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CAMBRIDGE O LEVEL MATHEMATICS

MICHAEL HANDBURY • JOHN JESKINS
JEAN MATTHEWS • HEATHER WEST

SERIES EDITOR: BRIAN SEAGER

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MATHEMATICS

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INTRODUCTION

For the teacher

This book is intended to be used by students preparing for Cambridge International Examinations O Level Mathematics (Syllabus D).

The structure of the book follows closely the contents of the syllabus, with one chapter devoted to each item of the syllabus. However, in some cases, the full development of a topic is not achieved until a later chapter. For example, further work relevant to Standard Form in Chapter 7 is contained in Chapter 13. It is possible to work through the chapters in order but more variety can be achieved by moving between different areas of the curriculum and the following features will help in planning this.

Each chapter starts with a ‘By the end of this chapter you will be able to:’ box, which repeats the relevant statements from the syllabus.

Immediately after this is a ‘Check you can:’ box, which lists the knowledge and skills required before commencing the chapter. This helps you decide whether topics from earlier in the book need to be covered or revised first. Sometimes, work from a later chapter may be required to complete the topic. If so, that part of the chapter can be delayed until later, or the extra knowledge or technique introduced early. It will then be consolidated in working through the later chapter.

Throughout each chapter there are numerous Examples with worked solutions, illustrating each aspect of the topic and helping students understand how to tackle the Exercises which follow. The questions in the exercises give considerable practice to reinforce the learning.*

The syllabus requires that some of the work be done without a calculator. This is clearly indicated in the book and questions in the exercises are indicated with 

There are also regular hints and guidance contained in the Note boxes, helping students avoid common pitfalls and clarifying what is in the text.

*Please note that answers to the questions in the exercises will appear on the website which accompanies this book. These are not endorsed by Cambridge International Examinations as a mark scheme but they will provide useful guidance when assessing student progress. Answers can be found at www.hoddereducation.com/Cambridgeextras

Brian Seager

For the student

How to use this book

The chapters in this book follow the syllabus and each begins with:

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- which lists what you will learn, as detailed in the syllabus.

Following these statements there is:

CHECK YOU CAN:

- which lists what you need to know before you start the chapter. If you are not certain, it is a good idea to revise these first.

Throughout a chapter there are:

Examples

Which show you how to apply what you are learning. In each case, a question is followed by a worked solution. These are a model to help you answer the questions in the Exercise which follows.

Another feature of the book is:

Note

These contain hints and advice about what you are learning.

1 NUMBER

Types of number

Integers

Integers are positive and negative whole numbers: ..., -2, -1, 0, 1, 2, ...

Natural numbers

Natural numbers are integers that can be used for counting: 1, 2, 3, 4, 5, ...

Rational numbers

A rational number is a number that can be written as a fraction $\frac{a}{b}$ where a and b are integers and $b \neq 0$.

Rational numbers include:

- all integers
- all terminating decimals, for example 5.81 can be written as $\frac{581}{100}$
- all recurring decimals, for example $0.\dot{6} = 0.666\ 666\dots$ can be written as $\frac{2}{3}$.

Irrational numbers

An irrational number is a number that cannot be written as a fraction such as $\sqrt{2}$ or π .

They give decimals that do not terminate or recur.

Real numbers

The set of real numbers is made up of all rational and irrational numbers.

Example 1.1

Question

Sort the numbers in the list below into rational and irrational numbers.

Show how you decide.

$$0.8652 \quad \sqrt{12} \quad 4\pi \quad \frac{67}{19} \quad \sqrt{64}$$

Solution

Rational numbers can be written as fractions, so 0.8652 , $\frac{67}{19}$ and $\sqrt{64}$ are rational.

$$0.8652 = \frac{8652}{10\ 000} \quad \sqrt{64} = 8$$

Irrational numbers cannot be written as fractions, so $\sqrt{12}$ and 4π are irrational.

$$\sqrt{12} = 3.464\ 101\dots \quad 4\pi = 12.566\ 370\dots$$

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- identify and use natural numbers, integers (positive, negative and zero), prime numbers, common factors and common multiples, rational and irrational numbers (e.g. π , $\sqrt{2}$), real numbers.

CHECK YOU CAN:

- recognise factors and multiples of a number
- write a product using index notation, for example $5 \times 5 \times 5 = 5^3$.

Note

Recurring decimals can be shown by placing dots above the digits, e.g.
 $0.\dot{3} = 0.333\ 333\dots$
 $1.\dot{3}0\dot{7} = 1.307\ 307\ 307\dots$

 **Exercise 1.1**

- 1** State which of these numbers are
- integers
 - natural numbers.
- 7 0.6 27 $\sqrt{8}$ 1534 0 $\frac{4}{5}$ -12
- 2** State which of these numbers are rational, showing how you decide.
- | | | |
|--------------------------------|-----------------------|---------------------------------|
| a $\frac{17}{20}$ | b 0.46 | c $\sqrt{\frac{2}{25}}$ |
| d 5π | e 3.141 59 | f -0.234 |
| g $\sqrt{\frac{4}{25}}$ | h $\sqrt{225}$ | i $2\sqrt{3} + \sqrt{3}$ |
- 3** State which of these numbers are rational, showing how you decide.
- | | | |
|-----------------------|--------------------------------|------------------------------|
| a $\sqrt{169}$ | b 0.49 | c $5 + \sqrt{3}$ |
| d -2.718 | e $5\pi + 2$ | f $\frac{4\pi}{3\pi}$ |
| g $\sqrt{27}$ | h $\sqrt{1\frac{7}{9}}$ | i $-6\sqrt{2}$ |
- 4** Write down an irrational number between each pair of numbers.
- | | | |
|------------------|--------------------|--------------------|
| a 3 and 4 | b 10 and 11 | c 19 and 20 |
|------------------|--------------------|--------------------|

Prime factors

The **factors** of a number are all of the numbers that divide exactly into that number.

A **prime number** is a number with only two factors.

The factors of 7 are 1 and 7, so 7 is a prime number.

The factors of 12 are 1, 2, 3, 4, 6 and 12, so 12 is not a prime number.

The only factor of 1 is 1, so 1 is not a prime number.

Any number that is not prime can be written as the product of its prime factors.

The prime factors of a number can be found either using a factor tree or by dividing repeatedly by prime numbers.

Example 1.2

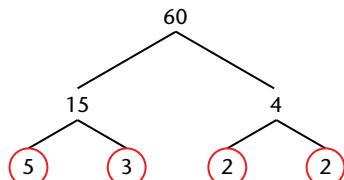
Question

Write 60 as the product of its prime factors.

Solution

Factor tree method

It doesn't matter how you start the factor tree, the ends of the branches will be the same.



The prime factors of 60 are 2, 3 and 5.

Written as a product of its prime factors, $60 = 2 \times 2 \times 3 \times 5$.

Using index notation, $60 = 2^2 \times 3 \times 5$.

Division method

2	60
2	30
3	15
5	5
	1

Divide by the prime numbers, starting with 2, until the result is 1.

Exercise 1.2

- 1 Write down all of the factors of each of these numbers.
a 8 b 15 c 27 d 54
- 2 Write down the first ten prime numbers.
- 3 Write down the prime factors of each of these numbers.
a 12 b 20 c 55 d 84
- 4 Express each of these numbers as the product of its prime factors.
a 48 b 72 c 210 d 350
e 75 f 275 g 120 h 198
- 5 Express each of these numbers as the product of its prime factors.
a 495 b 260 c 2700 d 1078
e 420 f 1125 g 112 h 1960
- 6 a Write each of these square numbers as the product of its prime factors.
i 25 ii 36 iii 100 iv 144
b Comment on what you notice about each of the products in a.
- 7 a Write 96 as the product of its prime factors.
b Find the smallest positive integer k such that $96k$ is a square number.
- 8 a Write 392 as the product of its prime factors.
b Find the smallest positive integer k such that $392k$ is a cube number.

Common factors and common multiples

A **common factor** of two numbers is a number that is a factor of *both* of them.

2 is a common factor of 8 and 12 because $8 \div 2 = 4$ and $12 \div 2 = 6$.

The **highest common factor (HCF)** of two numbers is the highest number that is a factor of both numbers.

The highest common factor of 8 and 12 is 4.

A **multiple** of a number is the product of the number and any integer.

A **common multiple** of two numbers is a number that is a multiple of *both* of them.

20 is a common multiple of 2 and 5 because $2 \times 10 = 20$ and $5 \times 4 = 20$.

The **lowest common multiple (LCM)** of two numbers is the lowest number that is a multiple of both numbers.

The lowest common multiple of 2 and 5 is 10.

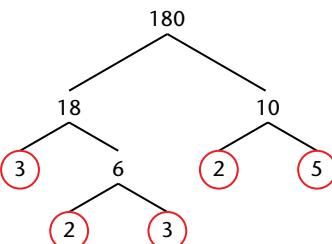
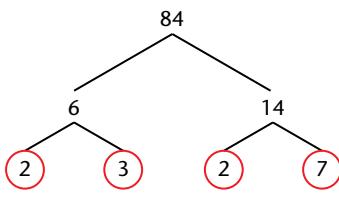
Example 1.3

Question

- Find the highest common factor (HCF) of 84 and 180.
- Find the lowest common multiple (LCM) of 84 and 180.

Solution

First write each number as the product of its prime factors.



$$84 = 2 \times 2 \times 3 \times 7 = 2^2 \times 3 \times 7$$

$$180 = 2 \times 2 \times 3 \times 3 \times 5 = 2^2 \times 3^2 \times 5$$

- To find the highest common factor, find all numbers that appear in *both* lists.

$$84 = 2 \times 2 \times 3 \times 7$$

The highest common factor will be no higher than the smaller of the two numbers.

$$180 = 2 \times 2 \times 3 \times 3 \times 5$$

The highest common factor of 84 and 180 is $2 \times 2 \times 3 = 12$.

This means that 12 is the highest number that is a factor of both 84 and 180.

- b** To find the lowest common multiple, find all of the prime numbers that appear in each list and use the higher power of each.

$$84 = 2^2 \times 3 \times 7$$

The lowest common multiple will be no lower than the larger of the two numbers.

$$180 = 2^2 \times 3^2 \times 5$$

The lowest common multiple of 84 and 180 is $2^2 \times 3^2 \times 5 \times 7 = 1260$.

This means that 1260 is the lowest number that is a multiple of both 84 and 180.

It is the lowest number that has both 84 and 180 as factors.

Exercise 1.3

- 1** For each pair of numbers:
- i express each number as the product of its prime factors
 - ii find the highest common factor (HCF)
 - iii find the lowest common multiple (LCM).
- a 18 and 24 b 64 and 100 c 50 and 350 d 72 and 126
- 2** Find the highest common factor (HCF) and lowest common multiple (LCM) of each pair of numbers.
- a 27 and 63 b 20 and 50 c 48 and 84 d 50 and 64 e 42 and 49
- 3** For each pair of numbers:
- i express each number as the product of its prime factors
 - ii find the highest common factor (HCF)
 - iii find the lowest common multiple (LCM).
- a 260 and 300 b 340 and 425 c 756 and 2100 d 1980 and 2376
- 4** Find the HCF and LCM of each pair of numbers.
- a 5544 and 2268 b 2016 and 10 584
- 5** a Find the highest common factor of 45, 60 and 75.
b Find the lowest common multiple of 45, 60 and 75.
- 6** A rectangle measures 240 mm by 204 mm.
It is split up into identical squares.
Find the largest possible side length for the squares.
- 7** There are two lighthouses near a port.
The light in the first lighthouse flashes every 22 seconds.
The light in the second lighthouse flashes every 16 seconds.
At 10 p.m. one evening both lights are switched on.
What is the next time that the lights flash at the same time?
- 8** Buses to Shenley leave the bus station every 40 minutes.
Buses to Winley leave every 15 minutes.
At 8.15 a.m. buses to both Shenley and Winley leave the bus station.
When is the next time that buses to both places leave at the same time?

2 SET LANGUAGE AND NOTATION

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- use language, notation and Venn diagrams to describe sets and represent relationships between sets.

CHECK YOU CAN:

- recall the meaning of the terms **integer**, **prime**, **multiple** and **factor**.

The definition of a set

A set is a collection of numbers, shapes, letters, points or other objects. They form a set because they fulfil certain conditions.

The notation for a set is a pair of curly brackets: $\{\dots\}$.

The individual members of a set are called **elements**.

A set can be defined by giving a rule which satisfies all the elements or by giving a list of the elements.

For example,

$$A = \{\text{integers from 1 to 10}\} \quad \text{or} \quad A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

Exercise 2.1

List the elements of the following sets.

- 1 {integers from 11 to 18}
- 2 {the first five prime numbers}
- 3 {the factors of 12}
- 4 {the multiples of 8 less than 50}
- 5 {vowels}

Note

If you are giving a rule for a set then it must be precise. In A above, do not just use 'integers'.

The universal set

All the sets in Exercise 2.1 are **finite sets**. They have a fixed number of elements.

Sets can also have an infinite number of elements.

For example,

$$\{\text{multiples of 8}\} \quad \text{or} \quad \{\text{prime numbers}\}$$

Most of the sets you will be dealing with will be finite sets.

Sometimes we make sure we are dealing with a finite set by defining a **universal set**.

A universal set is a set from which – for a particular situation – all other sets will be taken.

For example, if you define the universal set as positive integers less than 50, then the set in Exercise 2.1 question 4 could simply be defined as $\{\text{multiples of 8}\}$ since you are only considering integers less than 50.

Similarly, if you define the universal set as positive integers less than 12, then the set in question 2 could have been defined as $\{\text{prime numbers}\}$.

Notation

The symbol for a universal set is \mathcal{E}

The symbol \in means ‘is an element of’.

The symbol \notin means ‘is not an element of’.

So $3 \in \{\text{prime numbers}\}$

and $4 \notin \{\text{prime numbers}\}$.

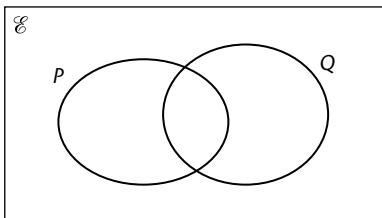
Venn diagrams

Venn diagrams are a way of showing sets and the relationships between sets.

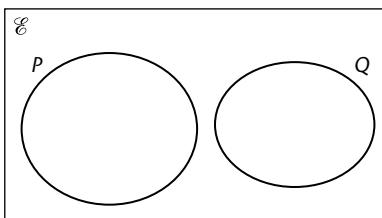
Venn diagrams were introduced in 1880 by John Venn.

In a Venn diagram the universal set is shown by a rectangle. Other sets are drawn as circles or ovals within the rectangle.

The diagrams here show three typical Venn diagrams.

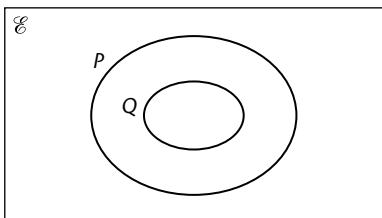


This Venn diagram shows two sets, P and Q , where there are some elements that are in both sets.



This Venn diagram shows two sets, P and Q , where there are no elements that are in both sets.

Sets P and Q are **disjoint**.

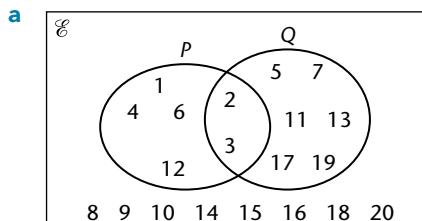


This Venn diagram shows two sets, P and Q , where all the elements of set Q are also in set P .

Set Q is a **subset** of set P .

Venn diagrams are not restricted to two sets. You may be asked to draw a Venn diagram for three or more sets.

You may be asked to draw a Venn diagram and fill in the elements of the sets.

Example 2.1 $\mathcal{E} = \{\text{integers from 1 to 20}\}$ $P = \{\text{factors of 12}\}$ $Q = \{\text{prime numbers}\}$ **Question**a Draw a Venn diagram to show the elements of the universal set and its subsets P and Q .b List the elements that are in both set P and set Q .**Solution**

b 2, 3

Note

Do not forget to fill in the elements of \mathcal{E} that are in neither P nor Q .

We do not put an element in a Venn diagram more than once.
So, for example, if

$$A = \{\text{letters in the word } \textit{label}\}$$

the set is $\{l, a, b, e\}$ and you put those four letters in the Venn diagram.
You would not put the letter l in twice.

Exercise 2.2

Draw a Venn diagram to show the elements of the universal set and its subsets P and Q .

1 $\mathcal{E} = \{\text{integers from 1 to 20}\}$
 $P = \{\text{factors of 18}\}$
 $Q = \{\text{odd numbers}\}$

2 $\mathcal{E} = \{\text{integers less than 25}\}$
 $P = \{\text{multiples of 4}\}$
 $Q = \{\text{prime numbers}\}$

3 $\mathcal{E} = \{\text{first 15 letters of the alphabet}\}$
 $P = \{\text{letters in the word } \textit{golf}\}$
 $Q = \{\text{letters in the word } \textit{beam}\}$

4 $\mathcal{E} = \{\text{positive integers less than 21}\}$
 $P = \{\text{even numbers}\}$
 $Q = \{\text{multiples of 4}\}$

5 $\mathcal{E} = \{\text{days of the week}\}$
 $P = \{\text{days with six letters}\}$
 $Q = \{\text{days beginning with } S\}$

6 $\mathcal{E} = \{\text{multiples of 3 less than 50}\}$
 $P = \{\text{factors of 36}\}$
 $Q = \{\text{odd numbers}\}$

The relationship between sets

The **intersection** of two sets, P and Q , is all the elements that are in both set P and set Q .

You can write this as $P \cap Q$.

In Example 2.1, $P \cap Q = \{2, 3\}$.

The **union** of two sets, P and Q , is all the elements that are in set P or set Q or both.

You can write this as $P \cup Q$.

In Example 2.1, $P \cup Q = \{1, 2, 3, 4, 5, 6, 7, 11, 12, 13, 17, 19\}$

If a set has no elements, it is called the **empty set**.

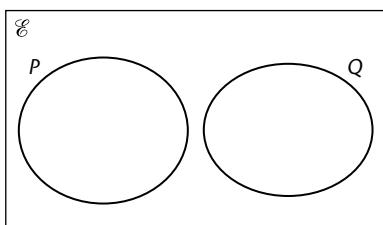
The symbol for the empty set is \emptyset .

It may seem a trivial idea but it is quite important when dealing with some sets.

For example, if

$$\mathcal{E} = \{\text{polygons}\} \quad P = \{\text{triangles}\} \quad Q = \{\text{quadrilaterals}\}$$

then the Venn diagram looks like this.



The sets are disjoint and so the intersection, $P \cap Q = \emptyset$.

Exercise 2.3

For each of the questions in Exercise 2.2, find

- a** $P \cap Q$ **b** $P \cup Q$.

The complement of a set

A' means those elements of the universal set which are *not* in set A .

It is called the **complement** of A .

For example, if

$$\mathcal{E} = \{\text{positive integers less than } 13\} \text{ and } A = \{\text{factors of } 12\} = \{1, 2, 3, 4, 6, 12\}$$

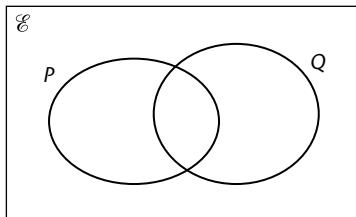
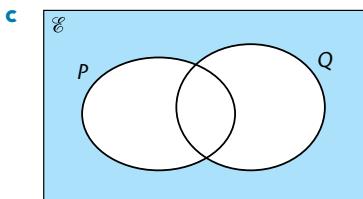
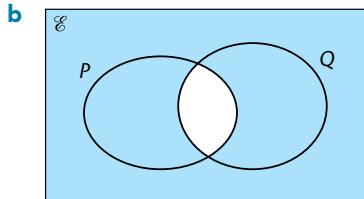
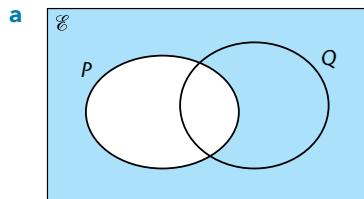
then

$$A' = \{5, 7, 8, 9, 10, 11\}$$

Example 2.2**Question**

On separate copies of the diagram shade these sets.

- P'
- $(P \cap Q)'$
- $(P \cup Q)'$

**Solution****Exercise 2.4**

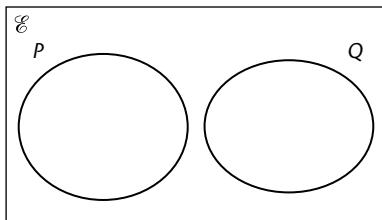
For each of the questions in Exercise 2.2, list the elements of these sets.

- Q'
- $(P \cup Q)'$

Exercise 2.5

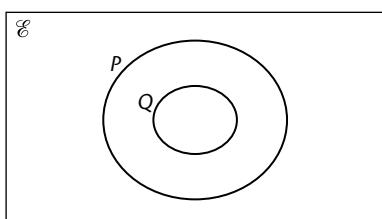
- On separate copies of the diagram shade these sets.

- P'
- $(P \cap Q)'$
- $(P \cup Q)'$



- On separate copies of the diagram shade these sets.

- $P \cap Q$
- $P \cup Q$
- P'



- $\mathcal{E} = \{\text{polygons}\}$

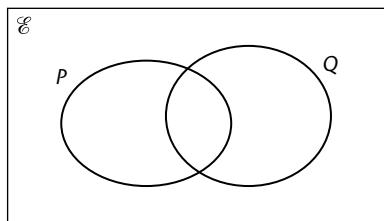
$A = \{\text{quadrilaterals}\}$

$B = \{\text{regular polygons}\}$

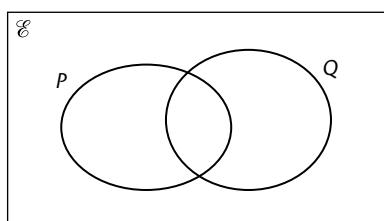
- Draw a Venn diagram to represent these sets.

- Describe in words the set $A \cap B$.

- 4 a** On separate copies of the diagram shade these sets.
i $(P \cap Q)'$ **ii** $P' \cup Q'$
b What do you notice?



- 5 a** On separate copies of the diagram shade these sets.
i $(P \cup Q)'$ **ii** $P' \cap Q'$
c What do you notice?



Subsets

Example 2.3

Question

Find all the subsets of $P = \{a, b, c, d\}$.

Solution

Subsets with one element: $\{a\}, \{b\}, \{c\}, \{d\}$

Subsets with two elements: $\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$

Subsets with three elements: $\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$

In addition to these 14 subsets,

the set P itself is regarded as a subset: $\{a, b, c, d\}$

the empty set also is regarded as a subset: \emptyset

This makes 16 subsets in total.

A set with 4 elements has 16 or 2^4 subsets.

In general, a set with n elements has 2^n subsets.

In Example 2.3, the first 14 subsets of set P and the empty set are known as **proper subsets**. The proper subsets *do not* include the set itself.

A set with n elements has $2^n - 1$ proper subsets

Notation

The symbol \subseteq means ‘is a subset of’.

So $A \subseteq B$ means set A is a subset of set B . It includes the possibility that set A could be the set B itself or the empty set.

The symbol $\not\subseteq$ means ‘is not a subset of’. So $A \not\subseteq B$ means set A is not a subset of set B .

The symbol \subset means ‘is a proper subset of’. So $A \subset B$ means set A is a proper subset of set B . It excludes the possibility that set A could be the set B itself.

The symbol $\not\subset$ means ‘is not a proper subset of’. So $A \not\subset B$ means set A is not a proper subset of set B .

Problem solving with Venn diagrams

If A is a set then $n(A)$ means the number of elements in set A .

For example, if $A = \{a, b, c, d\}$, then $n(A) = 4$.

When you are using a Venn diagram to solve a problem you can write the number of elements in the subsets rather than filling in all the elements.

Example 2.4

Question

There are 32 students in a class.

They can choose to study history (H) or geography (G) or both or neither:

- 18 study history
- 20 study geography
- 8 study both history and geography.

- Draw a Venn diagram to show this information.
- Find the number of students who study neither history nor geography.

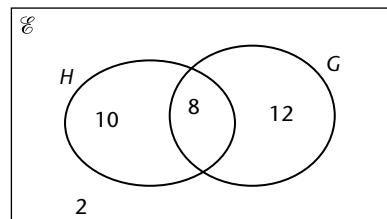
Solution

- Since 8 students study both,
10 students study history but not geography
12 students study geography but not history.
- The number of students who study history or geography or both is

$$10 + 8 + 12 = 30.$$

So the number of students who study neither history nor geography is

$$32 - 30 = 2.$$



Alternative ways to define a set

If a set is finite, you can define it by listing its elements.

For example, $P = \{1, 4, 9, 16, 25\}$.

However, if a set is infinite, you cannot list all the elements so you define it by giving the rule used to form it.

There are various ways to do this. For example,

$$A = \{x : x \text{ is a natural number}\}$$

$$B = \{(x, y) : y = 2x + 1\}$$

$$C = \{x : 2 \leq x \leq 5\}$$

Exercise 2.6

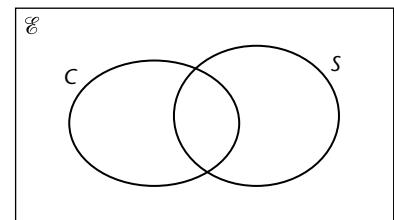
- 1** Find all the subsets of $\{p, q, r\}$.
- 2** How many subsets does the set $\{5, 6, 7, 8, 9, 10\}$ have?
- 3** Muna has 35 books in her electronic book reader
 - 20 are crime stories (C)
 - 12 are books of short stories (S)
 - 7 are books of short crime stories.
 a Copy and complete this Venn diagram to show the number of books of each type in Muna's e-reader.
 b Find
 - i $n(C \cap S')$
 - ii $n(C \cup S')$.
- 4** In a class of 30 students
 - 22 study physics (P)
 - 19 study chemistry (C)
 - 6 study neither physics nor chemistry.
 a Find the number of students who study both physics and chemistry.
 b Show the information in a Venn diagram.
- 5** In a sports club with 130 members
 - 50 play tennis but not football
 - 40 play football but not tennis
 - 15 play neither football nor tennis.
 a Draw a Venn diagram to show this information.
 b Find the number of members who play both tennis and football.
- 6** For two sets A and B , $n(A) = 20$, $n(B) = 30$ and $n(A \cup B) = 38$.
By drawing a Venn diagram or otherwise, find $n(A \cap B)$.
- 7** For two sets A and B , $n(A) = 20$, $n(B) = 30$ and $n(A \cap B) = 11$.
By drawing a Venn diagram or otherwise, find $n(A \cup B)$.
- 8** Two sets P and Q are such that $n(P \cap Q) = 0$.
Show the sets P and Q on a Venn diagram.
- 9** Two sets P and Q are such that $n(P \cap Q) = n(P)$.
Show the sets P and Q on a Venn diagram.
- 10** Two sets P and Q are such that $n(P \cup Q) = n(P)$.
Show the sets P and Q on a Venn diagram.
- 11** $\mathcal{E} = \{x : x \text{ is an integer}\}$ and $P = \{x : -2 \leq x < 4\}$.
List the elements of P .

Note

In definitions like this the colon (:) is read as 'where'.

So $A = \{1, 2, 3, 4, 5, \dots\}$ and is an infinite set.

$B = \{(x, y) : y = 2x + 1\}$ is read as 'set B consists of the points (x, y) where $y = 2x + 1$ '.



Three-set problems

Example 2.5

Question

In a sixth form of 200 students, three of the subjects students can study are mathematics (M), technology (T) and psychology (P).

110 students study mathematics, 85 study technology and 70 study psychology

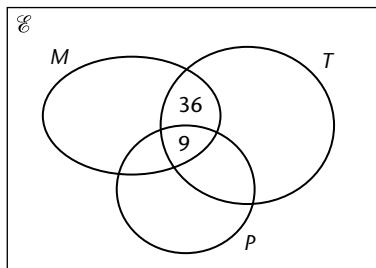
45 study mathematics and technology

35 study mathematics and psychology

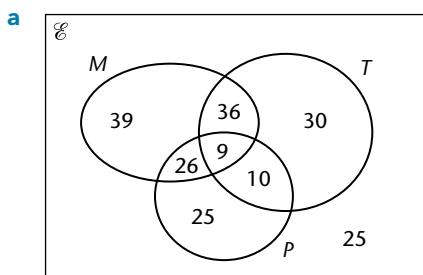
19 study technology and psychology

9 study all three subjects.

- a Copy and complete this Venn diagram.
- b Find
 - i the number of students who study none of the three subjects
 - ii $n(M \cap T \cap P')$
 - iii $n[M \cap (T \cup P)]$.



Solution



- b i $200 - (39 + 36 + 9 + 26 + 30 + 10 + 25) = 200 - 175 = 25$
 ii 36
 iii $36 + 9 + 26 = 71$

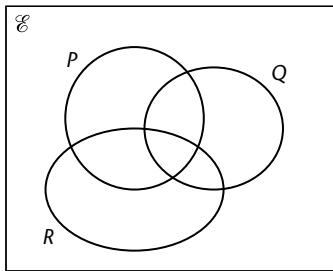
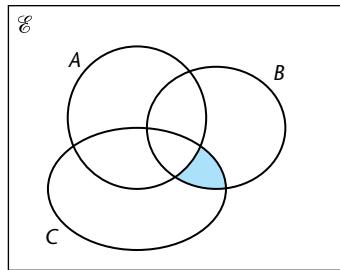
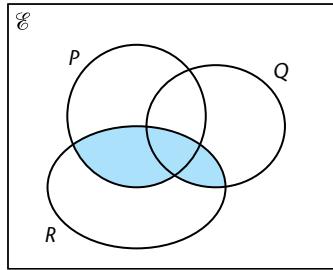
Exercise 2.7

- 1 $\mathcal{E} = \{\text{integers from 2 to 15 inclusive}\}$
 $A = \{\text{prime numbers}\}$
 $B = \{\text{multiples of 4}\}$
 $C = \{\text{multiples of 2}\}$
 - a Draw a Venn diagram to represent the sets \mathcal{E}, A, B and C .
 - b Find
 - i $n(B \cup C)'$
 - ii $n(A \cup B) \cap C'$.

2 Copy the diagram.

Insert a , b , c and d in the correct subsets in the Venn diagram given the following information.

- i** $a \in P \cap Q \cap R$
- ii** $b \in (P \cup Q \cup R)'$
- iii** $c \in (P \cup Q)' \cap R$
- iv** $d \in P \cap Q \cap R'$

**3** Use set notation to describe the sets represented by the shaded area in these Venn diagrams.**a****b****4** $\mathcal{E} = \{\text{integers}\}$

$$E = \{\text{multiples of } 2\}$$

$$T = \{\text{multiples of } 3\}$$

$$F = \{\text{multiples of } 4\}$$

- a** Use set notation to express $M = \{\text{multiples of } 12\}$, as simply as possible, in terms of E , T and F .

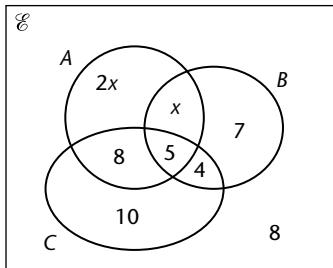
- b** Simplify $E \cup F$.

5 The Venn diagram shows the number of elements in each subset.

- a** Find $n(B \cap C)$.

- b** You are given that $n(A \cup B) = n(C)$.

Find x .

**6** The sets P , Q and R are subsets of the universal set \mathcal{E} .

$$Q \subset P$$

$$Q \cap R = \emptyset$$

$$P \cap R \neq \emptyset$$

Draw a Venn diagram to show the sets P , Q , R and \mathcal{E} .

7 Copy and complete these statements about sets P , Q and R .

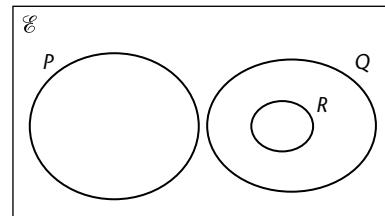
$$\mathbf{a} \quad P \cap \dots = \emptyset$$

$$\mathbf{b} \quad R \dots Q$$

$$\mathbf{c} \quad R \cup \dots = Q$$

$$\mathbf{d} \quad n(R \cap Q) = n(\dots)$$

$$\mathbf{e} \quad n(P) + n(\dots) = n(P \cup R)$$



3 SQUares, Square Roots, Cubes and Cube Roots

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- calculate squares, square roots, cubes and cube roots of numbers.

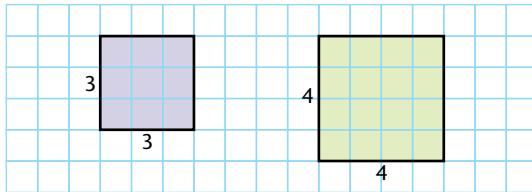
CHECK YOU CAN:

- multiply numbers, with and without a calculator
- find the area of a square.

You should already have met the topics in this chapter. This chapter is a quick reminder.

Squares and square roots

As you can see in the diagram, the square with side 3 has an area of $3 \times 3 = 9$ squares and the square with side 4 has an area of $4 \times 4 = 16$ squares.



Note

You can say that the square of 3 is 9 or 3 squared is 9 and write it as $3^2 = 9$.

The integers 1, 4, 9, 16, 25, ... are the squares of the integers 1, 2, 3, 4, 5,

Because $16 = 4^2$, the positive square root of 16 is 4. It is written as $\sqrt{16} = 4$.

Similarly $\sqrt{36} = 6$ and $\sqrt{81} = 9$.

You can use your calculator to find squares and square roots.

Example 3.1

Question

Work out these using a calculator.

a 47^2 b $\sqrt{729}$

Solution

a $47^2 = 2209$

On your calculator you need to press **4** **7** **x^2** **=**

b $\sqrt{729} = 27$

On your calculator you need to press **\sqrt{x}** **7** **2** **9** **=**

Note

Some calculators operate slightly differently. Learn which keys you need to use to do these operations on your calculator.

You also need to be able to find squares and square roots without a calculator.

Here is a list of the squares of the integers 1 to 15. You should learn these square numbers. You will also need them for finding square roots without a calculator.

Integer	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Square number	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225

Example 3.2**Question**Work out $8^2 - 4^2$.**Solution**

Work out the squares first.

$$8^2 = 8 \times 8 = 64$$

$$4^2 = 4 \times 4 = 16$$

$$\text{So } 8^2 - 4^2 = 64 - 16$$

$$= 48$$

Exercise 3.1

- 1 Write down the square of these numbers.

a 7	b 12	c 5	d 10	e 9
f 8	g 11	h 3	i 6	j 4

- 2 Write down the positive square root of these numbers.

a 49	b 121	c 81	d 36	e 25
f 169	g 144	h 225	i 100	j 196

- 3 Work out these.

a 13^2	b 11^2	c 14^2	d 6^2	e 9^2
----------	----------	----------	---------	---------

- 4 Work out these.

a $\sqrt{81}$	b $\sqrt{144}$	c $\sqrt{16}$	d $\sqrt{100}$	e $\sqrt{64}$
---------------	----------------	---------------	----------------	---------------

- 5 Work out these.

a $\sqrt{529}$	b $\sqrt{256}$	c $\sqrt{324}$	d $\sqrt{841}$	e $\sqrt{784}$
----------------	----------------	----------------	----------------	----------------

- 6 Work out these.

a 20^2	b 25^2	c 13^2	d 24^2	e 33^2
----------	----------	----------	----------	----------

- 7 Work out these.

a $6^2 - 5^2$	b $2^2 + 3^2$	c $7^2 - 4^2$	d $3^2 - 2^2$
e $4^2 + 5^2$	f $6^2 - 3^2$	g $5^2 - 4^2 - 3^2$	h $13^2 - 5^2 - 9^2$

- 8 Work out these.

a $21^2 - 9^2$	b $24^2 + 7^2 - 10^2$	c $17^2 - 15^2 + 11^2$	d $20^2 + 21^2 + 22^2$
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Cubes and cube roots

The cube in the diagram has a volume of $2 \times 2 \times 2 = 8$.

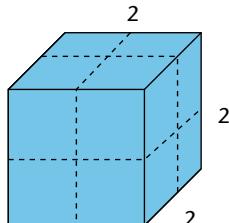
The cube of a number is the number multiplied by itself, and then by itself again.

The integers 1, 8, 27, 64, 125, 216, ... are the cubes of the integers 1, 2, 3, 4, 5, 6,

Because $8 = 2^3$ the cube root of 8 is 2. It is written

as $\sqrt[3]{8} = 2$.

Similarly $\sqrt[3]{27} = 3$ and $\sqrt[3]{64} = 4$.

**Note**

You can say that the cube of 2 is 8 or 2 cubed is 8 and write it as $2^3 = 2 \times 2 \times 2 = 8$.

You need to be able to find cubes and cube roots without a calculator.

Here is a list of the cubes of the integers 1 to 10. You should learn these cube numbers.

Integer	1	2	3	4	5	6	7	8	9	10
Cube number	1	8	27	64	125	216	343	512	729	1000

Example 3.3

Question

Work out $\sqrt[3]{216}$ without a calculator.

Solution

$\sqrt[3]{216} = 6$ You know that $6^3 = 216$, so you also know that the cube root of 216 is 6.

You can also use your calculator to find cubes and cube roots.

Example 3.4

Question

Work out these using a calculator.

a 15^3 b $\sqrt[3]{4913}$

Solution

a $15^3 = 3375$ On your calculator press $1 \boxed{5} x^3 =$

b $\sqrt[3]{4913} = 17$ On your calculator press $\sqrt[3]{\boxed{4} \boxed{9} \boxed{1} \boxed{3}} =$

Note

Some calculators operate slightly differently.

Learn which keys you need to use to do these operations on your calculator.

Exercise 3.2

- 1 Write down the cube of these numbers.
a 4 b 5 c 3 d 10 e 8
- 2 Write down the cube root of these numbers.
a 1 b 64 c 1000
- 3 Work out these.
a 7^3 b 9^3 c 20^3 d 25^3 e 1.5^3
f 2.7^3 g 5.4^3
- 4 Work out these.
a $\sqrt[3]{343}$ b $\sqrt[3]{729}$ c $\sqrt[3]{1331}$ d $\sqrt[3]{1000000}$
e $\sqrt[3]{216}$ f $\sqrt[3]{1728}$ g $\sqrt[3]{512}$
- 5 Work these out. Give your answers to 2 decimal places.
a $\sqrt[3]{56}$ b $\sqrt[3]{230}$ c $\sqrt[3]{529}$ d $\sqrt[3]{1100}$ e $\sqrt[3]{7526}$
- 6 Find the length of a cube whose volume is 45 cm^3 .
Give your answer to 2 decimal places.

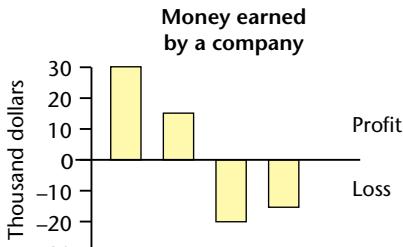
4 DIRECTED NUMBERS

Numbers below zero

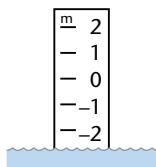
Some numbers are less than zero. These are called **negative numbers**. They are written as ordinary numbers with a negative sign in front.

Negative numbers are used in many situations.

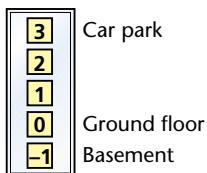
Thermometer measuring temperature



Depth of water in a river



Buttons in a lift



BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- use directed numbers in practical situations.

CHECK YOU CAN:

- add and subtract whole numbers
- read whole numbers from a scale.

Note

Remember that zero is neither positive nor negative.

Example 4.1

Question

Suravi measured the daytime and night-time temperatures in her garden for 2 days.

Here are her results.

Day	Monday daytime	Monday night-time	Tuesday daytime	Tuesday night-time
Temperature ($^{\circ}\text{C}$)	7	-2	3	-5

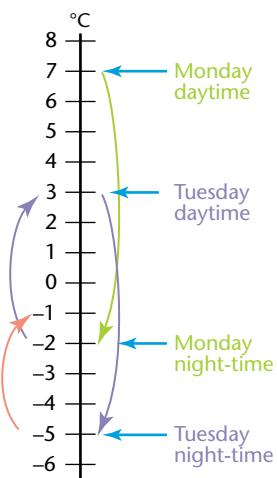
- How much did the temperature change between each reading?
- The daytime temperature on Wednesday was 4°C warmer than the Tuesday night-time temperature.

What was the Wednesday daytime temperature?

Use the temperature scale to help you.

Solution

- From 7°C to -2°C you go down 9°C .
From -2°C to 3°C you go up 5°C .
From 3°C to -5°C you go down 8°C .
- Find -5°C on the scale and move 4°C up.
You get to -1°C .



Example 4.2**Question**

Abal's bank account is overdrawn by \$75.

How much must he put in this account for it to be \$160 in credit?

Solution

Abal's bank account is overdrawn by \$75.

This means Abal owes the bank \$75. It can be shown as $-$75$.

\$160 in credit means that there is \$160 in the account. It can be shown as $+$160$.

From -75 to $+160$ is 235, so he must put \$235 into the account.

Exercise 4.1

- 1** Copy and complete the table.

	Start temperature/ $^{\circ}\text{C}$	Move/ $^{\circ}\text{C}$	End temperature/ $^{\circ}\text{C}$
a	4	Up 3	
b	-2	Down 4	
c	10	Down 14	
d	-5	Down 3	
e	-10		-2
f	10		-9
g	-4		2
h		Up 7	10
i		Down 6	-9
j		Up 2	-8

- 2** The floors in a shopping centre are numbered $-3, -2, -1, 0, 1, 2, 3, 4, 5$.

a Ali parks his car on floor -2 . He takes the lift and goes up 6 floors.
Which floor is he now on?

b While shopping, Ubah goes down 3 floors ending on floor -1 .
Which floor did she start from?

- 3** What is the difference in temperature between each of these?

- | | |
|--|--|
| a 17°C and -1°C | b -8°C and 12°C |
| c -19°C and -5°C | d 30°C and -18°C |
| e 13°C and -5°C | f -10°C and 15°C |
| g -20°C and -2°C | h 25°C and -25°C |

- 4** To the nearest degree, the hottest temperature ever recorded on Earth was 58°C in 1922 and the coldest ever recorded was -89°C in 1983.
What is the difference between these temperatures?

- 5** Geta's bank account is \$221 in credit.
She puts \$155 into the account.
Later she spends \$97 on clothes and \$445 on some electronic goods.
What will Geta's bank account show now?

- 6 The table shows the heights above sea level, in metres, of seven places.

Place	Height/m
Mount Everest	8 863
Bottom of Lake Baikal	-1 484
Bottom of Dead Sea	-792
Ben Nevis	1 344
Mariana Trench	-11 022
Mont Blanc	4 807
World's deepest cave	-1 602

- What is the difference in height between the highest and lowest places?
- 7 The highest temperature ever recorded in England was 38.5°C in 2006 and the lowest ever recorded was -26.1°C in 1982. What is the difference between these temperatures?
- 8 The table shows the coldest temperatures ever recorded on each continent.

Continent	Lowest temperature
Africa	-23.9°C
Antarctica	-89.2°C
Asia	-71.2°C
Australia	-23.0°C
North America	-60.0°C
South America	-32.8°C
Europe	-58.1°C

- What is the difference between the highest and lowest of these temperatures?
- 9 This is an extract from a tide table for Dungeness, Washington State, USA.

Day	High/Low	Tide time	Height/feet above normal
1	High	2.03 a.m.	7.2
	Low	9.28 a.m.	-1.8
	High	2.27 p.m.	7.5
	Low	9.53 p.m.	5.6
2	High	2.48 a.m.	7.2
	Low	10.09 a.m.	-2.2
	High	3.00 p.m.	7.7
	Low	10.42 p.m.	5.4

What is the difference in height between the highest high tide and the lowest low tide?

5 VULGAR AND DECIMAL FRACTIONS AND PERCENTAGES

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- use the language and notation of simple vulgar and decimal fractions and percentages in appropriate contexts
- recognise equivalence and convert between these forms.

CHECK YOU CAN:

- identify a fraction of a shape
- understand and use decimal notation
- find the highest common factor of two numbers.

Note

The denominator of the mixed number is the same as the denominator of the improper fraction.

Fractions

A fraction is a number written in the form $\frac{a}{b}$, where a and b are integers.

The value on the top of the fraction is known as the **numerator**.

The value on the bottom of the fraction is known as the **denominator**.

The fraction $\frac{4}{7}$ is a **proper fraction** because the numerator is smaller than the denominator.

The fraction $\frac{7}{4}$ is an **improper fraction** because the numerator is larger than the denominator.

The fraction $1\frac{3}{4}$ is a **mixed number** because it is formed from an integer and a proper fraction.

An improper fraction can be written as a mixed number.

$$\frac{7}{4} = \frac{4}{4} + \frac{3}{4} = 1\frac{3}{4}$$

Example 5.1

Question

- a Write $\frac{17}{6}$ as a mixed number. b Write $3\frac{4}{5}$ as an improper fraction.

Solution

- a Divide the numerator by the denominator and write the remainder as a fraction over the denominator.

$$17 \div 6 = 2 \text{ remainder } 5$$

$$\text{so } \frac{17}{6} = 2\frac{5}{6}$$

- b Multiply the integer by the denominator of the fraction and add the numerator.

$$3 \times 5 + 4 = 19$$

$$\text{so } 3\frac{4}{5} = \frac{19}{5}$$

Exercise 5.1

- 1 State whether each of these is a proper fraction, an improper fraction or a mixed number.

a $\frac{19}{18}$ b $1\frac{2}{3}$ c $\frac{6}{8}$ d $8\frac{5}{6}$ e $\frac{15}{16}$

- 2 Write each of these improper fractions as a mixed number.

a $\frac{11}{8}$ b $\frac{11}{5}$ c $\frac{9}{4}$ d $\frac{7}{2}$ e $\frac{15}{7}$
f $\frac{10}{3}$ g $\frac{19}{8}$ h $\frac{23}{4}$ i $\frac{33}{10}$ j $\frac{37}{9}$

- 3 Write each of these mixed numbers as an improper fraction.

a $1\frac{1}{8}$

b $2\frac{5}{8}$

c $3\frac{3}{4}$

d $5\frac{1}{2}$

e $3\frac{2}{9}$

f $2\frac{2}{5}$

g $3\frac{2}{3}$

h $2\frac{1}{10}$

i $2\frac{3}{8}$

j $4\frac{6}{7}$

Fraction of a quantity

A fraction can be used to describe a share of a quantity.

The denominator shows how many parts the quantity is divided into.

The numerator shows how many of those parts are required.

Example 5.2

Question

a Work out $\frac{1}{8}$ of 56.

b Work out $\frac{5}{8}$ of 56.

Solution

a Finding $\frac{1}{8}$ of 56 is the same as dividing 56 into 8 parts.

So $\frac{1}{8}$ of 56 = $56 \div 8 = 7$

b $\frac{5}{8}$ means $5 \times \frac{1}{8}$.

So $\frac{5}{8}$ of 56 = $5 \times 7 = 35$

To find a fraction of a quantity, divide by the denominator and multiply by the numerator.

Exercise 5.2



- 1 Work out these.

a $\frac{3}{4}$ of 64 b $\frac{2}{3}$ of 96 c $\frac{5}{7}$ of 35 d $\frac{9}{10}$ of 160 e $\frac{11}{12}$ of 180

- 2 In a school of 858 students, $\frac{6}{11}$ are boys.
How many boys are there?

- 3 In an election, the winning party got $\frac{5}{9}$ of the votes.
There were 28 134 votes in total.
How many votes did the winning party get?



- 4 A shop offers $\frac{1}{4}$ off everything as a special offer.
A mobile phone normally costs \$168.

How much does it cost with the special offer?



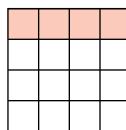
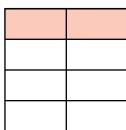
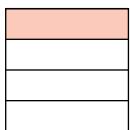
- 5 Which is larger, a $\frac{7}{10}$ share of \$120 or a $\frac{7}{8}$ share of \$104?
Show how you decide.



- 6 Which is larger, a $\frac{3}{8}$ share of \$192 or a $\frac{2}{5}$ share of \$180?
Show how you decide.

Equivalent fractions

These squares can be divided into equal parts in different ways.



The fraction represented by the shaded parts is $\frac{1}{4}$ or $\frac{2}{8}$ or $\frac{4}{16}$.

These three fractions are equal in value and are **equivalent**.

$\frac{1}{4} = \frac{2}{8} = \frac{4}{16}$ but $\frac{1}{4}$ is in its **simplest form** or in its **lowest terms**.

Example 5.3

Question

Write each fraction in its lowest terms.

a $\frac{6}{10}$

b $\frac{18}{24}$

Solution

- a The highest common factor of 6 and 10 is 2, so divide the numerator and denominator by 2.

$\frac{6}{10} = \frac{3}{5}$ in its lowest terms.

- b The highest common factor of 18 and 24 is 6, so divide the numerator and denominator by 6.

$\frac{18}{24} = \frac{3}{4}$ in its lowest terms.

Note

18 and 24 also have common factors of both 2 and 3, so $\frac{9}{12}$ and $\frac{6}{8}$ are also equivalent to $\frac{18}{24}$, but $\frac{3}{4}$ is the simplest form of this fraction.



Exercise 5.3

- 1 Fill in the missing numbers in each set of equivalent fractions.

a $\frac{1}{4} = \frac{\square}{8} = \frac{\square}{12} = \frac{5}{\square}$

b $\frac{1}{5} = \frac{\square}{10} = \frac{\square}{20} = \frac{7}{\square}$

c $\frac{2}{5} = \frac{4}{\square} = \frac{\square}{25} = \frac{12}{\square}$

d $\frac{2}{9} = \frac{4}{\square} = \frac{\square}{36} = \frac{6}{\square}$

e $\frac{1}{7} = \frac{2}{\square} = \frac{\square}{35}$

f $\frac{4}{9} = \frac{16}{\square} = \frac{\square}{72}$

- 2 Express each fraction in its lowest terms.

a $\frac{8}{10}$

b $\frac{2}{12}$

c $\frac{15}{21}$

d $\frac{12}{16}$

e $\frac{14}{21}$

f $\frac{25}{30}$

g $\frac{20}{40}$

h $\frac{18}{30}$

i $\frac{16}{24}$

j $\frac{150}{300}$

k $\frac{20}{120}$

l $\frac{500}{1000}$

m $\frac{56}{70}$

n $\frac{64}{72}$

o $\frac{60}{84}$

p $\frac{120}{180}$

- 3 A bag contains 96 balls.

36 of the balls are red.

What fraction of the balls are red? Give your answer in its lowest terms.

- 4** Akbar drove 64 km of a 120 km journey on motorways.
 What fraction of the journey did he drive on motorways?
 Give your answer in its lowest terms.
- 5** The table gives information about the members of a club.

	Male	Female
Adult	90	60
Child	55	45

Answer these questions, giving each answer as a fraction in its lowest terms.

- a** What fraction of the members are adult males?
b What fraction of the members are female?
c What fraction of the members are children?

Fractions and decimals

We can use place value in a decimal to convert the decimal to a fraction.

Example 5.4

Question

Convert the decimal 0.245 to a fraction in its lowest terms.

Solution

Units	.	Tenths	Hundredths	Thousands
0	.	2	4	5

$$0.245 = \frac{2}{10} + \frac{4}{100} + \frac{5}{1000}$$

$$= \frac{200}{1000} + \frac{40}{1000} + \frac{5}{1000} \quad \text{Convert each fraction to its equivalent with a denominator of 1000.}$$

$$= \frac{245}{1000} \quad \text{The HCF of 245 and 1000 is 5, so divide the numerator and the denominator by 5 to simplify the fraction.}$$

$$= \frac{49}{200}$$

0.245 is equivalent to $\frac{49}{200}$

Note

You can use the place value of the final digit to write the decimal directly as a single fraction. The final digit here, 5, represents thousandths so 0.245 is equivalent to $\frac{245}{1000}$.

Exercise 5.4

Convert each of these decimals to a fraction in its lowest terms.

- | | | | | |
|----------------|----------------|---------------|----------------|------------------|
| 1 0.7 | 2 0.29 | 3 0.85 | 4 0.07 | 5 0.312 |
| 6 0.056 | 7 0.008 | 8 0.02 | 9 0.545 | 10 0.1345 |

Terminating and recurring decimals

In Chapter 1, you learnt that both terminating and recurring decimals were rational numbers.

So both terminating and recurring decimals can be written as fractions.

Also all fractions can be written as a terminating or a recurring decimal.

You can convert the fraction $\frac{5}{8}$ to a decimal using division.

$$\frac{5}{8} = 5 \div 8 = 0.625$$

This is a **terminating decimal** because it finishes at the digit 5.

You can convert the fraction $\frac{1}{6}$ to a decimal using division.

$$\frac{1}{6} = 1 \div 6 = 0.166\overline{666\dots}$$

This is a **recurring decimal** because the digit 6 repeats indefinitely.

Dot notation for recurring decimals

Dot notation can be used when writing recurring decimals.

Dots are placed over the digits that recur.

For example,

$\frac{1}{3} = 0.333\overline{333\dots}$ is written using dot notation as $0.\dot{3}$.

$\frac{1}{6} = 0.166\overline{666\dots}$ is written using dot notation as $0.1\dot{6}$.

$\frac{124}{999} = 0.124\overline{124124\dots}$ is written using dot notation as $0.\overline{124}$.

Example 5.5

Question

- a Convert $\frac{13}{25}$ to a decimal.
- b Convert $\frac{7}{11}$ to a decimal.
- c Convert $0.\dot{6}$ to a fraction using $\frac{1}{3} = 0.\dot{3}$.

Solution

- a $\frac{13}{25} = 13 \div 25 = 0.52$
- b $\frac{7}{11} = 7 \div 11 = 0.636363\dots$ In this case the digits 6 and 3 recur so you can write the answer using dot notation.
 $\frac{7}{11} = 0.\overline{63}$
- c $0.\dot{6} = 2 \times 0.\dot{3} = 2 \times \frac{1}{3} = \frac{2}{3}$

 **Exercise 5.5**

- 1 Convert each of these fractions to a decimal.

a $\frac{3}{8}$

b $\frac{5}{16}$

c $\frac{11}{40}$

d $\frac{79}{250}$

e $\frac{3}{80}$

- 2 Convert each of these fractions to a recurring decimal. Write your answers using dot notation.

a $\frac{2}{3}$

b $\frac{5}{6}$

c $\frac{1}{12}$

d $\frac{4}{15}$

e $\frac{16}{33}$

- 3 Given that $0.\dot{1} = \frac{1}{9}$, write each of these as a fraction.

a $0.\dot{2}$

b $0.\dot{3}$

c $0.\dot{5}$

- 4 Given that $\frac{1}{27} = 0.\dot{0}3\dot{7}$ and $\frac{1}{11} = 0.\dot{0}\dot{9}$, find the decimal equivalent of each of these.

a $\frac{2}{27}$

b $\frac{5}{27}$

c $\frac{10}{27}$

d $\frac{2}{11}$

e $\frac{6}{11}$

Fractions, decimals and percentages

The term **per cent** means ‘out of 100’.

For example 75% means 75 out of every 100 or $\frac{75}{100}$.

$\frac{75}{100}$ can be written in decimal form as 0.75.

So 75% is equivalent to $\frac{75}{100}$ and 0.75.

You can find fraction and decimal equivalents of all percentages.

There are some fraction, decimal and percentage equivalents that are useful to remember.

Fraction	Decimal	Percentage
$\frac{1}{2}$	0.5	50%
$\frac{1}{4}$	0.25	25%
$\frac{3}{4}$	0.75	75%
$\frac{1}{10}$	0.1	10%
$\frac{1}{5}$	0.2	20%

Example 5.6**Question**

- a Convert $\frac{3}{8}$ to a percentage.

- b Convert 65% to a fraction in its lowest terms.

Solution

a $\frac{3}{8} = 3 \div 8 = 0.375$ Convert to a decimal by dividing.

$0.375 \times 100 = 37.5$ Multiply by 100 for percentage.

$$\text{So } \frac{3}{8} = 37.5\%$$

b $65\% = \frac{65}{100}$ The HCF of 65 and 100 is 5, so divide the numerator and the denominator by 5 to simplify the fraction.

Note

You can convert directly from a fraction to a percentage using multiplication.

Exercise 5.6

- 1 Convert each of these percentages to a fraction. Write your answers in their lowest terms.

- a 35% b 65% c 8% d 120%

- 2 Convert each of these percentages to a decimal.

- a 16% b 27% c 83% d 7%
 e 31% f 4% g 17% h 2%
 i 150% j 250% k 9% l 12.5%

- 3 Convert each of these fractions to a decimal.

- a $\frac{1}{100}$ b $\frac{17}{100}$ c $\frac{2}{50}$ d $\frac{8}{5}$ e $\frac{1}{8}$
 f $\frac{5}{8}$ g $\frac{3}{20}$ h $\frac{17}{40}$ i $\frac{5}{16}$

- 4 Convert each of the decimals you found in question 3 to a percentage.

- 5 Convert each of these fractions to a percentage. Give your answers correct to 1 decimal place.

- a $\frac{1}{6}$ b $\frac{5}{6}$ c $\frac{1}{12}$ d $\frac{5}{12}$ e $\frac{3}{70}$

- 6 Write three fractions that are equivalent to 40%.

- 7 Write three fractions that are equivalent to 0.125.

- 8 a Convert 160% to a decimal.

- b Convert 160% to a mixed number in its lowest terms.

- 9 The winning party in an election gained $\frac{7}{12}$ of the votes.
 What percentage is this? Give your answer correct to the nearest 1%.

- 10 Imran saves $\frac{2}{9}$ of his earnings.

- What percentage is this? Give your answer correct to the nearest 1%.

- 11 In a survey about the colour of cars, 22% of the people said they preferred red cars, $\frac{3}{20}$ of the people said they preferred silver cars and $\frac{6}{25}$ said they preferred black cars.

- Which colour car was most popular? Show how you decide.

- 12 In class P, $\frac{3}{7}$ of the students are boys.

- In class Q, 45% of the students are boys.

- Which class has the higher proportion of boys? Show how you decide.

6 ORDERING

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- order quantities by magnitude and demonstrate familiarity with the symbols $=$, \neq , $>$, $<$, \geq , \leq .

CHECK YOU CAN:

- use the symbols $=$ and \neq correctly
- order whole numbers
- work with negative numbers
- convert between metric units:
 - m, cm and mm
 - kg and g
 - litres (l), cl and ml
- convert between fractions, decimals and percentages.

Ordering integers

Example 6.1

Question

Put these masses in order, smallest first.

1.2 kg 1500 g 175 g 2 kg 0.8 kg

Solution

Write each value with the same units first. (It is usually easier to use the smallest unit.)

1200 g 1500 g 175 g 2000 g 800 g

Then order them, writing your answer using the original units.

So order is

175 g 0.8 kg 1.2 kg 1500 g 2 kg.

Exercise 6.1

1 Write each set of temperatures in order, lowest first.

- a -2°C 7°C 0°C -5°C 3°C
- b -2°C 5°C 1°C 2°C -1°C
- c 7°C -7°C 4°C -9°C -3°C
- d 9°C 4°C -2°C 7°C -8°C
- e -4°C 5°C -2°C 3°C -7°C

2 Write each set of lengths in order of size, smallest first.

- a 2.42 m 1600 mm 284 cm 9 m 31 cm
- b 423 cm 6100 mm 804 cm 3.2 m 105 mm

3 Write each set of masses in order of size, smallest first.

- a 4000 g 52 000 g 9.4 kg 874 g 1.7 kg
- b 4123 g 2104 g 3.4 kg 0.174 kg 2.79 kg

4 Write each set of capacities in order of size, smallest first.

- a 2.4 litres 1600 ml 80 cl 9 litres 51 cl
- b 3.1 litres 1500 ml 180 cl 1 litre 51.5 ml

Inequalities

$a < b$ means ‘ a is less than b ’.

$a \leq b$ means ‘ a is less than or equal to b ’.

$a > b$ means ‘ a is greater than b ’.

$a \geq b$ means ‘ a is greater than or equal to b ’.

Exercise 6.2

- 1 Rewrite these: insert $>$ or $<$ in each part as appropriate.

- | | | | |
|----------|-----------------------|----------|-------------------------|
| a | 7°C is ... than 2°C. | b | -10°C is ... than 5°C. |
| c | -3°C is ... than 0°C. | d | -7°C is ... than -12°C. |
| e | 10°C is ... than 0°C. | f | -2°C is ... than -5°C. |
| g | 4°C is ... than -1°C. | h | -2°C is ... than 2°C. |

Ordering decimals

Example 6.2

Question

Put these decimals in order of size, smallest first.

0.412 0.0059 0.325 0.046 0.012

Solution

Add zeros to make the decimals all the same length.

0.4120 0.0059 0.3250 0.0460 0.0120

Now, remove the decimal points and any zeros in front of the digits.

4120 59 3250 460 120

The order in size of these values is the order in size of the decimals.

The order is

0.0059 0.012 0.046 0.325 0.412

Note

As you get used to this, you can omit the second step.

Exercise 6.3

- 1 Put these numbers in order of size, smallest first.

- a** 462, 321, 197, 358, 426, 411
- b** 89 125, 39 171, 4621, 59 042, 6317, 9981
- c** 124, 1792, 75, 631, 12, 415
- d** 9425, 4257, 7034, 5218, 6641, 1611
- e** 1 050 403, 1 030 504, 1 020 504, 1 040 501, 1 060 504, 1 010 701

- 2 Put these decimals in order of size, smallest first.

- a** 0.123, 0.456, 0.231, 0.201, 0.102
- b** 0.01, 0.003, 0.1, 0.056, 0.066
- c** 0.0404, 0.404, 0.004 04, 0.044, 0.0044
- d** 0.71, 0.51, 0.112, 0.149, 0.2
- e** 0.913, 0.0946, 0.009 16, 0.090 11, 0.091

- 3 Put these numbers in order of size, smallest first.

- a** 3.12, 3.21, 3.001, 3.102, 3.201
- b** 1.21, 2.12, 12.1, 121, 0.12
- c** 7.023, 7.69, 7.015, 7.105, 7.41
- d** 5.321, 5.001, 5.0102, 5.0201, 5.02
- e** 0.01, 12.02, 0.0121, 1.201, 0.0012
- f** 8.097, 8.79, 8.01, 8.1, 8.04

Ordering fractions

To put fractions in order, convert them to equivalent fractions all with the same denominator, and order them by the numerator.

Example 6.3

Question

Which is the bigger, $\frac{3}{4}$ or $\frac{5}{6}$?

Solution

First find a common denominator. 24 is an obvious one, as $4 \times 6 = 24$, but a smaller one is 12. $\frac{3}{4} = \frac{9}{12}$ $\frac{5}{6} = \frac{10}{12}$
 $\frac{10}{12}$ is bigger than $\frac{9}{12}$, so $\frac{5}{6}$ is bigger than $\frac{3}{4}$.

Note

Multiplying the two denominators together will always work to find a common denominator, but the lowest common multiple is sometimes smaller.

Alternatively, you can convert each fraction to a decimal and compare the decimals as before.

Example 6.4

Question

Put these fractions in order, smallest first.

$$\frac{3}{10} \quad \frac{1}{4} \quad \frac{9}{20} \quad \frac{2}{5} \quad \frac{1}{2}$$

Solution

Use division to convert the fractions to decimals ($3 \div 10$, $1 \div 4$, etc.).

$$0.3 \quad 0.25 \quad 0.45 \quad 0.4 \quad 0.5$$

Now make the number of decimal places the same.

$$0.30 \quad 0.25 \quad 0.45 \quad 0.40 \quad 0.50$$

So order is $\frac{1}{4}, \frac{3}{10}, \frac{2}{5}, \frac{9}{20}, \frac{1}{2}$.

Exercise 6.4

- 1 Write each pair of fractions, inserting $>$ or $<$ as appropriate.

a $\frac{2}{3} \dots \frac{7}{9}$ b $\frac{5}{6} \dots \frac{7}{8}$ c $\frac{3}{8} \dots \frac{7}{20}$ d $\frac{3}{4} \dots \frac{5}{8}$ e $\frac{7}{9} \dots \frac{5}{6}$ f $\frac{3}{10} \dots \frac{4}{15}$

- 2 Write each of these sets of fractions in order, smallest first.

a $\frac{7}{10}, \frac{3}{4}, \frac{11}{20}, \frac{3}{5}$	b $\frac{7}{12}, \frac{3}{4}, \frac{7}{8}, \frac{5}{6}$	c $\frac{13}{15}, \frac{2}{3}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}$
d $\frac{13}{16}, \frac{5}{8}, \frac{3}{4}, \frac{7}{16}, \frac{1}{2}$	e $\frac{2}{5}, \frac{1}{2}, \frac{9}{20}, \frac{17}{40}, \frac{3}{8}$	f $\frac{11}{16}, \frac{7}{8}, \frac{3}{4}, \frac{17}{32}$

Note

Try both methods shown in the examples. The first is useful when calculators are not allowed.

Ordering fractions, decimals and percentages

Example 6.5

Question

Put these numbers in order, smallest first.

$$\frac{1}{4} \quad 3\% \quad 0.41 \quad \frac{11}{40} \quad 0.35$$

Solution

Convert them all to decimals.

$$0.25, \quad 0.03, \quad 0.41, \quad 0.275, \quad 0.35$$

So the order is

$$0.03 \quad 0.25 \quad 0.275 \quad 0.35 \quad 0.41.$$

Write the numbers in their original form.

$$3\% \quad \frac{1}{4} \quad \frac{11}{40} \quad 0.35 \quad 0.41$$

Note

A common error is to write 3% as 0.3, rather than 0.03.

Exercise 6.5

- 1 Put these numbers in order, smallest first.

$$\frac{4}{5}, 88\%, 0.83, \frac{17}{20}, \frac{7}{10}$$

- 2 Put these numbers in order, smallest first.

$$\frac{3}{8}, 35\%, 0.45, \frac{2}{5}, \frac{5}{12}$$

- 3 Put these numbers in order, smallest first.

$$\frac{3}{5}, 30\%, 0.7, \frac{3}{4}, \frac{2}{3}$$

- 4 Soccer teams United, City and Rovers have all played the same number of games.

United have won $\frac{3}{8}$ of the games they have played, City have won 35% and Rovers have won 0.4.

Put the teams in order of the number of matches they have won.

- 5 A group of boys were asked to name their favourite sport.

$\frac{2}{7}$ chose football, 0.27 chose rugby and 28% chose gymnastics.

List the sports in the order of their popularity, the most popular first.

7 STANDARD FORM

Standard form

Standard form is a way of making very large numbers and very small numbers easy to deal with.

In standard form, numbers are written as a number between 1 and 10 multiplied by a power of 10, for example 6.3×10^5 .

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- use the standard form $A \times 10^n$ where n is a positive or negative integer, and $1 \leq A < 10$.

Large numbers

Example 7.1

Write these numbers in standard form.

Question

- a 500 000
b 6 300 000
c 45 600

Solution

- a $500\ 000 = 5 \times 100\ 000 = 5 \times 10^5$
b $6\ 300\ 000 = 6.3 \times 1\ 000\ 000 = 6.3 \times 10^6$
c $45\ 600 = 4.56 \times 10\ 000 = 4.56 \times 10^4$

CHECK YOU CAN:

- understand and use index notation.

Note

You can write down the answer without any intermediate steps.

Move the decimal point until the number is between 1 and 10.

Count the number of places the point has moved: that is the power of 10.

Small numbers

Example 7.2

Question

Write these numbers in standard form.

- a 0.000 003 b 0.000 056 c 0.000 726

Solution

- a $0.000\ 003 = \frac{3}{1\ 000\ 000} = 3 \times \frac{1}{1\ 000\ 000} = 3 \times \frac{1}{10^6} = 3 \times 10^{-6}$
b $0.000\ 056 = \frac{5.6}{100\ 000} = 5.6 \times \frac{1}{100\ 000} = 5.6 \times \frac{1}{10^5} = 5.6 \times 10^{-5}$
c $0.000\ 726 = \frac{7.26}{10\ 000} = 7.26 \times \frac{1}{10\ 000} = 7.26 \times \frac{1}{10^4} = 7.26 \times 10^{-4}$

Note

You can write down the answer without any intermediate steps.

Move the decimal point until the number is between 1 and 10.

Count the number of places the point has moved, put a minus sign in front and that is the power of 10.

Exercise 7.1

- 1** Write these numbers in standard form.

a 7000	b 84 000	c 563
d 6 500 000	e 723 000	f 27
g 53 400	h 693	i 4390
j 412 300 000	k 8 million	l 39.2 million
- 2** Write these numbers in standard form.

a 0.003	b 0.056	c 0.0008
d 0.000 0063	e 0.000 082	f 0.0060
g 0.000 000 38	h 0.78	i 0.003 69
j 0.000 658	k 0.000 000 000 56	l 0.000 007 23
- 3** These numbers are in standard form. Write them as ordinary numbers.

a 5×10^4	b 3.7×10^5	c 7×10^{-4}	d 6.9×10^6
e 6.1×10^{-3}	f 4.73×10^4	g 2.79×10^7	h 4.83×10^{-5}
i 1.03×10^{-2}	j 9.89×10^8	k 2.61×10^{-6}	l 3.7×10^2
m 3.69×10^3	n 6.07×10^{-4}	o 5.48×10^{-7}	p 1.98×10^9
- 4** A billion is a thousand million.
In 2015 the population of the world was approximately 7.2 billion.
Write the population of the world in standard form.

Calculating with numbers in standard form

When you need to multiply or divide numbers in standard form you can use your knowledge of the laws of indices.

Example 7.3

Question

Work out these. Give your answers in standard form.

- a** $(7 \times 10^3) \times (4 \times 10^4)$
- b** $(3 \times 10^8) \div (5 \times 10^3)$

Solution

- a**
$$\begin{aligned}(7 \times 10^3) \times (4 \times 10^4) &= 7 \times 4 \times 10^3 \times 10^4 \\ &= 28 \times 10^7 \\ &= 2.8 \times 10^8\end{aligned}$$
- b**
$$\begin{aligned}(3 \times 10^8) \div (5 \times 10^3) &= \frac{3 \times 10^8}{5 \times 10^3} \\ &= 0.6 \times 10^5 \\ &= 6 \times 10^4\end{aligned}$$

Note

$$\begin{aligned}10^3 \times 10^4 &= 1000 \times 10000 \\ &= 10000000 \\ &= 10^7\end{aligned}$$

Note

$$\begin{aligned}\frac{3}{5} &= 0.6 \quad \text{and} \\ \frac{10^8}{10^3} &= \frac{100000000}{1000} \\ &= 100000 \\ &= 10^5\end{aligned}$$

When you need to add or subtract numbers in standard form it is much safer to change to ordinary numbers first.

Example 7.4

Question

Work out these. Give your answers in standard form.

- $(7 \times 10^3) + (1.4 \times 10^4)$
- $(7.2 \times 10^5) + (2.5 \times 10^4)$
- $(5.3 \times 10^{-3}) - (4.9 \times 10^{-4})$

Solution

- $$\begin{array}{r}
 7000 \\
 + 14000 \\
 \hline
 21000 = 2.1 \times 10^4
 \end{array}$$
- $$\begin{array}{r}
 720000 \\
 + 25000 \\
 \hline
 745000 = 7.45 \times 10^5
 \end{array}$$
- $$\begin{array}{r}
 0.00530 \\
 - 0.00049 \\
 \hline
 0.00481 = 4.81 \times 10^{-3}
 \end{array}$$

Exercise 7.2

1 Work out these. Give your answers in standard form.

- | | |
|--|---|
| a $(4 \times 10^3) \times (2 \times 10^4)$ | b $(6 \times 10^7) \times (2 \times 10^3)$ |
| c $(7 \times 10^3) \times (8 \times 10^2)$ | d $(9 \times 10^7) \div (3 \times 10^4)$ |
| e $(4 \times 10^3) \times (1.3 \times 10^4)$ | f $(4.8 \times 10^3) \div (1.2 \times 10^{-2})$ |
| g $(8 \times 10^6) \times (9 \times 10^{-2})$ | h $(4 \times 10^8) \div (8 \times 10^2)$ |
| i $(7 \times 10^{-4}) \times (8 \times 10^{-3})$ | j $(5 \times 10^{-5}) \div (2 \times 10^4)$ |
| k $(4 \times 10^3) + (6 \times 10^4)$ | l $(7 \times 10^6) - (3 \times 10^3)$ |
| m $(6.2 \times 10^5) - (3.7 \times 10^4)$ | n $(4.2 \times 10^9) + (3.6 \times 10^8)$ |
| o $(7.2 \times 10^6) - (4.2 \times 10^4)$ | p $(7.8 \times 10^{-5}) + (6.1 \times 10^{-4})$ |
- 2 Work out these. Give your answers in standard form.
- | | |
|---|--|
| a $(6.2 \times 10^5) \times (3.8 \times 10^7)$ | b $(6.3 \times 10^7) \div (4.2 \times 10^2)$ |
| c $(6.67 \times 10^8) \div (4.6 \times 10^{-3})$ | d $(3.7 \times 10^{-4}) \times (2.9 \times 10^{-3})$ |
| e $(1.69 \times 10^8) \div (5.2 \times 10^3)$ | f $(5.8 \times 10^5) \times (3.5 \times 10^3)$ |
| g $(5.2 \times 10^6)^2$ | h $(3.1 \times 10^{-4})^2$ |
| i $(3.72 \times 10^6) - (2.8 \times 10^4)$ | j $(7.63 \times 10^5) + (3.89 \times 10^4)$ |
| k $(5.63 \times 10^{-3}) - (4.28 \times 10^{-4})$ | l $(6.72 \times 10^{-3}) + (2.84 \times 10^{-5})$ |
| m $(4.32 \times 10^{-5}) - (4.28 \times 10^{-3})$ | n $(7.28 \times 10^8) + (3.64 \times 10^6)$ |

8 THE FOUR OPERATIONS

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- use the four operations for calculations with whole numbers, decimals and vulgar (and mixed) fractions, including correct ordering of operations and use of brackets.

CHECK YOU CAN:

- add, subtract, multiply and divide integers without a calculator
- add and subtract decimals without a calculator
- find equivalent fractions.

Calculating with negative numbers

You should know the rules for working with negative numbers. These are covered in detail in Chapter 18 but summarised here.

Adding a negative number is the same as subtracting a positive number.

$5 + -7$ is the same as $5 - 7$

so $5 + -7 = -2$

Subtracting a negative number is the same as adding a positive number.

$5 - -7$ is the same as $5 + 7$

so $5 - -7 = 12$

When you multiply or divide two negative numbers, the result is positive.

$-2 \times -7 = 14$

$-15 \div -3 = 5$

When you multiply or divide one positive number and one negative number, the result is negative.

$2 \times -7 = -14$

$-15 \div 3 = -5$

You can extend the rules for multiplying and dividing to calculations with more than two numbers.

If the number of negative signs is **even**, the result is **positive**.

If the number of negative signs is **odd**, the result is **negative**.

Example 8.1

Question

Work out these.

a $4 - 5 + 3 - 6$ b -7×-7 c $-3 \times -6 \div -2$

Solution

a $4 - 5 + 3 - 6$

$= 4 + 3 - 5 - 6$

$= 7 - 11$

$= -4$

b -7×-7

$= 7 \times 7$

$= 49$

First collect the numbers to add and the numbers to subtract.

Find separate totals for each.

Finally do the subtraction, making sure that the sign is correct.

There are two negative signs, so the answer will be positive.

This calculation is equivalent to the line above.

$$\begin{aligned} \text{c} \quad & -3 \times -6 \div -2 \\ & = 18 \div -2 \\ & = -9 \end{aligned}$$

There are three negative signs, so the answer will be negative.

This calculation is equivalent to the line above.

The answer to part b is an important result. It shows that the square root of 49 can be negative as well as positive.

All positive numbers have two square roots.

Exercise 8.1

Work out these.

- | | | | | | |
|----|------------------------|----|------------------------|----|-----------------------|
| 1 | 4 + 3 - 2 - 1 | 2 | 6 - 3 - 5 + 4 | 3 | 4 + 3 + 2 - 5 |
| 4 | 9 - 2 - 3 + 2 | 5 | -4 + 2 - 2 + 5 | 6 | 4 - 1 - 3 - 4 |
| 7 | 2 - 3 + 4 - 7 | 8 | 2 - 3 - 5 - 6 + 4 + 1 | 9 | 7 + 3 + 2 - 5 - 4 + 3 |
| 10 | -4 - 3 + 2 + 7 - 5 - 1 | 11 | -5 × 4 | 12 | -6 × -5 |
| 13 | 24 ÷ -6 | 14 | -32 ÷ 4 | 15 | -45 ÷ -5 |
| 16 | 6 × 10 ÷ -5 | 17 | -3 × -4 ÷ -6 | 18 | -84 ÷ -12 × -3 |
| 19 | 6 × -4 × 3 ÷ -2 | 20 | -1 × -2 × -3 × -4 × -5 | | |

Order of operations

If you are asked to work out the answer to a calculation involving more than one operation, it is important that you use the correct order of operations.

For example, if you want to work out $2 + 3 \times 4$, should you carry out the addition or the multiplication first?

This is the correct order of operations when carrying out any calculation:

- first work out anything in brackets
- then work out any powers (such as squares or square roots)
- then do any multiplication or division
- finally do any addition or subtraction.

So to work out $2 + 3 \times 4$, you should first do the multiplication, then the addition.

$$2 + 3 \times 4 = 2 + 12 = 14$$

If you want the addition to be done first, then brackets are needed in the calculation.

$$(2 + 3) \times 4 = 5 \times 4 = 20$$

Some calculations are written like fractions, for example $\frac{6 \times 4}{5 + 3}$.

In this case the fraction line works in the same way as brackets.

First evaluate the numerator, then the denominator and then do the division.

$$\frac{6 \times 4}{5 + 3} = \frac{24}{8} = 3$$

This calculation could also be written as $6 \times 4 \div (5 + 3)$.

Example 8.2**Question**

Work out these.

- a** $7 + 4 \div 2 - 5$
b $(6 - 2)^2 + 5 \times -3$

Solution

a $7 + 4 \div 2 - 5$

$$= 7 + 2 - 5$$

$$= 4$$

First work out the division.

The addition and subtraction can be done in a single step.

b $(6 - 2)^2 + 5 \times -3$

$$= 4^2 + 5 \times -3$$

$$= 16 + 5 \times -3$$

$$= 16 + -15$$

$$= 1$$

First work out the brackets.

Next work out the power.

Then the multiplication.

And finally the addition.

 **Exercise 8.2**

Work out these.

1 $2 \times 5 + 4$

4 $2 + 5^2$

7 $(13 - 2) \times 5$

10 $(5 - 3) \times (8 - 3)$

13 $\frac{12}{4 \times 3}$

16 $\frac{20}{5 + 3}$

19 $(20 - 2 \times 6)^2$

22 3×2^3

25 $\frac{9 \times 4}{-3 + -6}$

28 Hassan says that $12 \div 2 + 4 = 2$.

a Explain what he has done wrong.

b Work out the correct answer.

29 Aisha says that $3 \times 2^2 = 36$.

a Explain what she has done wrong.

b Work out the correct answer.

2 $2 \times (5 + 4)$

5 5×4^2

8 2×3^3

11 $6 + \frac{8}{2}$

14 $\frac{12}{4} \times 3$

17 $\frac{20}{5} + 3$

20 $12 - 6 \times 3$

23 $\frac{-7 \times -12}{-8 + 4}$

26 $4 \times 2 - 5 \times -4$

3 $2 + 5 \times 4$

6 $13 - 2 \times 5$

9 $(2 \times 5)^2$

12 $\frac{6+8}{2}$

15 $\frac{6+4}{5} - 2$

18 $3 \times 5 - \frac{6 \times 4}{8}$

21 $4 \times 3 - 5 \times 4$

24 $\frac{8 \times -6}{-4 + -8}$

27 $4 + 5 \times -2 - 6$

30 Write out each of the following calculations, with brackets if necessary, to give the answers stated.

- a $3 + 6 \times 5 - 1$ to give
i 44 ii 32 iii 27
- b $6 + 4^2 - 16 \div 2$ to give
i 6 ii 3 iii 92
- c $12 - 8 \div 4 + 4$ to give
i 11 ii 5 iii 14
- d $18 + 12 \div 6 - 3$ to give
i 2 ii 17 iii 22 iv 10

Multiplying integers

There are a number of methods that can be used for multiplying three-digit by two-digit integers. These methods can be adapted for use with integers of any size.

Example 8.3

Question

Work out 352×47 .

Solution

Method 1: Long multiplication

$$\begin{array}{r}
 & 3 & 5 & 2 \\
 \times & & 4 & 7 \\
 \hline
 & 1 & 4 & 0 & 8 & 0 & \text{(} 352 \times 40 \text{)} \\
 & & 2 & 4 & 6 & 4 & \text{(} 352 \times 7 \text{)} \\
 \hline
 & 1 & 6 & 5 & 4 & 4
 \end{array}$$

Method 2: Grid method

x	300	50	2	
40	12000	2000	80	= 14080
7	2100	350	14	= 2464
				<u>16544</u>

Exercise 8.3

Work out these.

- 1 138×13 2 581×23 3 614×14 4 705×32 5 146×79
 6 615×46 7 254×82 8 422×65 9 428×64 10 624×75

11 A theatre has 42 rows of seats. There are 28 seats in each row.

How many seats are there altogether?

12 A packet contains 36 pens. A container holds 175 of these packets.

How many pens are in the container?

Multiplying decimals

Use this method for multiplying two decimal numbers:

- count the number of decimal places in the numbers you are multiplying
- ignore the decimal points and do the multiplication
- put a decimal point in your result so that there are the same number of decimal places in your answer as in the original calculation.

Example 8.4

Question

Work out these.

a 0.4×0.05

b 4.36×0.52

Solution

a $0.\textcircled{4} \times 0.\textcircled{05}$

There are three decimal places in the calculation, so there will be three in the answer.

$4 \times 5 = 20$

Multiply the figures, ignoring the decimal points.

$0.4 \times 0.05 = 0.\textcircled{020}$

Insert a decimal point so that there are three decimal places in the answer. In this case we need to add an extra zero between the decimal point and the 2.

$0.4 \times 0.05 = 0.02$

b $4.\textcircled{36} \times 0.\textcircled{52}$

There are four decimal places in the calculation, so there will be four in the answer.

$436 \times 52 = 22\,672$

Multiply the figures, ignoring the decimal points. Use whichever method you prefer.

$4.36 \times 0.52 = 2.2672$

Insert a decimal point so that there are four decimal places in the answer.

Exercise 8.4

- 1 Given that $63 \times 231 = 14\,553$, write down the answers to these.

- | | | |
|---------------------|------------------------|----------------------|
| a 6.3×2.31 | b 63×23.1 | c 0.63×23.1 |
| d 63×0.231 | e $6.3 \times 23\,100$ | |

- 2 Given that $12.4 \times 8.5 = 105.4$, write down the answers to these.

- | | | |
|---------------------|----------------------|----------------------|
| a 124×8.5 | b 12.4×0.85 | c 0.124×8.5 |
| d 1.24×8.5 | e 0.124×850 | |

- 3 Work out these.

- | | | |
|---------------------|--------------------|----------------------|
| a 0.3×5 | b 0.6×0.8 | c 0.2×0.4 |
| d 0.02×0.7 | e 0.006×5 | f 0.07×0.09 |

- 4 Work out these.

- | | | |
|---------------------|---------------------|---------------------|
| a 42×1.5 | b 5.9×6.1 | c 10.9×2.4 |
| d 2.34×0.8 | e 5.46×0.7 | f 6.23×1.6 |

- 5 Work out these.

- | | | |
|--------------------------|-------------------------------------|--|
| a $0.5 + 0.2 \times 0.3$ | b $0.1 \times 0.8 - 0.03$ | |
| c $0.2 + 0.4^2$ | d $0.9 \times 0.8 - 0.2 \times 0.7$ | |

Dividing integers

There are a number of methods that can be used for dividing three-digit by two-digit integers. These methods can be adapted for use with integers of any size.

Example 8.5

Question

Work out $816 \div 34$.

Solution

Method 1: Long division

$$816 \div 34 = 24$$

$$\begin{array}{r} 24 \\ 34 \overline{)816} \\ - 680 \quad (34 \times 20) \\ \hline 136 \\ - 136 \quad (34 \times 4) \\ \hline 0 \end{array}$$

The answer is 24.

Method 2: Chunking

$$816 \div 34 = 24$$

$$\begin{array}{r} 816 \\ - 340 \quad (34 \times 10) \\ \hline 476 \\ - 340 \quad (34 \times 10) \\ \hline 136 \\ - 136 \quad (34 \times 4) \\ \hline 0 \end{array}$$

The answer is $10 + 10 + 4 = 24$.

Exercise 8.5

Work out these.

- 1** $987 \div 21$ **2** $684 \div 18$ **3** $864 \div 16$ **4** $924 \div 28$ **5** $544 \div 32$
6 $352 \div 22$ **7** $855 \div 45$ **8** $992 \div 31$ **9** $918 \div 27$ **10** $576 \div 18$

11 Eggs are packed in boxes of 12.

How many boxes are needed for 828 eggs?

12 A group of 640 people are going on a bus trip.

Each bus can carry 54 people.

How many buses are needed?

Dividing decimals

Use this method for dividing by a decimal:

- first write the division as a fraction
- make the denominator an integer by multiplying the numerator and the denominator by the same power of 10
- cancel the fraction to its lowest terms if possible; this gives you easier numbers to divide
- divide to find the answer.

Example 8.6**Question**

Work out these.

a $0.9 \div 1.5$

b $25.7 \div 0.08$

Solution

a $0.9 \div 1.5 = \frac{0.9}{1.5}$

$$= \frac{9}{15}$$

$$= \frac{3}{5}$$

$$\frac{3}{5} = \frac{0.6}{5} \overline{)3.0}$$

Write the division as a fraction.

Make the denominator an integer by multiplying both the numerator and the denominator by 10.

Simplify.

Divide, adding an extra zero after the decimal point to complete the division.

$$0.9 \div 1.5 = 0.6$$

b $25.7 \div 0.08 = \frac{25.7}{0.08}$

$$= \frac{2570}{8}$$

$$= \frac{1285}{4}$$

$$\frac{1285}{4} = \frac{321.25}{4} \overline{)1285.00}$$

Write the division as a fraction.

