqrt{x+1}\ dx = \frac{2}{3}(x+1)^{\frac{3}{2}} + c$.

b $\int \sqrt{2x+1}\ dx = \int (2x+1)^{\frac{1}{2}}\ dx$

If $y = (2x+1)^{\frac{3}{2}}$, then

$$\frac{dy}{dx} = \frac{3}{2} \times (2x+1)^{\frac{1}{2}} \times 2 = 3(2x+1)^{\frac{1}{2}}.$$

Hence $\int \sqrt{2x+1}\ dx = \frac{1}{3}(2x+1)^{\frac{3}{2}} + c$.

c $\int 3\sqrt{4x-2}\ dx = \int 3(4x-2)^{\frac{1}{2}}\ dx$

If $y = (4x-2)^{\frac{3}{2}}$, then

$$\frac{dy}{dx} = \frac{3}{2} \times (4x-2)^{\frac{1}{2}} \times 4 = 6(4x-2)^{\frac{1}{2}}.$$

Hence $\int 3\sqrt{4x-2}\ dx = 3 \times \frac{1}{6}(4x-2)^{\frac{3}{2}} + c$

$$= \frac{1}{2}(4x-2)^{\frac{3}{2}} + c.$$

5 a $\int \sqrt[3]{0.6x+5}\ dx = \int (0.6x+5)^{\frac{1}{3}}\ dx$

If $y = (0.6x+5)^{\frac{4}{3}}$, then

$$\frac{dy}{dx} = \frac{4}{3} \times (0.6x+5)^{\frac{1}{3}} \times 0.6 = 0.8(0.6x+5)^{\frac{1}{3}}.$$

Hence $\int \sqrt[3]{0.6x+5}\ dx = \frac{1}{0.8}(0.6x+5)^{\frac{4}{3}} + c$

$$= 1.25(0.6x+5)^{\frac{4}{3}} + c.$$

b $\int \frac{1}{\sqrt[3]{0.6x+5}} dx = \int (0.6x+5)^{-\frac{1}{3}}\ dx$

$$-\frac{1}{3} + 1 = \frac{2}{3}$$

If $y = (0.6x+5)^{\frac{2}{3}}$, then

$$\frac{dy}{dx} = \frac{2}{3} \times (0.6x+5)^{-\frac{1}{3}} \times 0.6 = 0.4(0.6x+5)^{-\frac{1}{3}}.$$

Hence $\int \frac{1}{\sqrt[3]{0.6x+5}} dx = \frac{1}{0.4}(0.6x+5)^{\frac{2}{3}} + c$

$$= 2.5(0.6x+5)^{\frac{2}{3}} + c.$$

6 a $\int (x+2)^5 dx = \frac{1}{6}(x+2)^6 + c$

b $\int (x+2)^6 dx = \frac{1}{7}(x+2)^7 + c$

c $(x+2)^6 = (x+2)\ (x+2)^5 = x\,(x+2)^5 + 2\,(x+2)^5$

Hence $x(x+2)^5 \equiv (x+2)^6 - 2(x+2)^5$.

d $\int x(x+2)^5 dx = \int \left\{(x+2)^6 - 2(x+2)^5\right\} dx$

$$= \int (x+2)^6 dx - 2\int (x+2)^5 dx$$

$$= \frac{1}{7}(x+2)^7 - 2 \times \frac{1}{6}(x+2)^6 + c$$

$$= \frac{1}{7}(x+2)^7 - \frac{1}{3}(x+2)^6 + c$$

7 a $\frac{x}{(x+1)^3} = \frac{x+1-1}{(x+1)^3} = \frac{x+1}{(x+1)^3} - \frac{1}{(x+1)^3}$

$$= \frac{1}{(x+1)^2} - \frac{1}{(x+1)^3}$$

b $\int \frac{x}{(x+1)^3} dx = \int \frac{1}{(x+1)^2} dx - \int \frac{1}{(x+1)^3} dx$

$$= \int (x+1)^{-2} dx - \int (x+1)^{-3} dx$$

$$= \frac{1}{-1}(x+1)^{-1} - \frac{1}{-2}(x+1)^{-2} + c$$

$$= -\frac{1}{x+1} + \frac{1}{2(x+1)^2} + c$$

8 a $\int \sqrt{10x+5}\ dx = \int (10x+5)^{\frac{1}{2}} dx$

$$= \frac{1}{10} \times \frac{2}{3}(10x+5)^{\frac{3}{2}} + c = \frac{1}{15}(10x+5)^{\frac{3}{2}} + c$$

b $\int \frac{1}{\sqrt{10x+5}}\ dx = \int (10x+5)^{-\frac{1}{2}} dx$

$$= \frac{1}{10} \times 2(10x+5)^{\frac{1}{2}} + c$$

$$= \frac{1}{5}\sqrt{10x+5} + c$$

c $\frac{x}{\sqrt{10x+5}} = \frac{\frac{1}{10}(10x+5) - \frac{1}{2}}{\sqrt{10x+5}}$

$$= \frac{1}{10}\sqrt{10x+5} - \frac{1}{2} \times \frac{1}{\sqrt{10x+5}}$$

Hence

$$\int \frac{x}{\sqrt{10x+5}} dx = \frac{1}{10}\int \sqrt{10x+5}\ dx - \frac{1}{2}\int \frac{1}{\sqrt{10x+5}} dx$$

$$= \frac{1}{10} \times \frac{1}{15}(10x+5)^{\frac{3}{2}} - \frac{1}{2} \times \frac{1}{5}\sqrt{10x+5} + c$$

$$= \frac{1}{150}(10x+5)^{\frac{3}{2}} - \frac{1}{10}\sqrt{10x+5} + c$$

9 $f'(x) = \int (ax+b)^3\ dx = \frac{1}{4a}(ax+b)^4 + c$

where c is a constant.

$$f(x) = \int f'(x)\,dx = \int \frac{1}{4a}(ax+b)^4 + c$$

$$= \frac{1}{4a} \times \frac{1}{5a}(ax+b)^5 + cx + d$$

$= \frac{1}{20a^2}(ax+b)^5 + cx + d$, where d is a constant.

There are two arbitrary constants.

10 a $\int\left(\sqrt[3]{2x+5}\right)^2 dx = \int(2x+5)^{\frac{2}{3}} dx = \frac{1}{2}\times\frac{3}{5}(2x+5)^{\frac{5}{3}} + c$

$= \frac{3}{10}(2x+5)^{\frac{5}{3}} + c$

b $\int\sqrt[3]{2x+5}\,dx = \int(2x+5)^{\frac{1}{3}} dx$

$= \frac{1}{2}\times\frac{3}{4}(2x+5)^{\frac{4}{3}} + c = \frac{3}{8}\left(\sqrt[3]{2x+5}\right)^4 + c$

$= \frac{3}{8}y^4 + c$

Exercise 7.3A

1 a $y = \int(4x-2)\,dx$

$= 2x^2 - 2x + c$

b When $x = 0$, $y = 3$.

$3 = 2\times 0^2 - 2\times 0 + c$

$= c$

The equation is $y = 2x^2 - 2x + 3$.

2 a $y = \int x^{\frac{1}{2}} dx$

$= \frac{2}{3}x^{\frac{3}{2}} + c$

b If $x = 9$, $y = 25$

$25 = \frac{2}{3}\times 9^{\frac{3}{2}} + c$

$= \frac{2}{3}\times 27 + c$

$= 18 + c$

$c = 7$

The equation is $y = \frac{2}{3}x^{\frac{3}{2}} + 7$.

3 $y = \int(0.4x+3)\,dx$

$= 0.2x^2 + 3x + c$

a $0 = 0 + c$

Hence the equation is $y = 0.2x^2 + 3x$.

b $5 = c$

$y = 0.2x^2 + 3x + 5$

c $0 = 0.2\times 5^2 + 3\times 5 + c$

$c = -20$

$y = 0.2x^2 + 3x - 20$

4 $\int\frac{x^2+10}{x^2}\,dx = \int 1 + 10x^{-2}\,dx$

$= x - 10x^{-1} + c$

$2 = 5 - \frac{10}{5} + c$

$c = -1$

The equation is $y = x - \frac{10}{x} - 1$.

5 a If $x \neq 0$, then x^2 is positive. This means that $\frac{2}{x^2}$ is always positive.

b $y = \int\frac{2}{x^2}\,dx = \int 2x^{-2}\,dx$

$= -2x^{-1} + c$

$4 = -1 + c$

$c = 5$

The equation is $y = -\frac{2}{x} + 5 = \frac{5x-2}{x}$.

6 a $y = \int(3x^2-3)\,dx$

$= x^3 - 3x + c$

$2 = c$

The curve is $y = x^3 - 3x + 2$.

b The turning points are where $3x^2 - 3 = 0$

$x^2 = 1$.

$x = 1$ or -1

There are 2 stationary points.

If $x = 1$, $y = 1 - 3 + 2 = 0$ so one t.p. is $(1, 0)$.

If $x = -1$, $y = -1 + 3 + 2 = 4$ so the other t.p. is $(-1, 4)$.

7 a $f(x) = \int\frac{20}{x^3}\,dx = \int 20x^{-3}\,dx = -10x^{-2} + c$

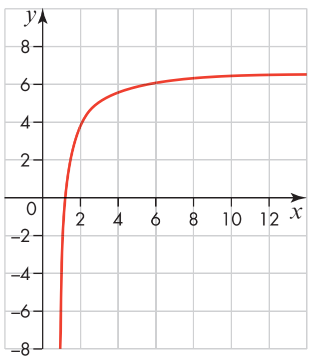
$f(2) = 4$

$4 = -\frac{10}{4} + c$

$c = 4 + 2.5 = 6.5$

$f(x) = -10x^{-2} + 6.5$ or $f(x) = 6.5 - \frac{10}{x^2}$

b



8 **a** $\int 0.03x^2(x-10)^2\,dx = \int 0.03x^2\left(x^2-20x+100\right)dx$

$= \int(0.03x^4 - 0.6x^3 + 3x^2)\,dx$

$= 0.006x^5 - 0.15x^4 + x^3 + c$

When $x = 0$, $y = 100$, so $c = 100$.

$y = 0.006x^5 - 0.15x^4 + x^3 + 100$

b After 10 seconds, $x = 10$ and

$y = 0.006 \times 10^5 - 0.15 \times 10^4 + 10^3 + 100$

$= 600 - 1500 + 1000 + 100 = 200$

The distance from A is 200 m.

9 $y = \int(3x^2 - 12x + 8)\,dx = x^3 - 6x^2 + 8x + c$

When $x = 0$, $y = 0$ so $c = 0$.

$y = x^3 - 6x^2 + 8x$

Where the curve crosses the x-axis, $x^3 - 6x^2 + 8x = 0$.

$x(x^2 - 6x + 8) = 0$;

$x(x-2)(x-4) = 0$

$x = 0$, 2 or 4.

The curve crosses the axes at (0, 0), (2, 0) and (4, 0).

10 **a** $y = \int 4x^{-\frac{2}{3}}dx = 4 \times 3x^{\frac{1}{3}} + c$

$y = 12x^{\frac{1}{3}} + c$

When $x = 8$, $y = 30$.

$30 = 12 \times 8^{\frac{1}{3}} + c$

$30 = 12 \times 2 + c$

$c = 6$

$y = 12x^{\frac{1}{3}} + 6$

b When $x = 20, y = 12 \times 20^{\frac{1}{3}} + 6 = 38.57\ldots$

The radius is 38.6 cm to 3 s.f.

11 **a** $f'(2) = a \times 2^2 + b \times 2 = 4a + 2b = -0.8$ ---------(1)

$f'(5) = a \times 5^2 + b \times 5 = 25a + 5b = 2.5$ ----------(2)

(1) × 5: $20a + 10b = -4$

(2) × 2: $50a + 10b = 5$

Subtract: $30a = 9$ so $a = 0.3$; then $1.2 + 2b = -0.8$ and $b = -1$.

b $f'(x) = 0.3x^2 - x$ so $f(x) = 0.1x^3 - 0.5x^2 + c$

$f(2) = 4$ so $0.1 \times 8 - 0.5 \times 4 + c = 4$; $c = 5.2$

The equation of the curve is $y = 0.1x^3 - 0.5x^2 + 5.2$.

12 $\frac{dy}{dx} = 4 - \frac{100}{x^2}$; at a stationary point $\frac{dy}{dx} = 0$

so $4 - \frac{100}{a^2} = 0$; $a^2 = 25$; $a = 5$.

The equation of the curve is $y = \int\left(4 - \frac{100}{x^2}\right)dx$

$= 4x + \frac{100}{x} + c.$

(5, 10) is on the curve so $10 = 20 + \frac{100}{5} + c$; $c = -30$.

The curve is $y = 4x + \frac{100}{x} - 30$.

13 $f'(x) = mx + c$; $f'(-2) = -2m + c = 7$

$f(x) = \frac{1}{2}mx^2 + cx + d$; $f(0) = d = 18$; $f(-2) = 2m - 2c + 18 = 8$; $2m - 2c = -10$

Add the two equations: $-c = -3$ so $c = 3$

Then $-2m + 3 = 7$ so $m = -2$.

The equation is $y = -x^2 + 3x + 18$.

14 **a** $\frac{dy}{dx} = \frac{k}{x^2}$ for some k. When $x = 4$, $\frac{dy}{dx} = -0.5$

so $\frac{k}{16} = -0.5$ and $k = -8$.

$\frac{dy}{dx} = -\frac{8}{x^2}$; $y = -\int 8x^{-2}\,dx$; $y = \frac{8}{x} + c$

When $x = 4$, $y = 6$ so $6 = \frac{8}{4} + c$ and $c = 4$;

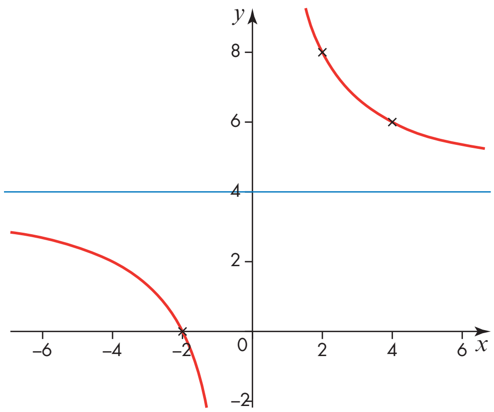
$y = \frac{8}{x} + 4.$

If the gradient is -2, $\frac{dy}{dx} = -\frac{8}{x^2} = -2$; $x^2 = 4$; $x = 2$ or -2.

If $x = 2$, $y = \frac{8}{2} + 4 = 8$; if $x = -2$, $y = \frac{8}{-2} + 4 = 0$.

The gradient is -2 at (2, 8) and $(-2, 0)$.

b



Exercise 7.4A

1 **a** $\int_1^2 3x^2\,dx = \left[x^3\right]_1^2 = 8 - 1 = 7$

b $\left[x^3\right]_2^4 = 64 - 8 = 56$

c $\left[x^3\right]_{-1}^3 = [27] - [-1] = 28$

2 **a** $\left[\frac{1}{5}x^5\right]_1^2 = \left[\frac{32}{5}\right] - \left[\frac{1}{5}\right] = \frac{31}{5} = 6.2$

b $\left[2.5x^4\right]_2^4 = [640] - [40] = 600$

c $\int_1^5 10x^{-2}dx = \left[-10x^{-1}\right]_1^5 = \left[-\frac{10}{5}\right] - [-10]$

$= -2 + 10 = 8$

3 **a** $\left[2x^3 - 2x^2\right]_3^4 = [128 - 32] - [54 - 18]$

$= 96 - 36 = 60$

b $\left[x^3 + 2x^2 + 3x\right]_0^2 = [8 + 8 + 6] - [0] = 22$

c $\int_2^5 3x^2 + 6x\,dx = \left[x^3 + 3x^2\right]_2^5$

$= [125 + 75] - [8 + 12]$

$= [200] - [20]$

$= 180$

4 **a** $\int_1^4 2x^{\frac{1}{2}}dx = \left[\frac{4}{3}x^{\frac{3}{2}}\right]_1^4 = \left[\frac{4}{3} \times 8\right] - \left[\frac{4}{3}\right] = \frac{28}{3}$ or $9\frac{1}{3}$

b $\int_1^4 20x^{-\frac{1}{2}}dx = \left[40x^{\frac{1}{2}}\right]_1^4 = [80] - [40] = 40$

c $\int_1^4 8x^{\frac{1}{3}}dx = \left[6x^{\frac{4}{3}}\right]_1^4 = \left[6 \times 4^{\frac{4}{3}}\right] - [6]$

$= 32.1$

5 **a** Try $y = \sqrt{x^2 + 9} = \left(x^2 + 9\right)^{\frac{1}{2}}$.

Then $\frac{dy}{dx} = \frac{1}{2}\left(x^2 + 9\right)^{-\frac{1}{2}} \times 2x = \frac{x}{\sqrt{x^2 + 9}}$.

Hence $\int_0^4 \frac{x}{\sqrt{x^2 + 9}}dx = \left[\sqrt{x^2 + 9}\right]_0^4$

$= \left[\sqrt{25}\right] - \left[\sqrt{9}\right] = 2.$

b Try $y = \left(x^2 + 9\right)^{\frac{3}{2}}$. Then $\frac{dy}{dx} = \frac{3}{2}\left(x^2 + 9\right)^{\frac{1}{2}} \times 2x$

$= 3x\left(x^2 + 9\right)^{\frac{1}{2}} = 3x\sqrt{x^2 + 9}$.

Hence $\int_0^4 x\sqrt{x^2 + 9}\;dx = \left[\frac{1}{3}\left(x^2 + 9\right)^{\frac{3}{2}}\right]_0^4$

$= \left[\frac{125}{3}\right] - [9] = 32\frac{2}{3}$.

6 Area $= \int_{-2}^2 (2 + 0.1x^4)\,dx = \left[2x + 0.02x^5\right]_{-2}^2$

$= [4.64] - [-4.64] = 9.28$

7 Area $= \int_0^4 (8 + 8x^{\frac{1}{2}} - 6x)\;dx = \left[8x + \frac{16}{3}x^{\frac{3}{2}} - 3x^2\right]_0^4$

$= \left[32 + 42\frac{2}{3} - 48\right] = 26\frac{2}{3}\text{ m}^2$

8 Suppose the coordinates of P are $(a, \sqrt{a})$.

Then the area of $OPA = \int_0^a x^{\frac{1}{2}}\,dx = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^a$

$= \frac{2}{3}a^{\frac{3}{2}} - 0 = \frac{2}{3}a^{\frac{3}{2}}$.

The area of $OAPB = OA \times OB = a \times \sqrt{a} = a \times a^{\frac{1}{2}} = a^{\frac{3}{2}}$.

This shows that area of OAP is $\frac{2}{3}$ of the area of $OAPB$.

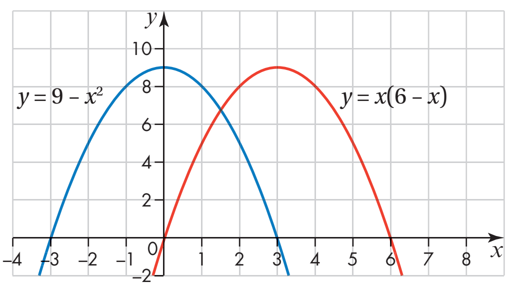
9 **a** $\int_{-3}^3 (9 - x^2)\,dx = \left[9x - \frac{1}{3}x^3\right]_{-3}^3$

$= [27 - 9] - [-27 + 9] = 36$

b $\int_0^6 (6x - x^2)\,dx = \left[3x^2 - \frac{1}{3}x^3\right]_0^6$

$= [108 - 72] - [0] = 36$

c



One graph is a translation of the other and the areas are the same.

10 a If $y = \sqrt{2x-2}$, then $y^2 = 2x - 2$ and $x = \frac{1}{2}y^2 + 1$.

The area is $\int_0^4 x \, dy = \int_0^4 \left(\frac{1}{2}y^2 + 1\right) dy = \left[\frac{1}{6}y^3 + y\right]_0^4$

$= \frac{64}{6} + 4 = 14\frac{2}{3}$.

b The area is the area of $OAPB - 14\frac{2}{3} = 36 - 14\frac{2}{3}$

$= 21\frac{1}{3}$.

11 a $\int_0^1 x^n \, dx = \left[\frac{1}{n+1}x^{n+1}\right]_0^1 = \frac{1}{n+1} - 0 = \frac{1}{n+1}$

b $\int_0^1 x^n(1-x) \, dx = \int_0^1 x^n - x^{n+1} \, dx$

$= \left[\frac{1}{n+1}x^{n+1} - \frac{1}{n+2}x^{n+2}\right]_0^1$

$= \frac{1}{n+1} - \frac{1}{n+2} = \frac{n+2-n-1}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)}$

c $\int_0^1 x^n(1-x^2) \, dx = \int_0^1 x^n - x^{n+2} \, dx$

$= \left[\frac{1}{n+1}x^{n+1} - \frac{1}{n+3}x^{n+3}\right]_0^1$

$= \frac{1}{n+1} - \frac{1}{n+3} = \frac{n+3-n-1}{(n+1)(n+3)} = \frac{2}{(n+1)(n+3)}$

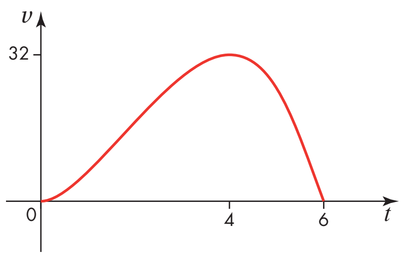
12 a $v = 6t^2 - t^3$; $\frac{dv}{dt} = 12t - 3t^2$; when the speed is a maximum, $\frac{dv}{dt} = 0$.

$12t - 3t^2 = 0$; $4t - t^2 = 0$; $t(4-t) = 0$; $t = 0$ or 4

When $t = 0$, $v = 0$ and that will be a minimum.
When $t = 4$, $v = 32$.

The maximum speed is 32 ms^{-1}.

b



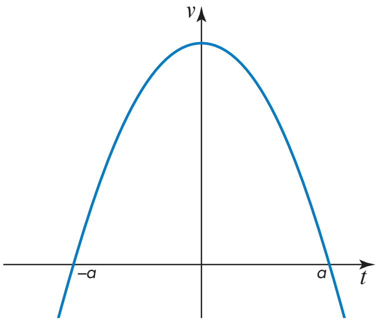
c The distance is $\int_0^6 \left(6t^2 - t^3\right) dt = \left[2t^3 - \frac{1}{4}t^4\right]_0^6$

$= [432 - 324] - 0 = 108$.

The distance is 108 m.

13 Where the curve crosses the x-axis, $a^2 - x^2 = 0$ so $x = \pm a$.

Sketch of the curve:



The area is $\int_{-a}^{a} \left(a^2 - x^2\right) dx = \left[a^2x - \frac{1}{3}x^3\right]_{-a}^{a}$

$= \left[a^3 - \frac{1}{3}a^3\right] - \left[-a^3 + \frac{1}{3}a^3\right]$

$= \frac{2}{3}a^3 + \frac{2}{3}a^3 = \frac{4}{3}a^3$.

14 a Where the curve meets the x-axis,

$(2x+16)^{\frac{3}{4}} = 0$ so $2x + 16 = 0$ and $x = -8$;

the coordinates are $(-8, 0)$.

Where it meets the y-axis, $x = 0$ and

$y = 16^{\frac{3}{4}} = 2^3 = 8$ so the coordinates are $(0, 8)$.

b The area is $\int_{-8}^{0} (2x+16)^{\frac{3}{4}} \, dx$

$= \left[\frac{1}{2} \times \frac{4}{7} \times (2x+16)^{\frac{7}{4}}\right]_{-8}^{0} = \left[\frac{2}{7} \times 16^{\frac{7}{4}}\right] - [0]$

$= \frac{2}{7} \times 128 = \frac{256}{7}$ or $36\frac{4}{7}$.

Exercise 7.5A

1 a $\int_0^2 (x^2 - 2x) \, dx = \left[\frac{1}{3}x^3 - x^2\right]_0^2 = \left[\frac{8}{3} - 4\right] - \lfloor 0 \rfloor = -\frac{4}{3}$

The area is $\frac{4}{3}$.

b Area $= \int_2^3 (x^2 - 2x) \, dx = \left[\frac{1}{3}x^3 - x^2\right]_2^3$

$= [9 - 9] - \left[\frac{8}{3} - 4\right] = 0 - \left[-\frac{4}{3}\right] = \frac{4}{3}$.

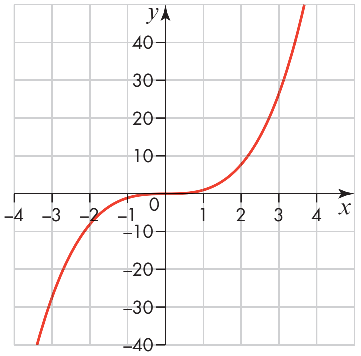
c Since the curve is symmetrical, this area is also $\frac{4}{3}$, the same as **part b**.

2 a i $\int_2^3 x^3 \, dx = \left[\frac{1}{4}x^4\right]_2^3 = \left[\frac{81}{4}\right] - [4] = 16\frac{1}{4}$

ii $\int_{-2}^{3} x^3 \, dx = \left[\frac{1}{4}x^4\right]_{-2}^{3} = \left[\frac{81}{4}\right] - [4] = 16\frac{1}{4}$

iii $\int_{-2}^{2} x^3 \, dx = \left[\frac{1}{4}x^4\right]_{-2}^{2} = [4] - [4] = 0$

b



The integral from –2 to 2 in **part iii** above is zero because the graph is symmetrical and the areas above and below the x-axis are the same.