Make the denominator an integer by multiplying both the numerator and the denominator by 100.

Simplify.

Use whichever method you prefer for the division.

Divide, adding extra zeros after the decimal point to complete the division.

$$25.7 \div 0.08 = 321.25$$

 **Exercise 8.6**

- 1 Given that $852 \div 16 = 53.25$, write down the answers to these.

a $852 \div 1.6$	b $8.52 \div 16$	c $85.2 \div 1.6$
d $85.2 \div 0.16$	e $852 \div 0.016$	
- 2 Given that $482 \div 25 = 19.28$, write down the answers to these.

a $4.82 \div 2.5$	b $0.482 \div 2.5$	c $48.2 \div 0.25$
d $4.82 \div 250$	e $4.82 \div 0.025$	
- 3 Work out these.

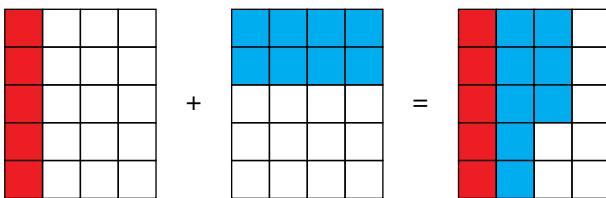
a $8 \div 0.2$	b $1.2 \div 0.3$	c $5.6 \div 0.7$
d $9 \div 0.3$	e $15 \div 0.03$	f $6.5 \div 1.3$
- 4 Work out these.

a $1.55 \div 0.05$	b $85.8 \div 0.11$	c $5.55 \div 1.5$
d $0.68 \div 1.6$	e $87.6 \div 0.24$	f $1.35 \div 1.8$
- 5 Work out these.

a $0.6 + 0.2 \div 0.1$	b $\frac{0.5 + 0.7}{0.3 \times 0.2}$	
c $\frac{1.8}{0.3^2}$	d $\frac{2.6 \times 0.5}{0.1^2}$	

Adding and subtracting fractions

In the diagram each rectangle is divided into 20 small squares.



The diagram shows $\frac{1}{4} + \frac{2}{5}$.

The result of the addition has 13 squares shaded or $\frac{13}{20}$.

Use this method to add two fractions:

- change the fractions to equivalent fractions with the same denominator
- add the numerators.

$$\frac{1}{4} = \frac{5}{20} \text{ and } \frac{2}{5} = \frac{8}{20}$$

$$\frac{1}{4} + \frac{2}{5} = \frac{5}{20} + \frac{8}{20} = \frac{13}{20}$$

The method is the same for subtracting two fractions. Change them to equivalent fractions with the same denominator then subtract the numerators.

If the fractions are mixed numbers, change them to improper fractions first.

Example 8.7

Question

Work out these.

a $\frac{2}{3} + \frac{5}{6}$

b $\frac{3}{4} - \frac{1}{3}$

c $2\frac{3}{5} + 1\frac{1}{3}$

d $3\frac{1}{4} - 1\frac{5}{6}$

Solution

a $\frac{2}{3} + \frac{5}{6} = \frac{4}{6} + \frac{5}{6}$

Write the fractions with a common denominator.

The lowest common multiple of 3 and 6 is 6, so use 6 as the common denominator.

$$= \frac{9}{6}$$

The result is an improper fraction and not in its lowest terms.

$$= \frac{3}{2} = 1\frac{1}{2}$$

Simplify the fraction and write it as a mixed number.

b $\frac{3}{4} - \frac{1}{3} = \frac{9}{12} - \frac{4}{12}$

The lowest common multiple of 4 and 3 is 12, so use 12 as the common denominator.

$$= \frac{5}{12}$$



c

$$\begin{aligned}2\frac{3}{5} + 1\frac{1}{3} &= \frac{13}{5} + \frac{4}{3} \\&= \frac{39}{15} + \frac{20}{15} \\&= \frac{59}{15} = 3\frac{14}{15}\end{aligned}$$

First change the mixed numbers to improper fractions.

The lowest common multiple of 5 and 3 is 15.

The result is an improper fraction, so write it as a mixed number.

d

$$\begin{aligned}3\frac{1}{4} - 1\frac{5}{6} &= \frac{13}{4} - \frac{11}{6} \\&= \frac{39}{12} - \frac{22}{12} \\&= \frac{17}{12} = 1\frac{5}{12}\end{aligned}$$

First change the mixed numbers to improper fractions.

The lowest common multiple of 4 and 6 is 12.

The result is an improper fraction, so write it as a mixed number.

Note

You should always give your final answer as a fraction in its lowest terms.

 **Exercise 8.7****1** Add these fractions.

a $\frac{2}{7} + \frac{4}{7}$

b $\frac{1}{3} + \frac{1}{6}$

c $\frac{2}{3} + \frac{1}{4}$

d $\frac{1}{5} + \frac{3}{4}$

e $\frac{3}{8} + \frac{1}{5}$

f $\frac{3}{4} + \frac{2}{5}$

2 Subtract these fractions.

a $\frac{2}{7} - \frac{1}{7}$

b $\frac{5}{6} - \frac{1}{3}$

c $\frac{2}{3} - \frac{1}{4}$

d $\frac{11}{12} - \frac{2}{3}$

e $\frac{5}{8} - \frac{1}{3}$

f $\frac{7}{9} - \frac{5}{12}$

3 Add these fractions. Write your answers as simply as possible.

a $\frac{3}{10} + 1\frac{2}{5}$

b $1\frac{1}{10} + \frac{3}{5}$

c $2\frac{1}{5} + 1\frac{1}{10}$

d $4\frac{1}{2} + 2\frac{3}{10}$

e $1\frac{3}{10} + \frac{2}{5}$

4 Subtract these fractions. Write your answers as simply as possible.

a $5\frac{1}{10} - \frac{7}{10}$

b $1\frac{5}{6} - \frac{2}{3}$

c $1\frac{1}{5} - \frac{7}{10}$

d $1\frac{1}{2} - \frac{3}{4}$

e $2\frac{7}{10} - 1\frac{4}{5}$

5 Work out these.

a $3\frac{1}{2} + 2\frac{1}{5}$

b $4\frac{7}{8} - 1\frac{3}{4}$

c $6\frac{5}{12} - 3\frac{1}{3}$

d $4\frac{3}{4} + 2\frac{5}{8}$

e $5\frac{5}{6} - 1\frac{1}{4}$

6 Work out these.

a $4\frac{7}{9} + 2\frac{5}{6}$

b $4\frac{7}{13} - 4\frac{1}{2}$

c $7\frac{2}{5} - 1\frac{3}{4}$

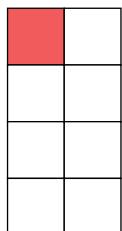
d $5\frac{2}{7} - 3\frac{1}{2}$

e $4\frac{1}{12} - 3\frac{1}{4}$

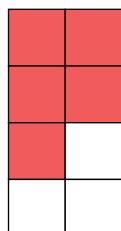
Multiplying fractions

Multiplying a fraction by a whole number

In this diagram, $\frac{1}{8}$ is red.



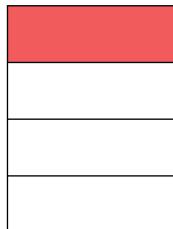
In this diagram, five times as much is red, so $\frac{1}{8} \times 5 = \frac{5}{8}$.



This shows that to multiply a fraction by an integer, you multiply the numerator by the integer. Then simplify by cancelling and changing to a mixed number if possible.

Multiplying a fraction by a fraction

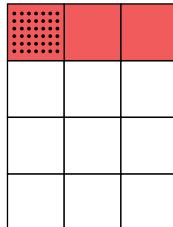
In this diagram, $\frac{1}{4}$ is red.



In this diagram, $\frac{1}{3}$ of the red is dotted.

This is $\frac{1}{12}$ of the original rectangle.

$$\text{So } \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}.$$



To multiply fractions, multiply the numerators and multiply the denominators, then simplify if possible.

Example 8.8**Question**

Work out these.

a $\frac{7}{12} \times 2$

b $\frac{3}{4} \times \frac{6}{7}$

c $2\frac{2}{3} \times 1\frac{5}{6}$

Solution

a $\frac{7}{12} \times 2 = \frac{14}{12}$

Multiply the numerator by 2.

$$= \frac{7}{6} = 1\frac{1}{6}$$

Simplify the fraction and write it as a mixed number.

b $\frac{3}{4} \times \frac{6^3}{7} = \frac{3}{2} \times \frac{3}{7}$

Cancel the common factor of 2 in the numerator and the denominator.

$$= \frac{9}{14}$$

The result is a proper fraction in its lowest terms.

c $2\frac{2}{3} \times 1\frac{5}{6} = \frac{8}{3} \times \frac{11}{6}$

First change the mixed numbers to improper fractions.

$$= \frac{8^4}{3} \times \frac{11}{6^3}$$

Cancel the common factor of 2 in the numerator and the denominator.

$$= \frac{4}{3} \times \frac{11}{3}$$

Multiply the numerators and multiply the denominators.

$$= \frac{44}{9} = 4\frac{8}{9}$$

The result is an improper fraction, so write it as a mixed number.

Note

Cancelling common factors before multiplying makes the arithmetic simpler.

If you multiply first, you may have to cancel to give a fraction in its lowest terms.

 **Exercise 8.8**

Work out these.

Write each answer as an integer, a proper fraction or a mixed number in its lowest terms.

1 $\frac{1}{2} \times 4$

2 $7 \times \frac{1}{2}$

3 $9 \times \frac{1}{3}$

4 $\frac{3}{4} \times 12$

5 $\frac{2}{5} \times 5$

6 $24 \times \frac{5}{12}$

7 $\frac{2}{3} \times 4$

8 $\frac{4}{9} \times 2$

9 $\frac{4}{5} \times 3$

10 $\frac{1}{5} \times 3$

11 $\frac{1}{4} \times \frac{2}{3}$

12 $\frac{2}{3} \times \frac{3}{5}$

13 $\frac{4}{9} \times \frac{1}{2}$

14 $\frac{1}{3} \times \frac{2}{3}$

15 $\frac{5}{6} \times \frac{3}{5}$

16 $\frac{3}{7} \times \frac{7}{9}$

17 $\frac{1}{2} \times \frac{5}{6}$

18 $\frac{3}{10} \times \frac{5}{11}$

19 $\frac{2}{3} \times \frac{5}{8}$

20 $\frac{3}{5} \times \frac{5}{12}$

21 $4\frac{1}{2} \times 2\frac{1}{6}$

22 $1\frac{1}{2} \times 3\frac{2}{3}$

23 $4\frac{1}{5} \times 1\frac{2}{3}$

24 $3\frac{1}{3} \times 2\frac{2}{5}$

25 $3\frac{1}{5} \times 1\frac{2}{3}$

Dividing fractions

When you work out $6 \div 3$, you are finding how many 3s there are in 6.

When you work out $6 \div \frac{1}{3}$, you are finding out how many $\frac{1}{3}$ s there are in 6.

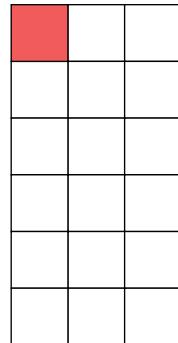
In this diagram, $\frac{1}{3}$ of the rectangle is red.



This diagram shows 6 of these rectangles.

You can see that 18 of the red squares will fit into this diagram.

$$\text{So } 6 \div \frac{1}{3} = 6 \times 3 = 18$$



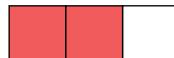
You can see that dividing by $\frac{1}{3}$ is the same as multiplying by 3.

$\frac{1}{3}$ is known as the **reciprocal** of 3.

This can be extended to division by a non-unit fraction.

For example, when you work out $4 \div \frac{2}{3}$, you are finding out how many $\frac{2}{3}$ s there are in 4.

In this diagram, $\frac{2}{3}$ of the rectangle is red.



This diagram shows 4 of these rectangles.

You can see that 6 of these 2-square shapes will fit into this diagram.

You can think of the calculation in two steps.

$$4 \div \frac{1}{3} = 4 \times 3 = 12 \text{ (there are 12 small squares in the 4 rectangles).}$$

$$\text{So } 4 \div \frac{2}{3} = 12 \div 2 = 6$$

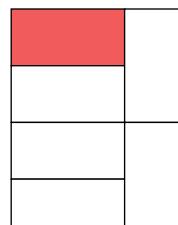
$$\text{Or } 4 \div \frac{2}{3} = 4 \times \frac{3}{2} = \frac{12}{2} = 6.$$

You can see that dividing by $\frac{2}{3}$ is the same as multiplying by $\frac{3}{2}$.

$\frac{2}{3}$ is known as the reciprocal of $\frac{3}{2}$.

To find the reciprocal of any fraction, turn it upside down.

The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.



Use this method for dividing by a fraction:

- find the reciprocal of the fraction you are dividing by
- multiply by the reciprocal.

Example 8.9**Question**

Work out these.

a $\frac{8}{9} \div 2$

b $\frac{9}{10} \div \frac{3}{4}$

c $2\frac{3}{4} \div 1\frac{5}{8}$

Solution

- a $\frac{8}{9} \div 2 = \frac{8}{9} \div \frac{2}{1}$ 2 is the same as $\frac{2}{1}$
 $= \frac{8}{9} \times \frac{1}{2}$ The reciprocal of $\frac{2}{1}$ is $\frac{1}{2}$, so multiply by this reciprocal.
 $= \frac{8^1}{9^1} \times \frac{1}{2^1}$ Cancel the common factor of 2 in the numerator and the denominator.
 $= \frac{4}{9} \times 1$
 $= \frac{4}{9}$
- b $\frac{9}{10} \div \frac{3}{4} = \frac{9}{10} \times \frac{4}{3}$ The reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$, so multiply by this reciprocal.
 $= \frac{9^3}{10^1} \times \frac{4^2}{3^1}$ Cancel the common factors of 2 and 3 in the numerator and the denominator.
 $= \frac{3}{5} \times \frac{2}{1}$
 $= \frac{6}{5} = 1\frac{1}{5}$ The result is an improper fraction, so write it as a mixed number.
- c $2\frac{3}{4} \div 1\frac{5}{8} = \frac{11}{4} \div \frac{13}{8}$ First change the mixed numbers to improper fractions.
 $= \frac{11}{4} \times \frac{8}{13}$ The reciprocal of $\frac{13}{8}$ is $\frac{8}{13}$, so multiply by this reciprocal.
 $= \frac{11}{4} \times \frac{8^2}{13^1}$ Cancel the common factor of 4 in the numerator and the denominator.
 $= \frac{22}{13} = 1\frac{9}{13}$ The result is an improper fraction, so write it as a mixed number.

Note

Remember that when you have a fraction calculation involving mixed numbers, you should first change them to improper fractions.

Always check that your final answer is given as a mixed number in its lowest terms.

**Exercise 8.9**

Work out these.

Write each answer as an integer, a proper fraction or a mixed number in its lowest terms.

1 $9 \div \frac{1}{3}$

2 $\frac{3}{4} \div \frac{1}{2}$

3 $\frac{2}{3} \div 3$

4 $\frac{1}{3} \div \frac{3}{4}$

5 $\frac{4}{9} \div 2$

6 $\frac{4}{5} \div 4$

7 $\frac{2}{3} \div \frac{1}{3}$

8 $\frac{5}{6} \div 10$

9 $\frac{7}{9} \div \frac{1}{9}$

10 $\frac{4}{5} \div \frac{3}{10}$

11 $\frac{1}{4} \div \frac{3}{8}$

12 $\frac{2}{5} \div \frac{1}{5}$

13 $\frac{2}{5} \div \frac{7}{10}$

14 $2\frac{1}{3} \div 1\frac{1}{3}$

15 $2\frac{2}{5} \div 1\frac{1}{2}$

16 $\frac{3}{18} \div 1\frac{1}{4}$

17 $2\frac{1}{4} \div 3\frac{1}{2}$

18 $(3\frac{1}{2} + 2\frac{4}{5}) \times 2\frac{1}{2}$

19 $5\frac{1}{3} \div 3\frac{3}{5} + 2\frac{1}{3}$

20 $(2\frac{4}{5} + 3\frac{1}{4}) \div (3\frac{2}{3} - 2\frac{3}{4})$

9 ESTIMATION

Estimating lengths

To estimate lengths, compare with measures you know.

Example 9.1

Question

Estimate the height of the lamp post.

Solution

7.5 m

Any answer from 6.5 to 8.5 m would be acceptable.

The lamp post is about four times the height of the man.



Note

The height to an adult person's waist is about 1 m (100 cm) and their full height is about 1.7 m (170 cm).

Exercise 9.1

1 Estimate

a the height of the fence



b the length of the fence.

2 Estimate the height of the tree.



3 The car is 3.8 m long. Estimate the length of the truck.



BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- make estimates of numbers, quantities and lengths
- give approximations to specified numbers of significant figures and decimal places
- round off answers to reasonable accuracy in the context of a given problem.

CHECK YOU CAN:

- use a protractor and ruler
- round to the nearest whole number, 10, 100, etc.
- use a calculator efficiently
- find the area of a rectangle
- find the volume of a cuboid.

Note

Make sure that you include units with your answer and that they are sensible. For example, the length of a motorway should be in kilometres, not metres.

Note

When estimating do not try to be too accurate. What is wanted is a rough estimate and anything about right is acceptable. Decide if it is bigger or smaller than a measure you know and work from there.

Rounding to a given number of decimal places

Find the digit in the **decimal place** to be rounded. Look at the next digit.

- If that is less than 5, leave the decimal as it is.
- If it is 5 or more, round the decimal up by 1.

You ignore any digits further to the right.

Example 9.2

Question

Write each of these numbers correct to 2 decimal places (2 d.p.).

a 9.368

b 0.0438

c 84.655

Solution

- a 9.37 The digits in the first two decimal places are 36.
 The digit in the third decimal place is 8, which is more than 5, so you add 1 to the 6.
- b 0.04 The digits in the first two decimal places are 04.
 The digit in the third decimal place is 3, which is less than 5, so you leave the 4 as it is.
- c 84.66 The digit in the third decimal place is 5, so you add 1 to the digit in the second decimal place.

Exercise 9.2

-  1 Write each of these numbers correct to 1 decimal place.
- | | | | | |
|----------|----------|----------|----------|----------|
| a 4.62 | b 5.47 | c 4.58 | d 8.41 | e 0.478 |
| f 0.1453 | g 82.16 | h 2.97 | i 6.15 | j 0.4512 |
| k 5.237 | l 48.024 | m 0.8945 | n 7.6666 | o 9.9876 |
-  2 Write each of these numbers correct to 2 decimal places.
- | | | | | |
|-----------|-----------|----------|----------|----------|
| a 5.481 | b 12.0782 | c 0.214 | d 0.5666 | e 9.017 |
| f 78.044 | g 7.0064 | h 0.0734 | i 1.5236 | j 2.1256 |
| k 9.424 | l 0.8413 | m 0.283 | n 0.851 | o 7.093 |
| p 18.6306 | q 7.1111 | r 8.081 | s 4.656 | t 3.725 |
- 3 Work out these. Give your answers to 2 decimal places.
- | | | | |
|----------------------|---------------------|--------------------|----------------------|
| a 1.38×6.77 | b 4.7×3.65 | c $9.125 \div 3.1$ | d 16.4×3.29 |
|----------------------|---------------------|--------------------|----------------------|
- 4 Work out these. Give your answers to 1 decimal place.
- | | | | |
|--------------|--------------|---------------|---------------|
| a $1 \div 8$ | b $3 \div 7$ | c $4 \div 11$ | d $5 \div 13$ |
|--------------|--------------|---------------|---------------|
- 5 Find the square root of 55. Write this value correct to 1 decimal place.
- 6 Work out these. Give your answers to 3 decimal places.
- | | | | | |
|--------------|--------------|---------------|---------------|---------------|
| a $1 \div 3$ | b $2 \div 7$ | c $3 \div 11$ | d $4 \div 13$ | e $28 \div 6$ |
|--------------|--------------|---------------|---------------|---------------|
- 7 Find the mean of these numbers.
 Give your answer correct to 1 decimal place.
 4, 6, 8, 9, 11, 3, 2, 15

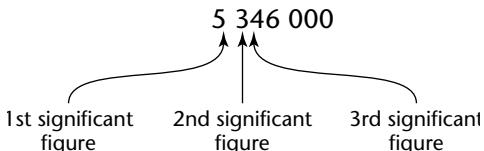
Note

The mean of a set of numbers is the sum of the numbers divided by how many numbers there are.

- 8** Find the mean of these numbers.
Give the answer to the nearest 10.
351, 521, 791, 831, 941, 1171, 1351
- 9** A rectangle is 3.4 cm by 5.2 cm. Work out the area correct to 1 decimal place.
- 10** A square has sides of 6.35 cm. Work out the area correct to 2 decimal places.
- 11** A cube has edges of 4.82 cm. Find its volume correct to 1 decimal place.
- 12** The river Nile is 4169 miles long. 1 mile is approximately 1.6 kilometres.
How long is the Nile in kilometres correct to the nearest kilometre?

Rounding to a given number of significant figures

Significant figures are counted from left to right, starting from the first non-zero digit.



You use a similar technique to the one you use for rounding to a given number of decimal places.

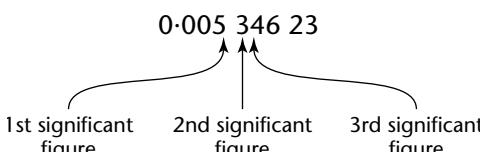
You look at the first digit that is not required. If this is less than 5, you round down. If it is 5 or more, you round up.

Rounding 5 346 000 to

1 significant figure	gives 5 000 000
2 significant figures	gives 5 300 000
3 significant figures	gives 5 350 000 ←

Note

Zeros are used to maintain the place value of the digits. In this case, 5 (millions), 3 (hundred thousands) and 5 (ten thousands).



Rounding 0.005 346 23 to

1 significant figure	gives 0.005
2 significant figures	gives 0.0053 ←
3 significant figures	gives 0.00535

Note

Again, zeros are used to maintain the place value of the digits. In this case, 5 (thousandths), 3 (ten thousandths).

Note

Zeros to the right of the first non-zero digit *are* significant. For example, 0.070 56 = 0.0706 to 3 significant figures.

Example 9.3**Question**

- a Round 52 617 to 2 significant figures.
- b Round 0.072 618 to 3 significant figures.
- c Round 17 082 to 3 significant figures.

Solution

a $52\ 617 = 53\ 000$ to 2 s.f.

b $0.072\ 618 = 0.0726$ to 3 s.f.

c $17\ 082 = 17\ 100$ to 3 s.f.

To round to 2 significant figures, look at the third significant figure.

It is 6, so the second significant figure changes from 2 to 3.

Remember to add zeros for placeholders.

The first significant figure is 7.

To round to 3 significant figures, look at the fourth significant figure.

It is 1, so the third significant figure does not change.

The 0 in the middle here is a significant figure.

To round to 3 significant figures, look at the fourth figure.

It is 8, so the third figure changes from 0 to 1.

Remember to add zeros for placeholders.

Note

Always state the accuracy of your answers when you have rounded them.

Example 9.4**Question**

Write each of these numbers to 1 significant figure.

- a 5126
- b 5.817
- c 0.04715
- d 146 500
- e 1968

Solution

- a 5000 The second significant figure is 1, so the 5 stays as it is.

You use three zeros to show the size.

- b 6 The second significant figure is 8, so you change the 5 to 6.
No zeros are needed.

- c 0.05 The second significant figure is 7, so the 4 becomes 5.
You keep the zero before the 5 to show the size.

- d 100 000 The second significant figure is 4 so the 1 stays as it is.
You use zeros to show the size.

- e 2000 The second figure is 9, so the 1 changes to 2.

Note

Zeros are used to show size. A common error is to use the wrong number of zeros.

Exercise 9.3

- 1 Write each of these to 1 significant figure.
- | | | | |
|-------|--------|--------|--------|
| a 4.2 | b 6.45 | c 7.9 | d 4.25 |
| e 5.6 | f 62.1 | g 45.9 | h 26.5 |
| i 314 | j 4579 | | |
- 2 Write each of these to 1 significant figure.
- | | | | |
|------------|----------|---------|-----------|
| a 4271 | b 26.45 | c 910.9 | d 64.25 |
| e 0.002156 | f 6.21 | g 4.59 | h 0.00265 |
| i 0.00314 | j 0.0459 | | |
- 3 Round each of these numbers to 2 significant figures.
- | | | | |
|----------|---------|----------|-----------|
| a 17.6 | b 184.2 | c 5672 | d 97520 |
| e 50.43 | f 0.172 | g 0.0387 | h 0.00612 |
| i 0.0307 | j 0.994 | | |
- 4 Round each of these numbers to 3 significant figures.
- | | | | |
|-----------|-----------|---------|----------|
| a 8.261 | b 69.77 | c 16285 | d 207.51 |
| e 12524 | f 7.103 | g 50.87 | h 0.4162 |
| i 0.03862 | j 3.14159 | | |
- 5 Over 3 days a shop had 4700 customers.
On average, how many customers were there each day correct to 2 significant figures?
- 6 At a soccer match the attendance was 15 870 men, 10 740 women and 8475 juniors.
A newspaper reported that there were 35 000 at the match.
Explain this.

Estimating answers to problems

To estimate answers to problems, you can round each number to 1 significant figure and then carry out the calculation.

Example 9.5

Question

Jim buys 24 golf balls at \$3.75 each.

What is the approximate cost?

Solution

$20 \times 4 = \$80$ Round to 1 significant figure.

Example 9.6**Question**

Estimate the result of each of these calculations by rounding to 1 significant figure.

a 3.7×2.1

b 47×63

c 4.32×8.76

d 2.95×0.321

e $0.815 \div 3.41$

Solution

a $4 \times 2 = 8$

b $50 \times 60 = 3000$

c $4 \times 9 = 36$

d $3 \times 0.3 = 0.9$

e $0.8 \div 3 = 0.266 = 0.27$ (2 s.f.) or 0.3 (1 s.f.)

Note

Do not try to be too accurate with your answer. Give the answer to 1 or 2 significant figures.

Example 9.7**Question**

Estimate the result of each of these calculations.

In each case, say whether the estimated result is bigger or smaller than the actual result and explain your answer.

a 4.7×3.9

b 516×2.34

c $79.6 \div 23.7$

Solution

a $5 \times 4 = 20$

The estimated result is bigger than the actual result because both the rounded values used in the calculation are bigger than the actual values.

b $500 \times 2 = 1000$

The estimated result is smaller than the actual result because both the rounded values used in the calculation are smaller than the actual values.

c $80 \div 20 = 4$

The estimated result is bigger than the actual result because the rounded value of the number being divided is bigger than the actual value and the rounded value of the number it is divided by is smaller than the actual value.

Exercise 9.4

In this exercise show the approximations you use to get your estimate.

- 1 At the school fair, Taha sold 245 ice-creams at 85c each.
Estimate his takings.
- 2 A rectangle measures 5.8 cm by 9.4 cm.
Estimate its area.
- 3 Salma bought 18 packets of crisps at 32c each.
Estimate how much she spent.
- 4 A cuboid has edges of 3.65 cm, 2.44 cm and 2.2 cm.
Estimate its volume.

- 5** A rectangle has an area of 63.53 cm^2 and its length is 9.61 cm .
Estimate the width of the rectangle.
- 6** An aeroplane flew 3123 km in 8.4 hours .
Estimate how far it flew on average in each hour.
- 7** Parsa went to see his local football team 23 times during the season.
Tickets cost $\$18.50$ for each game.
Estimate how much he spent.
- 8** At Duha's ice-cream stall, he sold 223 ice-creams at 67c each.
Estimate how much he took.
- 9** To find the area of a circle of radius 2.68 metres , Rihanna worked out
 $3.14 \times 2.68 \times 2.68$.
Estimate the result of this calculation.
- 10** Estimate the result of each of these calculations.
- | | | |
|------------------------------|------------------------------|-------------------------------|
| a 5.89×1.86 | b $19.25 \div 3.8$ | c 36.87×22.87 |
| d $9.7 \div 3.5$ | e 2.14×0.82 | f 3.14×7.92 |
| g 113.5×2.99 | h 4.93×0.025 | |
- 11** Estimate the result of each of these calculations.
- | | | |
|---|-------------------------------|---|
| a $3.6 \times 14.9 \times 21.5$ | b 0.89×5.2 | c 61.33×11.79 |
| d $198.5 \times 63.1 \times 2.8$ | e 9.87×0.0657 | f 0.246×0.789 |
| g 46×82 | h 7.05^2 | i $43.7 \times 18.9 \times 29.3$ |
| j 917×38 | k 3.1×14.9 | l $47 \times (21.7 + 39.2)$ |
- 12** Estimate the result of each of these calculations.
In each case, say whether the estimated result is bigger or smaller than the actual result and explain your answer.
- | | | |
|-----------------------------|--------------------------|----------------------------|
| a 4.21×81.6 | b $189 \div 11.4$ | c 16.3×897 |
|-----------------------------|--------------------------|----------------------------|
- 13 a** Estimate the area of a rectangle with sides of length 4.7 cm and 6.8 cm .
b Is the estimated area greater or smaller than the actual area?
Explain your answer.
- 14** The area of a rectangle is 17.3 cm^2 and the length is 6.3 cm .
a Estimate the width.
b Is the estimated width greater or less than the actual width?
Explain your answer.

Estimating answers to problems involving division and square roots

It is sometimes more convenient to use numbers other than the given values rounded to 1 significant figure.

When division is involved, you can round the numbers so that there are common factors.

When rounding numbers to estimate a square root, round to the nearest perfect square.

Example 9.8**Question**

Find approximate answers to these.

a $83.7 \div 2.87$ b $0.73 \times \sqrt{18.2}$

Solution

- a Round to $90 \div 3 = 30$
 or $81 \div 3 = 27 = 30$ to 1 s.f.
 (Rounding to $80 \div 3$ is more difficult but would still give 30 to 1 s.f.)
 b Round to $0.7 \times \sqrt{16} = 0.7 \times 4 = 2.8 = 3$ to 1 s.f.

Exercise 9.5

Estimate the result of each of these calculations.

Show the approximations you use to get your estimate.

- | | | |
|--|---------------------------------------|---|
| 1 $31.3 \div 4.85$ | 2 $289 \div 86$ | 3 $44.669 \div 8.77$ |
| 4 $45 \div 6.8$ | 5 $(1.8 \times 2.9) \div 3.2$ | 6 $\frac{14.56 \times 22.4}{59.78}$ |
| 7 $\sqrt{4.9 \times 5.2}$ | 8 $\frac{\sqrt{8.1 \times 1.9}}{1.9}$ | 9 $(0.35 \times 86.3) \div 7.9$ |
| 10 $0.95 \div 4.8$ | 11 $32 \times \sqrt{124}$ | 12 $\frac{62 \times 9.7}{10.12 \times 5.1}$ |
| 13 $44.555 \div 0.086$ | 14 $\sqrt{84}$ | 15 $\frac{1083}{8.2}$ |
| 16 $\frac{2.46}{18.5}$ | 17 $\frac{29}{41.6}$ | 18 $\frac{283 \times 97}{724}$ |
| 19 $\frac{614 \times 0.83}{3.7 \times 2.18}$ | 20 $\frac{6.72}{0.051 \times 39.7}$ | 21 $\sqrt{39 \times 80}$ |

Working to a sensible degree of accuracy

Measurements and calculations should always be expressed to a suitable degree of accuracy.

For example, it would be silly to say that a train journey took 4 hours 46 minutes and 13 seconds, but sensible to say that it took $4\frac{3}{4}$ hours or about five hours.

In the same way, saying that the distance the train travelled was 93 kilometres 484 metres and 78 centimetres would be giving the measurement to an unnecessary degree of accuracy.

It would more sensibly be stated as 93 km.

As a general rule the answer you give after a calculation should not be given to a greater degree of accuracy than any of the values used in the calculation.

Example 9.9**Question**

Bilal measured the length and width of a table as 1.84 m and 1.32 m.

He calculated the area as $1.84 \times 1.32 = 2.4288 \text{ m}^2$.

How should he have given the answer?

Solution

Bilal's answer has four places of decimals (4 d.p.) so it is more accurate than the measurements he took.

His answer should be 2.43 m^2 .

Exercise 9.6

- 1** Write down sensible values for each of these measurements.
 - a** 3 minutes 24.8 seconds to boil an egg
 - b** 2 weeks, 5 days, 3 hours and 13 minutes to paint a house
 - c** A book weighing 2.853 kg
 - d** The height of a door is 2 metres 12 centimetres and 54 millimetres
- 2** Work out these and give the answers to a reasonable degree of accuracy.
 - a** Find the length of the side of a square field with area 33 m^2 .
 - b** A book has 228 pages and is 18 mm thick.
How thick is Chapter 1 which has 35 pages?
 - c** It takes 12 hours to fly between two cities on an aeroplane travelling at 554 km/h.
How far apart are the cities?
 - d** The total weight of 13 people in a lift is 879 kg.
What is their average weight?
 - e** The length of a strip of card is 2.36 cm, correct to 2 decimal places.
The width is 0.041 cm, correct to 3 decimal places.
Calculate the area of the card.

10 LIMITS OF ACCURACY

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- give appropriate upper and lower bounds for data given to a specified accuracy
- obtain appropriate upper and lower bounds to solutions of simple problems given data to a specified accuracy

CHECK YOU CAN:

- round to a given number of decimal places
- round to a given number of significant figures
- convert one metric unit to another.

Note

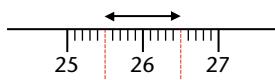
Many people are confused about the upper bound. The convention is that the lower bound is contained in the interval and the upper bound is in the next, higher interval.

Bounds of measurement

Ali measures the length of a sheet of paper. He says the length is 26 cm to the nearest centimetre. What does this mean?

It means the length is nearer to 26 cm than the closest measurements on each side, 25 cm and 27 cm.

Any measurement that is nearer to 26 cm than to 25 cm or 27 cm will be counted as 26 cm. This is the marked interval on the number line.



The boundaries of this interval are 25.5 cm and 26.5 cm. These values are exactly halfway between one measurement and the next. Usually when rounding to a given number of decimal places or significant figures, you would round 25.5 up to 26 and 26.5 up to 27.

So this gives

- The interval for 26 cm to the nearest centimetre is m cm where $25.5 \leq m < 26.5$.
- 25.5 cm is called the **lower bound** of the interval.
- 26.5 cm is called the **upper bound** of the interval but it is not actually included in the interval.

Example 10.1

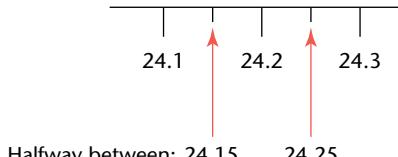
Question

Por won the 200 m race in a time of 24.2 seconds to the nearest tenth of a second. Complete the sentence below.

Por's time was between ... seconds and ... seconds.

Solution

As the measurement is stated to the nearest tenth of a second, the next possible times are 24.1 seconds and 24.3 seconds.



Por's time was between 24.15 seconds and 24.25 seconds.

Exercise 10.1

- 1 Give the upper and lower bounds of each of these measurements.
 - a Given to the nearest centimetre

i 27 cm	ii 30 cm	iii 128 cm
---------	----------	------------
 - b Given to the nearest 10 cm

i 10 cm	ii 30 cm	iii 150 cm
---------	----------	------------
 - c Given to the nearest millimetre

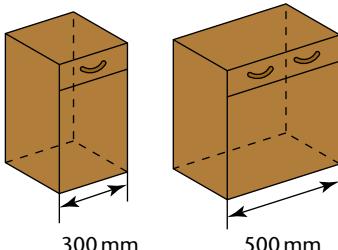
i 5.6 cm	ii 0.8 cm	iii 12.0 cm
----------	-----------	-------------
 - d Given to the nearest centimetre

i 1.23 m	ii 0.45 m	iii 9.08 m
----------	-----------	------------
 - e Given to the nearest hundredth of a second

i 10.62 seconds	ii 9.81 seconds	iii 48.10 seconds
-----------------	-----------------	-------------------
- 2 Complete each of these sentences.
 - a A mass given as 57 kg to the nearest kilogram is between ... kg and ... kg.
 - b A height given as 4.7 m to 1 decimal place is between ... m and ... m.
 - c A volume given as 468 ml to the nearest millilitre is between ... ml and ... ml.
 - d A winning time given as 34.91 seconds to the nearest hundredth of a second is between ... seconds and ... seconds.
 - e A mass given as 0.634 kg to the nearest gram is between ... kg and ... kg.
- 3 A drain cleaner uses a pole made up of ten identical flexible pieces. Each piece is 1 metre long, measured to the nearest centimetre. What length of drain can you be sure that he could reach?
- 4 A rectangle measures 12 cm by 5 cm, with both measurements correct to the nearest centimetre.
 - a Work out the greatest possible perimeter of the rectangle.
 - b Work out the smallest possible area of the rectangle.

Sums and differences of measurements

Two kitchen cupboards have widths of 300 mm and 500 mm correct to the nearest millimetre.



Lower bound of width: 299.5 mm
Upper bound of width: 300.5 mm

499.5 mm
500.5 mm

If the cupboards are put next to each other then

- the lower bound of w , their joint width = $299.5 + 499.5 = 799$ mm
- the upper bound of w , their joint width = $300.5 + 500.5 = 801$ mm
- so $799 \text{ mm} \leq w \leq 801 \text{ mm}$.

To find the lower bound of a sum, add the lower bounds.

To find the upper bound of a sum, add the upper bounds.

However, the difference between the widths of the kitchen cupboards is largest when the smallest possible width of the smaller cupboard is subtracted from the largest possible width of the larger cupboard.

The difference is least when the largest possible width of the smaller cupboard is subtracted from the smallest possible width of the larger cupboard. So

- the upper bound of the difference in their widths = $500.5 - 299.5 = 201$ mm
- the lower bound of the difference in their widths = $499.5 - 300.5 = 199$ mm.

To find the upper bound of a difference, subtract the lower bound of the smaller value from the upper bound of the larger value.

To find the lower bound of a difference, subtract the upper bound of the smaller value from the lower bound of the larger value.

Example 10.2

Question

A piece of red string is 35.2 cm long to the nearest millimetre.

A piece of blue string is 12.6 cm long to the nearest millimetre.

- What is the minimum length of the two pieces laid end to end?
- What is the lower bound of the difference in the lengths?

Solution

For the red string: LB = 35.15 cm UB = 35.25 cm

For the blue string: LB = 12.55 cm UB = 12.65 cm

- Minimum total length = sum of lower bounds

$$= 35.15 + 12.55 = 47.7 \text{ cm}$$
- LB of difference in lengths = LB of longer piece – UB of shorter piece

$$= 35.15 - 12.65 = 22.5 \text{ cm}$$

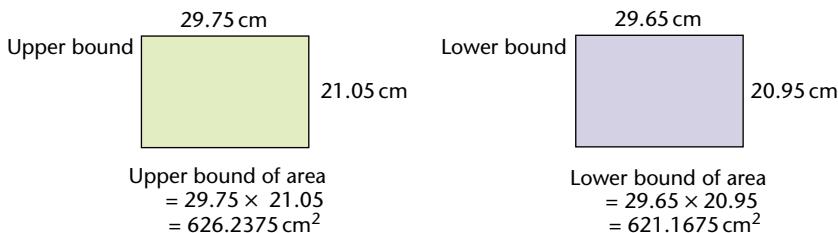
Exercise 10.2

- Find the upper bound of the sum of each of these pairs of measurements.
 - 29.7 seconds and 31.4 seconds (both to 3 significant figures)
 - 11.04 s and 13.46 s (both to the nearest hundredth of a second)
 - 6.42 m and 5.97 m (both to the nearest centimetre)
 - 1.248 kg and 0.498 kg (both to the nearest gram)
 - 86 mm and 98 mm (both to the nearest millimetre)
- Find the lower bound of the sum of measurements in question 1.
- Find the upper bound of the difference between each of these pairs of measurements.
 - 947 g and 1650 g (both to the nearest g)
 - 16.4 cm and 9.8 cm (both to the nearest mm)
 - 24.1 s and 19.8 s (both to the nearest 0.1 s)

- d** 14.86 s and 15.01 s (both to the nearest $\frac{1}{100}$ second)
- e** 12 700 m and 3800 m (both to the nearest 100 m)
- 4** Find the lower bound of the difference between measurements in question 3.
- 5** A piece of paper 21.0 cm long is taped on to the end of another piece 29.7 cm long.
Both measurements are given to the nearest millimetre.
What is the upper bound of the total length?
- 6** Two stages of a relay race are run in times of 14.07 seconds and 15.12 seconds, both to the nearest 0.01 second.
- Calculate the upper bound of the total time for these two stages.
 - Calculate the upper bound of the difference between the times for these two stages.
- 7** A triangle has sides of length 7 cm, 8 cm and 10 cm.
All measurements are correct to the nearest centimetre.
Work out the upper and lower bounds of the perimeter of the triangle.
- 8** Given that $p = 5.1$ and $q = 8.6$, both correct to 1 decimal place, work out the largest and smallest possible values of
- $p + q$
 - $q - p$.
- 9** Fah is fitting a new kitchen.
She has an oven which is 595 mm wide, to the nearest millimetre.
Will it definitely fit in a gap which is 60 cm wide, to the nearest centimetre?

Multiplying and dividing measurements

The measurements of a piece of A4 paper are given as 21.0 cm and 29.7 cm to the nearest millimetre. What are the upper and lower bounds of the area of the piece of paper?



When multiplying

- to find the upper bound, multiply the upper bounds
- to find the lower bound, multiply the lower bounds.

When dividing, however, the situation is different.

Dividing by a larger number makes the answer smaller.

When dividing

- to find the upper bound, divide the upper bound by the lower bound
- to find the lower bound, divide the lower bound by the upper bound.

Example 10.3

Question

Gan cycles 14.2 km (to 3 significant figures) in a time of 46 minutes (to the nearest minute).

What are the upper and lower bounds of his average speed in kilometres per hour?

Give your answer to 3 significant figures.

Solution

Gan cycles 14.2 km.

This distance lies between 14.15 km and 14.25 km.

Gan takes 46 minutes.

This time lies between 45.5 minutes and 46.5 minutes.

$$\text{Upper bound of speed} = \frac{\text{upper bound of distance}}{\text{lower bound of time}}$$

$$= \frac{14.25}{45.5} \text{ km/minute}$$

$$= \frac{14.25}{45.5 \div 60} \text{ km/h}$$

$$= 18.8 \text{ km/h to 3 significant figures}$$

$$\text{Lower bound of speed} = \frac{\text{lower bound of distance}}{\text{upper bound of time}}$$

$$= \frac{14.15}{46.5 \div 60} \text{ km/h}$$

$$= 18.3 \text{ km/h to 3 significant figures}$$

Note

To find the upper bound of any combined measurement, work out which of the upper and lower bounds of the given measurements you need to use to give you the greatest result. If you aren't sure, experiment!

Exercise 10.3

- 1 Pencils have a width of 8 mm, to the nearest millimetre.
What is the smallest total width of 10 pencils?
- 2 Find the upper and lower bounds of the floor area of a rectangular room with these measurements.
 - a 5.26 m by 3.89 m to the nearest centimetre
 - b 7.6 m by 5.2 m to the nearest 10 cm
- 3 Calculate the upper and lower bounds of the distance travelled for each of these pairs of speed and time. Give your answers to 3 significant figures.
 - a 92.4 cm/second for 12.3 seconds (both values to 3 s.f.)
 - b 1.54 m/second for 8.20 seconds (both values to 3 s.f.)
 - c 57 km/h for 2.5 hours (both values to 2 s.f.)
 - d 5.61 m/second for 2.08 seconds (both values to 3 s.f.)
- 4 Find the upper and lower bounds of the average speed for these measured times and distances. Give your answers to 3 significant figures.
 - a 6.4 cm in 1.2 seconds (both values to 2 s.f.)
 - b 106 m in 10.0 seconds (both values to 3 s.f.)

- 5 Calculate, to 3 significant figures, the minimum width of a rectangle with these dimensions.
- Area = 210 cm^2 to 3 significant figures,
length = 17.89 cm to the nearest millimetre
 - Area = 210 cm^2 to 2 significant figures,
length = 19.2 cm to the nearest millimetre
- 6 The population of a town is 108 000 to the nearest 1000.
Its area is given as 129 km^2 to the nearest km^2 .
Calculate the upper and lower bounds of its population density, giving your answers to 3 significant figures.
- 7 Work out the largest and smallest possible areas of a rectangle measuring 27 cm by 19 cm , where both lengths are correct to the nearest centimetre.
- 8 A bar of gold is a cuboid and measures 8.6 cm by 4.1 cm by 2.4 cm .
All the measurements are correct to the nearest millimetre.
- Work out the largest and smallest possible volumes of the bar of gold.
 - The density of gold is 19.3 g/cm^3 , correct to the nearest 0.1 g/cm^3 .
Work out the greatest and least possible masses of the gold bar.
- 9 Use the formula $P = \frac{V^2}{R}$ to work out the upper and lower bounds of P when $V = 6$ and $R = 1$ and both values are correct to the nearest whole number.
- 10 In May 1976 Don Quarrie ran 100 m in 9.9 s (correct to 1 d.p.) and in July 1983 Calvin Smith ran it in 9.93 s (correct to 2 d.p.).
Using these figures, explain which runner may have run the faster time.
- 11 A water tank measures 80 cm by 75 cm by 90 cm , correct to the nearest centimetre.
Could it hold exactly 550 litres?

11 RATIO, PROPORTION, RATE

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- demonstrate an understanding of ratio and proportion
- increase and decrease a quantity by a given ratio
- use common measures of rate
- solve problems involving average speed.

CHECK YOU CAN:

- find common factors
- simplify fractions
- understand enlargement
- change between metric units.

Ratio

Simplifying ratios

When something is divided into a number of parts, ratios are used to compare the sizes of those parts.

For example, if ten sweets are divided between two people so that one person gets six sweets and the other person gets four sweets, the sweets have been divided in the ratio 6 : 4.

As with fractions, ratios can be simplified.

For example, the fraction $\frac{6}{4} = \frac{3}{2}$; similarly, the ratio $6:4 = 3:2$.

When simplifying ratios you need to look for common factors.

When the ratio you are using involves measurements, the units must be the same for both measurements when simplifying.

Example 11.1

Question

Write each of these ratios in its lowest terms.

- a 1 millilitre : 1 litre
b 1 kilogram : 200 grams

Note

'In its lowest terms' means simplify as much as possible.

When the units are the same, you do not need to include them in the ratio.

Solution

- a $1 \text{ millilitre} : 1 \text{ litre} = 1 \text{ millilitre} : 1000 \text{ millilitres}$ Same units.
 $= 1 : 1000$
- b $1 \text{ kilogram} : 200 \text{ grams}$ Same units.
 $= 1000 \text{ grams} : 200 \text{ grams}$
 $= 5 : 1$ Divide each by 200.

Exercise 11.1

- 1 Write each of these ratios in its simplest form.

- a 6 : 3 b 25 : 75 c 30 : 6 d 10 : 15 e 7 : 35
f 15 : 12 g 24 : 8 h 4 : 48 i 3 : 27 j 9 : 81

- 2 Write each of these ratios in its simplest form.

- a 20 minutes : 1 hour b 50 g : 1 kg c 300 ml : 1 litre
d 2 kg : 600 g e 2 minutes : 30 seconds

- 3** Write each of these ratios in its simplest form.
- 50 g : 1000 g
 - 30 cents : \$2
 - 4 m : 75 cm
 - 300 ml : 2 litres
- 4** Write each of these ratios in its simplest form.
- 5 : 15 : 25
 - 6 : 12 : 8
 - 8 : 32 : 40
- 5** Write each of these ratios in its simplest form.
- 50 cents : \$2.50 : \$5
 - 20 cm : 80 cm : 1.20 m
 - 600 g : 750 g : 1 g
- 6** Ali, Pali and Dave invest \$500, \$800 and \$1000 respectively in a business.
Write the ratio of their investments in its simplest form.

Writing a ratio in the form $1:n$

It is sometimes useful to have a ratio with 1 on the left.

A common scale for a scale model is $1:24$.

The scale of a map or enlargement is often given as $1:n$.

To change a ratio to this form, divide both numbers by the number on the left.

Example 11.2

Question

Write these ratios in the form $1:n$.

a $2:5$ b $8\text{ mm}:3\text{ cm}$ c $25\text{ mm}:1.25\text{ km}$

Solution

a $2:5 = 1:2.5$	Divide each side by 2.
b $8\text{ mm}:3\text{ cm} = 8\text{ mm}:30\text{ mm}$ $= 1:3.75$	Write each side in the same units. Divide each side by 8.
c $25\text{ mm}:1.25\text{ km} = 25:1250000$ $= 1:50000$	Write each side in the same units. Divide each side by 25.

Note

$1:50000$ is a common map scale. It means that 1 cm on the map represents 50 000 cm, or 500 m, on the ground.

Exercise 11.2

- 1** Write each of these ratios in the form $1:n$.
- $2:6$
 - $3:15$
 - $6:15$
 - $4:7$
 - 20 cents : \$1.50
 - $4\text{ cm}:5\text{ m}$
 - $10:2$
 - $2\text{ mm}:1\text{ km}$
- 2** Write each of these ratios in the form $1:n$.
- $2:8$
 - $5:12$
 - $2\text{ mm}:10\text{ cm}$
 - $2\text{ cm}:5\text{ km}$
 - $100:40$
- 3** On a map a distance of 8 mm represents a distance of 2 km.
What is the scale of the map in the form $1:n$?
- 4** A passport photo is 35 mm long.
An enlargement of the photo is 21 cm long.
What is the ratio of the passport photo to the enlargement in the form $1:n$?

Using ratio to find an unknown quantity

Sometimes you know one of the quantities in the ratio, but not the other.

If the ratio is in the form $1:n$, you can work out the second quantity by multiplying the first quantity by n . You can work out the first quantity by dividing the second quantity by n .

Example 11.3

Question

A map is drawn to a scale of 1 cm : 2 km.

- a On the map, the distance between Amhope and Didburn is 5.4 cm.
What is the real distance in kilometres?
- b The length of a straight railway track between two stations is 7.8 km.
How long is this track, in centimetres, on the map?

Solution

- a The real distance, in kilometres, is twice as large as the map distance, in centimetres.

So multiply by 2.

$$2 \times 5.4 = 10.8$$

Real distance = 10.8 km

- b The map distance, in centimetres, is half of the real distance, in kilometres.

So divide by 2.

$$7.8 \div 2 = 3.9$$

Map distance = 3.9 cm

Sometimes you have to work out quantities using a ratio that is not in the form $1:n$.

To work out an unknown quantity, convert to an equivalent ratio involving the known quantity.

Example 11.4**Question**

Two photos are in the ratio 2:5.

- What is the height of the larger photo?
- What is the width of the smaller photo?

**Solution**

- a The multiplier for the ratio of the heights is 4, since for the smaller photo $2 \times 4 = 8$. Use the same multiplier for the larger photo.

$$\begin{array}{c} 2:5 \\ \times 4 \quad \curvearrowright \\ 8:20 \end{array}$$

Height of the larger photo = 20 cm.

- b The multiplier for the ratio of the widths is 3, since for the larger photo $5 \times 3 = 15$. Use the same multiplier for the smaller photo.

$$\begin{array}{c} 2:5 \\ \times 3 \quad \curvearrowright \\ 6:15 \end{array}$$

Width of the smaller photo = 6 cm.

Exercise 11.3

- The ratio of helpers to babies in a crèche must be 1:4.
 - There are six helpers on Tuesday.
How many babies can there be?
 - There are 36 babies on Thursday.
How many helpers must there be?
- Sanjay is mixing pink paint. To get the shade he wants, he mixes red and white paint in the ratio 1:3.
 - How much white paint should he mix with 2 litres of red paint?
 - How much red paint should he mix with 12 litres of white paint?
- For a recipe, Chetan mixes water and lemon curd in the ratio 2:3.
 - How much lemon curd should he mix with 20 ml of water?
 - How much water should he mix with 15 teaspoons of lemon curd?
- To make a solution of a chemical, a scientist mixes 3 parts of the chemical with 20 parts of water.
 - How much water should she mix with 15 ml of the chemical?
 - How much of the chemical should she mix with 240 ml of water?
- An alloy is made by mixing 2 parts of silver with 5 parts of nickel.
 - How much nickel must be mixed with 60 g of silver?
 - How much silver must be mixed with 120 g of nickel?
- Sachin and Rehan share a flat.
They agree to share the rent in the same ratio as their wages.
Sachin earns \$300 a month and Rehan earns \$400 a month.
Sachin pays \$90. How much does Rehan pay?

Dividing a quantity in a given ratio

Use this method to divide a quantity in a given ratio:

- find the total number of shares by adding the parts of the ratio
- divide the quantity by the total number of shares to find the multiplier
- multiply each part of the ratio by the multiplier.

Example 11.5

Question

To make fruit punch, orange juice and grapefruit juice are mixed in the ratio 5 : 3.

Jo wants to make 1 litre of punch.

- How much orange juice does she need, in millilitres?
- How much grapefruit juice does she need, in millilitres?

Solution

$$5 + 3 = 8 \quad \text{First work out the total number of shares.}$$

$$1000 \div 8 = 125 \quad \text{Convert 1 litre to millilitres and divide by 8 to find the multiplier.}$$

A table is often helpful for this sort of question.

	Orange	Grapefruit
Ratio	5	3
Amount	$5 \times 125 = 625 \text{ ml}$	$3 \times 125 = 375 \text{ ml}$

- She needs 625 ml of orange juice.
- She needs 375 ml of grapefruit juice.

Note

To check your answers, add the parts together.

Together they should equal the total quantity.

For example, $625 \text{ ml} + 375 \text{ ml} = 1000 \text{ ml} \checkmark$

Exercise 11.4

- 1 Paint is mixed in the ratio 3 parts red to 5 parts white to make 40 litres of pink paint.
 - How much red paint is used?
 - How much white paint is used?
- 2 To make a solution of a chemical, a scientist mixes 1 part of the chemical with 5 parts of water. She makes 300 ml of the solution.
 - How much of the chemical does she use?
 - How much water does she use?
- 3 Amit, Bree and Chris share \$1600 between them in the ratio 2 : 5 : 3.
How much does each receive?
- 4 A 600 g bar of brass is made using the metals copper and zinc in the ratio 2 : 1.
How much of each metal is used?
- 5 In a class of 36 students the ratio of boys to girls is 5 : 4.
How many boys are there?
- 6 To get home from work Mehr runs and walks.
The distances he runs and walks are in the ratio 2 : 3.
He works 2 km from home.
How far does he run?

-  7 Orange squash needs to be mixed in the ratio 1 part concentrate to 6 parts water.
How much concentrate is needed to make 3.5 litres of orange squash?
- 8 There are 572 senators in a national assembly.
The numbers of senators in the Blue, Orange and Green parties are in the ratio 6 : 3 : 2.
How many senators are there in each of the parties?
- 9 Sally makes breakfast cereal by mixing bran, currants and wheat germ in the ratio 8 : 3 : 1 by mass.
- How much bran does she use to make 600 g of the cereal?
 - One day, she has only 20 g of currants.
How much cereal can she make? She has plenty of bran and wheat germ.

Increasing and decreasing in a given ratio

A multiplier method is often used to increase or decrease sizes.

Example 11.6

Question

- Increase 12 cm in the ratio 5 : 2.
- Decrease \$30 in the ratio 3 : 8.

Solution

- a $12 \text{ cm} \times \frac{5}{2} = 30 \text{ cm}$ You need the answer to be an increase so the numerator is the larger part of the ratio.
 b $\$30 \times \frac{3}{8} = \11.25 You need the answer to be a decrease so the numerator is the smaller part of the ratio.

Exercise 11.5

- Increase 20 in the ratio

a 5 : 4	b 7 : 2	c 8 : 5
----------------	----------------	----------------
- Increase 48 in the ratio

a 5 : 3	b 9 : 8	c 6 : 5
----------------	----------------	----------------
- Decrease 50 in the ratio

a 7 : 10	b 3 : 4	c 5 : 8
-----------------	----------------	----------------
- Decrease 75 in the ratio

a 2 : 3	b 7 : 10	c 5 : 6
----------------	-----------------	----------------

Proportion

Direct proportion

In direct proportion, both quantities increase at the same rate.

For example, if you need 200 g of flour to make 10 cupcakes, you need 400 g of flour to make 20 cupcakes.

The quantity of flour is multiplied by 2.

The number of cupcakes is multiplied by 2.

Both quantities are multiplied by the same number, called the **multiplier**.

To find the multiplier, you divide either pair of quantities.

For example, $\frac{400}{200} = 2$ or $\frac{20}{10} = 2$.

So a quick way to solve this type of problem is to find the multiplier and use it to find the unknown quantity.

Example 11.7

Question

A car uses 20 litres of fuel when making a journey of 160 kilometres.

How many litres of fuel would it use to make a similar journey of 360 kilometres?

Solution

You could solve this problem using ratios, but here is a method just using proportion.

Write the distance travelled in the two journeys as a fraction.

You want to find the fuel needed for a 360 kilometre journey, so make 360 the numerator and 160 the denominator.

Cancel the fraction so that it is in its simplest form.

$$\frac{360}{160} = \frac{9}{4}$$

$$20 \times \frac{9}{4} = 45$$

Multiply the amount of fuel needed for the first journey by the multiplier.

45 litres of fuel would be used to make a journey of 360 kilometres.

Exercise 11.6

For each of these questions

- a write down the multiplier
- b calculate the required quantity.

- 1 An express train travels 165 metres in 3 seconds.
How far would it travel in 8 seconds?
- 2 An aeroplane travels 216 kilometres in 27 minutes.
How far did it travel in 12 minutes?
- 3 £50 is worth \$90.
How much is £175 worth?
- 4 A ladder that is 7 metres long has 28 rungs.
How many rungs would there be in a ladder that is 5 metres long?
- 5 A piece of string 27 metres long has a mass of 351 grams.
What is the mass of 15 metres of the string?
- 6 A piece of wood with a volume of 2.5 m^3 has a mass of 495 kilograms.
What is the mass of a piece of this wood with a volume of 0.9 m^3 ?

Inverse proportion

In inverse proportion, as one quantity increases, the other decreases.

In such cases, you need to divide by the multiplier, rather than multiply.

Example 11.8**Question**

Three excavators can dig a hole in 8 hours.

How long would it take four excavators to dig the hole?

Solution

Clearly, as more excavators are to be used, the digging will take less time.

The multiplier is $\frac{4}{3}$.

$$8 \div \frac{4}{3} = 6 \quad \text{Divide the known time by the multiplier to find the unknown time.}$$

It will take four excavators 6 hours to dig the hole.

Exercise 11.7

For each of these questions

- a write down the multiplier
- b calculate the required quantity.

- 1 A journey takes 18 minutes at a constant speed of 32 kilometres per hour.
How long would the journey take at a constant speed of 48 kilometres per hour?
- 2 It takes a team of 8 workers six weeks to paint a bridge.
How long would take 12 workers to paint the same bridge?
- 3 A pool is normally filled using four inlet valves in a period of 18 hours.
One of the inlet valves is broken.
How long will it take to fill the pool using just three inlet valves?
- 4 A journey can be completed in 44 minutes at an average speed of 50 kilometres per hour.
How long would the same journey take at an average speed of 40 kilometres per hour?
- 5 A quantity of hay is enough to feed 12 horses for 15 days.
How long would the same quantity of hay feed 20 horses?
- 6 It takes three harvesters 6 hours to harvest a crop of wheat.
How long would it take to harvest the wheat with only two harvesters?
- 7 On the outward leg of a journey a cyclist travels at an average speed of 12 kilometres per hour for a period of 4 hours.
The return journey took 3 hours.
What was the cyclist's average speed for the return journey?
- 8 It takes a team of 18 people 21 weeks to dig a canal.
How long would it take 14 people to dig the canal?
- 9 A tank can be emptied using six pumps in a period of 18 hours.
How long will it take to empty the tank using eight pumps?
- 10 A gang of 9 bricklayers can build a wall in 20 days.
How long would a gang of 15 bricklayers take to build the wall?

Rate

In this section you will look at rates such as rate of pay, rate of flow and average speed.

Rate of flow is a **compound measure** linking volume or mass with time.

$$\text{Average rate of flow} = \frac{\text{volume or mass}}{\text{total time taken}}$$

Rates of flow are measured in units such as kilograms per hour (kg/h) or litres per hour (l/h).

Example 11.9

Question

It takes half an hour to fill a 1000 litre tank.

What is the average rate of flow?

Solution

$$\text{Rate of flow} = \frac{1000 \text{ l}}{0.5 \text{ h}} = 2000 \text{ l/h}$$

Speed is a compound measure linking distance and time.

$$\text{Average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

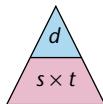
The units of your answer will depend on the units you begin with. Speed has units of ‘distance per time’, for example km/h.

The formula for speed can be rearranged to find the distance travelled or the time taken for a journey.

$$\text{Distance} = \text{speed} \times \text{time} \quad \text{Time} = \frac{\text{distance}}{\text{speed}}$$

Note

You may find the d.s.t. triangle helpful. Cover up the quantity you are trying to find. What is left shows how to multiply or divide.



Example 11.10

Question

How many minutes does it take to walk 2 km at a speed of 5 km/h?

Solution

$$\text{Time} = \frac{\text{distance}}{\text{speed}} = \frac{2}{5} \text{ hour}$$

$$\frac{2}{5} \text{ hour} = \frac{2}{5} \times 60 \text{ minutes} = 24 \text{ minutes}$$

Exercise 11.8

- 1 Water from a tap flows at 25 litres/minute.
How long will it take for this tap to fill a bath holding 500 litres?
- 2 Flour is loaded into a lorry which holds 20 tonnes.
It takes 40 minutes to load the lorry.
What is the rate of flow in kilograms per minute?
- 3 An oil tank contains 1500 litres.
Oil is used at a rate of 2 litres/hour.
How much is left in the tank after 7 days?
- 4 Hanif is paid at a rate of \$5.70 per hour.
One week he works 42 hours.
How much does he earn in this week?
- 5 Jo receives \$216 one week when she works 25 hours.
 - a What is her rate of pay per hour?
 - b At this rate how much would she earn for a week in which she works 16 hours?
- 6 Find the average speed of a car which travels 75 kilometres in $1\frac{1}{2}$ hours.
- 7 Find the average speed of a runner who covers 180 m in 40 seconds.
- 8 A cyclist rides 0.6 kilometres in 3 minutes.
Calculate her average speed in kilometres per hour.
- 9 A bus travels at 5 m/s on average.
How many kilometres per hour is this?
- 10 A runner's average speed in a 80 m race is 7 m/s.
Find the time he takes for the race, to the nearest 0.1 second.

12 PERCENTAGES

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- calculate a given percentage of a quantity
- express one quantity as a percentage of another
- calculate percentage increase or decrease
- carry out calculations involving reverse percentages.

CHECK YOU CAN:

- understand what is meant by a percentage
- convert between fractions, decimals and percentages.

Fractions, decimals and percentages

Since ‘per cent’ means ‘out of 100’, the fraction equivalent to a percentage is found by writing the percentage as a fraction with a denominator of 100.

So $13\% = \frac{13}{100}$, $5\% = \frac{5}{100}$, $140\% = \frac{140}{100}$, $2.3\% = \frac{2.3}{100}$, etc.

Sometimes you need to write equivalent fractions to give a fraction in its simplest form.

So $13\% = \frac{13}{100}$, but $5\% = \frac{1}{20}$, $140\% = 1\frac{2}{5}$, $2.3\% = \frac{23}{1000}$, etc.

The decimal equivalents can be found by carrying out the division shown in the fraction.

So $13\% = \frac{13}{100} = 0.13$, $5\% = \frac{5}{100} = 0.05$, $140\% = \frac{140}{100} = 1.4$,
 $2.3\% = \frac{2.3}{100} = 0.023$

Exercise 12.1

- Convert each of these percentages to a fraction in its lowest terms.
a 75% **b** 12% **c** 6% **d** 80%
- Convert each of these percentages to a decimal.
a 27% **b** 86% **c** 3% **d** 8%
e 150% **f** 140% **g** 250% **h** 174%
i 2.7% **j** 3.6% **k** 12.8% **l** 17.3%

Percentage of a quantity

Using fractions

This is often the best method to use when calculating percentages without a calculator.

When you are calculating simple percentages such as 10%, 20% or 5%, first find 10% and then multiply or divide the result to find the percentage you want.

Example 12.1**Question**

- a Find 15% of \$240.

- b Find 31% of \$6.80.

Solution

$$\begin{array}{ll}
 \text{a} & 10\% \text{ of } \$240 = 240 \div 10 \\
 & = \$24 \\
 5\% \text{ of } \$240 & = \text{half of } \$24 \\
 & = \$12 \\
 \text{So } 15\% \text{ of } \$240 & = \$24 + \$12 \\
 & = \$36
 \end{array}
 \quad
 \begin{array}{ll}
 \text{b} & 10\% \text{ of } 680c = 680 \div 10 \\
 & = 68c \\
 30\% \text{ of } 680 & = 68 \times 3 \\
 & = 204 \\
 \text{So } 30\% \text{ of } \$6.80 & = 204c \\
 1\% \text{ of } 680c & = 680 \div 100 = 6.8c \\
 \text{So } 31\% \text{ of } \$6.80 & = 204c + 6.8c \\
 & = 210.8c \\
 & = \$2.11 \text{ (to the nearest cent)}
 \end{array}$$

Sometimes it is easier to use the facts that $25\% = \frac{1}{4}$ and $50\% = \frac{1}{2}$ to work out percentages.

Example 12.2**Question**

- a Find 25% of \$72.

- b Find 75% of \$90.

Solution

$$\begin{array}{ll}
 \text{a} & 25\% \text{ of } \$72 = 72 \div 4 \\
 & = \$18 \\
 \text{b} & 50\% \text{ of } \$90 = 90 \div 2 \\
 & = \$45 \\
 & 25\% \text{ of } \$90 = \$45 \div 2 \\
 & = \$22.50 \\
 \text{So } 75\% \text{ of } \$90 & = \$45 + \$22.50 \\
 & = \$67.50
 \end{array}$$

Using decimals

To calculate a percentage of an amount, multiply the amount by the decimal equivalent of the percentage.

Example 12.3**Question**

- a Find 37% of \$48.

- b Find 2.4% of \$36 000.

Solution

$$\begin{array}{ll}
 \text{a} & 37\% \text{ of } \$48 = \$48 \times 0.37 \\
 & = \$17.76 \\
 \text{b} & 2.4\% \text{ of } \$36\,000 = \$36\,000 \times 0.024 \\
 & = \$864
 \end{array}$$

 **Exercise 12.2**

- 1** Work out these.
- | | | |
|-----------------------|----------------------|------------------------|
| a 10% of \$200 | b 10% of \$6 | c 10% of \$7.20 |
| d 20% of \$30 | e 30% of \$60 | f 20% of \$5 |
| g 20% of \$300 | h 30% of \$18 | i 30% of \$8.60 |
| j 5% of \$40 | k 5% of \$300 | l 5% of \$6.80 |
- 2** A sales representative gets 15% commission on sales of \$600.
How much is his commission?
- 3** Which is more, 15% of \$20 or 20% of \$15?
Show your working.
- 4** A shop increases its DVD prices by 5%.
What is the increase on a DVD which originally cost \$20?
- 5** A charity spends 5% of their income on administration.
One year they receive \$8000.
How much is spent on administration?
- 6** 20% of the students in a school arrive by bus.
There are 800 students in the school.
How many arrive by bus?
- 7** 15% of the 720 students in a school are left-footed.
How many students is this?
- 8** A school gave 30% of the money raised from a sponsored bike ride to a charity. The total raised was \$260.
How much did the charity receive?
- 9** In a year, a family spends 5% of their income on fuel.
They have an income of \$23 000.
How much do they spend on fuel?
- 10** Selina paid 23% of her income in tax.
How much tax did she pay in the month when she earned \$1284?
- 11** Work out these.
- | | | |
|-----------------------|---------------------|-----------------------|
| a 63% of 475 | b 75% of 27 | c 32% of \$720 |
| d 37% of 950 m | e 46% of 246 | f 99% of \$172 |
| g 13% of 156 | h 6% of 46 | i 8% of \$32 |
| j 3.5% of \$60 | | |
- 12** Beardalls' Jewellers offered 37.5% off the price of all rings.
How much did they offer off a ring priced at \$420?
- 13** A bus company claim that over 85% of the seats in their buses are fitted with safety belts.
What is the smallest number of seats that must have seat belts fitted in a 53-seater bus?

Expressing one quantity as a percentage of another

To express one quantity as a percentage of another, first write a fraction with the first quantity as the numerator of a fraction and the second quantity as the denominator.

Then multiply the fraction by 100.

Example 12.4**Question**

- Express 4 as a percentage of 5.
- Express \$5 as a percentage of \$30.
- Express 70 cm as a percentage of 2.3 m.

Solution

- a** $\frac{4}{5}$ Write as a fraction.
 $\frac{4}{5} \times 100 = \frac{400}{5} = 80\%$ Multiply by 100.
 Or: $\frac{4}{5} \times 100 = 0.8 \times 100 = 80\%$
- b** $\frac{5}{30} = 0.166\dots = 16.7\%$ to 1 decimal place
- c** $2.3 \text{ m} = 230 \text{ cm}$ Change 2.3 m to centimetres.
 $\frac{70}{230} = 0.304\dots = 30.4\%$ to 1 decimal place

Exercise 12.3

- 1** Find these.
- | | |
|--------------------------------------|--------------------------------------|
| a 12 as a percentage of 100 | b 4 as a percentage of 50 |
| c 80 as a percentage of 200 | d \$5 as a percentage of \$20 |
| e 4 m as a percentage of 10 m | f 30c as percentage of \$2 |
- 2** In each case, express the first quantity as a percentage of the second.

	a	b	c	d	e	f	g	h	i	j
First quantity	16	12	1 m	\$3	73c	8c	\$1.80	40 cm	50c	\$2.60
Second quantity	100	50	4 m	\$10	\$1	\$1	2	2 m	\$10	\$2

- 3** In Class 7B, 14 of the 25 students are boys.
 What percentage is this?
- 4** A sailor has a rope 50 m long. He cuts off 12 m.
 What percentage of the rope has he cut off?
- 5** In a sales promotion a company offers an extra 150 ml of cola free.
 The bottles usually hold 1 litre.
 What percentage of 1 litre is 150 ml?
- 6** Selma spends \$4 of her \$20 spending money on a visit to the cinema.
 What percentage is this?
- 7** At a keep-fit club, 8 of the 25 members are men.
 What percentage is this?
- 8** Derek scored 44 runs out of his team's total of 80.
 What percentage of the runs did he score?
- 9** Work out these.
- | | |
|--|--|
| a 25 as a percentage of 200 | b 40 as a percentage of 150 |
| c 76c as a percentage of \$1.60 | d \$1.70 as a percentage of \$2 |
| e 19 as a percentage of 24 | f 213 as a percentage of 321 |

- 10** Halima scores 17 out of 40 in a test.
What percentage is this?
- 11** Salma is given a discount of \$2.50 on a meal that would have cost \$17.
What percentage discount is she given?
- 12** During the 30 days of September in 2014, it rained on 28 of the days.
On what percentage of the days did it rain?
- 13** Last Saturday 6198 fans booked online for a festival and 4232 fans booked by phone. What percentage of the fans made online bookings?
- 14** A mathematics test has 45 questions. Abigail gets 42 right.
What percentage does Abigail get right?
- 15** A photo frame 20 cm by 30 cm contains a photo 14 cm by 18 cm.
What percentage of the whole area is the photo?

Percentage change

Percentage increases and decreases are worked out as percentages of the original amount, not the new amount.

Percentage profit or loss is worked out as a percentage of the cost price, not the selling price.

$$\text{Percentage change} = \frac{\text{change}}{\text{original amount}} \times 100$$

Example 12.5

Question

An art dealer buys a painting for \$45 and sells it for \$72.

What percentage profit is this?

Solution

$$\text{Profit} = \$72 - \$45 = \$27$$

$$\text{Percentage profit} = \frac{27}{45} \times 100 = 60\%$$

Example 12.6

Question

The value of a computer drops from \$1200 to \$700 in a year.

What percentage decrease is this?

Solution

$$\text{Decrease in value} = \$1200 - \$700 = \$500$$

$$\text{Percentage decrease} = \frac{500}{1200} \times 100 = 41.7\% \text{ to 1 decimal place}$$

Exercise 12.4

- 1** A shopkeeper buys an article for \$10 and sells it for \$12.
What percentage profit does she make?
- 2** Adam earned \$5 per hour. His pay is increased to \$5.60 per hour.
What is the percentage increase in Adam's pay?
- 3** Rahat bought a car for \$1250. A year later he sold it for \$600.
What percentage of the value did he lose?
- 4** Umar buys a signed photo for \$22 and sells it for \$28.
What is his percentage profit? Give your answer to the nearest 1%.
- 5** Hannah reduces the time it takes her to swim 30 lengths from 55 minutes to 47 minutes.
What percentage reduction is this? Give your answer to the nearest 1%.

Percentage increase and decrease

To find the amount after a percentage increase, work out the increase and add it on to the original amount.

To find the amount after a percentage decrease, work out the decrease and subtract it from the original amount.

Example 12.7

Question

An engineer receives an increase of 3% in her annual salary.

Her salary was \$24 000 before the increase.

What is her new salary?

Solution

$$3\% = 0.03$$

$$\text{Increase} = \$24\,000 \times 0.03 = \$720 \quad \text{Find the increase.}$$

$$\text{New salary} = \$24\,000 + \$720 = \$24\,720 \quad \text{Add the increase to the original amount.}$$

Example 12.8

Question

In a sale all prices are reduced by 15%.

Find the sale price of an article previously priced at \$17.60.

Solution

$$15\% = 0.15$$

$$17.60 \times 0.15 = 2.64 \quad \text{Find the decrease.}$$

$$\$17.60 - 2.64 = \$14.96 \quad \text{Subtract the decrease from the original amount.}$$

Exercise 12.5

- 1** Increase \$400 by each of these percentages.
a 20% **b** 45% **c** 6% **d** 80%
- 2** Decrease \$200 by each of these percentages.
a 30% **b** 15% **c** 3% **d** 60%
- 3** The rent for a shop was \$25 000 a year and went up by 20%.
 What is the new rent?
- 4** Simon earns \$12 000 per year.
 He receives a salary increase of 4%.
 Find his new salary.
- 5** Prices are reduced by 20% in a sale.
 The original price of a top was \$13.
 What is the sale price?
- 6** Mike puts \$1200 into a savings account.
 7.5% interest is added to the account at the end of the year.
 How much is in the account at the end of the year?
- 7** To test the strength of a piece of wire, it is stretched by 12% of its original length.
 The piece of wire was originally 1.5 m long.
 How long will it be after stretching?
- 8** Rushna pays 6% of her pay into a pension fund.
 She earns \$185 per week.
 What will her pay be after taking off her pension payment?
- 9** A drum kit costs \$280 before sales tax is added.
 What will it cost after sales tax at 17% has been added?
- 10** A savings account pays 6.2% per annum.
 Sohail put \$2500 in this account and left it for a year.
 How much was in the account at the end of the year?

Percentage increase and decrease using a multiplier

You can find a quantity after a percentage increase or decrease in one step using a multiplier.

Example 12.9

Question

Amir's salary is \$17 000 per year.

He receives a 3% increase.

Find his new salary.

Solution

Amir's new salary is 103% of his original salary.

So the multiplier is 1.03.

$$\text{New salary} = \$17\,000 \times 1.03 = \$17\,510.$$

Example 12.10**Question**

In a sale all prices are reduced by 15%.

Rushna buys a tablet in the sale.

The original price was \$155.

Calculate the sale price.

Solution

A 15% decrease is the same as $100\% - 15\% = 85\%$.

So the multiplier is 0.85.

$$\text{Sale price} = \$155 \times 0.85 = \$131.75$$

Exercise 12.6

- 1** Write down the multiplier that will increase an amount by each of these percentages.
 a 13% b 20% c 68% d 8%
 e 2% f 17.5% g 150%
- 2** Write down the multiplier that will decrease an amount by each of these percentages.
 a 14% b 20% c 45% d 7%
 e 3% f 23% g 16.5%
- 3** Sanjay used to earn \$4.60 per hour.
 He received a 4% increase.
 How much does he earn now? Give your answer to the nearest cent.
- 4** In a sale, everything is reduced by 30%.
 Ali buys a pair of shoes in the sale.
 The original price was \$42.
 What is the sale price?
- 5** A shop increases its prices by 8%.
 What is the new price of a skirt which previously cost \$30?

Finding the original quantity

To increase a quantity by 20%, you calculate

$$\text{new amount} = \text{original amount} \times 1.20.$$

It follows that

$$\text{original amount} = \text{new amount} \div 1.20.$$

Example 12.11**Question**

Jared received a salary increase of 20%.

After the increase his salary is \$31 260.

What was his salary before the increase?

Solution

$$\text{New salary} = \text{old salary} \times 1.20$$

$$\text{Old salary} = \text{new salary} \div 1.20$$

$$= \$31\,260 \div 1.20$$

$$= \$26\,050$$

To decrease a quantity by 10%, you calculate

$$\text{new amount} = \text{original amount} \times 0.90.$$

It follows that

$$\text{original amount} = \text{new amount} \div 0.90.$$

Example 12.12**Question**

In a sale all prices are reduced by 10%.

Selma paid \$38.70 for a dress in the sale.

What was the original price of the dress?

Solution

$$\text{New price} = \text{original price} \times 0.90$$

$$\text{Original price} = \text{new price} \div 0.90$$

$$= \$38.70 \div 0.90$$

$$= \$43$$

Exercise 12.7

- 1** After an increase of 12%, a quantity is 84 tonnes.
What was it before the increase?
- 2** A quantity is decreased by 3%. It is now 38.8.
What was it to start with?
- 3** In a sale everything is reduced by 5%.
A pair of shoes costs \$47.50 in the sale.
How much did they cost before the sale?
- 4** A newspaper increases the average number of copies it sells in a day by 3%.
The new number of copies sold is 58 195.
What was it before the increase?

- 5 Mr Lee made a profit of \$13 250 in the year 2015.
This was an increase of 6% on his profit in 2014.
What was his profit in 2014?
- 6 Santos sold his car for \$8520.
This was 40% less than he paid for it five years before.
What did he pay for it?
- 7 A charity's income in 2015 was \$8580.
This was a decrease of $2\frac{1}{2}\%$ compared with 2014.
What was its income in 2014?
- 8 Stephen was given a rise of 7%.
His salary after the rise was \$28 890.
What was it before the rise?
- 9 Between 2012 and 2013 house prices increased by 12.5%.
In 2013 the value of a house was \$27 000.
What was its value in 2012?
- 10 A television costs \$564 including a sales tax of 17.5%.
What is the cost without the sales tax?

13

USE OF AN ELECTRONIC CALCULATOR

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- use a calculator efficiently
- apply appropriate checks of accuracy
- enter a range of measures including 'time'
- interpret the calculator display appropriately.

CHECK YOU CAN:

- use the correct order of operations in calculations
- work with numbers in standard form
- use a calculator to find squares, square roots, cubes and cube roots
- round numbers to a given degree of accuracy
- estimate the answers to calculations by rounding to one significant figure.

Order of operations

You should know that this is the correct order of operations when carrying out any calculation:

- First work out anything in brackets.
- Then work out any powers (such as squares or square roots).
- Then do any multiplication or division.
- Finally do any addition or subtraction.

A scientific calculator should follow this order of operations.

Using the correct order of operations, $2 + 3 \times 4 = 14$.

Enter the calculation $\boxed{2} + \boxed{3} \times \boxed{4}$ on your calculator and check that you get the answer 14.

When a calculation is written as a fraction, you will need to use brackets to ensure that the calculator divides the result of the calculation in the numerator by the result of the calculation in the denominator.

Using the correct order of operations, $\frac{9+3}{4+2} = 2$.

On a calculator you will need to enter $(\boxed{9} + \boxed{3}) \div (\boxed{4} + \boxed{2})$ for this calculation. Check that you get the answer 2 when you do this.

A scientific calculator has function keys that can be used in calculations.

You need to find out what these keys look like on your calculator.

Your calculator will have keys for squares, cubes, square roots and cube roots.

You should make sure that you can use these keys in calculations.

Example 13.1

Question

Use a calculator to work out these, giving each answer correct to 3 significant figures.

a $\frac{4.65 + 5.72}{0.651 \times 12.7}$

b $\sqrt{3.65^2 - 1.83^2}$

Solution

- a In this calculation the fraction line means that you must divide the result of the calculation in the numerator by the result of the calculation in the denominator.

You use brackets so that the calculator does this.



$$(4.65 + 5.72) \div (0.651 \times 12.7) \\ = 1.254\ 278\ 699 \\ = 1.25$$

This calculation is entered on the calculator.
Round the result to 3 significant figures.

Check: $(5 + 6) \div (0.7 \times 10) = 11 \div 7 = 1.5\dots$

The estimate is close to the calculated answer which suggests that the correct order of operations has been used.

- b In this calculation, the complete subtraction is inside the square root sign.

$$\sqrt{(3.65^2 - 1.83^2)}$$

This calculation is entered on the calculator. Use the square key and the square root key. You may need to use brackets.

$$= 3.158\ 100\ 695 \\ = 3.16$$

Round the result to 3 significant figures.

Check: $\sqrt{(4^2 - 2^2)} = \sqrt{(16 - 4)} = \sqrt{12}$ $\sqrt{12}$ is close to the calculated answer, which suggests that the correct order of operations has been used.

Note

It is a good idea to check that your answer is approximately correct by rounding the values to one significant figure and working out an estimate of the answer. If the estimated answer is not close to the calculated answer, this suggests that you have not used the correct order of operations.

Exercise 13.1

Use a calculator to work out these. Give your answers correct to 3 significant figures where appropriate.

1 $\frac{-4.7 + 2.6}{-5.7}$

2 $\frac{-4.72}{-1.4} \times \frac{8.61}{-7.21}$

3 $\frac{7.92 \times 1.71}{-4.2 + 3.6}$

4 $\frac{3.14 - 8.16}{-8.25 \times 3.18}$

5 $\frac{5.2 + 10.3}{3.1}$

6 $\frac{127 - 31}{25}$

7 $\frac{9.3 + 12.3}{8.2 - 3.4}$

8 $6.2 + \frac{7.2}{2.4}$

9 $2.8 \times (5.2 - 3.6)$

10 $\frac{5.3}{2.6 \times 1.7}$

11 $\frac{5.3}{2.6 + 1.7}$

12 $2.7^3 + 3.8^2$

13 $\sqrt{1.2^2 + 0.5^2}$

14 $\sqrt{15.7 - 3.8}$

15 $\frac{2.6^2}{1.7 + 0.82}$

16 $\frac{6.2 \times 3.8}{22.7 - 13.8}$

17 $\frac{5.3}{\sqrt{6.2 + 2.7}}$

18 $\frac{5 + \sqrt{25 + 12}}{6}$

19 $2.7^2 + 3.6^2 - 2 \times 2.7 \times 3.6 \times 0.146$

20 $\frac{6.2}{2.6} + \frac{5.4}{3.9}$

21 $\frac{2.6 + 4.2}{7.8 \times 3.6^2}$

22 $\sqrt[3]{2.5^2 - 1.8^2}$

Standard form on your calculator

You can do calculations involving standard form on your calculator using the **EXP** or **$\times 10^x$** key.

Check which of these your calculator has.

In following work **EXP** is used, but if your calculator has **$\times 10^x$** use that instead.

Example 13.2**Question**

Find $7 \times 10^7 \div 2 \times 10^{-3}$ using your calculator.

Solution

These are the keys to press on your calculator.

7 EXP 7 ÷ 2 EXP (-) 3 =

You should see 3.5×10^{10} .

Note

When using the **EXP** key, do not enter 10 as well.

Sometimes your calculator will display an ordinary number which you will have to write in standard form, if necessary. Otherwise your calculator will give you the answer in standard form.

Modern calculators usually give the correct version of standard form, for example 3.5×10^{10} .

Older calculators often give a calculator version such as 3.5^{10} . You must write this in proper standard form, 3.5×10^{10} , for your answer.

Some graphical calculators display standard form as, for example, 3.5 E 10.

Again you must write this in proper standard form for your answer.

Practise by checking Example 13.3 on your calculator.

Example 13.3**Question**

The radius of Neptune is 2.48×10^4 km.

Assume that Neptune is a sphere.

The surface area of a sphere is given by $4\pi r^2$.

Calculate the surface area of Neptune.

Give your answer in standard form, correct to 2 significant figures.

Solution

$$A = 4\pi r^2 = 4 \times \pi \times (2.48 \times 10^4)^2$$

This is the sequence of keys to press.

4 × SHIFT π × (2 . 4 8 EXP 4) x^2 =

The result on your calculator should be 7 728 820 583.

So the answer is 7.7×10^9 km 2 correct to 2 significant figures.

Exercise 13.2

- 1** Work out these. Give your answers in standard form correct to 3 significant figures.
- a $6.21 \times 10^5 \times 3.78 \times 10^7$ b $8.34 \times 10^7 \div 1.78 \times 10^2$
 c $5.92 \times 10^8 \div 3.16 \times 10^{-3}$ d $6.27 \times 10^{-4} \times 4.06 \times 10^{-3}$
 e $9.46 \times 10^8 \div 3.63 \times 10^3$ f $7.3 \times 10^4 \times 3.78 \times 10^3$
 g $(5.63 \times 10^5)^2$ h $(8.76 \times 10^{-4})^2$
- 2** The radius of Jupiter is 7.14×10^4 km.
 Assume that Jupiter is a sphere.
 The surface area of a sphere is given by $4\pi r^2$.
 Calculate the surface area of Jupiter.
 Give your answer in standard form, correct to 2 significant figures.
- 3** The population of the UK in 2011 was approximately 63 200 000.
- a Write 63 200 000 in standard form.
 b The area of the UK is approximately 2.44×10^5 km².
 Calculate the average number of people per km² in the UK in 2011.
 Give your answer to a suitable degree of accuracy.
- 4** a The planet Neptune is 4.5×10^9 km from the Sun.
 Light travels at 3×10^5 km/s.
 How many seconds does it take light to travel from the Sun to Neptune?
 Give your answer in standard form.
 b Earth is 1.5×10^8 km from the Sun.
 How much further from the Sun is Neptune than Earth.
 Give your answer in standard form.
- 5** The population of Asia in 2013 was estimated to be 4.299 billion.
- a Write 4.299 billion in standard form.
 b The area of Asia is approximately 4.34×10^7 km².
 Calculate the number of people per km² in Asia.
 Give your answer to a suitable degree of accuracy.
- 6** Light takes approximately 3.3×10^{-9} seconds to travel 1 metre.
 The distance from the Earth to the Sun is 150 000 000 km.
 a Write 150 000 000 km in metres using standard form.
 b How long does it take for light to reach the Earth from the Sun?