3 **a** Where they intersect, $x^2 = 2 - x$.

$x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$

$x = -2$ or 1

If $x = -2$, then $y = 4$

If $x = 1$, then $y = 1$

The points are $(-2, 4)$ and $(1, 1)$.

b The area between them is $\int_{-2}^{1}(2 - x)\,dx - \int_{-2}^{1}x^2\,dx$

$= \left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-2}^{1}$

$= \left[2 - \frac{1}{2} - \frac{1}{3}\right] - \left[-4 - 2 + \frac{8}{3}\right] = \frac{7}{6} + \frac{10}{3} = 4\frac{1}{2}.$

4 **a** Where the lines cross, $4 - x = x(4 - x)$.

$4 - x = 4x - x^2$.

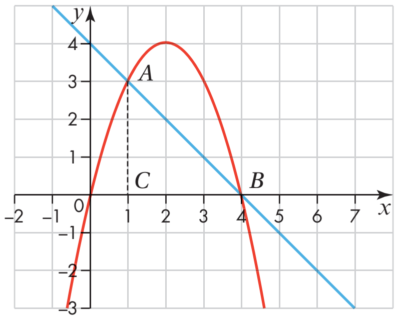
$x^2 - 5x + 4 = 0$

Factorise: $(x - 1)(x - 4) = 0$.

$x = 1$ or 4

Points are $(1, 3)$ and $(4, 0)$.

b



The area required is the difference between the area under the curve between A and B, and the area of triangle ABC.

Area under curve $= \int_{1}^{4}(4x - x^2)\,dx = \left[2x^2 - \frac{1}{3}x^3\right]_{1}^{4}$

$= \left[32 - \frac{64}{3}\right] - \left[2 - \frac{1}{3}\right]$

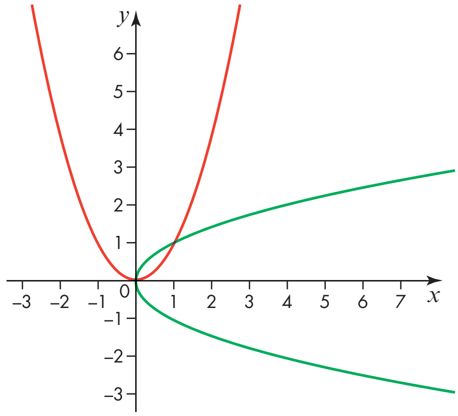
$= \left[10\frac{2}{3}\right] - \left[1\frac{2}{3}\right] = 9.$

Area of triangle $= \frac{1}{2} \times 3 \times 3 = 4\frac{1}{2}.$

Area between the curve and the straight line

$= 9 - 4\frac{1}{2} = 4\frac{1}{2}.$

5 **a** The graphs look like this and cross at $(0, 0)$ and $(1, 1)$.



The area between them is

$\int_{0}^{1}(x^{\frac{1}{2}} - x^2)\,dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_{0}^{1} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$

b Area $= \int_{0}^{1}(x^{\frac{1}{n}} - x^n)\,dx = \left[\frac{n}{n+1}x^{\frac{n+1}{n}} - \frac{1}{n+1}x^{n+1}\right]_{0}^{1}$

$= \frac{n}{n+1} - \frac{1}{n+1} = \frac{n-1}{n+1}.$

c When n is large the area is close to, but less than, one.

6 Where they cross, $4x - x^2 = x^2 - 4x + 6$.

Therefore $2x^2 - 8x + 6 = 0$.

$x^2 - 4x + 3 = 0.$

Factorise: $(x - 3)(x - 1) = 0$.

The curves cross where $x = 1$ or 3.

Area between $= \int_{1}^{3}\left(\left(4x - x^2\right) - \left(x^2 - 4x + 6\right)\right)dx$

$= \int_{1}^{3}8x - 2x^2 - 6)\,dx$

$= \left[4x^2 - \frac{2}{3}x^3 - 6x\right]_{1}^{3} = [36 - 18 - 18]$

$-\left[4 - \frac{2}{3} - 6\right] = [0] - \left[-\frac{8}{3}\right] = \frac{8}{3}$ or $2\frac{2}{3}$.

7 Where the curves cross, $(x-2)^2 = (x-10)(2-x)$.

Hence $x^2 - 4x + 4 = -x^2 + 12x - 20$.

Hence $2x^2 - 16x + 24 = 0$.

Divide by 2: $x^2 - 8x + 12 = 0$

Factorise: $(x-2)(x-6) = 0$.

Hence $x = 2$ or 6.

The required area is

$$\int_2^6 (x-2)^2\,dx + \int_6^{10} (x-10)(2-x)\,dx.$$

$$\int_2^6 (x-2)^2\,dx = \left[\frac{1}{3}(x-2)^3\right]_2^6 = \left[\frac{64}{3}\right] - [0] = 21\frac{1}{3}$$

$$\int_6^{10} (x-10)(2-x)\,dx = \int_6^{10} -x^2 + 12x - 20\,dx$$

$$= \left[-\frac{1}{3}x^3 + 6x^2 - 20x\right]_6^{10}$$

$$= \left[-\frac{1000}{3} + 600 - 200\right] - [-72 + 216 - 120]$$

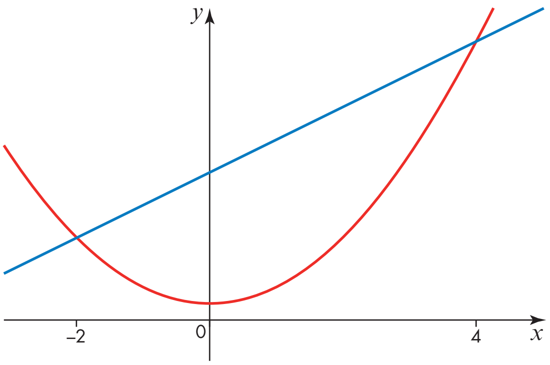
$$= \left[66\frac{2}{3}\right] - [24] = 42\frac{2}{3}$$

The required area $= 21\frac{1}{3} + 42\frac{2}{3} = 64$.

8 **a** Where they cross, $x^2 + 1 = 2x + 9$; $x^2 - 2x - 8 = 0$; $(x-4)(x+2) = 0$.

$x = 4$ or -2. If $x = 4$, then $y = 17$; if $x = -2$, then $y = 5$. The points are (4, 17) and (−2, 5).

b Here is a sketch:



Area under straight line =

$$\int_{-2}^{4} (2x+9)\,dx = \left[x^2 + 9x\right]_{-2}^{4} = [52] - [-14] = 66$$

(Alternatively, this is a trapezium and the area is $\frac{1}{2}(5+17) \times 6 = 66$, the same answer.)

Area under curve $= \int_{-2}^{4} (x^2+1)\,dx$

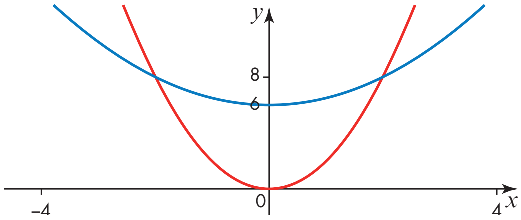
$$= \left[\frac{1}{3}x^3 + x\right]_{-2}^{4} = \left[\frac{76}{3}\right] - \left[-\frac{14}{3}\right] = 30.$$

The area between is $66 - 30 = 36$.

9 **a** Where they cross $0.5x^2 + 6 = 2x^2$; $1.5x^2 = 6$; $x^2 = 4$; $x = 2$ or -2.

If $x = 2$ or -2, $y = 8$; the curves cross at (2, 8) and (−2, 8).

b



c Area $= \int_{-2}^{2} (0.5x^2 + 6)\,dx - \int_{-2}^{2} 2x^2\,dx$

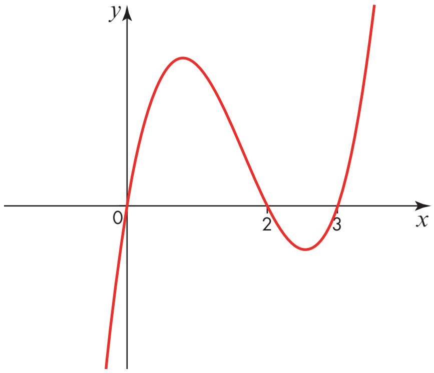
$$= \left[\frac{1}{6}x^3 + 6x\right]_{-2}^{2} - \left[\frac{2}{3}x^3\right]_{-2}^{2}$$

$$\left[\frac{40}{3}\right] - \left[-\frac{40}{3}\right] - \left\{\left[\frac{16}{3}\right] - \left[-\frac{16}{3}\right]\right\} = 16.$$

10 **a** If $y = 0$, then $x^3 - 5x^2 + 6x = 0$; $x(x^2 - 5x + 6) = 0$; $x(x-2)(x-3) = 0$;

$x = 0$, 2 or 3 and the coordinates are (0, 0), (2, 0) and (3, 0).

b



c Area between 0 and 2

$$= \int_0^2 (x^3 - 5x^2 + 6x)\,dx = \left[\frac{1}{4}x^4 - \frac{5}{3}x^3 + 3x^2\right]_0^2$$

$$= \left[4 - \frac{40}{3} + 12\right] - [0] = \frac{8}{3}$$

$$\int_2^3 (x^3 - 5x^2 + 6x)\,dx = \left[\frac{1}{4}x^4 - \frac{5}{3}x^3 + 3x^2\right]_2^3$$

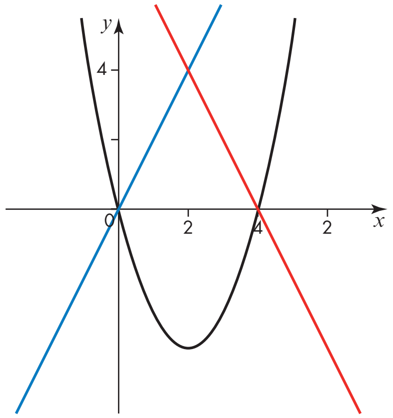
$$= \left[\frac{81}{4} - 45 + 27\right] - \left[\frac{8}{3}\right] = \frac{9}{4} - \frac{8}{3} = -\frac{5}{12}$$

The total area is $\frac{8}{3} + \frac{5}{12} = \frac{37}{12}$ or $3\frac{1}{12}$.

11 a $y = x(x - 4)$ is a parabola that crosses the x-axis at 0 and 4.

The straight lines cross where $2x = 8 - 2x$; $4x = 8$; $x = 2$; cross at (2, 4).

The lines look like this:



b $\int_0^4 x(x-4)\,dx = \int_0^4 \left(x^2 - 4x\right)dx$

$= \left[\frac{1}{3}x^3 - 2x^2\right]_0^4 = \left[\frac{64}{3} - 32\right] - [0] = -\frac{32}{3}$

The area below the x-axis is $\frac{32}{3}$.

The area of the triangle above the x-axis is $\frac{1}{2} \times 4 \times 4 = 8$.

The total area is $8 + \frac{32}{3} = 18\frac{2}{3}$.

12 a $\frac{dy}{dx} = 2x - 4$; at a stationary point $2x - 4 = 0$; $x = 2$ and then $y = 4 - 8 + 6 = 2$.

$\frac{d^2y}{dx^2} = 2 > 0$ so (2, 2) is a minimum point.

b At (1, 3) $\frac{dy}{dx} = 2 - 4 = -2$; the gradient of the tangent is -2; the gradient of the normal is $\frac{1}{2}$.

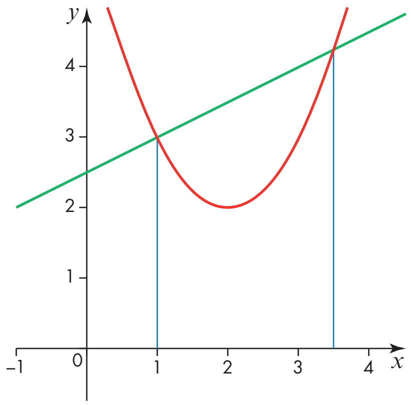
The equation of the normal is $y - 3 = \frac{1}{2}(x - 1)$ or $y = \frac{1}{2}x + \frac{5}{2}$.

Where the curve and the normal meet, $x^2 - 4x + 6 = \frac{1}{2}x + \frac{5}{2}$; $2x^2 - 8x + 12 = x + 5$;

$2x^2 - 9x + 7 = 0$; $(x - 1)(2x - 7) = 0$; $x = 1$ or $\frac{7}{2}$

At P, $x = \frac{7}{2}$ and $y = \frac{1}{2} \times \frac{7}{2} + \frac{5}{2} = \frac{17}{4}$; P is $\left(\frac{7}{2}, \frac{17}{4}\right)$ or $\left(3\frac{1}{2}, 4\frac{1}{4}\right)$.

c The lines look like this:



Area under curve $= \int_1^{3.5}\left(x^2 - 4x + 6\right)dx$

$= \left[\frac{1}{3}x^3 - 2x^2 + 6x\right]_1^{3.5}$

$= [10.792] - [4.333] = 6.4583.$

Area of trapezium $= \frac{1}{2} \times (3 + 4.25) \times 2.5 = 9.0625$

(or by integrating, area $= \int_1^{3.5}(0.5x + 2.5)\,dx$

$= \left[0.25x^2 + 2.5x\right]_1^{3.5} = [11.8125] - [2.75] = 9.0625$

which is the same answer).

Area enclosed $= 9.0625 - 6.4583 = 2.60$ to 3 s.f.

Exercise 7.6A

1 Area $= \int_0^{25} \frac{10}{\sqrt{x}}\,dx = \int_0^{25} 10x^{-\frac{1}{2}}\,dx = \left[10 \times 2x^{\frac{1}{2}}\right]_0^{25}$

$= \left[20\sqrt{x}\right]_0^{25} = [100] - [0] = 100.$

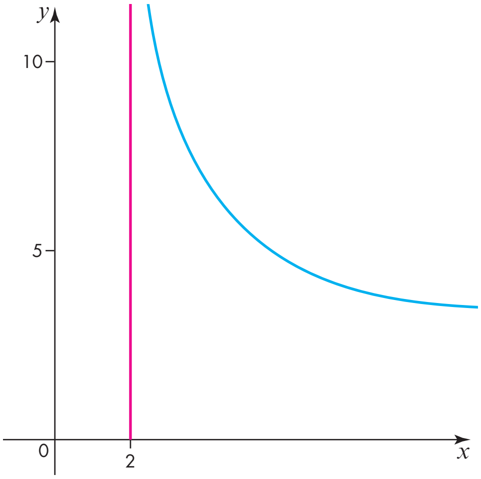
2 The area $= \int_2^{\infty} \frac{20}{x^3}\,dx = \int_2^{\infty} 20x^{-3}\,dx = \left[-10x^{-2}\right]_2^{\infty}$

$= [0] - [-2.5] = 2.5.$

3 The area $= \int_0^{\infty} \frac{8}{(x+2)^2}\,dx = \int_0^{\infty} 8(x+2)^{-2}\,dx$

$= \left[-8(x+2)^{-1}\right]_0^{\infty} = [0] - \left[-\frac{8}{2}\right] = 4.$

4 a



b Area $= \int_2^{11} \frac{10}{\sqrt{x-2}}\,dx = \int_2^{11} 10(x-2)^{-\frac{1}{2}}\,dx$

$= \left[10 \times 2(x-2)^{\frac{1}{2}}\right]_2^{11} = \left[20\sqrt{x-2}\right]_2^{11}$

$= \left[20\sqrt{9}\right] - (0) = 60.$

5 The area is $= \int_a^{\infty} 24x^{-\frac{3}{2}}\,dx = \left[-48x^{-\frac{1}{2}}\right]_a^{\infty} = \left[-\frac{48}{\sqrt{x}}\right]_0^{\infty}$

$= [0] - \left[-\frac{48}{\sqrt{a}}\right] = \frac{48}{\sqrt{a}}.$

The area is 1 and so $\frac{48}{\sqrt{a}} = 1.$

Hence $\sqrt{a} = 48$ and $a = 48^2 = 2304.$

6 Area $= \int_0^{\infty} \frac{x}{\left(1+x^2\right)^2}\,dx.$

Try $\frac{1}{1+x^2} = \left(1+x^2\right)^{-1}.$

Then $\frac{dy}{dx} = -1 \times \left(1+x^2\right)^{-2} \times 2x = -\frac{2x}{\left(1+x^2\right)^2}.$

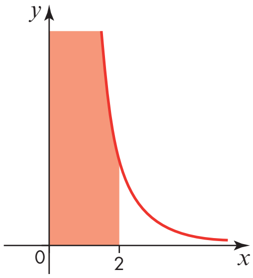
Hence $\int_0^{\infty} \frac{x}{\left(1+x^2\right)^2}\,dx = \left[-\frac{1}{2} \times \frac{1}{1+x^2}\right]_0^{\infty}$

$= [0] - \left[-\frac{1}{2}\right] = \frac{1}{2}.$

7 a $\int_2^{\infty} 100x^{-3}\,dx = \left[\frac{100}{-2}x^{-2}\right]_2^{\infty} = \left[-\frac{50}{x^2}\right]_2^{\infty}$

$= 0 - [-12.5] = 12.5$

b



The integral is represented by the unshaded area.

c To evaluate $\left[-\frac{50}{x^2}\right]_0^2$ is impossible because you cannot divide by 0. The area between 0 and 2 is infinite.

8 a $\int_2^4 \frac{20}{x^2}\,dx = \left[-\frac{20}{x}\right]_2^4 = \left[-\frac{20}{4}\right] - \left[-\frac{20}{2}\right]$

$= -5 + 10 = 5$

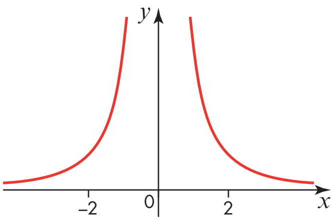
b $\int_4^{\infty} \frac{20}{x^2}\,dx = \left[-\frac{20}{x}\right]_4^{\infty} = [0] - \left[-\frac{20}{4}\right] = 5$

c The student calculated the answer like this:

$\int_{-2}^{2} \frac{20}{x^2}\,dx = \left[-\frac{20}{x}\right]_{-2}^{2} = \left[-\frac{20}{2}\right] - \left[-\frac{20}{-2}\right]$

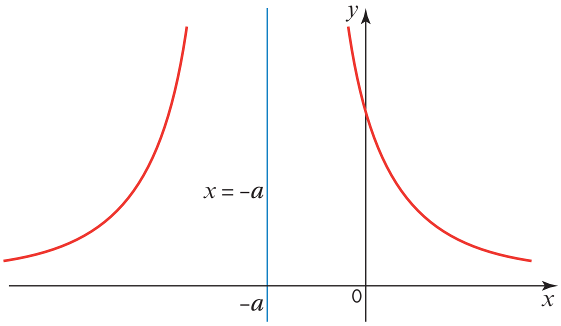
$= -10 - 10 = -20$

However, the graph looks like this:



The area between $x = 0$ and $x = 2$ is infinite because $\left[-\frac{20}{x}\right]_0^2$ cannot be evaluated.

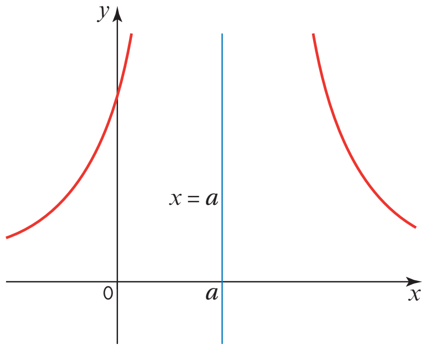
9 a If $x = 0$, then $y = \frac{24}{a^2}$ and the point is $\left(0, \frac{24}{a^2}\right)$.



b The area is $\int_0^{\infty} 24(x+a)^{-2}\,dx = \left[-24(x+a)^{-1}\right]_0^{\infty}$

$= \left[-\dfrac{24}{x+a}\right]_0^{\infty} = [0] - \left[-\dfrac{24}{a}\right] = \dfrac{24}{a}.$

c If a is negative the graph looks like this:



In this case the area is infinite because $\dfrac{24}{x+a}$ cannot be evaluated when $x = -a$.

10 a Area $= \int_4^{\infty} x\,dy$ and $\sqrt{x} = \dfrac{8}{y}$ so $x = \dfrac{64}{y^2}$;

$\int_4^{\infty} x\,dy = \int_4^{\infty} 64y^{-2}\,dy$

$= \left[-\dfrac{64}{y}\right]_4^{\infty} = [0] - [-16] = 16$

b If the area if finite, it is $\int_4^{\infty} y\,dx = \int_4^{\infty} 8x^{-\frac{1}{2}}\,dx$

$= \left[16x^{\frac{1}{2}}\right]_0^{\infty} = \left[16\sqrt{x}\right]_0^{\infty}.$

However, this cannot be evaluated because $\sqrt{\infty}$ cannot be found. The area is infinite.

Exercise 7.7A

1 Volume $= \int_0^4 \pi y^2 dx = \int_0^4 \pi x\,dx = \left[\dfrac{\pi}{2}x^2\right]_0^4$

$= [8\pi] - [0] = 8\pi$.

2 Volume $= \int_2^5 \pi y^2\;dx = \int_2^5 \pi \times 0.09x^2\,dx = \left[0.03\pi x^3\right]_2^5$

$= [3.75\pi] - [0.24\pi] = 3.51\pi.$

3 Volume $= \int_0^2 \pi y^2 dx = \int_0^2 \pi\left(\dfrac{1}{8}x^3\right)^2 dx = \int_0^2 \dfrac{\pi}{64}x^6 dx$

$= \left[\dfrac{\pi}{64} \times \dfrac{x^7}{7}\right]_0^2$

$= \left[\dfrac{\pi}{64} \times \dfrac{128}{7}\right] - [0] = \dfrac{2}{7}\pi\cdot$

4 Volume $= \int_{-1}^{1} \pi y^2 dx = \int_{-1}^{1} \pi\left(x^2+3\right)^2 dx$

$= \int_{-1}^{1} \pi\left(x^4+6x^2+9\right) dx$

$= \left[\pi\left(\dfrac{1}{5}x^5 + 2x^3 + 9x\right)\right]_{-1}^{1} = \left[\pi\left(\dfrac{1}{5}+2+9\right)\right]$

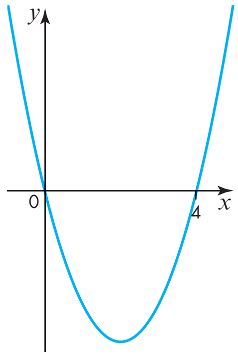
$-\left[\pi\left(-\dfrac{1}{5}-2-9\right)\right] = 11.2\,\pi + 11.2\pi = 22.4\pi.$

5 If $y = 0.4x^2$, then $x^2 = 2.5y$.

Volume $= \int_2^4 \pi x^2\,dy = \int_2^4 \pi \times 2.5y\;dy = \left[1.25\pi y^2\right]_2^4$

$= [20\pi] - [5\pi] = 15\pi.$

6 The curve is a parabola that looks like this:



The curve meets the x-axis at 0 and 4.

Volume $= \int_0^4 \pi y^2 dx = \int_0^4 \pi\left(x(x-4)\right)^2 dx$

$= \int_0^4 \pi\left(x^2-4x\right)^2 dx$

$= \int_0^4 \pi\left(x^4 - 8x^3 + 16x^2\right)dx = \left[\pi\left(\dfrac{1}{5}x^5 - 2x^4 + \dfrac{16}{3}x^3\right)\right]_0^4$

$= \left[\pi\left(\dfrac{1024}{5} - 512 + \dfrac{1024}{3}\right)\right] - [0] = 34\dfrac{2}{15}\pi$ or 34.13π.

7 a Volume $= \int_a^{\infty} \pi y^2\,dx = \int_a^{\infty} \pi x^{-2} dx = \left[-\pi x^{-1}\right]_a^{\infty}$

$= \left[-\dfrac{\pi}{x}\right]_a^{\infty} = [0] - \left[-\dfrac{\pi}{a}\right] = \dfrac{\pi}{a}.$

b Volume $= \int_0^a \pi y^2\,dx = \left[-\dfrac{\pi}{x}\right]_0^a.$

However, if $x \to 0$, then $\dfrac{\pi}{x} \to \infty$ and the volume is infinite.

8 $\dfrac{x^2}{225}+\dfrac{y^2}{100}=1.$

Hence $\dfrac{y^2}{100}=1-\dfrac{x^2}{225}$ and $y^2=100-\dfrac{4}{9}x^2.$

Volume $=\int_{-15}^{15}\pi y^2 dx=\int_{-15}^{15}\pi\left(100-\dfrac{4}{9}x^2\right)dx$

$=\left[\pi\left(100x-\dfrac{4}{27}x^3\right)\right]_{-15}^{15}$

$=[\pi(1500-500)]-[\pi(-1500+500)]$

$=1000\pi+1000\pi=2000\pi.$

The volume of the ball is 6280 cm³ to 3 s.f.

9 Where the circle and the ring cross, $y=4$ and $x^2+4^2=25$ hence $x^2=9$ and $x=\pm 3$.

The volume of the ring is the difference between the volumes of the shapes formed by rotating each of the lines.

Volume formed by the arc of the circle $=\int_{-3}^{3}\pi y^2\,dx$

$=\int_{-3}^{3}\pi y^2\,dx=\int_{-3}^{3}\pi\left(25-x^2\right)dx$

$=\left[\pi\left(25x-\dfrac{1}{3}x^3\right)\right]_{-3}^{3}=\left[\pi(75-9)\right]-\left[\pi(-75+9)\right]$

$=66\pi+66\pi=132\pi.$

The volume formed by the straight line is

$\int_{-3}^{3}\pi y^2\,dx=\int_{-3}^{3}16\pi\,dx$

$=[16\pi x]_{-3}^{3}=[48\pi]-[-48\pi]=96\pi.$

(Alternatively, as this is a cylinder, the volume is $\pi r^2 h=\pi\times 4^2\times 6=96\pi$.)