Checking accuracy

When you have solved a problem, it is useful to check whether the answer is sensible.

When the problem is set in a real-life context you may be able to use your experience to help you check. For example, if the question is about a shopping bill think about whether the total you have found is realistic.

If the question is not set in context, you need to use other methods to check your answer.

You have checked answers by rounding the numbers in the calculation to one significant figure and finding an estimate for the result. There are also other methods that you can use.

Using number facts

Look at these calculations.

$$40 \times 5 = 200 \qquad \qquad 40 \times 0.5 = 20$$

$$40 \times 2 = 80 \qquad \qquad 40 \times 0.8 = 32$$

$$40 \times 1.1 = 44 \qquad \qquad 40 \times 0.9 = 36$$

You can see that when 40 is multiplied by a number greater than 1, the result is greater than 40.

When 40 is multiplied by a number less than 1, the result is smaller than 40.

Look at these calculations.

$$40 \div 5 = 8 \qquad \qquad 40 \div 0.5 = 80$$

$$40 \div 2 = 20 \qquad \qquad 40 \div 0.8 = 50$$

$$40 \div 1.1 = 36.36\dots \qquad 40 \div 0.9 = 44.44\dots$$

You can see that when 40 is divided by a number greater than 1, the result is smaller than 40.

When 40 is divided by a number less than 1, the result is greater than 40.

These results can be applied to any multiplication or division.

Starting with any positive number,

- multiplying by a number greater than 1 gives a result that is larger than the starting number
- multiplying by a number between 0 and 1 gives a result that is smaller than the starting number
- dividing by a number greater than 1 gives a result that is smaller than the starting number
- dividing by a number between 0 and 1 gives a result that is larger than the starting number.

Using inverse operations

Inverse operations can be used to check the result of a calculation.

You know that subtraction is the inverse of addition and that division is the inverse of multiplication.

You can use this to check the result of a calculation.

Work out $6.9 \div 75$ on your calculator. The answer is 0.092.

You can check this answer using the multiplication 0.092×75 .

The answer to the multiplication is 6.9, so the answer to the division was correct.

Finding the square root is the inverse of finding the square of a number.

You can use these facts to check the result of a calculation.

Example 13.4

Question

Explain how you can tell the answer to each of these calculations is wrong.

- a $297 \div 2.8 = 10.6$ to 3 significant figures
- b $752 \div 24 = 18\ 048$
- c $\sqrt{35} = 9.52$ to 2 decimal places

Solution

- a Rounding to 1 significant figure gives $300 \div 3 = 100$, so the answer 10.6 is too small.
- b 752 is divided by a number greater than 1, so the result should be less than 752 but 18 048 is greater than 752.
- c $6^2 = 36$, so $\sqrt{35}$ must be less than 6 but 9.52 is greater than 6.
Alternatively, $9^2 = 81$, so $\sqrt{35}$ must be much less than 9.

Exercise 13.3

- 1 Which of these answers might be correct and which are definitely wrong?

Show how you decide.

- a $39.6 \times 18.1 = 716.76$
- b $175 \div 1.013 = 177.275$
- c $8400 \times 9 = 756\ 000$
- d An elevator can hold 9 people so a group of 110 people will need 12 trips.
- e Musa has \$100 to spend. He thinks that he has enough money to buy 5 DVDs costing \$17.99 each.

- 2 Look at these calculations. The answers are all wrong.

For each calculation, show how you can tell this quickly, without using a calculator to work it out.

- | | |
|----------------------------|--------------------------------|
| a $-6.2 \div -2 = -3.1$ | b $12.4 \times 0.7 = 86.8$ |
| c $31.2 \times 40 = 124.8$ | d $\sqrt{72} = 9.49$ to 2 d.p. |
| e $0.3^2 = 0.9$ | f $16.2 \div 8.1 = 20$ |
| g $125 \div 0.5 = 25$ | h $6.4 \times -4 = 25.6$ |
| i $24.7 + 6.2 = 30.8$ | j $76 \div 0.5 = 38$ |

Interpreting the calculator display

When a calculation involves units, you need to make sure that you use the same units for all of the quantities and that you give the correct units for your answer.

It is also important that you think about the accuracy needed when you give your answer.

If the answer is not exact, think about what degree of accuracy is sensible.

If the question is set in context, the context may help you to decide on what accuracy is appropriate. For example, in a problem about money the answer should be given to two decimal places if it is not exact.

If the problem is not set in context and the question does not tell you what accuracy to use, then you should give your answer correct to three significant figures.

If you need to use the result of a calculation in a second calculation, then you should use the unrounded value in the second calculation.

Example 13.5

Question

- a \$18 is shared equally between 10 people. How much does each person receive?
- b \$18 is shared equally between 13 people. How much does each person receive?

Solution

- | | |
|----------------------------------|---|
| a $18 \div 10 = 1.8$ | The calculator answer is 1.8. |
| Each person receives \$1.80. | Money should be written with 2 decimal places, so add the final zero. |
| b $18 \div 13 = 1.384\ 615\dots$ | The answer is not exact. |
| Each person receives \$1.38. | Round the answer to 2 decimal places. |

Exercise 13.4

- 1 These are the amounts of water Dilip drinks each day for 4 days.
1.5 litres 800 ml 2 litres 1 litre 250 ml
How much water did he drink in total?
- 2 A piece of string 4 m long is cut into seven equal pieces.
How long is each piece?
- 3 Work out how much each person receives in each case.
 - a \$250 is divided equally between 4 people.
 - b \$67 is divided equally between 8 people.
 - c \$137 is divided equally between 12 people.
 - d \$475 is divided equally between 17 people.
- 4 Narinder is paid \$12.45 per hour.
Work out how much he is paid for 8 hours' work.

Calculating with time

When a calculation involves time, you need to make sure that you use your calculator correctly.

Entering time on a calculator

There are 60 minutes in an hour, so a time of 2 hours 30 minutes is the same as $2\frac{1}{2}$ hours.

You enter 2.5 on your calculator.

If you need to enter a time such as 1 hour 12 minutes, first convert the minutes to a fraction of an hour.

12 minutes is $\frac{12}{60}$ of an hour or, as a decimal, 0.2 hour.

So 1 hour 12 minutes is 1.2 hours.

You can use the same method to enter times given in minutes and seconds on your calculator.

Interpreting the result of a time calculation

You are often asked to give an answer to a time calculation in hours and minutes or minutes and seconds. If you have used a calculator, then the answer will be a decimal which you need to convert to the correct units.

Some conversions can be done mentally.

For example, you know that 1.5 minutes is the same as $1\frac{1}{2}$ minutes.

There are 60 seconds in a minute, so 1.5 minutes is the same as 1 minute 30 seconds.

Similarly, you know that 1.25 minutes is the same as $1\frac{1}{4}$ minutes.

There are 60 seconds in a minute, so 1.25 minutes is the same as 1 minute 15 seconds.

Example 13.6

Question

- Convert 8 minutes 36 seconds into minutes.
- Write 6.4 hours in hours and minutes.

Solution

- 8 minutes 36 seconds is $8\frac{36}{60}$ minutes.
 $36 \div 60 = 0.6$ Convert the fraction to a decimal using division.
8 minutes 36 seconds is 8.6 minutes.
- $0.4 \times 60 = 24$ Multiply the decimal by 60 to convert to minutes.
0.4 hour is 24 minutes
6.4 hours is 6 hours 24 minutes.

Exercise 13.5

- 1 Convert each of these times into hours.
 - a 1 hour 45 minutes
 - b 2 hours 12 minutes
 - c 39 minutes
- 2 Convert each of these times into minutes.
 - a 5 minutes 15 seconds
 - b 3 minutes 27 seconds
 - c 42 seconds
- 3 Write each of these times in hours and minutes.
 - a 3.5 hours
 - b 1.3 hours
 - c 4.85 hours
- 4 Write each of these times in minutes and seconds.
 - a 3.75 minutes
 - b 2.4 minutes
 - c 1.9 minutes
- 5 Asma takes 8 minutes to make one birthday card.
If she works at the same rate, how long will she take to make 20 cards?
Give your answer in hours and minutes.

Hours, minutes and seconds on a calculator

Some calculators have a $\textcircled{.} \textcircled{''}$ key.

You can use this for working with time on your calculator.

When you do a calculation and the answer is a time in hours but your calculator shows a decimal, you change it to hours and minutes by pressing the $\textcircled{.} \textcircled{''}$ key and then the $=$ key.

To convert back to a decimal, press the $\textcircled{.} \textcircled{''}$ key again.

To enter a time of 8 hours 32 minutes on your calculator, press this sequence of keys.

8 $\textcircled{.} \textcircled{''}$ **3** **2** $\textcircled{.} \textcircled{''}$ **=**

The display should look like this.

8°32°0

You may wish to experiment with this key and learn how to use it to enter and convert times.

Fractions on a calculator

You need to be able to calculate with fractions without a calculator. However, when a calculator is allowed you can use the fraction key.

You may wish to experiment with calculations using the fraction key on your calculator.

The following instructions are for a calculator with a fraction key that looks like this. $\text{a}^{\text{b}}/\text{c}$

To enter a fraction such as $\frac{2}{5}$ into your calculator you press this sequence of keys.

2 $\text{a}^{\text{b}}/\text{c}$ **5** **=**

The display will look like this. 

This is the sequence of keys to press to do the calculation $\frac{2}{5} + \frac{1}{2}$.

=

The display should look like this. 

This is the sequence of keys to press to enter a mixed number such as

 into your calculator.

=

Your display will look like this. 

You can also cancel a fraction on your calculator.

When you press =, you should see: 

When you press  again, the display changes to .

When you do calculations with fractions on your calculator, it will automatically give the answer as a fraction in its lowest terms.

Similarly, if you enter an improper fraction into your calculator and press the = key, the calculator will automatically change it to a mixed number.

14 TIME

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- calculate times in terms of the 24-hour and 12-hour clock
- read clocks, dials and timetables.

CHECK YOU CAN:

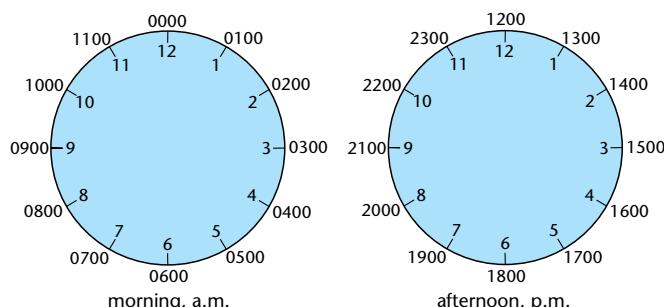
- calculate speed.

The 24-hour clock

The time of the day can be given as a.m., p.m. or on the 24-hour clock.

Digital recorders use the 24-hour clock and so you are likely to be familiar with this.

You also need to be able to change from one to the other.



Example 14.1

Question

Write these times using the 24-hour clock.

- a 11.32 a.m. b 9.14 a.m. c 12.18 a.m. d 6.55 p.m.

Solution

- a 11 32. Times using a.m. are the same as on the 24-hour clock.
You don't need to write a full stop or a colon when using the 24-hour clock, but 11:32 is also acceptable.
b 09 14. For times before 10 a.m. you need to add a zero at the start so that there are four figures.
c 00 18. You need to add two zeros.
d 18 55. To change times using p.m. to the 24-hour clock, add 12 to the hours.

Note

Times between midnight and 1 a.m. start with 00 on the 24-hour clock.

Example 14.2

Question

Write these times using the 12-hour clock.

- a 11 46 b 04 21 c 14 22 d 23 05

Solution

- a 11.46 a.m. Add a full stop and write a.m. 11:46 a.m. is also acceptable.
b 4.21 a.m. As it is in the morning, just add a.m. and drop the first zero.
c 2.22 p.m. As it is in the afternoon, subtract 12 hours and add p.m.
d 11.05 p.m.

Note

Times between 12 noon (midday) and 1 p.m. do not have 12 subtracted from them for p.m. times.

Exercise 14.1

1 Write these times using a.m. notation.

- | | | | | |
|--------|--------|--------|--------|--------|
| a 1145 | b 0553 | c 0140 | d 0920 | e 0010 |
| f 1041 | g 0232 | h 0430 | i 1120 | j 0048 |

2 Write these times using p.m. notation.

- | | | | | |
|--------|--------|--------|--------|--------|
| a 1345 | b 1553 | c 2140 | d 2259 | e 1210 |
| f 1440 | g 1723 | h 1940 | i 2019 | j 1203 |

3 Write these times using the 24-hour clock.

- | | | | |
|--------------|--------------|--------------|--------------|
| a 1.50 a.m. | b 2.40 p.m. | c 11.49 a.m. | d 6.30 p.m. |
| e 12.02 a.m. | f 3.20 a.m. | g 2.08 p.m. | h 12.49 a.m. |
| i 9.35 a.m. | j 11.02 p.m. | | |

4 Write these times using the 12-hour clock.

- | | | | | |
|--------|--------|--------|--------|--------|
| a 0345 | b 1456 | c 2340 | d 1159 | e 1255 |
| f 0435 | g 1516 | h 2140 | i 0159 | j 1452 |

Calculating with time

When calculating with time, remember that there are 60 minutes in an hour and 60 seconds in a minute.

Example 14.3

Question

Mavis left for school at 7.45 a.m. and arrived 40 minutes later.

At what time did she arrive?

Solution

$$\begin{aligned} 7.45 + 40 \text{ minutes} &= 7 \text{ hours} + 85 \text{ minutes} && 85 \text{ minutes is 1 hour and 25 minutes} \\ &= 8 \text{ hours } 25 \text{ minutes} \end{aligned}$$

She arrived at school at 8.25 a.m.

Example 14.4

Question

Carla set her digital recorder to start recording at 1520 and to stop at 1710.

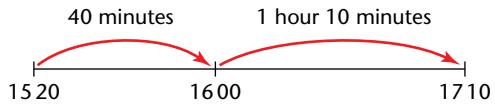
For how long was it recording?

Solution

Count to the next hour, 1600. This is 40 minutes.

Then, to get to 1710, a further 1 hour and 10 minutes are needed.

So the total time is 40 minutes + 1 hour 10 minutes = 1 hour 50 minutes.



Example 14.5**Question**

It takes Hollie 35 minutes to prepare some food for cooking.

She is ready to start cooking at 6.15 p.m.

At what time did she start preparing the food?

Solution

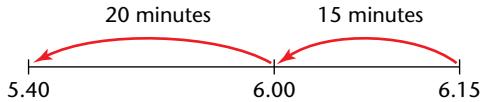
Work backwards from 6.15 p.m.

15 minutes before 6.15 is 6.00 p.m.

That leaves $35 - 15 = 20$ minutes still to take off.

20 minutes before 6.00 is 5.40 p.m.

So she started preparing the food at 5.40 p.m.

**Timetables****Example 14.6****Question**

This is the timetable for some trains from Delhi to Agra.

Delhi (depart)	0600	0835	1005	1555	2055	2115	2230
Agra (arrive)	0806	1100	1243	1808	2335	0002	0107

Work out how long these trains take to do the journey.

- a The 0835 from Delhi
- b The 1555 from Delhi
- c The 2230 from Delhi

Solution

- a From 0835 to 0900 is 25 minutes.
0900 to 1100 is 2 hours.
So the 0835 from Delhi takes 2 hours 35 minutes.
- b From 1555 to 1600 is 5 minutes.
From 1600 to 1800 is 2 hours.
From 1800 to 1808 is 8 minutes.
So the 1555 from Delhi takes 2 hours 13 minutes.
- c From 2230 to 2300 is 30 minutes.
From 2300 to 0100 is 2 hours.
From 0100 to 0107 is 7 minutes.
So the 2230 from Delhi takes 2 hours 37 minutes.

Exercise 14.2

- 1 A train due at 1440 was 50 minutes late.
What time did it arrive?
- 2 A ferry left Liverpool at 1420 and arrived at Belfast at 2105.
How long did it take?
- 3 Kylie left home at 7.50 a.m. and took 2 hours 45 minutes to reach her friend's house.
What time did she arrive at her friend's house?
- 4 Sam set his oven to switch on at 3.45 p.m. and switch off at 6.20 p.m.
How long was the oven on?
- 5 Jo set her digital recorder to switch on at 8.35 p.m. and switch off at 10.20 p.m.
How long was the recorder on?
- 6 A bus left Barnmouth at 1547 and arrived in Manton at 1820.
How long did it take?
- 7 Here is part of the Howerdale railway timetable.

Stafford	0815
Luker	0905
Melpett	0940
Haker	1120
Golpath	1135
Oldway	1320

- a How long does it take to get from
 - i Stafford to Melpett
 - ii Haker to Oldway
 - iii Luker to Golpath
 - iv Melpett to Oldway?
- b One day the train is delayed at Luker for 35 minutes.
At what time did it arrive at
 - i Golpath
 - ii Oldway?
- 8 Ciara's favourite film ended at 8.50 p.m.
The film lasted for 1 hour and 45 minutes.
What time did it start?
- 9 A train journey lasts 2 hours and 40 minutes.
The train arrives at 10.35 a.m.
What time did it start its journey?
- 10 A ferry leaves port at 0835 on a 55-minute journey.
The ferry arrives 20 minutes late due to bad weather.
What time does it arrive at its destination?

Converting between hours and minutes and hours written as a decimal

When you are calculating with time, your calculator will often give times in hours using decimals but many questions ask for answers in hours and minutes.

Similarly a time in a question may be given in hours and minutes but you need to change this to a decimal number to do the calculation.

Example 14.7**Question**

Write 15.36 hours in hours and minutes correct to the nearest minute.

Solution

$$0.36 \text{ hours} = 0.36 \times 60 \quad \text{To convert hours to minutes multiply by 60.}$$

$$= 21.6 \text{ minutes}$$

So 15.36 hours = 15 hours 22 minutes correct to the nearest minute.

Example 14.8**Question**

The Beijing to Shanghai train travels 1318 km in 5 hours 37 minutes.

Find the average speed of the train correct to the nearest km/h.

Solution

$$37 \text{ minutes} = 37 \div 60 \quad \text{To convert minutes to hours divide by 60.}$$

$$= 0.6166\ldots \text{ hours}$$

So the total time is $5 + 0.6166\ldots = 5.6166\ldots$ hours.

Speed = distance \div time

$$\text{Speed} = 1318 \div 5.6166\ldots = 234.658\ldots$$

$$= 235 \text{ km/h correct to the nearest km/h}$$

Time zones

The time of day varies in different countries depending on their longitude.

In large countries there may even be differences within the country.

Usually these differences are a whole number of hours.

For example, the local time in Sydney is 6 hours ahead of the local time in Islamabad.

This means that when it is 11:00 in Islamabad it is 17:00 in Sydney.

The local time in Dubai is 5 hours behind the local time in Tokyo.

This means that when it is 21:00 in Tokyo it is 16:00 in Dubai.

Example 14.9

Question

An aircraft leaves Dubai at 08 55 local time and flies to Tokyo.

The flight takes 9 hours 25 minutes.

Local time in Tokyo is 5 hours ahead of Dubai.

What is the local time in Tokyo when the aircraft lands?

Solution

$$\begin{aligned} 08\ 55 + 9 \text{ hours } 25 \text{ minutes} &= 17\ 55 + 25 \text{ minutes} \\ &= 18\ 20 \text{ (Dubai time)} \end{aligned}$$

$$18\ 20 + 5 \text{ hours} = 23\ 20 \text{ (Tokyo time)}$$

Exercise 14.3

- 1** Write these times in hours and minutes.

Where necessary give the time to the nearest minute.

- a** 5.4 hours
- b** 3.26 hours
- c** 2.84 hours
- d** 12.76 hours

- 2** Write these times in hours using decimals.

Where necessary give the time to 2 decimal places.

- a** 4 hours 15 minutes
- b** 1 hour 50 minutes
- c** 45 minutes
- d** 10 hours 39 minutes
- e** 3 hours 38 minutes

- 3** Write these times in minutes and seconds.

- a** 4.7 minutes
- b** 5.25 minutes
- c** 25.3 minutes
- d** 0.4 minutes

- 4** Assad leaves home at 13 48 and drives for 4.8 hours

What time does he arrive?

- 5** Soraya drives for 3 hours 17 minutes and travels 140 km.

What is her average speed?

- 6** A train travels at an average speed of 85 km/h.

The journey distance is 495 km.

The train leaves at 13 35.

What time does it arrive? Give your answer to the nearest minute.

- 7** An aircraft leaves Karachi at 11 35, local time, to fly to London.

The flight time is 8 hours 30 minutes.

The time in London is 4 hours behind the time in Karachi.

What is the local time in London when the aircraft arrives?

- 8** An aircraft leaves Dubai at 01 50 local time and arrives in Sydney at 22 30 local time.

The flight time is 13 hours 40 minutes.

How many hours ahead of Dubai time is Sydney time?

15 MONEY

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- solve problems involving money and convert from one currency to another.

CHECK YOU CAN:

- convert between metric units
- work with measures of rate.

Note

Whichever method you use, make sure that you make the right conclusion.

The better value is the one which costs less for each kilogram, gram, litre or millilitre or the one where you get more for each dollar or cent.

Value for money

When shopping it is easy to assume that the biggest size is the best value for money but it is not always the case.

Example 15.1

Question

A packet of Wheat-o-Flakes weighing 1.5 kg costs \$4.99.

A packet of Wheat-o-Flakes weighing 500 g costs \$1.70.

Which is better value for money?

Solution

Method 1

Find the price for each kilogram.

For the larger packet: $4.99 \div 1.5 = 3.326\ldots$ \$/kg

For the smaller packet: $1.70 \div 0.5 = 3.40$ \$/kg

The larger packet costs slightly less for each kilogram so is better value for money.

Method 2

Find how many grams you get for each cent.

For the larger packet: $1500 \div 499 = 3.006\ldots$ grams per cent

For the smaller packet: $500 \div 170 = 2.941\ldots$ grams per cent

You get slightly more for each cent with the larger packet so it is better value for money.

Although the methods shown in Example 15.1 will always work, sometimes there is an obvious easier method.

For example, in Example 15.1, the larger packet weighs three times as much as the smaller packet.

When you buy three of the smaller packets, you get $0.5 \text{ kg} \times 3 = 1.5 \text{ kg}$ for $\$1.70 \times 3 = \5.10 .

\$4.99 is less than \$5.10 so the larger packet is better value for money.

Example 15.2

Question

At Mega CD, if you buy two CDs at \$7.99 each, you get a third CD free.

At CD World, CDs cost \$5.50 each if you buy two or more.

Explain which is the better offer if you want to buy

a three CDs

b four CDs.



Solution

- a At Mega CD three CDs cost $2 \times \$7.99 = \15.98
 At CD World three CDs cost $3 \times \$5.50 = \16.50
 So Mega CD has the better offer.
- b At Mega CD you need to buy the fourth CD at \$7.99.
 Total cost = $\$15.98 + \$7.99 = \$23.97$
 At CD World four CDs cost $4 \times \$5.50 = \22.00
 So CD World has the better offer.

Exercise 15.1

- 1 Show which is the better value, 5 litres of water for \$1.29 or 2 litres for 87c.
- 2 Here are some prices for bottles of cola.
 3 litres for \$3.98 2 litres \$2.70 1 litre \$1.14 500 ml 65c
 Which size gives the best value?
- 3 A brand of shaving gel costs \$2.38 for the 75 ml bottle and \$5.78 for the 200 ml bottle. Show which is the better value.
- 4 Here are the prices of some packs of soda.
 12 × 150 ml cans for \$5.90
 6 × 330 ml cans for \$3.18
 12 × 330 ml cans for \$5.98
 Which pack gives the most soda for your money?
- 5 An 800 g loaf of bread costs 96c.
 A 1.2 kg loaf costs \$1.41.
 Show which is the better value for money.
- 6 A 420 g bag of chocolate bars costs \$3.18.
 A 325 g bag of chocolate bars costs \$2.18.
 Show which bag is the better value for money.
- 7 A 680 g block of cheese costs \$5.18.
 A 500 g block of cheese costs \$3.98.
 Show which block is the better value for money.
- 8 Li and Chan want to go out for a meal on Wednesday.
 Their two local restaurants have special offers.
 At restaurant A if you buy a meal for \$12.50, you get a second meal half price.
 At restaurant B meals cost \$9.50 on Monday to Thursday.
 Which restaurant will be cheaper for them?

Currency conversion

Currency exchange rates tell you what one of the units of one currency is equal to in the other currency.

For example, if the exchange rate between US dollars (\$) and euros (€) is

$\$1 = €0.88$, each US dollar is worth €0.88.

Exchange rates can change daily.

Example 15.3

Question

The exchange rate between dollars and euros is \$1 = €0.88.

- a Sam changes \$135 into euros.
How much will he receive in euros?

- b Françoise changes €340 into dollars.
How much will she receive in dollars?

Solution

- a You know what each dollar is worth in euros so you multiply.
 $\$135 \times 0.88 = €118.80$

- b You do not know how much in dollars each euro is worth.

You need to know how many 0.88s there are in \$340 so you divide.

$$€340 \div 0.88 = \$386.36 \text{ correct to the nearest cent.}$$

Note

The answer on the calculator was 118.8. When you write answers to money questions, you use two decimal places as the two decimal places give the number of the smaller unit, in this case, cents.

Note

When you are given an exchange rate, the only decision is whether to multiply or divide by that rate. You can check you have chosen correctly. If \$1 = €0.88 then the number of dollars will be greater than the number of euros.

Exercise 15.2

- 1 The exchange rate between US dollars (USD) and Pakistani rupees (PKR) is 1 USD = 101.42 PKR. Change
 - a \$240 to Pakistani rupees
 - b 3500 PKR to US dollars.
- 2 The exchange rate between euros (EUR) and United Arab Emirates dirhams (AED) is 1 EUR = 4.16 AED. Change
 - a €280 to dirhams
 - b 850 AED to euros.
- 3 Jackie goes on holiday from Australia to Japan.
 - a She changes 1250 Australian dollars (AUD) into Japanese yen (JPY).
The exchange rate is 1 AUD = 91.89 JPY.
How many Japanese yen does she receive?
 - b When she returns, she changes 15 000 Japanese yen back into Australian dollars.
The exchange rate is now 1 AUD = 93.15 JPY.
How many Australian dollars does she receive?
- 4 Brian travels from London to Mauritius. He changes £450 (GBP) into Mauritian rupees (MUR).
The exchange rate is 1 GBP = 50.98 MUR.
 - a Calculate how much he receives in Mauritian rupees.
Whilst there he buys presents worth 3500 MUR.
 - b Calculate how much these are worth in British pounds.

- 5** Khalid is buying a laptop computer. He can buy it in Pakistan for 54 500 rupees. He sees an advert on the internet for the same laptop. If he buys it on the internet, it will cost \$450 + \$80 for delivery. The exchange rate is 1 USD = 101.50 PKR.
- a Is it cheaper for Khalid to buy the laptop in Pakistan or on the internet?
- b What is the difference in the cost? Give your answer in
i Pakistani rupees ii dollars.
- 6** Here are two exchange rates.
 $\text{£}1 = \text{€}1.35$ $\text{£}1 = 2.08$ Singapore dollars
- a Assuming proportional exchange rates
- i How many Singapore dollars will you get for €1?
ii How many euros will you get for 1 Singapore dollar?
- b A camera costs 240 Singapore dollars.
Calculate its cost in euros.

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- use given data to solve problems on personal and small business finance involving earnings, simple interest and compound interest
- extract data from tables and charts.

CHECK YOU CAN:

- find a percentage of a quantity
- express one quantity as a percentage of another
- find percentage increases and decreases using a multiplier
- understand and use index notation.

Extracting data from tables and charts

Example 16.1

Question

Here are the prices to visit a museum.

	Midweek (\$)	Weekend (\$)
Adult	5.25	6.75
Child (under 16)	2.80	3.60
Family ticket (2 adults and 2 children)	13.50	16.50

Mr and Mrs Singh are taking their two children to the museum at the weekend.

How much will they save by buying a family ticket?

Solution

2 adults	$2 \times \$6.75 = \13.50
2 children	$2 \times \$3.60 = \7.20
Total	$\$13.50 + \$7.20 = \$20.70$
Saving	$\$20.70 - \$16.50 = \$4.20$

Exercise 16.1

- 1 The table shows the daily hire prices for a car rental company.

Car type	Number of seats	Price per day (\$)		
		Number of days		
		1 to 6	7 to 13	14+
A	4	50	44	42
B	5	60	52	50
C	4 to 5	68	58	54
D	5 to 6	80	70	66

The Lee family want to rent a type C car for a 17-day holiday.
How much will the hire charge be?

- 2** The table shows the prices for a holiday.
All prices are in dollars (\$) per person.

Departures on or between	7 nights	14 nights
April 18 to May 1	399	569
May 2 to May 15	429	599
May 16 to May 29	449	639
May 30 to June 16	479	679
Discount if you do not need travel insurance	30	40

Mr and Mrs Green want to book a 7-night holiday departing on 20th May.
They have their own travel insurance.

What is the total cost of the holiday?

- 3** The table shows an order form for trousers.

Trouser sizes	Size	Colour	Quantity	Price	Total
10, 12, 14, 16, 18				\$14.99	
20, 22, 24, 26				\$16.99	
Postage & packing					\$1.95
Total					

Jane and Rushna want to order one pair of red size 14 trousers and two pairs of blue size 20 trousers.

Copy and complete the order form.

- 4** Dilshan goes to the supermarket.

Copy and complete his bill.

Item	Cost	Total
2.4 kg potatoes	\$1.60 per kg	\$.....
..... oranges	26c each	\$1.56
1.4 kg onions	\$..... per kg	\$3.08
8 bananasc each	\$.....
	Total	\$10.96

Wages and salaries

A **wage** or **wages** is an amount you earn, usually according to how many hours or days you work each week or month. It may be paid weekly or monthly.

A **salary** is a fixed amount of money you earn each month or year from your job. It is paid monthly.

What you earn is called your **gross pay**.

Tax and other amounts may be deducted from your gross pay.

The amount actually received by the employee is often called the **take-home pay**.

Some employees, particularly sales staff, are paid a basic wage or salary plus a **commission**, which is usually a percentage of their sales.

Example 16.2

Question

- a Jenny's annual salary is \$8500.

She is paid monthly.

What is her monthly gross pay?

- b Rob's basic annual pay is \$9600.

He also earns 5% commission on the sales he makes each month.

In May his sales are \$7500. What is his gross pay for May?

Solution

$$\begin{aligned} \text{a} \quad \text{Jenny's monthly gross pay} &= \$8500 \div 12 = \$708.333\dots \\ &= \$708.33 \end{aligned}$$

$$\text{b} \quad \text{Basic monthly pay} = \$9600 \div 12 = \$800$$

Commission = $\$7500 \times 0.05 = \375 Use your preferred method to find a multiplier.

$$\text{Total pay for May} = \$800 + \$375 = \$1175$$

Discount, profit and loss

A **discount** is a reduction in the price of something.

This is often a percentage of the original price but can be a fixed amount.

Example 16.3

Question

A jacket is priced at \$38. The shopkeeper offers Rashid a discount of 15%.

How much does Rashid pay for the jacket?

Solution

$$\begin{aligned} \text{Rashid will pay } 100 - 15 &= 85\% \text{ of the original price} \\ &\quad \text{Use your preferred method to find a percentage of an amount.} \end{aligned}$$

$$\text{Rashid pays } \$38 \times 0.85 = \$32.30$$

Percentage profit and loss are always worked out as a percentage of the price paid by the **seller**.

Example 16.4**Question**

A shopkeeper buys potatoes at \$1.40 per kilogram.
He sells them for \$1.80 per kilogram.
Find his percentage profit.

Solution

$$\text{Profit} = \$1.80 - \$1.40 = \$0.40$$

$$\text{Percentage profit} = \frac{0.40}{1.40} \times 100 = 28.6\% \text{ to 1 decimal place}$$

Example 16.5**Question**

Françoise buys a car for \$9500.
A year later she sells it for \$8300.
Calculate her percentage loss.

Solution

$$\text{Loss} = \$9500 - \$8300 = \$1200$$

$$\text{Percentage loss} = \frac{1200}{9500} \times 100 = 12.6\% \text{ to 1 decimal place}$$

Exercise 16.2

- 1 a Pali earns an annual salary of \$10 980. How much is this a month?
b Steve's wage is \$195 per week. How much does he earn in a year?
- 2 In a special promotion, a health club offers 20% discount for membership.
Membership usually costs \$35.
How much does it cost with the discount?
- 3 Ali earns £12 000 per year plus commission of 7.5% on his sales.
Last year he sold \$76 000 of goods.
What was his average monthly pay last year?
- 4 A shopkeeper buys an article for \$22 and sells it for \$27.
What is his percentage profit?
- 5 A1 Electrics buys washing machines for \$255 and sells them for \$310.
Bob's Budget Bargains buys washing machines for \$270 and sells them for \$330.
Which company makes the greater percentage profit and by how much?
- 6 A trader buys an item for \$12.50 and sells it at a loss for \$5.
What is his percentage loss?
- 7 A shopkeeper buys 100 cups for \$1.20 each.
He sells 80 of them, making a profit of 15% on each.
He sells the remaining 20 at 95c each.
What is his overall percentage profit?

Repeated percentage change

In Chapter 12, you were shown a single-step method for increasing a quantity by a percentage.

For example, to increase a quantity by 18%, you multiply by 1.18.

Similarly, to increase a quantity by 7%, you multiply by 1.07.

This method is particularly useful when calculating repeated increases.

Example 16.6

Question

Due to inflation, prices increase by 5% per year.

An item costs \$12 now.

What will it cost in 2 years' time?

Solution

A 5% increase means that the new amount is $100 + 5 = 105\%$ of the old amount each year.

To find 105%, you multiply by 1.05.

In 1 year the price will be $\$12 \times 1.05 = \12.60 .

In 2 years the price will be $\$12.60 \times 1.05 = \13.23 .

Alternatively, this calculation can be worked out as

$$\$12 \times 1.05 \times 1.05 = \$13.23$$

$$\text{or } \$12 \times 1.05^2 = \$13.23$$

Example 16.7

Question

Selena invests \$4000 and receives 5% interest a year which is added to the amount each year.

How much is her investment worth in total after

- a 1 year b 6 years?

Solution

a After one year the investment is worth $\$4000 \times 1.05 = \4200 .

b After 6 years the investment is worth

$$\begin{aligned} & 4000 \times 1.05 \times 1.05 \times 1.05 \times 1.05 \times 1.05 \\ &= 4000 \times (1.05)^6 \\ &= \$5360.38 \text{ (to the nearest cent)} \end{aligned}$$

Note

On a calculator, the calculation for part b can be done using the power key.

This is usually labelled x but may be labelled y^x or x^y or x .

Example 16.8

Question

Each year a car decreases by 12% of its value at the beginning of the year.

At the beginning of the year, a car was valued at \$9000.

Find its value after 3 years.

Solution

$$100\% - 12\% = 88\%$$

$$\text{Value after three years} = \$9000 \times (0.88)^3 = \$6133.25 \text{ (to the nearest cent).}$$

Note

When the value of something decreases, as in this Example it is often called **depreciation**.

Compound and simple interest

When you invest money in a bank, you may receive interest.

There are two types of interest.

Simple interest is calculated as a percentage of the original amount.

If you invest \$4000 for 2 years in an account that pays 5% simple interest annually, you will receive

$\$4000 \times 0.05 = \200 at the end of the first year and the same again each year.

By contrast, **compound interest** is added to the original amount so that the amount of interest paid each year increases.

Example 16.7 is an example of compound interest.

At the end of the first year the interest is

5% of \$4000 = \$200.

This is added to the original amount. At the end of the second year the interest is calculated as a percentage of this new amount.

5% of \$4200 = \$210

So each year the interest received increases.

Compound interest is just one example of repeated percentage change and you can calculate the new amount in the same way as you calculate other repeated percentage changes.

However, because it has such wide application in everyday life, it has its own formula for working out the value of the investment after a given time.

$$A = P \left(1 + \frac{r}{100}\right)^n$$

In this formula

- A is the value of the investment after the given time
- P is the amount invested and is called the **principal**
- r is the percentage rate of interest (usually per year but it can be per month)
- n is the time (usually in years) for which the money is invested.

You can use either the method used in Example 16.7 or the formula above.

Example 16.9

Question

Salim has \$4000 to invest for 5 years.

The bank offers him two options.

- 3.5% simple interest or
- 3.25% compound interest

Which is the better option?

How much more will his investment be worth with the better option?

Solution

3.5% simple interest

$$\text{Interest for one year} = 0.035 \times 4000 = \$140$$

$$\text{Interest for 5 years} = 5 \times \$140 = \$700$$

$$\text{Value of investment at the end of 5 years} = \$4000 + \$700 = \$4700$$

3.25% compound interest

The multiplier for a 3.25% increase is 1.0325.

$$\text{Value of investment at the end of 5 years} = \$4000 \times 1.0325^5 = \$4693.65 \text{ (to the nearest cent).}$$

Alternatively, using the formula

$$\begin{aligned} A &= P \left(1 + \frac{r}{100}\right)^n \\ &= 4000 \times \left(1 + \frac{3.25}{100}\right)^5 \\ &= \$4693.65 \text{ (to the nearest cent)} \end{aligned}$$

So 3.5% simple interest is the better option.

$$\text{Difference} = \$4700 - \$4693.65 = \$6.35 \text{ (to the nearest cent)}$$

Exercise 16.3

- 1 Craig puts \$240 into a savings account that earns compound interest of 6% annually.
What will his savings be worth after 3 years?
Give your answer to the nearest cent.
- 2 Calculate the amount each of these items will be worth if they reduce in value each year by the given percentage for the given number of years.

	Original value	% reduction	Number of years
a	\$250	45	5
b	\$3500	11	7
c	£1400	15	4
d	\$10 500	12	10

- 3** Each year a car depreciates by 11% of its value at the start of the year.
 At the beginning of the year, it is worth \$8000.
 What will it be worth after 2 years?
- 4** \$1500 is invested in an account that earns compound interest of 4% annually.
 What is the value of the investment after four years?
- 5** Calculate the final value of the investment when \$3000 is invested with compound interest in these cases. Give your answers to the nearest cent.
a 5% for 4 years **b** 6% for 20 years **c** 3.5% for 10 years
- 6** Martin had shares worth \$8000.
 They increased in value by 7.5% each year.
 What was their value after 10 years?
 Give your answer to the nearest dollar.
- 7** A population of bacteria is estimated to increase by 12% every 24 hours.
 The population was 2000 at midnight on Friday.
 What was the population (to the nearest whole number) by midnight the following Wednesday?
- 8** Tony says his boat is increasing in value by 6% a year.
 It is valued at \$25 000.
 How much would it be worth, to the nearest hundred dollars, after six years if he is correct?
- 9** Mr Costa was given an 8% pay rise every year.
 His starting salary was \$28 500.
 How much does he earn after four rises? (Give your answer to the nearest dollar.)
- 10** Pulova has high inflation. In 2014, prices increased by 15% a month for the first six months and 12.5% for the next six months.
 A car cost 78 000 dubs (their unit of currency) in January 2014.
 How much did it cost
a after six months?
b in January 2015?
 Give your answers to the nearest whole number.
- 11** Andrew invested \$3500 in a six-year bond that paid 5% compound interest for the first three years and 7.5% compound interest for the next three years.
 What is the amount in the bond, to the nearest cent
a after three years? **b** after six years?
- 12** Find the difference in interest earned by investing \$500 for three years at 12% simple interest or for three years at 10% compound interest.
- 13** Is it better to invest \$1000 for five years at 8% compound interest or for four years at 9% compound interest?
- 14** A painting was worth \$15 000 in 2009.
 The painting increased in value by 15% every year for 6 years.
 How much was it worth at the end of the 6 years?
 Give your answer to the nearest dollar.
- 15** Yassim invested \$500 at 4% compound interest.
 How many years will it take for the value of the investment to be over \$600?

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- use letters to express generalised numbers and express basic arithmetic processes algebraically
- substitute numbers for words and letters in formulae
- construct and transform formulae and equations.

CHECK THAT YOU CAN:

- add, subtract, multiply and divide whole numbers and decimals
- add, subtract, multiply and divide negative numbers
- add, subtract, multiply and divide fractions
- use the correct order of operations in arithmetic.

For transforming formulae, you will also need to be able to:

- solve linear equations
- factorise expressions such as $ax + bx$.

Letters for unknowns

Imagine you had a job where you were paid by the hour. You would receive the same amount for each hour you worked. How could you work out how much you will earn in a week?

You would need to work it out as the number of hours you worked multiplied by the amount you are paid for each hour.

This is a formula in words.

If you work 35 hours at \$4.50 an hour, it is easy to work out $35 \times \$4.50$, but what if the numbers change?

The calculation ' $35 \times \$4.50$ ' is only correct if you work 35 hours at \$4.50 an hour. Suppose you move to another job where you are paid more for each hour.

You need a simple formula that always works. You can use symbols to stand for the numbers that can change.

You could use ? or \square , but it is less confusing to use letters.

Let the number of hours worked be N

the amount you are paid each hour be P

the amount you earn in a week be W .

Then $W = N \times P$.

Example 17.1

Question

Use the formula $W = N \times P$ to find the amount earned in a week when $N = 40$ and $P = 5$.

Solution

$$\begin{aligned} W &= N \times P \\ &= 40 \times 5 \\ &= 200 \end{aligned}$$

The amount earned in a week is \$200.

Some rules of algebra

You do not need to write the \times sign.

$4 \times t$ is written $4t$.

In multiplications, the number is always written in front of the letter.

$p \times 6 - 30$ is written as $6p - 30$.

You often start a formula with the single letter you are finding.

$2 \times l + 2 \times w = P$ is written as $P = 2l + 2w$.

When there is a division in a formula, it is usually written as a fraction.

$y = k \div 6$ can be written as $y = \frac{k}{6}$.

Substituting numbers into algebraic expressions

Numbers that can be substituted into algebraic expressions can be positive, negative, decimals or fractions.

Example 17.2

Question

- a Find the value of $4x + 3$ when $x = 2$.
- b Find the value of $3x^2 + 4$ when $x = 3$.
- c Find the value of $2x^2 + 6$ when $x = -2$.

Solution

- | | |
|--|--|
| a $4x + 3 = 4 \times 2 + 3$
$= 8 + 3$
$= 11$ | Work out each term separately and then collect together. |
| b $3x^2 + 4 = 3 \times 3^2 + 4$
$= 3 \times 9 + 4$
$= 27 + 4$
$= 31$ | Remember $3x^2$ means $3 \times x \times x$ not $3 \times x \times 3 \times x$. |
| c $2x^2 + 6 = 2 \times (-2)^2 + 6$
$= 2 \times 4 + 6$
$= 8 + 6$
$= 14$ | Take special care when negative numbers are involved. |

Exercise 17.1

- 1 Find the value of these expressions when $a = 5$, $b = 4$ and $c = 2$.

a $a + b$	b $b + c$	c $a - c$
d $a + b + c$	e $2a$	f $3b$
g $5c$	h $3a + b$	i $3c - b$
j $a + 6c$	k $4a + 2b$	l $2b + 3c$
m $a - 2c$	n $8c - 2b$	o bc
p $4ac$	q abc	r $ac + bc$
s $ab - bc - ca$	t a^2	u $\frac{a+b}{3}$
v $\frac{ab}{c}$	w $b^2 + c^2$	x $3c^2$
y a^2b	z c^3	

- 2** Find the value of these expressions when $t = 3$.
- a** $t + 2$ **b** $t - 4$ **c** $5t$ **d** $4t - 7$ **e** $2 + 3t$
f $10 - 2t$ **g** t^2 **h** $10t^2$ **i** $t^2 + 2t$ **j** t^3
- 3** Use the formula $A = 5k + 4$ to find A for these values of k .
- a** $k = 3$ **b** $k = 7$ **c** $k = 0$ **d** $k = \frac{1}{2}$ **e** $k = 2.1$
- 4** Use the formula $y = mx + c$ to find y in these.
- a** $m = 3, x = 2, c = 4$
b $m = \frac{1}{2}, x = 8, c = 6$
c $m = 10, x = 15, c = 82$
- 5** Use the formula $D = \frac{m}{v}$ to find D in these.
- a** $m = 24, v = 6$ **b** $m = 150, v = 25$
c $m = 17, v = 2$ **d** $m = 4, v = \frac{1}{2}$

Using harder numbers when substituting

You need to be able to work with positive and negative integers, decimals or fractions, and to do so with or without a calculator.

Example 17.3

Question

Find the value of

- a** $a + 5b^3$ when $a = 2.6$ and $b = 3.2$.
b $5cd$ when $c = 2.4$ and $d = 3.2$.



Solution

$$\begin{aligned} \text{a } a + 5b^3 &= 2.6 + 5 \times 3.2^3 \\ &= 2.6 + 5 \times 32.768 \\ &= 2.6 + 163.84 \\ &= 166.44 \end{aligned}$$

$$\begin{aligned} \text{b } 5cd &= 5 \times 2.4 \times 3.2 \\ &= 12 \times 3.2 \\ &= 10 \times 3.2 + 2 \times 3.2 \\ &= 32 + 6.4 \\ &= 38.4 \end{aligned}$$

Note

The working is shown here to remind you of the order of operations, but you should be able to do this calculation in one step on your calculator.

Note

You could work out 2.4×3.2 first, but this would be harder – use your number facts to help you see the best way to do non-calculator calculations.

Example 17.4

Question

$P = ab + 4b^2$. Find P when $a = \frac{4}{5}$ and $b = \frac{3}{8}$, giving your answer as a fraction.

Solution

$$P = \left(\frac{4}{5} \times \frac{3}{8}\right) + \left(4 \times \frac{3}{8} \times \frac{3}{8}\right) = \frac{3}{10} + \frac{9}{16} = \frac{24}{80} + \frac{45}{80} = \frac{69}{80}$$

Exercise 17.2

 1 Work out these.

- $V = ab - ac$ when $a = 3$, $b = -2$ and $c = 5$
- $P = 2rv + 3r^2$ when $r = 5$ and $v = -2$
- $T = 5s^2 - 2t^2$ when $s = -2$ and $t = 3$
- $M = 2a(3b + 4c)$ when $a = 5$, $b = 3$ and $c = -2$
- $R = \frac{2qv}{q+v}$ when $q = 3$ and $v = -4$
- $L = 2n + m$ when $n = \frac{2}{3}$ and $m = \frac{5}{6}$
- $D = a^2 - 2b^2$ when $a = \frac{4}{5}$ and $b = \frac{2}{5}$
- $A = a^2 + b^2$ when $a = 5$ and $b = -3$
- $P = 2c^2 - 3cd$ when $c = 2$ and $d = -5$
- $B = p^2 - 3q^2$ when $p = -4$ and $q = -2$

- 2 Find the value of $M = \frac{ab}{(2a + b^2)}$ when $a = 2.75$ and $b = 3.12$.

Give your answer correct to 2 decimal places.

- 3 The distance S metres fallen by a pebble is given by the formula $S = \frac{1}{2}gt^2$, where t is in seconds.

Find S in each of these cases.

- $g = 10$ and $t = 12$
 - $g = 9.8$ and $t = 2.5$
- 4 The surface area ($A \text{ cm}^2$) of a cuboid with sides x , y and $z \text{ cm}$ is given by the formula $A = 2xy + 2yz + 2xz$.
Find A when $x = 5$, $y = 4.5$ and $z = 3.5$.
- 5 The elastic energy of an elastic string is given by the formula $E = \frac{\lambda x^2}{2a}$.
Find E when $\lambda = 3.4$, $x = 5.7$ and $a = 2.5$.
Give your answer correct to 1 decimal place.
- 6 The focal length of a lens is given by the formula $f = \frac{uv}{u+v}$.
Find the focal length when $u = 6$ and $v = -7$.

Writing formulae

Example 17.5

Question

Write a formula for the mean height, M_{cm} , of a group of n people whose heights total $h \text{ cm}$.

Solution

To find the mean height, you divide the total of the heights by the number of people, so the formula for M is

$$M = h \div n \text{ or } M = \frac{h}{n}$$

Note

If you are not sure whether to multiply or divide, try an example with numbers first.

Example 17.6**Question**

The cost ($\$C$) of hiring a car is a fixed charge ($\$f$), plus the number of days (n) multiplied by the daily rate ($\$d$).

Write a formula for C .

Solution

$$C = f + n \times d \text{ or } C = f + nd$$

Exercise 17.3

Write a formula for each of these, using the letters given.

- 1** The cost (C cents) of x pencils at y cents each.
- 2** The area ($A \text{ cm}^2$) of a rectangle $m \text{ cm}$ long and $n \text{ cm}$ wide.
- 3** The height ($h \text{ cm}$) of a stack of n tins each $t \text{ cm}$ high.
- 4** The temperature in $^{\circ}\text{F}$ (F) is 32, plus 1.8 times the temperature in $^{\circ}\text{C}$ (C).
- 5** The area ($A \text{ cm}^2$) of a triangle is half the base ($b \text{ cm}$) times the height ($h \text{ cm}$).
- 6** The current in a circuit (I) is the voltage (V) divided by the resistance (R).
- 7** The cost of fuel ($\$C$) is the number of litres (n) multiplied by the price of fuel per litre ($\$p$).
- 8** The total of the wages ($\$w$) in a supermarket is the number of workers (n) multiplied by the weekly wage ($\$q$).
- 9** The number of books (N) that can fit on a shelf is the length ($L \text{ cm}$) of the shelf divided by the thickness ($t \text{ cm}$) of each book.
- 10** The number of posts (Q) for a fence is the length of the fence ($R \text{ metres}$) divided by 2, plus 1.

Before you do Exercise 17.4, check your answers to Exercise 17.3.

Exercise 17.4

Use the formulae from Exercise 17.3 to find each of these.

- 1** C when $x = 15$ and $y = 12$
- 2** A when $m = 7$ and $n = 6$
- 3** h when $n = 20$ and $t = 17$
- 4** F when $C = 40$
- 5** A when $b = 5$ and $h = 6$
- 6** I when $V = 13.6$ and $R = 2.5$
- 7** C when $n = 50$ and $p = 70$
- 8** w when $n = 200$ and $q = 150$
- 9** N when $L = 90$ and $t = 3$
- 10** Q when $R = 36$

Writing equations

Some everyday problems can be solved by writing equations and solving them.

Example 17.7

Question

The length of a rectangle is a cm.

The width is 15 cm shorter.

The length is three times the width.

Write down an equation in a and solve it to find the length and width of the rectangle.

Solution

If the length = a , the width = $a - 15$.

Also the length = $3 \times$ width = $3(a - 15)$.

The equation is $a = 3(a - 15)$.

$$a = 3(a - 15)$$

$a = 3a - 45$ Multiply out the bracket.

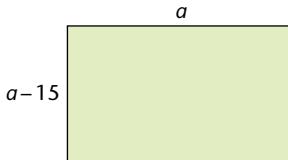
$a + 45 = 3a$ Add 45 to each side.

$45 = 2a$ Subtract a from each side.

$22.5 = a$ Divide each side by 2.

$a = 22.5$ Rewrite with the subject on the left-hand side.

So the length = 22.5 cm and the width = 7.5 cm.



Exercise 17.5

- 1 Eira is x years old and Jaan is three years older than Eira.
Their ages add up to 23.
Write an equation in x and solve it to find their ages.
- 2 Two angles of a triangle are the same and the other is 15° bigger.
Call the two equal angles a .
Write an equation and solve it to find all the angles.
- 3 Chan and Ali go shopping.
Chan spends \$ x and Ali spends twice as much.
They spend \$45 altogether.
Write an equation and solve it to find how much each spends.
- 4 A pentagon has two angles of 150° , two of x° and one of $(x + 30)^\circ$.
The sum of the angles in a pentagon is 540° .
 - a Write down an equation in x and solve it.
 - b State the size of each of the angles.
- 5 On a school trip to China, there are 15 more girls than boys.
Altogether 53 students go on the trip.
 - a If the number of boys is x , write an equation in x and solve it.
 - b How many boys and how many girls go on the trip?

- 6** A company employs 140 people, of whom x are men.
There are ten fewer women than men.
Use algebra to find how many men and how many women work for the company.
- 7** Two angles in a triangle are x° and $(2x - 30)^\circ$.
The first angle is twice the size of the second.
Write an equation and solve it to find the size of the two angles.
- 8** The width of a rectangle is 3 cm and the length is $x + 4$ cm.
The area is 27 cm².
Write an equation and solve it to find x .
- 9** Sabir thinks of a number.
If he doubles the number and then subtracts 5, he gets the same answer as if he subtracts 2 from the number and then multiplies by 3.
Let the number be n .
Write an equation and solve it to find n .
- 10** On a bus trip, each child pays \$ c and each adult pays \$12 more than this.
There are 28 children and 4 adults on the bus.
The same amount of money is collected from all the children as from all the adults.
Write an equation and solve it to find how much each child and each adult pays.

Transforming formulae

When solving equations, you simplify and collect terms if necessary, then use inverse operations to get the unknown on its own.

Formulae can be treated in the same way as equations. This means they can be rearranged to change the subject. Use the same steps as you would if the formula was an equation.

Example 17.8

Question

Rearrange each of these formulae to make the letter in the bracket the subject.

a $a = b + c$ (b) **b** $a = bx + c$ (b) **c** $n = m - 3s$ (s) **d** $p = \frac{q+r}{s}$ (r)

Solution

a $a = b + c$

$a - c = b$ Subtract c from both sides.

$b = a - c$

Reverse to get b on the left.

b $a = bx + c$

$a - c = bx$ Subtract c from both sides.

$\frac{a-c}{x} = b$

Divide both sides by x .

$b = \frac{a-c}{x}$

Reverse to get b on the left.



c $n = m - 3s$

$n + 3s = m$ Add $3s$ to both sides.

$3s = m - n$ Subtract n from both sides.

$s = \frac{m - n}{3}$ Divide both sides by 3.

d $p = \frac{q+r}{s}$

$sp = q + r$ Multiply both sides by s .

$sp - q = r$ Subtract q from both sides.

$r = sp - q$ Reverse to get r on the left.

Note

If you are not sure whether to multiply or divide, try an example with numbers first.

Example 17.9

Question

The formula for the total cost, T , of entry to the cinema for three adults and three children is given by the formula $T = 3(a + c)$, where a is the price of an adult ticket and c is the price of a child ticket.

- Make c the subject of the formula.
- Find the cost for a child when the cost for an adult is \$5 and the total cost is \$24.

Solution

a $T = 3(a + c)$

$T = 3a + 3c$ Multiply out the bracket.

$T - 3a = 3c$ Subtract $3a$ from both sides.

$\frac{T - 3a}{3} = c$ Divide both sides by 3.

$c = \frac{T - 3a}{3}$ Reverse to get c on the left.

b $c = \frac{24 - 3 \times 5}{3}$

$= \frac{24 - 15}{3}$

$= \frac{9}{3} = 3$

The cost for a child is \$3.

Exercise 17.6

- 1 Rearrange each formula to make the letter in the bracket the subject.

a $a = b - c$ (b)

b $3a = wx + y$ (x)

c $v = u + at$ (t)

d $A = \frac{T}{H}$ (T)

e $C = P - 3T$ (T)

f $P = \frac{u + v}{2}$ (u)

g $C = 2\pi r$ (r)

h $A = p(q + r)$ (q)

i $p = q + 2r$ (q)

j $B = s + 5r$ (r)

k $s = 2u - t$ (t)

l $m = \frac{pqr}{s}$ (q)

m $L = 2G - 2F$ (G)

n $F = \frac{m + 4n}{t}$ (n)

o $T = \frac{S}{2a}$ (S)

p $A = t(x - 2y)$ (y)

- 2 The formula for finding the perimeter P of a rectangle of length l and width w is $P = 2(l + w)$.

- a Rearrange the formula to make l the subject.

- b Find the length of a rectangle of width 8 metres and perimeter 44 metres.

- 3** The cost ($\$C$) of catering for a conference is given by the formula $C = A + 32n$, where A is the cost of the room and n is the number of delegates.
- Rearrange the formula to make n the subject.
 - Work out the number of delegates when A is \$120 and the total cost C is \$1912.
- 4** The cooking time, T minutes, for w kg of meat is given by $T = 45w + 40$.
- Make w the subject of this formula.
 - What is the value of w when the cooking time is 2 hours and 28 minutes?
- 5** The curved surface area (Scm^2) of a cylinder of radius r cm and height h cm is given by $S = 2\pi rh$.
- Rearrange this formula to make r the subject.
 - Find the radius of a cylinder of height 4 cm which has a curved surface area of 60.3 cm^2 .
- 6** The formula for the volume V of a cone of height h and base radius r is $V = \frac{1}{3}\pi r^2 h$.
- Rearrange the formula to make h the subject.
 - Find the height of a cone with base radius 9 cm and volume 2290 cm^3 .
- 7** The cost ($\$C$) of booking a bus for n people is $C = 40 + 5n$.
- Make n the subject of this formula.
 - Find the number of people when the cost is \$235.
- 8** The total surface area (Scm^2) of a cylinder of radius r cm and height h cm is given by $S = 2\pi rh + 2\pi r^2$.
- Rearrange this formula to make h the subject.
 - Find the height of a cylinder of radius 6 cm which has a total surface area of 500 cm^2 .

Transforming harder formulae

All the formulae that you have rearranged up to this point have contained the new subject only once, and also the subject has not been raised to a power. However, this is now extended in the following examples.

Example 17.10

Question

Rearrange the formula $V = \frac{4}{3}\pi r^3$ to make r the subject.

Solution

$$V = \frac{4}{3}\pi r^3$$

$3V = 4\pi r^3$ Multiply both sides by 3.

$4\pi r^3 = 3V$ Rearrange to get all terms involving r on the left.

$$r^3 = \frac{3V}{4\pi}$$

Divide both sides by 4π .

$$r = \sqrt[3]{\frac{3V}{4\pi}}$$

Take the cube root of both sides.

Example 17.11**Question**

Rearrange the formula $a = x + \frac{cx}{d}$ to make x the subject.

Solution

$$a = x + \frac{cx}{d}$$

$ad = dx + cx$ Multiply both sides by d .

$dx + cx = ad$ Rearrange to get all terms involving x (the subject) on the left.

$x(d + c) = ad$ Factorise the left-hand side, taking out the factor x .

$$x = \frac{ad}{d + c}$$

Divide both sides by the bracket $(d + c)$.

Example 17.12**Question**

Rearrange the equation $ax + by = cy - ad$ to make a the subject.

Solution

$$ax + by = cy - ad$$

$ax + ad = cy - by$ Rearrange to get all terms involving a on the left and all the other terms on the right. This is done by adding ad to both sides and subtracting by from both sides.

$a(x + d) = cy - by$ Factorise the left-hand side, taking out the factor a .

$$a = \frac{cy - by}{x + d}$$

Divide both sides by the bracket $(x + d)$.

Example 17.13**Question**

Rearrange the formula $a = \frac{1}{p} + \frac{1}{q}$ to make p the subject.

Solution

$$a = \frac{1}{p} + \frac{1}{q}$$

$apq = q + p$ Multiply through by pq .

$apq - p = q$ Collect all the terms in p on the left.

$p(aq - 1) = q$ Factorise.

$$p = \frac{q}{aq - 1}$$

Divide both sides by the bracket $(aq - 1)$.

Exercise 17.7

For questions 1–25 rearrange the formula to make the letter in the bracket the subject.

- | | | | |
|--|-----|---|-----|
| 1 $s = at + 2bt$ | (t) | 2 $s = ab - bc$ | (b) |
| 3 $P = t - \frac{at}{b}$ | (t) | 4 $v^2 = u^2 + 2as$ | (u) |
| 5 $A = 4\pi r^2$ | (r) | 6 $3(a + y) = by + 7$ | (y) |
| 7 $ab - cd = ac$ | (a) | 8 $2(a - 1) = b(1 - 2a)$ | (a) |
| 9 $s = 2r^2 - 1$ | (r) | 10 $s - 2ax = b(x - s)$ | (x) |
| 11 $a(b + d) = c(b - d)$ | (d) | 12 $a = \frac{t}{b} - st$ | (t) |
| 13 $V = 5ab^2 + 3c^3$ | (c) | 14 $A = P + \frac{PRT}{100}$ | (P) |
| 15 $s = \frac{uv}{u + v}$ | (v) | 16 $s = \frac{1}{a} + b$ | (a) |
| 17 $a = \frac{1}{b + c}$ | (c) | 18 $a = b + \frac{c}{d + 1}$ | (d) |
| 19 $m = \frac{100(a - b)}{b}$ | (b) | 20 $\frac{a}{p} = \frac{1}{1 + p}$ | (p) |
| 21 $\frac{a}{x + 1} = \frac{b}{2x + 1}$ | (x) | 22 $T = 2\pi \sqrt{\frac{L}{g}}$ | (L) |
| 23 $y = 3x^2 - 4$ | (x) | 24 $V = \frac{1}{3}\pi r^2 h$ | (r) |

- 25** The formula for finding the length, d , of the diagonal of a cuboid whose dimensions are x , y and z is $d = \sqrt{x^2 + y^2 + z^2}$.

- a** Find d when $x = 2$, $y = 3$ and $z = 4$.
- b** How long is the diagonal of a cuboid block of concrete with dimensions 2 m, 3 m and 75 cm?
- c** Rearrange the formula to make x the subject.
- d** Find x when $d = 0.86$ m, $y = 0.25$ m, and $z = 0.41$ m.

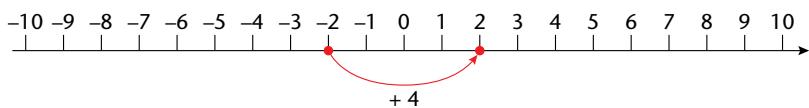
18 ALGEBRAIC MANIPULATION

Adding and subtracting with negative numbers

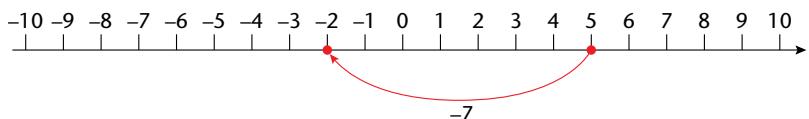
A negative number is a number less than zero.

A number line is very useful when adding or subtracting with negative numbers.

This number line shows $-2 + 4 = 2$



This number line shows $5 - 7 = -2$



Patterns can also help.

$$-2 + 4 = 2 \quad -2 + 3 = 1 \quad -2 + 2 = 0 \quad -2 + 1 = -1$$

$$-2 + 0 = -2 \quad -2 - 1 = -3 \quad -2 - 2 = -4$$

$$1 - 4 = -3 \quad 1 - 3 = -2 \quad 1 - 2 = -1 \quad 1 - 1 = 0$$

$$1 - 0 = 1 \quad 1 - -1 = 2 \quad 1 - -2 = 3 \quad 1 - -3 = 4$$

This helps to shows that

- adding a negative number is the same as subtracting a positive number
- subtracting a negative number is the same as adding a positive number.

Example 18.1

Question

Work out these.

a $-2 + -6$ b $-2 - -5$

Solution

a $-2 + -6 = -8$ Adding a negative number is like subtracting a positive number.

So $-2 + -6$ is the same as $-2 - 6$.

b $-2 - -5 = 3$ Subtracting a negative number is like adding a positive number.

So $-2 - -5$ is the same as $-2 + 5$.

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- manipulate directed numbers
- use brackets and extract common factors
- expand products of algebraic expressions
- factorise where possible expressions of the form:
 $ax + bx + kay + kby$
 $a^2x^2 - b^2y^2$
 $a^2 + 2ab + b^2$
 $ax^2 + bx + c$
- manipulate algebraic fractions
- factorise and simplify rational expressions.

CHECK YOU CAN:

- add, subtract, multiply and divide natural numbers
- order integers on a number line
- understand that in algebra letters stand for numbers and obey the rules of arithmetic
- express basic arithmetic processes algebraically
- understand and use index notation.

 **Exercise 18.1**

Work out these.

1 $4 + -5$

4 $6 - -3$

7 $4 + 3 - -2 - 5$

10 $2 + -3 - -4 - 7$

2 $6 + -2$

5 $2 + -1$

8 $9 + -2 - 3 - -2$

3 $-5 - -6$

6 $-1 - -4$

9 $-4 - -2 - 2 - -5$

When adding and subtracting many numbers, it is best to add up the positive numbers and the negative numbers separately, then find the difference between the totals and give the answer the sign of the larger total.

Multiplying and dividing with negative numbers

Look at these patterns.

$$5 \times 4 = 20$$

$$5 \times 3 = 15$$

$$5 \times 2 = 10$$

$$5 \times 1 = 5$$

$$5 \times 0 = 0$$

$$5 \times -1 = -5$$

$$5 \times -2 = -10$$

$$5 \times -3 = -15$$

$$5 \times -4 = -20$$

$$-3 \times 4 = -12$$

$$-3 \times 3 = -9$$

$$-3 \times 2 = -6$$

$$-3 \times 1 = -3$$

$$-3 \times 0 = 0$$

$$-3 \times -1 = 3$$

$$-3 \times -2 = 6$$

$$-3 \times -3 = 9$$

$$-3 \times -4 = 12$$

This suggest these rules.

$$+ \times + = +$$

and

$$- \times - = +$$

$$+ \times - = -$$

and

$$- \times + = -$$

Example 18.2

Questions

Work out these.

a 6×-4

b -7×-3

c -5×8

Solution

a $6 \times 4 = 24$ $(+ \times - = -)$

So $6 \times -4 = -24$

b $7 \times 3 = 21$ $(- \times - = +)$

So $-7 \times -3 = +21 = 21$

c $5 \times 8 = 40$ $(- \times + = -)$

So $-5 \times 8 = -40$

From the above page, you can see that $-3 \times 4 = -12$.

It follows that $-12 \div 4 = -3$ and $-12 \div -3 = 4$.

This suggests these rules.

$+ \div + = +$	and	$- \div - = +$
$+ \div - = -$	and	$- \div + = -$

You now have a complete set of rules for multiplying and dividing positive and negative numbers.

$+ \times + = +$	$- \times - = +$	$+ \div + = +$	$- \div - = +$
$+ \times - = -$	$- \times + = -$	$+ \div - = -$	$- \div + = -$

Here is another way of thinking of these rules.

Signs different – answer is negative.

Signs the same – answer is positive.

Note

These only apply to multiplying and dividing, not to adding and subtracting.

You can extend the rules to calculations with more than two numbers.

If there is an even number of negative signs, the answer is positive.

If there is an odd number of negative signs, the answer is negative.

Combining operations

As with positive numbers, multiplying and dividing is done before addition and subtraction unless there are brackets.

Example 18.3

Question

Work out these.

a $(-3 \times -4) + (-2 \times 3)$

b $6 - 2 \times 3$

c $\frac{5 \times -4 + 3 \times -2}{-6 + 4}$

Solution

a $(-3 \times -4) + (-2 \times 3) = +12 + -6$ Brackets are not really needed here, as multiplication should be done first.

$$\begin{aligned} &= 12 - 6 \\ &= +6 = 6 \end{aligned}$$

b $6 - 2 \times 3 = 6 - 6 = 0$ Do the multiplication first.

c $\frac{5 \times -4 + 3 \times -2}{-6 + 4} = \frac{-20 + -6}{-2}$ Work out the numerator and the denominator separately.

$$\begin{aligned} &= \frac{-26}{-2} \\ &= 13 \end{aligned}$$

Exercise 18.2

Work out these.

1 $(-4 \times -3) - (-2 \times 1)$

2 $(-7 \times -2) + (4 \times -2)$

3 $(-15 \div 2) - (4 \times -6)$

4 $-4 + 3 + 2 + 3 + 4 - 5 - 6 - 9 + 1$

5 $\frac{-2 + 12}{-5}$

6 $\frac{-4 \times -3}{-4 + 3}$

7 $\frac{-4 \times 5}{-6 + 4}$

8 $(16 \div 2) - (-2 \times 4)$

9 $-2 \times 3 + -3 \times 4$

10 $-1 \times -4 + -7 \times -8$

11 $(24 \div -3) - (-5 \times -4)$

12 $-6 - 2 - 3 + 5 - 7 + 4 - 2 + 8$

13 $-12 \div -4 - 24 \div -6$

14 $6 + (5 \times -2 \times -5)$

15 $\frac{-3 \times 7}{-2}$

16 $\frac{-7 \times -12}{-8 + 4}$

17 $\frac{8 \times -6}{-4 + -8}$

18 $\frac{9 \times 4}{-3 \times -6}$

19 $4 \times 2 - 5 \times -4$

20 $4 + 5 \times -2 - 6$

Simplifying algebraic expressions

These methods can also be used in algebra to **simplify** an expression.

Collect all the positive terms and all the negative terms separately.

Then add up each set and find the difference.

The sign of the answer is the sign of the larger total.

Example 18.4**Question**Simplify $5b - 4b - 8b + 2b$.**Solution**

$5b + 2b - 4b - 8b$

Collect all the positive terms and all the negative terms separately.

$= 7b - 12b$

Add up each set and find the difference.

$= -5b$

You can only add or subtract **like terms**.

Like terms use only the same letter or the same combination of letters.

Example 18.5**Question**

Simplify $3a + 2b - 2a + 5b$.

Solution

$$3a + 2b - 2a + 5b$$

$$= 3a - 2a + 2b + 5b \quad \text{Reorder with the like terms together.}$$

$$= a + 7b$$

Algebraic terms can be multiplied together.

Remember that in algebra, you do not need to write the multiplication signs.

However, writing them in your working can help.

Example 18.6**Question**

a Simplify $2a \times 3b$.

b Simplify $p \times q \times p^2 \times q$.

Solution

a $2a \times 3b$

$$= 2 \times a \times 3 \times b$$

$$= 2 \times 3 \times a \times b$$

Reorder with the numbers together.

$$= 6ab$$

b $p \times q \times p^2 \times q$

$$= p \times p^2 \times q \times q$$

Reorder with the like terms together.

$$= p^3 \times q^2$$

$$p \times p^2 = p \times p \times p = p^3.$$

You say this as 'p cubed q squared'.

Note

$1a$ or $1 \times a$ is always written as just a .

Some people confuse $2a$ and a^2 .

$2a = 2 \times a = a + a$, while $a^2 = a \times a$.

Example 18.7**Question**

A rectangle has length $3a$ and width $2b$.

Find and simplify an expression for the perimeter of the rectangle.

Solution

$$\text{Perimeter} = 3a + 2b + 3a + 2b$$

$$= 6a + 4b$$

Exercise 18.3

Simplify the expressions in questions 1 to 16.

1 $x + x + x + x + x$

3 $x + x + y + y$

5 $5 \times x$

7 $b + 2b + 3b$

9 $s + 2s + s$

11 $a + 2b + 2a + b$

13 $5x - 3x + 2y - y$

15 $3p \times 4q$

17 A square has sides $2a$ long.

Find and simplify an expression for its perimeter.

18 A quadrilateral has sides of $2a$, $3b$, $4a$ and $6b$.

Find and simplify an expression for its perimeter.

19 A pentagon has two sides x cm long and two sides $2x$ cm long.

The perimeter of the pentagon is $9x$.

What is the length of the fifth side?

20 Parvez walked c miles on Friday and d miles on Saturday.

Liz walked twice as far as Parvez on Friday and three times as far on Saturday.

Find and simplify an expression for the total distance they walked.

Note

Errors are often made by trying to go too far.

For example, $2a + 3b$ cannot be simplified any further.

Another common error is to work out $4a^2 - a^2$ as 4. The answer is $3a^2$.

Simplifying more complex algebraic expressions

Terms such as ab^2 , a^2b and a^3 cannot be collected together unless they are exactly the same type.

Example 18.8

Question

Simplify each of these expressions by collecting together the like terms.

a $2x^2 - 3xy + 2yx + 3y^2$

b $3a^2 + 4ab - 2a^2 - 3b^2 - 2ab$

c $3 + 5a - 2b + 2 + 8a - 7b$

Solution

a $2x^2 - 3xy + 2yx + 3y^2$
 $= 2x^2 - xy + 3y^2$

The middle two terms are like terms because xy is the same as yx .

b $3a^2 + 4ab - 2a^2 - 3b^2 - 2ab$
 $= 3a^2 - 2a^2 + 4ab - 2ab - 3b^2$
 $= a^2 + 2ab - 3b^2$

Here the a^2 terms and the ab terms can be collected but they cannot be combined with each other or with the single b^2 term.

c $3 + 5a - 2b + 2 + 8a - 7b$
 $= 3 + 2 + 5a + 8a - 2b - 7b$
 $= 5 + 13a - 9b$

Here there are three different types of terms, which can be collected separately but not combined together.

Exercise 18.4

Simplify these where possible.

- 1** $3ab + 2ab - ab$
- 2** $5ac + 2ab - 3ac + 4ab$
- 3** $3abc + 2abc - 5abc$
- 4** $4ab - 3ac + 2ab - ac$
- 5** $a^2 + 3b^2 - 2a^2 - b^2$
- 6** $2x^2 - 3xy - xy + y^2$
- 7** $4b^2 + 3a^2 - 2b^2 - 4a^2$
- 8** $9a^2 - 3ab + 5ab - 6b^2$
- 9** $4ab + 2bc - 3ba - bc$
- 10** $2p^2 - 3pq + 4pq - 5p^2$
- 11** $3ab + 2ac + ad$
- 12** $9ab - 2bc + 3bc - 7ab$
- 13** $a^3 + 3a^3 - 6a^3$
- 14** $5a^3 + 4a^2 + 3a$
- 15** $a^3 + 3a^2 + 2a^3 + 4a^2$
- 16** $3ab^2 - 4ba^2 + 7a^2b$
- 17** $3x^3 - 2x^2 + 4x^2 - 3x^3$
- 18** $8a^3 - 4a^2 + 5a^3 - 2a^2$
- 19** $abc + cab - 3abc + 2bac$
- 20** Pali, Gemma and Saad all drew rectangles.
Pali's was a cm long and $2b$ cm wide.
Gemma's was $3a$ cm long and b cm wide.
Saad's was $2a$ cm long and $3b$ cm wide.
Find and simplify an expression for the total area of the three rectangles.

Expanding a single bracket

What is $2 \times 3 + 4$? Is it 14? Is it 10?

The rule is ‘do the multiplication first’, so the answer is 10.

If you want the answer to be 14, you need to add 3 and 4 first.

You use a bracket to show this $2 \times (3 + 4)$.

Notice that this is equal to $2 \times 3 + 2 \times 4$.

It is the same in algebra.

$a(b + c)$ means ‘add b and c and then multiply by a ’ and this is the same as ‘multiply b by a , multiply c by a and then add the results’.

So $a(b + c) = ab + ac$.

This is called expanding the bracket.

Note

Remember to multiply each of the terms inside the bracket by the number or letter outside the brackets.

Example 18.9

Question

Expand these brackets.

a $5(2x + 3)$

b $4(2x - 1)$

Solution

a $5(2x + 3) = 5 \times 2x + 5 \times 3 = 10x + 15$

b $4(2x - 1) = 4 \times 2x + 4 \times -1 = 8x - 4$

Note

Remember about multiplying with negative numbers. For example,

$$-3 \times x = -3x$$

$$-3 \times -x = 3x.$$

Exercise 18.5

Expand these brackets.

- | | | | | | | | |
|-----------|--------------|-----------|---------------|-----------|-------------------|-----------|-------------------|
| 1 | $2(a + b)$ | 2 | $4(2x + 1)$ | 3 | $2(p + 3)$ | 4 | $3(3x - 1)$ |
| 5 | $2(2x - 3)$ | 6 | $7(3y + z)$ | 7 | $4(3 - 8a)$ | 8 | $10(2a + 3b)$ |
| 9 | $5(3e - 8f)$ | 10 | $2(1 - x)$ | 11 | $5(p - q)$ | 12 | $a(a + 2)$ |
| 13 | $y(y - 1)$ | 14 | $y(3 - 2y)$ | 15 | $x(2 - x)$ | 16 | $-y(2 + y)$ |
| 17 | $3c(c + 4)$ | 18 | $-2x(5x - 3)$ | 19 | $2(3i + 4j - 5k)$ | 20 | $4(5m - 3n + 2p)$ |

Factorising algebraic expressions

Factors are numbers or letters which will divide into an expression.

The factors of 6 are 1, 2, 3 and 6.

The factors of p^2 are 1, p and p^2 .

To factorise an expression, look for common factors.

For example, the common factors of $2a^2$ and $6a$ are 2, a and $2a$.

Example 18.10

Question

Factorise these fully.

a $4p + 6$ **b** $2a^2 - 3a$ **c** $8x^2 - 12x$

Solution

- a** The only common factor is 2.

You divide each term by the common factor, 2.

$$4p \div 2 = 2p \quad \text{and} \quad 6 \div 2 = 3$$

$$4p + 6 = 2(2p + 3)$$

- b** The only common factor is a .

You divide each term by the common factor, a .

$$2a^2 \div a = 2a \quad \text{and} \quad 3a \div a = 3$$

$$2a^2 - 3a = a(2a - 3)$$

- c** Think about the numbers and the letters separately and then combine them.

The highest common factor of 8 and 12 is 4 and the common factor of x^2 and x is x . Therefore, the common factor of $8x^2$ and $12x$ is $4 \times x = 4x$.

$4x(\dots)$ You write this factor outside the bracket.

You then divide each term by the common factor, $4x$.

$$8x^2 \div 4x = 2x \quad \text{and} \quad 12x \div 4x = 3$$

$$8x^2 - 12x = 4x(2x - 3)$$

Note

Make sure that you have found all the common factors. Check that the expression in the brackets will not factorise further.

Exercise 18.6

For questions 1 to 8, copy and complete the factorisations.

- | | | |
|--|---|--|
| 1 $12a + 3 = 3(\square + 1)$ | 2 $9a + 18 = 9(a + \square)$ | 3 $5y - 30 = 5(y - \square)$ |
| 4 $6b - 4 = \square(3b - 2)$ | 5 $4x + 16 = \square(x + 4)$ | 6 $y^2 + 2y = y(\square + \square)$ |
| 7 $2b + 6b^2 = 2b(\square + \square)$ | 8 $8a^2 + 20a = \square(2a + 5)$ | |

For questions 9 to 31, factorise each of the expressions fully.

- | | | |
|-------------------------------------|----------------------------------|--------------------------------|
| 9 $2x + 6$ | 10 $4x - 20$ | 11 $9 - 12x$ |
| 12 $3x^2 + 5x$ | 13 $5a^2 + 10b$ | 14 $24 + 36a^2$ |
| 15 $10x^2 - 100x$ | 16 $24x + 32y$ | 17 $15ab - 20ac$ |
| 18 $30f^2 - 18fg$ | 19 $42ab + 35a^2$ | 20 $5a^2b + 10ab^2$ |
| 21 $3ab - 2ac + 3ad$ | 22 $2x^2y^2 - 3x^3y$ | 23 $5x^2 - 15x + 15$ |
| 24 $12x - 6y + 8z$ | 25 $9ab + 6b^2$ | 26 $4a^2c - 2ac^2$ |
| 27 $12x^2y + 8xy - 4xy^2$ | 28 $3a^2b - 9a^3b^2$ | 29 $5a^2b^2c^2 - 10abc$ |
| 30 $2a^2b - 3a^2b^3 + 7a^4b$ | 31 $4abc - 3ac^2 + 2a^2b$ | |

Factorising expressions of the form $ax + bx + kay + kby$

Example 18.11

Question

Factorise $x(a + b) + 2y(a + b)$.

Solution

The bracket $(a + b)$ is a common factor so this can be factorised as $(a + b)(x + 2y)$.

Example 18.12

Question

Factorise $x^2 - 3x + 2ax - 6ax$.

Solution

There is no common factor for all four terms but the first pair of terms have a common factor and the second pair of terms also have a common factor.

Factorising in pairs gives $x(x - 3) + 2a(x - 3)$.

The bracket $(x - 3)$ is a common factor so

$$x(x - 3) + 2a(x - 3) = (x - 3)(x + 2a)$$

Sometimes you have to change the order of the terms, as in this example.

Example 18.13

Question

Factorise $5ax - 9b + 3bx - 15a$.

Solution

The pairs of terms in this order do not have common factors.

Changing the order gives $5ax + 3bx - 15a - 9b$.

You can then factorise in pairs.

$$x(5a + 3b) - 3(5a + 3b) = (5a + 3b)(x - 3)$$

Note

Taking the factor -3 out means the sign of $3b$ is $+$ since $-3 \times +3b = -9b$.

Exercise 18.7

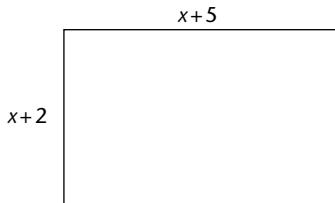
Factorise these.

- | | | | |
|-----------|-------------------------------|-----------|-----------------------------|
| 1 | $a(2x + 3) + b(2x + 3)$ | 2 | $5x(a - 2b) - 3(a - 2b)$ |
| 3 | $p^2 + pq + pr + qr$ | 4 | $ax - ay + bx - by$ |
| 5 | $a^2 + ab - ac - bc$ | 6 | $a^2 - ab - ac + bc$ |
| 7 | $4ab + 12a + 3b + 9$ | 8 | $2ax - 4ay + 3x - 6y$ |
| 9 | $5ax^2 - 5bx - 2ax + 2b$ | 10 | $3a^3 + 6ab - 2a^2b - 4b^2$ |
| 11 | $10ax - 5ay + 6bx - 3by$ | 12 | $8ac - 6ad - 12bc + 9bd$ |
| 13 | $8a^2 - 6ab - 15bc + 20ac$ | 14 | $a^2 + ab + a + b$ |
| 15 | $ac + 6bd + 3ad + 2bc$ | 16 | $10ac + 3bd - 2ad - 15bc$ |
| 17 | $10ab - 5ac - 3cd + 6bd$ | 18 | $6x^2 + 3yz - 2xz - 9xy$ |
| 19 | $10x^2 + 15y^2 - 25x^2y - 6y$ | 20 | $2x^2 - 6x + 3 - x$ |

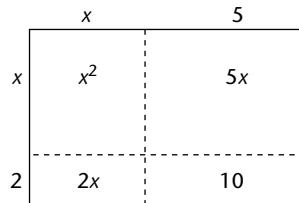
Expanding a pair of brackets

You expand a pair of brackets by multiplying one bracket by the other bracket.

Look at this rectangle.



Now split the rectangle up.



The area of the rectangle is $(x + 5)(x + 2)$.

$$\text{So } (x + 5)(x + 2) = x^2 + 2x + 5x + 10 = x^2 + 7x + 10$$

When multiplying out a pair of brackets you must multiply each term inside the second bracket by each term inside the first bracket.

The example which follows shows three methods for doing this.

Choose the method you prefer and stick to it.

Example 18.14

Question

Expand these.

a $(a + 2)(a + 5)$ b $(b + 4)(b - 1)$ c $(m - 5)(m - 4)$

Solution

a Method 1

Expand the first bracket and then multiply out.

$$\begin{aligned}(a + 2)(a + 5) &= a(a + 5) + 2(a + 5) \\&= a^2 + 5a + 2a + 10 \\&= a^2 + 7a + 10\end{aligned}$$

Notice that the middle two terms are like terms and so can be collected.

b Method 2

x	b	+4
b	b^2	+4b
-1	-1b	-4

$$\begin{aligned}(b + 4)(b - 1) &= b^2 + 4b - 1b - 4 \\&= b^2 + 3b - 4\end{aligned}$$

c Method 3

Use the word FOIL to make sure you multiply each term in the second bracket by each term in the first.

If you draw arrows to show the multiplications, you can think of a smiley face.



$$\begin{aligned}(m - 5)(m - 4) &= m \times m + m \times -4 - 5 \times m - 5 \times -4 \\&= m^2 - 4m - 5m + 20 \\&= m^2 - 9m + 20\end{aligned}$$

Note

You may be asked to 'multiply out', 'simplify', 'expand' or 'remove' the brackets. These all mean the same.

Note

Most errors are made in expanding brackets when negative signs are involved.

F: first \times first
O: outer \times outer
I: inner \times inner
L: last \times last

There are two special types of expansion of a pair of brackets that you need to know.

The first is when a bracket is squared, as in parts a and b of the next example. The important thing with this type of expansion is to make sure that you write the brackets separately and that you end up with three terms.

The second type is shown in part **c** of the example. With this type of expansion, you get only two terms because the middle terms cancel each other out. This type is known as **the difference of two squares** because $(A - B)(A + B) = A^2 - B^2$.

Example 18.15

Question

Expand these.

a $(x + 3)^2$ **b** $(x - 3)^2$ **c** $(x + 3)(x - 3)$

Solution

a $(x + 3)^2 = (x + 3)(x + 3)$ The method shown is method 1 but you can use any of the methods shown in Example 18.14. You will get the same answer.

$$\begin{aligned} &= x(x + 3) + 3(x + 3) \\ &= x^2 + 3x + 3x + 9 \\ &= x^2 + 6x + 9 \end{aligned}$$

b $(x - 3)^2 = (x - 3)(x - 3)$

$$\begin{aligned} &= x(x - 3) - 3(x - 3) \\ &= x^2 - 3x - 3x + 9 \\ &= x^2 - 6x + 9 \end{aligned}$$

c $(x + 3)(x - 3) = x(x - 3) + 3(x - 3)$

$$\begin{aligned} &= x^2 - 3x + 3x - 9 \\ &= x^2 - 9 \end{aligned}$$

Note

Take care with the negative signs.

Exercise 18.8

Expand these.