The volume of the ring is $132\pi-96\pi=36\pi$.

10 a The cylinder has a radius of 5 and height of 6. Volume $=\pi r^2 h=\pi\times 5^2\times 6=150\pi$

b When the curve is rotated the volume is $\int_2^6\pi y^2\,dx.$

$y^2=\left(2\sqrt{x-2}\right)^2=4(x-2)=4x-8$

Volume $=\int_2^6\pi(4x-8)\,dx=\pi\left[2x^2-8x\right]_2^6$

$=\pi[24]-\pi[-8]=32\pi.$

The volume of the shape is the difference between these answers $=150\pi-32\pi=118\pi$.

11 a Where they cross $0.5x^2=0.5x+3$;
$0.5x^2-0.5x-3=0$; $x^2-x-6=0$
$(x-3)(x+2)=0$; $x=3$ or -2; the coordinates are $(3, 4.5)$ and $(-2, 2)$

b The volume formed by the curved line is

$\int_0^{4.5}\pi x^2\,dy=\int_0^{4.5}\pi\times 2y\,dy=\left[\pi y^2\right]_0^{4.5}=20.25\pi.$

The volume formed by the straight line is

$\int_0^{4.5}\pi x^2\,dy$

where $y=0.5x+3$ so $2y=x+6$; $x=2y-6$ and hence $x^2=4y^2-24y+36$.

The volume is $\int_3^{4.5}\pi x^2\,dy$

$=\int_3^{4.5}\pi\left(4y^2-24y+36\right)dy$

$=\pi\left[\dfrac{4}{3}y^3-12y^2+36y\right]_3^{4.5}$

$=40.5\pi-36\pi=4.5\pi$

(or use the fact that it is a cone with radius = 3 and height = 1.5 so the volume is $\dfrac{1}{3}\pi r^2 h$
$=\dfrac{1}{3}\pi\times 3^2\times 1.5=4.5\pi$).

The volume of the shape is $20.25\pi-4.5\pi=15.75\pi$

12 a If $x=2$ then $\dfrac{1}{2}x^3=4$ and $(x-4)^2=4$ so both curves pass through $(2, 4)$.

b The volume is in two parts.

For the curve $y=\dfrac{1}{2}x^3$ the volume

$=\int_0^2\pi y^2\,dx=\pi\int_0^2\dfrac{1}{4}x^6\,dx=\pi\left[\dfrac{1}{28}x^7\right]_0^2$

$=\dfrac{128}{28}\pi=\dfrac{32}{7}\pi.$

The curve $y=(x-4)^2$ meets the x-axis at 4 so for this part the volume is

$\int_2^4\pi y^2\,dx=\int_2^4\pi(x-4)^4\,dx=\pi\left[\dfrac{1}{5}(x-4)^5\right]_2^4$

$=\pi[0]-\pi\left[-\dfrac{32}{5}\right]=\dfrac{32}{5}\pi.$

The total volume is $\dfrac{32}{7}\pi+\dfrac{32}{5}\pi=10\dfrac{34}{35}\pi$ or 11.0π (3 s.f.).

Exam-style questions

1 $y=\int(8x^3-12x^2)\,dx$

$=2x^4-4x^3+c$

When $x=2$, $y=-3$

$32-32+c=-3$

So $y=2x^4-4x^3-3$.

2 Distance $= \int_0^{10}\left(15 - 0.1x^2 dx\right) = \left[15x - \frac{0.1}{3}x^3\right]_0^{10}$

$$= \left[150 - 33\tfrac{1}{3}\right] - [0] = 116\tfrac{2}{3} \text{ m.}$$

3 a $\int\left(5x^{-2} + x^{-\frac{3}{2}}\right)dx = -5x^{-1} - 2x^{-\frac{1}{2}} + c$

b $\int_1^{\infty} \frac{5+\sqrt{x}}{x^2}\,dx = \left[-\frac{5}{x} - \frac{2}{\sqrt{x}}\right]_1^{\infty} = [0] - [-5-2] = 7$

4 $y = \int -16x^{-2}\,dx = 16x^{-1} + c$

$= \frac{16}{x} + c$

When $x = 4$, $y = 6$.

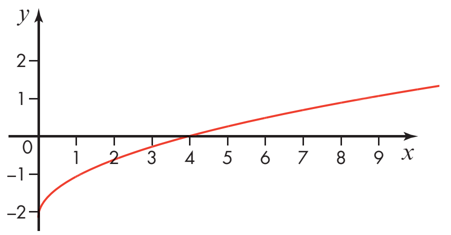
$6 = \frac{16}{4} + c$

$c = 6 - 4 = 2$

The equation is $y = \frac{16}{x} + 2$.

5 a Find some points on the curve:

x	0	1	4	9
y	–2	–1	0	1



b $\int_0^4\left(x^{\frac{1}{2}} - 2\right)dx = \left[\frac{2}{3}x^{\frac{3}{2}} - 2x\right]_0^4 = \left[\frac{16}{3} - 8\right] - [0]$

$= -\frac{8}{3}$ so the area below the x-axis is $\frac{8}{3}$.

$\int_4^9\left(x^{\frac{1}{2}} - 2\right)dx = \left[\frac{2}{3}x^{\frac{3}{2}} - 2x\right]_4^9$

$= [18 - 18] - \left[-\frac{8}{3}\right] = \frac{8}{3}$ so the area above the x-axis is $\frac{8}{3}$.

Hence the total area is $\frac{8}{3} + \frac{8}{3} = \frac{16}{3}$ or $5\frac{1}{3}$.

6 a $V = \int(0.6t - 10)\,dt = 0.3t^2 - 10t + c$

When $t = 0$, $V = 100$ so $c = 100$.

$V = 0.3t^2 - 10t + 100$

b When $t = 20$, $\frac{dV}{dt} = 0.6 \times 20 - 10 = 2$.

This is positive and implies that the volume is increasing. This cannot be the case. If there is a leak, $\frac{dV}{dt}$ will be negative.

7 Area $= \int_0^{\infty} 24(x+2)^{-2}\,dx = \left[-24(x+2)^{-1}\right]_0^{\infty}$

$= \left[-\frac{24}{x+2}\right]_0^{\infty} = [0] - [-12] = 12.$

8 $y = \int(2x+6)\,dx$

$= x^2 + 6x + c$

At a turning point, $\frac{dy}{dx} = 0$

$2x + 6 = 0$

$x = -3$

The turning point is on the x-axis so the coordinates are $(-3, 0)$.

Put these into the equation for y.

$0 = 9 - 18 + c$

$c = 9$

The equation of the curve is $y = x^2 + 6x + 9$.

9 a If $f(x) = \left(4x^2+5\right)^{\frac{3}{2}}$, then $f'(x) = \frac{3}{2}\left(4x^2+5\right)^{\frac{1}{2}} \times 8x$

$= 12x\sqrt{4x^2+5}$

b If $y = \left(4x^2+5\right)^{\frac{3}{2}}$, then $\frac{dy}{dx} = \frac{3}{2}\left(4x^2+5\right)^{\frac{1}{2}} \times 8x$

$= 12x\sqrt{4x^2+5}.$

Hence $\int x\sqrt{4x^2+5}\,dx = \frac{1}{12}\left(4x^2+5\right)^{\frac{3}{2}} + c.$

10 Where the curve crosses the x-axis, $x(4 - x) = 0$.

$x = 0$ or 4

The area $= \int_0^4\left(4kx - kx^2\right)dx = \left[2kx^2 - \frac{1}{3}kx^3\right]_0^4$

$= \left[32k - 21\frac{1}{3}k\right] = 10\frac{2}{3}k.$

$10\frac{2}{3}k = 32$

$k = 3$

11 Area beneath the curve between A and B is:

$\int_{-a}^{a} x^2\,dx = \left[\frac{1}{3}x^3\right]_{-a}^{a} = \left[\frac{1}{3}a^3\right] - \left[-\frac{1}{3}a^3\right] = \frac{2}{3}a^3$

The area of rectangle $ABCD = 2a \times a^2 = 2a^3$.

The area between AB and the curve is

$2a^3 - \frac{2}{3}a^3 = \frac{4}{3}a^3$ and this is $\frac{2}{3}$ of the area of the rectangle.

12 a First, find the coordinates of the points where the lines cross.

Where $x^2 - x + 4 = 2x + 8$

$x^2 - 3x - 4 = 0$

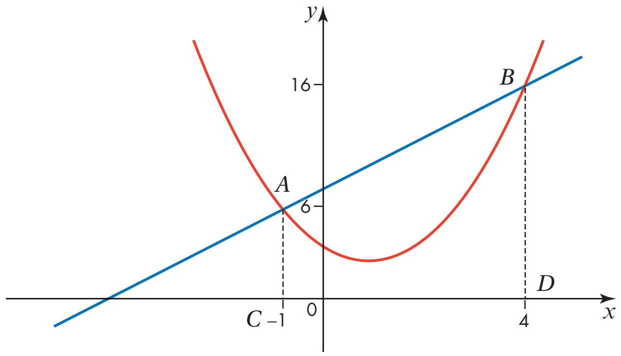
$(x - 4)(x + 1) = 0$

$x = -1$ or 4

If $x = -1$, $y = 1 + 1 + 4 = 6$.

If $x = 4$, $y = 16 - 4 + 4 = 16$.

The points are $(-1, 6)$ and $(4, 16)$.



b You want the difference between the area of the trapezium *ABCD* and the area under the curve between *A* and *B*.

Area under curve $= \int_{-1}^{4}\left(x^2 - x + 4\right)\mathrm{d}x$

$= \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + 4x\right]_{-1}^{4}$

$= \left[\frac{64}{3} - 8 + 16\right] - \left[-\frac{1}{3} - \frac{1}{2} - 4\right] = 29\frac{1}{3} + 4\frac{5}{6} = 34\frac{1}{6}.$

Area of trapezium $= \frac{6 + 16}{2} \times 5 = 55.$

The required area is $55 - 34\frac{1}{6} = 20\frac{5}{6}$.

13 Where they cross, $\frac{1}{16}x^3 = 2x^{\frac{1}{2}}$ and either $x = 0$ or $x^{\frac{5}{2}} = 32$ so $x = 4$.

Area between $y = \frac{1}{16}x^3$ and the x-axis is

$\int_0^4 \frac{1}{16}x^3\,\mathrm{d}x = \left[\frac{1}{64}x^4\right]_0^4 = 4 - 0 = 4.$

Area between $y = 2\sqrt{x}$ and the x-axis is

$\int_0^4 2x^{\frac{1}{2}}\,\mathrm{d}x = \left[\frac{4}{3}x^{\frac{3}{2}}\right]_0^4 = \frac{32}{3} - 0 = 10\frac{2}{3}.$

Area between curves $= 10\frac{2}{3} - 4 = 6\frac{2}{3}$.

14 Volume $= \int_2^5 \pi y^2\,\mathrm{d}x$.

$y = \sqrt{4x + 1}$

Thus $y^2 = 4x + 1$.

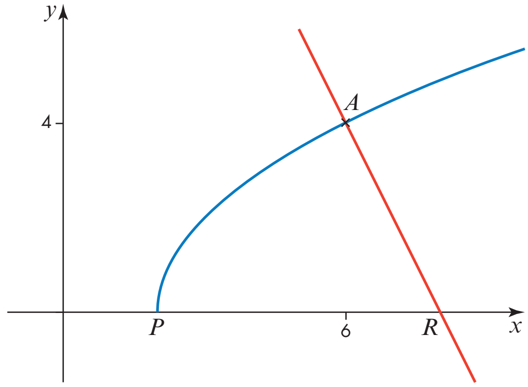
Volume $= \int_2^5 \pi(4x + 1)\,\mathrm{d}x = \left[\pi\left(2x^2 + x\right)\right]_2^5$

$= [55\pi] - [10\pi] = 45\pi.$

15 a $y = (4x - 8)^{\frac{1}{2}}$; $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(4x - 8)^{-\frac{1}{2}} \times 4 = \frac{2}{\sqrt{4x - 8}}$;

if $x = 6$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{\sqrt{16}} = \frac{1}{2}$; the gradient of the normal is -2 and the equation is $y - 4 = -2(x - 6)$ or $y = -2x + 16$.

b



At P $4x - 8 = 0$ so $x = 2$; at R $-2x + 16 = 0$ so $x = 8$.

Volume formed by rotating curve $= \int_2^6 \pi y^2\,\mathrm{d}x$

$= \pi\int_2^6 (4x - 8)\,\mathrm{d}x = \pi\left[2x^2 - 8x\right]_2^6$

$= [24\pi] - [-8\pi] = 32\pi.$

Volume formed by rotating the normal is a cone with volume $\frac{1}{3}\pi \times 4^2 \times 2 = \frac{32}{3}\pi$

(or) $\int_6^8 \pi(16 - 2x)^2\,\mathrm{d}x = \pi\left[-\frac{1}{6}(16 - 2x)^3\right]_6^8$

$= [0] - \pi\left[-\frac{64}{6}\right] = \frac{32}{3}\pi.$

The total volume is $32\pi + \frac{32}{3}\pi = 42\frac{2}{3}\pi.$

16 a Where the curves cross, $x^2 - 10x + 25 = 5 + 4x - x^2$;

$2x^2 - 14x + 20 = 0$;

$(x - 2)(x - 5) = 0$; $x = 2$ or 5.

When $x = 2$, $y = 9$ so $A = (2, 9)$.

When $x = 5$, $y = 0$ so $B = (5, 0)$.

b The volume is

$\int_2^5 \pi\left(5 + 4x - x^2\right)^2\mathrm{d}x - \int_2^5 \pi\left(x^2 - 10x + 25\right)^2\mathrm{d}x$

$= 129.6\pi - 48.6\pi = 81\pi = 254.$

17 a Where they cross, $x^4 + 2 = x^2 + 14$;

$x^4 - x^2 - 12 = 0$; $(x^2 - 4)(x^2 + 3) = 0$

$x^2 = 4$ or -3

$x^2 = -3$ has no solution; if $x^2 = 4$, then $x = 2$ or -2; in either case, $y = 2^4 + 2 = 18$ and the coordinates are $B\ (2, 18)$ and $A\ (-2, 18)$.

b The minimum points are where the curves cross the y-axis.

$y = x^4 + 2$ is at $(0, 2)$ and $y = x^2 + 14$ is at $(0, 14)$

c If $y = x^4 + 2$, then $\sqrt{y-2} = x^2$ and the volume generated by $y = x^4 + 2$ is $\int_2^{18} \pi x^2\, dy$

$$= \int_2^{18} \pi (y-2)^{\frac{1}{2}}\, dy = \pi\left[\frac{2}{3}(y-2)^{\frac{3}{2}}\right]_2^{18}$$

$$= \left[\frac{128}{3}\pi\right] - [0] = \frac{128}{3}\pi.$$

If $y = x^2 + 14$, then $y - 14 = x^2$ and the volume generated by $y = x^2 + 14$ is $\int_{14}^{18} \pi x^2\, dy$

$$= \int_{14}^{18} \pi(y-14)\, dy = \pi\left[\frac{1}{2}y^2 - 14y\right]_{14}^{18}$$

$= [-90\pi] - [-98\pi] = 8\pi$.

The volume required $= \frac{128}{3}\pi - 8\pi = \frac{104}{3}\pi$ or $34\frac{2}{3}\pi$.

18 a $$\mathrm{f}(x) = \int \frac{1}{\sqrt{ax+3}}\, dx = \int (ax+3)^{-\frac{1}{2}}\, dx$$

$$= \frac{2}{a}(ax+3)^{\frac{1}{2}} + c = \frac{2}{a}\sqrt{ax+3} + c$$

b $\mathrm{f}(0) = 0 \rightarrow \frac{2}{a} \times \sqrt{3} + c = 0;\ c = -\frac{2\sqrt{3}}{a}$ and

$$\mathrm{f}(x) = \frac{2}{a}\sqrt{ax+3} - \frac{2\sqrt{3}}{a}$$

$$\mathrm{f}(a) = 2\sqrt{2} - 2 \rightarrow \frac{2}{a}\sqrt{a^2+3} - \frac{2\sqrt{3}}{a} = 2\sqrt{2} - 2;$$

From which

$$\sqrt{a^2+3} = a\left(\sqrt{2}-1\right) + \sqrt{3}$$

$$a^2 + 3 = a^2\left(3 - 2\sqrt{2}\right) + 2a\sqrt{3}\left(\sqrt{2}-1\right) + 3$$

$$2a - 2a\sqrt{2} + 2\sqrt{6} - 2\sqrt{3} = 0$$

$$a = \frac{2\sqrt{3} - 2\sqrt{6}}{2 - 2\sqrt{2}} = \sqrt{3}$$

19 a $$\int_a^{2a} \frac{100}{x^2}\, dx = \left[-\frac{100}{x}\right]_a^{2a} = \left[-\frac{100}{2a}\right] - \left[-\frac{100}{a}\right]$$

$$= -\frac{50}{a} + \frac{100}{a} = \frac{50}{a}$$

b The areas are given by the formula in part a with $a = 1, 2, 4, 8$, etc.

Total area $= \frac{50}{1} + \frac{50}{2} + \frac{50}{4} + \frac{50}{8} + \ldots$

This is a geometric series with $a = 50$ and $r = \frac{1}{2}$.

Then $S_\infty = \frac{a}{1-r} = \frac{50}{1-\frac{1}{2}} = 100$.

20 a Where it crosses the x-axis, $y = 0$; $\frac{x^2}{a^2} = 1$; $x^2 = a^2$;

$x = \pm a$; points $(a, 0)$ and $(-a, 0)$.

Where it crosses the y-axis, $x = 0$; $\frac{y^2}{b^2} = 1$;

$y^2 = b^2$; $y = \pm b$; points $(0, b)$ and $(0, -b)$.

b Volume $= \int_{-a}^{a} \pi y^2\, dx$, where $\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$ and

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right).$$

So volume $= \int_{-a}^{a} \pi b^2\left(1 - \frac{x^2}{a^2}\right) dx$

$$= \pi b^2\left[x - \frac{x^3}{3a^2}\right]_{-a}^{a}$$

$$= \pi b^2\left[a - \frac{a}{3}\right] - \pi b^2\left[-a + \frac{a}{3}\right]$$

$$= \pi b^2 \times \frac{2}{3}a + \pi b^2 \times \frac{2}{3}a = \frac{4}{3}\pi a b^2.$$

Mathematics in life and work

Assume that the segment is from $x = -a$ to $x = a$.

The volume is $= \int_{-a}^{a} \pi\left(8 - 0.009x^2\right)^2 dx$

$$= \int_{-a}^{a} \pi\left(64 - 0.144x^2 + 0.000\,081x^4\right) dx$$

$$= \pi\left[64x - 0.048x^3 + 0.000\,016\,2x^5\right]_{-a}^{a}$$

$$= 2\pi\left[64a - 0.048a^3 + 0.000\,016\,2a^5\right].$$

If the length is 15.6 cm then $a = 7.8$ and a volume of 2996 cm³ is just under 3 litres.

Summary Review

Please note: Full worked solutions are provided as an aid to learning, and represent one approach to answering the question. In some cases, alternative methods are shown for contrast.

All sample answers have been written by the authors. Cambridge Assessment International Education bears no responsibility for the example answers to questions taken from its past question papers, which are contained in this publication.

Non-exact numerical answers should be given correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

Warm-up Questions

1 **a** **i** 27, 38

ii Add consecutive odd numbers to get the next term.

b 1, 5, 9

2 $5x + 2y = 16$ ①

$3x - 4y = 7$ ②

① × 2:

$10x + 4y = 32$ ③

② + ③:

$13x = 39$

$x = 3$

In ①:

$15 + 2y = 16$

$2y = 1$

$y = 0.5$

3 $r = 7.5\,\text{cm}$

Area of circle $= \pi \times 7.5^2 = 177\,\text{cm}^2$.

A Level Questions

1 $f'(x) = 3x^2 - 7$

Integrating: $f(x) = x^3 - 7x + c$

Substitute $f(3) = 5 \Rightarrow 27 - 21 + c = 5 \Rightarrow c = -1$

$\therefore f(x) = x^3 - 7x - 1$

2 **i** $\sin\theta = k \Rightarrow \sin^2\theta = k^2$

$\sin^2\theta + \cos^2\theta = 1$

$k^2 + \cos^2\theta = 1$

$\cos^2\theta = 1 - k^2$

$\cos\theta = -\sqrt{1 - k^2}$

Negative because θ is obtuse.

ii $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = -\dfrac{k}{\sqrt{1-k^2}}$

iii $\sin(\theta + \pi) = -k$

3 For 'x': $\binom{7}{1}2^6(ax)^1 = 448ax$.

For 'x^2': $\binom{7}{2}2^5(ax)^2 = 672a^2x^2$.

$448a = 672a^2$

$0 = 672a^2 - 448a$

$= 3a^2 - 2a$

$= a(3a - 2)$

$a = \dfrac{2}{3}$

4 **i** A translation $\begin{pmatrix} -\frac{\pi}{2} \\ 0 \end{pmatrix}$ and a stretch in the y-direction, scale factor 3.

ii The minimum point of $f(x)$ is $\left(\frac{3\pi}{2}, -1\right)$. After the translation, this becomes $(\pi, -1)$. After the stretch, this becomes $(\pi, -3)$. So the minimum point of $g(x)$ is $(\pi, -3)$.

5 $\tan 2x = 2$

$2x = 63.4\ldots, 243.4\ldots$

$x = 31.7°, 121.7°$

6 **i** Area $OAB = \dfrac{1}{2} \times 8^2\alpha = 32\alpha$.

Area $OCA = \dfrac{\pi \times 4^2}{2} = 8\pi$

Therefore $64\alpha = 8\pi$

$\alpha = \dfrac{\pi}{8}$

ii $BA = 8\alpha$ and $OCA = 4\pi$

$P = 8\alpha + 4\pi + 8$

$P = 8 \times \dfrac{\pi}{8} + 4\pi + 8$

$P = 5\pi + 8$

7 $\dfrac{dy}{dx} = -8x^{-3} - 1$

$y = 4x^{-2} - x + c$

Substitute (2, 4) $4 = 1 - 2 + c$

$c = 5$

$y = \dfrac{4}{x^2} - x + 5$

8 i Let C be the centre of the circle. So $A(-7, -7)$ and $C(-1, 6)$. B is at the opposite side of the circle, so you can conclude $B(5, 19)$.

ii AC is a radius of the circle. By Pythagoras,

$AC = r = \sqrt{6^2 + 13^2} = \sqrt{205}$

iii Gradient of AC is $\frac{13}{6} \Rightarrow$ gradient of perpendicular bisector is $-\frac{6}{13}$ and it passes through C.

$$y - 6 = -\frac{6}{13}(x+1)$$

$$13y - 78 = -6x - 6$$

$$6x + 13y - 72 = 0$$

9 i $\tan\alpha = \frac{CB}{4} \quad \Rightarrow \quad CB = 4\tan\alpha$

Area of triangle $= \frac{1}{2} \times 4 \times 4\tan\alpha = 8\tan\alpha$.

Sector area $= \frac{1}{2} \times 2^2\alpha = 2\alpha$.

Shaded area $= 8\tan\alpha - 2\alpha$.

ii $\cos\alpha = \frac{4}{AC} \qquad AC = \frac{4}{\cos\alpha} \qquad DC = \frac{4}{\cos\alpha} - 2$

Arc length $= 2\alpha$

$$\text{Perimeter} = \frac{4}{\cos\alpha} - 2 + 2\alpha + 2 + 4\tan\alpha$$

$$= \frac{4}{\cos\alpha} + 4\tan\alpha + 2\alpha$$

10 i $y - 2t = -2(x - 3t)$

$y = -2x + 8t$

At A, $y = 0 \quad \Rightarrow \quad x = 4t$.

At B, $x = 0 \quad \Rightarrow \quad y = 8t$.

Area of triangle $OAB = \frac{4t \times 8t}{2} = 16t^2$.

ii Equation of perpendicular line is $y - 2t = \frac{1}{2}(x - 3t)$.

$2y = x + t$

At C, $y = 0 \quad \Rightarrow \quad x = -t$.

So $C(-t, 0)$ and $P(3t, 2t)$.

Mid-point of CP is $\left(\frac{3t - t}{2}, \frac{2t}{2}\right) = (t, t)$, which is on the line $y = x$.

11 i $y = 8x^{-1} + 2x$

$$\frac{dy}{dx} = -8x^{-2} + 2 = -\frac{8}{x^2} + 2$$

$$\frac{d^2y}{d^2x} = 16x^{-3} = \frac{16}{x^{-3}}$$

ii $0 = \frac{-8}{x^2} + 2$

$= -8 + 2x^2$

$x = \pm 2$

Substituting into original equation gives $y = \pm 8$.

So the stationary points are $(2,8)$ and $(-2, -8)$.

When $x = 2$, $\frac{d^2y}{d^2x} = 2 > 0 \quad \Rightarrow$ minimum.