1 $(x + 2)(x + 3)$

4 $(x + 5)(x - 2)$

7 $(x - 3)(x + 6)$

10 $(a + 3)(a + 3)$

13 $(p + 4)(p - 2)$

16 $(x - 9)(x - 3)$

19 $(a - 5)^2$

22 $(a + 2)^2$

25 $(b - 7)^2$

28 $(x + 6)(x - 6)$

31 $(x + 10)(x - 10)$

2 $(a + 4)(a + 3)$

5 $(x + 7)(x - 3)$

8 $(x - 5)(x - 4)$

11 $(a + 2)(a + 1)$

14 $(a + 7)(a + 8)$

17 $(x + 10)(x - 1)$

20 $(b + 1)^2$

23 $(x - 10)^2$

26 $(x + 4)^2$

29 $(x - 4)(x + 4)$

32 $(x + 7)(x - 7)$

3 $(a + 2)(a + 1)$

6 $(x - 5)(x - 6)$

9 $(x + 1)(x + 3)$

12 $(x - 2)(x + 1)$

15 $(x - 6)(x + 4)$

18 $(x + 3)^2$

21 $(x - 2)^2$

24 $(x + 8)^2$

27 $(x + 2)(x - 2)$

30 $(x + 1)(x - 1)$

Expanding more complex brackets

The same methods can be used for brackets containing more complex expressions.

This is shown in the following example.

Example 18.16**Question**

Expand and simplify these.

a $(3x + 2)(5x + 1)$

b $(5a - 2b)(3a - b)$

Solution

$$\begin{aligned} \text{a } (3x + 2)(5x + 1) &= 3x(5x + 1) + 2(5x + 1) \\ &= 15x^2 + 3x + 10x + 2 \\ &= 15x^2 + 13x + 2 \end{aligned}$$

$$\begin{aligned} \text{b } (5a - 2b)(3a - b) &= 5a \times 3a + 5a \times -b - 2b \times 3a - 2b \times -b \\ &= 15a^2 - 5ab - 6ab + 2b^2 \\ &= 15a^2 - 11ab + 2b^2 \end{aligned}$$


Exercise 18.9

Expand these.

1 $(5+x)(x-6)$

4 $(5x+6)(2x-3)$

7 $(2x-3y)(x-2y)$

10 $(3j-8m)(2j-7m)$

13 $(2q+7p)(2q-9p)$

2 $(5x-1)(2x-4)$

5 $(x+y)(2x+y)$

8 $(7x+8y)(6x-4y)$

11 $(2m+7n)(5m-6n)$

14 $(3r-8s)(2r-7s)$

3 $(2x-5)(3x-2)$

6 $(3x-5y)(x-4y)$

9 $(2g-3h)(2g-7h)$

12 $(2r+3n)(3r-5n)$

15 $(2s-3t)(2s-7t)$

Factorising expressions of the form $a^2x^2 - b^2y^2$

When expanding brackets earlier in this chapter, you met expressions of the form $a^2 - b^2$.

They are called **the difference of two squares**.

We have already seen that $(x-3)(x+3) = x^2 - 9$.

So, following this pattern we see that factorising $x^2 - a^2$ gives $(x-a)(x+a)$.

Note

It is worth learning this so that you recognise it when you see it!

Example 18.17**Question**Factorise $x^2 - 16$.**Solution**

$$\begin{aligned} x^2 - 16 &= x^2 - 4^2 \\ &= (x-4)(x+4) \end{aligned}$$

In fact *any* expression which can be written as the difference of two squares can be factorised in this way.

Example 18.18**Question**

Factorise each of these expressions.

a $25x^2 - 1$

b $9x^2 - 4y^2$

Solution

a $25x^2 - 1 = 5^2x^2 - 1$
 $= (5x - 1)(5x + 1)$

b $9x^2 - 4y^2 = 3^2x^2 - 2^2y^2$
 $= (3x - 2y)(3x + 2y)$

When a numerical term is not a square number, check to see whether a common factor can be extracted, leaving an expression in a form that allows you to use the difference of two squares method.

Example 18.19**Question**

Factorise $3x^2 - 12$.

Solution

$$\begin{aligned} 3x^2 - 12 &= 3(x^2 - 4) && \text{Take out the common factor 3.} \\ &= 3(x^2 - 2^2) \\ &= 3(x - 2)(x + 2) \end{aligned}$$

Exercise 18.10

Factorise each of these expressions.

1 $x^2 - 25$

5 $25x^2 - 49y^2$

9 $x^2y^2 - 16a^2$

13 $3x^2 - 192$

17 $5x^2 - 45$

2 $x^2 - 4$

6 $9x^2 - 64$

10 $y^2 - 169$

14 $45 - 20x^2$

18 $7x^2 - 343$

3 $4a^2 - b^2$

7 $1 - 49t^2$

11 $121x^2 - 144y^2$

15 $25x^2y^2 - 100$

19 $10x^2 - 4000$

4 $9 - 16y^2$

8 $100x^2 - 1$

12 $8 - 2x^2$

16 $3x^2 - 12$

20 $8x^2 - 200$

Factorising expressions of the form $x^2 + bx + c$

In the expression $x^2 + bx + c$, c is the numerical term and b is called the coefficient of x .

Factorising quadratic expressions with a positive numerical term

The expression $(x + 6)(x + 2)$ can be multiplied out and simplified to $x^2 + 8x + 12$.

Therefore $x^2 + 8x + 12$ can be factorised as a product of two brackets, by reversing the process.

The order in which you write the brackets does not matter.

Example 18.20**Question**

Factorise $x^2 + 7x + 12$.

Solution

This will factorise into two brackets with x as the first term in each.

$$x^2 + 7x + 12 = (x \quad)(x \quad)$$

As both the signs are positive, both the numbers will be positive.

You need to find two numbers that multiply to give 12 and add to give 7.

These are +3 and +4.

So $x^2 + 7x + 12 = (x + 3)(x + 4)$

or $x^2 + 7x + 12 = (x + 4)(x + 3)$.

Example 18.21**Question**

Factorise $x^2 - 3x + 2$.

Solution

You need to find two negative numbers that multiply to give 2 and add to give -3. They are -2 and -1.

$$x^2 - 3x + 2 = (x - 2)(x - 1)$$

Note

If the numerical term is positive, both the numbers in the brackets must have the same sign as the coefficient of x .

If the coefficient of x is negative and the numerical term is positive, both the numbers in the brackets will be negative.

Exercise 18.11

Factorise these expressions.

- | | | | |
|----------------------------|----------------------------|---------------------------|----------------------------|
| 1 $x^2 + 5x + 6$ | 2 $x^2 + 6x + 5$ | 3 $x^2 + 4x + 3$ | 4 $x^2 + 6x + 8$ |
| 5 $x^2 + 5x + 4$ | 6 $x^2 + 9x + 20$ | 7 $x^2 + 2x + 1$ | 8 $x^2 - 7x + 6$ |
| 9 $x^2 - 9x + 18$ | 10 $x^2 - 7x + 10$ | 11 $x^2 - 4x + 3$ | 12 $a^2 - 2a + 1$ |
| 13 $y^2 - 9y + 14$ | 14 $x^2 - 6x + 8$ | 15 $a^2 + 8a + 12$ | 16 $a^2 - 6a + 9$ |
| 17 $b^2 - 12b + 32$ | 18 $x^2 + 11x + 24$ | 19 $x^2 - 9x + 20$ | 20 $x^2 - 15x + 56$ |

Factorising quadratic expressions with a negative numerical term**Example 18.22****Question**

Factorise $x^2 - 3x - 10$.

Solution

As the numerical term is negative, you need two numbers, with opposite signs, that multiply to give -10 and add to give -3.

The numbers are -5 and +2.

$$x^2 - 3x - 10 = (x - 5)(x + 2)$$

Example 18.23**Question**

Factorise $x^2 + 4x - 12$.

Solution

As the numerical term is negative, you need two numbers, with opposite signs, that multiply to give -12 and add to give $+4$.

The numbers are $+6$ and -2 .

$$x^2 + 4x - 12 = (x + 6)(x - 2)$$

Note

Remember that if the numerical term is negative, the two numbers in the brackets have different signs and the larger number has the sign of the coefficient of x .

It is easy to make a mistake when factorising. Always check by multiplying out the brackets.

Note

The first and third terms are squares and the middle term is twice the product of the quantities that are squared.

Note

If the middle term is negative the sign in the bracket will be negative.

Exercise 18.12

Factorise these expressions.

- | | | | | | | | |
|-----------|------------------|-----------|------------------|-----------|-----------------|-----------|-----------------|
| 1 | $x^2 - 2x - 8$ | 2 | $x^2 + 4x - 5$ | 3 | $x^2 - x - 6$ | 4 | $x^2 + 5x - 6$ |
| 5 | $x^2 + 2x - 3$ | 6 | $x^2 - 3x - 18$ | 7 | $x^2 - 3x - 28$ | 8 | $x^2 - 9x - 10$ |
| 9 | $x^2 - 9x - 22$ | 10 | $x^2 + 4x - 32$ | 11 | $y^2 + 9y - 22$ | 12 | $x^2 + x - 12$ |
| 13 | $x^2 + x - 20$ | 14 | $a^2 + 8a - 20$ | 15 | $a^2 - 6a - 27$ | 16 | $a^2 - 6a - 16$ |
| 17 | $b^2 + 19b - 20$ | 18 | $x^2 + 11x - 26$ | 19 | $x^2 + 7x - 30$ | 20 | $x^2 - 7x - 18$ |

Factorising expressions of the form $a^2 + 2ab + b^2$

We have found that $(x + 3)^2 = x^2 + 6x + 9$ and $(x - 3)^2 = x^2 - 6x + 9$.

These are examples of **perfect squares**.

In general

$$(a + b)^2 = (a + b)(a + b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2.$$

The result can be used in reverse to factorise expressions of the form $a^2 + 2ab + b^2$.

$$\text{So } a^2 + 2ab + b^2 = (a + b)^2.$$

Example 18.24**Question**

Factorise $x^2 + 8x + 16$.

Solution

The 1st and 3rd terms are x^2 and 4^2 and the middle term is $8x = 2 \times 4 \times x$.
 $x^2 + 8x + 16 = (x + 4)^2$

Example 18.25**Question**

Factorise $9x^2 - 30xy + 25y^2$.

Solution

The first and third terms are $(3x)^2$ and $(5y)^2$ and the middle term
 $30xy = 2 \times 3x \times 5y$.

The middle term is negative so the sign in the bracket will be negative.

$$9x^2 - 30xy + 25y^2 = (3x - 5y)^2$$

Exercise 18.13

Factorise these expressions.

- | | | | |
|----------|-----------------------|-----------|------------------------|
| 1 | $x^2 + 2x + 1$ | 2 | $x^2 - 2x + 4$ |
| 3 | $x^2 - 10x + 25$ | 4 | $a^2 + 20a + 100$ |
| 5 | $9 - 12y + 4y^2$ | 6 | $4x^2 + 4xy + y^2$ |
| 7 | $49a^2 + 28a + 4$ | 8 | $36x^2 - 60xy + 25y^2$ |
| 9 | $16a^2 - 24ab + 9b^2$ | 10 | $4a^2b^2 + 4abc + c^2$ |

Factorising quadratic expressions of the form $ax^2 + bx + c$

You have already learned how to factorise simple quadratic expressions such as $x^2 + bx + c$.

- If c is positive, you find two numbers with the same sign that multiply to give c and add to give b .
- If c is negative, you find two numbers with different signs that multiply to give c and add to give b .

The expression $ax^2 + bx + c$, when factorised, will be

$$(px + q)(rx + s) = prx^2 + (ps + qr)x + qs.$$

So $pr = a$, $qs = c$ and $ps + qr = b$.

It is easiest to look at examples.

Factorising more complex quadratic expressions with a positive numerical term

Example 18.26

Question

Factorise $3x^2 + 11x + 6$.

Solution

As the numerical term is positive, both numerical terms in the brackets have the same sign, and as the coefficient of x is positive, they are both positive.

The only numbers that multiply to give 3 are 3 and 1.

So, as a start, $(3x + \dots)(x + \dots)$.

The numbers that multiply to give 6 are either 3 and 2 or 6 and 1.

So the possible answers are

$(3x + 2)(x + 3)$ or $(3x + 3)(x + 2)$ or $(3x + 6)(x + 1)$ or $(3x + 1)(x + 6)$.

By expanding the brackets it can be seen that the first one is correct.

$$3x^2 + 11x + 6 = (3x + 2)(x + 3)$$

The coefficient of x is $3 \times 3 + 2 \times 1 = 9 + 2 = 11$.

It is very useful to check completely by multiplying out the two brackets fully.

Writing out all the possible brackets can be a long process and it is quicker to test the possibilities until you find the correct coefficient of x and then multiply out the brackets to check.

Example 18.27

Question

Factorise $4x^2 - 14x + 6$.

Solution

First, check whether there is a common factor.

Here the common factor is 2.

$$4x^2 - 14x + 6 = 2(2x^2 - 7x + 3)$$

Now look at the quadratic expression.

As the numerical term is positive and the coefficient of x is negative, both the numerical terms in the brackets will be negative.

The possibilities for the coefficients of the x -terms in the brackets are 2 and 1.

The possibilities for the numerical terms in the brackets are -1 and -3 or -3 and -1 .

The coefficient of x is thus $2 \times -1 + 1 \times -3 = -5$ or $2 \times -3 + 1 \times -1 = -7$.

The second is correct so $2x^2 - 7x + 3 = (2x - 1)(x - 3)$.

$$\text{So } 4x^2 - 14x + 6 = 2(2x - 1)(x - 3).$$

Exercise 18.14

Factorise each of these expressions.

1 $2x^2 + 6x + 4$

4 $2x^2 + 7x + 6$

7 $3x^2 - 11x + 6$

10 $4x^2 - 16x + 15$

13 $5x^2 - 13x + 6$

16 $8x^2 - 18x + 9$

19 $3x^2 + 13x + 4$

22 $3x^2 + 11x + 10$

2 $3x^2 + 7x + 2$

5 $6x^2 - 15x + 6$

8 $3x^2 - 13x + 10$

11 $4x^2 + 8x + 3$

14 $5x^2 - 22x + 8$

17 $3x^2 + 17x + 20$

20 $5x^2 + 18x + 9$

23 $2x^2 + 5x + 2$

3 $2x^2 + 9x + 4$

6 $3x^2 - 12x + 12$

9 $3x^2 - 11x + 10$

12 $7x^2 + 10x + 3$

15 $6x^2 - 19x + 10$

18 $2x^2 + 7x + 6$

21 $4x^2 + 6x + 2$

24 $4x^2 + 17x + 15$

Note

First look for any common factor, then try the most obvious pairs first and remember, if the sign of c (the numerical term) is positive, both the numerical terms in the brackets have the same sign as b (the coefficient of x).

Factorising more complex quadratic expressions with a negative numerical term

The examples looked at so far had a positive numerical term.

When c (the numerical term) is negative the signs of the numerical terms in the brackets are different.

These are, again, best shown by examples.

Example 18.28

Question

Factorise $3x^2 - 7x - 6$.

Solution

There is no common factor.

The signs in the numerical terms in the brackets are different.

The coefficients of the x -terms in the brackets are 3 and 1.

The possibilities for the numerical terms are 1 and -6 or -1 and 6 , 6 and -1 or -6 and 1 , 3 and -2 or -3 and 2 , or 2 and -3 or -2 and 3 .

Test the possibilities until you find the correct coefficient of x .

$$3 \times 1 + 1 \times -6 = -3$$

$$3 \times -1 + 1 \times 6 = 3$$

$$3 \times 6 + 1 \times -1 = 17$$

$$3 \times -6 + 1 \times 1 = -17 \quad \text{You don't need to test this pair as you can see that the result will be } -17.$$

$$3 \times 3 + 1 \times -2 = 7$$

$$3 \times -3 + 1 \times 2 = -7 \quad \text{You can see from the previous calculation that this pair will give the required result.}$$

$$\text{So} \quad 3x^2 - 7x - 6 = (3x + 2)(x - 3).$$

Multiply out to check.

Example 18.29**Question**

Factorise $6x^2 + 3x - 30$.

Solution

There is a common factor of 3.

$$6x^2 + 3x - 30 = 3(2x^2 + x - 10)$$

Look at the quadratic expression.

The signs in the numerical terms in the brackets are different.

The coefficients of the x -terms in the brackets are 2 and 1.

The possibilities for the numerical terms are 1 and -10 or -1 and 10 , 10 and -1 or -10 and 1 , 5 and -2 or -5 and 2 , or 2 and -5 or -2 and 5 .

Try the combinations you think may give the correct result first.

$$2 \times 5 + 1 \times -2 = 8 \quad \text{So neither } 5 \text{ and } -2 \text{ nor } -5 \text{ and } 2 \text{ is correct.}$$

$$2 \times 2 + 1 \times -5 = -1 \quad \text{Which is the correct number but the wrong sign.}$$

$$2 \times -2 + 1 \times 5 = 1 \quad \text{So the numerical terms in the brackets are } -2 \text{ and } 5.$$

$$\text{So } 6x^2 + 3x - 30 = 3(2x + 5)(x - 2).$$

Check by multiplying out.

Note

First look for any common factor. If the sign of c (the numerical term) is negative, the numerical terms in the brackets will have different signs. Always check by multiplying out.

Exercise 18.15

Factorise each of these expressions.

- | | | | | | | | |
|-----------|------------------|-----------|------------------|-----------|-------------------|-----------|-------------------|
| 1 | $3x^2 + x - 10$ | 2 | $2x^2 + 5x - 3$ | 3 | $3x^2 - 2x - 8$ | 4 | $3x^2 - 11x - 4$ |
| 5 | $2x^2 + 9x - 5$ | 6 | $3x^2 + 4x - 15$ | 7 | $5x^2 - 15x - 50$ | 8 | $5x^2 + 13x - 6$ |
| 9 | $4x^2 - 4x - 3$ | 10 | $7x^2 + 10x - 8$ | 11 | $3x^2 - x - 14$ | 12 | $3x^2 - 11x - 20$ |
| 13 | $2x^2 - x - 21$ | 14 | $2x^2 - 15x - 8$ | 15 | $6x^2 - 17x - 14$ | 16 | $6x^2 - 13x - 15$ |
| 17 | $2x^2 - x - 15$ | 18 | $3x^2 + x - 14$ | 19 | $5x^2 - 17x - 12$ | 20 | $3x^2 - 5x - 12$ |
| 21 | $4x^2 - 3x - 10$ | 22 | $2x^2 - 7x - 15$ | 23 | $4x^2 - 7x - 2$ | 24 | $3x^2 - 16x - 12$ |

Simplifying algebraic fractions

To simplify a numerical fraction, you cancel any common factors of both the numerator and the denominator to write it in its simplest terms.

In the same way, to simplify an algebraic fraction, you cancel any common factors of both the numerator and the denominator.

Factors can be numbers, letters or brackets.

Example 18.30**Question**

$$\text{Simplify } \frac{4ab^2}{3c^2} \times \frac{9c^2}{2a^2b}.$$

Solution

$$\frac{4ab^2}{3c^2} \times \frac{9c^2}{2a^2b} = \frac{\cancel{4}\cancel{a}^1\cancel{b}^2 \times \cancel{9}^3\cancel{c}^2}{\cancel{3}\cancel{c}^2 \times \cancel{2}\cancel{a}^2\cancel{b}} = \frac{6b}{a}$$

2, 3, a, b and c^2 are all factors of both the numerator and the denominator so these cancel.

Example 18.31**Question**

$$\text{Simplify } \frac{x^2 + x}{x^2 - 2x - 3}.$$

Solution

As it stands this fraction cannot be cancelled.

First both numerator and denominator must be factorised.

$$\frac{x^2 + x}{x^2 - 2x - 3} = \frac{x(x+1)}{(x-3)(x+1)}$$

$(x+1)$ is a factor of both the numerator and the denominator so it cancels.

$$= \frac{x}{(x-3)}$$

Note

Errors often occur by cancelling individual terms. Only factors, which can be individual numbers, letters or brackets, can be cancelled.

Exercise 18.16

Simplify these expressions.

- | | | | | | | | |
|----|--|----|---|----|--|----|--|
| 1 | $\frac{15a^2}{6} \times \frac{b^2}{a}$ | 2 | $\frac{12abc}{a^2b} \times \frac{a^3b}{4c}$ | 3 | $\frac{x^3y^2}{10x} \times \frac{15xy}{10y^2}$ | 4 | $\frac{2x^2y^2}{xy} \times \frac{3xy^3}{4x^2}$ |
| 5 | $\frac{5x^2 - 20x}{10x^2}$ | 6 | $\frac{3x^2 - 6x}{x^2 + x - 6}$ | 7 | $\frac{x^2 + 2x + 1}{x^2 - 1}$ | 8 | $\frac{3x^2 - x - 4}{5x^2 - 5}$ |
| 9 | $\frac{3x + 15}{x^2 + 3x - 10}$ | 10 | $\frac{6x - 18}{x^2 - x - 6}$ | 11 | $\frac{x^2 - 5x + 6}{x^2 - 4x + 3}$ | 12 | $\frac{x^2 - 3x - 4}{x^2 - 4x - 5}$ |
| 13 | $\frac{x^2 - 2x - 3}{x^2 - 9}$ | 14 | $\frac{3x^2 - 12}{x^2 + 2x - 8}$ | 15 | $\frac{3x^2 + 5x + 2}{2x^2 - x - 3}$ | 16 | $\frac{2x^2 + x - 6}{x^2 + x - 2}$ |
| 17 | $\frac{6x^2 - 3x}{(2x - 1)^2}$ | 18 | $\frac{5(x + 3)^2}{x^2 - 9}$ | 19 | $\frac{(x - 3)(x + 2)^2}{x^2 - x - 6}$ | 20 | $\frac{2x^2 + x - 6}{(2x - 3)^2}$ |

Adding and subtracting algebraic fractions

To add and subtract numerical fractions, you first write all the fractions with a common denominator.

In the same way, to add and subtract algebraic fractions, you first write all the fractions with a common denominator.

Example 18.32

Question

Simplify $\frac{x+3}{2} - \frac{x-3}{3}$.

Solution

The common denominator is 6.

So $(x+3)$ is multiplied by $6 \div 2 = 3$ and $(x-3)$ is multiplied by $6 \div 3 = 2$.

$$\begin{aligned}\frac{x+3}{2} - \frac{x-3}{3} &= \frac{3(x+3) - 2(x-3)}{6} \\ &= \frac{3x+9-2x+6}{6} \\ &= \frac{x+15}{6}\end{aligned}$$

Note

Do not miss out the first step. Most errors occur because of expanding the brackets wrongly.

The procedure is still the same when the denominators involve algebraic expressions.

Example 18.33

Question

Simplify $\frac{3}{x+1} - \frac{2}{x}$.

Solution

The common denominator is $x(x+1)$.

So 3 is multiplied by $x(x+1) \div (x+1) = x$ and 2 is multiplied by $x(x+1) \div x = (x+1)$.

$$\begin{aligned}\frac{3}{x+1} - \frac{2}{x} &= \frac{3x-2(x+1)}{x(x+1)} \\ &= \frac{3x-2x-2}{x(x+1)} \\ &= \frac{x-2}{x(x+1)}\end{aligned}$$

Exercise 18.17

Simplify these.

1 $\frac{x}{2} + \frac{3x}{5}$

2 $\frac{2x}{3} - \frac{3x}{5}$

3 $\frac{x-1}{2} - \frac{x-3}{5}$

4 $\frac{x-3}{5} + \frac{2x}{3} - \frac{3x-2}{10}$

5 $\frac{2x-1}{6} + \frac{3x}{4} - \frac{x-2}{12}$

6 $\frac{1}{x} + \frac{2}{x-1}$

7 $\frac{3}{x} + \frac{2}{x+1}$

8 $\frac{2x}{x+1} - \frac{x-1}{x+3}$

9 $\frac{x+1}{x-1} + \frac{3x-1}{x+2}$

10 $\frac{2x}{x-1} - \frac{3x+2}{x+2}$

11 $\frac{x}{x+1} - \frac{3}{5} + \frac{x-2}{x}$

12 $\frac{x}{x+1} + \frac{3}{5} - \frac{x+3}{x}$

13 $\frac{2x}{x-3} + \frac{x-1}{x+2} - \frac{4}{9}$

14 $\frac{2}{2x+1} + \frac{3x+5}{x+2}$

15 $\frac{4x+17}{x+3} - \frac{2x-15}{x-3}$

19 INDICES

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- understand and use the rules of indices
- use and interpret positive, negative, fractional and zero indices.

CHECK YOU CAN:

- work with squares, cubes, square roots and cube roots
- write a fraction in its simplest form.

Simplifying numbers with indices

Indices (or powers) are a form of mathematical shorthand.

You know that 4^2 means 4×4 .

Note

The plural of 'index' is 'indices'.

The notation 4^2 is known as **index notation**.

2 is the **index** and 4 is the **base**.

Index notation can be used as shorthand for any power of a number.

$3 \times 3 \times 3 \times 3$ can be written as 3^4 using index notation.

$2 \times 2 \times 2$ can be written as 2^8 using index notation.

Example 19.1

Question

Write each of these using index notation.

- a $5 \times 5 \times 5 \times 5$
b $6 \times 6 \times 6 \times 4 \times 6 \times 4 \times 6 \times 4$

Solution

- a $5 \times 5 \times 5 \times 5 = 5^4$
5 is multiplied 4 times, so the index is 4 and the base is 5.
b $6 \times 6 \times 6 \times 4 \times 6 \times 4 \times 6 \times 4 = 6^5 \times 4^3$
6 is multiplied 5 times so the index is 5 and the base is 6.
4 is multiplied 3 times so the index is 3 and the base is 4.
The two terms with different bases cannot be combined.

Exercise 19.1

- 1 Write these using index notation.
- a $6 \times 6 \times 6 \times 6$ b $7 \times 7 \times 7$
c $8 \times 8 \times 8 \times 8 \times 8$ d $4 \times 4 \times 4 \times 4$
e $2 \times 2 \times 2 \times 2 \times 2 \times 2$ f 10×10
- 2 Write these using index notation.
- a $5 \times 4 \times 4 \times 4 \times 5$
b $3 \times 3 \times 5 \times 5 \times 5$
c $2 \times 2 \times 2 \times 3 \times 3 \times 4 \times 4 \times 4 \times 4 \times 4$
d $7 \times 7 \times 7 \times 8 \times 8 \times 9 \times 9 \times 9$
- 3 Work out the value of these.
- a 4^3 b 2^6 c 3^4 d 10^5

Multiplying numbers in index form

When calculations involve more than one term with the same base they can be simplified.

$$\begin{aligned}3^4 \times 3^8 &= (3 \times 3 \times 3 \times 3) \times (3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3) \\&= 3 \times 3 \\&= 3^{12}\end{aligned}$$

The indices are added.

$$\begin{aligned}3^4 \times 3^8 &= 3^{4+8} \\&= 3^{12}\end{aligned}$$

This gives a rule for multiplying numbers in index form.

$$n^a \times n^b = n^{a+b}$$

Example 19.2

Question

Write these in a simpler form using indices.

a $4^3 \times 4^8$ **b** $9^2 \times 9 \times 9^4$ **c** $2^5 \times 3^2 \times 2^4$

Solution

- | | |
|--|--|
| a $4^3 \times 4^8 = 4^{3+8} = 4^{11}$ | The indices are added. |
| b $9^2 \times 9 \times 9^4 = 9^{2+1+4} = 9^7$ | Note that 9 is the same as 9^1 .
The bases are all the same so the three indices are added. |
| c $2^5 \times 3^2 \times 2^4 = 2^{5+4} \times 3^2 = 2^9 \times 3^2$ | Only terms with the same base can be combined.
There are two terms in the final answer. |

Exercise 19.2

Write these in a simpler form using indices.

- | | | |
|---------------------------------------|--|---|
| 1 $5^2 \times 5^3$ | 2 $6^2 \times 6^7$ | 3 $10^3 \times 10^4$ |
| 4 $3^6 \times 3^5$ | 5 $8^3 \times 8^2$ | 6 $4^2 \times 4^3$ |
| 7 $9^2 \times 9^7$ | 8 $6^2 \times 6^6$ | 9 $4^2 \times 4^5 \times 4$ |
| 10 $5^2 \times 6^2 \times 5^5$ | 11 $2^5 \times 3 \times 2^4 \times 3^7$ | 12 $7 \times 8^2 \times 7^3 \times 8^3 \times 7^4$ |

Dividing numbers in index form

Divisions can be simplified by cancelling common factors.

$$\begin{aligned}2^6 \div 2^4 &= \frac{2 \times 2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2 \times 2} \div 2 \times 2 \\&= 2 \times 2 \\&= 2^2\end{aligned}$$

The indices are subtracted.

$$\begin{aligned} 2^6 \div 2^4 &= 2^{6-4} \\ &= 2^2 \end{aligned}$$

This gives a rule for dividing numbers in index form.

$$n^a \div n^b = n^{a-b}$$

Example 19.3

Question

Write these in a simpler form using indices.

a $8^6 \div 8^2$ b $\frac{7^9}{7 \times 7^3}$

Solution

a $8^6 \div 8^2 = 8^{6-2} = 8^4$ The indices are subtracted.

b $\frac{7^9}{7 \times 7^3} = \frac{7^9}{7^4}$ First simplify the denominator by adding the indices.
 $= 7^{9-4}$ Then subtract the indices.
 $= 7^5$

Exercise 19.3

1 Write these in a simpler form using indices.

a $10^5 \div 10^2$	b $3^5 \div 3^2$	c $8^4 \div 8^2$
d $7^5 \div 7^3$	e $6^3 \div 6^2$	

2 Work out these, giving your answers in index form.

a $\frac{3^9}{3^4 \times 3^2}$	b $\frac{2^4 \times 2^3}{2^5}$	c $\frac{5^4 \times 5^5}{5^2 \times 5^3}$
d $\frac{4^{12}}{4^5 \times 4^4}$	e $\frac{2^5 \times 2^6}{2^4}$	f $\frac{6^5 \times 6^4}{6^2 \times 6}$

Simplifying algebraic expressions using indices

Indices can be used with algebraic expressions as well as numerical expressions.

You know that $a \times a \times a = a^3$ and $a \times a \times a \times a \times a = a^5$

So $a^3 \times a^5 = (a \times a \times a) \times (a \times a \times a \times a \times a) = a^8$

The indices are added $a^3 \times a^5 = a^{3+5} = a^8$

Similarly $a^5 \div a^3 = (a \times a \times a \times a \times a) \div (a \times a \times a)$
 $= \frac{a \times a \times a \times a \times a}{a \times a \times a}$
 $= a \times a$
 $= a^2$

The indices are subtracted $a^5 \div a^3 = a^{5-3} = a^2$

The rules for multiplying and dividing numbers in index form can also be applied to algebraic expressions.

These are the general laws for indices.

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

Using the first law, you can see that $(a^2)^3 = a^2 \times a^2 \times a^2 = a^{2+2+2} = a^6$

This is the same as $(a^2)^3 = a^{2 \times 3} = a^6$

This gives another general law.

$$(a^n)^m = a^{n \times m}$$

Using the second law, you can see that $a^3 \div a^3 = a^{3-3} = a^0$

But $a^3 \div a^3 = 1$

This gives a fourth general law.

$$a^0 = 1$$

These four laws can be used to simplify algebraic expressions involving indices.

Example 19.4

Question

Simplify these expressions where possible.

a $3a^2 \times 4a^3$

b $\frac{6a^5}{2a^3}$

c $(2x^3)^4$

d $\frac{12ab^3 \times 3a^2b}{2a^3b^2}$

e $4a^2 + 3a^3$

Solution

a $3a^2 \times 4a^3 = 12a^5$

The numbers are multiplied and the indices are added.

b $\frac{6a^5}{2a^3} = 3a^2$

The numbers are divided and the indices are subtracted.

c $(2x^3)^4 = 16x^{12}$

The number is raised to the power 4 and the indices are multiplied.

d $\frac{12ab^3 \times 3a^2b}{2a^3b^2} = \frac{36a^3b^4}{2a^3b^2}$

First simplify the numerator by multiplying the numbers and adding the indices for the a terms and the b terms separately.

$= 18a^0b^2$ Then divide the numbers and subtract the indices for the a terms and the b terms separately.

$= 18b^2$ Remember that $a^0 = 1$.

e $4a^2 + 3a^3$

This expression cannot be simplified.

The terms are not like terms so cannot be added.

Exercise 19.4

- 1 Write these as a single power of a .
a $a^2 \times a^3$ **b** $a^4 \times a^5$ **c** $a^4 \times a^2$ **d** $a^3 \times a^6$
- 2 Write these as a single power of a .
a $a^6 \div a^4$ **b** $a^7 \div a^3$ **c** $a^8 \div a^2$ **d** $a^5 \div a^2$
- 3 Simplify these.
a $2a^2 \times 3a^3$ **b** $4a^4 \times 3a^5$ **c** $3a^4 \times 4a^2$ **d** $5a^3 \times 3a^6$
- 4 Simplify these.
a $6a^6 \div 2a^4$ **b** $10a^7 \div 5a^3$ **c** $6a^8 \div 3a^2$ **d** $12a^5 \div 4a^2$
- 5 Simplify these.
a $(3a^3)^2$ **b** $(2a)^3$ **c** $(5a^4)^2$ **d** $(2a^3)^5$
- 6 Write these as a single power of x .
a $\frac{x^2 \times x^5}{x^3}$ **b** $\frac{x^7}{x^2 \times x^3}$ **c** $\frac{x \times x^8}{x^3 \times x^4}$ **d** $\left(\frac{x^5}{x^2}\right)^4$
- 7 Simplify these where possible.
a $2a^2b \times 3a^3b^2$ **b** $3a^2b^3 \times 2a^3b^4$ **c** $4a^2b - 2ab^2$
- 8 Simplify these.
a $\frac{15a^2b^3 \times 3a^2b}{9a^3b^2}$ **b** $\frac{4abc \times 3a^2bc^2}{6a^2b^2c^2}$ **c** $\frac{(3a^2b^2)^3}{(a^3b)^2}$
- 9 Simplify these.
a $\frac{6a^4b}{5c^2} \times \frac{10c^3}{9a^2b^3}$ **b** $\frac{12x^5y^4}{7z^5} \times \frac{14z^3}{15x^3y^2}$ **c** $\frac{8a^7b^5}{15c^4} \div \frac{4b^2}{9c}$

Negative indices

The index laws can also be used for numbers with negative indices.

Using the laws of indices

$$5^0 \div 5^3 = 5^{0-3} = 5^{-3}.$$

You also know that $5^0 = 1$, so

$$5^0 \div 5^3 = 1 \div 5^3 = \frac{1}{5^3}.$$

It follows that $5^{-3} = \frac{1}{5^3}$.

This gives a general law.

$$a^{-n} = \frac{1}{a^n}$$

Example 19.5**Question**

Write these as fractions.

a 6^{-2}

b 16^{-1}

c n^{-3}

d $\left(\frac{1}{4}\right)^{-2}$

Solution

a $6^{-2} = \frac{1}{6^2} = \frac{1}{36}$

b $16^{-1} = \frac{1}{16^1} = \frac{1}{16}$

c $n^{-3} = \frac{1}{n^3}$

d $\left(\frac{1}{4}\right)^{-2} = 4^2 = 16$

Exercise 19.5

- 1 Write each of these as an integer or a fraction.

a 6^{-1}

b 3^{-2}

c 1^{-5}

d 5^{-2}

e 10^{-3}

- 2 Write each of these as an integer or a fraction.

a $\left(\frac{1}{2}\right)^{-1}$

b $\left(\frac{1}{3}\right)^{-2}$

c $\left(\frac{2}{3}\right)^{-1}$

d $\left(\frac{3}{4}\right)^{-2}$

e $\left(\frac{2}{5}\right)^{-3}$

- 3 Evaluate these.

a $3^4 \times 3^{-3}$

b $5^2 \times 5^{-1}$

c $6^8 \times 6^{-6}$

d $2^{-3} \times 2^0$

Fractional indices

Is there an index for a square or cube root?

Suppose that $a^b = \sqrt[3]{a}$.

Then $(a^b)^3 = (\sqrt[3]{a})^3$ Cube both sides.

The cube of the cube root of a is equal to a so the right-hand side of the equation is equivalent to a .

$$a^{3b} = a = a^1$$

Use the index law for powers on the left-hand side of the equation.

So $3b = 1$

Equate the powers.

$$b = \frac{1}{3}$$

So $a^{\frac{1}{3}} = \sqrt[3]{a}$.

This can be extended to give a general law.

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

The index laws can be combined to work with more complex fractional powers.

For example, $a^{\frac{3}{2}} = \left(a^{\frac{1}{2}}\right)^3 = (\sqrt{a})^3$

Also $a^{\frac{3}{2}} = (a^3)^{\frac{1}{2}} = \sqrt{a^3}$

A scientific calculator has a power key. It may look like this .

Find out how to use this key on your calculator to evaluate fractional and negative indices.

Example 19.6

Question

- a Write $\sqrt[3]{n}$ using index notation.
- b Evaluate $16^{\frac{1}{4}}$.
- c Evaluate $25^{-\frac{1}{2}}$.
- d Evaluate $125^{\frac{2}{3}}$.

Solution

- a $\sqrt[3]{n} = n^{\frac{1}{3}}$
- b $16^{\frac{1}{4}} = 2$ because $2^4 = 16$ so $16^{\frac{1}{4}} = (2^4)^{\frac{1}{4}} = 2^1 = 2$ Check this result on your calculator.
- c $25^{-\frac{1}{2}} = \left(25^{\frac{1}{2}}\right)^{-1} = 5^{-1} = \frac{1}{5}$ Check this result on your calculator.
- d $125^{\frac{2}{3}} = \left(\sqrt[3]{125}\right)^2 = 5^2 = 25$ Check this result on your calculator.

Note

Either the cube root or the square can be calculated first, the final result will be the same. It is often easier to work out the root first.

Exercise 19.6

- 1 Write each of these in index form.

- a the cube root of n
- b $\sqrt[6]{x}$
- c $\sqrt[3]{m^5}$
- d $\sqrt[5]{p^4}$

Work out these. Give your answers as whole numbers or fractions.

 2 a 4^{-1}	b $4^{\frac{1}{2}}$	c 4^0	d 4^{-2}	e $4^{\frac{3}{2}}$
 3 a $8^{\frac{1}{3}}$	b 8^{-1}	c $8^{\frac{4}{3}}$	d $\left(\frac{1}{8}\right)^{-2}$	e 8^1
 4 a 9^{-1}	b $9^{\frac{1}{2}}$	c 9^0	d 9^{-2}	e $9^{\frac{3}{2}}$
 5 a $27^{\frac{1}{3}}$	b $27^{\frac{4}{3}}$	c 27^{-1}	d $\left(\frac{1}{27}\right)^{\frac{1}{3}}$	e 27^0
 6 a $64^{\frac{1}{2}}$	b $64^{-\frac{1}{3}}$	c 64^0	d $64^{-\frac{2}{3}}$	e $64^{\frac{5}{6}}$
 7 a $16^{\frac{1}{2}}$	b $16^{-\frac{1}{4}}$	c 16^0	d $16^{\frac{3}{2}}$	e $16^{-\frac{7}{4}}$
 8 a $3^0 + 3^3$	b $4 + 4^{\frac{1}{2}}$	c $2^{-1} - 4^{-1}$	d $36^{\frac{1}{2}} - 36^0$	

-  **9** a $2^2 \times 9^{\frac{1}{2}}$ b $2^5 \times 8^{\frac{1}{3}}$ c $81^{\frac{1}{4}} \times 3^{-2}$ d $9^{\frac{1}{2}} \times 6^2 \times 4^{-1}$
 **10** a $2^2 + 3^0 + 16^{\frac{1}{2}}$ b $\left(\frac{3}{4}\right)^{-2} \times 27^{\frac{2}{3}}$ c $4^2 \div 9^{\frac{1}{2}}$ d $4^2 - 8^{\frac{1}{3}} + 9^0$

Use your calculator to work out these. Give the answers as whole numbers or fractions.

- 11** a $25^{\frac{3}{2}}$ b $36^{\frac{1}{2}}$ c $125^{\frac{2}{3}} \times 8^{\frac{2}{3}}$ d $49^{\frac{3}{2}} \times 81^{-\frac{1}{4}}$
12 a $5^{-2} \times 10^5 \times 16^{-\frac{1}{2}}$ b $\left(\frac{4}{5}\right)^2 \times 128^{-\frac{3}{7}}$
 c $5^3 - 25^{\frac{1}{2}} - \left(\frac{2}{5}\right)^{-2}$ d $125^{-\frac{1}{3}} - 121^{\frac{1}{2}} + 216^{\frac{1}{3}}$

Using the laws of indices with numerical and algebraic expressions

You have met these laws of indices.

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(a^n)^m = a^{n \times m}$$

$$a^0 = 1$$

$$a^{-n} = \frac{1}{a^n}$$

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

These laws can be used together to simplify expressions involving numbers or letters.

Example 19.7

Question

Write each of these as a single power of 2, where possible.

a $2\sqrt{2}$

b $(\sqrt[3]{2})^2$

c $2^3 \div 2^{\frac{1}{2}}$

d $2^3 + 2^4$

e $8^{\frac{3}{4}}$

f $2^3 \times 4^{\frac{3}{2}}$

g $2^n \times 4^3$

Solution

a $2\sqrt{2} = 2^1 \times 2^{\frac{1}{2}} = 2^{\frac{3}{2}}$

b $(\sqrt[3]{2})^2 = (2^{\frac{1}{3}})^2 = 2^{\frac{2}{3}}$

c $2^3 \div 2^{\frac{1}{2}} = 2^{3-\frac{1}{2}} = 2^{\frac{5}{2}}$

d $2^3 + 2^4$ These powers cannot be added.

e $8^{\frac{3}{4}} = (2^3)^{\frac{3}{4}} = 2^{\frac{3 \times 3}{4}} = 2^{\frac{9}{4}}$

f $2^3 \times 4^{\frac{3}{2}} = 2^3 \times (2^2)^{\frac{3}{2}} = 2^3 \times 2^3 = 2^6$

g $2^n \times 4^3 = 2^n \times (2^2)^3 = 2^n \times 2^6 = 2^{n+6}$

 **Exercise 19.7**

1 Write each of these in the form 2^k .

a $2^2 \times 2^{\frac{1}{2}}$

b $2^{-2} \times 2^{\frac{1}{4}}$

c $(2^{-3})^2$

d $(2^0)^2 \times 2^{-3}$

e $\left(\frac{1}{\sqrt[3]{2}}\right)^2$

f $\frac{(\sqrt{2})^3}{2}$

2 Write each of these numbers as a power of 3, as simply as possible.

a 27

b $\frac{1}{3}$

c $3 \times \sqrt{3}$

d $81^{\frac{3}{2}}$

e $3^4 \times 9^{-1}$

f $9^n \times 27^{3n}$

3 Write each of these numbers as a power of 2, as simply as possible.

a 32

b $8^{\frac{2}{3}}$

c $2 \times \sqrt[3]{64}$

d 0.25

e $2^{2n} \times 4^{\frac{n}{2}}$

f $2^{3n} \times 16^{-2}$

4 Write each of these numbers as a power of 5, as simply as possible.

a 625

b $25^{-\frac{1}{2}} \times 5^3$

c 0.2

d $125^{\frac{3}{2}} \times 5^{-3} \div 25^2$

e $5^4 \div 5^n$

f $25^{3n} \times 125^{\frac{n}{3}}$

5 Write each of these in the form $2^a \times 3^b$.

a 24

b $6^2 \times 4^2$

c $18^{\frac{1}{3}}$

d $\frac{4}{9}$

e $13\frac{1}{2}$

f 12^{2n}

6 Write each of these expressions as a power of x , as simply as possible.

a $x^3 \times x^2$

b $x^n \times x^n$

c $x^{\frac{3}{2}} \div x^{-\frac{1}{2}}$

d $\left(x^{\frac{1}{2}}\right)^5$

e $\frac{x^4 \times x \times x^{\frac{1}{2}}}{\sqrt[3]{x^2}}$

f $\sqrt{\frac{\sqrt{x} \times x^3}{x}}$

7 Solve these equations.

a $2^x = 16$

b $64^x = 4$

c $3^x = 81$

d $\left(\frac{1}{9}\right)^x = 3$

e $2^x = \frac{1}{8}$

f $16^x = \frac{1}{4}$

Note

Solve means find the value of x that makes the statement true.

8 Simplify these.

a $2a^{-3} \times 5a^2$

b $\frac{3}{4}a^5 \times \frac{2}{3}a^{-2}$

c $\frac{3}{10}a^{\frac{1}{2}} \times 5a^{-1}$

d $\frac{4}{9}a^{-5} \times 6a^{\frac{3}{2}}$

9 Simplify these.

a $\left(\frac{9ab^2}{a^3}\right)^{\frac{1}{2}}$

b $\left(\frac{a^5b^2}{27a^2b^8}\right)^{\frac{1}{3}}$

c $\left(\frac{5ab^2}{15a^2b}\right)^{-2}$

d $\left(\frac{75a^8b}{3a^2b^5}\right)^{-\frac{1}{2}}$

10 Simplify.

a $4y^2 \div 2y^{-2}$

b $\frac{2}{5}y^{-3} \div \frac{1}{5}y^4$

c $5y^{\frac{1}{2}} \div 10y^{-1}$

d $\frac{3}{4}y^{-2} \div \frac{3}{5}y^{\frac{1}{2}}$

20 SOLUTION OF EQUATIONS AND INEQUALITIES

Solving simple linear equations

When you solve an equation, you always do the same thing to both sides of the equation.

Here is a reminder about solving simple linear equations.

Example 20.1

Question

Solve $5x + 2 = 12$.

Solution

$$5x + 2 = 12$$

$$5x = 10$$

$$x = 2$$

Subtract 2 from both sides.

Divide both sides by 5.

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- solve simple linear equations in one unknown
- solve fractional equations with numerical and linear algebraic denominators
- solve simultaneous linear equations in two unknowns
- solve quadratic equations by factorisation, completing the square or by use of the formula
- solve simple linear inequalities.

Exercise 20.1

Solve these equations.

1 $3x + 4 = 16$

2 $5x - 4 = 16$

3 $7x - 3 = 18$

4 $9a + 4 = 40$

5 $8y + 3 = 27$

Solving equations with a bracket

Sometimes the equation formed to solve a problem will involve a bracket.

Example 20.2

Question

Solve $4(x - 5) = 18$.

Solution

By expanding the bracket, you will have an equation like that in Example 20.1.

Method 1

$$4(x - 5) = 18$$

$$4x - 20 = 18$$

$$4x = 38$$

$$x = 9\frac{1}{2}$$

Expand the brackets.

Add 20 to both sides.

Divide both sides by 4.

Note

You may find it easier to use Method 1.

CHECK YOU CAN:

- add and subtract, multiply and divide numerical fractions
- simplify algebraic expressions by collecting like terms
- multiply out expressions such as $3(2x - 5)$
- simplify algebraic fractions
- factorise quadratic expressions when this is possible.

Method 2

$$4(x - 5) = 18$$

$x - 5 = 4\frac{1}{2}$ Divide both sides by 4.

$x = 9\frac{1}{2}$ Add 5 to both sides.

Note

Method 2 is usually shorter.

Exercise 20.2

Solve these equations.

1 $2(x + 1) = 10$

4 $2(x + 4) = 8$

7 $2(x - 7) = 3$

10 $2(5 + 2x) = 17$

13 $2(4x + 7) = 12$

16 $5(3x - 1) = 40$

19 $3(5x - 12) = 24$

2 $4(x - 1) = 12$

5 $2(x - 4) = 8$

8 $2(2x + 3) = 18$

11 $4(3x - 1) = 20$

14 $5(x - 6) = 20$

17 $2(5x - 3) = 14$

20 $2(2x - 5) = 12$

3 $5(x + 6) = 20$

6 $3(x + 7) = 9$

9 $5(x - 1) = 12$

12 $3(5x - 13) = 21$

15 $7(x + 3) = 28$

18 $7(x - 4) = 28$

Solving equations with the unknown on both sides

Sometimes the equation formed to solve a problem will have the unknown on both sides.

Example 20.3**Question**

Solve $2(3x - 1) = 3(x - 2)$.

Solution

$$2(3x - 1) = 3(x - 2)$$

$6x - 2 = 3x - 6$ Expand the brackets.

$6x = 3x - 4$ Add 2 to both sides.

$3x = -4$ Subtract $3x$ from both sides.

$x = -1\frac{1}{3}$ Divide both sides by 3.

Exercise 20.3

Solve these equations.

1 $2x - 1 = x + 3$

4 $4x - 1 = 3x + 7$

7 $5(2x - 1) = 3x + 9$

10 $2x - 3 = 7 - 3x$

13 $x - 5 = 2x - 9$

16 $49 - 3x = x + 21$

19 $3(4x - 1) = 5(2x + 3)$

2 $5x - 6 = 3x$

5 $4x - 3 = x$

8 $2(x - 1) = x + 2$

11 $4x - 1 = 2 + x$

14 $3x - 4 = 2 - 3x$

17 $3(x - 1) = 2(x + 1)$

20 $7(x - 2) = 3(2x - 7)$

3 $4x + 1 = x - 8$

6 $2(x + 3) = x + 7$

9 $2(2x + 3) = 3x - 7$

12 $3x - 2 = x + 7$

15 $5x - 6 = 16 - 6x$

18 $3(4x - 3) = 10x - 1$

Solving equations involving fractions

You may have to solve equations involving fractions.

The first step is to eliminate the fraction or fractions.

Example 20.4

Question

Solve $\frac{x}{3} = 2x - 3$.

Solution

$$\begin{aligned}\frac{x}{3} &= 2x - 3 && \text{Multiply both sides by 3.} \\ x &= 3(2x - 3) && \text{Expand the bracket.} \\ x &= 6x - 9 && \text{Add 9 to both sides.} \\ x + 9 &= 6x && \text{Subtract } x \text{ from both sides.} \\ 9 &= 5x\end{aligned}$$

Collecting the x -terms on the right-hand side of the equation makes the final x -term positive.

$$x = \frac{9}{5} \text{ or } 1\frac{4}{5} \text{ or } 1.8 \quad \text{Divide both sides by 5.}$$

Note

A common error when multiplying through by a number or letter is to multiply just the first term. Use a bracket to make sure.

Note

Another common error in examples like this would be to give the answer as $\frac{5}{9}$ rather than $\frac{9}{5}$.

Example 20.5

Question

Solve the equation $\frac{400}{x} = 8$.

Solution

$$\begin{aligned}\frac{400}{x} &= 8 && \text{Multiply both sides by } x. \\ 400 &= 8x && \text{Divide both sides by 8.} \\ x &= 50\end{aligned}$$

Note

It is much easier to collect all the x -terms on to the side where the final x -term will be positive.

Example 20.6

Question

Solve $\frac{x}{3} + \frac{x}{2} = 5$.

Solution

$$\begin{aligned}\frac{x}{3} + \frac{x}{2} &= 5 && \text{Multiply every term by 6 because 6 is the common denominator of } \frac{x}{3} \text{ and } \frac{x}{2}. \\ \frac{6x}{3} + \frac{6x}{2} &= 5 \times 6 \\ 2x + 3x &= 30 && \text{Cancel the common factor in the numerator and denominator of each fraction.} \\ 5x &= 30 && \text{This will eliminate both fractions.} \\ x &= 6 && \text{Divide both sides by 5.}\end{aligned}$$

Note

It is much easier to multiply both sides of the equation by a number that will eliminate the fraction than to expand the bracket.

Exercise 20.4

Solve these equations.

$$1 \quad \frac{x}{2} = 3x - 10$$

$$4 \quad \frac{3x}{5} = 4 - x$$

$$7 \quad \frac{2x}{7} - 2 = 5$$

$$10 \quad \frac{39}{2x} = 3$$

$$13 \quad \frac{5x - 12}{2} = x$$

$$16 \quad \frac{7}{2x} - 2 = 3$$

$$19 \quad \frac{3x}{2} - \frac{x}{6} = 12$$

$$2 \quad \frac{2x}{3} = x - 2$$

$$5 \quad \frac{5x}{3} = 4x - 2$$

$$8 \quad \frac{300}{x} = 15$$

$$11 \quad \frac{1}{3}(x - 2) = 7$$

$$14 \quad \frac{5}{x} + 3 = 7$$

$$17 \quad \frac{5}{3x} + 7 = 10$$

$$20 \quad \frac{x}{2} - 4 = \frac{x}{3}$$

$$3 \quad \frac{2x}{5} = x - 3$$

$$6 \quad \frac{x}{3} + 5 = 9$$

$$9 \quad \frac{200}{x} = 4$$

$$12 \quad \frac{3x - 1}{5} = 4$$

$$15 \quad \frac{6}{x} - 5 = -1$$

$$18 \quad \frac{x}{2} - \frac{x}{4} = 3$$

Solving more complex equations involving fractions

The following example illustrates how to solve more complex equations involving fractions.

Example 20.7

Question

Solve $\frac{x-2}{4} - \frac{x+3}{5} = 1$.

Solution

$$\frac{x-2}{4} - \frac{x+3}{5} = 1$$

The common denominator of 4 and 5 is 20.

$$\frac{20(x-2)}{4} - \frac{20(x+3)}{5} = 20$$

Multiply every term by the common denominator.

$$\cancel{^5} \frac{20(x-2)}{\cancel{4}_1} - \cancel{^4} \frac{20(x+3)}{\cancel{5}_1} = 20$$

Cancel the common factors in the numerator and denominator of each fraction.

$$5(x-2) - 4(x+3) = 20$$

This will eliminate the fractions.

$$5x - 10 - 4x - 12 = 20$$

Expand the bracket. Take care with the signs.

$$x - 22 = 20$$

Collect like terms.

$$x = 42$$

Add 22 to both sides.

Note

The most common error is in the signs when the left-hand side is a subtraction. Make sure you put the brackets in.

Exercise 20.5

Solve these equations.

$$\text{1} \quad \frac{x+2}{3} + \frac{x-3}{2} = 5$$

$$\text{2} \quad \frac{3x}{2} = \frac{3x-2}{5} + 4$$

$$\text{3} \quad \frac{2x-1}{5} - \frac{x-3}{3} = \frac{5}{2}$$

$$\text{4} \quad \frac{2x-1}{6} = \frac{x-3}{2} + \frac{2}{3}$$

$$\text{5} \quad \frac{x-2}{3} + \frac{2x-1}{2} = \frac{17}{6}$$

$$\text{6} \quad \frac{1}{3}(x+2) - \frac{1}{2}(x-3) = 2$$

$$\text{7} \quad \frac{2x-3}{3} - \frac{2x+1}{6} + \frac{3}{2} = 0$$

$$\text{8} \quad \frac{3x-2}{2} = \frac{x-3}{3} + \frac{7}{6}$$

$$\text{9} \quad \frac{x-1}{3} - \frac{2x}{5} = \frac{3x-1}{5}$$

$$\text{10} \quad \frac{1}{5}(x+2) - \frac{1}{3}(x-5) = 2$$

$$\text{11} \quad \frac{1}{2}(x-1) - \frac{1}{5}(x-3) = 1$$

$$\text{12} \quad \frac{2x-1}{6} + \frac{3x}{4} = \frac{x-2}{12}$$

$$\text{13} \quad \frac{1}{7}(x-5) + \frac{1}{2}(x-8) = 1$$

$$\text{14} \quad \frac{x-3}{2} + \frac{2x+3}{5} = \frac{4x-5}{4}$$

$$\text{15} \quad \frac{1}{5}(x+3) - \frac{1}{2}(2x-1) = \frac{1}{3}(11-x)$$

Solving inequalities

$a < b$ means ‘ a is less than b ’

$a \leq b$ means ‘ a is less than or equal to b ’

$a > b$ means ‘ a is greater than b ’

$a \geq b$ means ‘ a is greater than or equal to b ’

Expressions involving these signs are called **inequalities**.

Example 20.8

Question

Find the integer values of x that satisfy each of these inequalities.

a $-3 < x \leq -1$ b $1 \leq x < 4$

Solution

a If $-3 < x \leq -1$, then $x = -2$ or -1 . Note that -3 is not included but -1 is.

b If $1 \leq x < 4$, then $x = 1, 2$ or 3 . Note that 1 is included but 4 is not.

In equations, if you always do the same thing to both sides the equation is still valid.

The same is usually true for inequalities, but there is one important exception.

If you multiply or divide an inequality by a negative number, you must reverse the inequality sign.

For example, $5 < 7$ but, if you multiply by -2 , -10 is greater than -14 (see number line).

Otherwise inequalities can be treated in the same way as equations.

Example 20.9

Question

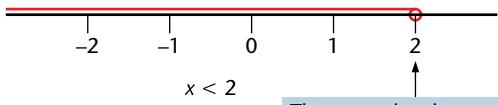
Solve each of these inequalities and show the solution on a number line.

- a $3x + 4 < 10$
- b $2x - 5 \leq 4 - 3x$
- c $x + 4 < 3x - 2$

Solution

a $3x + 4 < 10$

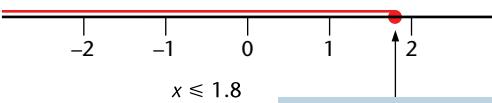
$$\begin{aligned} 3x &< 6 && \text{Subtract 4 from both sides.} \\ x &< 2 && \text{Divide both sides by 3.} \end{aligned}$$



The open dot shows that 2 is not included.

b $2x - 5 \leq 4 - 3x$

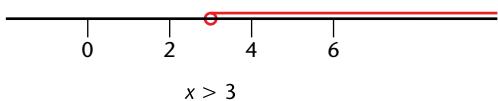
$$\begin{aligned} 2x &\leq 9 - 3x && \text{Add 5 to both sides.} \\ 5x &\leq 9 && \text{Add } 3x \text{ to both sides.} \\ x &\leq 1.8 && \text{Divide both sides by 5.} \end{aligned}$$



The closed dot shows that 1.8 is included.

c $x + 4 < 3x - 2$

$$\begin{aligned} x &< 3x - 6 && \text{Subtract 4 from both sides.} \\ -2x &< -6 && \text{Subtract } 3x \text{ from both sides.} \\ x &> 3 && \text{Divide both sides by } -2 \text{ and change } < \text{ to } > \text{ (when dividing by } -2\text{).} \end{aligned}$$



Example 20.10

Question

Solve this inequality.

$$5 < 2x - 1 \leq 10$$

Solution

$$5 < 2x - 1 \leq 10$$

$$6 < 2x \leq 11$$

$$3 < x \leq 5.5$$

Add 1 to each part of the inequality.

Divide each part of the inequality by 2.

Exercise 20.6

- 1 Write down two values of x that satisfy the inequality $x < -2$.
- 2 Write down the integer values of x that satisfy each of these inequalities.

a $-4 \leq x < 0$	b $1 < x \leq 5$
c $1 < x \leq 4$	d $-5 < x \leq -1$

Solve the inequalities in questions 3 to 29.

For questions 3 to 7, represent your solution on a number line.

- | | | |
|--|----------------------------------|-------------------------------------|
| 3 $x - 3 \leq 4$ | 4 $2x - 3 < 5$ | 5 $3x + 4 \leq 7$ |
| 6 $2x + 3 > 6$ | 7 $3x - 4 \leq 8$ | 8 $2x \geq x + 5$ |
| 9 $5x > 3 - x$ | 10 $4x > 2x + 5$ | 11 $3x - 6 \geq x + 2$ |
| 12 $5a - 3 > 2a$ | 13 $2x - 3 < x + 1$ | 14 $3x + 7 < x + 3$ |
| 15 $8x - 10 > 3x + 25$ | 16 $6x + 11 \leq 18 - x$ | 17 $2x + 9 > 4x + 5$ |
| 18 $3(2x - 1) \geq 11 - x$ | 19 $x + 4 > 2x$ | 20 $2x - 5 < 4x + 1$ |
| 21 $3(x - 4) > 5(x + 1)$ | 22 $x - 2 < 2x + 4$ | 23 $2x - 1 > x - 4$ |
| 24 $3(x + 3) \geq 2x - 1$ | 25 $3(2x - 4) < 5(x - 6)$ | 26 $\frac{2x + 7}{3} \leq 5$ |
| 27 $\frac{x + 5}{2} > \frac{2x - 1}{3}$ | 28 $6 < 3x < 15$ | 29 $-2 < 5x + 3 \leq 15$ |

- 30** Find the integer values of x that satisfy each of these inequalities.

a $3 \leq 2x - 1 \leq 5$ **b** $-4 \leq 3x + 2 \leq 11$

Solving linear simultaneous equations

An equation in two unknowns does not have a unique solution.

For example, the graph of the equation $x + y = 4$ is a straight line. Every point on the line will have coordinates that satisfy the equation.

When you are given two equations in two unknowns, such as x and y , they usually have a common solution where the two lines meet at a point.

These are called **simultaneous equations**.

It is possible to solve simultaneous equations graphically. However, it can be time-consuming and you do not always obtain an accurate solution.

It is often preferable, therefore, to solve them accurately using algebra.

If two quantities A and B are equal and two other quantities C and D are equal, it follows logically that

$$A + C = B + D \text{ and}$$

$$A - C = B - D$$

Therefore you can add the left-hand sides of two simultaneous equations and the result will equal the two right-hand sides added together.

Similarly you can subtract the left-hand sides of two simultaneous equations and the result will equal the two right-hand sides subtracted.

This is shown in the following examples.

Example 20.11

Question

Solve the simultaneous equations $2x + 5y = 9$ and $2x - y = 3$.

Solution

$$2x + 5y = 9 \quad (1) \quad \text{Write the two equations, one under the other, and label them.}$$

$$2x - y = 3 \quad (2)$$

This time $(+)$ $2x$ appears in each equation, so subtract to eliminate the x -terms.

$$2x - 2x + 5y - (-y) = 9 - 3 \quad (1) - (2). \text{ Take care with the signs.}$$

$$\begin{aligned} 6y &= 6 \\ y &= 1 \end{aligned}$$

$$2x + 5 = 9 \quad \text{Substitute } y = 1 \text{ in (1). } 5y \text{ is replaced by } 5 \times 1 = 5.$$

$$2x = 4$$

$$x = 2$$

The solution is $x = 2, y = 1$.

Check in equation (2): the left-hand side is $2x - y = 4 - 1 = 3$ which is correct.

Note

When eliminating, if the signs of the letter to be eliminated are the same, subtract. If they are different, add.

Note

When subtracting, take great care with the signs. If your check is wrong, see if you have made an error with any signs.

Example 20.12

Question

Solve the simultaneous equations $x + 3y = 10$ and $3x + 2y = 16$.

Solution

$$x + 3y = 10 \quad (1) \quad \text{Write the two equations, one under the other, and label them.}$$

$$3x + 2y = 16 \quad (2)$$

The coefficients of x and y are different in the two equations.

Multiply (1) by 3 to make the coefficient of x the same as in equation (2).

$$3x + 9y = 30 \quad (3) \quad (1) \times 3$$

$$3x + 2y = 16 \quad (2)$$

Now $(+)$ $3x$ appears in both equations, so subtract.

$$3x - 3x + 9y - 2y = 30 - 16 \quad (3) - (2)$$

$$\begin{aligned} 7y &= 14 \\ y &= 2 \end{aligned}$$

$$x + 6 = 10 \quad \text{Substitute } y = 2 \text{ in (1).}$$

$$x = 4$$

The solution is $x = 4, y = 2$.

Check in equation (2): the left-hand side is $3x + 2y = 12 + 4 = 16$ which is correct.

Note

When subtracting equations, you can do equation (1) – equation (2) or equation (2) – equation (1). It is better to make the letter positive. Always write down clearly what you are doing.

Exercise 20.7

Solve these simultaneous equations.

1 $x + y = 5$
 $2x - y = 7$

4 $2x - y = 7$
 $3x + y = 13$

7 $3x - y = 11$
 $3x - 5y = 7$

10 $2x - 3y = 0$
 $3x + y = 11$

13 $3x + 2y = 13$
 $x + 3y = 16$

16 $x + y = 4$
 $4x - 2y = 7$

19 $2x - 3y = 9$
 $3x + y = 8$

2 $3x + y = 9$
 $2x + y = 7$

5 $2x + y = 7$
 $4x - y = 5$

8 $2x + y = 6$
 $3x + 2y = 10$

11 $x + 2y = 19$
 $3x - y = 8$

14 $2x + 3y = 7$
 $3x - y = 5$

17 $2x + 4y = 11$
 $x + 3y = 8$

20 $2x - y = -4$
 $x + 3y = -9$

3 $2x + y = 6$
 $2x - y = 2$

6 $2x + 3y = 13$
 $x - y = 4$

9 $3x + y = 7$
 $2x + 3y = 7$

12 $x + 2y = 6$
 $2x - 3y = 5$

15 $2x + y = 3$
 $3x - 2y = 8$

18 $2x + 2y = 7$
 $4x - 3y = 7$

21 $x + y = 0$
 $2x + 4y = 3$

Solving more complex simultaneous equations

Sometimes the letters in the equations are not in the same order, so the first thing to do is to rearrange them.

Sometimes each of the equations needs to be multiplied by a different number.

Example 20.13

Question

Solve simultaneously the equations $3y = 4 - 4x$ and $6x + 2y = 11$.

Solution

$$4x + 3y = 4 \quad (1) \quad \text{Rearrange the first equation.}$$

$$6x + 2y = 11 \quad (2)$$

To eliminate x multiply (1) by 3 and (2) by 2 and subtract, or to eliminate y multiply (1) by 2 and (2) by 3 and subtract.

$$12x + 9y = 12 \quad (3) \quad (1) \times 3$$

$$12x + 4y = 22 \quad (4) \quad (2) \times 2$$

$$5y = -10 \quad (3) - (4) \quad \text{Eliminate } x.$$

$$y = -2$$

$$4x - 6 = 4 \quad \text{Substitute in (1). } 3y \text{ is replaced by } -6.$$

$$4x = 10$$

$$x = \frac{10}{4} = \frac{5}{2} = 2\frac{1}{2}$$

The solution is $x = 2\frac{1}{2}$, $y = -2$.

Check in equation (2): LHS = $6x + 2y = 15 - 4 = 11$ which is correct.

Note

If the equations are not already in the form $ax + by = c$, rearrange them so that they are.

Solving simultaneous equations using substitution

If x or y is the subject of one of the equations, there is an easier method to eliminate one of them.

This method is called substitution and is shown in Example 20.14.

Example 20.14

Question

Solve simultaneously the equations $3x - 2y = 6$ and $y = 2x - 5$.

Solution

Since y is the subject of the second equation we can substitute it in the first.

$$3x - 2y = 6 \quad (1)$$

$$y = 2x - 5 \quad (2)$$

$3x - 2(2x - 5) = 6$ Substitute (2) in (1). Replace y in equation (1) by $2x - 5$.

$3x - 2(2x - 5) = 6$ Solve the resulting equation.

$3x - 4x + 10 = 6$ Expand the bracket.

$$-x = -4$$

$$x = 4$$

$y = 2 \times 4 - 5$ Now substitute $x = 4$ into equation (2).

$$y = 3$$

So the solution is $x = 4$, $y = 3$.

Note

Always substitute the answer you have found first into the equation with x or y as the subject to find the other value.

The most common error is to forget the brackets when making the substitution.

Note

If there is a choice whether to add or subtract, it is usually easier to add.

Exercise 20.8

Solve these simultaneous equations

1 $y = 2x - 1$
 $x + 2y = 8$

4 $y = 3x - 3$
 $2x + 3y = 13$

7 $3x - 2y = 11$
 $2x + 3y = 16$

10 $4x - 3y = 1$
 $5x + 2y = -16$

13 $4x - 2y = 3$
 $5y = 23 - 3x$

2 $y = 3 - 2x$
 $3x - 3y = 0$

5 $3x + 2y = 7$
 $2x - 3y = -4$

8 $2x - 3y = 5$
 $3x + 4y = 16$

11 $4x + 3y = 1$
 $3x + 2y = 0$

14 $x = 2y + 9$
 $3x - 2y = 19$

3 $5y = x + 1$
 $2x + 2y = 10$

6 $3x - 2y = 3$
 $2x - y = 4$

9 $3x + 4y = 5$
 $2x + 3y = 4$

12 $y = x + 2$
 $2x - 4y = -9$

15 $4x + 2y = 18$
 $y = 3x - 4$

Solving quadratic equations of the form $x^2 + bx + c = 0$ by factorisation

For any two numbers, if $A \times B = 0$, then either $A = 0$ or $B = 0$.

If $(x - 3)(x - 2) = 0$ then either $(x - 3) = 0$ or $(x - 2) = 0$.

To solve a quadratic equation, factorise it into two brackets and then use this fact.

Remember, to factorise $x^2 + bx + c$:

- if c is positive, you find two numbers with the same sign as b that multiply to give c and add to give b
- if c is negative, you find two numbers with different signs that multiply to give c and add to give b .

If an equation is written as $x^2 + bx = c$ or $x^2 = bx + c$, first rearrange it so that all the terms are on one side, leaving zero on the other side.

Example 20.15

Question

Solve the equation $x^2 = 4x - 4$ by factorisation.

Solution

$x^2 - 4x + 4 = 0$ Rearrange so that all the terms are on the same side.

$(x - 2)(x - 2) = 0$ Factorising: c is positive and b is negative, $-2 \times -2 = 4$, $-2 + -2 = -4$.

$x - 2 = 0$ or $x - 2 = 0$

The solution is $x = 2$ (repeated).

Note

There are always two solutions, so if they are both the same write 'repeated'.

Example 20.16

Question

Solve the equation $x^2 + 5x = 0$.

Solution

$x(x + 5) = 0$ Factorising with x as a common factor.

$x = 0$ or $x + 5 = 0$

The solution is $x = 0$ or $x = -5$.

Example 20.17**Question**

Solve the equation $x^2 - 49 = 0$.

Solution

$$(x + 7)(x - 7) = 0$$

Factorising by 'difference of two squares'.

$$x + 7 = 0 \text{ or } x - 7 = 0$$

The solution is $x = -7$ or $x = 7$. This can be written as $x = \pm 7$.

You can use an alternative method for equations like this.

$$x^2 - 49 = 0$$

$$x^2 = 49$$

Add 49 to both sides.

$$x = \pm 7$$

Take the square root of both sides, remembering that this can give 7 or -7.

This method is perhaps simpler but it is easy to forget the negative solution.

Exercise 20.9

Solve these equations by factorisation.

$$\mathbf{1} \quad x^2 - 5x + 6 = 0$$

$$\mathbf{4} \quad x^2 - 100 = 0$$

$$\mathbf{7} \quad x^2 + 9x + 20 = 0$$

$$\mathbf{10} \quad x^2 - 7x + 6 = 0$$

$$\mathbf{13} \quad x^2 + 7x + 12 = 0$$

$$\mathbf{16} \quad x^2 - 10x + 24 = 0$$

$$\mathbf{19} \quad x^2 - 225 = 0$$

$$\mathbf{22} \quad x^2 = 10x$$

$$\mathbf{25} \quad x^2 = x$$

$$\mathbf{28} \quad x^2 - 9x = 10$$

$$\mathbf{2} \quad x^2 - 6x + 5 = 0$$

$$\mathbf{5} \quad x^2 + 6x + 8 = 0$$

$$\mathbf{8} \quad x^2 - 25 = 0$$

$$\mathbf{11} \quad x^2 - 9x + 18 = 0$$

$$\mathbf{14} \quad x^2 + 3x = 0$$

$$\mathbf{17} \quad x^2 - 6x + 8 = 0$$

$$\mathbf{20} \quad x^2 + 2x - 3 = 0$$

$$\mathbf{23} \quad x^2 - x - 12 = 0$$

$$\mathbf{26} \quad x^2 - 2x - 15 = 0$$

$$\mathbf{29} \quad x^2 - 17x + 30 = 0$$

$$\mathbf{3} \quad x^2 - 4x + 3 = 0$$

$$\mathbf{6} \quad x^2 + 5x + 4 = 0$$

$$\mathbf{9} \quad x^2 + 2x + 1 = 0$$

$$\mathbf{12} \quad x^2 - 8x = 0$$

$$\mathbf{15} \quad x^2 + 6x = 0$$

$$\mathbf{18} \quad x^2 - 169 = 0$$

$$\mathbf{21} \quad x^2 + 4x - 5 = 0$$

$$\mathbf{24} \quad x^2 + 5x - 6 = 0$$

$$\mathbf{27} \quad x^2 - 3x = 18$$

$$\mathbf{30} \quad x^2 + 4x = 32$$

Solving quadratic equations of the form $ax^2 + bx + c = 0$ by factorisation

The following example has a question where the coefficient of x^2 is not 1.

Example 20.18**Question**

Solve the equation $3x^2 + 11x + 6 = 0$.

Solution

$$3x^2 + 11x + 6 = 0$$

$$(3x + 2)(x + 3) = 0$$

$$(3x + 2) = 0 \quad \text{or} \quad (x + 3) = 0$$

$$3x = -2 \quad \text{or} \quad x = -3$$

$$x = -\frac{2}{3} \quad \text{or} \quad x = -3$$

Exercise 20.10

Solve these equations.

- | | | |
|--------------------------------|--------------------------------|---------------------------------|
| 1 $2x^2 - 5x - 12 = 0$ | 2 $3x^2 - 10x - 8 = 0$ | 3 $2x^2 + 5x + 3 = 0$ |
| 4 $2x^2 - 3x - 5 = 0$ | 5 $3x^2 + 2x - 1 = 0$ | 6 $2x^2 + 11x + 5 = 0$ |
| 7 $2x^2 - 13x + 15 = 0$ | 8 $12x^2 + 10x - 8 = 0$ | 9 $2x^2 + 2x - 60 = 0$ |
| 10 $3x^2 - 12x + 9 = 0$ | 11 $2x^2 - 2x - 12 = 0$ | 12 $3x^2 - 14x - 24 = 0$ |
| 13 $2x^2 - 8 = 0$ | 14 $3x^2 - 27 = 0$ | 15 $5x^2 - 125 = 0$ |

Solving quadratic equations by completing the square

This quadratic equation factorises.

$$x^2 - 4x + 3 = 0$$

This one does not.

$$x^2 - 4x + 1 = 0$$

$x^2 - 4x$ is part of the expansion $(x - 2)^2 = x^2 - 4x + 4$.

In this method for solving a quadratic equation you ‘complete the square’ so that the left-hand side of the equation is a square $(mx + k)^2$.

When the coefficient of x^2 is 1,

$$(x + k)^2 = x^2 + 2kx + k^2$$

so to find the number to add to complete the square you halve the coefficient of x and square it.

Example 20.19

Question

Solve $x^2 - 4x + 1 = 0$.

Give your answer correct to 2 decimal places.

Solution

$$x^2 - 4x + 1 = 0$$

$$x^2 - 4x + 1 + 3 = 0 + 3$$

$$x^2 - 4x + 4 = 3$$

$$(x - 2)^2 = 3$$

$$x - 2 = \sqrt{3} \text{ or } -\sqrt{3}$$

$$x = 2 + \sqrt{3} \text{ or } 2 - \sqrt{3}$$

$$x = 3.73 \text{ or } 0.27 \text{ to 2 decimal places}$$

Note

Look carefully at the question. If it asks for an exact answer then you must give $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

If it asks for the answer to a given number of decimal places or significant figures then you must give the decimal correctly rounded.

Completing the square may also be used to write quadratic expressions in completed square form.

For example,

$$x^2 - 4x + 1 = (x - 2)^2 - 3.$$

Since the least value that a square can be is zero, this gives the information that the least value of the expression is -3 and this occurs when $x = 2$.

Example 20.20

Question

Write these quadratic expressions in completed square form, $a(x + b)^2 + c$.

- a** $x^2 + 6x - 3$
- b** $3x^2 + 15x + 14$

Solution

a $x^2 + 6x - 3 = (x + 3)^2 - 9 - 3$ $x^2 + 6x$ is part of the expansion of $(x + 3)^2$ but without the $+9$ term, so subtract 9.

$$= (x + 3)^2 - 12$$

b $3x^2 + 15x + 14$ Take out a factor of 3 from the terms involving x .

$$= 3(x^2 + 5x) + 14$$

$$= 3\left(\left(x + \frac{5}{2}\right)^2 - \frac{25}{4}\right) + 14$$

Recognise that $(x^2 + 5x)$ is part of the expansion of $\left(x + \frac{5}{2}\right)^2$ but without the term $\left(\frac{5}{2}\right)^2$.

Remember that you need to multiply the $-\frac{25}{4}$ term by 3.

$$= 3\left(x + \frac{5}{2}\right)^2 - \frac{75}{4} + 14$$

$$= 3\left(x + \frac{5}{2}\right)^2 - \frac{19}{4}$$

Exercise 20.11

Solve the equations in questions **1** to **14** by completing the square.

- | | |
|-------------------------------|---------------------------------|
| 1 $x^2 - 6x + 4 = 0$ | 2 $x^2 + 10x + 5 = 0$ |
| 3 $x^2 - 7x + 2 = 0$ | 4 $x^2 + 6x - 4 = 0$ |
| 5 $x^2 - 10x + 6 = 0$ | 6 $x^2 - 5x + 2 = 0$ |
| 7 $4x^2 - 6x + 1 = 0$ | 8 $3x^2 - 4x - 5 = 0$ |
| 9 $2x^2 + 12x + 3 = 0$ | 10 $3x^2 + 2x - 2 = 0$ |
| 11 $5x^2 - 6x - 4 = 0$ | 12 $2x^2 - 5x - 2 = 0$ |
| 13 $5x^2 - x - 1 = 0$ | 14 $16x^2 + 12x + 1 = 0$ |

- 15 a** Write $x^2 + 12x + 12$ in the form $(x + m)^2 - n$.
b Hence state the minimum value of $y = x^2 + 12x + 12$.
c Using your answer to part **a**, or otherwise, solve the equation $x^2 + 12x + 12 = 0$.
 Give your answer in the form $p \pm q\sqrt{6}$.
- 16** Write these quadratic expressions in the form $a(x + b)^2 + c$.
a $2x^2 + 8x + 5$ **b** $5x^2 + 5x - 2$ **c** $2x^2 + 18x + 7$

Solving quadratic equations using the quadratic formula

Using the method of completing the square it may be shown that

$$\text{the equation } ax^2 + bx + c = 0 \text{ has roots } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 20.21

Question

Solve the equation $3x^2 + 4x - 2 = 0$.

Give your answers to 2 decimal places.

Solution

In the equation $3x^2 + 4x - 2 = 0$, $a = 3$, $b = 4$, $c = -2$.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-4 \pm \sqrt{16 - 4 \times 3 \times (-2)}}{2 \times 3} \\ &= \frac{-4 \pm \sqrt{16 + 24}}{6} \\ &= \frac{-4 \pm \sqrt{40}}{6} \\ &= \frac{-4 + 6.324...}{6} \\ &= \frac{-4 + 6.324}{6} \text{ or } \frac{-4 - 6.324}{6} \end{aligned}$$

$$x = 0.39 \text{ or } -1.72 \text{ to 2 decimal places}$$

Note

The main errors that occur in using the formula are

- errors with the signs, especially with $-4ac$
- failure to divide the whole expression by $2a$.

Exercise 20.12

Use the formula to solve the equations in questions 1 to 12.

Give your answers correct to 2 decimal places.

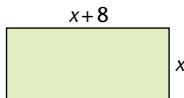
1 $x^2 + 8x + 6 = 0$ 2 $2x^2 - 2x - 3 = 0$ 3 $3x^2 + 5x - 1 = 0$

4 $5x^2 - 12x + 5 = 0$ 5 $5x^2 + 9x - 6 = 0$ 6 $x^2 - 5x - 1 = 0$

7 $3x^2 + 9x + 5 = 0$ 8 $x^2 + 7x + 4 = 0$ 9 $2x^2 - 3x - 4 = 0$

10 $3x^2 + 2x - 2 = 0$ 11 $5x^2 - 13x + 7 = 0$ 12 $5x^2 + 9x + 3 = 0$

- 13 A garden is 8 m longer than it is wide and it has an area of 25 m².



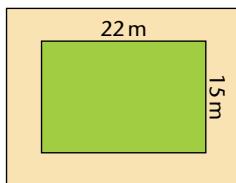
Write down an equation and solve it to find the dimensions correct to the nearest centimetre.

- 14 A rectangular pen is constructed along an existing wall using 20 m of fencing.



- a The width of the pen is x m.
Write an expression for the area enclosed.
- b Write an equation and solve it to find the dimensions to give an area of 40 m².
- c By completing the square, find the maximum possible area of a pen constructed with 20 m of fencing.

- 15 A rectangular lawn measuring 22 m by 15 m is surrounded by a path x m wide.



- a Form and simplify an expression for the total area of lawn and path.
- b Write an equation and solve it to find the width of the path correct to the nearest centimetre when the total area is 400 m².

Solving equations involving algebraic fractions

You need to be able to solve equations involving fractions with algebraic expressions in the denominator.

You deal with these in the same way as you dealt with fractions with numerical denominators.

You multiply every term by the common denominator to eliminate the fractions.

The resulting equation may be linear or quadratic.

Example 20.22

Question

$$\text{Solve } \frac{3}{x+1} - \frac{2}{x} = \frac{1}{x-2}.$$

Solution

$$\frac{3}{x+1} - \frac{2}{x} = \frac{1}{x-2}$$

The common denominator of $x+1$, x and $x-2$ is $x(x+1)(x-2)$.

$$3x(x-2) - 2(x+1)(x-2) = x(x+1)$$

Multiply by $x(x+1)(x-2)$.

Make sure you multiply every expression on both sides.

$$3x^2 - 6x - 2(x^2 - x - 2) = x^2 + x$$

$$3x^2 - 6x - 2x^2 + 2x + 4 - x^2 - x = 0$$

Expand the bracket and collect on one side.

$$-5x + 4 = 0$$

Collect like terms.

$$4 = 5x$$

$$x = \frac{4}{5}$$

Note

In equations with algebraic fractions, multiply through by the common denominator. This eliminates the fractions.

Exercise 20.13

Solve these.

$$1 \quad \frac{2x}{3} - \frac{3x}{5} = \frac{1}{3}$$

$$2 \quad \frac{x-1}{2} - \frac{x-3}{5} = 1$$

$$3 \quad \frac{2x-1}{6} + \frac{3x}{4} = \frac{x-2}{12}$$

$$4 \quad \frac{x-1}{3} + \frac{2x}{5} = \frac{3x+1}{5}$$

$$5 \quad \frac{3}{x} - \frac{2}{x+1} = 0$$

$$6 \quad \frac{5}{x} - \frac{2}{x-3} = 0$$

$$7 \quad \frac{5}{6x} - \frac{1}{x+1} = \frac{1}{3x}$$

$$8 \quad \frac{4}{x-2} - \frac{1}{x+1} = \frac{3}{x}$$

$$9 \quad 4x = \frac{3}{x} - 1$$

$$10 \quad 4x + \frac{3}{x} = 7$$

$$11 \quad 2x^2 - \frac{x}{3} = 5$$

$$12 \quad 2x + \frac{4}{x} = 9$$

$$13 \quad \frac{1}{x-1} - \frac{3}{x+2} = \frac{1}{4}$$

$$14 \quad \frac{2x}{3x+1} - \frac{5}{x+3} = 0$$

$$15 \quad \frac{2x}{x-3} - \frac{x}{x-2} = 3$$

$$16 \quad \frac{x}{x-2} - 2x = 3$$

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- represent linear inequalities graphically.

CHECK YOU CAN:

- use the inequality signs, $<$, $>$, \leq and \geq , correctly
- draw straight-line graphs.

Showing regions on graphs

It is often possible to show the region on a graph that satisfies an **inequality**.

An inequality involving the signs $<$ or $>$ is represented on a graph using a dashed line.

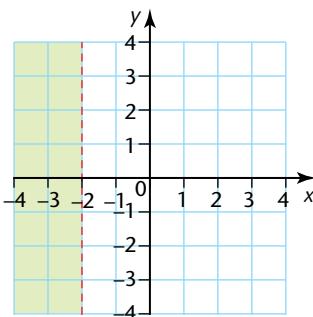
An inequality involving the signs \leq or \geq is represented by an solid line.

Example 21.1

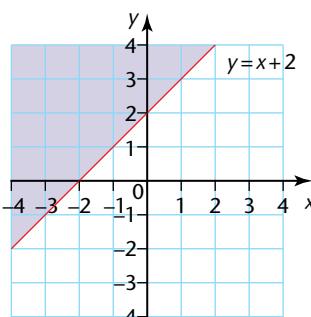
Question

Write down the inequality that describes the region shaded in each graph.

a



b



Solution

a $x < -2$

The line is dashed, so the inequality is either $x < -2$ or $x > -2$.

Choose any point in the shaded region, for example, $(-3, 0)$.

Substitute for x in the inequality.

$-3 < -2$, so the region represents $x < -2$.

b $y \geq x + 2$

The line is solid, so the inequality is either $y \geq x + 2$ or $y \leq x + 2$.

Choose any point in the shaded region, for example, $(0, 3)$.

Substitute for x and y .

$3 \geq 0 + 2$, so the region represents $y \geq x + 2$

Example 21.2**Question**

On separate grids shade each of these regions.

- a $y \geq 2$ b $y < 2x - 3$

Solution

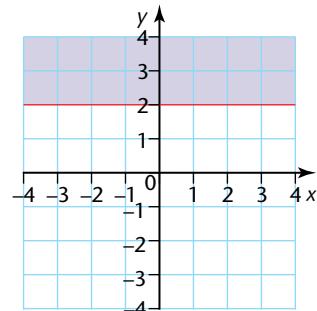
- a Draw the solid line $y = 2$.

Choose a point on one side of the line, for example, $(0, 0)$.

Substitute into the inequality.

$$0 \geq 2$$

Since this is *not* a true statement, $(0, 0)$ is not in the required region.



- b Draw the dashed line $y = 2x - 3$.

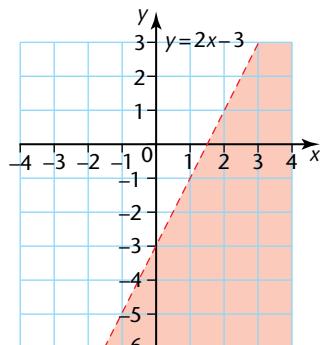
Choose a point on one side of the line, for example, $(3, 0)$.

Substitute into the inequality.

$$0 < 2 \times 3 - 3$$

$$0 < 3$$

Since this is a true statement, $(3, 0)$ is in the required region.

**Note**

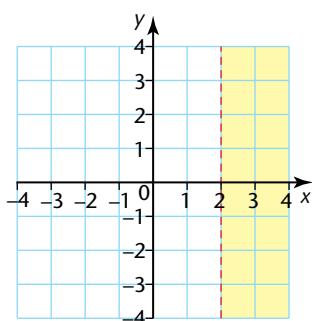
When testing a point, if possible use $(0, 0)$.