When $x = -2$, $\frac{d^2y}{d^2x} = -2 < 0 \quad \Rightarrow$ maximum

12 i The centre of the circle is the origin $O(0, 0)$. The line and the circle intersect when

$x^2 + (x + 4)^2 = 8$

$x^2 + 4x + 4 = 0$

$(x + 2)^2 = 0$

$x = -2 \quad \Rightarrow \quad y = 2$

The line and the circle intersect once only at $P(-2, 2)$ so the line is a tangent.

ii The second tangent is at the opposite side of the circle and passes through $(2, -2)$. The equation is

$y + 2 = 1\,(x - 2) \quad \Rightarrow \quad y = x - 4$

13 $4\sin^2 x + 8\cos x - 7 = 0$

$4(1 - \cos^2 x) + 8\cos x - 7 = 0$

$4\cos^2 x - 8\cos x + 3 = 0$

$(2\cos x - 1)(2\cos x - 3) = 0$

$\cos x = \frac{3}{2} \quad \Rightarrow \quad$ no solutions

$\cos x = \frac{1}{2} \quad \Rightarrow \quad x = 60°, 300°$

14 i $f(x) = x^2 + 1$

$y = x^2 + 1$

$y - 1 = x^2$

$x = \sqrt{y - 1}$

$f^{-1}(x) = \sqrt{x - 1} \quad (x > 1)$

ii $f(x^2 + 1) = (x^2 + 1)^2 + 1 = \frac{185}{16}$

$(x^2 + 1)^2 = \frac{169}{16}$

$x^2 + 1 = \pm\frac{13}{4}$

$x = \frac{3}{2}$

Only one solution, since $x \geqslant 0$

15 i $(2x - x^2)^6 = \binom{6}{0}(2x)^6 + \binom{6}{1}(2x)^5(-x^2) + \binom{6}{2}$

$(2x)^4(-x^2)^2 + \ldots = 64x^6 - 192x^7 + 240x^8 + \ldots$

ii $(2 + x)(2x - x^2)^6$

For 'x^8':

$2 \times 240x^8 = 480x^8$ and $x \times -192x^7 = -192x^8$

So the coefficient of x^8 is 288.

16 Completing the square: $x^2 + y^2 - 2x - 16y + 40 = 0$ becomes $(x - 1)^2 + (y - 8)^2 = 25$

This is a circle with centre (1, 8) and radius 5.

The translation $\begin{pmatrix} -6 \\ -5 \end{pmatrix}$ moves the centre to $(-5, 3)$ and the radius remains as 5.

17 i $-x^2 + 6x - 5 = -1(x - 3)^2 + 4$ by completing the square

ii $m = 3$

iii
$$y = -(x - 3)^2 + 4$$
$$(x - 3)^2 = 4 - y$$
$$x = 3 + \sqrt{4 - y}$$
$$f^{-1}(x) = 3 + \sqrt{4 - x}$$

The domain is $x \leqslant 0$

18 i $S = \dfrac{a}{1 - r}$ and $3S = \dfrac{a}{1 - 2r}$

$$\frac{3a}{1 - r} = \frac{a}{1 - 2r}$$
$$3(1 - 2r) = 1 - r$$
$$r = \frac{2}{5}$$

ii $84 = 7 + (n - 1)d \quad \Rightarrow \quad d = \dfrac{77}{n - 1}$ ①

$245 = 7 + (3n - 1)d \quad \Rightarrow \quad d = \dfrac{238}{3n - 1}$ ②

Equating ① and ②:
$$238(n - 1) = 77(3n - 1)$$
$$238n - 238 = 231n - 77$$
$$7n = 161$$
$$n = 23$$

19 i Perimeter $= 400 \quad \Rightarrow \quad 400 = 2\pi r + 2x$

$\Rightarrow \quad 2x = 400 - 2\pi r$ ①

$A = \pi r^2 + 2xr$

Substituting ①:

$A = \pi r^2 + r(400 - 2\pi r)$

$A = \pi r^2 + 400r - 2\pi r^2$

$A = 400r - \pi r^2$

ii $\dfrac{dA}{dr} = 400 - 2\pi r$

$0 = 400 - 2\pi r$

$2\pi r = 400$

Substituting into ①:

$2x = 400 - 400$

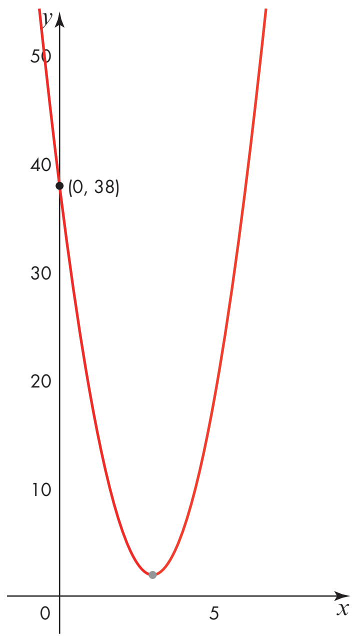
$x = 0$

Therefore there are no straight sections.

$\dfrac{d^2A}{dr^2} = -2\pi < 0 \quad \Rightarrow$ maximum

20 i $f(x) = 4(x - 3)^2 + 2$

ii



The minimum point is (3, 2)

iii $f(2x)$ moves (3, 2) to (1.5, 2)

$f(2x - 3)$ moves (3, 2) to (4.5, 2)

$-f(2x - 3)$ moves (3, 2) to (4.5, –2)

So the maximum point of $-f(2x - 3)$ is (4.5, –2)

21 i $ar^2 = -108$ ①

$ar^5 = 32$ ②

② ÷ ①:

$$r^3 = -\frac{8}{27}$$
$$r = -\frac{2}{3}$$

ii Substituting value of r into ① gives:

$$\frac{4}{9}a = -108$$
$$a = -243$$

iii $S_\infty = \dfrac{-243}{1 - -\dfrac{2}{3}}$

$= -145.8$

22 i $x = x^2 - 4x + 4$

$0 = x^2 - 5x + 4$

$0 = (x - 1)(x - 4)$

So $x = 1, y = 1$ $\quad A(1, 1)$.

And $x = 4, y = 4$ $\quad B(4, 4)$.

Therefore, mid-point of AB is (2.5, 2.5).

ii $\dfrac{dy}{dx} = 2x - 4$

$x^2 - 4x + 4 = (2x - 4)x$

$x^2 = 4$

$x = \pm 2$

$x = 2 \quad \Rightarrow \quad y = 0 \quad \Rightarrow \quad m = 0$ Ignore this solution since $m = 0$.

$x = -2 \quad \Rightarrow \quad y = 16 \quad \Rightarrow \quad m = -8$

So the tangent meets the curve at (–2, 16).

23 i $4x^2 - 24x + 11 = 4(x-3)^2 - 25$

Vertex is at (3, –25).

ii $g(1) = 4 - 24 + 11 = -9$

Therefore $g(x) \geqslant -9$.

iii $y = 4(x-3)^2 - 25$

$$(x-3)^2 = \frac{y+25}{4}$$

$$x = 3 + \sqrt{\frac{y+25}{4}}$$

$$= 3 + \frac{1}{2}\sqrt{y+25}$$

$$g^{-1}(x) = 3 + \frac{1}{2}\sqrt{x+25}$$

The domain is $x \geqslant -9$.

24 By Pythagoras:

i $AB = \sqrt{12^2 + 10^2} = \sqrt{244}$

Similarly, $AC = \sqrt{1^2 + 11^2} = \sqrt{122}$ and

$BC = \sqrt{11^2 + 1^2} = \sqrt{122}$

Therefore, $(AC)^2 + (BC)^2 = (AB)^2 \Leftrightarrow ABC$ is a right-angled triangle.

ii $(AC)^2 + (BC)^2 = (AB)^2 \Rightarrow AB$ is the hypotenuse

$\Rightarrow AB$ is the diameter of the circle

$\therefore$ The centre of the circle is the midpoint of AB. The centre is (–4, 2) and the radius is

$$\frac{\sqrt{244}}{2} = \sqrt{61}$$

So the equation of the circle is

$(x+4)^2 + (y-2)^2 = 61$

25 i Gradient of $CE = \frac{4 - 1.75}{4 - 3.5} = 4.5$ and gradient of

$DE = \frac{4 - 3.51}{4 - 3.9} = 4.9$

ii The sequence of chords have gradients: 3, 4, 4.5, 4.9. This indicates that as the chords approach E, the gradients are converging towards 5. It indicates that f'(4) = 5.

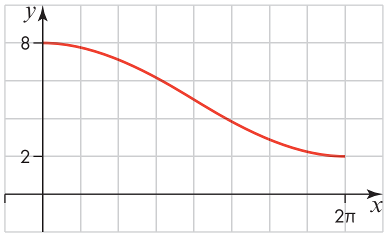
26 i $5 + 3\cos\left(\frac{x}{2}\right) = 7$

$$\cos\left(\frac{x}{2}\right) = \frac{2}{3}$$

$$\frac{x}{2} = 0.841\ldots$$

$$x = 1.68$$

ii



iii f has an inverse because it is a one–one function (or it has no turning point).

iv
$$y = 5 + 3\cos\left(\frac{x}{2}\right)$$

$$\frac{y-5}{3} = \cos\left(\frac{x}{2}\right)$$

$$2\cos^{-1}\left(\frac{y-5}{3}\right) = x$$

$$f^{-1}(x) = 2\cos^{-1}\left(\frac{x-5}{3}\right)$$

27 i $y = 8 - 2x - x^2$

$\frac{dy}{dx} = -2 - 2x$

$2 = -2 - 2x$

$x = -2$

$y = 8$

Therefore

$8 = -4 + c$

$c = 12$

ii $8 - 2x - x^2 = 2x + 11$

$0 = x^2 + 4x + 3$

$= (x+1)(x+3)$

$x = -1$ or $x = -3$

$$\text{Area} = \int_{-3}^{-1}(8 - 2x - x^2)\,dx - \int_{-3}^{-1}(2x+11)\,dx.$$

$$\text{Area} = \left[8x - x^2 - \frac{x^3}{3}\right]_{-3}^{-1} - \left[x^2 + 11x\right]_{-3}^{-1} = \frac{4}{3}.$$

28 i Completing the square: $x^2 + y^2 - 8x - 6y + 21 = 0$

$\Rightarrow (x-4)^2 + (y-3)^2 = 4$.

Therefore, the centre is (4, 3).

ii The radius is 2.

iii When $x = 0 \quad \Rightarrow \quad (y-3)^2 = -12 \quad \Rightarrow$ no real solutions $\Rightarrow$ does not intersect y-axis.

When $y = 0 \quad \Rightarrow \quad (x-4)^2 = -5 \quad \Rightarrow$ no real solutions $\Rightarrow$ does not intersect x-axis.

iv The diameter lies on the line AB.

The gradient of the tangent is $\frac{1}{3}$, so the gradient of the diameter (and hence AB) is –3. AB also passes through the centre. So the equation is:

$y-3=-3(x-4)$
$y-3=-3x+12$
$3x+y-15=0$

29 i $0=3u+3u^{-1}-10$
$0=3u^2-10u+3$
$0=(3u-1)(u-3)$
$u=\frac{1}{3} \quad x=\frac{1}{9}$
$u=3 \quad x=9$

ii $f''(x)=\frac{3}{2}x^{-\frac{1}{2}}-\frac{3}{2}x^{-\frac{3}{2}}$

$f''\left(\frac{1}{9}\right)=-36<0 \Rightarrow$ maximum

$f''(9)=\frac{4}{9}>0 \Rightarrow$ minimum

iii $f(x)=2x^{\frac{3}{2}}+6x^{\frac{1}{2}}-10x+c$

$-7=2(4)^{\frac{3}{2}}+6(4)^{\frac{1}{2}}-10(4)+c$
$c=5$

$f(x)=2x^{\frac{3}{2}}+6x^{\frac{1}{2}}-10x+5$

30 i $y=7 \Rightarrow (x+2)^2+9=25 \Rightarrow (x+2)^2=16$

So $x+2=\pm 4 \Rightarrow x=2$ or $x=-6$

Therefore, $P(2, 7)$.

ii Let C be the centre of the circle. The gradient of CP is $\frac{7-4}{2--2}=\frac{3}{4}$.

The tangent is perpendicular to CP, so it has gradient $-\frac{4}{3}$.

The equation of the tangent is

$y-7=-\frac{4}{3}(x-2) \Rightarrow 4x+3y-29=0$

iii For the tangent line, $y=0 \Rightarrow 4x=29 \Rightarrow x=7.25 \Rightarrow Q(7.25, 0)$

By Pythagoras, $PQ=\sqrt{5.25^2+7^2}=8.75$

So, the area of triangle CPQ is

$\frac{5\times 8.75}{2}=21.875$ units^2

31 i $y=(3-2x)^3$

$\frac{dy}{dx}=3(3-2x)^2(-2)$
$=-6(3-2x)^2$

When $=\frac{1}{2}, \frac{dy}{dx}=-6(3-1)^2=-24.$

So the equation of the tangent is:

$y-8=-24\left(x-\frac{1}{2}\right)$

$y-8=-24x+12$
$y=-24x+20$

ii Area $=\int_0^{\frac{1}{2}}(3-2x)^3\,dx-\int_0^{\frac{1}{2}}(-24x+20)\,dx$

$=\left[-\frac{1}{8}(3-2x)^4\right]_0^{\frac{1}{2}}-\left[-12x^2+20x\right]_0^{\frac{1}{2}}$

$=\frac{81}{8}-9$

$=\frac{9}{8}$

32 The graphs intersect when $6x-x^2=8$

So $x^2-6x+8=0$

$(x-2)(x-4)=0$

$x=2$ or $x=4$

$V=\pi\int_2^4(6x-x^2)^2\,dx-\pi\int_2^4 8^2\,dx$

$V=\pi\int_2^4(36x^2-12x^3+x^4)\,dx-\pi\int_2^4 64\,dx$

$V=\pi\left[12x^3-3x^4+\frac{1}{5}x^5\right]_2^4-\pi[64x]_2^4$

$V=\pi[[204.8]-[54.4]]-128\pi$

$V=22.4\pi\left(\text{or } \frac{112\pi}{5}\right)$

Extension Questions

1 i Radius and tangent meet at B, so angle ABC is a right-angle and ABC is a right-angled triangle.