If the line goes through $(0, 0)$, choose a point with positive coordinates, for example $(1, 0)$, to test in the inequality.

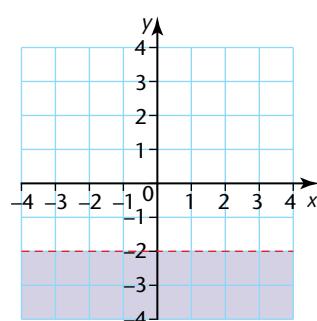
Exercise 21.1

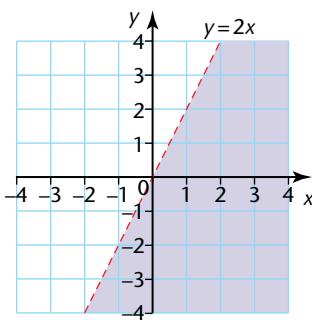
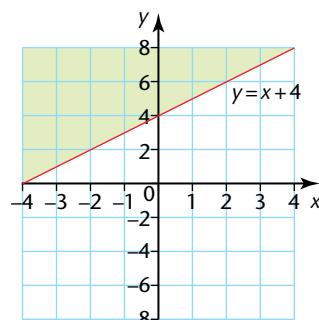
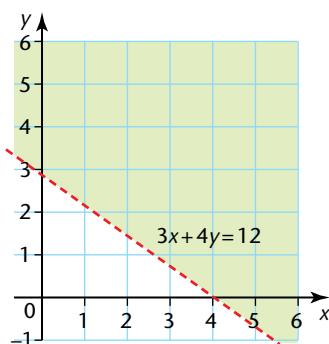
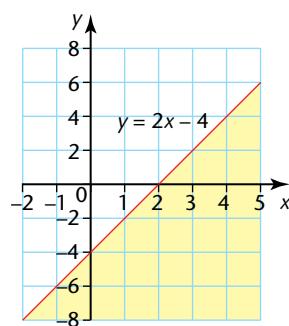
- 1 Write down the inequality that describes the shaded region in each of these diagrams.

a



b



c**d****e****f**

- 2** Draw a set of axes and label them from -4 to 4 for x and y . Shade the region $y > -3$.
- 3** Draw a set of axes and label them from -4 to 4 for x and y . Shade the region $x \geq -1$.
- 4** Draw a set of axes and label them from -4 to 4 for x and y . Shade the region $y < 3 - x$.
- 5** Draw a set of axes and label them from -4 to 4 for x and y . Shade the region $y \geq 2x + 1$.
- 6** Draw a set of axes and label them from -3 to 6 for x and from -3 to 5 for y . Shade the region $2x + 5y < 10$.
- 7** Draw a set of axes and label them from -1 to 6 for x and from -2 to 5 for y . Shade the region $4x + 5y \leq 20$.
- 8** Draw a set of axes and label them from 0 to 5 for x and y . Shade the region $3x + 5y > 15$.

Note

When plotting a line like $3x + 4y = 12$, choose $x = 0$ to give $4y = 12$ and $y = 3$.

Then choose $y = 0$ to give $3x = 12$ and $x = 4$.

This gives the points where the line crosses the axes, $(0, 3)$ and $(4, 0)$.

Representing regions satisfying more than one inequality

When you represent a region that satisfies more than one inequality, it is often better to shade the region that *does not* satisfy the inequality; the region that *does* satisfy the inequality is left unshaded.

Several inequalities can be represented on the same axes and the region where the values of x and y satisfy all of them can be found.

Example 21.3

Question

Draw a set of axes and label them from 0 to 8 for both x and y .

Show, by shading, the region where $x \geq 0$, $y \geq 0$ and $x + 2y \leq 8$.

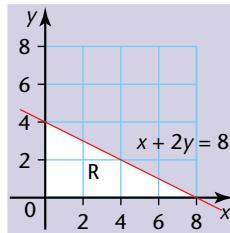
Solution

First draw the line $x + 2y = 8$.

Then shade the regions $x < 0$, $y < 0$ and $x + 2y > 8$.

These are the regions where the values of x and y *do not* satisfy the inequalities.

The required region, where the values of x and y satisfy all three inequalities, is left unshaded. It is labelled R.



Exercise 21.2

- 1 Draw a set of axes and label them from 0 to 6 for both x and y .
Show, by shading, the region where $x \geq 0$, $y \geq 0$ and $x + 2y \leq 6$.
- 2 Draw a set of axes and label them from 0 to 12 for x and from 0 to 8 for y .
Show, by shading, the region where $y > 0$, $x > 0$ and $3x + 5y < 30$.
- 3 Draw a set of axes. Label them from -1 to 4 for x and from -3 to 4 for y .
Show, by shading, the region where $x \geq 0$, $y \leq 3$ and $y \geq 2x - 3$.
- 4 Draw a set of axes and label them from -4 to 4 for both x and y .
Show, by shading, the region where $x > -2$, $y < 3$ and $y > 2x$.
- 5 Draw a set of axes and label them from 0 to 6 for both x and y .
Show, by shading, the region where $x < 4$, $y < 3$ and $3x + 4y < 12$.
- 6 Draw a set of axes and label them -1 to 5 for both x and y .
Show, by shading, the region where $y \geq 0$, $y \leq x + 1$ and $3x + 5y < 15$.
- 7 Draw a set of axes and label them from -1 to 8 for x and from -1 to 7 for y .
Show, by shading, the region where $x > 0$, $3x + 8y > 24$ and $5x + 4y < 20$.

22 SEQUENCES

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- continue a given number sequence
- recognise patterns in sequences and relationships between different sequences
- generalise sequences as simple algebraic statements.

CHECK YOU CAN:

- add, subtract, multiply and divide whole numbers
- substitute numbers into a formula
- solve simultaneous equations.

Number patterns

Here is a number pattern.

6 11 16 21 26 31 ...

You can see that, to get from one number to the next, you add 5.

Here is another number pattern.

1 2 4 8 16 32 ...

To get from one number to the next this time you multiply by 2.

1 $\xrightarrow{\times 2}$ 2 $\xrightarrow{\times 2}$ 4 $\xrightarrow{\times 2}$ 8 $\xrightarrow{\times 2}$ 16 $\xrightarrow{\times 2}$ 32 ...

Number patterns like these are called **sequences**.

The rule for finding the next number in a sequence from the previous one is called the **term-to-term rule**.

Example 22.1

Question

Find the term-to-term rule for each of these sequences and give the next number.

a 1 4 7 10 13 16 ...

b 63 56 49 42 35 28 ...

c 1 000 000 100 000 10 000 1 000 100 ...

Solution

- a To get from one number to the next you add 3.

1 $\xrightarrow{+ 3}$ 4 $\xrightarrow{+ 3}$ 7 $\xrightarrow{+ 3}$ 10 $\xrightarrow{+ 3}$ 13 $\xrightarrow{+ 3}$ 16 ...

The next number is $16 + 3 = 19$.

- b To get from one number to the next you subtract 7.

63 $\xrightarrow{- 7}$ 56 $\xrightarrow{- 7}$ 49 $\xrightarrow{- 7}$ 42 $\xrightarrow{- 7}$ 35 $\xrightarrow{- 7}$ 28 ...

The next number is $28 - 7 = 21$.

- c To get from one number to the next you divide by 10.

1 000 000 $\xrightarrow{\div 10}$ 100 000 $\xrightarrow{\div 10}$ 10 000 $\xrightarrow{\div 10}$ 1 000 $\xrightarrow{\div 10}$ 100 ...

The next number is $100 \div 10 = 10$.

In the sequences in the next example, some of the numbers are missing.

It is possible to work out what they should be by looking at the numbers in the sequence and working out how to get from one number to the next.

Example 22.2

Question

Find the missing numbers in each of these sequences.

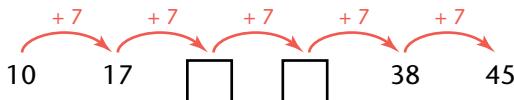
a 10 17 \square \square 38 45

b 35 \square 23 17 \square 5

c \square 15 \square 23 27 31

Solution

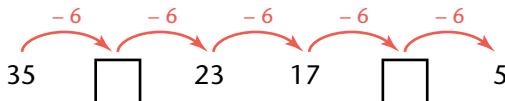
a The term-to-term rule is 'add 7'.



So the first missing number is $17 + 7 = 24$.

The second missing number is $24 + 7 = 31$.

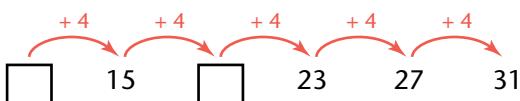
b The term-to-term rule is 'subtract 6'.



So the first missing number is $35 - 6 = 29$.

The second missing number is $17 - 6 = 11$.

c The term-to-term rule is 'add 4'.



To get to the previous number you must subtract 4.

So the first missing number is $15 - 4 = 11$.

You find the second missing number by adding 4 in the usual way.

The second missing number is $15 + 4 = 19$.

Exercise 22.1

1 Find the term-to-term rule for each of these sequences and give the next number.

a 2 6 10 14 18 22 ...

b 3 11 19 27 35 43 ...

c 4 9 14 19 24 29 ...

- 2** Find the term-to-term rule for each of these sequences and give the next number.

a 1 3 9 27 81 243 ...
b 15 625 3125 625 125 25 ...
c 1 2 4 7 11 16 ...

- 3** Find the missing numbers in each of these sequences.

a 7 12 17 \square \square 32 \square
b 25 \square 19 16 \square 10 \square
c 1 4 16 \square \square 1024 4096

- 4** Explain how you know whether or not 77 is in each of these sequences.

a 7 14 21 28 35 42 ...
b 14 18 22 26 30 34 ...

- 5** Look at this sequence of numbers.

1 4 9 16 25 36 ...

- a** Find the next number in the sequence.
b Explain how you get each number of the sequence.

Linear sequences

Look at this sequence.

2, 5, 8, 11, 14, ...

The terms of the sequence increase by 3 each time.

When the increase is constant like this, the sequence is called a **linear sequence**.

In the same way, the sequence 11, 9, 7, 5, 3, 1, -1, ... is also linear, since the terms decrease by the same amount (2) each time.

Another way of thinking of this is that you are adding -2 each time.

This idea will help with finding the n th term.

Exercise 22.2

- a** Which of these sequences are linear and which are not?
b For each sequence, write down the next two terms.

- 1** 3, 5, 7, 9, ...
2 43, 40, 38, 35, 32, ...
3 78, 75, 72, 69, ...
4 1, 5, 9, 13, ...
5 4, 9, 16, 25, ...
6 1, 6, 15, 28, ...
7 2, 9, 16, 23, ...

The n th term

It is often possible to find a formula to give the terms of a sequence.

Example 22.3

Question

The formula for the n th term of a sequence is $2n + 1$. Find each of these terms.

a u_1 b u_2 c u_3

Solution

a $u_1 = 2 \times 1 + 1 = 3$

b $u_2 = 2 \times 2 + 1 = 5$

c $u_3 = 2 \times 3 + 1 = 7$

Note

You usually use n to stand for the number of a term and u_n for the n th term.

Exercise 22.3

Each of these is the formula for the n th term of a sequence.

Find the first four terms of each sequence.

- | | | | | |
|--------------------|---------------------|---------------------|-------------------|---------------------|
| 1 $n + 1$ | 2 $2n$ | 3 $2n - 1$ | 4 $n + 5$ | 5 $3n$ |
| 6 $3n + 1$ | 7 $5n - 3$ | 8 $10n$ | 9 $7n - 7$ | 10 $2 - n$ |
| 11 n | 12 $n + 3$ | 13 $4n$ | 14 $n - 1$ | 15 $2n + 1$ |
| 16 $3n - 1$ | 17 $6n + 5$ | 18 $2n - 3$ | 19 $5 - n$ | 20 $10 - 2n$ |
| 21 n^2 | 22 $n^2 + 2$ | 23 $n^2 - 5$ | 24 $3n^2$ | 25 n^3 |

Finding the formula for the n th term

Look back at the formulae in Exercise 22.3 questions **1–20** and the sequences in your answers, and notice how they are connected.

All the sequences in questions **1–20** are linear.

If the formula contains the expression $2n$, the terms increase by 2 each time.

If it contains the expression $5n$, the terms increase by 5 each time.

Similarly, if it contains the expression $-3n$, the terms increase by -3 (or decrease by 3) each time.

So to find a formula for the n th term of a given sequence, find how much more (or less) each term is than the one before it.

Note

This will always work if the differences are constant.

The difference between terms in a linear sequence is always constant.

Example 22.4

Question

Find the formula for the n th term of this sequence.

3, 5, 7, 9, ...

Solution

The difference between each term is 2, so the formula will include the expression $2n$.

When $n = 1$, $2n = 2$.

But the first term of the sequence is 3, which is 1 more.

So the formula for the n th term of this sequence is $2n + 1$.

You can also use algebra to find the formula for the n th term.

The n th term can be written as T_n or u_n .

Example 22.5

Question

Find the formula for the n th term of this sequence.

20, 17, 14, 11, ...

Solution

As the difference between the terms is constant, this is a linear sequence.

The general formula for the n th term of a linear sequence is

$$T_n = an + b.$$

$$T_1 = 20 \text{ so } 20 = a + b$$

$$T_2 = 17 \text{ so } 17 = 2a + b$$

Solving the simultaneous equations gives

$$a = -3 \text{ and } b = 23$$

$$\text{So } T_n = 23 - 3n$$

Note

Notice the similarity between this and the equation of a straight line, $y = mx + c$.

Exercise 22.4

Find the formula for the n th term of each of these sequences.

Note

Check that your formula for the n th term is correct by trying it out for the first few terms. Substitute $n = 1, 2, 3, \dots$, into your formula.

1 1, 2, 3, 4, ... **2** 4, 6, 8, 10, ... **3** 4, 8, 12, 16, ...

4 0, 2, 4, 6, ... **5** 7, 11, 15, 19, ... **6** 1, 7, 13, 19, ...

7 11, 21, 31, 41, ... **8** 5, 8, 11, 14, ... **9** 101, 201, 301, 401, ...

10 0, 1, 2, 3, ... **11** 2, 5, 8, 11, ... **12** 7, 9, 11, 13, ...

13 4, 9, 14, 19, ... **14** 15, 20, 25, 30, ... **15** -1, 3, 7, 11, ...

16 5, 7, 9, 11, ... **17** 101, 102, 103, 104, ... **18** 4, 3, 2, 1, ...

19 7, 4, 1, -2, ... **20** 25, 23, 21, 19, ...

The n th term of quadratic, cubic and exponential sequences

This is the sequence of square numbers.

1^2	2^2	3^2	4^2	5^2	...	n^2	...
↓	↓	↓	↓	↓		↓	
1	4	9	16	25	...	n^2	...

This is the sequence of cube numbers.

1^3	2^3	3^3	4^3	5^3	...	n^3	...
↓	↓	↓	↓	↓	...	↓	...
1	8	27	64	125	...	n^3	...

This is an exponential sequence.

3^1	3^2	3^3	3^4	3^5	...	3^n
↓	↓	↓	↓	↓	...	↓
3	9	27	81	243	...	3^n

You can find the formula for the n th term of a quadratic, cubic or exponential sequence by comparing the sequence with the sequence of square numbers, cube numbers or exponential numbers.

Example 22.6

Question

Find the formula for the n th term of this cubic sequence.

-1	6	25	62	123	...
----	---	----	----	-----	-----

Solution

Compare the given sequence with the sequence of cube numbers.

1	8	27	64	125	...	n^3
↓-2	↓-2	↓-2	↓-2	↓-2	...	↓-2
-1	6	25	62	123	...	$n^3 - 2$

Notice that every term is 2 smaller than the corresponding term in the sequence of cube numbers.

So the formula is $n^3 - 2$.

Example 22.7

Question

Find the formula for the n th term of this quadratic sequence.

$$8 \quad 13 \quad 20 \quad 29 \quad 40 \quad \dots$$

Solution

Compare the given sequence with the sequence of square numbers.

$$\begin{array}{ccccccc} 1 & 4 & 9 & 16 & 25 & \dots & n^2 \\ \downarrow +7 & \downarrow +9 & \downarrow +11 & \downarrow +13 & \downarrow +15 & \dots & \downarrow +2n+5 \\ 8 & 13 & 20 & 29 & 40 & \dots & n^2 + 2n + 5 \end{array}$$

Notice that the differences between the given sequence and the sequence of square numbers are the sequence 7, 9, 11, 13, 15,

The formula for the n th term of the sequence of differences is $2n + 5$.

So the formula for the n th term of the sequence 8, 13, 20, 29, 40, ... is therefore $n^2 + 2n + 5$.

Example 22.8

Question

The n th term of a sequence is found using the formula $u_n = An^2 + B^n$.

$$u_1 = 3 \text{ and } u_2 = 8.$$

Find u_3 .

Solution

Substitute for $n = 1$ and $n = 2$ in the formula.

$$3 = A + B \quad (1)$$

$$8 = 4A + B^2 \quad (2)$$

Solve the simultaneous equations.

$$12 = 4A + 4B \quad (1) \times 4$$

$$8 = 4A + B^2$$

$$4 = 4B - B^2 \quad \text{Subtract.}$$

$$B^2 - 4B + 4 = 0 \quad \text{Solve the quadratic equation.}$$

$$(B - 2)^2 = 0$$

$$\text{So } B = 2$$

Substituting in equation (1) gives $A = 1$.

$$\text{So } u_n = n^2 + 2^n$$

$$u_3 = 3^2 + 2^3 = 9 + 8 = 17$$

Note

Remember to check your answer by substituting in the formula.

Exercise 22.5

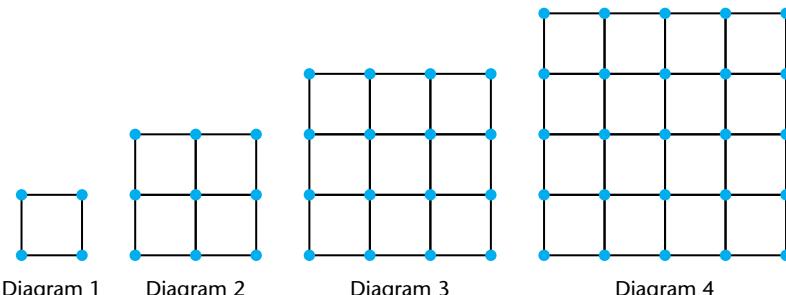
- 1** Find the formula for the n th term of each of these quadratic sequences.
- 3, 6, 11, 18, 27, ...
 - 4, -1, 4, 11, 20, ...
 - 2, 6, 12, 20, 30, ...
 - 2, 8, 18, 32, 50, ...
 - 3, 9, 17, 27, 39, ...
 - 3, 4, 7, 12, 19, ...
- 2** Find the formula for the n th term of each of these cubic sequences.
- 11, 18, 37, 74, 135, ...
 - 8, -1, 18, 55, 116, ...
 - 2, 16, 54, 128, 250, ...
 - 2, 12, 36, 80, 150, ...
 - 0, 6, 24, 60, 120, ...
- 3** The formula for the n th term of the sequence 2, 4, 8, 16, 32, ... is 2^n . Use this to find the formula for the n th term of each of these sequences.
- 1, 3, 7, 15, 31, ...
 - 14, 16, 20, 28, 44, ...
 - 3, 6, 11, 20, 37, ...
 - 0, 0, 2, 8, 22, ...
- 4** The n th term of a sequence is found using the formula $T_n = An^3 + Bn^2$. $T_1 = 1$ and $T_2 = 12$.
- Show that $A + B = 1$ and $8A + 4B = 12$.
 - Solve the simultaneous equations to find A and B .
 - Find T_3 and T_4 .

- 5** The first five terms of a sequence are

$$\frac{2}{3}, \frac{8}{5}, \frac{18}{7}, \frac{32}{9}, \frac{50}{11}$$

Find a formula for the n th term.

- 6** Look at this sequence of diagrams made from lines and dots.



Copy and complete the table.

Diagram (n)	1	2	3	4	n
Number of small squares	1	4	9		
Number of dots	4	9			
Number of lines	4	12			

23 VARIATION

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- express direct and inverse variation in algebraic terms and use this form of expression to find unknown quantities.

CHECK YOU CAN:

- find and use multipliers
- manipulate simple formulae
- substitute numbers into simple formulae.

Note

You can easily tell whether the proportion is direct or inverse – in direct proportion, both variables change in the same way, either up or down; in inverse proportion, when one variable goes up, the other will go down.

Proportion

In Chapter 11 you solved problems involving proportion. When two quantities are in **direct proportion**, as one quantity increases, so does the other.

For example, a truck travels x km using y litres of fuel.

In this case, y varies in the same way as x . If you double x , you also double y .

It is possible to express this in symbols as $y \propto x$, which is read as ‘ y varies as x ’ or ‘ y is proportional to x ’.

When two quantities are in **indirect proportion**, as one quantity increases, the other decreases.

For example, x workers can dig a hole in y days.

In this case, y varies inversely as x . If you double x , you halve y .

This is the same as ‘ y varies as $\frac{1}{x}$ ’, which is written in symbols as $y \propto \frac{1}{x}$.

Exercise 23.1

- Describe the variation in each of these situations, using the symbol \propto .
 - The time the journey takes, t , and the speed of the train, s .
 - The number of pages in a book, p , and the number of words, w .
 - The depth of water in a rectangular tank, d , and the length of time it has been filling, t .
 - The number of buses, b , needed to carry 2000 people and the number of seats on a bus, s .
 - The time a journey takes, t , at a fixed speed and the distance covered, d .
 - The number of ice creams you can buy, c , and the amount of money you have, m .
- Describe the variation shown in each of these tables of values. Use the symbol \propto .

a	<table border="1"><tr><td>x</td><td>3</td><td>15</td></tr><tr><td>y</td><td>1</td><td>5</td></tr></table>	x	3	15	y	1	5
x	3	15					
y	1	5					

b	<table border="1"><tr><td>x</td><td>4</td><td>15</td></tr><tr><td>y</td><td>28</td><td>105</td></tr></table>	x	4	15	y	28	105
x	4	15					
y	28	105					

c	<table border="1"><tr><td>x</td><td>8</td><td>20</td></tr><tr><td>y</td><td>10</td><td>4</td></tr></table>	x	8	20	y	10	4
x	8	20					
y	10	4					

d	<table border="1"><tr><td>x</td><td>10</td><td>12</td></tr><tr><td>y</td><td>50</td><td>60</td></tr></table>	x	10	12	y	50	60
x	10	12					
y	50	60					

e	<table border="1"><tr><td>x</td><td>24</td><td>4.8</td></tr><tr><td>y</td><td>16</td><td>3.2</td></tr></table>	x	24	4.8	y	16	3.2
x	24	4.8					
y	16	3.2					

f	<table border="1"><tr><td>x</td><td>3</td><td>15</td></tr><tr><td>y</td><td>5</td><td>1</td></tr></table>	x	3	15	y	5	1
x	3	15					
y	5	1					

g	<table border="1"><tr><td>x</td><td>3</td><td>15</td></tr><tr><td>y</td><td>2</td><td>10</td></tr></table>	x	3	15	y	2	10
x	3	15					
y	2	10					

h	<table border="1"><tr><td>x</td><td>8</td><td>20</td></tr><tr><td>y</td><td>10</td><td>25</td></tr></table>	x	8	20	y	10	25
x	8	20					
y	10	25					

i	x	1	0.1
	y	50	500

j	x	16	56
	y	6.4	22.4

Variation as a formula

Example 23.1

Question

Find the formula connecting x and y .

x	5	15
y	3	9

Solution

You can see that $y \propto x$.

So the formula will be $y = kx$.

When $x = 5$, $y = 3$.

Substituting these values in the equation gives $3 = k \times 5$, so $k = \frac{3}{5}$.

This gives the formula $y = \frac{3}{5}x$.

Example 23.2

Question

Find the formula connecting x and y .

x	4	20
y	10	2

Solution

$$y \propto \frac{1}{x}$$

$$\text{So } y = k \times \frac{1}{x} \text{ or } y = \frac{k}{x}.$$

When $x = 4$, $y = 10$.

Substituting these values in the formula gives $10 = \frac{k}{4}$ so $k = 10 \times 4 = 40$.

This gives the formula $y = \frac{40}{x}$.

Check the formula works for the other pair of values in the table: $y = \frac{40}{x} = \frac{40}{20} = 2$, which is correct.

Exercise 23.2

- 1 Find a formula for each variation in Exercise 23.1, question 2.
- 2 Ohm's law states that if a voltage, V volts, is applied across the terminals of an electrical component, and if the current flowing in the component is I amperes, then $I \propto V$.
When $V = 15$, $I = 2.5$.
 - a Find the formula for the variation.
 - b Find I when $V = 60$ volts.

- 3** If w m is the wavelength of a sound wave and f hertz is the frequency of the wave, then $w \propto \frac{1}{f}$.
 When $f = 300$ hertz, $w = 1.1$ m.
- Find the formula for the variation.
 - Find w when $f = 660$ hertz.
- 4** The variables A and B are such that $B = 50$ when $A = 20$.
- i Write down a formula where A is directly proportional to B .
 - ii Write down a formula where A is inversely proportional to B .
 - Calculate B when $A = 25$ for each of the two formulae you have found.

Other types of variation

The examples you have met in this chapter so far have been for variation: $y \propto x$ and $y \propto \frac{1}{x}$.

Sometimes two variables can be related in more complicated ways.

For example, $y \propto x^2$, $y \propto \frac{1}{x^2}$, $y \propto \sqrt{x}$, $y \propto \frac{1}{\sqrt{x}}$, $y \propto x^3$ and $y \propto \frac{1}{x^3}$.

This is the method you use for finding the formula for the variation.
 It is the same method you used previously.

- First write the proportionality using the \propto sign. For example, $y \propto x^2$.
- Replace the \propto sign by $=$ and multiply the right-hand side by k .
 For example, $y = kx^2$.
- Substitute a known pair of values for x and y .
- Solve the equation to find k .
- Rewrite the formula using the found value of k .

Example 23.3

Question

y is proportional to x^2 and when $x = 5$, $y = 10$.

a Find a formula connecting x and y .

b Find the value of y when $x = 15$.

Solution

a $y \propto x^2$

So $y = kx^2$

$$10 = k \times 5^2$$

$$k = \frac{10}{25} = \frac{2}{5}$$

Substitute $x = 5$, $y = 10$.

Solve for k .

$$y = \frac{2}{5}x^2$$

Substitute for k in the formula.

b $y = \frac{2}{5} \times 15^2 = 90$

Substitute $x = 15$ into the formula.

Note

There are different ways of talking about variation.

' y varies directly as x^2 ' means the same as ' y is proportional to x^2 '.

Example 23.4

Question

y is inversely proportional to x^2 and when $x = 5$, $y = 10$.

- Find a formula connecting x and y .
- Find the value of y when $x = 10$.
- Find the value of x when $y = 1000$

Solution

a $y \propto \frac{1}{x^2}$

$$\text{So } y = k \times \frac{1}{x^2} = \frac{k}{x^2}$$

$$10 = \frac{k}{5^2}$$

$$k = 10 \times 25 = 250$$

$$y = \frac{250}{x^2}$$

b $y = \frac{250}{10^2} = 2.5$

c $1000 = \frac{250}{x^2}$

$$1000x^2 = 250$$

$$x^2 = \frac{250}{1000}$$

$$x^2 = \frac{1}{4}$$

$$x = \pm \sqrt{\frac{1}{4}} = \pm \frac{1}{2}$$

Note

For inverse variation you can go straight to the fraction version with k as the numerator.

Example 23.5

Question

$y \propto \frac{1}{x^2}$ and $y = 7$ when $x = 7$.

Find y when $x = 14$.

Solution

Although you can use the formula method, this is quicker.

x has been **multiplied** by 2.

Since y is **inversely** proportional to x^2 , y will be **divided** by 2^2 .

$$y = \frac{7}{2^2} = 1.75$$

Example 23.6**Question**

y varies inversely as the square root of x .

When $x = 4$, $y = 12$.

Find the value of y when $x = 9$.

Solution

First find the formula connecting x and y .

$$y \propto \frac{1}{\sqrt{x}}$$

$$\text{So } y = k \times \frac{1}{\sqrt{x}} = \frac{k}{\sqrt{x}}$$

$$12 = \frac{k}{\sqrt{4}} \quad \text{Substitute } x = 4, y = 12.$$

$$k = 12 \times 2 = 24 \quad \text{Solve for } k.$$

$$y = \frac{24}{\sqrt{x}} \quad \text{Substitute for } k \text{ in the formula.}$$

$$y = \frac{24}{\sqrt{9}} = 8 \quad \text{Substitute } x = 9 \text{ into the formula.}$$

Example 23.7**Question**

y is inversely proportional to x^2 .

When x takes a certain value, y is 18.

Find the value of y when x is multiplied by 3.

Solution

x is **multiplied** by 3.

Since y is **inversely** proportional to x^2 , y will be **divided** by 3^2 .

$$y = \frac{18}{3^2} = 2$$

Exercise 23.3

- 1 $y \propto x^2$ and $y = 3$ when $x = 6$.
 - a Find a formula connecting x and y .
 - b Find y when $x = 12$.
 - c Find x when $y = 48$.
- 2 $y \propto \frac{1}{x^2}$ and $y = 4$ when $x = 4$.
 - a Find a formula connecting x and y .
 - b Find y when $x = 8$.
 - c Find x when $y = 25$.

- 3** y is directly proportional to the square root of x and $y = 8$ when $x = 9$.
- Find a formula connecting x and y .
 - Find y when $x = 36$.
 - Find x when $y = 4$.
- 4** y varies inversely as the cube of x and $y = 5$ when $x = 2$.
- Find a formula connecting x and y .
 - Find y when $x = 5$.
 - Find x when $y = 320$.
- 5** y is directly proportional to x^3 and $y = 20$ when $x = 2$.
- Find a formula connecting x and y .
 - Find y when $x = 4$.
 - Find x when $y = 540$.
- 6** $y \propto x^2$ and $y = 9$ when $x = 7.5$.
Find y when $x = 5$.
- 7** $y \propto \frac{1}{x^2}$ and $y = 10$ when $x = 3$.
Find y when $x = 12$.
- 8** $y \propto \sqrt{x}$ and $y = 6$ when $x = 25$.
Find y when $x = 16$.
- 9** $y \propto \frac{1}{x^3}$ and $y = \frac{1}{2}$ when $x = 2$.
Find y when $x = 1$.
- 10** y is inversely proportional to x^2 .
 $y = 4$ when $x = 3$.
Find y when $x = 10$.
- 11** A car skids to a halt in an accident.
Investigators measure the length, l metres, of the skid to estimate the speed, v m/s, at which the car had been travelling before the accident.
The speed is proportional to the square root of the length of the skid.
- Given that a car travelling at 20 m/s skids 25 m find a formula for v in terms of l .
 - Use your formula to find the speed of a car which skids 100 m.
- 12** Seven people can paint a bridge in 15 days.
- How long will it take 3 people to paint the bridge?
 - The bridge was painted in t days.
Write down an expression, in terms of t , for the number of people needed to paint the bridge.
- 13** The force, F , between two particles is inversely proportional to the square of the distance between them.
The force is 36 newtons when the distance between the particles is r metres.
Find the force when the distance between the particles is $3r$ metres.
- 14** The braking distance of a car is directly proportional to the square of its speed.
When the speed is p metres per second, the braking distance is 6 m.
When the speed is increased by 300%, find
- an expression for the speed of the car
 - the braking distance
 - the percentage increase in the braking distance.

- 15** Describe the variation shown in each of these tables of values.
Use the symbol \propto .

a

x	5	25
y	5	125

b

x	5	10
y	5	1.25

c

x	5	2.5
y	5	10

d

x	4	6
y	18	8

e

x	2	6
y	7	63

f

x	1	0.25
y	1	4

g

x	54	21.6
y	33	13.2

h

x	24	48
y	4	1

Conversion graphs

A conversion graph can be used to convert quantities from one unit to another.

They are often used to convert between different currencies.

Note

Conversion graphs will only give approximate values. If you know the conversion rate then you can find a more accurate answer by calculation.

Example 24.1

Question

The exchange rate between pounds and dollars is £1 = \$1.60.

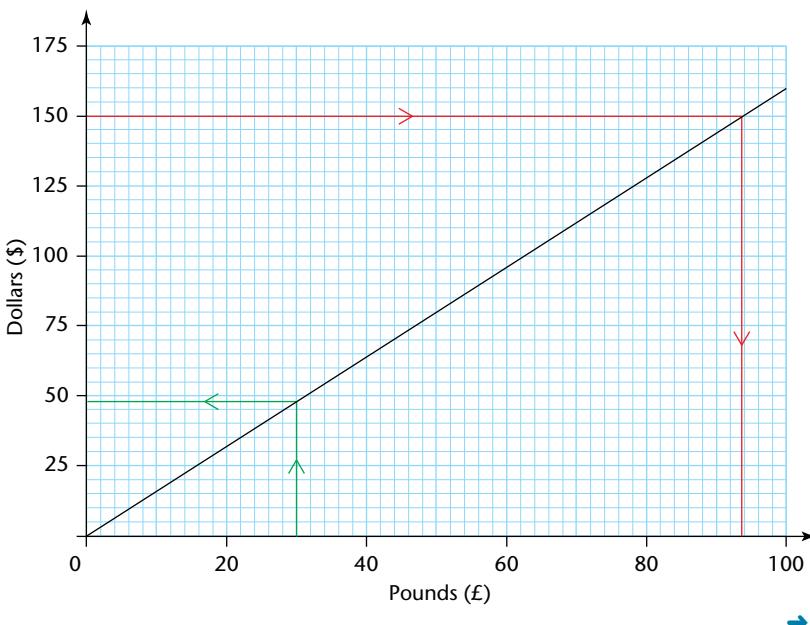
- Draw a conversion graph for pounds to dollars.
- Use your graph to convert
 - £30 to dollars
 - \$150 to pounds.

Solution

- As £1 is equivalent to \$1.60, then £100 is equivalent to \$160.

Draw a straight line from (0, 0) to (100, 160).

Label each axis with the currency.



BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- interpret and use graphs in practical situations including travel graphs and conversion graphs
- draw graphs from given data
- apply the idea of rate of change to easy kinematics involving distance-time and speed-time graphs, acceleration and deceleration
- calculate distance travelled as area under a linear speed-time graph.

CHECK YOU CAN:

- plot points and read values from a graph
- work with the formula for speed
- find the area of a rectangle and a triangle.

- b i** Reading from the graph, £30 = \$48

This is an estimated value.

The line meets the axis between \$45 and \$50, closer to \$50 so it is estimated at \$48.

- ii** Reading from the graph, \$150 = £94

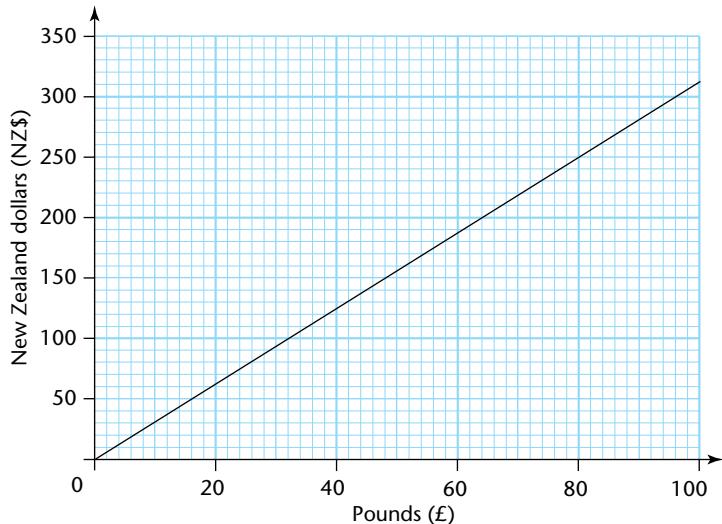
This is an estimated value.

The line meets the axis between £92 and £94, but very close to £94, so the answer is given correct to the nearest pound.

Exercise 24.1

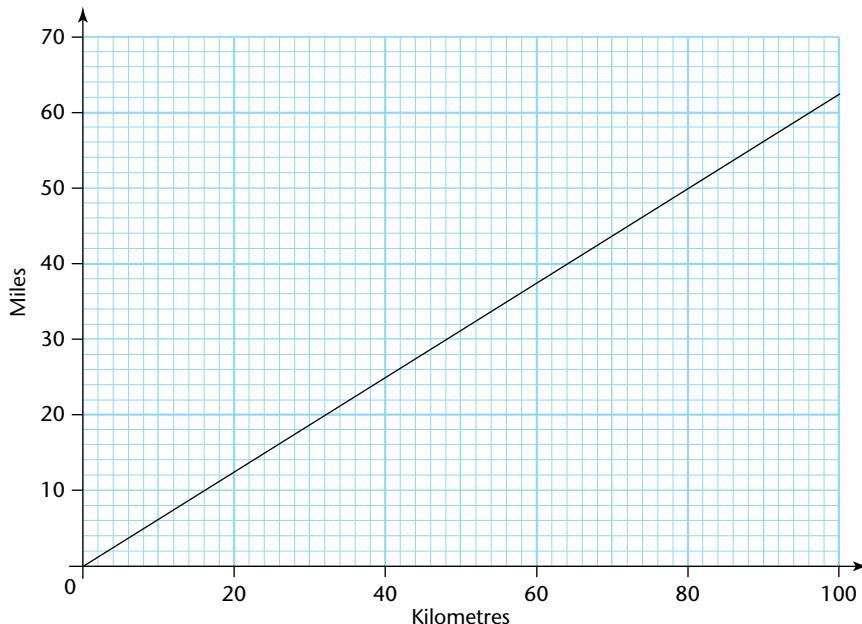


- 1** The exchange rate between dollars and euros (€) is \$1 = €0.70.
- Draw a conversion graph between dollars and euros, for up to \$100.
 - Use your graph to convert
 - i \$35 to euros
 - ii €60 to dollars.
- 2** This conversion graph is for pounds (£) to New Zealand dollars (NZ\$) for amounts up to £100.



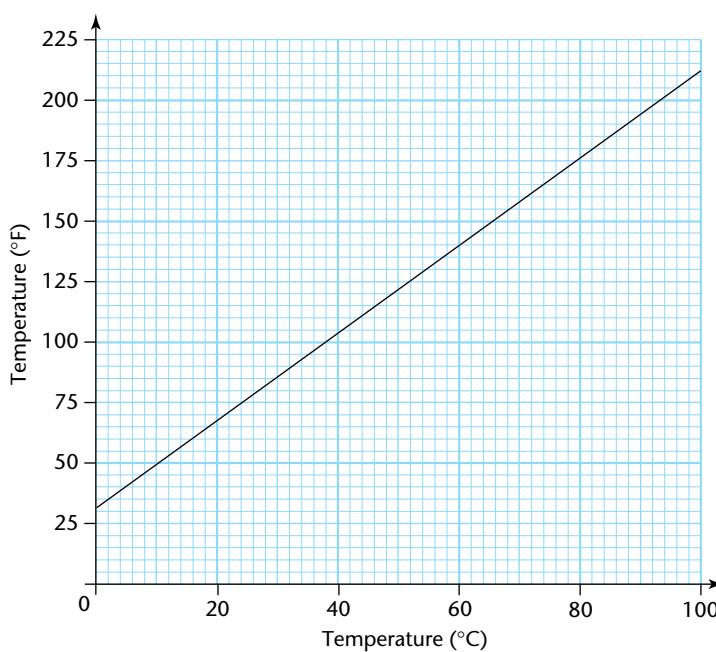
- Use the graph to find the number of New Zealand dollars equal to
 - i £20
 - ii £85.
- Use the graph to find the number of pounds equal to
 - i NZ\$100
 - ii NZ\$250.
- How many pounds are equal to NZ\$600?

-  3 This conversion graph is for the number of kilometres (km) equal to miles up to 100 km.



- a Use the graph to find the number of miles equal to
 - i 40 km
 - ii 65 km.
- b Use the graph to find the number of kilometres equal to
 - i 10 miles
 - ii 56 miles.
- c How many kilometres are equal to 150 miles?

-  4 This conversion graph is for temperatures in °C to °F, from 0 to 100 °C.



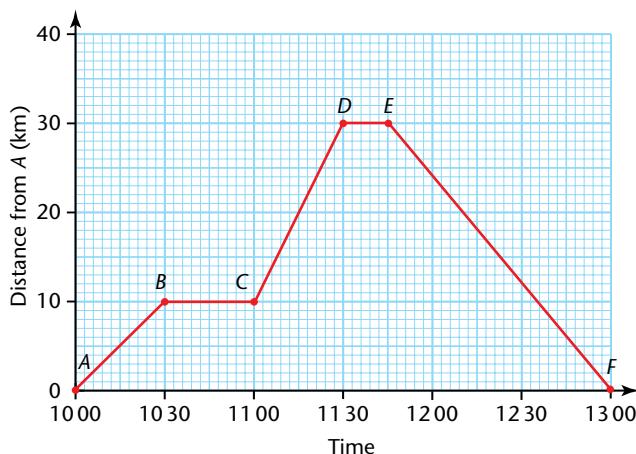
- a Use the graph to find the temperature in °F equal to
 - i 0°C
 - ii 85°C .
- b Use the graph to find the temperature in °C equal to
 - i 100°F
 - ii 170°F .

Travel graphs

A travel graph is a graph showing a journey.

Time is plotted on the horizontal axis and distance is plotted on the vertical axis.

This graph shows the distance, in kilometres, of a truck from point *A* at a given time.



Note

Take care when reading the time scale. There are 60 minutes in an hour, so each small square on this graph represents 3 minutes.

The graph can be used to describe the journey.

A to *B* The truck travels 10 km at constant speed away from *A*. It takes 30 minutes.

B to *C* The truck is stationary for 30 minutes.

C to *D* The truck travels 20 km at constant speed away from *A*. It takes 30 minutes. The speed is greater than the speed from *A* to *B*.

D to *E* The truck is stationary for 15 minutes.

E to *F* The truck travels 30 km at constant speed back to *A*. It takes 1 hour 15 minutes.

Example 24.2

Question

Lalith ran the first 3 kilometres to school at a speed of 12 km/h.

She then waited 5 minutes for her friend.

They walked the last 2 kilometres to school together, taking 20 minutes.

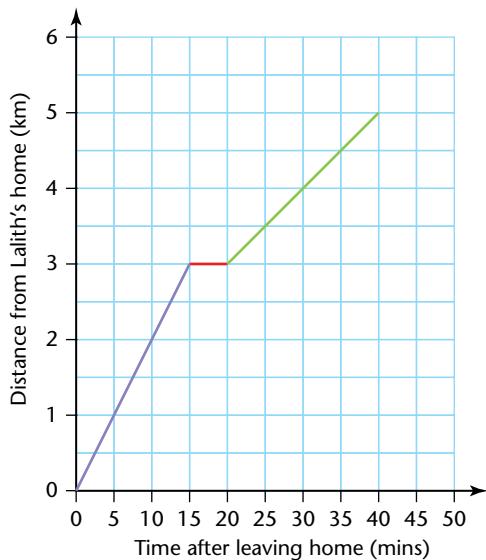
Draw a graph to show Lalith's journey.

Solution

It takes 15 minutes to run 3 kilometres at 12 km/h, so the first part of the graph is a straight line from the origin to (15, 3).

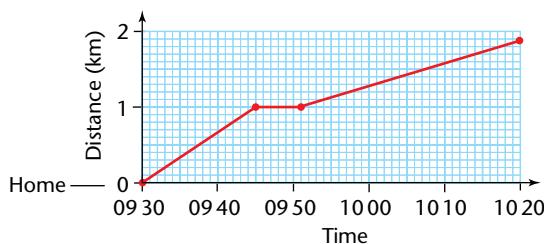
The second part of the graph is horizontal from (15, 3) to (20, 3) as Lalith is not moving for 5 minutes.

The final part of the journey covers a further 2 kilometres in 20 minutes, so is a straight line from (20, 3) to (40, 5). This assumes that they walk at a constant speed.



Exercise 24.2

- 1 Amit left home and went for a walk, stopping at the supermarket on the way and ending at his brother's house.
This graph shows his journey.



Note

The slope of the line represents the speed.

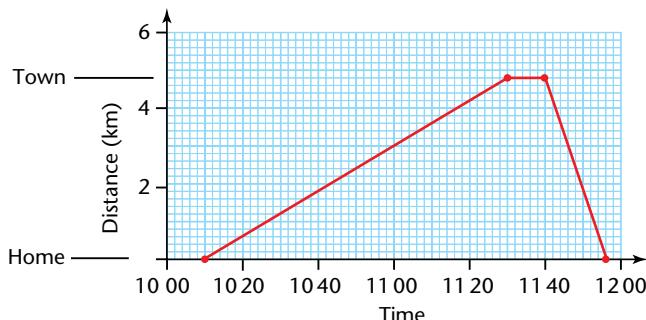
If the line is horizontal, the person is not moving.

If the line is straight, the speed is constant.

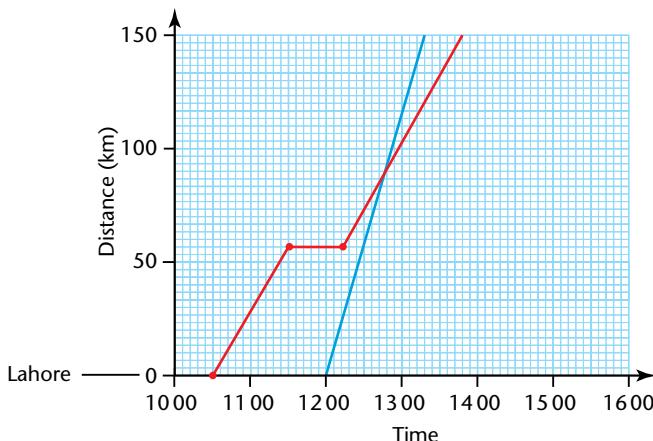
The steeper the slope of the line, the faster the person is moving.

- a At what time did Amit reach the supermarket?
- b How long did he stop at the supermarket?
- c How far is it from his house to his brother's house?
- d On which part of the walk did he walk faster?
- e What was his speed on the first part of his journey?

- 2 This graph shows Nasreen's walk to town from home one day and her return on the bus.



- a At what time did Nasreen leave home?
 b How far did she walk?
 c How long did she stop in town for?
 d How long did the bus journey take?
- 3 This graph shows the journeys of two men, Yousef and Usman, who both left Lahore. Yousef left first.



- a How much later than Yousef did Usman leave?
 b How long did Yousef stop for?
 c How far had they travelled when Usman passed Yousef?
 d At what time did Usman pass Yousef?
- 4 Mr Khan left home at 7.15 a.m. and walked the 500 m to the station in 10 minutes.

He then waited for the train which left at 7.46 a.m.

The train took 10 minutes to cover the 12 km to the next station, where Mr Khan got off.

He then took a taxi for 2 km to his office, taking 8 minutes.

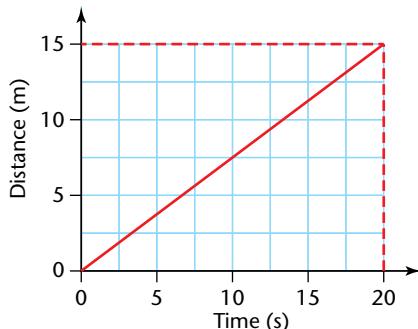
- a Draw a distance–time graph to show this journey.
 Show time from 7 a.m. to 8.40 a.m. and use a scale of 2 cm for 20 minutes.

Show distance from 0 to 15 km and use a scale of 2 cm for 5 km.

- b How long did Mr Khan wait at the station?
 c At what time did he arrive at his office?

Rate of change on a distance–time graph

The **gradient** of a graph is a measure of its steepness or rate of change.



For this distance–time graph

$$\text{gradient} = \frac{\text{change in distance}}{\text{change in time}}$$

$$= \frac{15 \text{ metres}}{20 \text{ seconds}}$$

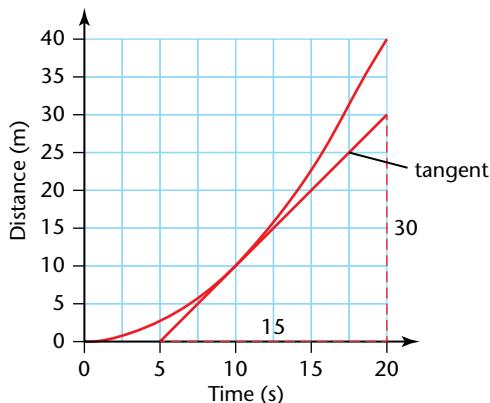
$$= 0.75 \text{ metres per second}$$

The units of the gradient in this case are metres per second or m/s.

The gradient gives the speed of the object.

The graph is a straight line, so the object is moving with a constant speed of 0.75 m/s.

If the distance–time graph is not a straight line, then the speed of the object is changing.



The graph is curving upwards, so the speed is increasing.

You can estimate the speed of the object at a given time by drawing a line which touches the curve at that point. This is called a **tangent**.

You then calculate the gradient of the tangent to the curve at that point.

The gradient of the tangent to the curve at time 10 seconds gives the speed of the object after 10 seconds.

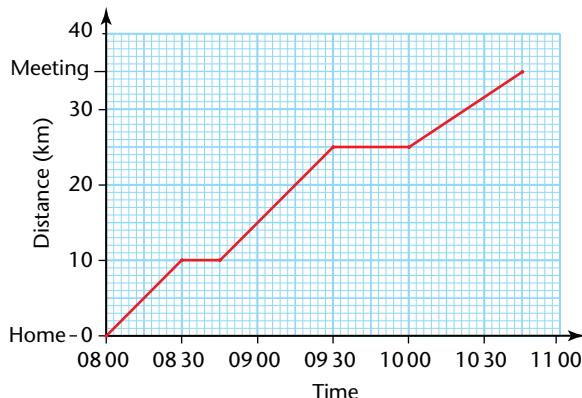
$$\begin{aligned}\text{gradient} &= \frac{\text{change in distance}}{\text{change in time}} \\ &= \frac{30 \text{ metres}}{15 \text{ seconds}} \\ &= 2 \text{ metres per second}\end{aligned}$$

The speed of the object after 10 seconds is 2 m/s.

Example 24.3

Question

This distance–time graph shows Manjit's journey from his home to a meeting.



- a Work out Manjit's speed, in kilometres per hour, for each stage of the journey.
- b Sketch a speed–time graph for Manjit's journey.

Solution

- a The graph is made up of five straight-line segments.

Each of these lines represents a stage with a constant speed.

To work out the speed, find the gradient of the line.

$$\text{From } 08:00 \text{ to } 08:30, \text{ speed} = \frac{10 \text{ km}}{0.5 \text{ hours}} = 20 \text{ km/h}$$

From 08:30 to 08:45, the line is horizontal so Manjit is not moving.

$$\text{From } 08:45 \text{ to } 09:30, \text{ speed} = \frac{(25 - 10) \text{ km}}{0.75 \text{ hours}} = 20 \text{ km/h}$$

From 09:30 to 10:00, the line is horizontal so Manjit is not moving.

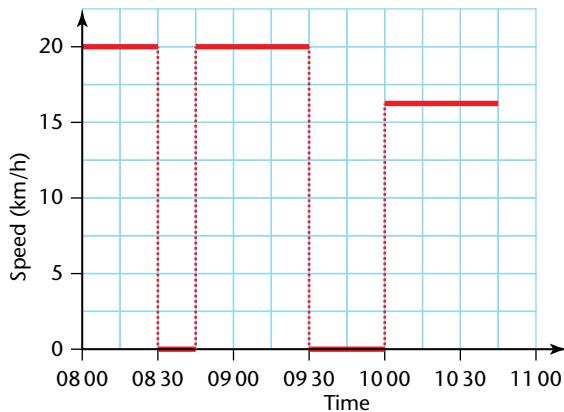
$$\text{From } 10:00 \text{ to } 10:45, \text{ speed} = \frac{(35 - 25) \text{ km}}{0.75 \text{ hours}} = 13.33\ldots \text{ km/h}$$

Note

The question asks for the speed in kilometres per hour so you must use the correct units in the gradient calculation.

You must convert 30 minutes to 0.5 hours and 45 minutes to 0.75 hours.

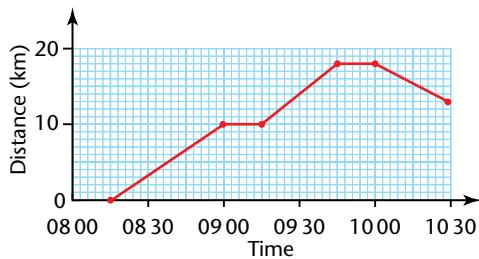
- b The speed–time graph is made up of five horizontal line segments.

**Note**

This is a sketch of the situation because the actual speed would not change instantaneously from 20 km/h to 0 km/h, for example.

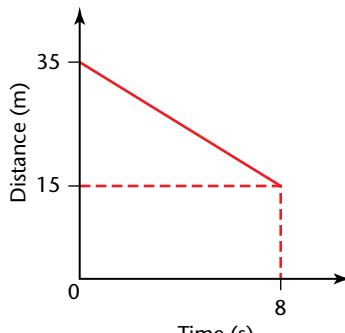
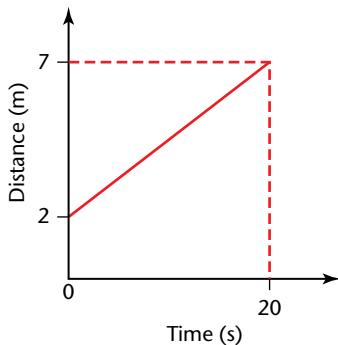
Exercise 24.3

- 1 This distance–time graph shows Baljit’s cycle ride.

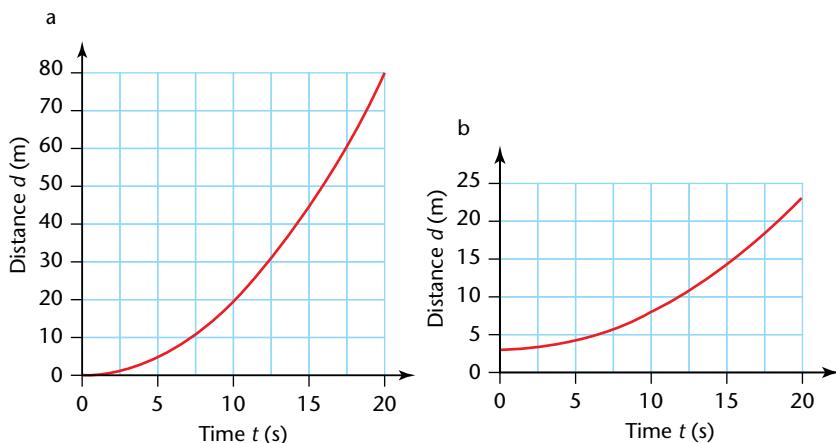


Work out Baljit’s speed, in km/h, for each stage of his journey.

- 2 Work out the speed, in m/s, shown on each of these distance–time graphs.

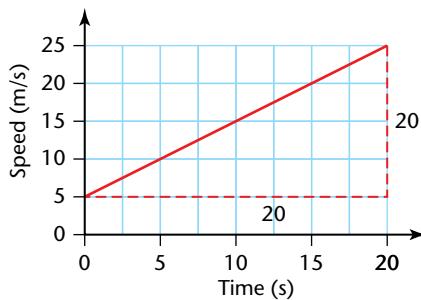


- 3 Use the graphs below to estimate the speed, in m/s, of each object when $t = 10$.



Rate of change on a speed–time graph

This speed–time graph shows the speed of an object during a given time period.



The gradient of the graph represents the rate of change of speed with time.

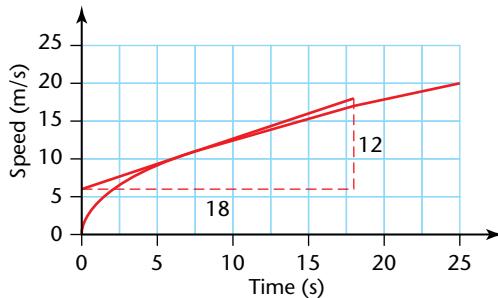
The graph is a straight line, so the speed is changing at a constant rate.

$$\text{gradient} = \frac{20 \text{ m/s}}{20 \text{ s}} = 1 \text{ m/s}^2$$

The rate of change of speed with time is the **acceleration** of the object.

The units of acceleration in this case are metres per second squared or m/s^2 .

If the speed–time graph is not a straight line, then the acceleration is changing.



The graph is curving downwards, so the acceleration is decreasing.

You can estimate the acceleration of the object at a given time by calculating the gradient of the tangent to the curve at that point.

The gradient of the tangent to the curve at time 7.5 seconds gives the speed of the object after 7.5 seconds.

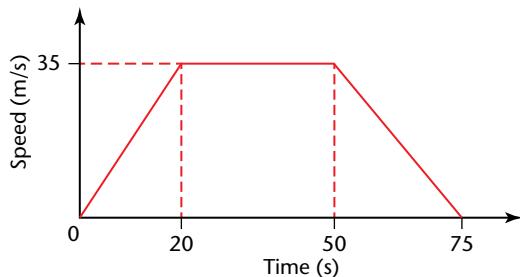
$$\begin{aligned}\text{gradient} &= \frac{\text{change in speed}}{\text{change in time}} \\ &= \frac{12 \text{ m/s}}{18 \text{ seconds}} \\ &= 0.67 \text{ m/s}^2 \text{ correct to 2 decimal places}\end{aligned}$$

The acceleration of the object after 7.5 seconds is 0.67 m/s^2 .

Example 24.4

Question

The speed–time graph shows part of a car’s journey.



- a Work out the acceleration, in m/s^2 , for each stage of this journey.
- b Calculate the speed of the car after 60 seconds.

Solution

- a The graph is made up of three straight-line segments.
Each of these lines represents a stage with constant acceleration.
To work out the acceleration, find the gradient of the line.
From 0 to 20 seconds, acceleration = $\frac{35 \text{ m/s}}{20 \text{ s}} = 1.75 \text{ m/s}^2$
From 20 to 50 seconds, the speed is constant, so acceleration = 0 m/s^2
From 50 to 75 seconds, acceleration = $\frac{-35 \text{ m/s}}{25 \text{ s}} = -1.4 \text{ m/s}^2$
- b You know that the speed of the car is 35 m/s after 50 seconds.
After this the car is decelerating at 1.4 m/s^2 .
In 10 seconds, the speed will decrease by $10 \times 1.4 = 14 \text{ m/s}$.
After 60 seconds, the speed is $35 - 14 = 21 \text{ m/s}$.

Note

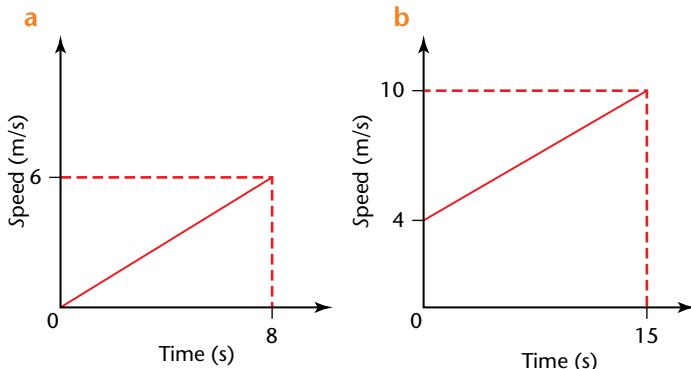
When the acceleration is negative, the car is slowing down.

Negative acceleration is known as **deceleration**.

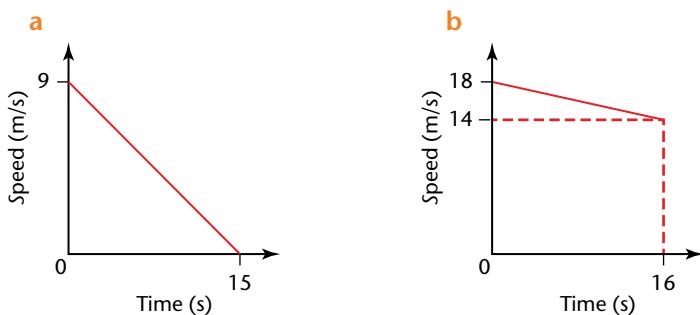
You can say that from 50 to 75 seconds the car has a deceleration of 1.4 m/s^2 .

Exercise 24.4

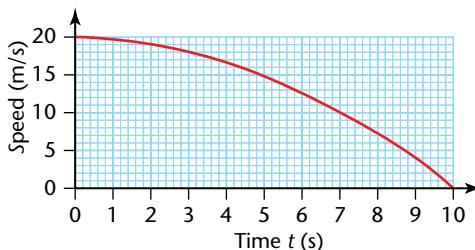
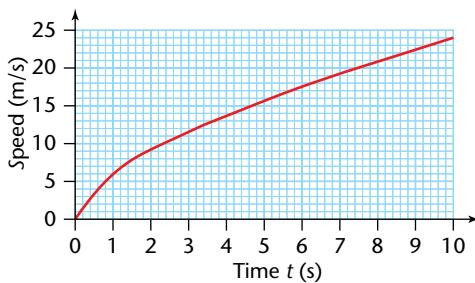
- 1** Calculate the acceleration, in m/s^2 , for each of these speed–time graphs.



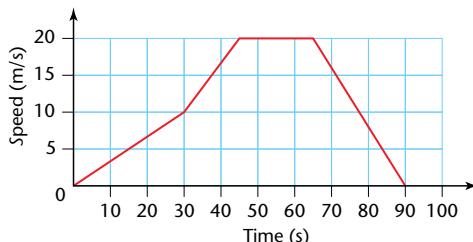
- 2** Calculate the deceleration, in m/s^2 , for each of these speed–time graphs.



- 3** Use the graphs to estimate the acceleration of each object when $t = 5$.



- 4 This is the speed–time graph for part of a car’s journey.



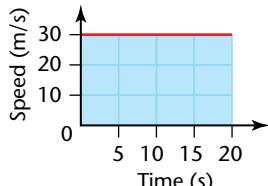
- a Calculate the acceleration, in m/s^2 , for each stage of the journey.
 - b Calculate the speed of the car when
 - i $t = 20$
 - ii $t = 75$.
- 5 A train slows down uniformly from a speed of 8 m/s to rest in 24 seconds.
- a Sketch a speed–time graph to show this information.
 - b Calculate the train’s deceleration in m/s^2 .
 - c Calculate the speed of the train after 9 seconds.
- 6 a An object moves from rest with constant acceleration of 1.2 m/s^2 .
 Calculate the speed of the object after 15 seconds.
 b An object slows down from a speed of 16 m/s with a constant
 deceleration of 0.8 m/s^2 .
 Calculate the time taken for the object to stop.

Area under a speed–time graph

The distance travelled by an object moving at 30 m/s for 20 seconds can be found using the formula $\text{distance} = \text{speed} \times \text{time}$.

$$\text{Distance} = 30 \times 20 = 600 \text{ metres}$$

This is the speed–time graph for an object moving with a constant speed of 30 m/s for 20 seconds.



The blue rectangle is the area under the graph.

$$\text{The area of the rectangle is } 30 \times 20 = 600.$$

The units of the area in this case are $\text{metres per second} \times \text{seconds} = \text{metres}$.

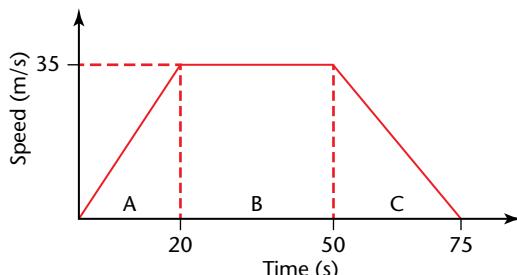
So the area under the graph represents 600 metres.

This is the distance travelled by the object.

The area under a speed–time graph gives the distance travelled by the object in that time.

Example 24.5**Question**

The speed–time graph shows part of a car’s journey.



Calculate the distance travelled by the car in these 75 seconds.

Solution

The area under the graph gives the distance travelled.

The area can be found by adding the areas of two triangles and a rectangle.

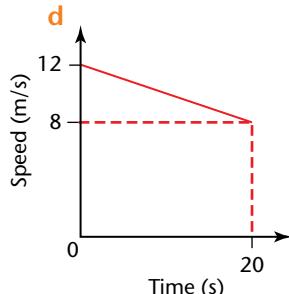
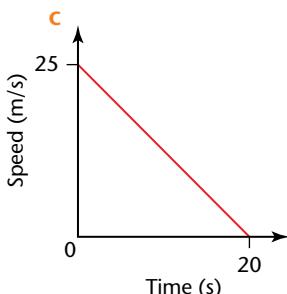
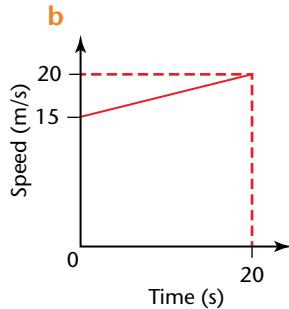
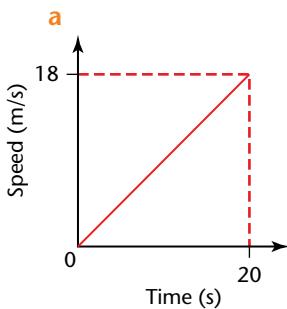
$$\text{Distance travelled} = \text{area of triangle A} + \text{area of rectangle B} + \text{area of triangle C}$$

$$= \frac{1}{2} \times 20 \times 35 + 30 \times 35 + \frac{1}{2} \times 25 \times 35$$

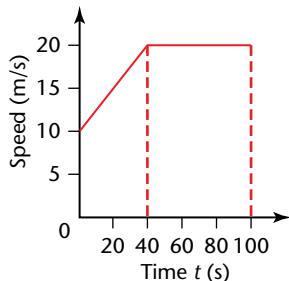
$$= 1837.5 \text{ metres}$$

**Exercise 24.5**

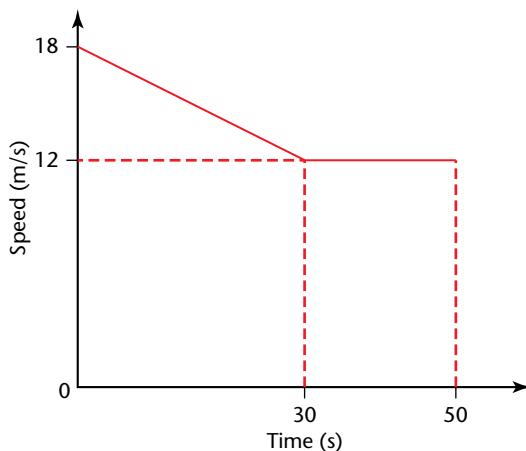
- 1 Calculate the distance travelled in 20 seconds from each of these speed–time graphs.



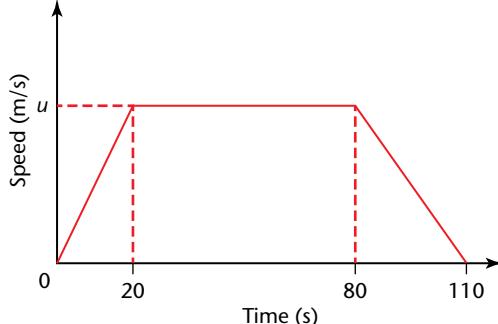
- 2 The speed–time graph shows 100 seconds of a car’s journey.



- a Calculate the acceleration of the car when $t = 20$.
 b i Calculate the distance travelled by the car in the first 40 seconds.
 ii Calculate the total distance travelled in 100 seconds.
- 3 The speed–time graph shows 50 seconds of a car’s journey.

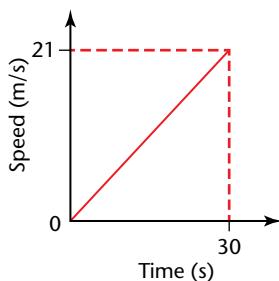


- a Calculate the distance travelled by the car in these 50 seconds.
 b After 50 s, the car slows down with a constant deceleration of 0.4 m/s^2 . Work out how many more seconds it takes for the car to come to rest.
- 4 The speed–time graph shows part of a train’s journey.



The train travels a total of 680 m in 110 seconds.
 Work out the value of u .

- 5 The speed–time graph shows the first 30 s of a car’s journey.



- a i Work out the speed of the car after 15 seconds.
ii Work out the distance travelled by the car in the first 15 seconds.
b Sketch a distance–time graph for the car’s journey.

25 GRAPHS OF FUNCTIONS

Quadratic graphs

Quadratic graphs are graphs of equations of the form $y = ax^2 + bx + c$ where a , b and c are constants, and b and c may be zero.

When the graph of a quadratic equation is drawn, it produces a curve called a **parabola**.

Example 25.1

Question

- Draw the graph of $y = x^2 - 2x - 3$ for values of x from -2 to 4 .
Label the axes from -2 to 4 for x and from -5 to 5 for y .
- Find the values of x for which $y = 0$.

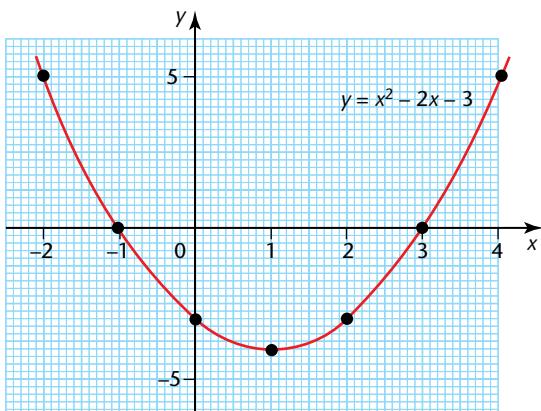
Solution

x	-2	-1	0	1	2	3	4
x^2	4	1	0	1	4	9	16
$-2x$	4	2	0	-2	-4	-6	-8
-3	-3	-3	-3	-3	-3	-3	-3
$y = x^2 - 2x - 3$	5	0	-3	-4	-3	0	5

To find the value of y , add together the values in rows 2, 3 and 4.

Now plot the points $(-2, 5)$, $(-1, 0)$, and so on. Join them with a smooth curve.

Label the curve.



- $y = 0$ when the curve crosses the x axis.

This is when $x = -1$ or $x = 3$.

Note

Sometimes a table will be given with only the two rows for the x and y values. You may find it useful to include all the rows and then just add the correct rows to get the values of y .

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- construct tables of values and draw graphs for functions of the form ax^n where a is a rational constant, and $n = -2, -1, 0, 1, 2, 3$, and simple sums of not more than three of these and for functions of the form ka^x where a is a positive integer
- interpret graphs of linear, quadratic, reciprocal and exponential functions
- solve associated equations approximately by graphical methods
- estimate gradients of curves by drawing tangents.

CHECK YOU CAN:

- draw the graph of a straight line, given its equation
- find the gradient of a straight line.

Note

Practise drawing smooth curves. Keep your pencil on the paper. Make the curve a thin single line. Look ahead to the next point as you draw the line. Keep your hand inside the curve by turning the paper.

Note

Complete a table of values to work out the points. If you are not given a table then make your own. A common error is to include the value in the x -row when adding to find y , so separate this row off clearly.

Note

If you find that you have two equal lowest (or highest) values for y , the curve will go below (or above) that value.

To find the y -value between the equal values, find the value of x halfway between the two points. Substitute to find the corresponding value of y .

Note

In question 2, to find the value of y , rows 3 and 4 are added.

Exercise 25.1

- 1 Copy and complete this table for $y = x^2 + 3x - 7$. Do not draw the graph. Notice the extra value at $x = -1.5$.

x	-4	-3	-2	-1	0	1	2	-1.5
x^2				4				
$+3x$				-6				
-7				-7				
$y = x^2 + 3x - 7$				-9				

- 2 Copy and complete this table for $y = 2x^2 - 8$. Do not draw the graph.

x	-3	-2	-1	0	1	2	3
x^2	9						
$2x^2$	18						
-8	-8						
$y = 2x^2 - 8$	10						

- 3 Copy and complete this table for $y = -x^2 - 5x + 6$. Do not draw the graph. Remember that $-x^2$ is always negative. Notice the extra value at -2.5 .

x	-6	-5	-4	-3	-2	-1	0	1	2	-2.5
$-x^2$				-16						-4
$-5x$				20						-10
$+6$				6						6
$y = -x^2 - 5x + 6$				10						-8

Use 2 mm graph paper to draw the graphs in questions 4 to 10.

- 4 a Copy and complete this table for $y = x^2 - 3x$.

x	-1	0	1	2	3	4	5	1.5
x^2								
$-3x$								
$y = x^2 - 3x$								

- b Draw the graph of $y = x^2 - 3x$ for values of x from -1 to 5. Label the x axis from -1 to 5 and the y axis from -5 to 10. Use a scale of 1 cm to 1 unit on the x axis and 2 cm to 5 units on the y axis.

- 5 Draw the graph of $y = x^2 - 6x + 5$ for values of x from -1 to 6.

- 6 Draw the graph of $y = -x^2 + 4x - 3$ for values of x from -1 to 5.

- 7 a Draw the graph of $y = x^2 - 5x + 2$ for values of x from -1 to 6.

- b Find the values of x on your graph when $y = 0$.

Give your answers correct to 1 decimal place.

Note

An extra point at $x = 2.5$ might be useful for question 7.

- 8 a** Draw the graph of $y = 2x^2 - 5x + 1$ for values of x from -2 to 4 .
b Find the value of x on your graph where $y = 0$.
 Give your answers correct to 1 decimal place.
- 9 a** Draw the graph of $y = 2x^2 - 12x$ for values of x from -1 to 7 .
b Write down the values of x where the curve crosses $y = 5$.
- 10** When a stone is dropped from the edge of a cliff, the distance, d metres, it falls is given by the formula $d = 5t^2$, where t is the time in seconds.
a Work out the values of d for values of t from 0 to 5 .
b Draw the graph for $t = 0$ to 5 .
c The cliff is 65 metres high.
 How long does it take the stone to reach the bottom of the cliff?
 Give your answer correct to 1 decimal place.

Note

Note that in some questions, the values of y are not in pairs so the lowest point is not exactly halfway between two of the given points.

Using graphs to solve equations

One way of finding solutions to quadratic equations is to draw and use a graph.

Example 25.2

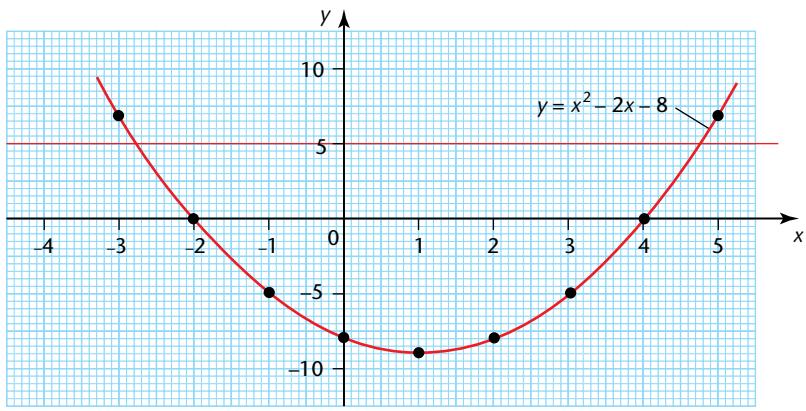
Question

- a** Draw the graph of $y = x^2 - 2x - 8$ for values of x from -3 to 5 .
b Solve the equation $x^2 - 2x - 8 = 0$.
c Solve the equation $x^2 - 2x - 8 = 5$.

Solution

a

x	-3	-2	-1	0	1	2	3	4	5
x^2	9	4	1	0	1	4	9	16	25
$-2x$	6	4	2	0	-2	-4	-6	-8	-10
-8	-8	-8	-8	-8	-8	-8	-8	-8	-8
$y = x^2 - 2x - 8$	7	0	-5	-8	-9	-8	-5	0	7



- b** The solution of $x^2 - 2x - 8 = 0$ is when $y = 0$, where the curve cuts the x axis. The solution is $x = -2$ or $x = 4$.
- c** The solution of $x^2 - 2x - 8 = 5$ is when $y = 5$.
Draw the line $y = 5$ on your graph and read off the values of x where the curve cuts the line.
The solution is $x = -2.7$ or $x = 4.7$, to 1 decimal place.

Sometimes you may have drawn a quadratic graph and then need to solve an equation that is different from the graph you have drawn. Rather than drawing another graph, it may be possible to rearrange the equation to obtain the one you have drawn.

Example 25.3

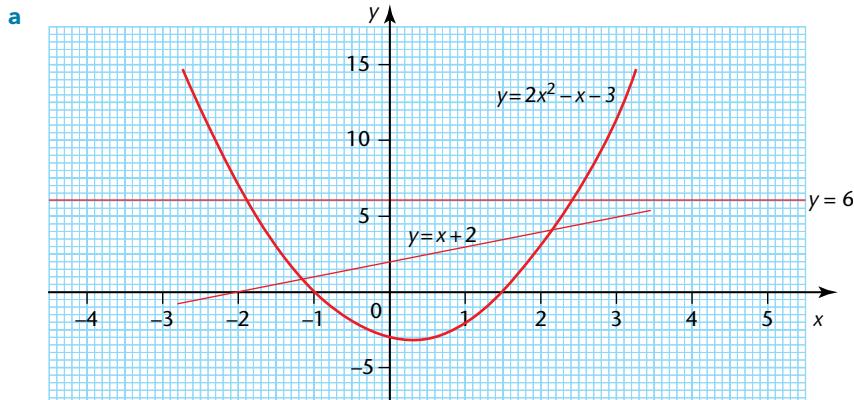
Question

- a Draw the graph of $y = 2x^2 - x - 3$ for values of x between -3 and 4 .
b Use this graph to solve these equations.

i $2x^2 - x - 3 = 6$

ii $2x^2 - x = x + 5$

Solution



- b i To solve $2x^2 - x - 3 = 6$ you draw the line $y = 6$ on your graph.
At the points of intersection of the line and the curve, $y = 6$ and $2x^2 - x - 3$ and therefore $2x^2 - x - 3 = 6$.
The curve and line cross at approximately $x = -1.9$ and $x = 2.4$.
- ii To solve $2x^2 - x = x + 5$ you must rearrange the equation so that you have $2x^2 - x - 3$ on the left-hand side.
You do this by subtracting 3 from both sides.
 $2x^2 - x = x + 5$
 $2x^2 - x - 3 = x + 2$
Now draw the line $y = x + 2$ on your graph.
The points where the line and the curve intersect are the solution to $2x^2 - x = x + 5$.
The curve and line cross at approximately $x = -1.2$ and $x = 2.2$.

Exercise 25.2

Save the graphs you draw in this exercise. You will use some of them again in Exercise 25.5.

- 1 a** Draw the graph of $y = x^2 - 4x + 3$ for values of x from -1 to 5 .
b Use your graph to solve the equation $x^2 - 4x + 3 = 0$.
- 2 a** Draw the graph of $y = x^2 - 3x$ for values of x from -2 to 5 .
b Use your graph to solve the equation $x^2 - 3x = 0$.
- 3 a** Draw the graph of $y = x^2 - 2x - 3$ for values of x from -2 to 4 .
b Use your graph to solve these equations.
i $x^2 - 2x - 3 = 0$ **ii** $x^2 - 2x - 3 = -2$
iii $x^2 - 2x - 3 = x$ **iv** $x^2 - 2x - 5 = 0$
- 4 a** Draw the graph of $y = x^2 - 2x + 2$ for values of x from -2 to 4 .
b Use your graph to solve these equations.
i $x^2 - 2x + 2 = 8$ **ii** $x^2 - 2x + 2 = 5 - x$
iii $x^2 - 2x - 5 = 0$
- 5 a** Draw the graph of $y = 2x^2 + 3x - 9$ for values of x from -3 to 2 .
b Use your graph to solve these equations.
i $2x^2 + 3x - 9 = -1$ **ii** $2x^2 + 3x - 4 = 0$
- 6 a** Draw the graph of $y = x^2 - 5x + 3$ for values of x from -2 to 8 .
b Use your graph to solve these equations.
i $x^2 - 5x + 3 = 0$ **ii** $x^2 - 5x + 3 = 5$
iii $x^2 - 7x + 3 = 0$
- 7** An archer shoots an arrow from ground level towards a castle 45 m high. The height of the arrow (h metres) above the ground after t seconds is given by

$$h = 30t - 5t^2$$
 - a** Draw the graph for values of t from 0 to 5 .
 - b** Can the arrow reach the top of the castle? Explain.
 - c** If the arrow does not hit anything, after how long will it be at ground level again?

For the remaining questions, do not draw the graphs.

- 8** The graph of $y = x^2 - 8x + 2$ has been drawn.
 What other line needs to be drawn to solve the equation $x^2 - 8x + 6 = 0$?
- 9** Find and simplify the equation in x that is solved by the intersections of the graph of $y = 2x^2 - 5x + 1$ and the graph of $y = 4x - 3$.
- 10** Find and simplify the equation in x that is solved by the intersections of the graph of $y = 3x^2 + 7x - 2$ and the graph of $y = 5x + 4$.

Cubic graphs

Cubic graphs are graphs of equations of the form

$$y = ax^3 + bx^2 + cx + d$$

where a, b, c and d are constants, and b, c and d may be zero.

Example 25.4**Question**

- a Draw the graph of $y = x^3$ for values of x from -3 to 3 .
 Label the x axis from -3 to 3 and the y axis from -30 to 30 .
- b Use your graph to solve the equation $x^3 = 12$. Give your answer to 1 decimal place.

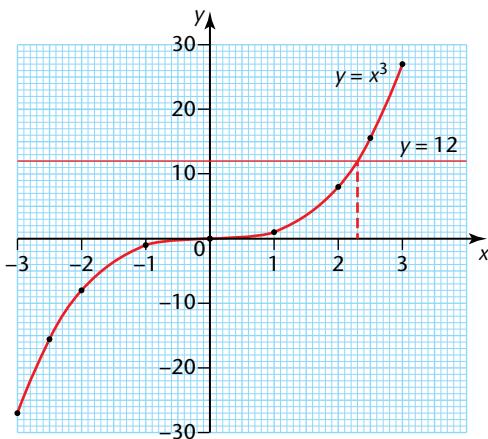
Solution

a

x	-3	-2.5	-2	-1	0	1	2	2.5	3
$y = x^3$	-27	-15.625	-8	-1	0	1	8	15.625	27

The outside points are a long way apart so plotting the values for $x = 2.5$ and $x = -2.5$ helps with the drawing of the curve.

- b To solve $x^3 = 12$, you need to draw the line $y = 12$ on your graph and find where it intersects the curve. The solution is $x = 2.3$, to 1 decimal place.



The curve goes from bottom left to top right and has a ‘double bend’ in the middle.

All cubic curves with a positive x^3 term have a similar shape.

If the x^3 term is negative the curve goes from top left to bottom right.

Example 25.5**Question**

- a Draw the graph of $y = x^3 - 2x$ for values of x from -2 to 2 .
 Label the x axis from -2 to 2 and the y axis from -4 to 4 .
- b Use the graph to solve the equation $x^3 - 2x = 0$. Give your answers correct to 1 decimal place.



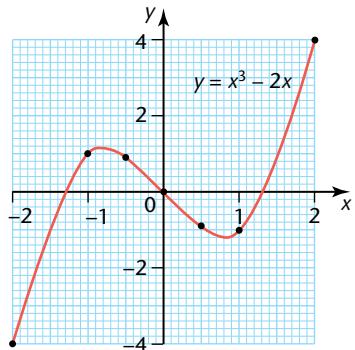
Solution**a**

x	-2	-1	-0.5	0	0.5	1	2
x^3	-8	-1	-0.125	0	0.125	1	8
$-2x$	4	2	1	0	-1	-2	-4
$y = x^3 - 2x$	-4	1	0.875	0	-0.875	-1	4

It helps to see more clearly where the curve is highest and lowest if you work out the values of y for $x = -0.5$ and $x = 0.5$.

- b** To solve $x^3 - 2x = 0$, you need to find where $y = x^3 - 2x$ crosses $y = 0$, which is the x axis.

The solution is $x = -1.3$ (to 1 d.p.) or $x = 0$ or $x = 1.3$ (to 1 d.p.).



The shape of the curve is similar to the previous one. The $-2x$ term makes the ‘double bend’ more pronounced.

Reciprocal graphs

Reciprocal graphs are graphs of equations of the form

$$y = \frac{a}{x}$$

where a is a non-zero constant.

Example 25.6

Question

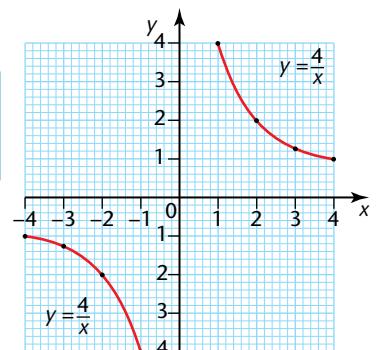
Draw the graph of $y = \frac{4}{x}$.

Solution

x	-4	-3	-2	-1.5	-1	1	1.5	2	3	4
$y = \frac{4}{x}$	-1	-1.33...	-2	-2.66...	-4	4	2.66...	2	1.33...	1

You cannot use 0 as a point in this type of graph, since you cannot divide 4 (or any other number) by 0.

Again it is helpful to work out extra points to give a better curve. In this case you might work out the value of y when $x = -1.5$ and 1.5 . It is also useful to have the same scale for both x and y in this case.



All equations of this type have graphs of the same shape, with two separate branches in opposite quadrants.

Plotting points for $x = 0.5$ and $x = 0.1$ would help to show that the curve gets closer to the y axis without ever meeting it.

Plotting points for $x = 5$ and $x = 6$ would help to show that the curve gets closer to the x axis without ever meeting it.

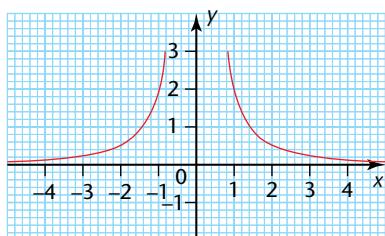
Example 25.7

Question

Draw the graph of $y = \frac{2}{x^2}$.

Solution

x	-4	-3	-2	-1.5	-1	1	1.5	2	3	4
$y = \frac{2}{x^2}$	0.125	0.22...	0.5	0.88...	2	2	0.88...	0.5	0.22...	0.125



All equations of this type have graphs of the same shape, with two separate branches side by side.