Area of triangle $ABD=\frac{3(BD)}{2}$

$\Rightarrow$ Area of kite $ABDC=3\,(BD)$

Therefore, $12=3\,(BD) \Rightarrow BD=4$ cm

By Pythagoras: $(AD)^2=3^2+4^2 \Rightarrow AD=5$ cm

ii The locus of points is a circle, centre (5, 8) and radius 5 cm. So the equation of the locus is:

$(x-5)^2+(y-8)^2=\sqrt{5}$

2 i Sometimes. The result is not valid for $a=0$.

ii Sometimes. The result is only true when $a=b$. The circle is in the positive quadrant and touches both the x-axis and the y-axis.

If $a>b$, the circle extends beneath the x-axis.

If $a<b$, the circle touches the y-axis only and does not reach the x-axis.

iii Sometimes. Solving the equation:

$\tan\theta=1 \Rightarrow \theta=\frac{\pi}{4}, \frac{5\pi}{4}, \ldots$

In general, the result is true when

$\theta=\frac{(4n-3)\pi}{4}$, where $n\in\mathbb{Z}$.

iv Always. This is an identity and should be written as $\sin^2\theta + \cos^2\theta \equiv 1$.

3 Solving the quadratic: $(3x+5)(x-2)=0$

$\Rightarrow x = -\frac{5}{3}$ or $x = 2$.

For the cubic: $\frac{dy}{dx} = 3x^2 - 8x + 6$. The discriminant is $8^2 - 4(3)(6) = -8 \quad \Rightarrow$ no turning points.

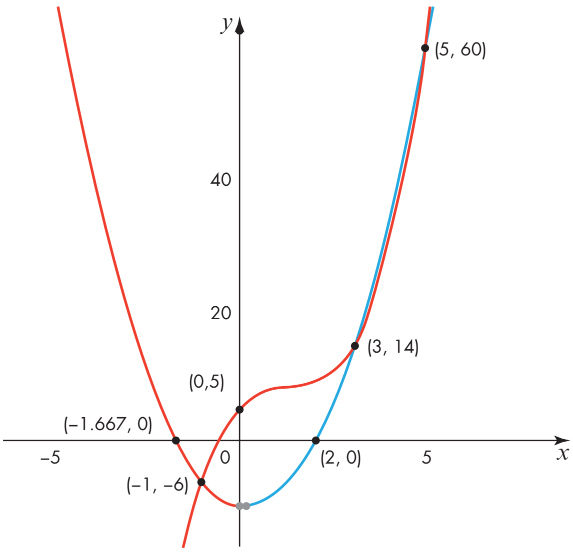
At the intersection of the two curves: $x^3 - 4x^2 + 6x + 5 = 3x^2 - x - 10 \quad \Rightarrow \quad x^3 - 7x^2 + 7x + 15 = 0$

$x = -1$ is a root $\quad \Leftrightarrow \quad (x+1)$ is a factor

By inspection: $(x+1)(x^2 - 8x + 15) = 0$

$\Leftrightarrow \quad (x+1)(x-3)(x-5) = 0$

This information allows a sketch to be drawn.



The area between the curves from (3, 14) to (5, 60) is the area we need to find.

$$\text{Shaded area} = \int_3^5 \left(\left(3x^2 - x - 10\right) - \left(x^3 - 4x^2 + 6x + 5\right)\right) dx$$

$$\text{Shaded area} = \int_3^5 \left(-x^3 + 7x^2 - 7x - 15\right) dx$$

$$= \left[-\frac{1}{4}x^4 + \frac{7}{3}x^3 - \frac{7}{2}x^2 - 15x\right]_3^5 = \frac{20}{3}\text{ units}^2$$

4 All of the different combinations that contribute to the coefficient of x^3 are:

$$\binom{5}{3}(x)^3(2)^2 \times (-3)^4 \times (-4)^6 = 13271040x^3$$

$$\binom{5}{2}(x)^2(2)^3 \times \binom{4}{1}(2x)(-3)^3 \times (-4)^6 = -70778880x^3$$

$$\binom{5}{2}(x)^2(2)^3 \times (-3)^4 \times \binom{6}{1}\left(\frac{x}{2}\right)(-4)^5 = -19906560x^3$$

$$\binom{5}{1}(x)(2)^4 \times \binom{4}{2}(2x)^2(-3)^2 \times (-4)^6 = 70778880x^3$$

$$\binom{5}{1}(x)(2)^4 \times (-3)^4 \times \binom{6}{2}\left(\frac{x}{2}\right)^2(-4)^4 = 6220800x^3$$

$$\binom{5}{1}(x)(2)^4 \times \binom{4}{1}(2x)(-3)^3 \times \binom{6}{1}\left(\frac{x}{2}\right)(-4)^5 = 53084160x^3$$

$$(2)^5 \times \binom{4}{3}(2x)^3(-3) \times (-4)^6 = -12582912x^3$$

$$(2)^5 \times \binom{4}{2}(2x)^2(-3)^2 \times \binom{6}{1}\left(\frac{x}{2}\right)(-4)^5 = -21233664x^3$$

$$(2)^5 \times \binom{4}{1}(2x)(-3)^3 \times \binom{6}{2}\left(\frac{x}{2}\right)^2(-4)^4 = -6635520x^3$$

$$(2)^5 \times (-3)^4 \times \binom{6}{3}\left(\frac{x}{2}\right)^3(-4)^3 = -414720x^3$$

Summing: the coefficient of x^3 is 11 802 624.

5 The errors in this solution include division by $\sin^2 x$ (which loses some solutions) and $\sin x = \frac{1}{2}$ has more solutions in the given range than shown.

The correct solutions is:

$\sin^2 x - \cos^2 x = 4\sin^3 x - 1$

$\sin^2 x - (1 - \sin^2 x) = 4\sin^3 x - 1$

$2\sin^2 x - 1 = 4\sin^3 x - 1$

$2\sin^2 x = 4\sin^3 x$

$\sin^2 x = 2\sin^3 x$

$2\sin^3 x - \sin^2 x = 0$

$\sin^2 x(2\sin x - 1) = 0$

$\sin x = 0$ or $\sin x = \frac{1}{2}$

$x = 0, \pi, 2\pi \quad x = \frac{\pi}{6}, \frac{5\pi}{6}$

6 **i** $ax^2 + bx + c = 0$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

ii

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} = -\frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

iii $x + \dfrac{b}{2a} = \pm\sqrt{\dfrac{b^2 - 4ac}{4a^2}}$

$x + \dfrac{b}{2a} = \dfrac{\pm\sqrt{b^2 - 4ac}}{2a}$

$x = -\dfrac{b}{2a}\dfrac{\pm\sqrt{b^2 - 4ac}}{2a}$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

7 i $gh(x) = g(\sin x) = \sin^2 x$, $0 \leqslant x \leqslant \dfrac{\pi}{2}$ and

$0 \leqslant gh(x) \leqslant 1$

ii $fgh(x) = f(\sin^2 x) = \sqrt{1 - \sin^2 x} = \cos x$

$0 \leqslant x \leqslant 1$ and $0 \leqslant fgh(x) \leqslant \cos(1)$

8 i Area of circle 1 is 100π and area of circle 2 is 25π.

So $a = 100\pi$ and $r = \dfrac{1}{4}$. The area is reduced by a scale factor of 4 for each consecutive circle. The radius is halved for each consecutive circle.

Therefore, the 4th circle has area $\dfrac{25\pi}{16}$ cm^2 from which its radius is $\dfrac{5}{4}$ cm.

The distance of the centre of the 4th circle from O is the radii of circles 1 and 4 and the diameters of circles 2 and 3 $\Rightarrow 10 + (2 \times 5) + (2 \times 2.5) + 1.25 = 26.25$.

So the centre of circle 4 is (26.25, 0) with radius $\dfrac{5}{4}$ cm.

ii $S_\infty = \dfrac{100\pi}{1 - \frac{1}{4}} = \dfrac{400\pi}{3}$ cm^2

iii The total area is found by doubling the sum to positive infinity and subtracting the area of the original circle as this will have been double-counted.

Total area $= 2\left(\dfrac{400\pi}{3}\right) - 100\pi = \dfrac{500\pi}{3}$ cm^2

9 i $\dfrac{dy}{dx} = 4x^3 - 9x^2 \quad \dfrac{d^2y}{dx^2} = 12x^2 - 18x$

At stationary points: $0 = 4x^3 - 9x^2 \Rightarrow x = 0$ or $x = \dfrac{9}{4}$.

When $x = \dfrac{9}{4}$, $\dfrac{d^2y}{dx^2} = \dfrac{81}{4} > 0 \Rightarrow$ minimum

ii When $x = 0$, $\dfrac{d^2y}{dx^2} = 0 \Rightarrow$ may be a point of inflection.

When $x = -0.1$, $\dfrac{dy}{dx} = -0.094$ and when $x = 0.1$, $\dfrac{dy}{dx} = -0.086$

The gradient either side of zero is negative and the curve is continuous. This suggests that $x = 0$ is a point of inflection.

iii Not always. A point of inflection occurs when the sign of the curvature changes. This can occur without the need for a stationary point.

10 $\cos x = \sin\left(\dfrac{\pi}{2} - x\right) \Rightarrow \sin\left(\cos\left(\dfrac{\pi}{6}\right)\right) = \sin\left(\dfrac{\pi}{2} - x\right)$

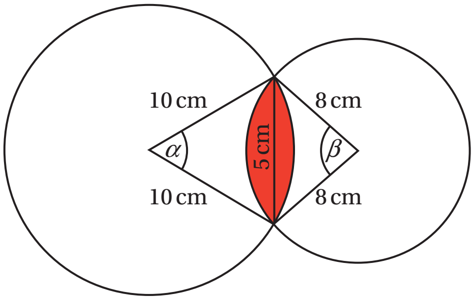
$\cos\left(\dfrac{\pi}{6}\right) = \dfrac{\pi}{2} - x$

$\dfrac{\sqrt{3}}{2} = \dfrac{\pi}{2} - x$

$x = \dfrac{\pi}{2} - \dfrac{\sqrt{3}}{2}$

$x = \dfrac{\pi - \sqrt{3}}{2}$

11



Use the cosine rule to find α and β.

$5^2 = 10^2 + 10^2 - 2(10)(10)\cos\alpha$

$25 = 200 - 200\cos\alpha$

$\cos\alpha = \dfrac{175}{200}$

$\alpha = 0.505$ radians

$5^2 = 8^2 + 8^2 - 2(8)(8)\cos\beta$

$25 = 128 - 128\cos\beta$

$\cos\beta = \dfrac{103}{128}$

$\beta = 0.636$ radians

The shaded area can be found in two parts. In each case, subtract the area of the isosceles triangle from the area of the sector.

Shaded area $= \left(\dfrac{1}{2} \times 10^2\alpha - \dfrac{1}{2} \times 10^2 \sin\alpha\right)$

$+ \left(\dfrac{1}{2} \times 8^2\beta - \dfrac{1}{2} \times 8^2 \sin\beta\right) = 2.404..$

The required area is the area of each circle minus twice the shaded area.

Required area $= 100\pi + 64\pi - 2 \times 2.404 \ldots = 510.41$ cm^2

12 i The terms can be written as: $2, 2 + 2d, 2 + 5d$.

Consecutive terms in a geometric sequence

$\Rightarrow \frac{2+2d}{2} = \frac{2+5d}{2+2d}$

Solving: $(2 + 2d)^2 = 2(2 + 5d)$

$4 + 8d + 4d^2 = 4 + 10d$

$4d^2 - 2d = 0$

$2d^2 - d = 0$

$d\,(2d - 1) = 0$

The non-trivial value of $d = \frac{1}{2}$

We know that

$r = \frac{2+2d}{2} = \frac{2+1}{2} = \frac{3}{2}$

ii For the arithmetic progressions, the 4th term is $2 + 3\left(\frac{1}{2}\right) = \frac{7}{2}$.

For the geometric progression, the 4th term is $a\left(\frac{3}{2}\right)^3 = \frac{27}{8}a$

Equating the terms:

$\frac{27}{8}a = \frac{7}{2}$

$54a = 56$

$a = \frac{28}{27}$