Plotting points for $x = 0.5$ and $x = 0.1$ would help to show that the curve gets closer to the y axis without ever meeting it.

Plotting points for $x = 5$ and $x = 6$ would help to show that the curve gets closer to the x axis without ever meeting it.

Exercise 25.3

Save the graphs you draw in this exercise. You will use some of them again in Exercise 25.5.

- 1 a Copy and complete this table for $y = x^3 + 5$.

x	-3	-2	-1	0	1	2	3
x^3	-27			0		8	
+5	5						
y	-22						

- b Draw the graph of $y = x^3 + 5$.
 c Use your graph to solve the equation $x^3 + 5 = 0$.

- 2 a** Copy and complete this table for $y = -x^3$.

x	-3	-2	-1	0	1	2	3
x^3	-27			0		8	
y	27						

- b** Draw the graph of $y = -x^3$.
c Use your graph to solve the equation $-x^3 = 6$.
- 3 a** Copy and complete this table for $y = \frac{1}{x}$.

x	-10	-5	-2	-1	-0.5	-0.1	0.1	0.5	1	2	5	10
y						-10					0.2	

- b** Draw the graph of $y = \frac{1}{x}$.
 Use a scale of 1 cm to 2 units on both axes.
c Use your graph to solve each of these equations.
 i $\frac{1}{x} = 0.3$ ii $\frac{1}{x} = -5$
- 4 a** Make a table of values for $y = x^3 - 12x + 2$ for $x = -3$ to 4.
b Draw the graph of $y = x^3 - 12x + 2$.
c Solve the equation $x^3 - 12x + 2 = 0$.
- 5 a** Make a table of values for $y = x^3 - x^2 - 6x$ for $x = -3$ to 4.
b Draw the graph of $y = x^3 - x^2 - 6x$.
c Use the graph to solve the equation $x^3 - x^2 - 6x = 0$.
- 6 a** Make a table of values for $y = \frac{8}{x}$ for $x = -8, -4, -2, -1, 1, 2, 4, 8$.
b Draw the graph of $y = \frac{8}{x}$.
 Use a scale of 1 cm to 1 unit on both axes.
- 7 a** Make a table of values for $y = \frac{12}{x^2}$ for $x = -12, -8, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 8, 12$.
b Draw the graph of $y = \frac{12}{x^2}$.
- 8 a** Make a table of values for $y = \frac{5}{x}$ for $x = -5, -4, -2.5, -2, -1, 1, 2, 2.5, 4, 5$.
b Draw the graph of $y = \frac{5}{x}$.
 Use a scale of 1 cm to 1 unit on both axes.
c On the same grid, draw the graph of $y = x$.
d Use your graph to solve $x^2 = 5$, giving your answers to 1 decimal place.
- 9** The temperature (y °C) of a cup of coffee t minutes after it was made is given by the equation $y = 20 + \frac{50}{t}$.
a Plot the equation for values of t between 1 and 10.
b Estimate how long it takes for the coffee to cool to 40 °C.
c Estimate the temperature after 30 minutes.
- 10** The graph of $y = x^3 - 2x^2$ has been drawn.
 What other curve needs to be drawn to solve the equation
 $x^3 - x^2 - 4x + 3 = 0$?

Note

Remember: always make a table of values.

Exponential graphs

Exponential graphs are graphs of equations of the form

$$y = ka^x.$$

For the graphs that you will meet in this chapter, a is a positive integer.

Example 25.8

Question

Plot a graph of $y = 3^x$ for values of x from -2 to 3 .

Use your graph to estimate

- a the value of y when $x = 2.4$
- b the solution to the equation $3^x = 20$.

Solution

Make a table of values and plot the graph.

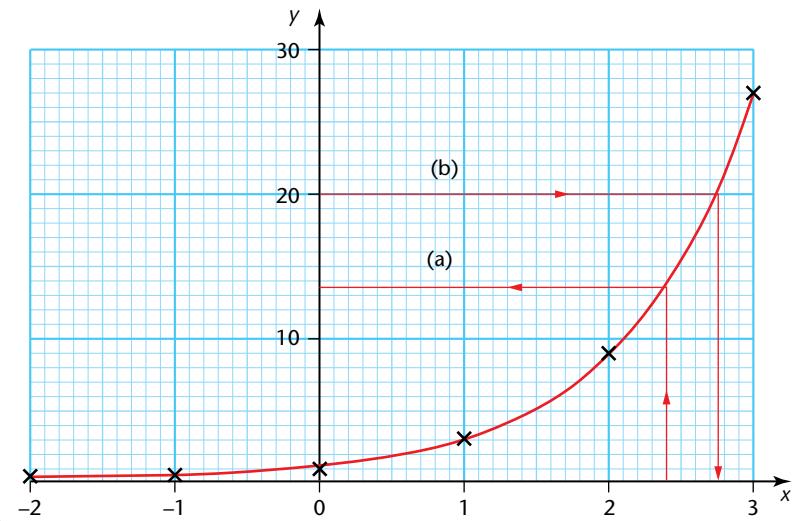
x	-2	-1	0	1	2	3
y	0.111...	0.333...	1	3	9	27

- a Draw the line $x = 2.4$ and find where it intersects the curve.

$$y = 13.5$$

- b Draw the line $y = 20$ and find where it intersects the curve.

$$x = 2.75$$



Exercise 25.4

- 1 Plot a graph of $y = 2^x$ for values of x from -2 to 5 .

Use a scale of 2 cm to 1 unit on the x axis and 2 cm to 5 units on the y axis.

Use your graph to estimate

- the value of y when $x = 3.2$
- the solution to the equation $2^x = 20$.

- 2 Plot a graph of $y = 1.5^x$ for values of x from -3 to 5 .

Use a scale of 2 cm to 1 unit on both axes.

Use your graph to estimate

- the value of y when $x = 2.4$
- the solution to the equation $1.5^x = 6$.

- 3 a Sketch these curves on the same axes.

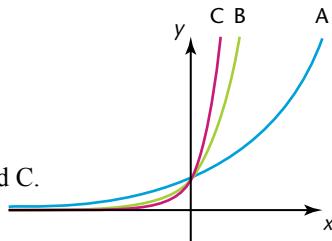
$$y = 2^x, y = 3.5^x \text{ and } y = 5^x$$

- b The graph shows three curves, A, B and C.

Match each curve with its equation.

$$y = 6^x \quad y = 1.5^x \quad y = 3^x$$

- 4 Without drawing the graph, write down how the graph of $y = 3 \times 2^x$ would look compared to the graph of $y = 2^x$.



Estimating the gradient to a curve

The gradient of a straight line is constant, and gradient = $\frac{\text{increase in } y}{\text{increase in } x}$.

The gradient of a curve varies. At any point the gradient of the curve is the gradient of the tangent to the curve at that point.

So to find the gradient of a curve at a given point, first draw the tangent at the point and then work out its gradient. Since you cannot know whether the tangent you have drawn is completely accurate, this method only gives an estimate of the gradient of the curve.

Example 25.9

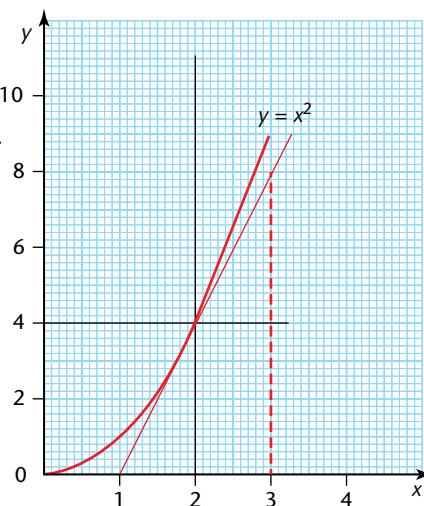
Question

Estimate the gradient of the curve $y = x^2$ at the point $(2, 4)$.

Solution

Plot the graph and draw a tangent to the curve at $(2, 4)$.

$$\begin{aligned} \text{Gradient} &= \frac{\text{increase in } y}{\text{increase in } x} \\ &= \frac{8 - 0}{3 - 1} \\ &= 4 \end{aligned}$$



Note

To draw as good a tangent as possible, try to move your ruler so you have an equal amount of space between the curve and the tangent on each side of the given point.

When choosing two points to work out the gradient of the tangent, it is easiest to make the x increase an easy number.

Exercise 25.5

- 1** Use the graph of $y = x^2 - 4x + 3$ that you drew in Exercise 25.2 question **1**.
Draw tangents to the curve at $x = 4$ and $x = 1$.
Use the tangents to estimate the gradient of the curve at $x = 4$ and $x = 1$.
- 2** Use the graph of $y = x^2 - 3x$ that you drew in Exercise 25.2 question **2**.
Draw tangents to the curve at $x = 4$ and $x = 1$.
Use the tangents to estimate the gradient of the curve at $x = 4$ and $x = 1$.
- 3** Use the graph of $y = x^3 + 5$ that you drew in Exercise 25.3 question **1**.
Draw tangents to the curve at $x = 2$ and $x = -1$.
Use the tangents to estimate the gradient of the curve at $x = 2$ and $x = -1$.
- 4** Use the graph of $y = x^3 - 12x + 2$ that you drew in Exercise 25.3 question **4**.
Draw tangents to the curve at $x = 0$, $x = 2$ and $x = 3$.
Use the tangents to estimate the gradient of the curve at $x = 0$, $x = 2$ and $x = 3$.
- 5** Look at the graph of $y = \frac{1}{x}$ that you drew in Exercise 25.3 question **3**.
 - a** Identify the x values of the points on the curve where the gradient is -1 .
 - b** Draw tangents to estimate the gradient of the curve at the points where $x = 2$ and $x = -\frac{1}{2}$.

26 FUNCTION NOTATION

Function notation

$f: x \mapsto 2x - 3$ means the **function** that maps x onto $2x - 3$.

This is often written instead as $f(x) = 2x - 3$.

$f(5)$ means the value of the function f when $x = 5$.

To find this, substitute 5 in the expression.

In the example above this will be $f(5) = 2 \times 5 - 3 = 7$.

Sometimes other letters, such as g and h , are used for functions.

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- use function notation, e.g. $f(x) = 3x - 5$, $f: x \mapsto 3x - 5$, to describe simple functions
- find inverse functions $f^{-1}(x)$.

Example 26.1

Question

You are given that $f: x \mapsto 3(x - 2)$.

- Find the value of $f(6)$.
- Solve the equation $f(x) = 6$.
- Find and simplify an expression for $f(x + 4)$.

Solution

a $f(6) = 3(6 - 2)$

$$= 3 \times 4$$

$$= 12$$

b $3(x - 2) = 6$

$$3x - 6 = 6$$

$$3x = 12$$

$$x = 4$$

- c To find $f(x + 4)$, substitute $x + 4$ instead of x in the expression for f .

$$f(x + 4) = 3(x + 4 - 2)$$

$$= 3(x + 2)$$

Note

You could start by dividing both sides by 3, giving $x - 2 = 2$.

CHECK YOU CAN:

- substitute numbers into algebraic expressions
- simplify algebraic expressions
- solve equations
- change the subject of a formula.

Exercise 26.1

- When $f: x \mapsto 4x + 3$, find
 - $f(1)$
 - $f(-2)$
 - $f(0)$.
- When $g(x) = 6 - 2x$, find
 - $g(2)$
 - $g(-3)$
 - $g(0.5)$.
- When $h(x) = \frac{x+6}{x}$, find
 - $h(3)$
 - $h(-4)$
 - $h\left(\frac{1}{2}\right)$.
- When $f(x) = x^2 + 7$, find
 - $f(4)$
 - $f(-3)$
 - $f\left(\frac{1}{2}\right)$.

- 5** When $g: x \mapsto 5(2 + 3x)$,
- find $g(-4)$
 - solve $g(x) = 0$
 - find an expression for $g(-x)$.
- 6** When $h(x) = \frac{12}{x+1}$,
- find $h(2)$
 - solve $h(x) = -6$
 - find and simplify an expression for $h(2x - 1)$.
- 7** When $f(x) = 2x - 3$ and $g(x) = 7 - 2x$,
- find $f(4)$
 - solve $g(x) = 0$
 - solve $f(x) = g(x)$.
- 8** When $h(x) = x^2 - 5$,
- find $h(4)$
 - find and simplify an expression for $h(2x)$
 - find the two values of x for which $h(x) = 31$.
- 9** When $f(x) = x(x - 5)$,
- find $f(4)$
 - solve $f(x) = 0$
 - solve $f(x) = 14$.
- 10** When $g(x) = \frac{2x+3}{5}$ and $h(x) = 4x - 1$,
- find $g(2)$
 - solve $h(x) = 6$
 - solve $g(x) = h(x)$.

Inverse functions

The **inverse** of a function f is the function that maps all the values of $f(x)$ back to their original x values.

The inverse of a function is written as $f^{-1}(x)$.

For example, when $f(x) = x + 2$, $f^{-1}(x) = x - 2$.

To find the inverse function, write y in place of $f(x)$.

$$y = x + 2$$

Rearrange the formula to make x the subject.

$$x = y - 2$$

Replace x with $f^{-1}(x)$ and y with x to give the expression for the inverse function.

$$f^{-1}(x) = x - 2$$

Example 26.2**Question**When $f(x) = 3x - 5$,

- a** find $f^{-1}(x)$ **b** find $f^{-1}(7)$.

Solution

a $y = 3x - 5$

Write y in place of $f(x)$.

$$y + 5 = 3x$$

Rearrange the formula to make x the subject.

$$x = \frac{y+5}{3}$$

$$f^{-1}(x) = \frac{x+5}{3}$$

Replace x with $f^{-1}(x)$ and y with x .

b $f^{-1}(7) = \frac{7+5}{3} = 4$

Note

Finding $f^{-1}(7)$ is the same as finding the value of x when $f(x) = 7$. We could solve the equation $3x - 5 = 7$ if we had not found $f^{-1}(x)$ first.

Exercise 26.2

- 1 When $f: x \mapsto 4x + 3$,
 - a** solve $f(x) = 5$
 - b** find $f^{-1}(x)$
 - c** use $f^{-1}(x)$ to find $f^{-1}(5)$.
- 2 When $g(x) = 6 - 2x$,
 - a** find $g^{-1}(x)$
 - b** use $g^{-1}(x)$ to find $g^{-1}(4)$ and check your work by substituting your answer into $g(x)$.
- 3 When $h: x \mapsto 3(x - 6)$, find

a $h^{-1}(x)$	b $h^{-1}(9)$	c $h^{-1}(0)$.
----------------------	----------------------	------------------------
- 4 When $f(x) = \frac{2x+1}{5}$, find

a $f^{-1}(x)$	b $f^{-1}(3)$	c $f^{-1}\left(\frac{4}{5}\right)$.
----------------------	----------------------	---
- 5 When $g(x) = \frac{x}{2} - 7$, find

a $g^{-1}(x)$	b $g^{-1}(-4)$	c $g^{-1}(12)$.
----------------------	-----------------------	-------------------------
- 6 When $h(x) = \frac{12}{x+1}$, find

a $h^{-1}(x)$	b $h^{-1}(-4)$	c $h^{-1}(2)$.
----------------------	-----------------------	------------------------
- 7 When $f(x) = 3(2x - 5)$, find

a $f^{-1}(9)$	b $f^{-1}(0)$	c $f^{-1}(-18)$.
----------------------	----------------------	--------------------------
- 8 When $g(x) = 7 - 3x$, find

a $g^{-1}(1)$	b $g^{-1}(-5)$	c $g^{-1}(-293)$.
----------------------	-----------------------	---------------------------
- 9 Find the inverse of each of these functions.

a $f(x) = \frac{6x-1}{3}$	b $g(x) = 2 - 5x$	c $h(x) = \frac{6}{x}$
d $f(x) = \frac{3x}{4} - 7$	e $g(x) = 4(2x + 1)$	f $h(x) = \frac{2x+1}{x}$

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- demonstrate familiarity with Cartesian coordinates in two dimensions
- find the gradient of a straight line
- calculate the gradient of a straight line from the coordinates of two points on it
- calculate the length and the coordinates of the midpoint of a line segment from the coordinates of its end points
- interpret and obtain the equation of a straight-line graph in the form $y = mx + c$
- determine the equation of a straight line parallel to a given line
- find the gradient of parallel and perpendicular lines.

CHECK YOU CAN:

- use coordinates in two dimensions
- draw a straight-line graph, given its equation
- use Pythagoras' Theorem to calculate lengths
- substitute into and rearrange simple formulae
- understand the meaning of the terms *parallel* and *perpendicular*.

The gradient of a straight-line graph

You have already found the **gradient** of a straight line as

$$\text{Gradient} = \frac{\text{increase in } y}{\text{increase in } x}$$

Remember that lines with a positive gradient slope up to the right and lines with a negative gradient slope down to the right.

If you know two points on the line then you do not have to draw it to work out the gradient.

For a line through two points (x_1, y_1) and (x_2, y_2) , gradient = $\frac{y_2 - y_1}{x_2 - x_1}$.

Example 27.1

Question

Find the gradient of the line joining the points $(3, 5)$ and $(8, 7)$.

Solution

$$\text{Gradient} = \frac{7 - 5}{8 - 3} = \frac{2}{5}$$

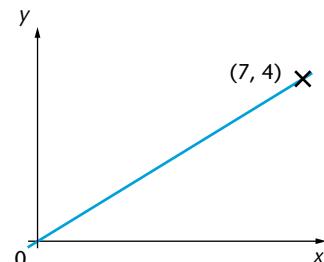
Note

You can use the points the other way round in the formula. The answer will be the same.

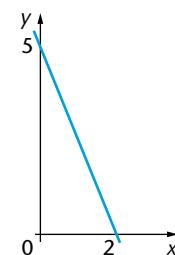
Exercise 27.1

- 1 Find the gradient of these lines.

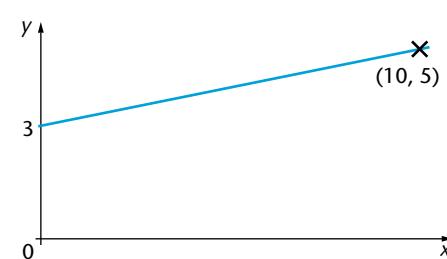
a



b



c



- 2** Calculate the gradient of the line joining each of these pairs of points.
- a** (1, 5) and (3, 5) **b** (-1, 1) and (3, 2) **c** (-2, 6) and (0, 4)
- 3** Calculate the gradient of the line joining each of these pairs of points.
- a** (0.6, 3) and (3.6, -9) **b** (2.5, 4) and (3.7, 4.9)
- c** (2.5, 7) and (4, 2.2)

Line segments

A line can be extended in either direction forever. It is infinite.

A **line segment** is the part of a line between two points.

The midpoint of a line segment

The coordinates of the midpoint of a line segment are the mean of the coordinates of the two end points.

The midpoint of the line joining the points (x_1, y_1) and (x_2, y_2)
is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$.

Example 27.2

Question

Find the coordinates of the midpoint of the line segment with these end points.

- a** A(2, 1) and B(6, 7) **b** C(-2, 1) and D(2, 5)

Solution

$$\begin{aligned} \text{a} \quad \text{Midpoint} &= \left(\frac{2+6}{2}, \frac{1+7}{2} \right) & \text{b} \quad \text{Midpoint} &= \left(\frac{-2+2}{2}, \frac{1+5}{2} \right) \\ &= (4, 4) & &= (0, 3) \end{aligned}$$

The length of a line segment

You can use Pythagoras' Theorem to find the length of a line segment.

The length of the horizontal side is $6 - 1 = 5$.

The length of the vertical side is $5 - 2 = 3$.

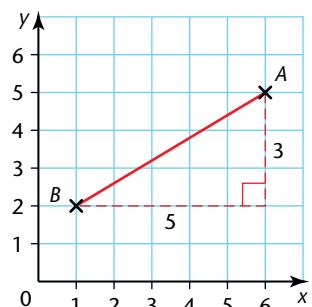
You can then use Pythagoras' Theorem to work out the length of AB .

$$AB^2 = 5^2 + 3^2$$

$$AB^2 = 25 + 9 = 34$$

$$AB = \sqrt{34}$$

$$AB = 5.83 \text{ units to 2 decimal places}$$



The length of a line segment with end points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Example 27.3

Question

A is the point $(-5, 4)$ and B is the point $(3, 2)$.

Find the length AB .

Solution

$$\begin{aligned} AB &= \sqrt{(-5-3)^2 + (4-2)^2} = \sqrt{(-8)^2 + 2^2} = \sqrt{64+4} = \sqrt{68} \\ &= 8.25 \text{ units to 2 decimal places} \end{aligned}$$

Exercise 27.2

- 1 For each of these line segments
 - i find the coordinates of the midpoint
 - ii calculate the length.

You can draw a diagram to help you.

 - a $A(1, 4)$ and $B(1, 8)$
 - b $C(1, 5)$ and $D(7, 3)$
 - c $E(2, 3)$ and $F(8, 6)$
 - d $G(3, 7)$ and $H(8, 2)$
 - e $I(-2, 3)$ and $J(4, 1)$
 - f $K(-4, -3)$ and $L(-6, -11)$
- 2 A is the point $(-2, -5)$.
The midpoint of AB is $(3, 1)$.
Find the coordinates of B .
- 3 A is the point $(-1, 5)$, B is the point $(7, 3)$ and C is the point $(1, -7)$.
 - a Find the length of AC .
 - b Find the length of the line joining the midpoints of BA and BC .
 - c What do you notice?

The general form of the equation of a straight line

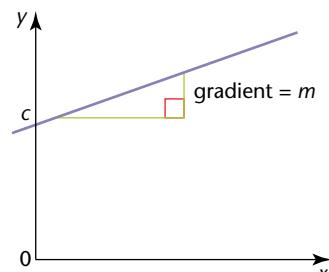
This line has gradient m and crosses the y axis at the point $(0, c)$.

c is called the **y intercept** because it is where the line intercepts, or crosses, the y axis.

The equation of the line is $y = mx + c$.

The equation of any straight line can be rearranged to give $y = mx + c$.

If you know the equation of a line, you can easily find its gradient and its y intercept.



Example 27.4**Question**

The equation of a straight line is $5x + 2y = 10$.

Find its gradient and y intercept.

Solution

$$5x + 2y = 10$$

$$2y = -5x + 10$$

Rearrange the equation into the form $y = mx + c$.

$$y = -2.5x + 5$$

So the gradient is -2.5 and the y intercept is 5 .

Finding the equation of a straight line

You can also use this method in reverse to find the equation of a line from its graph.

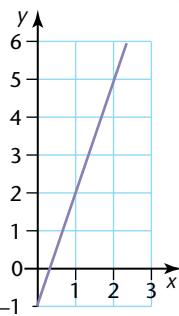
Example 27.5**Question**

Find the equation of this straight line.

Solution

$$\text{The gradient of a line, } m = \frac{\text{increase in } y}{\text{increase in } x} \\ = \frac{6 - (-1)}{2 - 0} = 3$$

The line passes through $(0, -1)$ so the y intercept, c , is -1 .
So the equation of the line is $y = 3x - 1$.

**Example 27.6****Question**

Find the equation of the line that passes through the points $(4, 6)$ and $(6, 2)$.

Solution

First find the gradient of the line.

$$\text{Gradient, } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 6}{6 - 4} = \frac{-4}{2} = -2$$

To find the y intercept, c , substitute the coordinates of either of the given points and the gradient into the general equation of a line.

$$y = mx + c$$

$$y = -2x + c \quad \text{The gradient is } -2.$$

$$6 = -2 \times 4 + c \quad \text{Substitute the coordinates of one of the given points.}$$

$$6 = -8 + c$$

$$6 + 8 = c$$

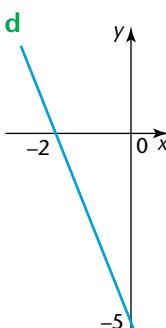
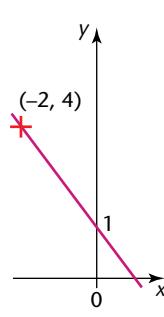
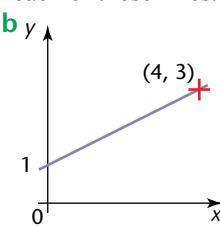
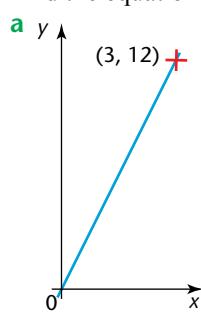
$$c = 14 \quad \text{Solve to find } c.$$

$$\text{So } y = -2x + 14$$

$$\text{Or } 2x + y = 14$$

Exercise 27.3

- 1 Find the gradient of each of these lines and their y intercept.
- $y = 3x - 2$
 - $y = 5x - 3$
 - $y = 2 + 5x$
 - $y = 7 + 2x$
 - $y = 7 - 2x$
 - $y = 9 - 3x$
- 2 Find the gradient of each of these lines and their y intercept.
- $y + 2x = 5$
 - $y - 5x = 1$
 - $4x + 2y = 7$
 - $3x + 2y = 8$
 - $6x + 5y = 10$
 - $2x + 5y = 15$
- 3 Write down the equation of each of these straight lines.
- Gradient 3 and passing through $(0, 2)$
 - Gradient -1 and passing through $(0, 4)$
 - Gradient 5 and passing through $(0, 0)$
 - Gradient 4 and passing through $(0, -1)$
 - Gradient -2 and passing through $(0, 5)$
 - Gradient 3 and passing through the origin
- 4 Find the equation of each of these lines.



- 5 Find the equation of each of these lines.
- A line with gradient $-\frac{3}{4}$, passing through the point $(3, 0)$.
 - A line passing through $(1, 4)$ and $(4, 7)$.
 - A line passing through $(2, 3)$ and $(5, 9)$.
 - A line passing through $(-1, 5)$ and $(3, -7)$.
 - A line passing through $(3, 1)$ and $(6, -1)$.

Parallel and perpendicular lines

Parallel lines have the same gradient.

There is also a connection between the gradients of **perpendicular** lines.

If a line has gradient m , a line perpendicular to this has gradient $-\frac{1}{m}$.

The product of the two gradients is -1 .

Note

$$m \times \frac{-1}{m} = -1$$

Example 27. 7

Question

Find the equation of the line that is parallel to the line $2y = 3x + 4$ and passes through the point (3, 2).

Solution

The gradient of the line $2y = 3x + 4$ is 1.5.

So the equation of a line parallel to this is $y = 1.5x + c$.

$$y = 1.5x + c$$

To find the y intercept, c , substitute the coordinates of the given point into the equation.

$$2 = 1.5 \times 3 + c$$

$$2 = 4.5 + c$$

$$c = 2 - 4.5$$

$$c = -2.5$$

$$\text{So } y = 1.5x - 2.5$$

$$\text{Or } 2y = 3x - 5$$

Multiply both sides of the equation by 2 to eliminate the decimals.

Example 27. 8

Question

Find the equation of the line that crosses the line $y = 2x - 5$ at right angles at the point (3, 1).

Solution

The gradient of the line $y = 2x - 5$ is 2.

The gradient of a line perpendicular to this is $-\frac{1}{m}$.

So the gradient of the perpendicular line is $-\frac{1}{2}$.

So the equation of the perpendicular line is $y = -\frac{1}{2}x + c$.

$$y = -\frac{1}{2}x + c$$

To find the y intercept, c , substitute the coordinates of the given point into the equation.

$$1 = -\frac{1}{2} \times 3 + c$$

$$1 = -\frac{3}{2} + c$$

$$c = 1 + \frac{3}{2}$$

$$c = 2\frac{1}{2}$$

$$\text{So } y = -\frac{1}{2}x + 2\frac{1}{2}$$

$$x + 2y = 5$$

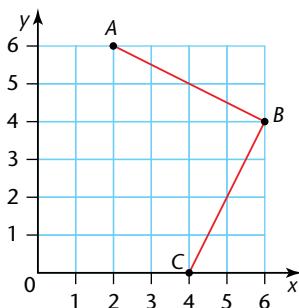
Multiply both sides of the equation by 2 to eliminate the fractions and rearrange to give a positive x term.

A line parallel to $y = mx + c$ will be of the form $y = mx + d$, where c and d are constants.

A line perpendicular to $y = mx + c$ will be of the form $y = -\frac{1}{m}x + e$, where c and e are constants.

Exercise 27.4

- 1** In the diagram AB and BC are two sides of a square, $ABCD$.
- Find the equation of the line AD .
 - Find the equation of the line DC .
 - Find the coordinates of D .



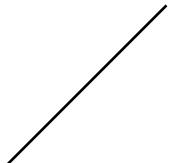
- 2** Which of these lines are
- parallel
 - perpendicular?
- $$y = 4x + 3 \quad 2y - 3x = 5 \quad 6y + 4x = 1 \quad 4x - y = 5$$
- 3** Find the equation of the line that passes through $(1, 5)$ and is parallel to $y = 3x - 3$.
- 4** Find the equation of the line that passes through $(0, 3)$ and is parallel to $3x + 2y = 7$.
- 5** Find the equation of the line that passes through $(1, 5)$ and is perpendicular to $y = 3x - 1$.
- 6** Find the equation of the line that passes through $(0, 3)$ and is perpendicular to $3x + 2y = 7$.
- 7** Find the gradient of a line perpendicular to the line joining each of these pairs of points.
- $(1, 2)$ and $(5, -6)$
 - $(2, 4)$ and $(3, 8)$
 - $(-2, 1)$ and $(2, -4)$
- 8** Two lines cross at right angles at the point $(5, 3)$. One passes through $(6, 0)$. What is the equation of the other line?

28 GEOMETRICAL TERMS

Dimensions

A **point** has no dimensions. .

A **line** has length but no width. It has one dimension.

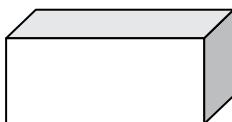


A **plane** has length and width but no height.

It is a flat **surface** with two dimensions.

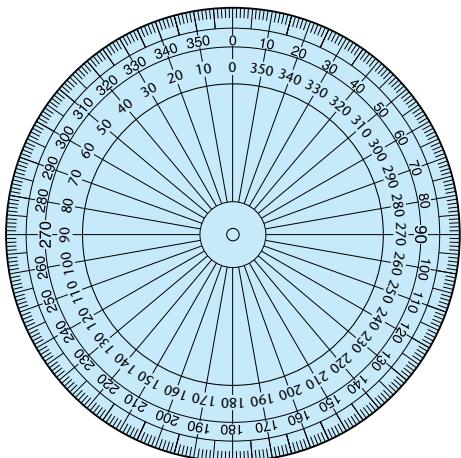


A shape which has length, width and height has three dimensions.



Angles

To describe and measure angles, we use a scale marked in degrees.



On this scale, one whole turn is equal to 360 degrees. This is written as 360° .

So a half turn is equal to 180° and a quarter turn is equal to 90° .

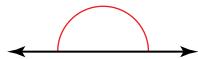
BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- use and interpret the geometrical terms: point; line; plane; parallel; perpendicular; bearing; right angle, acute, obtuse and reflex angles; interior and exterior angles; similarity and congruence
- use and interpret vocabulary of triangles, special quadrilaterals, circles, polygons and simple solid figures
- understand and use the terms: centre, radius, chord, diameter, circumference, tangent, arc, sector and segment
- use and interpret nets.

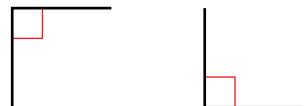
CHECK YOU CAN:

- read scales
- use a ruler and compasses
- draw diagrams accurately on a square grid.

An angle of a half turn (180°) is a straight line.



Angles of a quarter turn (90°) are called **right angles**.



Angles of less than 90° are called **acute angles**.



Angles of between 90° and 180° are called **obtuse angles**.



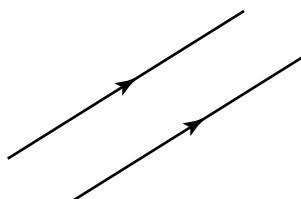
Angles of more than 180° are called **reflex angles**.



Lines

Two lines that never meet are **parallel** to each other.

Arrows are used to show that two lines are parallel.

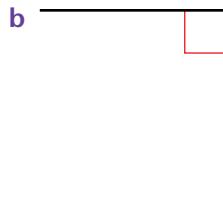
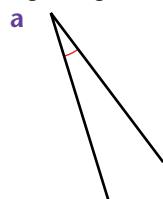


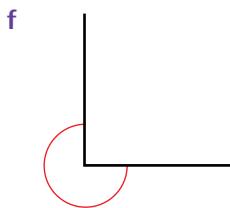
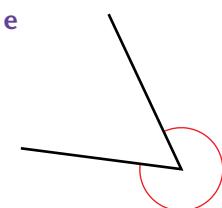
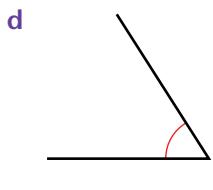
Two lines that meet or cross at 90° are **perpendicular** to each other.



Exercise 28.1

- 1 For each of these angles, say whether it is acute, obtuse, reflex or a right angle.





2 What sort of angle is each of these sizes?

- a 145°
- b 86°
- c 350°
- d 190°
- e 126°
- f 226°
- g 26°
- h 90°

3 ABCD is a rectangle.



Which side is

- a perpendicular to AB
- b parallel to DC ?

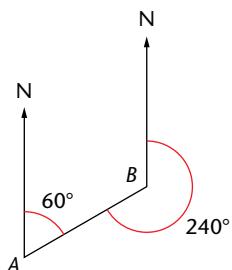
Bearings

Bearings are

- used to describe a direction
- measured in degrees from north clockwise
- made up of three digits.

Example 28.1

Question



Find the bearing of

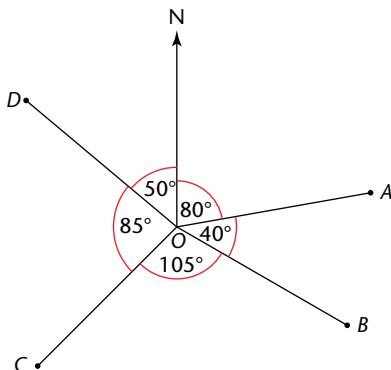
- a B from A
- b A from B.

Solution

- a The angle from the north line at A to the point B is 60° so the bearing of B from A is 060° .
- b The angle from the north line at B to the point A is 240° so the bearing of A from B is 240° .

Exercise 28.2

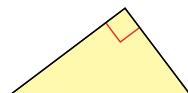
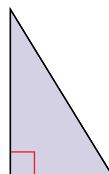
- 1 Write down the bearings for these directions.
 a due east b due south c due west
 d north-east e south-east
- 2 Write these angles as bearings.
 a 24° b 101° c 3°
- 3 Find the bearings from O of the points A, B, C, D .



Triangles

A **triangle** is a three-sided shape.

These are **right-angled triangles**.

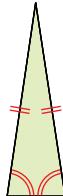
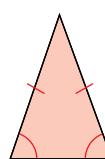


These are **isosceles triangles**.

They have two sides the same length.

They have two angles the same size.

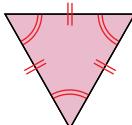
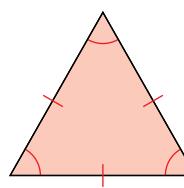
The equal sides are marked with lines.



These are **equilateral triangles**.

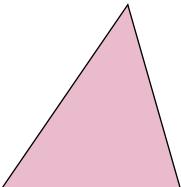
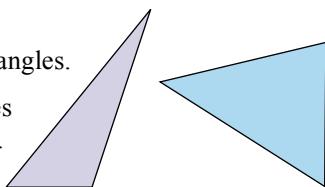
All their sides are the same length.

All their angles are the same size.



These triangles are **scalene** triangles.

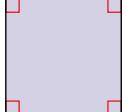
In scalene triangles all the sides and all the angles are different.



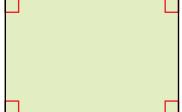
Quadrilaterals

A **quadrilateral** is a four-sided shape.

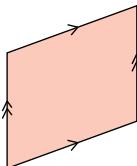
Here are seven different types.



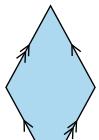
Square



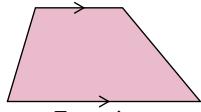
Rectangle



Parallelogram



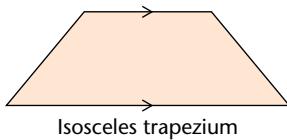
Rhombus



Trapezium



Kite



Isosceles trapezium

The table shows the names and geometrical properties of the seven types of quadrilaterals shown in the diagram.

Name	Angles	Sides		Diagonals
		Lengths	Parallel	
Square	All 90°	All equal	Opposite sides parallel	Equal length Bisect at 90°
Rectangle	All 90°	Opposite sides equal	Opposite sides parallel	Equal length Bisect but not at 90°
Parallelogram	Opposite angles equal	Opposite sides equal	Opposite sides parallel	Not equal Bisect but not at 90°
Rhombus	Opposite angles equal	All equal	Opposite sides parallel	Not equal Bisect at 90°
Trapezium	Can be different	Can be different	One pair of sides parallel	Nothing special
Kite	One pair of opposite angles equal	Two pairs of adjacent sides equal	None parallel	Not equal Only one bisected by the other They cross at 90°
Isosceles trapezium	Two pairs of adjacent angles equal	One pair of opposite sides equal	Other pair of opposite sides parallel	Equal length Do not bisect or cross at 90°

Example 28.2**Question**

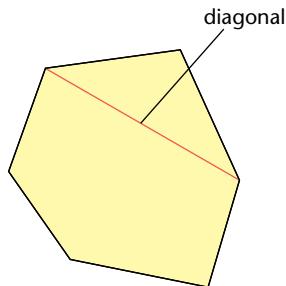
Which quadrilaterals have

- a both pairs of opposite sides equal
- b just one pair of opposite angles equal?

Solution

- a Square, rectangle, parallelogram, rhombus
- b Kite

Polygons



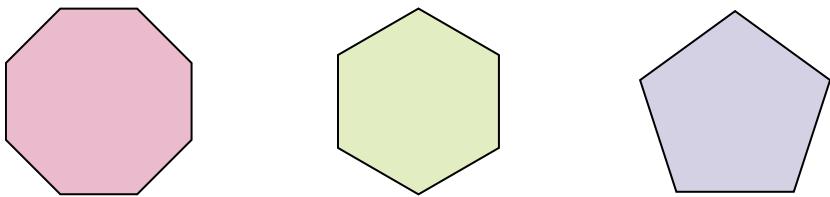
A line joining two corners of a polygon is called a **diagonal**.

A **polygon** is a many-sided shape.

Here are some common ones, apart from the triangles and quadrilaterals you have met already.

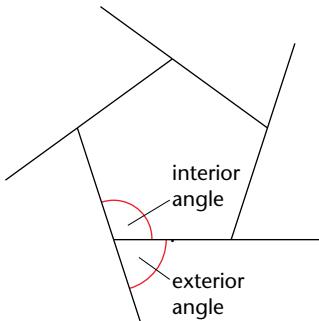
Shape	Pentagon	Hexagon	Octagon	Decagon
Number of sides	5	6	8	10

When the sides of a polygon are all the same length and the angles of the polygon are all the same, it is called a **regular** polygon. Otherwise, the polygon is **irregular**.



The angles inside a polygon are called **interior angles**.

If a side of the shape is continued outside the shape, the angle made is called an **exterior angle**.

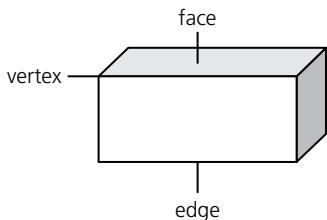


Exercise 28.3

- 1 Draw any pentagon.
Draw diagonals across the pentagon from each vertex (corner of the pentagon) to another vertex.
How many diagonals can you draw in the pentagon altogether?
- 2 Which quadrilaterals have all four angles the same?
- 3 Which quadrilaterals have opposite sides the same, but not all four sides the same?
- 4 A rectangle can be described as a special type of parallelogram.
What else is true about a rectangle?
- 5 Which quadrilaterals have all four sides the same?
- 6 Which quadrilaterals have opposite angles the same, but not 90° ?
- 7 A rhombus can be described as a special type of parallelogram.
What else is true about a rhombus?
- 8 Which quadrilaterals have diagonals that cross at 90° ?
- 9 A quadrilateral has diagonals the same length.
What types of quadrilateral could it be?
- 10 List all the quadrilaterals that have both pairs of opposite sides parallel and equal.

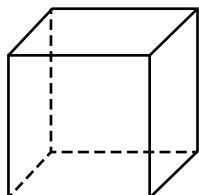
Solids

Here is a **cuboid**.



On this diagram of the cuboid, there are three hidden **edges**, three hidden **faces** and one hidden **vertex**.

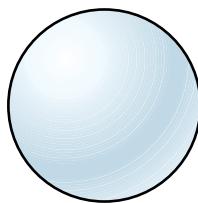
On this diagram of a **cube**, the hidden edges have been shown with dashed lines. All the faces of a cube are squares.



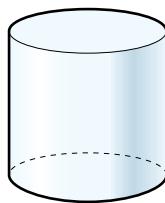
The cube and cuboid are examples of three-dimensional **solids** where all the faces are **planes**.

Some solids have curved **surfaces**.

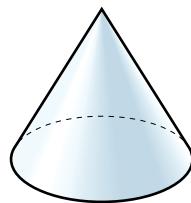
A **sphere** has only a curved surface but a **cylinder** and a **cone** each have both flat and curved surfaces.



Sphere



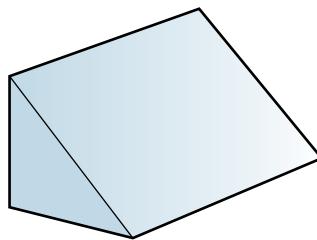
Cylinder



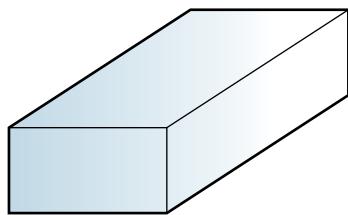
Cone

Prisms and **pyramids** are also three-dimensional solid shapes.

Here are two examples of **prisms**.



Triangular prism



Rectangular prism

Look at the shapes of the faces.

The shape on each end of the triangular prism is a triangle.

All the other faces are rectangles.

If you cut through the triangular prism at right angles to one of the rectangular faces you will see its cross-section. It is the same shape as the triangle at either end of the prism.

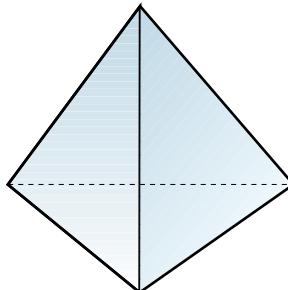
The cross-section of a prism is the same all the way through.

The cross-section of a prism can be any shape.

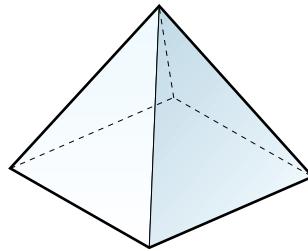
The cross-section of the rectangular prism shown earlier is a rectangle.

A rectangular prism is called a cuboid.

A **pyramid** has a base and triangular faces. The base can have any number of sides.



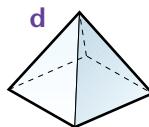
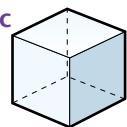
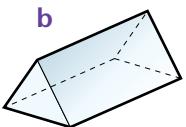
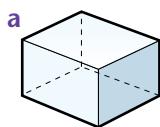
Triangle-based pyramid



Square-based pyramid

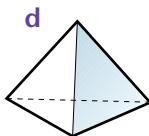
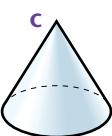
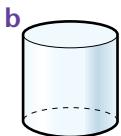
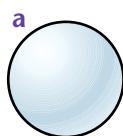
Exercise 28.4

- 1** Name each of these shapes.



- 2** How many faces and how many vertices do each of the shapes in question **1** have?

- 3** Name each of these shapes.



- 4** Which of the shapes in questions **1** and **3** are also prisms?

Congruence and similarity

Shapes that are exactly the same size and shape are said to be **congruent**.

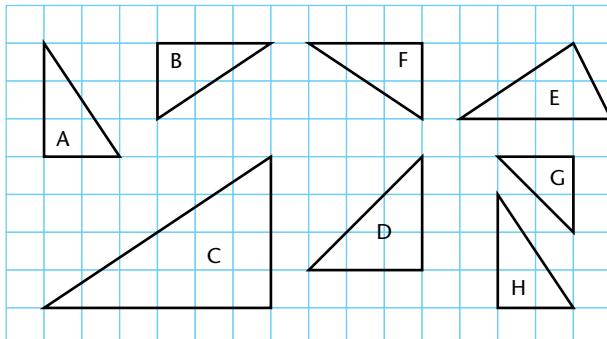
If you cut out two congruent shapes one shape would fit exactly on top of the other.

The corresponding sides are equal and the corresponding angles are equal.

Example 28.3

Question

Which of these triangles are congruent to triangle A?

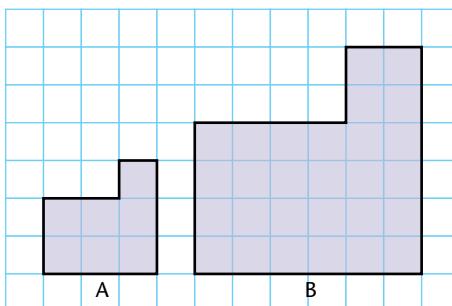


Solution

B, F and H.

They are exactly the same with corresponding sides equal, but are a different way round.

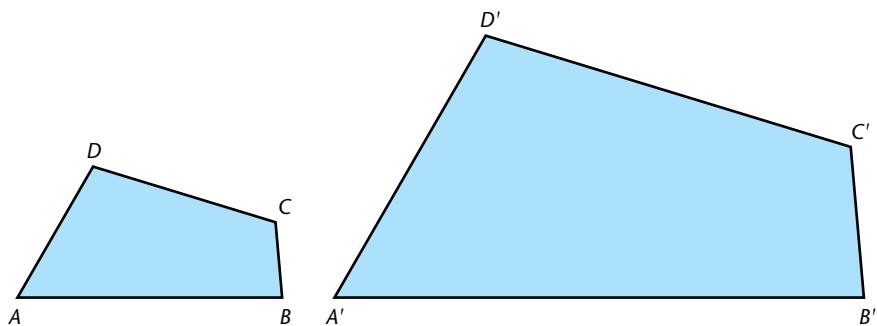
Every length in shape B is two times as long as the corresponding length in the shape A.



Shape B is a 2 times **enlargement** of shape A.

We can also say that shapes A and B are **similar**.

Shapes like these two quadrilaterals, which have the same angles and side lengths that are in proportion, are **similar**. One is an enlargement of the other.

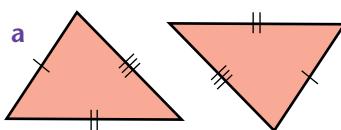


Exercise 28.5

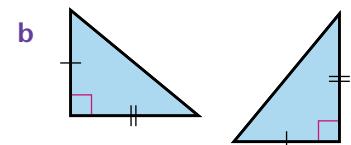
- 1 In this question, sides that are equal are marked with corresponding lines and angles that are equal are marked with corresponding arcs.

Which of these pairs of triangles are congruent?

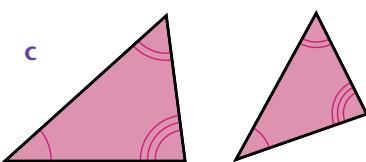
a



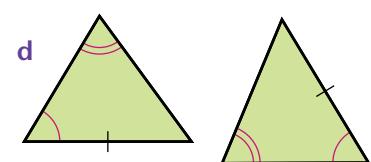
b



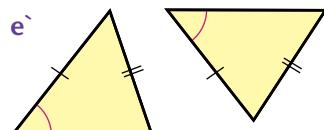
c



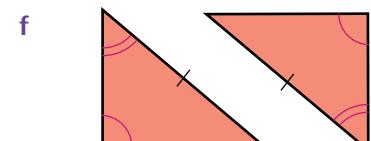
d



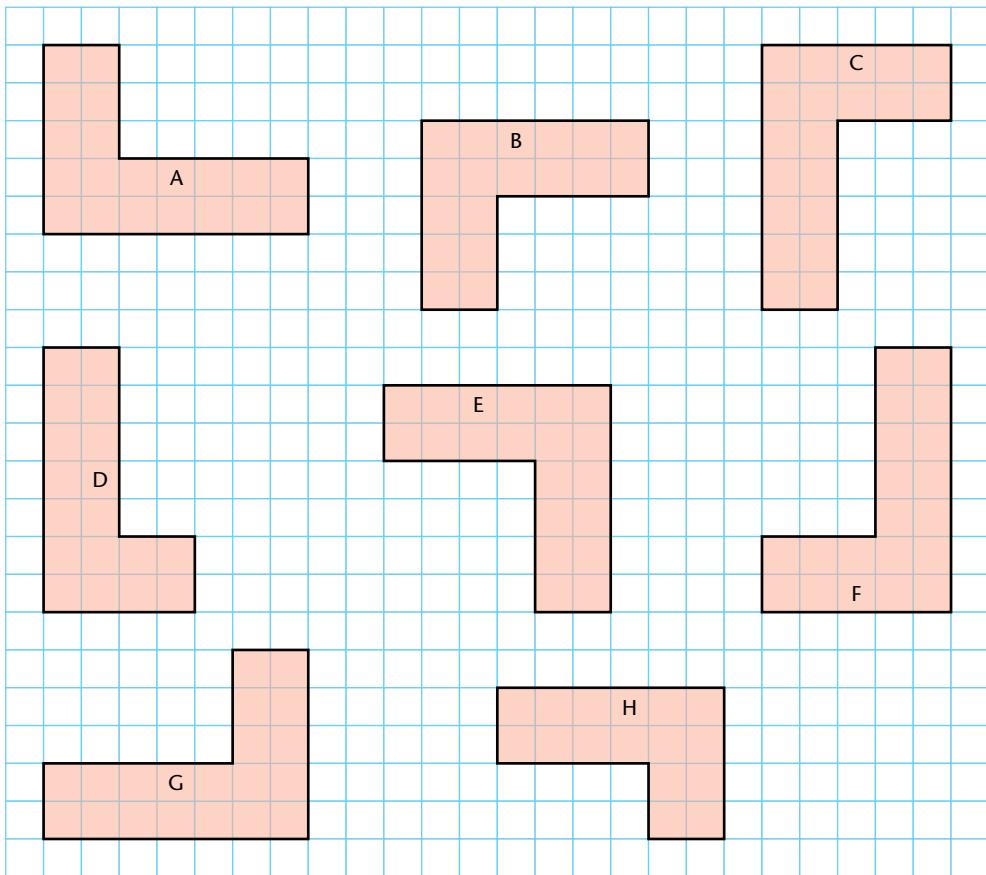
e



f



- 2** Which of these shapes are congruent to shape A?



- 3** Two triangles are congruent.

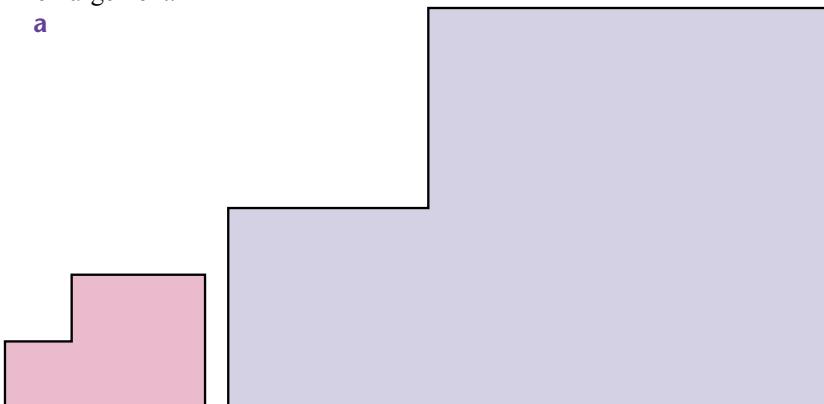
The first triangle has angles of 35° , 75° and 70° .

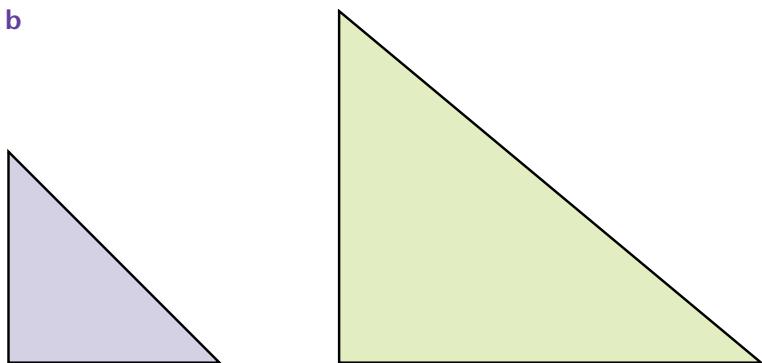
What are the sizes of the angles in the second triangle?

- 4** Measure the lengths of the two shapes in each diagram.

Are the shapes similar? If they are, state the scale factor of the enlargement.

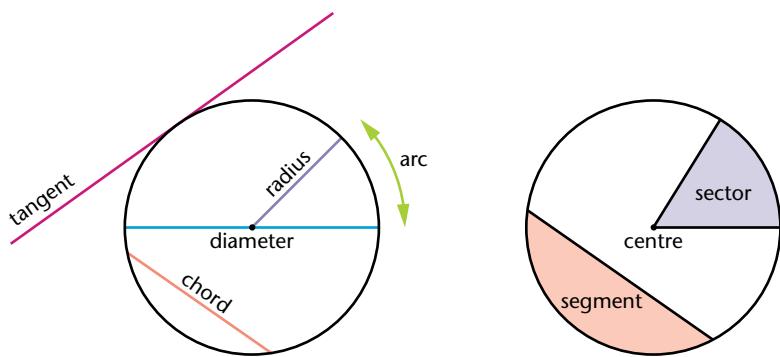
a





Circles

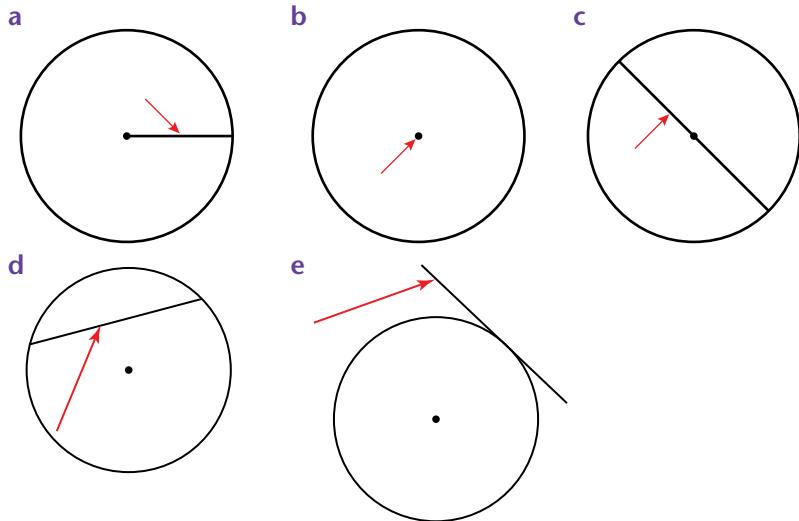
You need to be able to identify the parts of a **circle**.



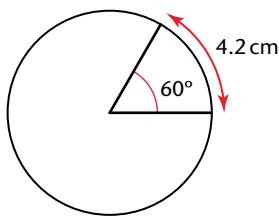
The **circumference** of a circle is the distance all the way round – the perimeter of the circle.

Exercise 28.6

- 1 Name each of these parts of a circle.



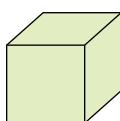
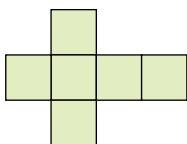
- 2 A circle has radius 5.7 cm.
How long is its diameter?
- 3 A sector of a circle has angle 60° at the centre of the circle.
The arc length of the sector is 4.2 cm.
Use the fact that a full turn is 360° to find the length of the circumference of the circle.



Nets of 3-D shapes

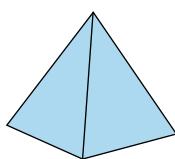
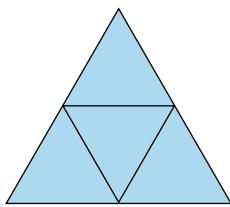
A **net** of a three-dimensional object is a flat shape that will fold to make that object.

This flat shape can be folded to make a cube – it is a net of the cube.



A regular triangular-based pyramid has edges that are all the same length.

Here is its net.



As a cuboid has six faces, its net needs six rectangles. But not every arrangement of the six rectangles will fold together to make the cuboid.

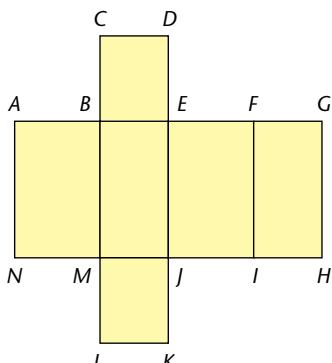
Example 28.4

Question

This net can be folded to make a cuboid.

When it is folded, which point or points will meet with

- a** point A **b** point D?



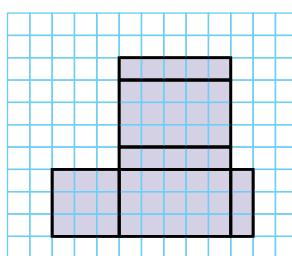
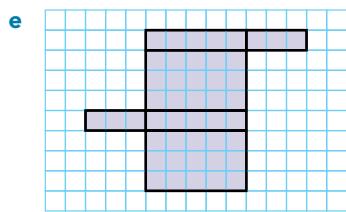
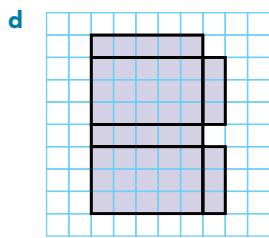
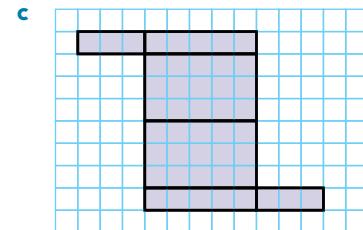
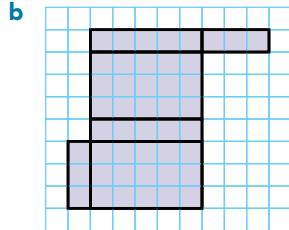
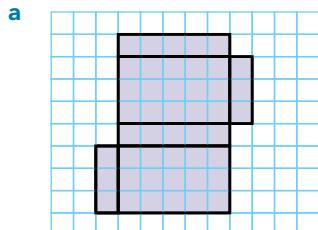
Solution

- a** Points C and G **b** Point F

Example 28.5**Question**

For each of these arrangements, say whether it is the net of a cuboid.

They are drawn on squared paper to size.

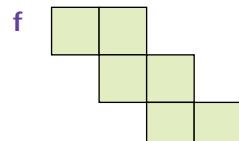
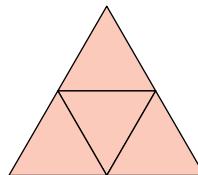
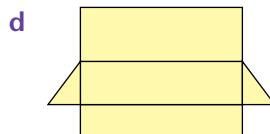
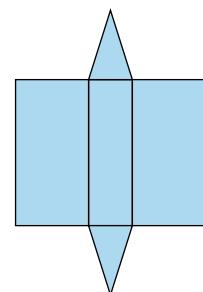
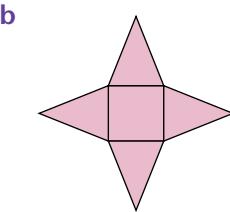
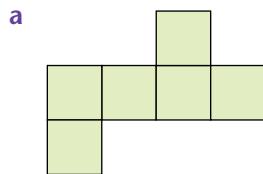
**Solution**

- a** Yes **b** Yes
d No **e** Yes

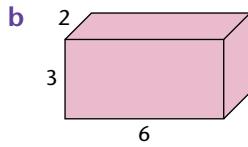
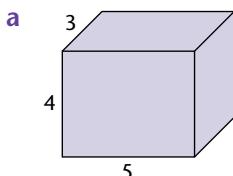
- c** No
f No

Exercise 28.7

- 1 Look at each of these nets and say what shape it makes.



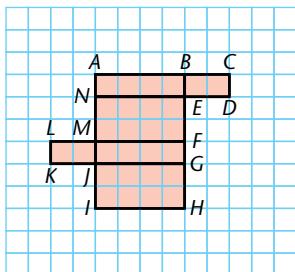
- 2** Draw a net for each of these cuboids on squared paper.
All lengths are given in centimetres.



Note

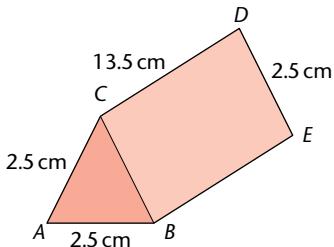
If you are not sure whether a net is correct, cut it out and try folding it.

- 3** For a cuboid, state
- the number of edges
 - the number of vertices
 - the number of faces.
- 4** When this net is folded to make a cuboid, which point or points will meet with
- point A
 - point D?



- 5** A box containing paper has a base 15 cm by 20 cm and it is 5 cm high.
It has no top.
Draw a net for the box.
Use a scale of 1 cm to 5 cm.

- 6** Construct a net of this triangular prism.



BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- measure lines and angles
- construct a triangle, given three sides, using a ruler and a pair of compasses only
- construct other simple geometrical figures from given data, using a ruler and protractor as necessary
- construct angle bisectors and perpendicular bisectors using protractors or set squares as necessary
- read and make scale drawings.

CHECK YOU CAN:

- read the scales on a ruler
- read the scales on a protractor.

Measuring angles

The instrument used to measure an angle is called a protractor or angle measurer.

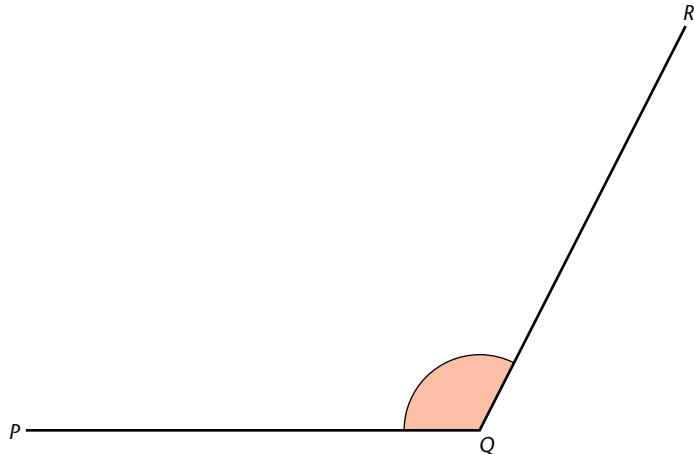
Some protractors are full circles and can be used to measure angles up to 360° .

Most protractors are semi-circular in shape and can be used to measure angles up to 180° .

Example 29.1

Question

Measure angle PQR .



Solution

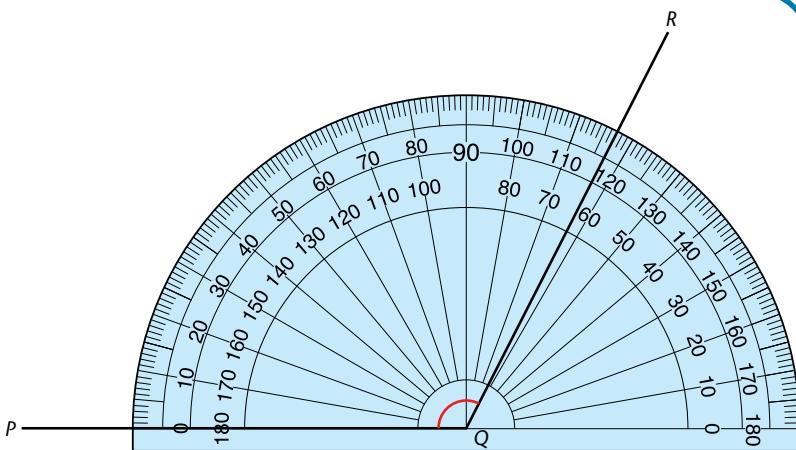
First make an estimate.

The angle is obtuse and so will be between 90° and 180° .

A rough estimate is about 120° .

Place your protractor so that the zero line is along one of the arms of the angle and the centre is at the point of the angle.





Start at zero.

Go round this scale until you reach the other arm of the angle.

Then read the size of the angle from the scale.

Angle $PQR = 117^\circ$

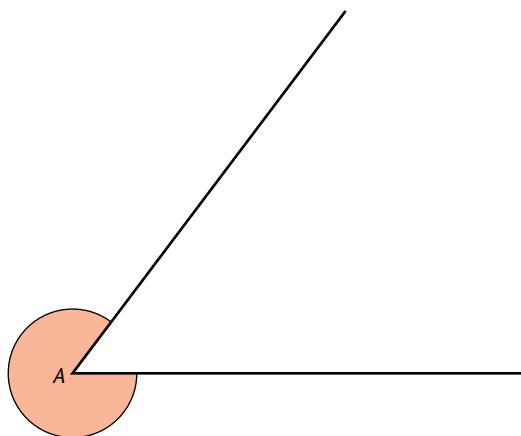
Note

A common error in measuring angles is to use the wrong scale of the two on the protractor. Make sure you choose the one that starts at zero. A useful further check is to estimate the angle first. Knowing approximately what the angle is should prevent you using the wrong scale.

Example 29.2

Question

Measure angle A.



Solution

A reflex angle is between 180° and 360° .

This reflex angle is over $\frac{3}{4}$ of a turn so it is bigger than 270° .

A rough estimate is 300° .

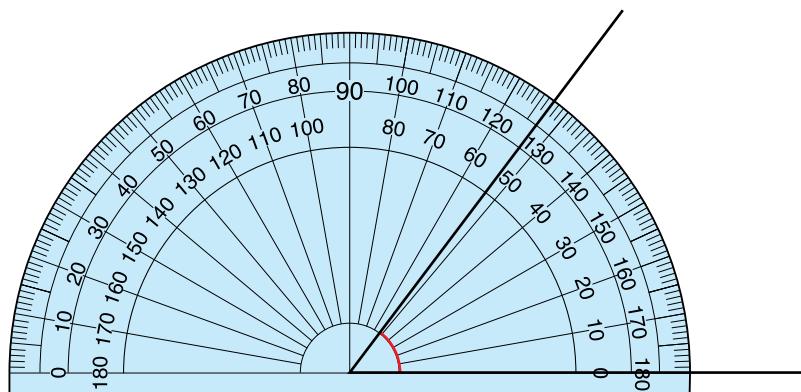
You can measure an angle of this size directly using a 360° circular angle measurer. However, the scale on a semi-circular protractor only goes up to 180° .



You need to do a calculation as well as measure an angle.

Measure the acute angle first.

The acute angle is 53° .



The acute angle and the reflex angle together make 360° .

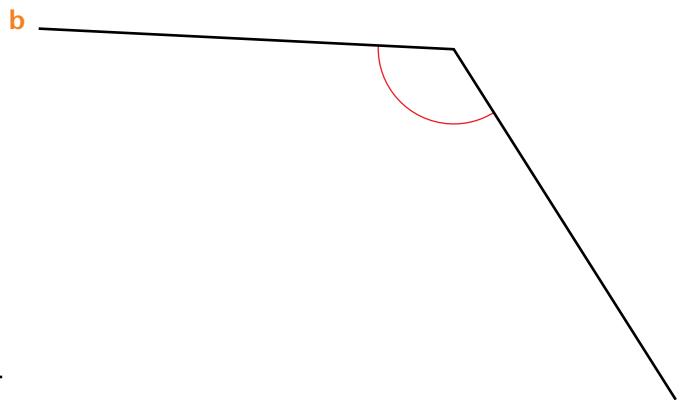
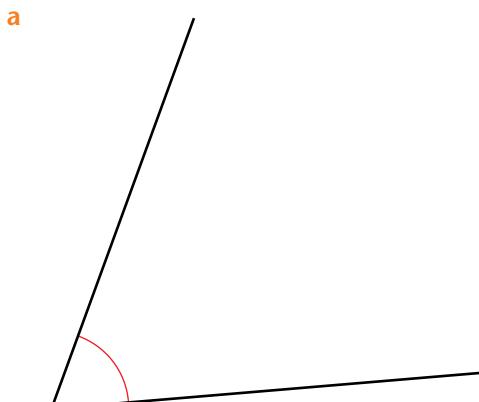
Use the fact that the two angles add up to 360° to calculate the reflex angle.

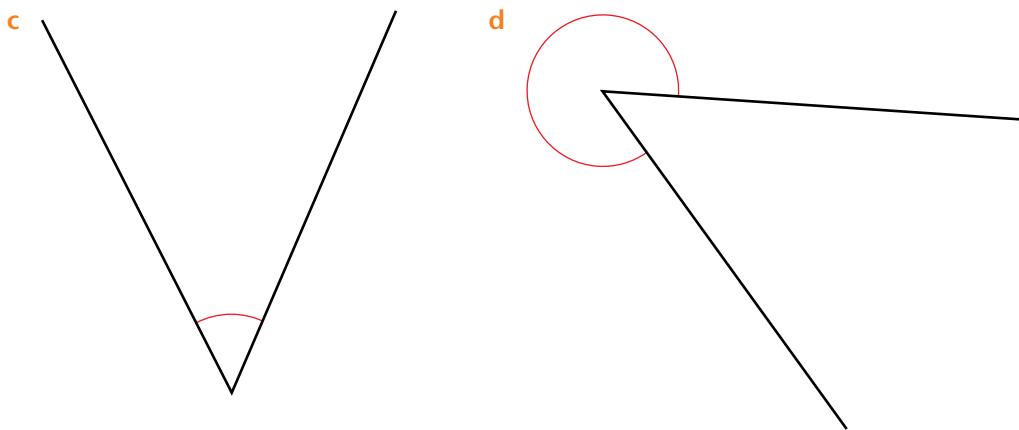
$$\begin{aligned} \text{Angle } A &= 360^\circ - 53^\circ \\ &= 307^\circ \end{aligned}$$

Exercise 29.1

- 1 Copy and complete this table by
 i estimating each angle ii measuring each angle.

	Estimated angle	Measured angle
a		
b		
c		
d		





Constructing a triangle using a ruler and protractor

To **construct** a geometrical figure such as a triangle, you need to measure and draw lines and angles accurately.

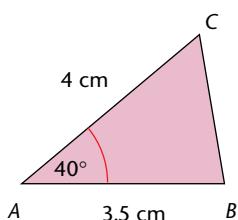
Constructing a triangle given two sides and the included angle

The *included angle* is the angle between the two given sides.

Example 29.3

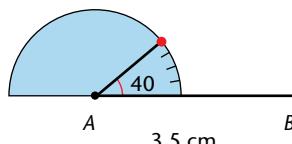
Question

Make an accurate drawing of this triangle.



Note

Make sure your pencil is sharp before you start making accurate constructions.

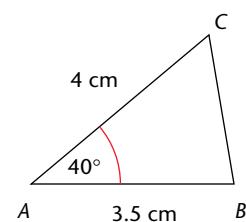
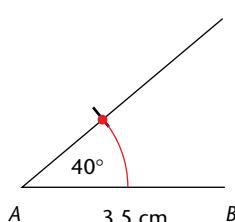


Solution

Draw the line AB 3.5 cm long. Measure and mark the 40° angle at A .

Draw a line from A through the point.

Mark the point C on this line, 4 cm from A . Join C to B .



Constructing a triangle given one side and two angles

Example 29.4

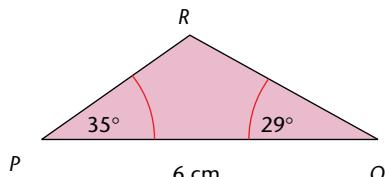
Question

In triangle PQR , $PQ = 6 \text{ cm}$, angle $RPQ = 35^\circ$ and angle $PQR = 29^\circ$.

Construct triangle PQR .

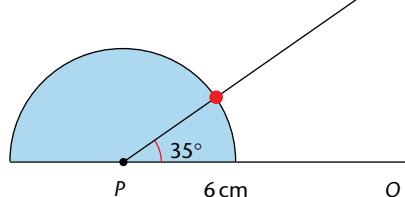
Solution

First draw a sketch.



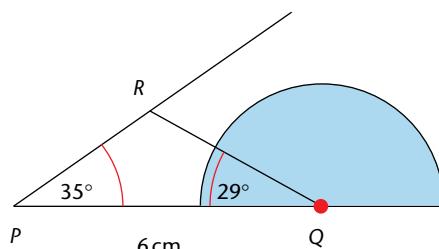
Draw the line PQ , 6 cm long.

Mark an angle of 35° at P and draw a line through the point, longer than PQ .



Mark an angle of 29° at Q .

Draw a line through the point to meet the previous line at R . This completes the triangle.



Note

Draw a sketch of the triangle first. When an angle is written as three letters, the middle letter indicates the vertex (corner) of the angle.

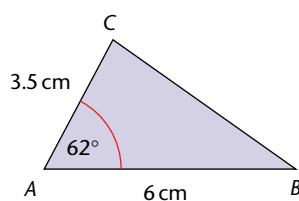
Note

If the two given angles were at R and Q , then angle P could be found by adding the two known angles and subtracting from 180° .

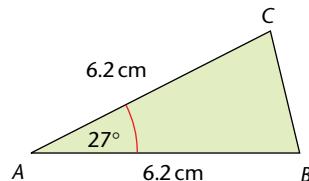
Exercise 29.2

- 1 Construct each of these triangles.

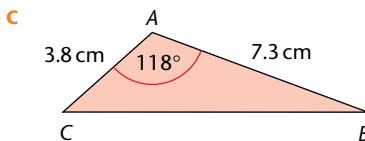
a



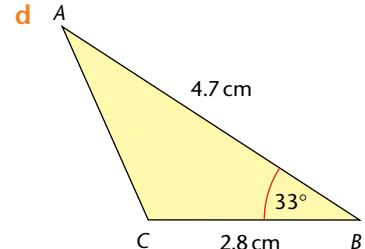
b



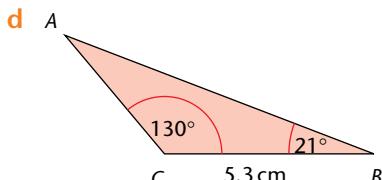
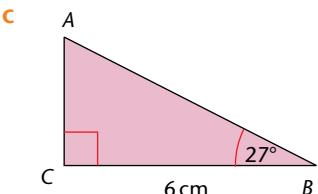
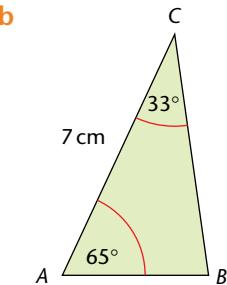
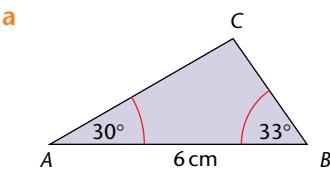
c



d



- 2 For each triangle in question 1, measure the unmarked side and the other two angles on your drawing.
- 3 Make an accurate full-size drawing of each of these triangles.



- 4 For each triangle in question 3, measure the unmarked sides and the other angle on your drawing.

Constructing a geometrical figure using compasses

Constructing a triangle given three sides

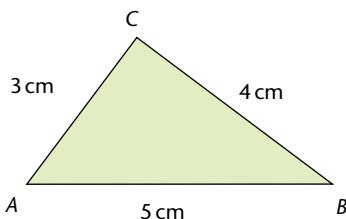
Example 29.5

Question

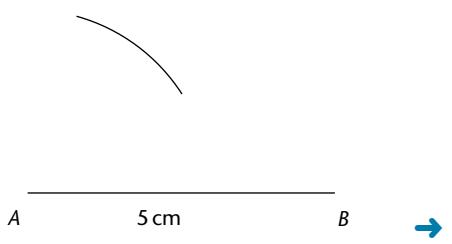
Construct triangle ABC, where $AB = 5 \text{ cm}$, $BC = 4 \text{ cm}$ and $AC = 3 \text{ cm}$.

Solution

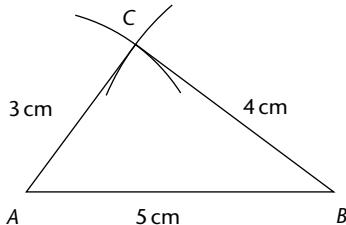
First draw a sketch.



Draw the line AB , 5 cm long. From A , with your compasses set to a radius of 3 cm , draw an arc above the line.



From B , with your compasses set to a radius of 4 cm, draw another arc to intersect the first. The point where the arcs meet is C . Join points A and B to point C to complete the triangle.



Constructing a triangle given two sides and a non-included angle

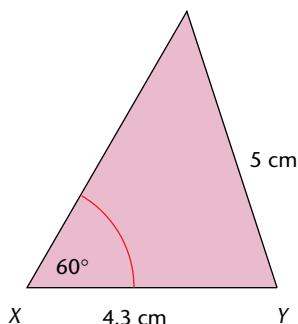
When you are given two sides and one of the angles which is not between them, you need to use a pair of compasses and a protractor to draw the triangle.

The following example shows the method.

Example 29.6

Question

Make an accurate drawing of this triangle.



Solution

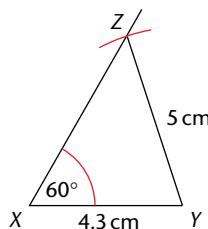
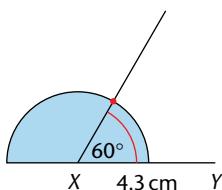
Draw the line XY 4.3 cm long.

Measure an angle of 60° at X and draw a line from X of any length, but longer than XY .

From Y , with your compasses set to a radius of 5 cm, draw an arc to intersect the line.

The point where the arc cuts the line is Z .

Join Y to Z to complete the triangle.



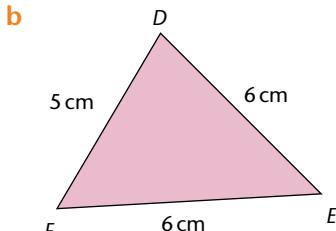
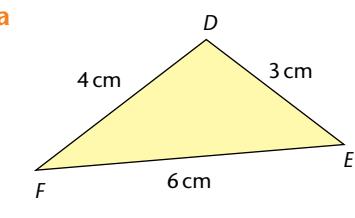
Note

When you are given two sides and a non-included angle, you can sometimes draw two different triangles.

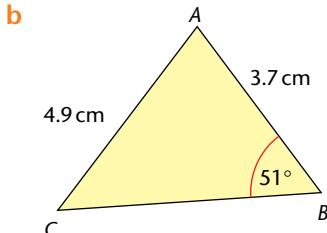
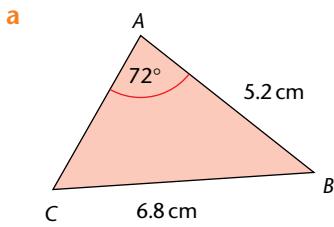
If the second side is shorter than the first, the arc you draw will cross the line you drew to make the given angle twice.

Exercise 29.3

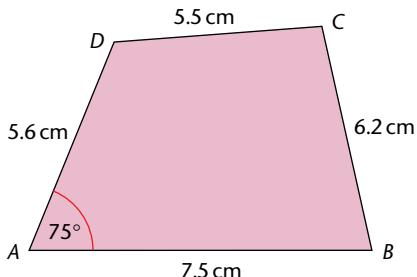
- 1** Construct each of these triangles.



- c** Triangle DEF with $DE = 4.3$ cm, $EF = 7.2$ cm and $FD = 6.5$ cm.
2 For each triangle in question **1**, measure the angles on your drawing.
3 Construct an equilateral triangle with sides of 4.2 cm.
4 Construct each of these triangles.



- 5** For each triangle in question **4**, measure the unmarked side and the other two angles on your drawing.
6 Draw the triangle ABC where $AB = 5$ cm, $BC = 4$ cm and angle $BAC = 40^\circ$.
 There are two possible solutions.
 Draw both of them.
7 **a** Use a ruler, protractor and pair of compasses to construct an accurate drawing of this quadrilateral.



- b** Measure angle D .
8 **a** Draw a rhombus $ABCD$ with diagonal $AC = 3.5$ cm and sides of length 5.3 cm.
b Measure the length of diagonal BD .

Constructing regular polygons in circles

There are several different methods to construct a regular polygon. Possibly the easiest method is to divide the angle at the centre of the circle, 360° , into a number of equal parts – the number of parts being equal to the number of sides of the polygon.

Example 29.7

Question

Construct a regular pentagon.

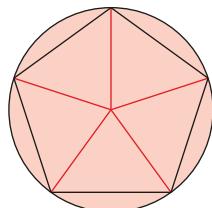
Solution

Mark a centre and draw a circle with a radius of, for example, 5 cm.

A pentagon has five sides, so divide the angle at the centre by five giving $360^\circ \div 5 = 72^\circ$.

Measure, using a protractor, angles of 72° around the centre, marking the points on the circumference.

Join the points on the circumference to form a pentagon.



Exercise 29.4

- 1 Construct a regular hexagon.
- 2 Construct a regular octagon.

Constructing angle bisectors and perpendicular bisectors

The bisector of an angle

This is the method you use to construct the bisector of an angle with a ruler and pair of compasses only.

Draw an angle and mark the vertex A .

Put the point of your compasses at A and draw an arc to cut the lines forming the angle at B and C .

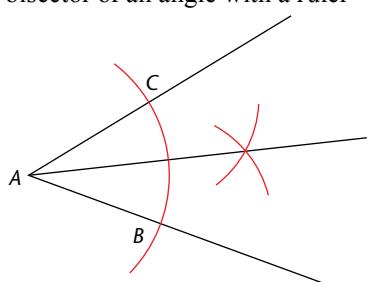
Put the point of the compasses at B and draw an arc in the angle.

Keep the compasses set to the same radius.

Put the point of the compasses at C and draw an arc in the angle to cut the arc just drawn.

Draw a straight line through A and the point where the arcs cross.

The bisector could be continued to the left of A . It bisects the reflex angle CAB .



Note

Alternatively, you can draw the angle bisector by measuring the angle CAB , halving it and then using a protractor to draw this angle from the line AB .

This is not as accurate as a carefully drawn construction.

The perpendicular bisector of a line

This is the method you use to construct the perpendicular bisector of a line with a ruler and pair of compasses only.

Draw the line AB .

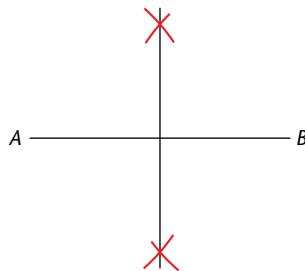
Open your compasses to a radius that is more than half the length of AB .

Put the point of the compasses at A and draw two arcs, above and below the line.

Keep the compasses set to the same radius.

Put the point of the compasses at B and draw two arcs, above and below the line.

Join the two points where the pairs of arcs cross.



Note

Alternatively, you could measure the line AB and mark the midpoint.

With one side of your set square on the line AB , line up the right angle corner at the midpoint and draw the top half of the perpendicular bisector.

Use your ruler to extend this line below AB .

This is not as accurate as a carefully drawn construction.

Exercise 29.5

In this exercise use only a ruler and a pair of compasses to make your constructions.

- 1 Draw an angle of 70° .
Construct the bisector of the angle.
- 2 Draw an angle of 104° .
Construct the bisector of the angle.
- 3 Draw a line 7 cm long.
Construct the perpendicular bisector of the line.
- 4 Draw a line AB 6.7 cm long.
Construct the perpendicular bisector of AB .
- 5 Draw a line 10.4 cm long.
Construct the perpendicular bisector of the line.
- 6 Construct triangle ABC with $AB = 8$ cm, $AC = 7$ cm and $BC = 5$ cm.
Bisect angle A .

Scale drawings and maps

A **scale drawing** is exactly the same shape as the original drawing but is different in size.

Large objects are scaled down in size so that, for example, they can fit on to the page of a book.

A map is a scale drawing of an area of land.

The scale of the drawing can be written as these examples.

1 cm to 2 m This means that 1 cm on the scale drawing represents 2 m in real life.

2 cm to 5 km This means that 2 cm on the scale drawing represents 5 km in real life.

Example 29.8**Question**

Here is a scale drawing of a truck.

The scale of the drawing is 1 cm to 2 m.

- How long is the truck?
- Will the truck go safely under a bridge 4 m high?
- The truck driver is 1.8 m tall.
How high will he be on the scale drawing?

**Solution**

- Measure the length of the truck on the scale drawing.

Length of truck on the drawing = 4 cm.

As 1 cm represents 2 m, multiply the length on the drawing by 2 and change the units.

Length of truck in real life = $4 \times 2 = 8$ m.

- Height of truck on the drawing = 2.5 cm.

Height of truck in real life = $2.5 \times 2 = 5$ m.

So the truck will not go under the bridge.

- To change from measurements in real life to measurements on the drawing you have to divide by 2 and change the units.

Height of driver in real life = 1.8 m.

Height of driver on the drawing = $1.8 \div 2 = 0.9$ cm.

Exercise 29.6

- Measure each of these lines as accurately as possible.

Using a scale of 1 cm to 4 m, work out the length that each line represents.

a _____

b _____

c _____

d _____

- Measure each of these lines as accurately as possible.

Using a scale of 1 cm to 10 km, work out the length that each line represents.

a _____

b _____

c _____

d _____

- Draw accurately the line to represent these actual lengths.

Use the scale given.

a 5 m Scale: 1 cm to 1 m

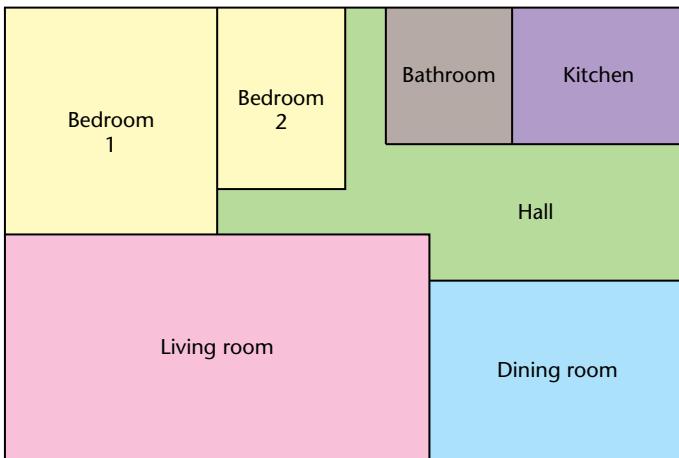
b 10 km Scale: 1 cm to 2 km

c 30 km Scale: 2 cm to 5 km

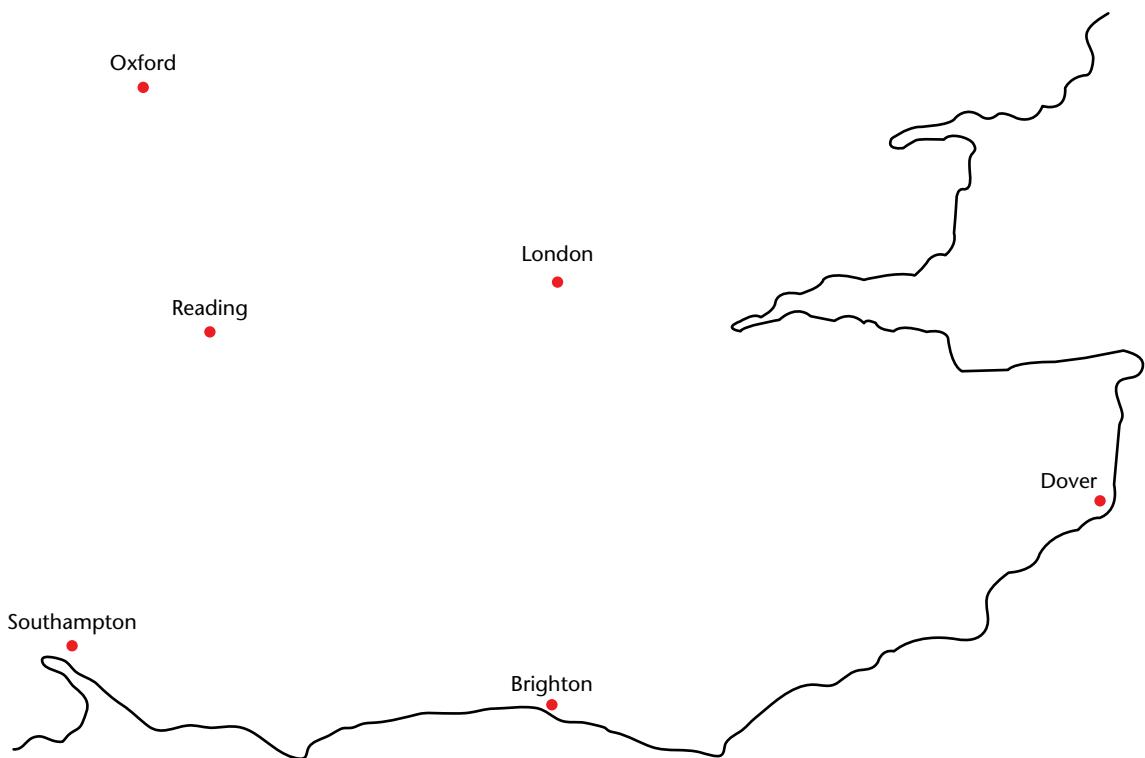
d 750 m Scale: 1 cm to 100 m

- 4 Here is a plan of a house.

The scale of the drawing is 1 cm to 2 m.



- a How long is the hall in real life?
 b Work out the length and width of each of the six rooms in real life.
 c The bungalow is on a plot of land measuring 26 m by 15 m.
 What will the measurements of the plot of land be on this scale drawing?
- 5 The map shows some towns and cities in the south-east of England.
 The scale of the map is 1 cm to 20 km.



- a** What is the real-life distance, in kilometres, between these towns?
 - i** London and Reading
 - ii** Southampton and Dover
 - iii** London and Brighton
 - iv** Oxford and Reading
 - v** Brighton and Southampton
 - vi** Dover and Oxford
- b** It is 90 km from London to Cambridge.
How many centimetres will this be on the map?

30 SIMILARITY AND CONGRUENCE

Congruence

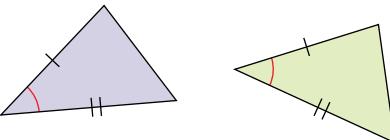
For any two shapes to be **congruent**, they must be the same shape and the same size.

If two shapes are congruent then

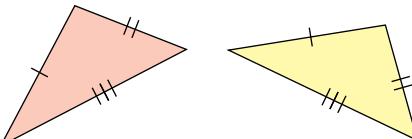
- all the angles in one shape equal the corresponding angles of the other shape. (This is also true for similar shapes.)
- all the sides in one shape equal the corresponding sides of the other shape. (This is not true for similar shapes.)

Two *triangles* are congruent if any of these sets of conditions is satisfied.

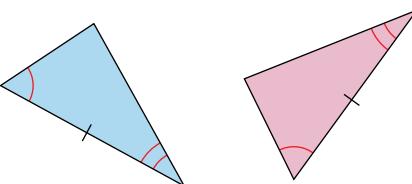
- Two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle (side, angle, side or SAS).



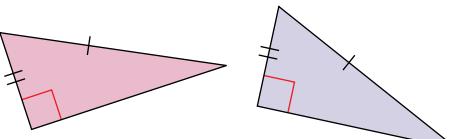
- The three sides of one triangle are equal to the three sides of the other triangle (side, side, side or SSS).



- Two angles and a side in one triangle are equal to the corresponding two angles and side in the other triangle (angle, side, angle or ASA).

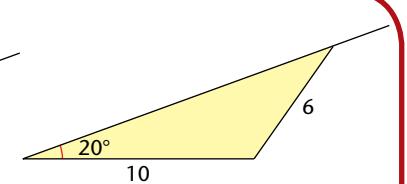
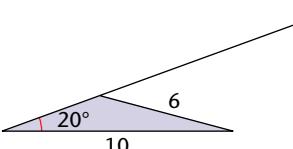


- Both triangles are right-angled and the hypotenuse and one side of one triangle are equal to the hypotenuse and corresponding side of the other triangle (right angle, hypotenuse, side or RHS).



Note

The test involving two sides and an angle is valid only if the angle is the included angle. If the given angle is not the included angle then there are two possible solutions – the ambiguous case.



BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- solve problems and give simple explanations involving similarity and congruence
- calculate lengths of similar figures
- use the relationships between areas of similar triangles, with corresponding results for similar figures, and extension to volumes and surface areas of similar solids.

CHECK YOU CAN:

- use the angle properties of triangles and quadrilaterals
- use the angle properties of parallel lines
- understand the term *scale factor*.

Example 30.1**Question**

Prove that the diagonal of a parallelogram splits the parallelogram into two congruent triangles.

Hence show that the opposite sides of a parallelogram are equal.

Solution

$B\hat{A}C = A\hat{C}D$ Alternate angles, AB and DC parallel.

$A\hat{C}B = D\hat{A}C$ Alternate angles, AD and BC parallel.

In triangles ACB and ACD , AC is common.

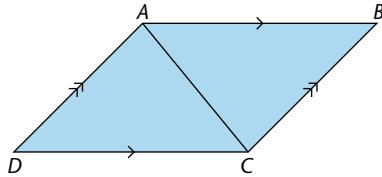
Hence these triangles are congruent (ASA).

Since the triangles are congruent, corresponding pairs of sides are equal.

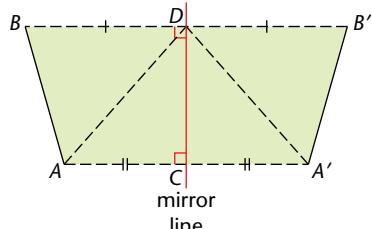
So $CB = AD$

and $AB = CD$.

That is, the opposite sides of a parallelogram are equal.

**Example 30.2****Question**

Use congruency to show that the reflection $A'B'$ of a line AB in any line has a length equal to AB .

**Solution**

Let AA' and BB' cross the mirror line at C and D respectively.

By the definition of reflection, $AC = A'C$ and AA' crosses the mirror line at right angles.

Since CD is common, triangles ACD and $A'CD$ are congruent (SAS).

This implies $AD = A'D$ and angle $ADC = \text{angle } A'DC$.

By the definition of reflection $BD = B'D$ and angle $BDC = \text{angle } B'DC = 90^\circ$.

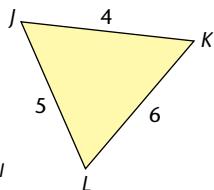
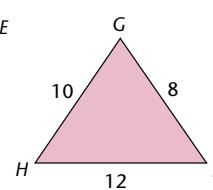
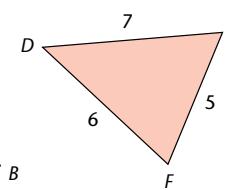
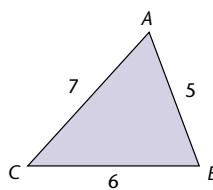
Angle $BDA = \text{angle } B'DA' = 90^\circ - \text{angle } CDA$.

So triangles BDA and $B'DA'$ are congruent (SAS) and, in particular, $AB = A'B'$ as required.

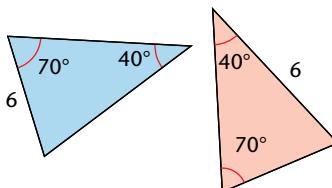
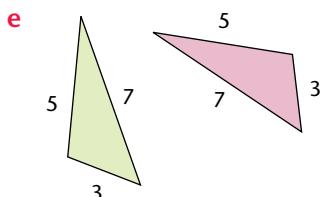
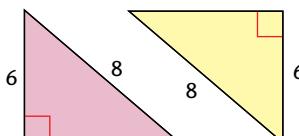
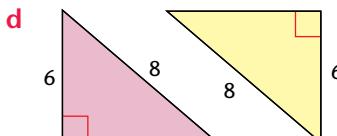
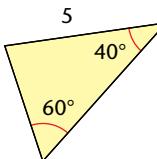
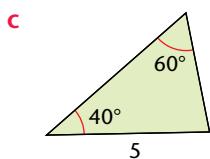
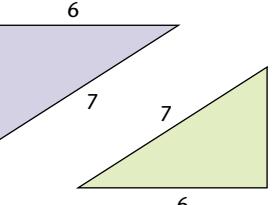
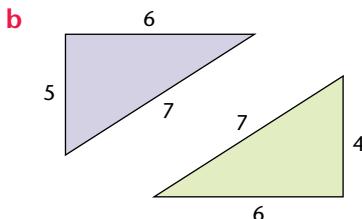
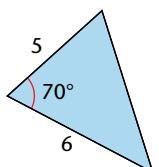
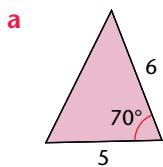
Exercise 30.1

- 1 Which of these triangles are congruent?

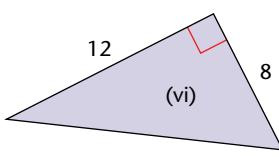
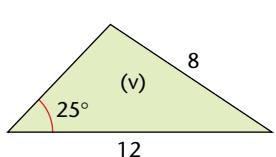
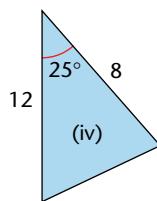
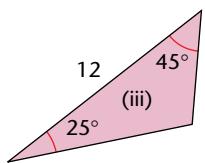
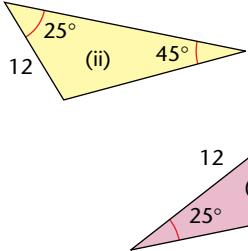
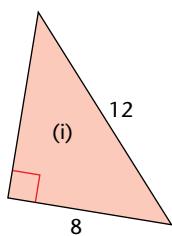
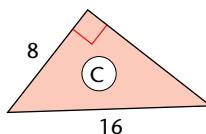
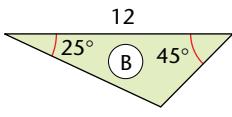
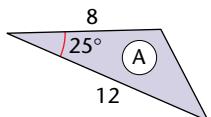
Give a reason or explanation for your answer.



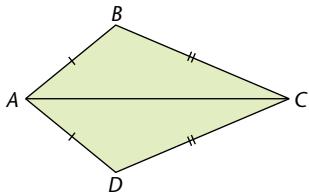
- 2** Which of these pairs of triangles are congruent?
Explain your answer in each case.



- 3** Which of the triangles i, ii, iii, iv, v and vi are congruent to any of triangles A, B or C?
Explain your answer in each case.



- 4** Prove that diagonal AC bisects the angles at A and C .



A sketch will prove useful for questions **5** to **10**.

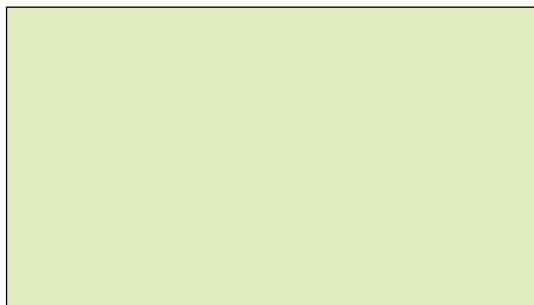
- 5** Prove that the angles opposite the equal sides in an isosceles triangle are equal.
- 6** Sketch an isosceles triangle ABC with the equal angles at B and C . Draw a straight line from the midpoint of side BC to the vertex A . Prove that this line bisects angle BAC and is perpendicular to side BC .
- 7** Prove that the diagonals of a rectangle are equal in length.
- 8** Use the properties of congruent triangles to prove that the diagonals of a rhombus bisect each other at right angles.
- 9** Prove that the longer diagonal of a kite bisects the shorter diagonal at right angles and bisects the unequal pair of angles.
- 10** Sketch an equilateral triangle. Join the midpoints of each side to make a second triangle. Prove that this triangle is equilateral.

Similarity

In mathematics the word ‘similar’ has a very exact meaning. It does *not* mean ‘roughly the same’ or ‘alike’.

For two shapes to be **similar**, each shape must be an exact enlargement of the other.

For example, look at these two rectangles.



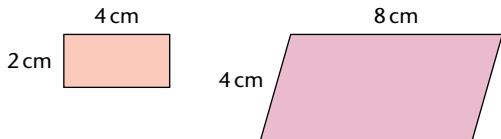
The first rectangle is 2 cm wide by 4 cm long.

The second rectangle is 4 cm wide by 7 cm long.

The rectangles are *not* similar because although the width of the large one is twice the width of the small one, the length of the large one is *not* twice the length of the small one.

If the length of the large one was 8 cm then the rectangles would be similar.

Now look at these two shapes.



Although the scale factor for both pairs of sides is 2, the shapes are not similar because corresponding angles are not the same.

For two shapes to be similar

- all corresponding sides must have proportional lengths
- all corresponding angles must be equal.

Because the lengths of three sides define a unique triangle, for two triangles to be similar only one of the tests here needs to be made.

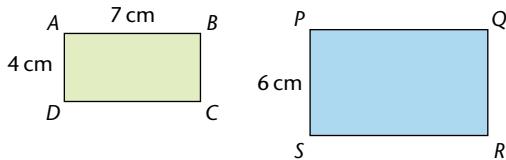
If you can establish that the angles are the same, you can conclude that the triangles are similar and carry out calculations to find the lengths of the sides.

Calculating lengths in similar shapes

Example 30.3

Question

The rectangles $ABCD$ and $PQRS$ are similar.



Find the length of PQ .

Solution

Since the widths of the rectangles are 6 cm and 4 cm, the scale factor is $6 \div 4 = 1.5$.

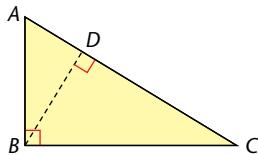
$$\text{Length of } PQ = 7 \times 1.5$$

$$= 10.5 \text{ cm}$$

Example 30.4

Question

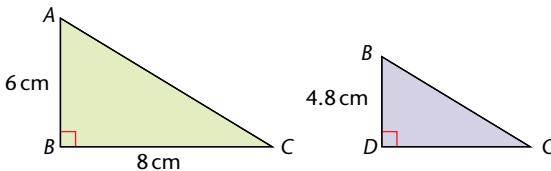
In the triangle, angle $ABC = \text{angle } BDC = 90^\circ$, $AB = 6 \text{ cm}$, $BC = 8 \text{ cm}$ and $BD = 4.8 \text{ cm}$.



- Explain why triangles ABC and BDC are similar.
- Calculate the length of DC .

Solution

- In the triangles ABC and BDC , angle $ABC = \text{angle } BDC = 90^\circ$ (given). The angle at C is in both triangles. Since the angle sum of a triangle is 180° , the third angles must be equal. So, since all the corresponding angles are equal, the triangles are similar.
- First redraw the triangles so they are the same way round as each other.



Since $AB = 6 \text{ cm}$ and $BD = 4.8 \text{ cm}$ the scale factor $= 4.8 \div 6 = 0.8$.

$$\begin{aligned} CD &= 8 \times 0.8 \\ &= 6.4 \text{ cm} \end{aligned}$$

Note

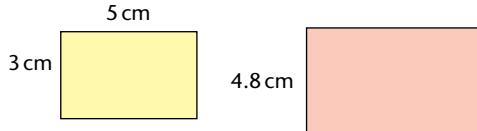
If you are asked to explain why two triangles are similar, look for reasons why the angles are equal. Usual reasons include 'opposite angles are equal', 'alternate angles are equal' and 'the angle is in both triangles (common angles)'.

Note

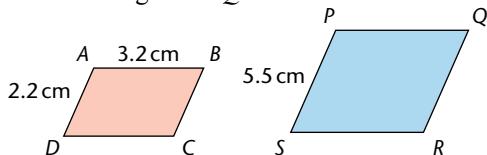
It is always easier to spot the corresponding sides if the shapes are the same way round. It is worth spending time redrawing the shapes separately and the same way round and marking the lengths on the new diagram.

Exercise 30.2

- The two rectangles in the diagram are similar. Find the length of the larger rectangle.



- These two parallelograms are similar. Find the length of PQ .



- 3** Josh bought a set of table mats.

The set contained mats of three different sizes.

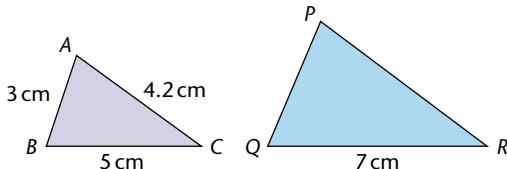
They were all rectangular and similar in shape.

The smallest one was 5 cm by 8 cm, the middle one was an enlargement of the smallest by scale factor 3, and the shorter side of the largest mat was 25 cm.

Find the dimensions of the two larger mats.

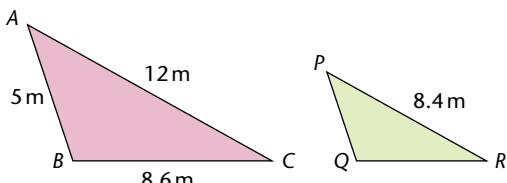
- 4** The triangles ABC and PQR are similar.

Calculate the lengths of PQ and PR .



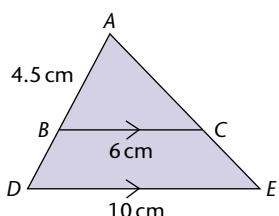
- 5** The triangles ABC and PQR are similar.

Calculate the lengths of PQ and QR .



- 6** In the diagram, DE is parallel to BC , $AB = 4.5 \text{ cm}$, $BC = 6 \text{ cm}$ and $DE = 10 \text{ cm}$.

Calculate the length of BD .

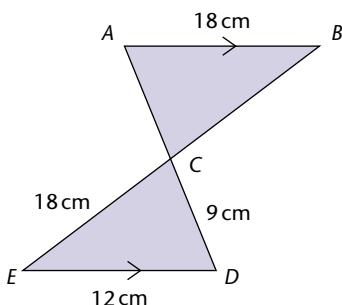


- 7** At noon a radio mast of height 12 m has a shadow length of 16 m.

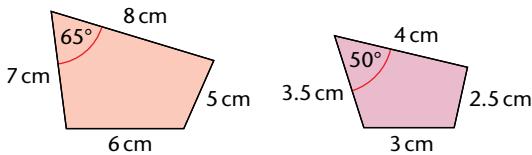
Calculate the height of a tower with a shadow length of 56 m at noon.

- 8 a** Explain how you know that the two triangles in the diagram are similar.

b Calculate the lengths of AC and CB .

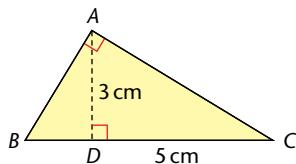


- 9 Explain why these quadrilaterals are *not* similar.



- 10 In the diagram, angle $BAC = \text{angle } ADC = 90^\circ$, $AD = 3 \text{ cm}$ and $DC = 5 \text{ cm}$.

- a Explain why triangles ADC and BDA are similar.
b Calculate the length of BD .

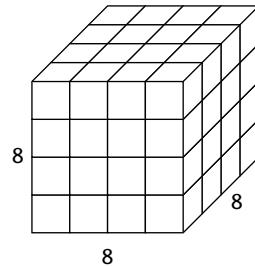


The areas and volumes of similar shapes

This cube has a volume of 8 cm^3 .



This cube has a volume of 512 cm^3 .



The lengths of the small cube have been enlarged by scale factor 4.

The volume has been enlarged by scale factor 64.

Since there are three dimensions for volume, and each dimension has been enlarged by scale factor 4, the volume scale factor = 4^3 .

Similarly, consider the area of the face of each cube.

For the small cube, the area is 4 cm^2 .

For the large cube, the area is 64 cm^2 .

The area has been enlarged by scale factor 16.

There are two dimensions for area, so the area scale factor is 4^2 .

For mathematically similar shapes

- area scale factor = (length scale factor)²
- volume scale factor = (length scale factor)³.

Example 30.5**Question**

A model aircraft is made to a scale of 1 : 50.

The area of the wing on the model is 18 cm^2 .

What is the area of the wing on the real aircraft?

Solution

Length scale factor = 50

Area scale factor = 50^2

$$\begin{aligned}\text{Area of real wing} &= 18 \times 50^2 \\ &= 45000 \text{ cm}^2 \\ &= 4.5 \text{ m}^2\end{aligned}$$

Note

Remember that

$$1 \text{ m} = 100 \text{ cm}$$

$$1 \text{ m}^2 = 10000 \text{ cm}^2$$

$$1 \text{ m}^3 = 1000000 \text{ cm}^3$$

Example 30.6**Question**

A jug holding 50 cl is 12 cm high.

A similar jug holds 2 litres.

What is its height?

Solution

50 cl = 0.5 litres

$$\text{Volume scale factor} = \frac{2}{0.5} = 4$$

$$\text{Length scale factor} = \sqrt[3]{4}$$

$$\begin{aligned}\text{Height of larger jug} &= 12 \text{ cm} \times \sqrt[3]{4} \\ &= 19.0 \text{ cm to 1 decimal place}\end{aligned}$$

Exercise 30.3

- 1** State the area scale factor for each of these length scale factors.
 a 2 b 3 c 5 d 4 e 6 f 10
- 2** State the volume scale factor for each of these length scale factors.
 a 10 b 4 c 5 d 2 e 3 f 8
- 3** State the length scale factor for each of these.
 a Area scale factor of 16
 b Volume scale factor of 216
 c Area scale factor of 64
 d Volume scale factor of 1000
- 4** A model of a building is made to a scale of 1 : 50.
 A room in the model is a cuboid with dimensions 7.4 cm by 9.8 cm by 6.5 cm high.
 Calculate the floor area of the room in
 - a the model
 - b the actual building.

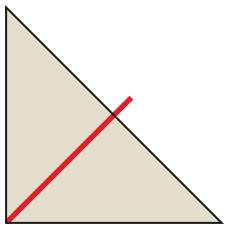
- 5** A glass holds 15 cl.
The heights of this and a larger similar glass are in the ratio 1 : 1.2.
Calculate the capacity of the larger glass.
- 6** A tray has an area of 160 cm^2 .
What is the area of a similar tray whose lengths are one and a half times as large?
- 7** To what scale is a model drawn if an area of 5 m^2 in real life is represented by 20 cm^2 on the model?
- 8** Three similar wooden boxes have heights in the ratio 3 : 4 : 5.
What is the ratio of their volumes?
- 9** The area of a rug is 2.4 times as large as the area of a similar rug.
The length of the smaller rug is 1.6 m.
Find the length of the larger rug.
- 10** An artist makes a model for a large sculpture.
It is 24 cm high.
The finished sculpture will be 3.6 m high.
a Find the length scale factor.
b Find the area scale factor.
c The model has a volume of 1340 cm^3 .
What will be the volume of the sculpture?
- 11** A glass is 12 cm high.
How tall is a similar glass that holds twice as much?
- 12** A model aircraft is made to a scale of 1 : 48.
The area of the wing of the real aircraft is 52 m^2 .
What is the area of the wing of the model?
- 13** A square is enlarged by increasing the length of its sides by 10%.
The length of the sides was originally 8 cm.
What is the area of the enlarged square?
- 14** Two similar containers hold 1 litre and 2 litres of milk.
The surface area of the larger container is 1100 cm^2 .
Calculate the surface area of the smaller one.

31 SYMMETRY

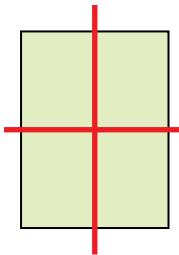
Line symmetry

A **line of symmetry** divides a shape into two identical parts.

A shape may have more than one line of symmetry.



The triangle has one line of symmetry.

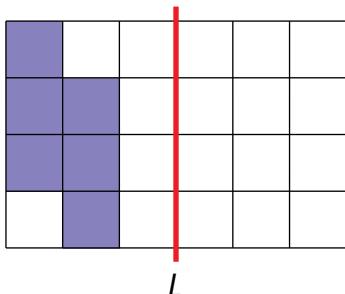


The rectangle has two lines of symmetry.

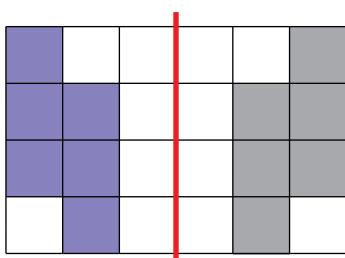
Example 31.1

Question

Complete the pattern so that L is a line of symmetry.



Solution



The image is the same distance from the line of symmetry as the original shape but on the opposite side of the line.

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

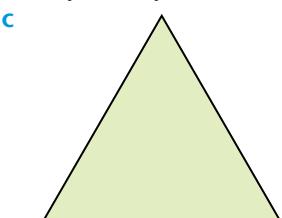
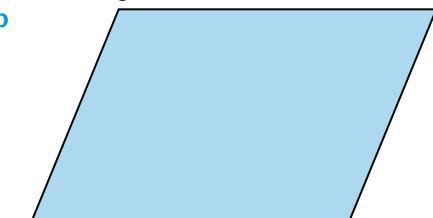
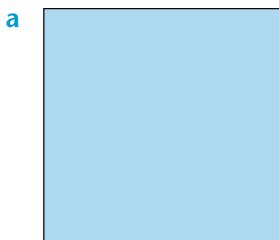
- recognise rotational and line symmetry (including order of rotational symmetry) in two dimensions
- recognise symmetry properties of the prism (including cylinder) and the pyramid (including cone)
- use the following symmetry properties of circles:
 - equal chords are equidistant from the centre
 - the perpendicular bisector of a chord passes through the centre
 - tangents from an external point are equal in length.

CHECK YOU CAN:

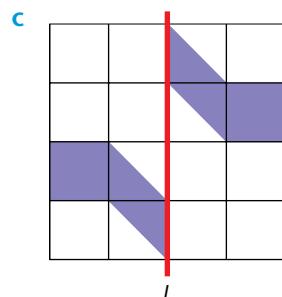
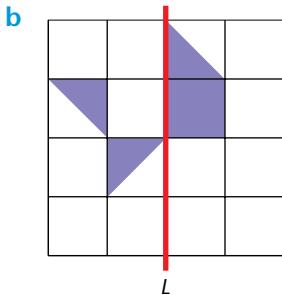
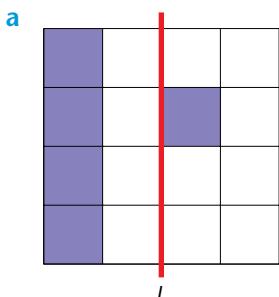
- use the vocabulary of shapes and solids
- show that triangles are congruent
- use the angle properties of a triangle.

Exercise 31.1

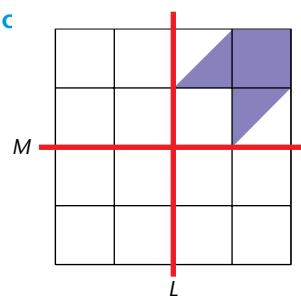
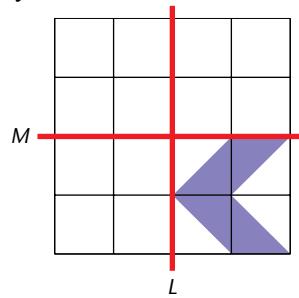
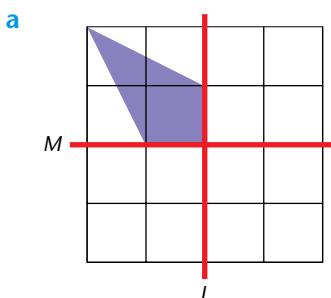
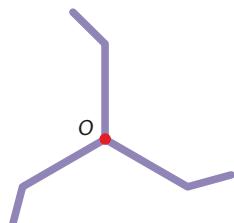
- 1** For each shape, write down the number of lines of symmetry.



- 2** Copy and complete each pattern so that L is a line of symmetry.



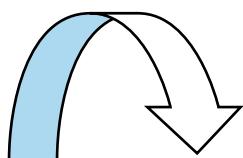
- 3** Copy and complete each pattern so that lines L and M are both lines of symmetry.

**Rotational symmetry**

This shape has rotational symmetry of order 3 about centre O .

There are three positions where the shape fits onto the original shape in one complete turn about centre O .

The **order of rotational symmetry** is the number of times that a shape fits onto itself in one complete turn.



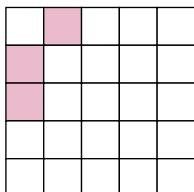
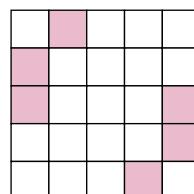
This shape has no rotational symmetry.

There is only one position where the shape fits onto the original in one complete turn.

It has rotational symmetry of order 1.

Example 31.2**Question**

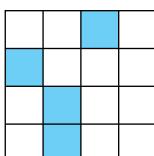
Shade three more squares in this shape so that the pattern has rotational symmetry of order 2.

**Solution**

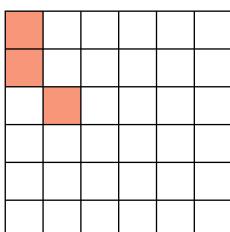
You can check that your diagram is correct by rotating it through a half turn.

Exercise 31.2

- 1** Copy this diagram and shade two more squares so that the pattern has rotational symmetry of order 2.



- 2** Copy this diagram and shade nine more squares so that the pattern has rotational symmetry of order 4.

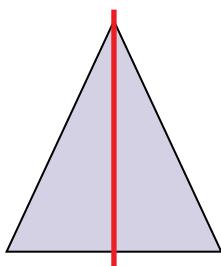


- 3** Shade squares on a 4×4 grid to make a pattern with rotational symmetry of order 2 but no lines of symmetry.
4 Copy and complete this diagram so that it has rotational symmetry of order 2.



Symmetry properties of shapes and solids

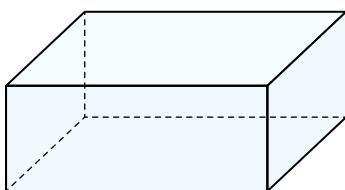
This is an isosceles triangle.



It has one line of symmetry and no rotational symmetry.

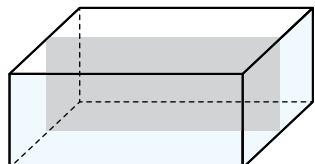
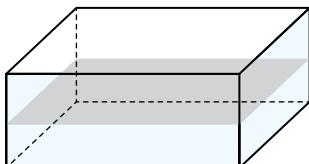
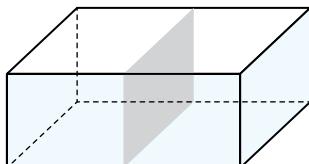
These symmetry properties are the same for any isosceles triangle.

This is a cuboid.

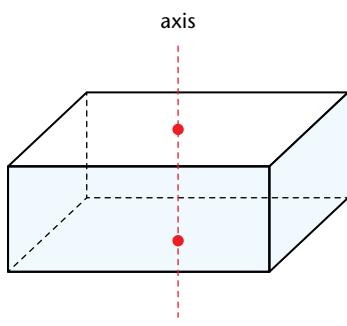


A plane of symmetry divides a solid into two identical parts.

The cuboid has three planes of symmetry.



The cuboid has rotational symmetry of order 2 about the axis shown.



The cuboid has three axes of rotational symmetry.

When you are asked to describe the symmetry of a shape, remember to describe both rotational and line symmetry.

Example 31.3**Question**

Describe the symmetry of

- a parallelogram
- b a square-based pyramid.

Solution

- a A parallelogram has no lines of symmetry and rotational symmetry of order 2.
- b A square-based pyramid has four planes of symmetry and one axis of rotational symmetry of order 4.

Exercise 31.3

- 1 Describe the symmetry of each of these shapes.
 - a a rectangle
 - b an equilateral triangle
 - c a regular pentagon
- 2 Write down the number of planes of symmetry of each of these solids.
 - a a cube
 - b a cone
- 3 Describe the symmetry of a prism which has an isosceles triangle for its cross-section.
- 4 Sketch an octagon with two lines of symmetry and rotational symmetry of order 2.
- 5 a Which special quadrilateral has four lines of symmetry and rotational symmetry of order 4?
b Which special quadrilateral has one line of symmetry and no rotational symmetry?
- 6 Which solid has an infinite number of planes of symmetry and an infinite number of axes of symmetry?

The symmetry properties of the circle**The perpendicular from the centre to a chord**

AB is a chord to the circle, centre O .

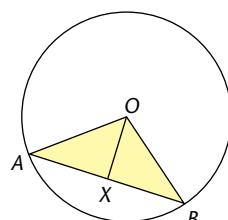
X is the midpoint of AB .

OA and OB are radii, so triangle OAB is isosceles.

OX is a line of symmetry for triangle OAB .

Therefore OX is perpendicular to AB .

This shows that



the perpendicular from the centre to a chord bisects the chord.

Equal chords

AB and CD are chords to the circle, centre O .

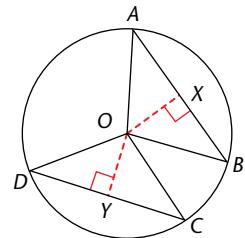
Chords AB and CD are of equal length.

X is the midpoint of AB and Y is the midpoint of CD .

OA , OB , OC and OD are radii so are equal, so triangles OAB and OCD are congruent (SSS).

Therefore $OX = OY$.

This shows that



equal chords are equidistant from the centre of the circle.

Tangents from an external point

T is a point outside a circle, centre O .

TX and TY are the tangents from T to the circle.

OX and OY are radii so are equal.

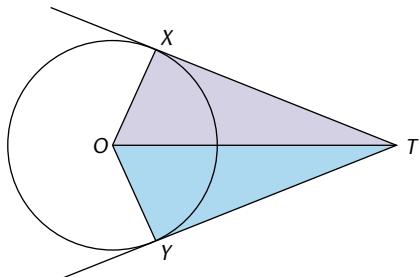
OT is a line of symmetry and bisects $X\hat{O}Y$, so $X\hat{O}T = Y\hat{O}T$.

OT is common to triangles OXT and OYT .

So triangles OXT and OYT are congruent (SAS).

Therefore $XT = YT$.

This shows that



tangents to a circle from an external point are equal in length.

These facts can be used to solve geometrical problems.

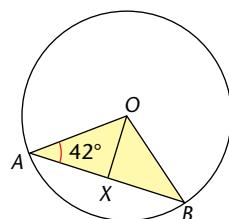
Example 31.4

Question

X is the midpoint of chord AB .

Given that $O\hat{A}B = 42^\circ$, find

- a $O\hat{B}A$ b $A\hat{O}B$ c $A\hat{O}X$.



Solution

a $O\hat{B}A = 42^\circ$ Triangles AOX and BOX are congruent.

b $A\hat{O}B = 96^\circ$ Angles in triangle AOB add up to 180° .

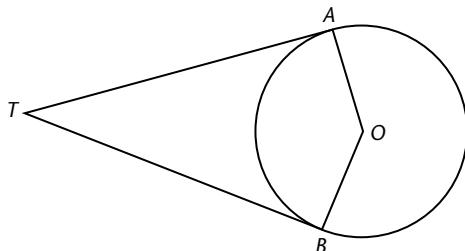
c $A\hat{O}X = 48^\circ$ By symmetry, $A\hat{O}X = B\hat{O}X$.

Exercise 31.4

- 1** AT and BT are tangents to the circle, centre O .

What is the mathematical name of quadrilateral $OATB$?

Explain how you know.

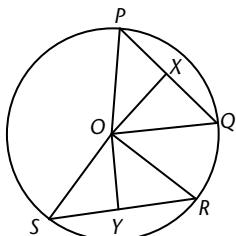


- 2** P, Q, R and S are points on the circle, centre O .

X is the midpoint of PQ , Y is the midpoint of RS and $OX = OY$.

Find three triangles that are congruent to triangle OPX .

Explain how you know.

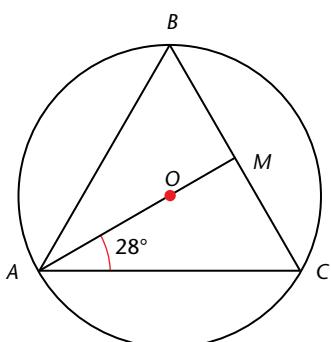


- 3** A, B and C are points on the circle, centre O .

M is the midpoint of BC and $\hat{CAM} = 28^\circ$.

Work out \hat{MCA} .

Give a reason for your answer.



32 ANGLES

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

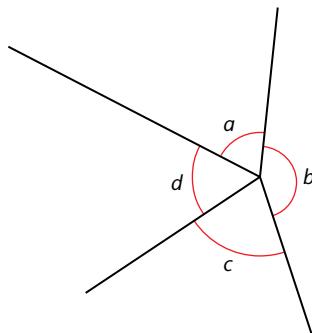
- calculate unknown angles and give simple explanations using the following geometrical properties:
- angles at a point
 - angles at a point on a straight line and intersecting straight lines
 - angles formed within parallel lines
 - angle properties of triangles and quadrilaterals
 - angle properties of regular and irregular polygons
 - angle in a semi-circle
 - angle between tangent and radius of a circle
 - angle at the centre of a circle is twice the angle at the circumference
 - angles in the same segment are equal
 - angles in opposite segments are supplementary.

CHECK YOU CAN:

- distinguish between acute, obtuse, reflex and right angles
- distinguish between equilateral, isosceles, scalene and right-angled triangles
- understand the meaning of the term *congruent*
- understand and use geometrical terms
- understand that the perpendicular bisector of a chord of a circle passes through the centre.

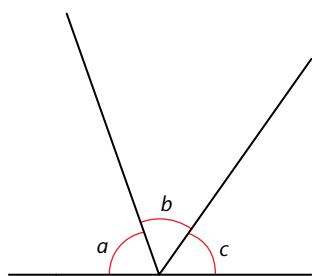
Angles formed by straight lines

The sum of the angles at a point is 360° .



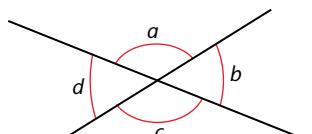
$$a + b + c + d = 360^\circ$$

The sum of the angles on a straight line is 180° .



$$a + b + c = 180^\circ$$

When two lines cross, vertically opposite angles are equal.

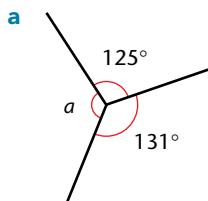


$$a = c \text{ and } b = d$$

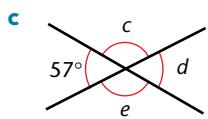
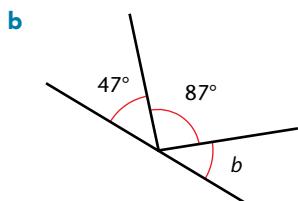
Example 32.1**Question**

Work out the size of angles a , b , c , d and e .

Give a reason for each answer.

**Note**

The diagrams are not drawn accurately, so do not try to measure the angles.

**Note**

Remember to give a reason for your answer.

State the angle fact that you have used.

Solution

a $a + 125^\circ + 131^\circ = 360^\circ$

$$a + 256^\circ = 360^\circ$$

$$a = 104^\circ$$

b $47^\circ + 87^\circ + b = 180^\circ$

$$134^\circ + b = 180^\circ$$

$$b = 46^\circ$$

c $c + 57^\circ = 180^\circ$

$$c = 123^\circ$$

$$d = 57^\circ$$

$$e = 123^\circ$$

The sum of the angles at a point is 360° .

The sum of angles on a straight line is 180° .

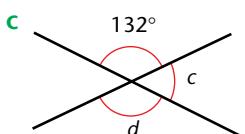
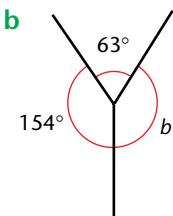
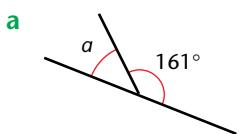
The sum of the angles on a straight line is 180° .

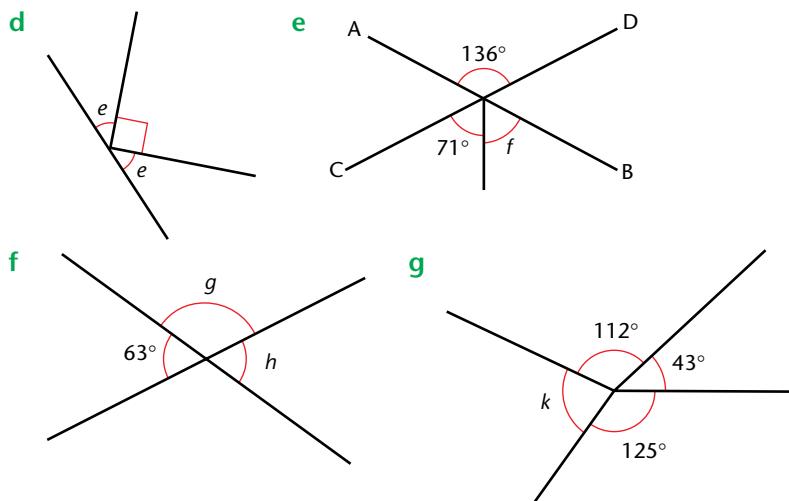
Vertically opposite angles are equal.

Vertically opposite angles are equal.

Exercise 32.1

- 1** Find the size of the angles marked with letters. Give reasons for your answers.





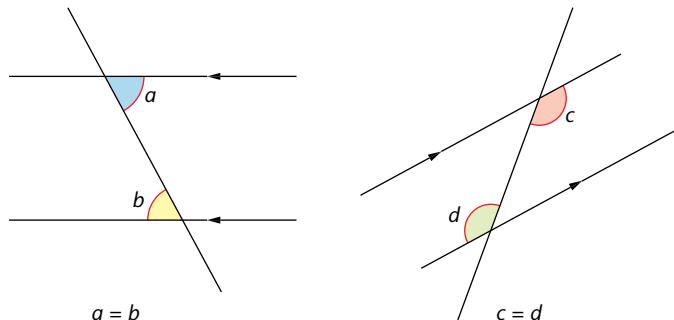
- 2 Three angles meet at a point. Karim measures them and says the angles are 113°, 123° and 134°.
Could he be correct?
Give a reason for your answer.

Angles formed within parallel lines

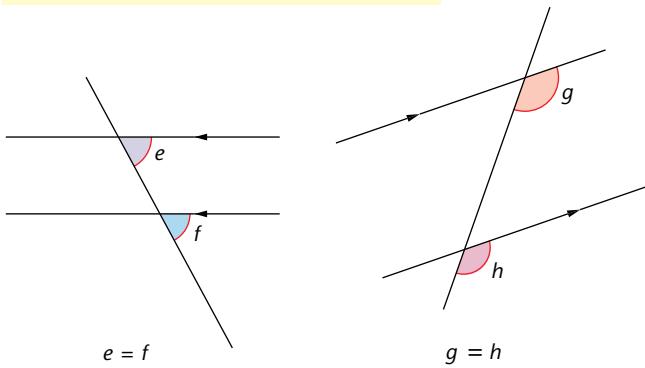
Note

Thinking of a Z-shape may help you to remember that alternate angles are equal.
Alternatively, turn the page round and look at the diagrams upside down and you will see the same shapes!

Alternate angles are equal.



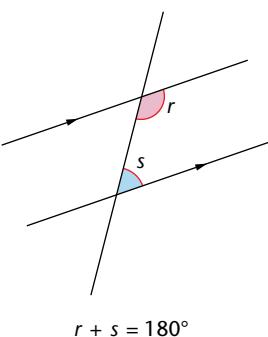
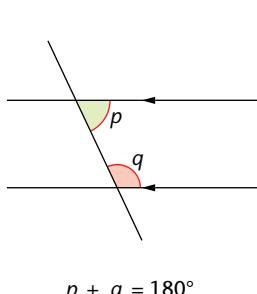
Corresponding angles are equal.



Note

Thinking of an F-shape or a translation may help you to remember that corresponding angles are equal.

Co-interior angles (sometimes called **allied angles**) add up to 180° .



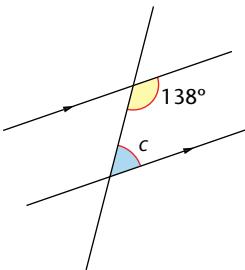
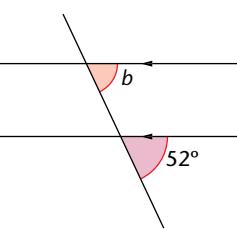
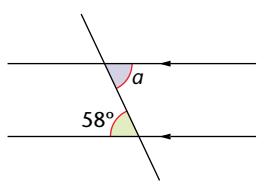
Note

Remembering a C-shape or using facts about angles on a straight line may help you to remember that allied angles add up to 180° .

Example 32.2

Question

Find the size of each of the lettered angles in these diagrams, giving your reasons.



Solution

$a = 58^\circ$ Alternate angles are equal.

$b = 52^\circ$ Corresponding angles are equal.

$c = 42^\circ$ Allied angles add up to 180° .

Note

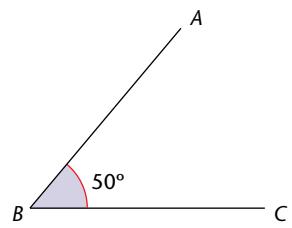
Single letters are often used for angles, as in this chapter so far.

Sometimes, the ends of a line are labelled with letters.

These can be used to name the angles.

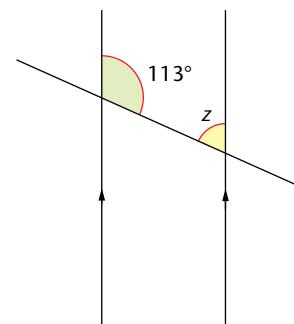
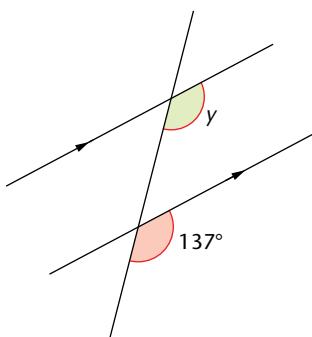
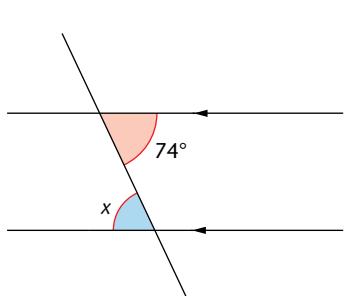
In the diagram, angle $ABC = 50^\circ$.

This may also be written $\hat{A}BC = 50^\circ$.

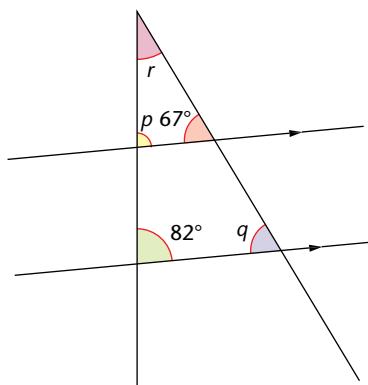
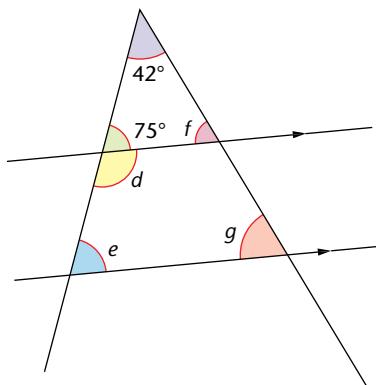
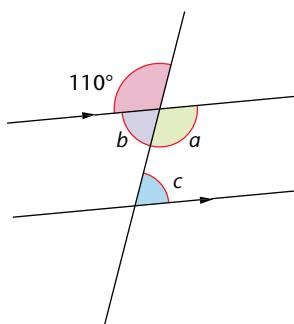


Exercise 32.2

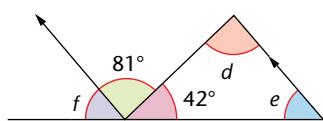
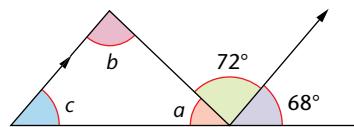
- Find the value of x , y and z in these diagrams, giving your reasons.



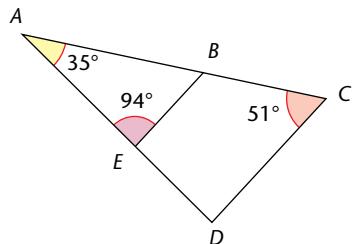
- 2 Find the size of each of the lettered angles in these diagrams.



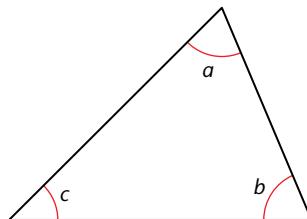
- 3 In a parallelogram $ABCD$, angle $ABC = 75^\circ$.
Make a sketch and mark this angle.
Calculate the other three angles and label them on your sketch.
- 4 In an isosceles trapezium $ABCD$, angle BCD is 127° .
Make a sketch of the trapezium and mark this angle.
Calculate the other three angles and label them on your sketch.
- 5 Find the size of each of the lettered angles in these diagrams and give a reason for your answers.



- 6 Use what you know about triangles to find the sizes of angles ABE and CDE in this diagram.
Show why BE and CD are parallel.



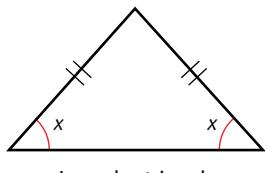
The angles in a triangle



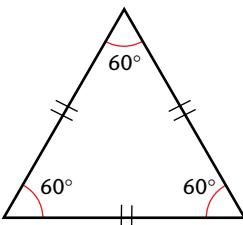
The angles in a triangle add up to 180° .

$$a + b + c = 180^\circ$$

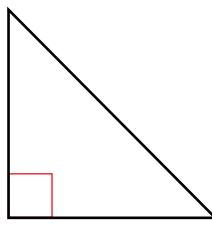
Here are the three special triangles you have already met.



Isosceles triangle



Equilateral triangle



Right-angled triangle

An isosceles triangle

has two angles the same
and two sides the same.

An equilateral triangle
has all three angles the same (60°)
and all three sides the same.

A right-angled triangle
has one angle of 90° .

A triangle that has all its angles different is called a **scalene triangle**.

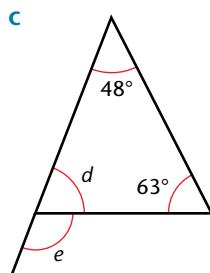
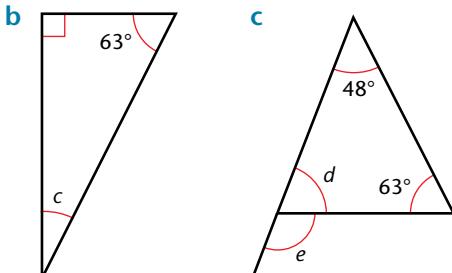
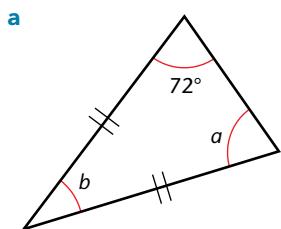
These facts can be used to find angles.

Example 32.3**Question**

Work out the sizes of the angles in these diagrams.

Sides the same length are marked with two short lines.

Give reasons for your answers.

**Solution**

a $a = 72^\circ$
 $b + 72^\circ + 72^\circ = 180^\circ$
 $b + 144^\circ = 180^\circ$
 $b = 36^\circ$

b $c + 63^\circ = 90^\circ$
 $c = 27^\circ$

c $d + 48^\circ + 63^\circ = 180^\circ$
 $d + 111^\circ = 180^\circ$
 $d = 69^\circ$
 $e + 69^\circ = 180^\circ$
 $e = 111^\circ$

Equal angles in an isosceles triangle.
Angles in a triangle add up to 180° .

Other angles in a right-angled triangle add up to 90° , or angles in a triangle add up to 180° .
Angles in a triangle add up to 180° .

The sum of angles on a straight line is 180° .

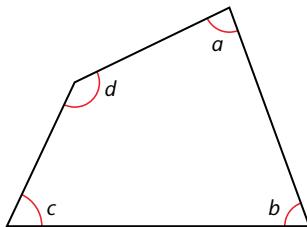
Note

Angle $e = 111^\circ$
 $= 63^\circ + 48^\circ$

So the exterior angle of a triangle is equal to the sum of the two opposite interior angles.

The angles in a quadrilateral

The angles in a quadrilateral add up to 360° .



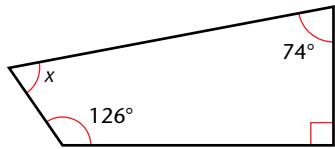
$$a + b + c + d = 360^\circ$$

Example 32.4

Question

Work out the size of angle x .

Give a reason for your answer.



Solution

$x = 360^\circ - (126^\circ + 90^\circ + 74^\circ)$ Angles in a quadrilateral add up to 360° .

$$x = 70^\circ$$

Note

Sides the same length are marked with two short lines.

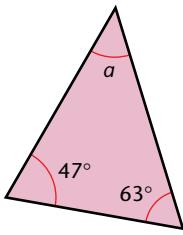
Unknown angles marked with the same letter in a diagram are equal in size.

Exercise 32.3

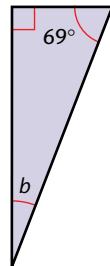
Calculate the size of the angles marked with letters in these shapes.

Give reasons for your answers.

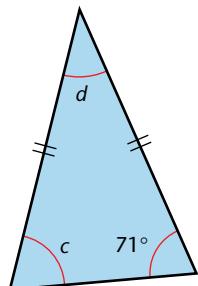
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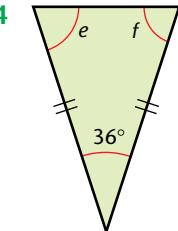
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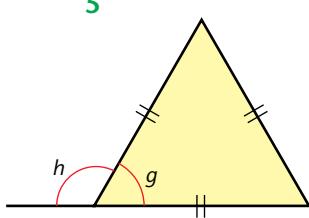
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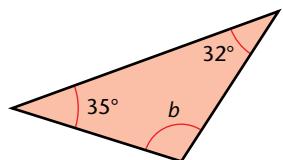
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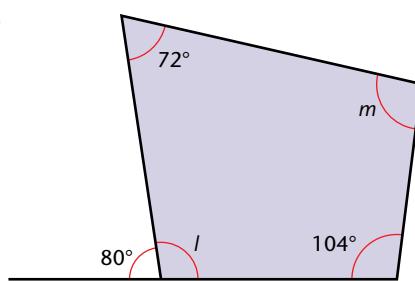
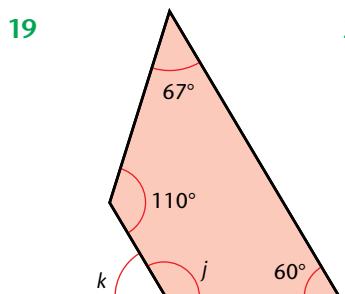
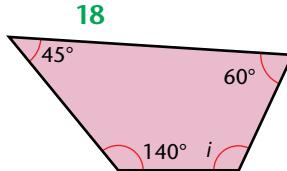
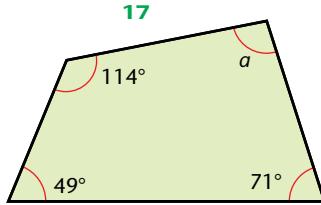
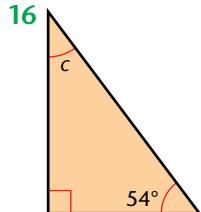
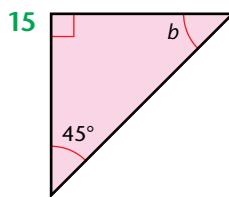
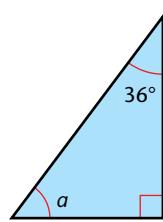
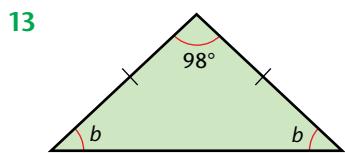
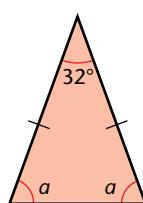
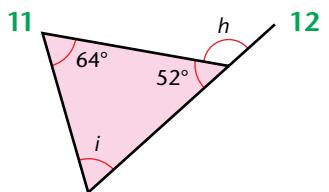
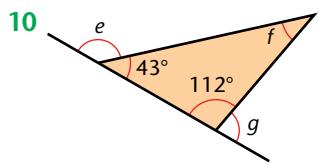
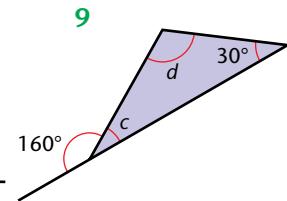
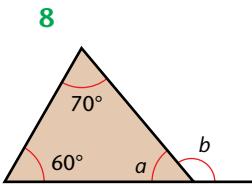
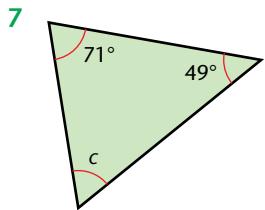


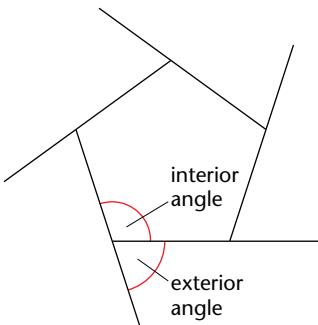
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6







The angles in a polygon

The sum of the exterior angles of any convex polygon is 360° .

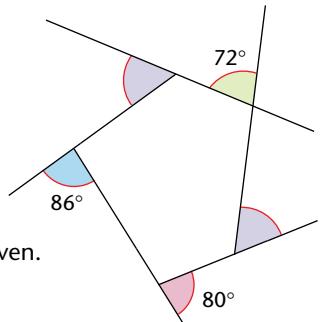
At each vertex, the interior and exterior angles make a straight line.

For any convex polygon, at any vertex: interior angle + exterior angle = 180° .

Example 32.5

Question

The two unlabelled exterior angles of this pentagon are equal.
Find their size.



Solution

First, find the sum of the angles that are given.

$$72^\circ + 80^\circ + 86^\circ = 238^\circ$$

The sum of all the exterior angles is 360° .

The sum of the remaining two angles = $360^\circ - 238^\circ = 122^\circ$.

So each angle is $122 \div 2 = 61^\circ$.

The two angle facts above can be used to find the sum of the interior angles of a polygon with n sides (and n angles).

At each vertex of a convex polygon, the interior angle + the exterior angle = 180° .

Sum of (interior + exterior angles) for the polygon = $180^\circ \times n$

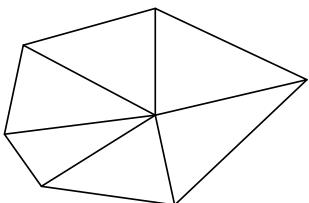
But the sum of the exterior angles is 360° .

For an n -sided convex polygon,
the sum of the interior angles = $(180n - 360^\circ) = 180^\circ(n - 2)$

Another way you can find the sum of the interior angles of any polygon is to divide it into triangles.

The sum of the angles in all the triangles = $180^\circ \times n$

The sum of the angles at the centre = 360° and this must be subtracted.



Example 32.6

Question

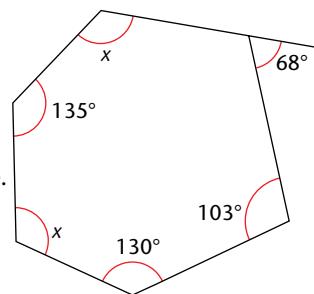
Find the value of x in this hexagon.

Solution

First, find the size of the missing interior angle.

$$\text{Interior angle} + \text{exterior angle} = 180^\circ$$

$$\begin{aligned}\text{Interior angle} &= 180^\circ - 68^\circ \\ &= 112^\circ\end{aligned}$$



Next, find the sum of the interior angles.

$$\begin{aligned}\text{Sum of interior angles} &= 112^\circ + 103^\circ + 130^\circ + 135^\circ + 2x \\ &= 480^\circ + 2x\end{aligned}$$

The sum of the interior angles of a hexagon = $180^\circ(n - 2)$

$$\begin{aligned}&= 180^\circ \times 4 \\ &= 720^\circ\end{aligned}$$

Therefore

$$480^\circ + 2x = 720^\circ$$

$$2x = 720^\circ - 480^\circ$$

$$2x = 240^\circ$$

$$x = 120^\circ$$

Example 32.7

Question

Find the interior angle of a regular hexagon.

Solution

For a regular hexagon, the exterior angle = $360^\circ \div 6 = 60^\circ$.

So the interior angle = $180^\circ - 60^\circ = 120^\circ$.

Example 32.8

Question

Find the number of sides of a regular polygon with an interior angle of 144° .

Solution

If the interior angle = 144° , the exterior angle = $180^\circ - 144^\circ = 36^\circ$.

The sum of the exterior angles = 360° .

Therefore the number of sides = $360^\circ \div 36^\circ = 10$.

Exercise 32.4

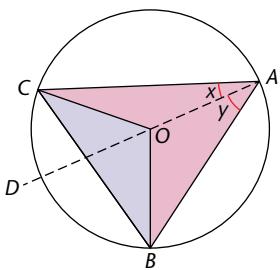
- 1 Four of the exterior angles of a pentagon are 70° , 59° , 83° and 90° .
 - Find the size of the other exterior angle.
 - Find the size of each interior angle.
- 2 Five of the exterior angles of a hexagon are 54° , 48° , 65° , 35° and 80° .
 - Find the size of the other exterior angle.
 - Find the size of each interior angle.
- 3 Four of the exterior angles of a hexagon are 67° , 43° , 91° and 37° .
 - Find the size of the other exterior angles, given that they are equal.
 - Find the size of each interior angle.

- 4** A regular polygon has nine sides.
Find the size of each of its exterior and interior angles.
- 5** Find the size of the interior angle of
 a a regular dodecagon (12 sides)
 b a regular 20-sided polygon.
- 6** A regular polygon has an exterior angle of 24° .
How many sides does it have?
- 7** Six of the angles of a heptagon are 122° , 141° , 137° , 103° , 164° and 126° .
Calculate the size of the remaining angle.
- 8** A regular polygon has 15 sides.
Find the size of each of its exterior and interior angles.
- 9** A regular polygon has an exterior angle of 30° .
How many sides does it have?
- 10** A polygon has 11 sides. Ten of its interior angles add up to 1490° .
Find the size of the remaining angle.

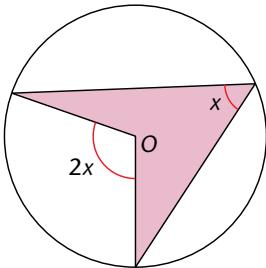
The angles in a circle

Note

An angle is formed or **subtended** at a particular point by an arc when straight lines from its ends are joined at that point.



Note



The angle subtended by an arc at the centre is twice the angle subtended at the circumference

BC is a chord of the circle whose centre is O .

Arc BC subtends angle BOC at the centre of the circle and angle BAC at the circumference and AD is a straight line.

Here is a simple proof that angle $COB =$ twice angle CAB .

Let angle $CAO = x$ and angle $BAO = y$.

Then angle $CAB = x + y$.

Triangles OAB and OAC are isosceles.

OA , OB and OC are radii of the circle.

Angle $ACO = x$ and angle $ABO = y$

Base angles of isosceles triangles.

Angle $DOC =$ angle $OAC +$ angle $OCA = 2x$

Angle $DOB =$ angle $OAB +$ angle $OBA = 2y$

The exterior angle of a triangle equals the sum of the opposite interior angles.

$$\text{Angle } COB = \text{angle } DOC + \text{angle } DOB$$

$$= 2x + 2y$$

$$= 2(x + y)$$

$$= 2 \times \text{angle } CAB$$

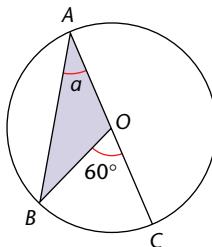
Example 32.9**Question**

In the diagram, O is the centre of the circle.

Calculate the size of angle a .

Solution

Angle $a = 30^\circ$ Angle at the centre = twice the angle at the circumference.



The angle subtended at the circumference in a semi-circle is a right angle

This is a special case of the angle subtended by an arc at the centre is twice the angle subtended at the circumference.

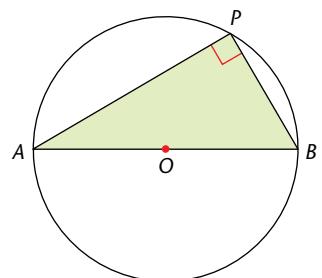
Angle $AOB = 2 \times$ angle APB The angle at the centre is twice the angle at the circumference.

Angle $AOB = 180^\circ$ AB is a diameter, that is, a straight line.

Angle $APB = 90^\circ$ Half of angle AOB .

So the angle in a semi-circle is 90° .

This theorem is often used in conjunction with the angle sum of a triangle, as is shown in the next example.

**Example 32.10****Question**

Work out the size of angle x .

Give a reason for each step of your work.

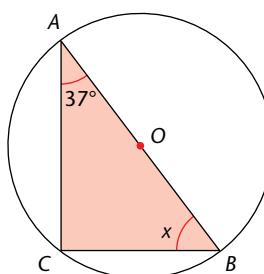
Solution

Angle $ACB = 90^\circ$

The angle in a semi-circle is a right angle.

$x = 180 - (90 + 37)$ The angle sum of a triangle is 180° .

$x = 53^\circ$



Note

When calculating an angle never assume something is true, for example 'because it looks like it' on the diagram. You must always give a reason for each step of your calculation.

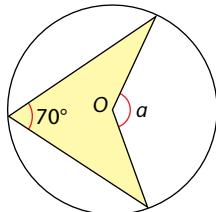
Exercise 32.5

In the diagrams, O is the centre of the circle.

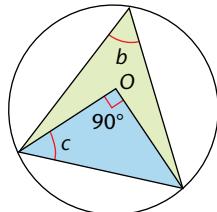
Find the size of each of the lettered angles.

Write down the reasons for each step of your calculations.

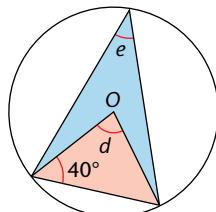
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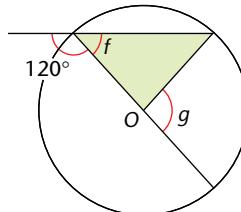
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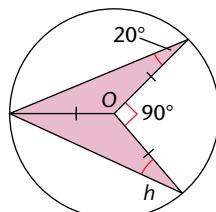
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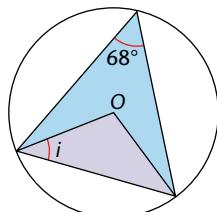
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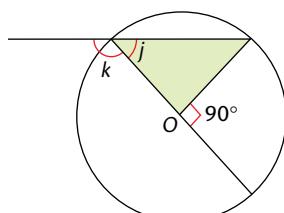
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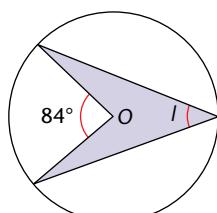
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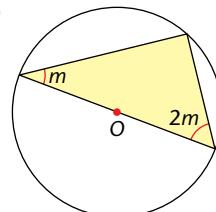
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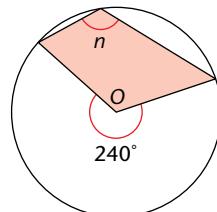
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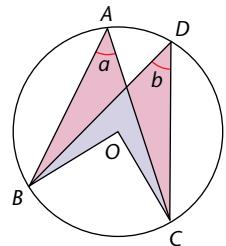
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Angles in the same segment of a circle are equal

In the diagram, $a = b$, as both are equal to half the angle subtended at the centre.

To use this property, you have to identify angles subtended by the same arc since these are in the same segment. This is shown in the next example.



Example 32.11

Question

Find the sizes of angles a , b and c .

Give a reason for each step of your work.

Solution

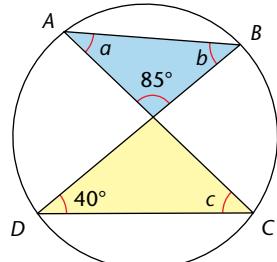
$a = \text{angle } BDC = 40^\circ$ Angles in the same segment are equal.

$b = 180 - (40 + 85)$

$$= 55^\circ$$

The angle sum of a triangle is 180° .

$c = b = 55^\circ$ Angles in the same segment are equal.



Angles in opposite segments are supplementary

This property is also derived from the fact that the angle at the centre is twice the angle at the circumference.

In the diagram

$\text{angle } ABC = \frac{1}{2}x$ The angle at the centre is twice the angle at the circumference.

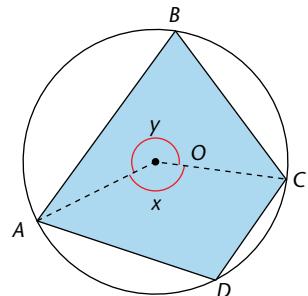
$\text{angle } ADC = \frac{1}{2}y$

$x + y = 360^\circ$ The angles around a point add up to 360° .

$\text{angle } ABC + \text{angle } ADC = \frac{1}{2}(x + y)$

$\text{angle } ABC + \text{angle } ADC = 180^\circ$

So the angles in opposite segments are supplementary.



Note

Two angles are *supplementary* if they add up to 180° .

Example 32.12**Question**

Find the sizes of angles c and d .

Give a reason for each step of your work.

Solution

$$c = 180 - 96 = 84^\circ$$

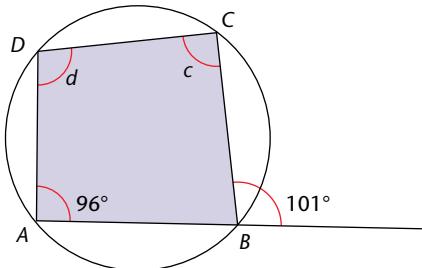
Angles in opposite segments
are supplementary.

$$\text{Angle } ABC = 180 - 101 = 79^\circ$$

Angles on a straight line add
up to 180° .

$$d = 180 - 79 = 101^\circ$$

Angles in opposite segments
are supplementary.

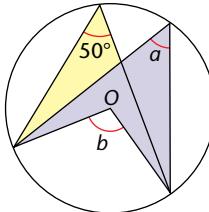
**Exercise 32.6**

In the diagrams, O is the centre of the circle.

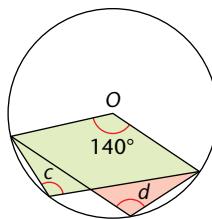
Find the size of each of the lettered angles.

Write down the reasons for each step of your calculations.

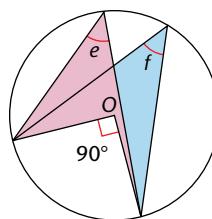
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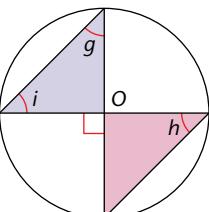
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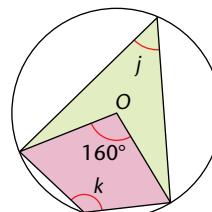
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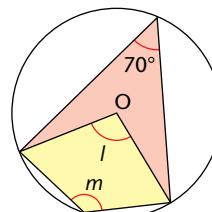
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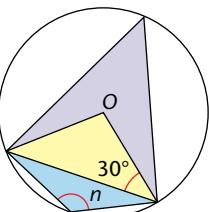
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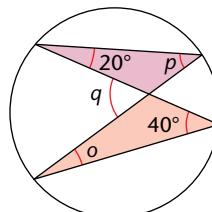
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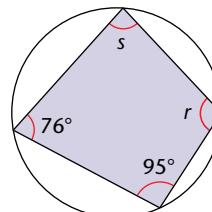
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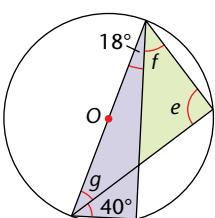
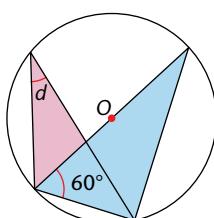
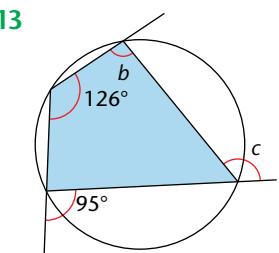
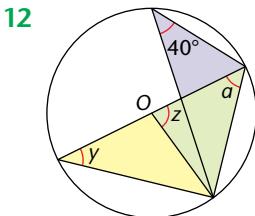
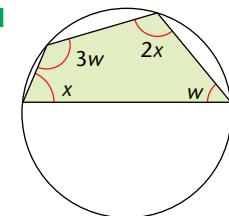
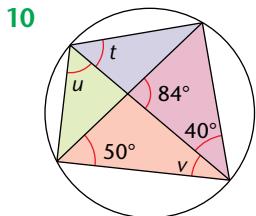


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You need to be able to identify when to use the circle theorems you have learned.

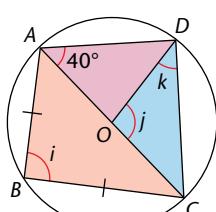
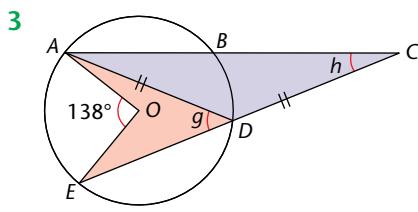
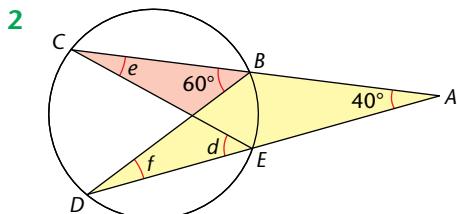
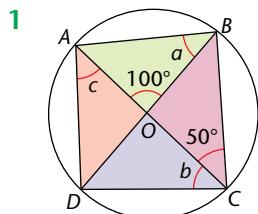
Use this exercise to practise.

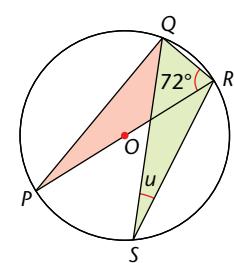
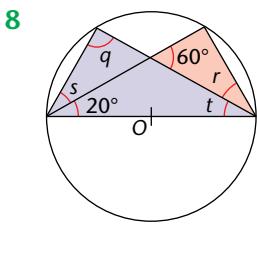
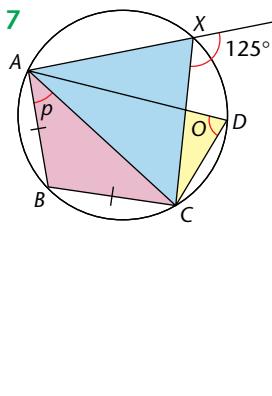
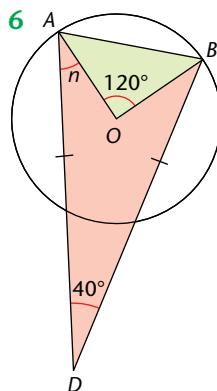
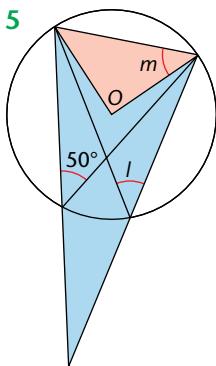
Exercise 32.7

Calculate the sizes of the angles marked with letters.

O is the centre of each circle.

Give the reasons for each step of your working.





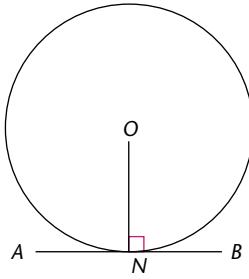
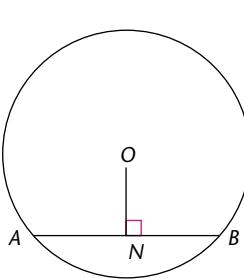
The tangent to a circle is at right angles to the radius at the point of contact

In Chapter 31 you saw that the perpendicular bisector of a chord passes through the centre of the circle.

So the line from the centre to the midpoint of the chord is at right angles to the chord.

If you move the chord further from the centre it will eventually become a tangent and ON will be a radius.

So the tangent to a circle is at right angles to the radius at the point of contact.



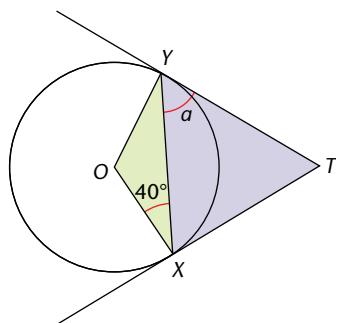
Exercise 32.8

In the diagrams, O is the centre of the circle and X and Y are the points of contact of the tangents to each circle.

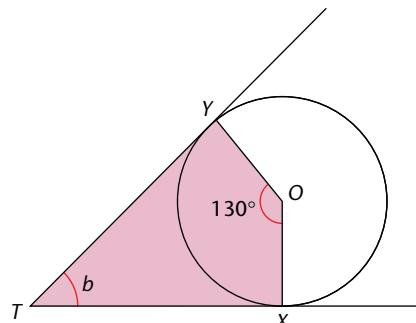
Find the size of each of the lettered angles.

Write down the reasons for each step of your calculations.

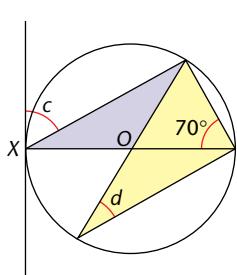
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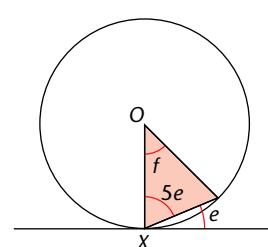
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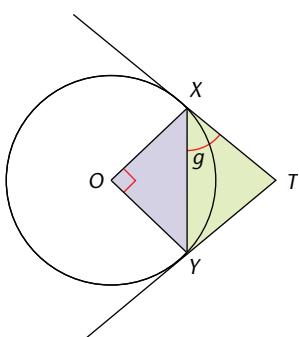
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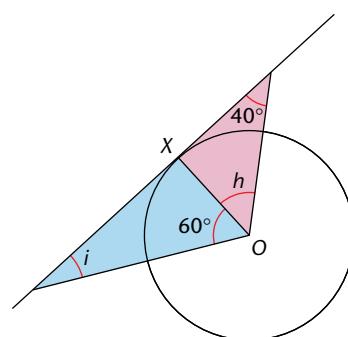
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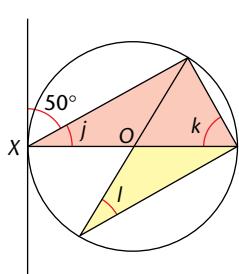
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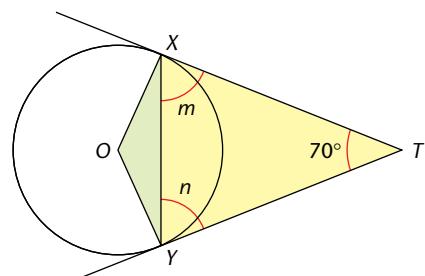
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33 LOCI

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- use the following loci and the method of intersecting loci for sets of points in two dimensions which are:
 - at a given distance from a given point
 - at a given distance from a given straight line
 - equidistant from two given points
 - equidistant from two given intersecting straight lines.

CHECK YOU CAN:

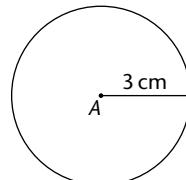
- use a protractor and compasses
- construct the perpendicular bisector of a line
- construct the bisector of an angle
- construct a triangle given
 - three sides
 - two sides and an angle
 - two angles and a side
- make scale drawings.

Simple loci

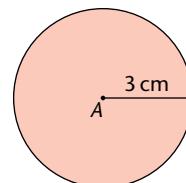
The **locus** of a point is the path or the region that the point covers as it moves according to a particular rule. The plural of locus is **loci**.

The locus of points that are a fixed distance from a given point is a circle, centred at the given point.

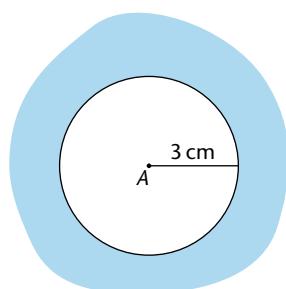
This means that the locus of a point 3 cm from A is a circle, centre A and radius 3 cm.



The locus of a point less than ($<$) 3 cm from A is the region inside a circle, centre A and radius 3 cm.

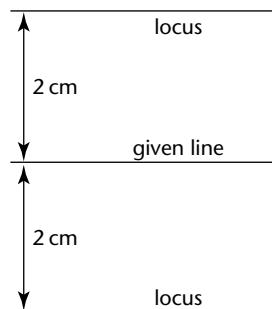


The locus of a point greater than ($>$) 3 cm from A is the region outside a circle, centre A and radius 3 cm.



The locus of points that are a fixed distance from a given line is a pair of parallel lines.

For example, the locus of a point 2 cm from a straight line is a pair of lines parallel to that line, 2 cm away from it on either side.

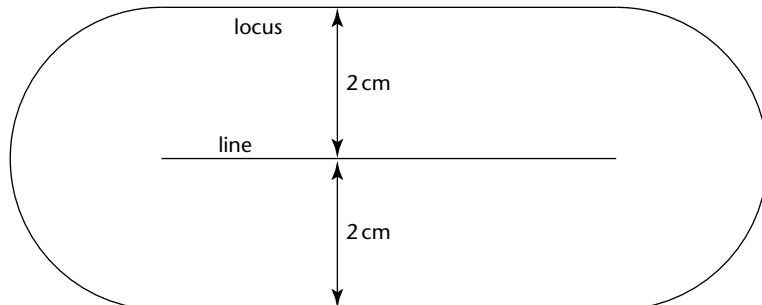


Example 33.1**Question**

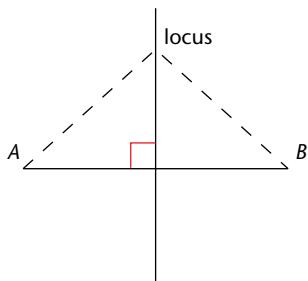
A line is 6 cm long. Construct the locus of all the points that are 2 cm from the line.

Solution

The locus is two parallel lines with a semi-circle joining them at each end.



The locus of a point equidistant from two points is the perpendicular bisector of the line joining the two points.

**Example 33.2****Question**

Two towns P and Q are 5 kilometres apart.

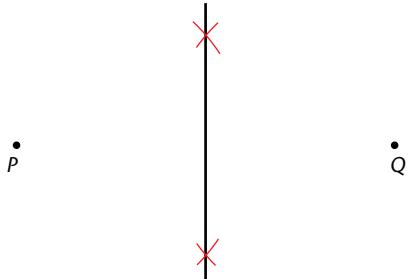
Draw a diagram and shade the region that is nearer to P than Q .

Use a scale of 1 cm to 1 km.

Solution

Mark P and Q 5 cm apart.

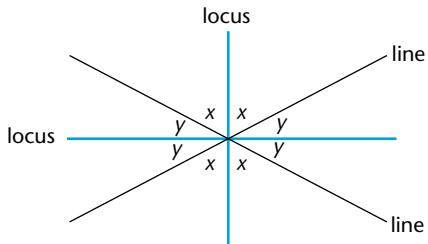
Draw the perpendicular bisector of the line PQ and shade the region on P 's side of the line. The shading could go beyond P and further up or down the page.



The locus of a point equidistant from two intersecting lines is the pair of lines that bisect the angles between the lines.

Note

The locus of the points equidistant from two intersecting lines is a pair of lines, but usually you only require one.



Example 33.3

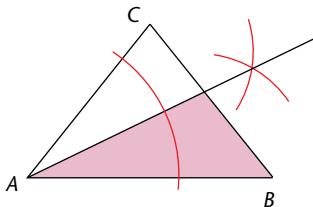
Question

Draw a triangle ABC with sides $AB = 5\text{ cm}$ and $AC = 4\text{ cm}$, and angle $A = 50^\circ$.

Use compasses to bisect angle A .

Shade in the locus of the points inside the triangle that are nearer to AB than AC .

Solution



Note

In most cases you will be asked to construct a locus either to size or to scale. Draw it as accurately as you can.

Note

When you do a construction, leave in your construction lines. Do not stop the line of a construction at the intersection of the arcs: draw it through the intersection.

Exercise 33.1

- 1 Draw a rectangle 4 cm by 5 cm . Draw the locus of the points outside the rectangle that are 1 cm from the perimeter of the rectangle.
- 2 Draw a rectangle $ABCD$ with $AB = 6\text{ cm}$ and $BC = 4\text{ cm}$. Draw the locus of the points inside the rectangle that are nearer to A than B .
- 3 Draw two parallel lines across the page, 4 cm apart. Draw the locus of the points that are 1 cm from the top line and 3 cm from the bottom line.
- 4 Show, by shading, the locus of the points that are more than 4 cm from a fixed point A .
- 5 Draw a line 6 cm long. Show, by shading, the locus of the points that are less than 2 cm from the line.
- 6 Construct a triangle ABC with $AB = 8\text{ cm}$, $AC = 7\text{ cm}$ and $BC = 5\text{ cm}$. Use compasses and a ruler to bisect angle A . Shade the locus of the points inside the triangle that are nearer to AB than AC .

- 7** Draw a square $ABCD$ with sides of 6 cm.
 Construct the locus of the points that are equidistant from A and C .
 What do you notice about the locus?
- 8** Draw a square with sides of 4 cm. Label one corner A .
 Show the locus of the points inside the square that are less than 3 cm
 from A .
- 9** Draw a rectangle $ABCD$ with sides $AB = 7$ cm and $BC = 5$ cm.
 Use compasses to construct the line equidistant from AB and AC .
- 10** Construct the triangle ABC with angle $A = 30^\circ$, angle $B = 50^\circ$ and
 $AB = 10$ cm.
 Construct the locus of the points equidistant from A and B .

Problems involving intersection of loci

Combining all you know about loci, you can answer more complicated questions involving more than one locus.

Example 33.4

Question

Construct triangle ABC with $AB = 7$ cm, $AC = 6$ cm and $BC = 4$ cm.

By using constructions, find the point that is equidistant from all three vertices. Label this point D .

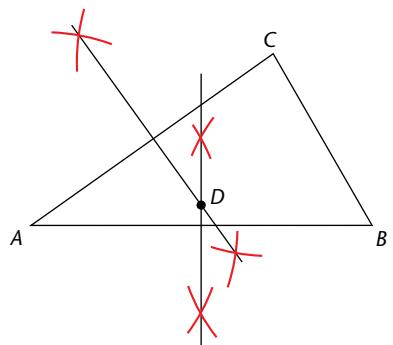
Solution

First you need the line equidistant from two vertices. If you choose vertices A and B , you need to construct the perpendicular bisector of AB .

Then you need to construct the perpendicular bisector of another side. Where they cross is the required point.

You could also bisect the third side, and that line would also pass through the same point.

This diagram is not to scale.



Example 33.5

Question

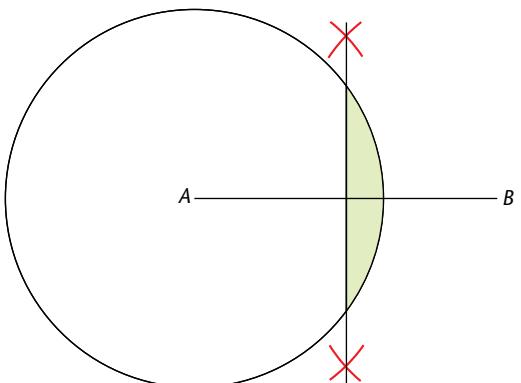
Two points A and B are 4 cm apart.

Show, by shading, the locus of the points that are less than 2.5 cm from A and nearer to B than A .

Solution

You need to draw a circle, radius 2.5 cm and centre A . You also need to draw the bisector of the line AB . The region you require is inside the circle and on the B side of the bisector.

The required region is shaded on the diagram.



Example 33.6

Question

Erica wants to put a chair in her room.

She wants the chair more than 0.5 m from a wall and less than 2 m from corner A.

This is a sketch of her room.

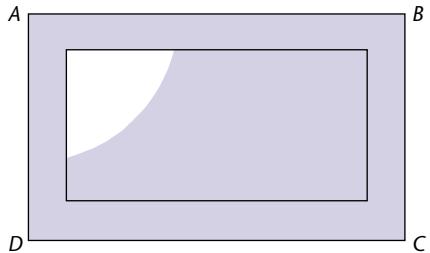
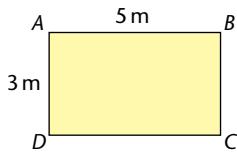
Using a scale of 1 cm : 1 m, make a scale drawing of the room and show, by shading, the region where the chair can be placed.

Solution

Draw the rectangle and then add lines 0.5 cm from each side.

Draw an arc of a circle, centre A and radius 2 cm.

In this diagram the regions not required are shaded, leaving the white region where the chair can be placed.

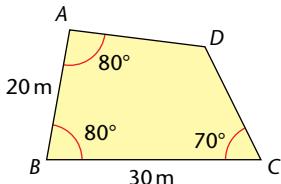


Exercise 33.2

Draw all of these accurately.

- 1 Two points A and B are 5 cm apart.
Show, by shading, the region that is less than 3 cm from A , and more than 4 cm from B .
- 2 A rectangle $ABCD$ has sides $AB = 5$ cm and $BC = 4$ cm.
Draw the rectangle and show, by shading, the region inside the rectangle that is nearer to AB than CD , and less than 3.5 cm from B .
- 3 Draw a triangle ABC with $AB = 6$ cm, angle $A = 60^\circ$ and angle $B = 55^\circ$.
Use constructions to find the point D that is equidistant from all three sides.
- 4 Draw a rectangle $ABCD$ with sides $AB = 4$ cm and $BC = 3$ cm.
Show the points that are equidistant from AB and BC , and 3.5 cm from A .
- 5 Draw a triangle ABC with sides $AB = 9$ cm, $BC = 6$ cm and $AC = 5$ cm.
Show, by shading, the region inside the triangle that is nearer to AB than BC , and more than 3 cm from C .
- 6 Show, by shading, the locus of the points that are more than 2 cm from a point A and less than 3 cm from point A .
- 7 Two points A and B are 4 cm apart.
Show, by shading, the locus of all the points that are less than 2.5 cm from A , and more than 3 cm from B .
- 8 Draw a triangle ABC with $AB = 6$ cm, angle $A = 40^\circ$ and angle $B = 35^\circ$.
Use constructions to find the point D that is equidistant from A and B , and 4 cm from C .
- 9 Draw a square with sides of 4 cm.
Show, by shading, the region within the square that is more than 2 cm from every vertex.
- 10 Draw a triangle ABC with $AB = 6$ cm, $AC = 5$ cm and angle $A = 55^\circ$.
Show, by shading, the region that is inside the triangle, nearer to AB than AC , and nearer to B than A .

- 11** Vijay's bedroom is rectangular with sides of 4 m and 6.5 m.
 He wants to put a desk within 1 metre of a longer wall and within 2.5 m of the centre of the window in the middle of one of the shorter walls.
 Using a scale of 1 cm : 1 m, make a scale drawing and show, by shading, the region where the desk can be placed.
- 12** A field is a quadrilateral $ABCD$ with $AB = 25$ m, $BC = 30$ m, angle $A = 90^\circ$, angle $B = 106^\circ$ and angle $C = 65^\circ$.
 The owner wants to put a post within 15 m of corner A and nearer to CD than CB .
 Using a scale of 1 cm to 5 m, draw the field and show, by shading, the region where the post can be placed.
- 13** Tariq's garden is a rectangle $ABCD$ with $AB = 10$ m and $BC = 4$ m.
 He wants to plant a tree in the garden.
 The tree must be more than 4 m from corner C and more than 1 m from side AB .
 Using a scale of 1 cm : 1 m, make a scale drawing of the garden and show where he can plant the tree.
- 14** This is a sketch of a playing field.



A new swing is to be placed in the field.
 It must be within 15 m of A and nearer to AB than AD .
 Use a scale of 1 cm : 5 m to make a scale drawing and show the region where the swing can be placed.

34 MEASURES

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- use current units of mass, length, area, volume and capacity in practical situations and express quantities in terms of larger or smaller units.

CHECK YOU CAN:

- multiply and divide by powers of 10
- add and subtract decimals
- find the volume of a cuboid.

Basic units of length, mass and capacity

These are the connections between the metric units of length.

1 kilometre = 1000 metres	1 km = 1000 m
1 metre = 1000 millimetres	1 m = 1000 mm
1 metre = 100 centimetres	1 m = 100 cm
1 centimetre = 10 millimetres	1 cm = 10 mm

These are the connections between the metric units of mass.

1 kilogram = 1000 grams	1 kg = 1000 g
1 tonne = 1000 kg	1 t = 1000 kg

When a volume is filled with liquid or gas, it is called the **capacity**.

These are the connections between the metric units of capacity.

1 litre = 1000 millilitres	1 l = 1000 ml
1 litre = 100 centilitres	1 l = 100 cl

A millilitre is exactly the same as a cm^3 , but is used for liquids rather than cm^3 .

Example 34.1

Question

Misha is baking some cakes.

One recipe needs 1.6 kg of flour, another needs $\frac{1}{2}$ kg of flour.

- How much flour does he need altogether?
- He has a new 3 kg bag of flour.
How much will be left after he has made the cakes?

Solution

- When you have to add fractions and/or decimal parts of a kilogram, it is usually easier to change all the weights to grams.

$$\begin{aligned}\text{Total} &= 1.6 \text{ kg} + \frac{1}{2} \text{ kg} = 1600 \text{ g} + 500 \text{ g} \\ &= 2100 \text{ g} \text{ or } 2.1 \text{ kg}\end{aligned}$$

- Amount of flour left = 3 kg - 2.1 kg = 3000 g - 2100 g
= 0.9 kg or 900 g

Exercise 34.1

- 1 Which metric unit would you use to measure each of these?

a The width of a book	b The height of a room
c The length of a classroom	d The length of a finger
- 2 Change each of these lengths to millimetres.

a 4.52 cm	b 2 cm	c 4.5 cm	d 9.35 cm
e 2.101 m	f 3 m	g 2.239 m	h 9.1 m
- 3 Change each of these lengths to centimetres.

a 52 m	b 5 m	c 2.32 m	d 18.16 m
e 660 mm	f 70 mm	g 310 mm	h 46 mm
- 4 Change each of these lengths to metres.

a 2146.3 mm	b 5142 mm	c 570 cm	d 1146 mm
-------------	-----------	----------	-----------
- 5 Write each set of lengths in order of size, smallest first.

a 2.42 m	1600 mm	284 cm	9 m	31 cm
b 423 cm	6100 mm	804 cm	3.2 m	105 mm
- 6 Marta runs for 3.5 km, walks for 800 m, then runs another 2.4 km.
How far has she gone altogether?
- 7 Change each of these masses to grams.

a 0.012 kg	b 7 kg	c 1.13 kg	d 2.14 kg
------------	--------	-----------	-----------
- 8 Change each of these masses to kilograms.

a 6600 g	b 8000 g	c 6300 g	d 5126 g
----------	----------	----------	----------
- 9 Write each set of masses in order of size, lightest first.

a 4000 g	52 000 g	9.4 kg	874 g	1.7 kg
b 4123 g	2104 g	3.4 kg	0.174 kg	2.79 kg
- 10 Nasrin uses 750 g of sugar from a $1\frac{1}{2}$ kg bag.
How much is left?
- 11 Change these capacities to millilitres.

a 52 cl	b 7 litres	c 1.52 litres	d 0.16 litres
---------	------------	---------------	---------------
- 12 Change these capacities to litres.

a 9503 cl	c 2000 ml	c 2341 ml	d 4642 cl
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- 13 Write these capacities in order of size, smallest first.

a 2.4 litres	1600 ml	80 cl	9 litres	51 cl
b 3.1 litres	1500 ml	180 cl	1 litre	51.5 ml
- 14 A bottle holds 1 litre of lemonade.
Ali drinks 350 ml.
How much is left in the bottle?

Area and volume measures

You can use the basic relationships between metric units of length to work out the relationships between metric units of area and volume.

For example:

$$1 \text{ cm}^2 = 1 \text{ cm} \times 1 \text{ cm} = 10 \text{ mm} \times 10 \text{ mm} = 100 \text{ mm}^2$$

$$1 \text{ cm}^3 = 1 \text{ cm} \times 1 \text{ cm} \times 1 \text{ cm} = 10 \text{ mm} \times 10 \text{ mm} \times 10 \text{ mm} = 1000 \text{ mm}^3$$

$$1 \text{ m}^2 = 1 \text{ m} \times 1 \text{ m} = 100 \text{ cm} \times 100 \text{ cm} = 10000 \text{ cm}^2$$

$$1 \text{ m}^3 = 1 \text{ m} \times 1 \text{ m} \times 1 \text{ m} = 100 \text{ cm} \times 100 \text{ cm} \times 100 \text{ cm} = 1000000 \text{ cm}^3$$

Note

Remember that volume and capacity units are related.

$$1 \text{ litre} = 1000 \text{ cm}^3$$

$$1 \text{ ml} = 1 \text{ cm}^3$$

Example 34.2**Question**

Change these units.

a 5 m^3 to cm^3 b 5600 cm^2 to m^2

Solution

a $5 \text{ m}^3 = 5 \times 1000000 \text{ cm}^3$ Convert 1 m^3 to cm^3 and multiply by 5.
 $= 5000000 \text{ cm}^3$

b $5600 \text{ cm}^2 = 5600 \div 10000 \text{ m}^2$ To convert from m^2 to cm^2 you multiply,
 $= 0.56 \text{ m}^2$ so to convert from cm^2 to m^2 you divide.

Note

Make sure you have done the right thing by checking that your answer makes sense. If you had multiplied by 10000, you would have got 5600000 m^2 , which is obviously a much larger area than 5600 cm^2 .

Exercise 34.2

1 Change these units.

a 3 m^2 to cm^2

c 9.52 m^2 to cm^2

b 2.3 cm^2 to mm^2

d 0.014 cm^2 to mm^2

2 Change these units.

a 90000 mm^2 to cm^2

c 7200000 cm^2 to m^2

b 8140 mm^2 to cm^2

d 94000 cm^2 to m^2

3 Change these units.

a 3.2 m^3 to cm^3

c 5000 cm^3 to m^3

b 42 cm^3 to m^3

d 6.42 m^3 to cm^3

4 Change these units.

a 2.61 litres to cm^3

c 2.4 litres to ml

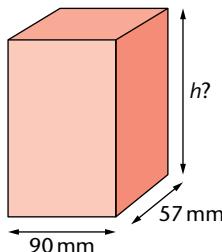
b 9500 ml to litres

d 910 ml to litres

5 What is wrong with this statement?

'The trench I have just dug is 5 m long, 2 m wide and 50 cm deep. To fill it in, I would need 500 m^3 of concrete.'

6 This carton holds 1 litre of juice. How high is it?



7 How many litres are there in 1 cubic metre?

8 $1 \text{ hectare} = 10000 \text{ m}^2$.

How many hectares are there in 1 km^2 ?

9 A sugar cube has sides of 15 mm.

Find how many will fit in a box measuring 11 cm by 11 cm by 5 cm.

10 Two football pitches have an area of 1 hectare.

Each pitch is 100 m long.

How wide is each pitch?

35 MENSURATION

The perimeter of a two-dimensional shape

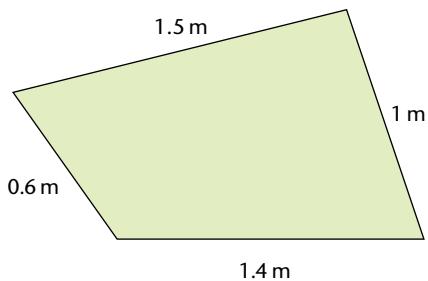
The **perimeter** of a shape is the distance all the way around the edge of the shape.

Since the perimeter of a shape is a length, you must use units such as centimetres (cm), metres (m) or kilometres (km).

Example 35.1

Question

Find the perimeter of this shape.



Solution

$$\text{Perimeter} = 0.6 + 1.4 + 1 + 1.5 = 4.5 \text{ m}$$

Note

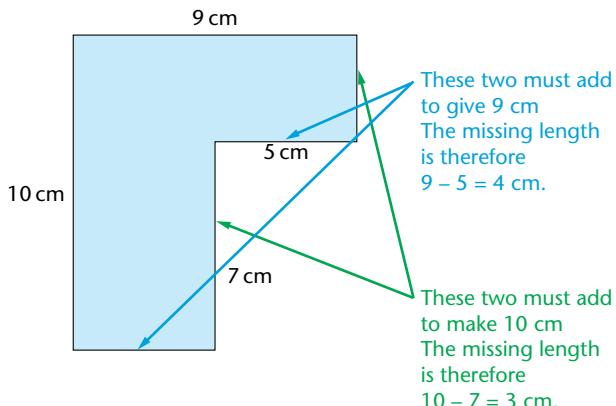
Although you don't have to give the units in your working, you must remember to give the units with your answer.

Sometimes not all of the lengths of the shape are given in the diagram. Before trying to find the perimeter, work out all the lengths.

Example 35.2

Question

Find the perimeter of this shape made from rectangles.



Solution

$$\text{Perimeter} = 9 + 10 + 4 + 7 + 5 + 3 = 38 \text{ cm}$$

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- solve problems involving:
 - the perimeter and area of a rectangle and triangle
 - the perimeter and area of a parallelogram and a trapezium
 - the circumference and area of a circle
 - arc length and sector area as fractions of the circumference and area of a circle
 - the surface area and volume of a cuboid, cylinder, prism, sphere, pyramid and cone
 - the areas and volumes of compound shapes.

CHECK YOU CAN:

- use metric units for length, area, volume and capacity
- understand the geometric terms used with shapes
- rearrange formulae and solve equations
- round numbers to a given number of decimal places or significant figures.

For calculating the surface area of a pyramid and a cone, you will also need to be able to

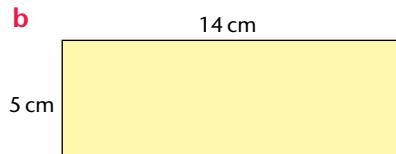
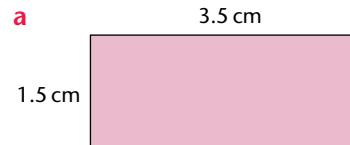
- use Pythagoras' Theorem.

For calculating the volume and surface area of more complex compound shapes, you will also need to be able to

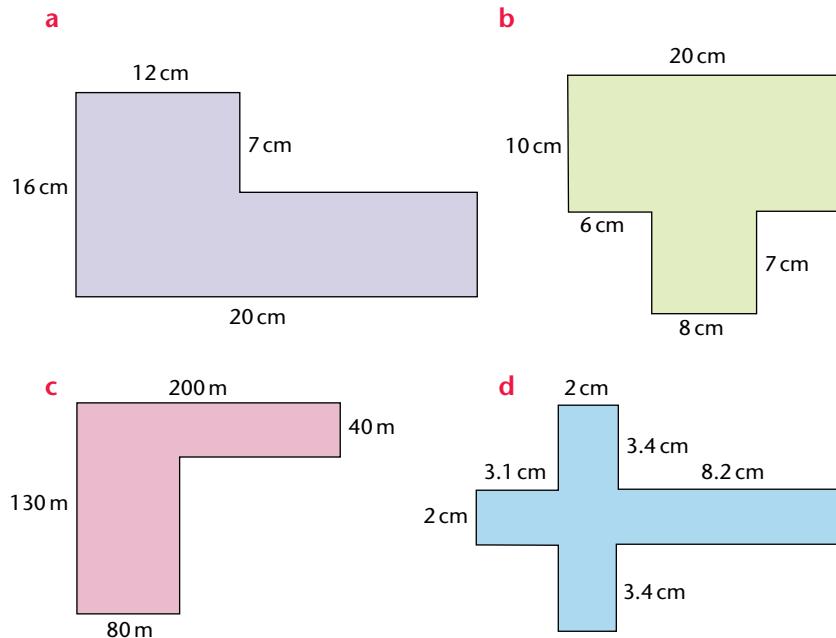
- use Pythagoras' Theorem
- calculate lengths in similar figures
- find the area of a non-right-angled triangle using the formula $\frac{1}{2} bc \sin A$.

Exercise 35.1

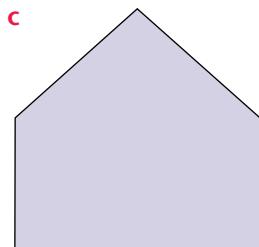
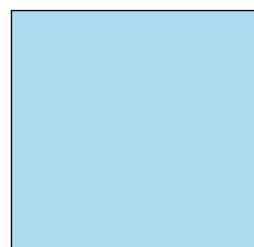
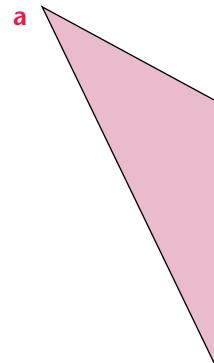
- 1** Find the perimeter of each of these rectangles.



- 2** Copy each of these diagrams, where the shapes are made from rectangles. Find any missing lengths and mark them on your diagram. Find the perimeter of each shape.



- 3** Measure each of these shapes accurately. Work out the perimeter of each shape.



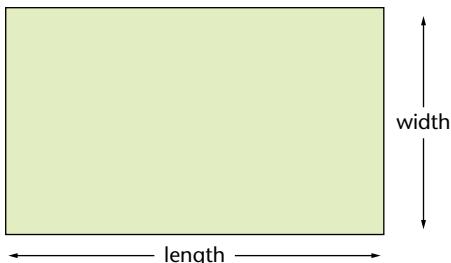
- 4 The perimeter of a rectangle is 26 cm.
Two sides are 8 cm long.
How long are the other sides?
- 5 A square has a perimeter of 120 cm.
How long is each side?
- 6 A rectangle has a perimeter of 60 cm.
The lengths of its sides are in whole centimetres.
What are the possible sizes of the rectangle?

The area of a rectangle

The **area** of a two-dimensional shape is the amount of flat space inside the shape.

For rectangles of any size,

$$\text{area of a rectangle} = \text{length} \times \text{width}$$



Note

Whenever you are giving an answer for an area, make sure you include the units.

If the lengths are in centimetres, the area will be in cm^2 .

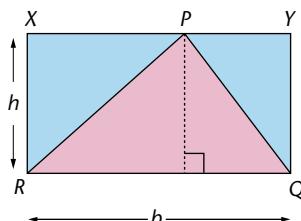
If the lengths are in metres, the area will be in m^2 .

Exercise 35.2

- 1 A rectangle measures 4.7 cm by 3.6 cm.
Find its area.
- 2 A square has sides 2.6 m long.
Find its area.
- 3 A rectangle measures 3.62 cm by 4.15 cm.
Find its area.
- 4 A rectangular pond measures 4.5 m by 8 m.
Find its area.
- 5 An airport is built on a rectangular piece of land 1.8 km long and 1.3 km wide.
 - a Calculate the area of the land.
 - b Find the length of fencing needed to go round the perimeter of the airport.
- 6 The perimeter of a square is 28 cm.
Calculate its area.
- 7 A rectangle has an area of 240 cm^2 .
One of its sides is 16 cm long.
Calculate the length of the other side of the rectangle.

- 8** A rectangular lawn measures 24 m by 18.5 m.
- Work out the area of the lawn.
Lawn weedkiller is spread on the lawn.
50 g of weedkiller is needed for every square metre.
 - How many kilograms of weedkiller are needed to treat the whole lawn?
 - Weedkiller is sold in 2.5 kg boxes.
How many boxes are needed?

The area of a triangle



Triangle PQR has base length b and perpendicular height h .

Rectangle $XYQR$ has the same base length and height.

The blue area is the same size as the pink area.

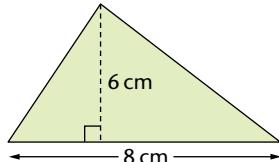
Area of the rectangle is bh , so the area of the triangle is $\frac{1}{2}bh$.

$$\text{Area of a triangle} = \frac{1}{2} \times \text{base} \times \text{perpendicular height} \text{ or } A = \frac{1}{2}bh$$

Example 35.3

Question

- a Find the area of this triangle.

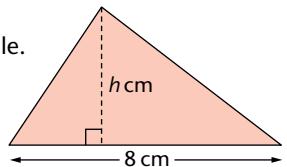


- b The area of this triangle is 20 cm^2 .

Find the perpendicular height of the triangle.

Solution

a Area $= \frac{1}{2}bh$
 $= \frac{1}{2} \times 8 \times 6$
 $= 24 \text{ cm}^2$



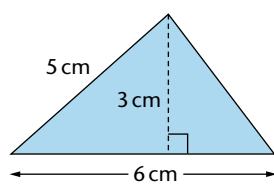
b Area $= \frac{1}{2}bh$
 $20 = \frac{1}{2} \times 8 \times h$
 $20 = 4h$
 $h = 5$

So the height is 5 cm.

Note

Always use the perpendicular height of the triangle, never the slant height.

In this triangle,
 $\text{area} = \frac{1}{2} \times 6 \times 3 = 9 \text{ cm}^2$.

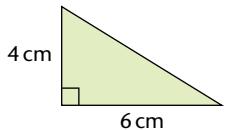
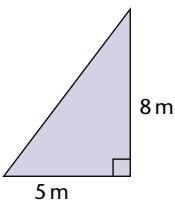
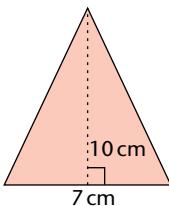
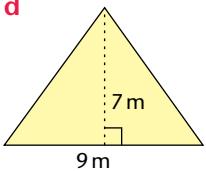
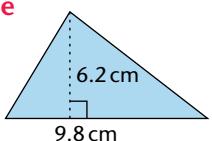
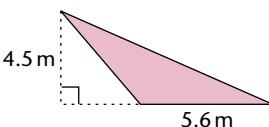
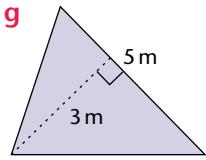
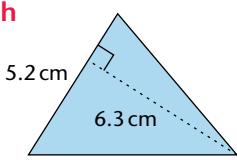
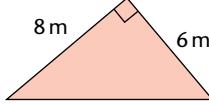


Remember that the units of area are always square units such as square centimetres or square metres, written cm^2 or m^2 .

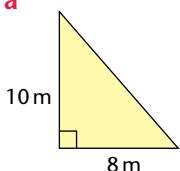
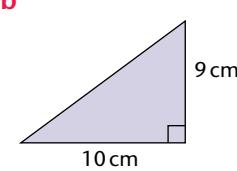
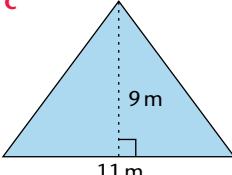
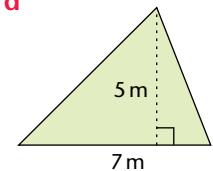
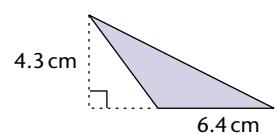
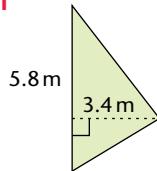
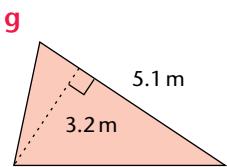
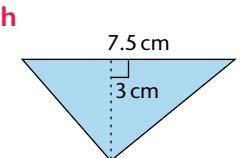
When using the formula, you can use any of the sides of the triangle as the base, provided you use the perpendicular height that goes with it.

Exercise 35.3

- 1** Find the area of each of these triangles.

a**b****c****d****e****f****g****h****i**

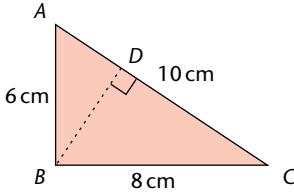
- 2** Find the area of each of these triangles.

a**b****c****d****e****f****g****h****i**

- 3** In triangle ABC , $AB = 6\text{cm}$, $BC = 8\text{cm}$ and $AC = 10\text{cm}$. Angle $ABC = 90^\circ$.

a Find the area of the triangle.

b Find the perpendicular height BD .



The area of a parallelogram

A parallelogram may be cut up and rearranged to form a rectangle or two congruent triangles.

$$\text{Area of a rectangle} = \text{base} \times \text{height}$$

$$\text{Area of each triangle} = \frac{1}{2} \times \text{base} \times \text{height}$$

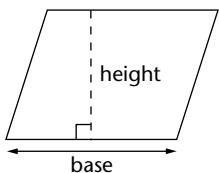


Both these ways of splitting a parallelogram show how to find its area.

Note

Make sure you use the perpendicular height and not the sloping edge when finding the area of a parallelogram.

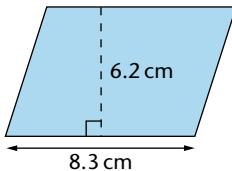
$$\text{Area of a parallelogram} = \text{base} \times \text{height}$$



Example 35.4

Question

Find the area of this parallelogram.



Solution

$$\text{Area of a parallelogram} = \text{base} \times \text{height}$$

$$= 8.3 \times 6.2$$

$$= 51.46 \text{ cm}^2$$

$$= 51.5 \text{ cm}^2 \text{ to 1 decimal place}$$

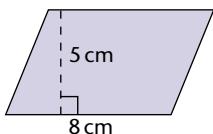
Note

Always give your final answer to a suitable degree of accuracy, but don't use rounded answers in your working.

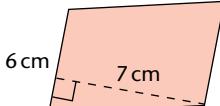
Exercise 35.4

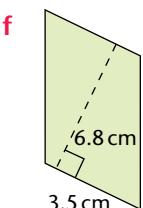
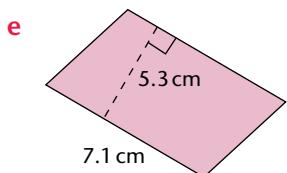
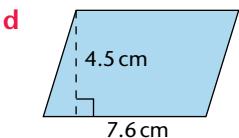
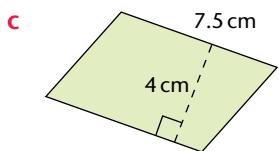
- 1 Find the area of each of these parallelograms.

a

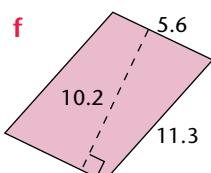
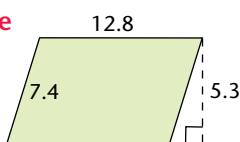
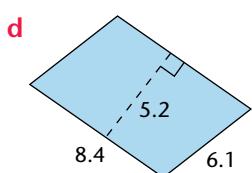
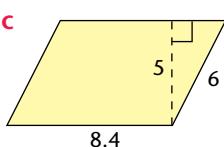
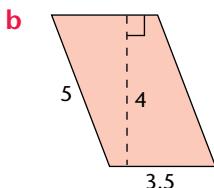
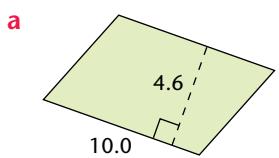


b

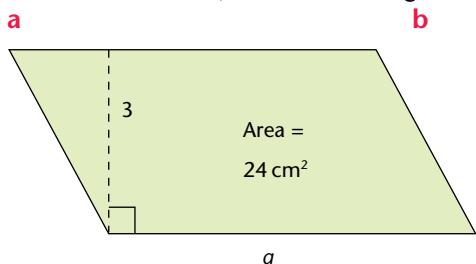
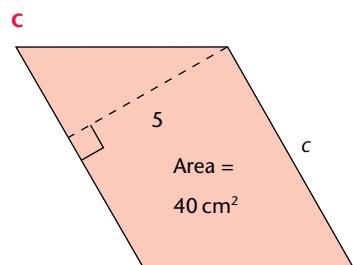
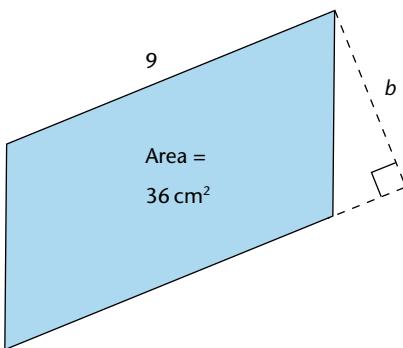




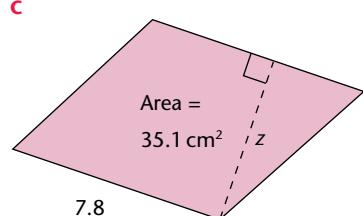
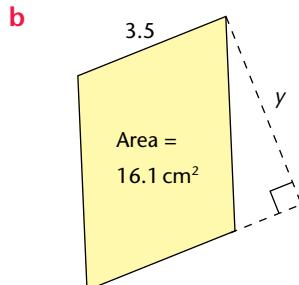
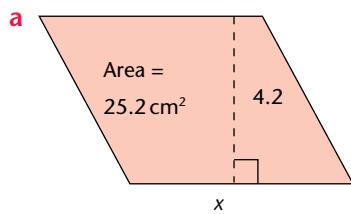
- 2 Find the area of each of these parallelograms. The lengths are in centimetres.



- 3 Find the values of a , b and c . The lengths are in centimetres.

**b**

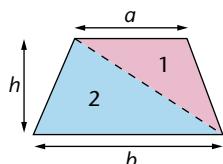
- 4 Find the values of x , y and z . The lengths are in centimetres.



The area of a trapezium

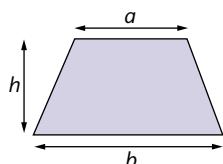
A trapezium has one pair of opposite sides parallel.

It can be split into two triangles.



$$\text{Area of triangle } 1 = \frac{1}{2} \times a \times h$$

$$\text{Area of triangle } 2 = \frac{1}{2} \times b \times h$$



$$\begin{aligned}\text{Area of trapezium} &= \frac{1}{2} \times a \times h + \frac{1}{2} \times b \times h \\ &= \frac{1}{2} \times (a + b) \times h \\ &= \frac{1}{2}(a + b)h\end{aligned}$$

You can remember the formula in words or algebraically.

$$\begin{aligned}\text{Area of a trapezium} &= \text{half the sum of the parallel sides} \times \text{the height} \\ &= \frac{1}{2}(a + b)h\end{aligned}$$

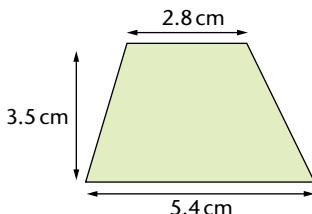
Example 35.5

Question

Calculate the area of this trapezium.

Solution

$$\begin{aligned}\text{Area of a trapezium} &= \frac{1}{2}(a + b)h \\ &= \frac{1}{2} \times (2.8 + 5.4) \times 3.5 \\ &= 14.35 \text{ cm}^2 \\ &= 14.4 \text{ cm}^2 \text{ to 1 decimal place}\end{aligned}$$

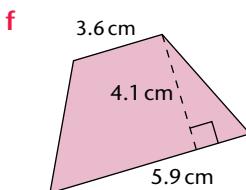
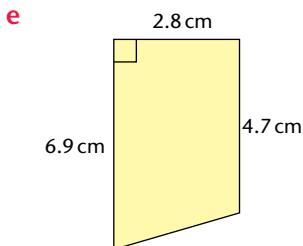
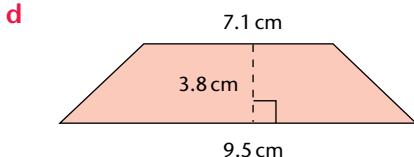
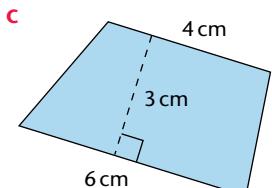
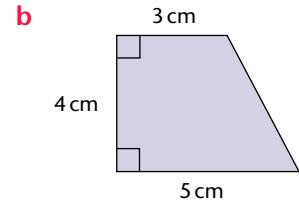
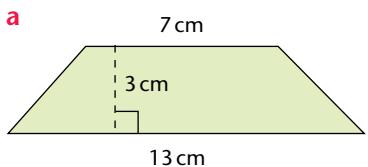


Note

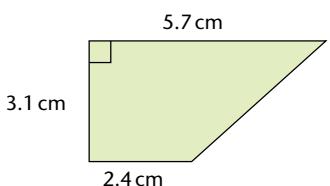
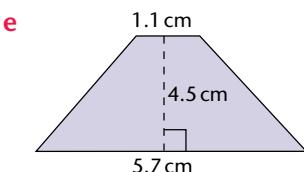
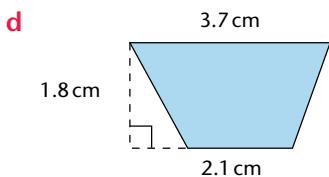
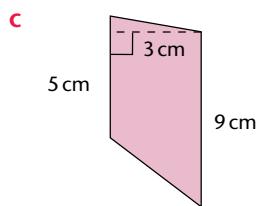
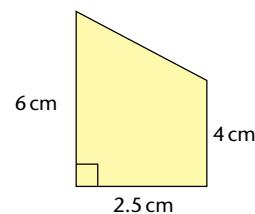
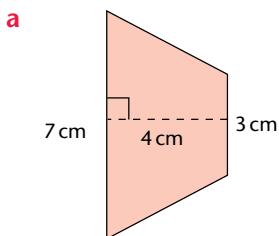
Use the brackets function on your calculator.
Without a calculator, remember to work out the brackets first.

Exercise 35.5

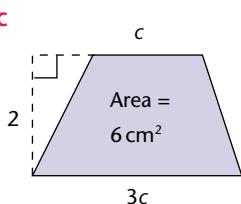
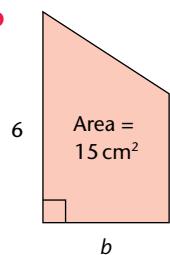
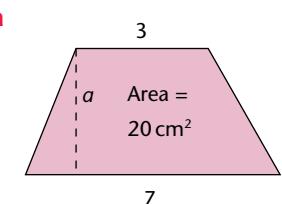
- 1** Find the area of each of these trapezia.



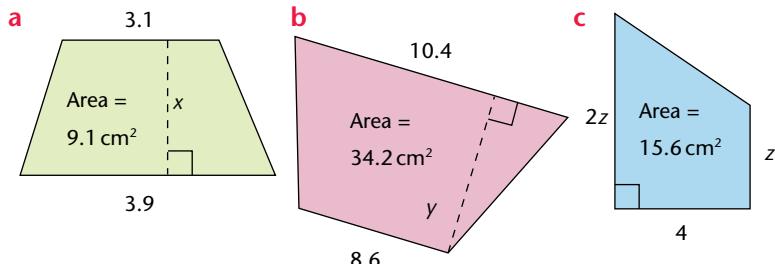
- 2** Find the area of each of these trapezia.



- 3** Find the values of a , b and c in these trapezia. The lengths are in centimetres.



- 4 Find the values of x , y and z in these trapezia. The lengths are in centimetres.



- 5 A trapezium has height 4 cm and area 28 cm^2 .

One of its parallel sides is 5 cm long.

How long is the other parallel side?

- 6 A trapezium has height 6.6 cm and area 42.9 cm^2 .

One of its parallel sides is 5 cm long.

How long is the other parallel side?

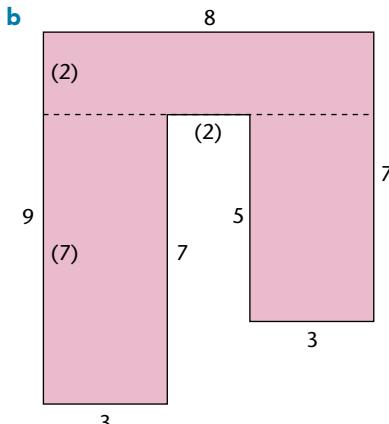
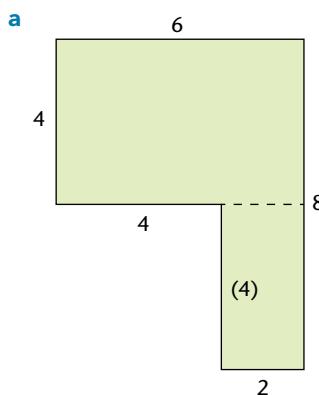
The area of shapes made from rectangles and triangles

A way to find the area of many **compound** shapes is to split them up into rectangles and triangles.

Example 35.6

Question

Work out the area of each of these shapes. All lengths are in centimetres.



(The lengths marked in brackets were not all given in the question, but have been worked out. The dashed lines have been added as part of the answer.)

Solution

- a The shape has been split into two rectangles by a horizontal dotted line.
(It could have been split in a different way, by a vertical line.)

$$\text{Area} = 6 \times 4 + 2 \times 4 = 24 + 8 = 32 \text{ cm}^2$$

- b The shape has been split into three rectangles.

$$\text{Area} = 8 \times 2 + 7 \times 3 + 5 \times 3 = 16 + 21 + 15 = 52 \text{ cm}^2$$

Note

A common error is to split the shape correctly but then multiply the wrong numbers to get the area.

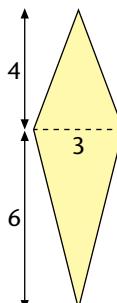
Example 35.7**Question**

Work out the area of this shape. All lengths are in centimetres.

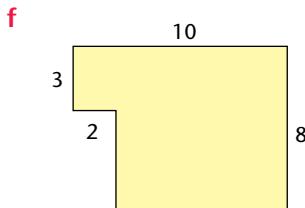
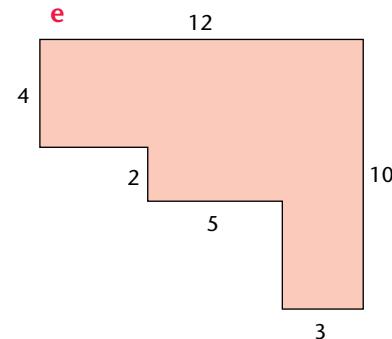
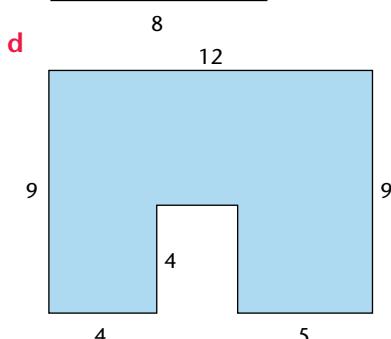
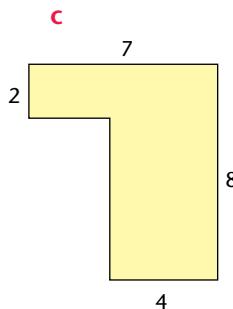
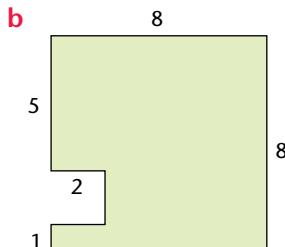
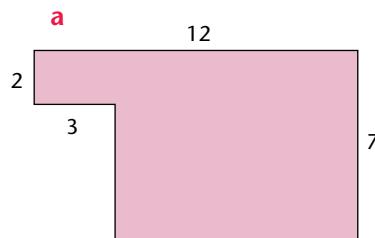
Solution

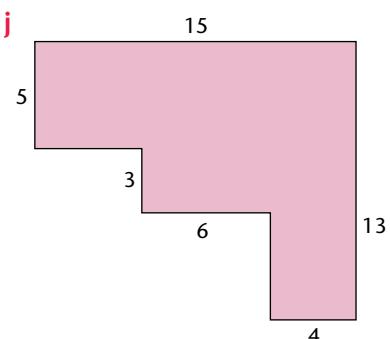
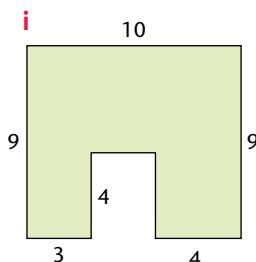
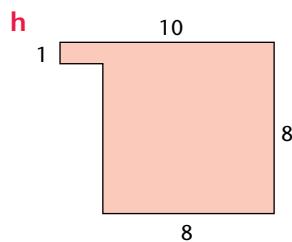
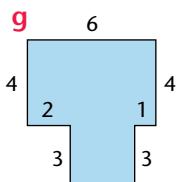
The shape has been split by the horizontal line into two triangles.

$$\begin{aligned}\text{Area} &= \frac{1}{2} \times 4 \times 3 + \frac{1}{2} \times 6 \times 3 \\ &= 6 + 9 \\ &= 15 \text{ cm}^2\end{aligned}$$

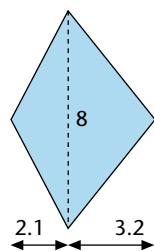
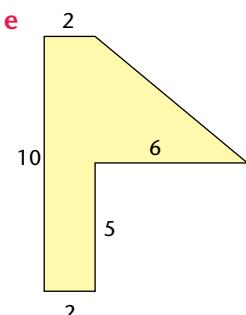
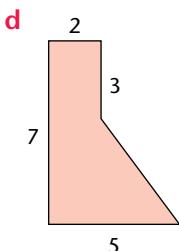
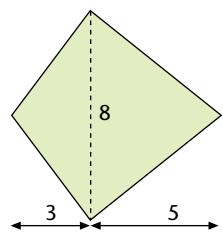
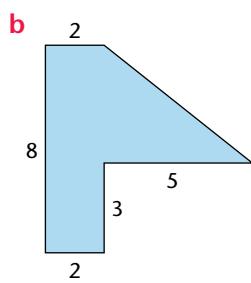
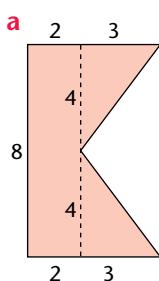
**Exercise 35.6**

- 1** Work out the area of each of these shapes. All lengths are in centimetres.





2 Work out the area of each of these shapes. All lengths are in centimetres.



The circumference of a circle

The fact that the **circumference** of a circle is approximately three times the **diameter** has been known for thousands of years. Accurate calculations have found this number to hundreds of decimal places. The number is a never-ending decimal, and is denoted by the Greek letter π .

π is an irrational number. Your calculator has the number which π represents stored in its memory.

The formula for the circumference is

$$C = \pi d$$

If you know the radius of the circle instead of the diameter, use the fact that the diameter is double the radius, $d = 2r$.

Note

If you are using your calculator, always use the π key rather than an approximation.

An alternative formula for the circumference of a circle is

$$C = 2\pi r$$

Example 35.8

Question

A circle has a radius of 8 cm.

Find its circumference.

Solution

$$\text{Circumference} = 2\pi r$$

$$= 2 \times \pi \times 8$$

$$= 50.265\dots$$

$$= 50.3 \text{ cm (to 1 d.p.)}$$

Example 35.9

Question

A circle has a circumference of 20 m. Circumference = πd

Find its diameter.

Solution

$$20 = \pi d$$

$$d = 20 \div \pi$$

$$= 6.366\dots$$

$$= 6.37 \text{ m (to 2 d.p.)}$$

Exercise 35.7

- 1 Calculate the circumference of the circles with these diameters, giving your answers correct to 1 decimal place.

- | | | | |
|----------|----------|----------|-----------|
| a 12 cm | b 9 cm | c 20 m | d 16.3 cm |
| e 15.2 m | f 25 m | g 0.3 cm | h 17 m |
| i 5.07 m | j 6.5 cm | | |

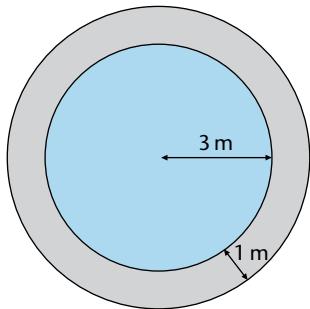
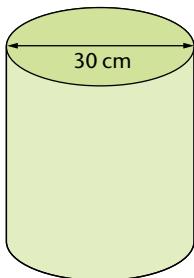
- 2 Find the circumference of circles with these radii, giving your answers correct to 1 decimal place.

- | | | | |
|---------|---------|----------|----------|
| a 5 cm | b 7 cm | c 16 m | d 18.1 m |
| e 5.3 m | f 28 cm | g 3.2 cm | h 60 m |
| i 1.9 m | j 73 cm | | |

- 3 The centre circle on a football pitch has a radius of 9.15 metres. Calculate the circumference of the circle.

- 4 The radius of the Earth at the equator is 6378 km. Calculate the circumference of the Earth at the equator.

- 5 The diagram shows a wastepaper bin in the shape of a cylinder.



- Calculate the circumference of the rim.
- 6 Calculate the diameter of the circles with these circumferences, giving your answers correct to 1 decimal place.
- a 75 cm b 18 cm c 50 cm
- 7 A circular racetrack is 300 metres in circumference.
Calculate the diameter of the racetrack.
- 8 Vivek has a circular pond of radius 3 m.
He wants to surround the pond with a gravel path 1 metre wide.
An edging strip is needed along each side of the gravel path.
What length of edging strip is needed altogether?

The area of a circle

The formula to calculate the area, A , of a circle of radius r is

$$A = \pi r^2$$

Example 35.10

Question

The radius of a circle is 4.3 m.
Calculate the area of the circle.

Solution

$$\begin{aligned} A &= \pi r^2 \\ &= \pi \times 4.3^2 \\ &= 58.1 \text{ m}^2 \text{ (to 1 d.p.)} \end{aligned}$$

Note

Make sure you can use the π key and the square key on your calculator.

Note

One of the most common errors is to mix up diameter and radius. Every time you do a calculation, make sure you have used the right one.

Example 35.11

Question

The diameter of a circle is 18.4 cm.
Calculate the area of the circle.

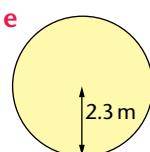
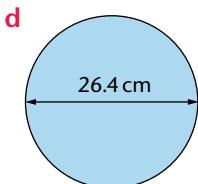
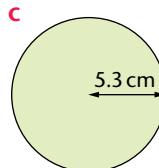
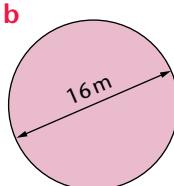
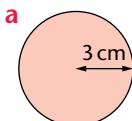
Solution

$$\begin{aligned} r &= 18.4 \div 2 \\ &= 9.2 \text{ cm} \\ A &= \pi r^2 \\ &= \pi \times 9.2^2 \\ &= 266 \text{ cm}^2 \text{ (to the nearest cm}^2\text{)} \end{aligned}$$

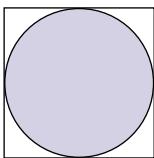
Exercise 35.8

In all of these questions, make sure you state the units of your answer.

- 1 Find the area of circles with these radii.
 a 4 cm b 16 m c 11.3 m d 13.6 m e 8.9 cm
- 2 Find the area of each of these circles.



- 3 The radius of a circular fish pond is 1.5 m.
Find the area of the surface of the water.
- 4 A circular mouse mat has a radius of 9 cm.
Find the area of the mouse mat.
- 5 The diameter of a circular table is 0.8 m.
Find the area of the table.
- 6 According to *Guinness World Records*, the largest pizza ever made was 37.4 m in diameter.
What was the area of the pizza?
- 7 To make a table mat, a circle of radius 12 cm is cut from a square of side 24 cm, as shown in the diagram.



- Calculate the area of the material that is wasted.
- 8 A square has a side of 3.5 cm and a circle has a radius of 2 cm.
Which has the bigger area?
Show your calculations.
 - 9 Cala is making a circular lawn with a radius of 15 m.
The packets of grass seed say ‘sufficient to cover 50 m²’.
How many packets will she need?
 - 10 Use your calculator to find the area of a circle with radius 6.8 cm.
Without using your calculator, do an approximate calculation to check your answer.

Note

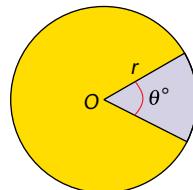
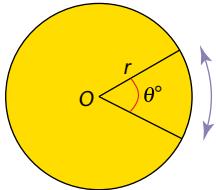
In an examination you will have to choose which formula to use.

Remember r^2 must give units such as cm² or m² and so πr^2 must be the area formula.

Arc length and sector area

A **sector** is a fraction of a circle.

It is $\frac{\theta}{360}$ of the circle, where θ° is the sector angle at the centre of the circle.



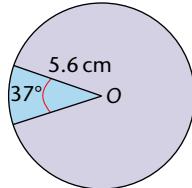
$$\begin{aligned}\text{Arc length} &= \frac{\theta}{360} \times \text{circumference} \\ &= \frac{\theta}{360} \times 2\pi r\end{aligned}$$

$$\begin{aligned}\text{Sector area} &= \frac{\theta}{360} \times \text{area of circle} \\ &= \frac{\theta}{360} \times \pi r^2\end{aligned}$$

Example 35.12

Question

Calculate the arc length and area of this sector.



Solution

$$\begin{aligned}\text{Arc length} &= \frac{\theta}{360} \times 2\pi r \\ &= \frac{37}{360} \times 2\pi \times 5.6 \\ &= 3.62 \text{ cm to 3 significant figures} \\ \text{Sector area} &= \frac{\theta}{360} \times \pi r^2 \\ &= \frac{37}{360} \times \pi \times 5.6^2 \\ &= 10.1 \text{ cm}^2 \text{ to 3 significant figures}\end{aligned}$$

Example 35.13

Question

Calculate the sector angle of a sector with arc length 6.2 cm in a circle with radius 7.5 cm.

Solution

$$\begin{aligned}\text{Arc length} &= \frac{\theta}{360} \times 2\pi r \\ 6.2 &= \frac{\theta}{360} \times 2\pi \times 7.5 \\ \theta &= \frac{6.2 \times 360}{2\pi \times 7.5} \\ &= 47.4^\circ \text{ to 3 significant figures}\end{aligned}$$

Example 35.14**Question**

A sector makes an angle of 54° at the centre of a circle.

The area of the sector is 15 cm^2 .

Calculate the radius of the circle.

Solution

$$\text{Sector area} = \frac{\theta}{360} \times \pi r^2$$

$$15 = \frac{54}{360} \times \pi r^2$$

$$r^2 = \frac{15 \times 360}{54 \times \pi}$$

$$r^2 = 31.830\dots$$

$$r = \sqrt{31.83\dots}$$

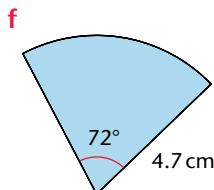
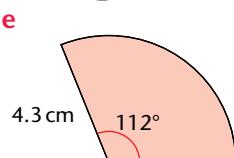
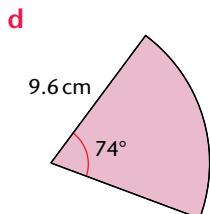
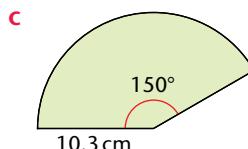
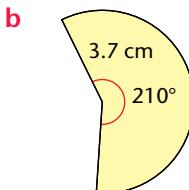
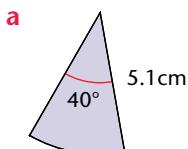
$= 5.64 \text{ cm}$ to 3 significant figures.

Note

You can rearrange the formula before you substitute, if you prefer.

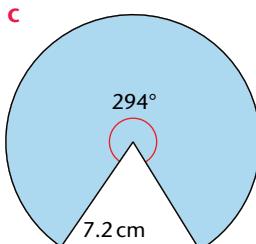
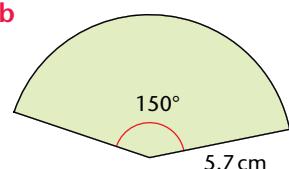
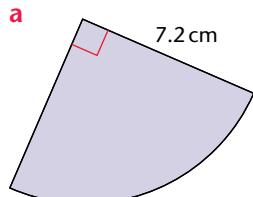
Exercise 35.9

- 1 Calculate the arc length of each of these sectors.
Give your answers to 3 significant figures.

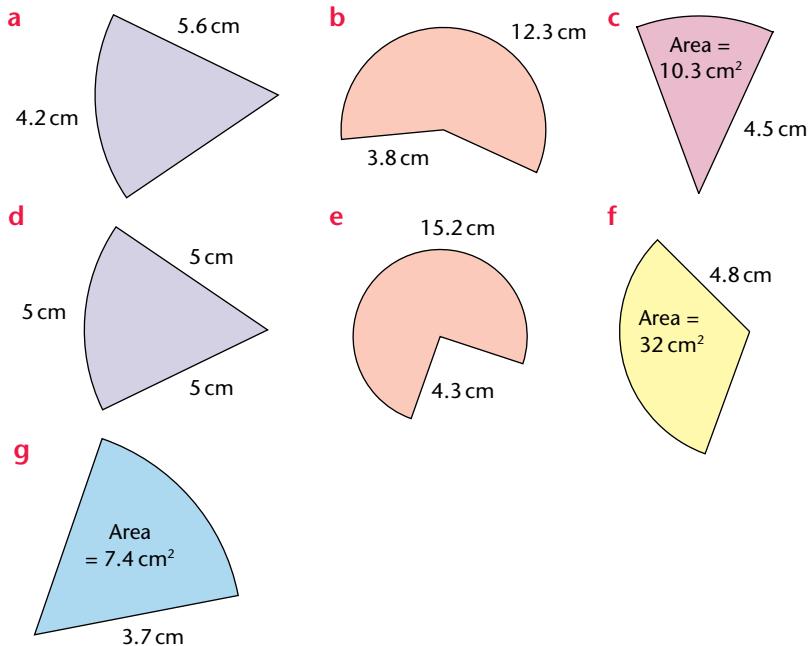


- 2 Calculate the area of each of the sectors in question 1.
Give your answers to 3 significant figures.

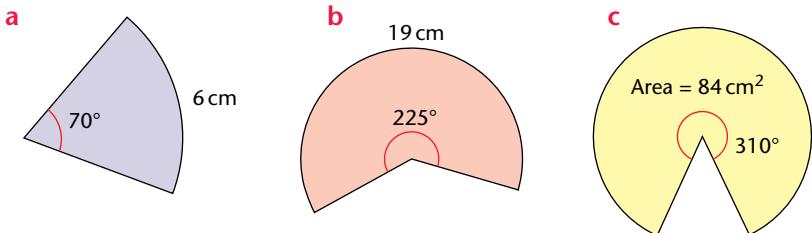
- 3 Calculate the perimeter of each of these sectors.
Give your answers to 3 significant figures.



- 4** Calculate the sector angle in each of these sectors.
Give your answers to the nearest degree.



- 5** Calculate the radius of each of these sectors.



- 6** Calculate the radius of each of these sectors.

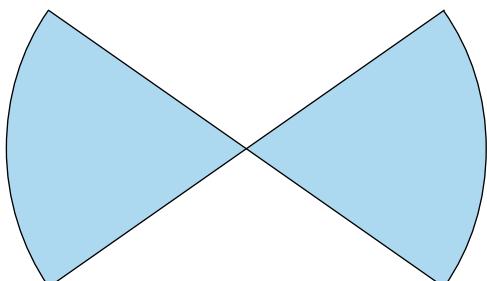
- a** Sector area = 19.7 cm^2 , sector angle = 52° .
b Sector area = 2.7 cm^2 , sector angle = 136° .
c Sector area = 6.2 m^2 , sector angle = 218° .

- 7** A shape consists of two separate sectors of a circle, each with angle 35° and radius 32 mm.

They are to be painted blue with a thin black border. Calculate the total area of blue and the total length of the black border.

Give your answers to an appropriate degree of accuracy.

- 8** A sector of a circle has its arc length equal to the radius of the circle. The sector angle is defined as 1 radian. What is this angle in degrees?



The volume of a prism

A **prism** is a three-dimensional shape that has the same cross-section throughout its length.

A **cuboid** is a prism with a rectangular cross-section.

$$\text{Volume of a cuboid} = \text{length} \times \text{width} \times \text{height}$$

You can also think of this as

$$\text{Volume of a cuboid} = \text{area of cross-section} \times \text{height}$$

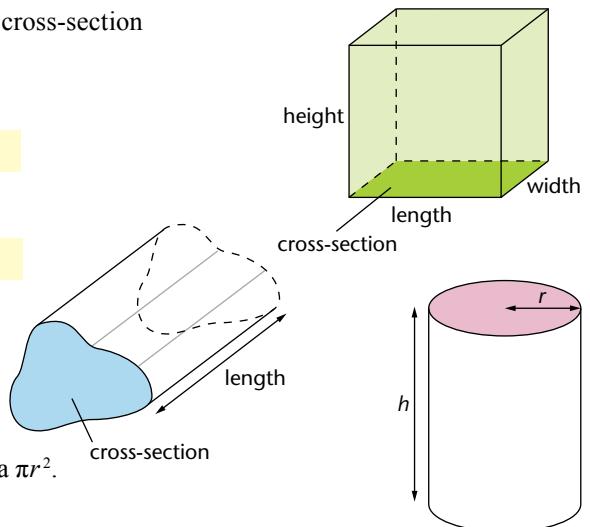
The general formula for the volume of a prism is

$$\text{Volume of a prism} = \text{area of cross-section} \times \text{length}$$

Another important prism is the **cylinder**.

The cross-section of a cylinder is a circle, which has area πr^2 .

$$\text{Volume of a cylinder} = \pi r^2 h$$



Example 35.15

Question

Calculate the volume of a cylinder with base diameter 15 cm and height 10 cm.

Solution

$$\text{Radius of base} = \frac{15}{2} = 7.5 \text{ cm.}$$

$$\text{Volume of a cylinder} = \pi r^2 h$$

$$= \pi \times 7.5^2 \times 10$$

$$= 1767 \text{ cm}^3, \text{ to the nearest whole number.}$$

Example 35.16

Question

A chocolate box is a prism with a trapezium as cross-section, as shown.

Calculate the volume of the prism.

Solution

$$\text{Area of a trapezium} = \frac{1}{2}(a + b)h$$

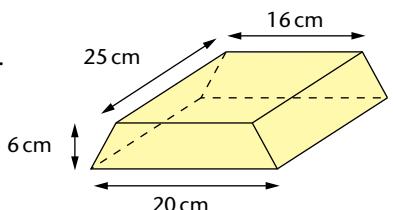
$$= \frac{1}{2}(20 + 16) \times 6$$

$$= 108 \text{ cm}^2$$

$$\text{Volume of a prism} = \text{area of cross-section} \times \text{length}$$

$$= 108 \times 25$$

$$= 2700 \text{ cm}^3$$



Example 35.17**Question**

A cylinder has volume 100 cm^3 and is 4.2 cm high.

Find the radius of its base.

Give your answer to the nearest millimetre.

Solution

$$\text{Volume of a cylinder} = \pi r^2 h$$

$$100 = \pi \times r^2 \times 4.2$$

$$r^2 = \frac{100}{\pi \times 4.2}$$

$$= 7.578\dots$$

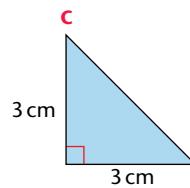
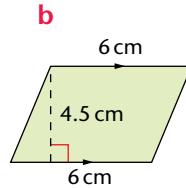
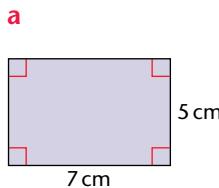
$$r = \sqrt{7.578}$$

$$= 2.752\dots$$

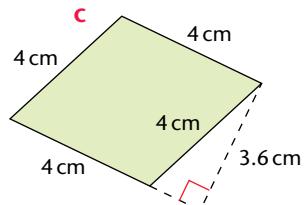
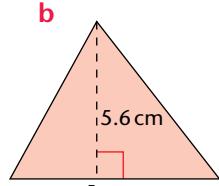
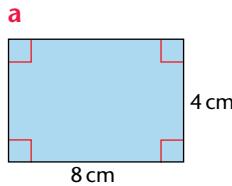
= 2.8 cm, to the nearest millimetre

Exercise 35.10

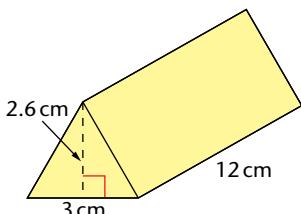
- 1 A cuboid has edges 3 cm, 5 cm and 2 cm long.
Work out the volume.
- 2 The edges of a cube are 2 cm long.
Work out the volume.
- 3 A rectangular cupboard is 2 m by 1 m by 1.5 m.
Find its volume.
- 4 A cuboid has a base 3 cm by 4 cm and its volume is 48 cm^3 .
What is the height of the cuboid?
- 5 Calculate the volume of a cylinder with base radius 5.6 cm and height 8.5 cm.
- 6 A cylinder is 12 cm long and has radius 2.4 cm.
Find its volume.
- 7 A cylinder has diameter 8 cm and height 8 cm.
Calculate its volume.
- 8 Calculate the volume of a prism 15 cm long with each of these cross-sections.



- 9 Calculate the volume of a prism 12 cm long with each of these cross-sections.

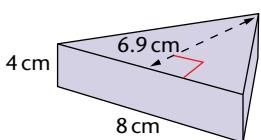


- 10** A chocolate bar is in the shape of a triangular prism.
Calculate its volume.



- 11** A cylindrical water tank is 4.2 m high and has radius 3.6 m.
Find its volume.

- 12** A gift box is a prism with a triangular base.
Calculate its volume.

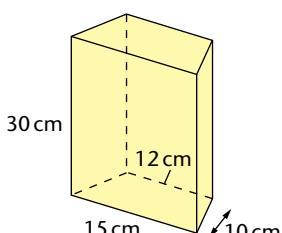


- 13** The area of cross-section of a prism is 75 cm^2 .
Its volume is 1200 cm^3 .
Calculate its length.

- 14** The volume of a cylinder is 800 cm^3 .
Its radius is 5.3 cm.
Calculate its height.

- 15** The volume of a cylindrical tank is 600 m^3 .
Its height is 4.6 m.
Calculate the radius of its base.

- 16** A vase is a prism with a trapezium as its base.
The internal measurements are as shown.



How much water can the vase hold?

Give your answer in litres.

(Remember that $1 \text{ litre} = 1000 \text{ cm}^3$.)

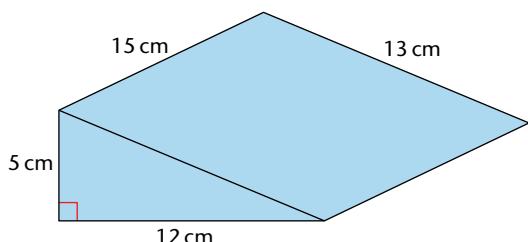
The surface area of a prism

To find the total surface area of a prism, you add up the surface area of all the individual surfaces.

Example 35.18

Question

Find the total surface area of this prism.



Solution

$$\text{Area of end} = \frac{1}{2} \times 12 \times 5 = 30 \text{ cm}^2$$

$$\text{Area of other end} = 30 \text{ cm}^2$$

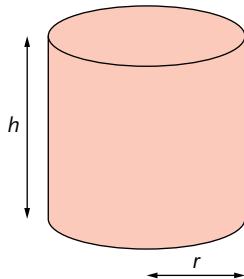
$$\text{Area of base} = 12 \times 15 = 180 \text{ cm}^2$$

$$\text{Area of top} = 13 \times 15 = 195 \text{ cm}^2$$

$$\text{Area of back} = 5 \times 15 = 75 \text{ cm}^2$$

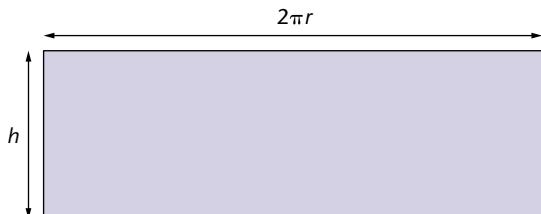
$$\text{Total surface area} = 510 \text{ cm}^2$$

For a cylinder there are three surfaces.



The two ends are circles and each have area πr^2 .

If the cylinder were made out of paper, the curved surface would open out to a rectangle.



The length of the rectangle is the circumference of the cylinder so is $2\pi r$.

The curved surface area is therefore $2\pi r \times h = 2\pi rh$.

$$\text{Total surface area of a cylinder} = 2\pi rh + 2\pi r^2$$

Example 35.19**Question**

Calculate the total surface area of a cylinder with base diameter 15 cm and height 10 cm.

Solution

$$\text{Radius of base} = \frac{15}{2} = 7.5 \text{ cm}$$

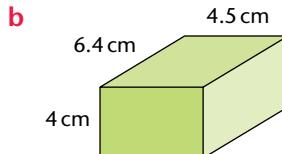
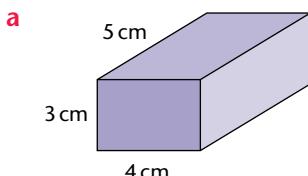
$$\text{Area of two ends} = 2 \times \pi r^2 = 2 \times \pi \times 7.5^2 = 353.429\ldots \text{cm}^2$$

$$\text{Curved surface area} = 2\pi rh = 2 \times \pi \times 7.5 \times 10 = 471.238\ldots \text{cm}^2$$

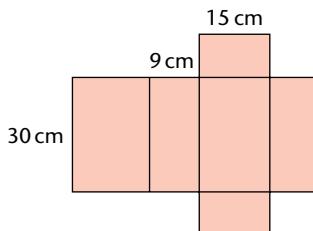
$$\text{Total surface area} = 353.429\ldots + 471.238\ldots = 825 \text{ cm}^2, \text{ to the nearest whole number}$$

Exercise 35.11

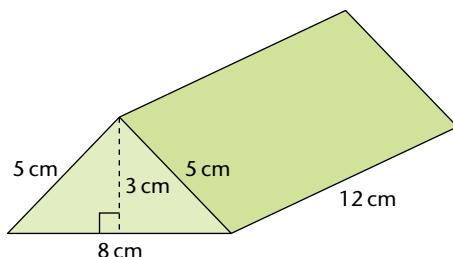
- 1 Find the surface area of these cuboids.



- 2 Find the surface area of a cube of side 20 cm.
3 The diagram shows the net of a shoe box.
Calculate its surface area.



- 4 A fish tank is 80 cm long, 45 cm wide and 40 cm high.
It does not have a lid.
Calculate the surface area of the glass used to make the fish tank.
5 Calculate the surface area of this triangular prism.

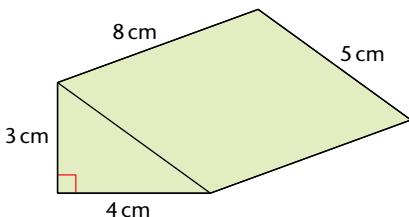
**Note**

You may find it helpful to make a rough sketch of the net of the prism before calculating its surface area. This should stop you missing out a face.

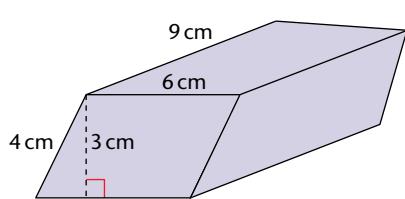
- 6 Find the total surface area of these shapes from Exercise 35.10.
- The cylinder in question 5
 - The cylinder in question 6
 - The cylinder in question 7
 - The prism in question 8a
 - The prism in question 9a

- 7 Find the total surface area of each of these prisms.

a



b



The volume of a pyramid, a cone and a sphere

Not all three-dimensional shapes are prisms.

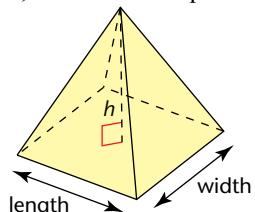
Some shapes have a cross-section which, though similar, decreases to a point.

Such shapes include the pyramid and the cone.

The volume of a pyramid or cone is given by

$$\text{Volume} = \frac{1}{3} \times \text{area of base} \times \text{height}$$

Because a cone has a circular base, for a cone with base radius r and height h ,

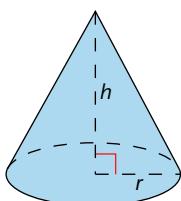
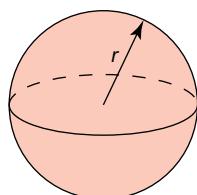


$$\text{Volume} = \frac{1}{3}\pi r^2 h$$

You also need to know about a different type of three-dimensional shape – the sphere.

For a sphere of radius r ,

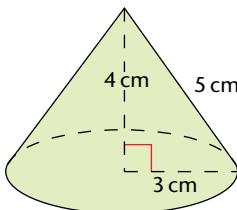
$$\text{Volume} = \frac{4}{3}\pi r^3$$



You do not need to learn these formulae but you do need to be able to use them.

Example 35.20**Question**

Find the volume of this cone.

**Solution**

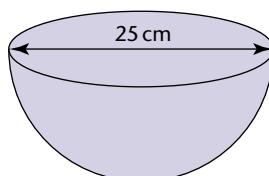
$$\begin{aligned}\text{Volume} &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3} \times \pi \times 3^2 \times 4 \\ &= 37.7 \text{ cm}^3 \text{ to 3 significant figures}\end{aligned}$$

Example 35.21**Question**

A bowl is in the shape of a hemisphere of diameter 25 cm.

How much water can the bowl hold?

Give your answer in litres.

**Note**

A hemisphere is half a sphere.

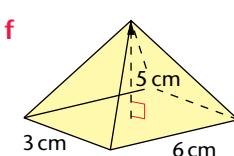
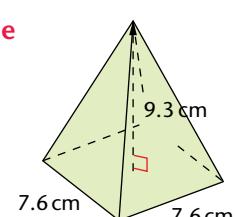
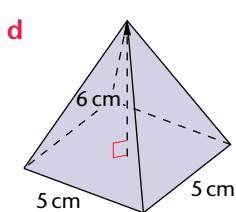
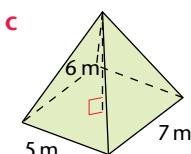
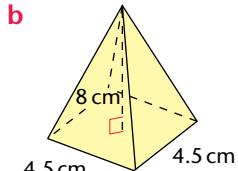
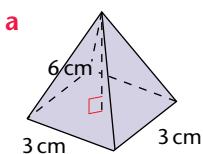
Solution

$$\begin{aligned}\text{Volume of hemisphere} &= \frac{1}{2} \times \text{volume of sphere} \\ &= \frac{1}{2} \times \frac{4}{3}\pi r^3 \\ &= \frac{2}{3}\pi r^3 \\ &= \frac{2}{3}\pi \times 12.5^3 \\ &= 4090.6\dots \text{ cm}^3 \quad 1 \text{ litre} = 1000 \text{ cm}^3 \\ &= 4.09 \text{ litres to 3 significant figures}\end{aligned}$$

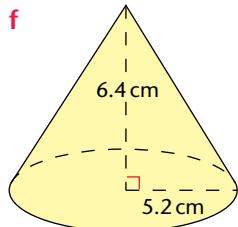
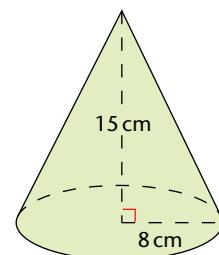
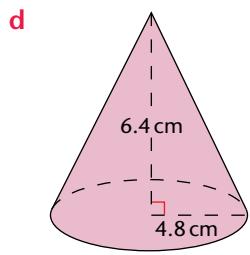
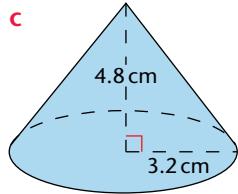
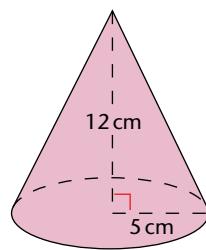
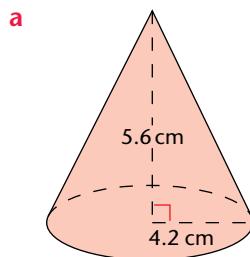
Exercise 35.12

Use the formulae given on page 324 to answer these questions.

- 1** Calculate the volume of each of these pyramids. Their bases are squares or rectangles.



- 2 Calculate the volume of each of these cones.



- 3 Find the volume of a sphere of these radii.

- a 5 cm b 6.2 cm c 2 mm
d 3 cm e 4.7 cm f 7.8 mm

- 4 A pyramid has a square base with sides of 8 cm.

Its volume is 256 cm^3 .

Find its height.

- 5 Find the radius of the base of a cone with these dimensions.

- a Volume 114 cm^3 , height 8.2 cm
b Volume 52.9 cm^3 , height 5.4 cm
c Volume 500 cm^3 , height 12.5 cm

- 6 Find the volume of a sphere with these dimensions.

- a Radius 5.1 cm b Radius 8.2 cm c Diameter 20 cm

- 7 Find the radius of a sphere of these volumes.

- a 1200 cm^3 b 8000 cm^3

- 8 How many ball bearings of radius 0.3 cm can be made from 10 cm^3 of metal when it is melted?

- 9 A solid cone and a solid cylinder both have base radius 6 cm.

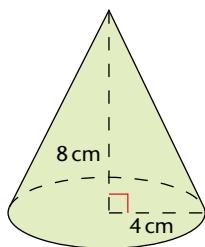
The height of the cylinder is 4 cm.

The cone and the cylinder both have the same volume.

Find the height of the cone.

- 10 A sphere has the same volume as this cone.

Calculate the radius of the sphere.



The surface area of a pyramid, a cone and a sphere

As for the surface area of a prism, the surface area of a pyramid, cone or sphere is the total area of all its surfaces.

All the surfaces of a pyramid that have a base in the shape of a polygon are flat.

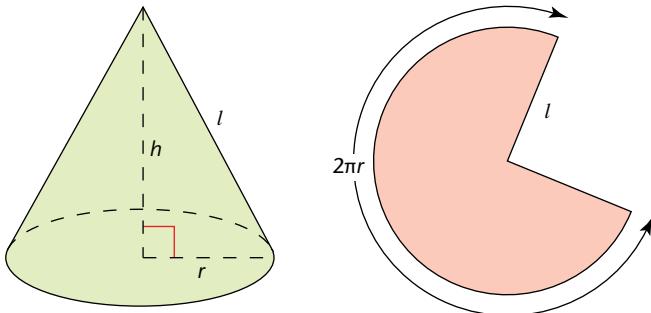
Apart from the base, the faces are triangular.

To find the total surface area, you find the area of each face and add them together.

A cone has a base in the shape of a circle and a curved surface.

The curved surface of a cone can be opened out to form a sector of a circle of radius l , where l is the slant height of the cone.

The arc length of the sector is the circumference of the base of the cone.



Note

Make sure you distinguish between the perpendicular height h and the slant height l of a cone – and don't read l as 1!

$$\text{Curved surface area of a cone} = \pi r l$$

All of the surface of a sphere is curved.

The formula for the surface area of a sphere of radius r is

$$\text{surface area of a sphere} = 4\pi r^2$$

You do not need to learn these formulae but you do need to be able to use them.

Example 35.22

Question

This cone has a solid base.

Find the total surface area of the cone.

Solution

$$\text{Curved surface area} = \pi r l$$

$$= \pi \times 3 \times 5 = 15\pi$$

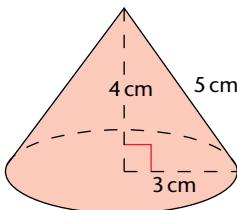
$$\text{Area of base} = \pi r^2$$

$$= \pi \times 3^2 = 9\pi$$

$$\text{Total surface area} = 15\pi + 9\pi$$

$$= 24\pi$$

$$= 75.4 \text{ cm}^2 \text{ to 3 significant figures}$$

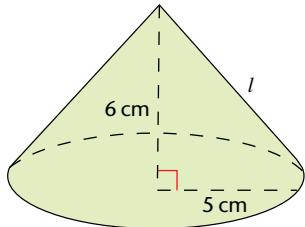


Note

Leaving the two areas in terms of π means that you are using exact values and avoids rounding errors.

Example 35.23**Question**

Calculate the curved surface area of this cone.

**Solution**

First the slant height l must be found.

Using Pythagoras' Theorem,

$$l^2 = 5^2 + 6^2$$

$$= 61$$

$$l = \sqrt{61}$$

$$\text{Curved surface area} = \pi r l$$

$$= \pi \times 5 \times \sqrt{61}$$

$$= 123 \text{ cm}^2 \text{ to 3 significant figures}$$

Note

Leaving the value for l in surd form means that you are using an exact value and avoids rounding errors.

Example 35.24**Question**

Calculate the surface area of this pyramid.

Solution

This is the section through midpoints of opposite sides of the base.

It is an isosceles triangle.

The height splits it into two right-angled triangles.

The hypotenuse of one of the right-angled triangles is the height of one of the triangular faces of the pyramid.

Using Pythagoras' Theorem,

$$\text{height}^2 = 4^2 + 3^2 = 25$$

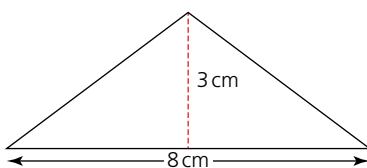
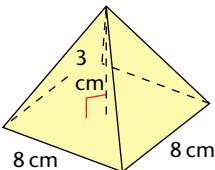
$$\text{height} = 5$$

$$\text{Area of one triangular face} = \frac{1}{2} \times 8 \times 5 = 20 \text{ cm}^2.$$

$$\text{Total surface area of the pyramid} = 4 \times \text{area of one triangular face} + \text{the area of the square base}$$

$$= 4 \times 20 + 8 \times 8$$

$$= 144 \text{ cm}^2$$

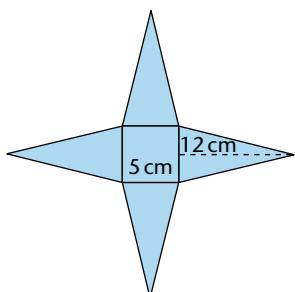
**Exercise 35.13**

Use the formulae given on page 327 to answer these questions.

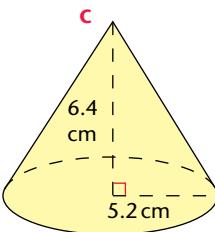
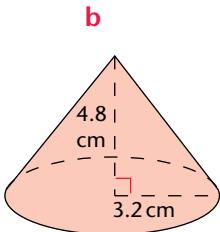
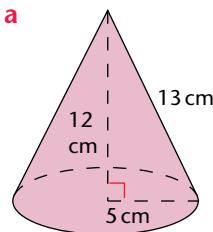
Some of the questions in this exercise also ask for the volume of the shape.

- 1 This is a sketch of the net for a square-based pyramid.

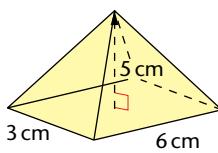
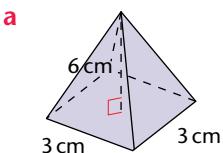
Calculate the surface area of the pyramid.



- 2** Calculate the curved surface area of each of these cones.



- 3** Calculate the surface area of these pyramids.



- 4** Calculate the surface area of a sphere of each of these radii.

a 5 cm

b 6.2 cm

c 2 mm

d 3 cm

e 4.7 cm

f 7.8 mm

- 5** A sphere has a surface area of 47.6 cm^2 .

Calculate its radius.

- 6** A solid cone has a base radius of 4.5 cm and a height of 6.3 cm.

Calculate its total surface area.

- 7** Calculate the total surface area of a solid cone with base radius 7.1 cm and slant height 9.7 cm.

- 8** Calculate the total surface area of a solid hemisphere of radius 5.2 cm.

- 9** Calculate the slant height of a cone of base radius 5 cm and curved surface area 120 cm^2 .

- 10** The flat surface of a hemisphere has an area of 85 cm^2 .

Calculate the curved surface area.

- 11** A sphere has a surface area of 157.6 cm^2 .

Calculate its radius.

- 12** Calculate the base area of a cone with slant height 8.2 cm and curved surface area 126 cm^2 .

- 13** A cone has slant height 7 cm and curved surface area 84 cm^2 .

Calculate the total surface area of the cone.

- 14** The radius of Jupiter is $7.14 \times 10^4 \text{ km}$.

Assume that Jupiter is a sphere.

Find the surface area of Jupiter.

Give your answer in standard form, correct to 2 significant figures.

- 15** A sphere containing liquid has a capacity of 1 litre.

Calculate the surface area of the sphere.

- 16** A spherical ball has a curved surface area of 120 cm^2 .

Calculate its volume.

- 17** A cone has a base of radius 4.2 cm and a slant height of 7.8 cm.

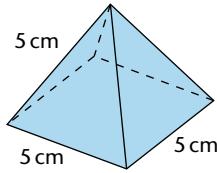
Calculate

a its surface area

b its height

c its volume.

- 18** All the edges of this square-based pyramid are 5 cm.
Calculate



- its surface area
- its height
- its volume.

- 19** A cone has height 6 cm and volume 70 cm^3 .

Calculate

- its base radius
- its curved surface area.

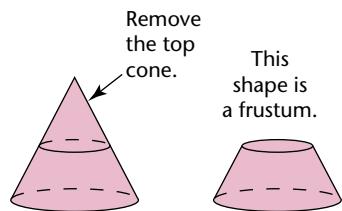
- 20** A cone has a base of radius 5 cm and a perpendicular height of 12 cm.
Calculate its curved surface area.

The area and volume of compound shapes

You have already met the areas of compound shapes made from rectangles and triangles.

In this section we look at examples of compound shapes made from other shapes you have met in this chapter.

The circle on the top of the frustum is in a plane parallel to the base.



$$\text{Volume of a frustum} = \text{volume of whole cone} - \text{volume of missing cone}$$

Example 35.25

Question

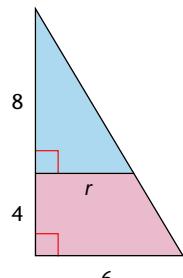
Find the volume of the frustum remaining when a cone of height 8 cm is removed from a cone of height 12 cm and base radius 6 cm.

Solution

First, use similar triangles to find the base radius, r cm, of the cone which has been removed.

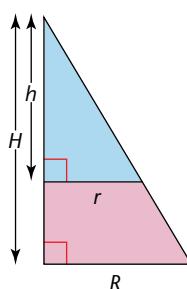
$$\frac{r}{8} = \frac{6}{12}$$

$$r = 4$$



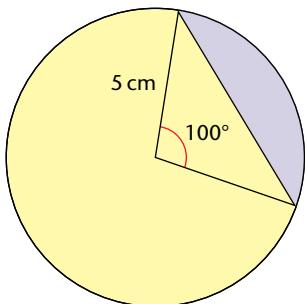
Then find the volume of the frustum.

$$\begin{aligned}\text{Volume of frustum} &= \text{volume of whole cone} - \text{volume of missing cone} \\ &= \frac{1}{3} \pi R^2 H - \frac{1}{3} \pi r^2 h \\ &= \frac{1}{3} \times \pi \times 6^2 \times 12 - \frac{1}{3} \times \pi \times 4^2 \times 8 \\ &= 318 \text{ cm}^3 \text{ to 3 significant figures}\end{aligned}$$



Example 35.26**Question**

Calculate the area of the purple segment.

**Solution**

Area of segment = area of sector – area of triangle

$$\begin{aligned}\text{Area of sector} &= \frac{\theta}{360} \pi r^2 \\ &= \frac{100}{360} \times \pi \times 5^2 \\ &= 21.816\dots \text{cm}^2\end{aligned}$$

$$\begin{aligned}\text{Area of triangle} &= \frac{1}{2} ab \sin C \\ &= \frac{1}{2} \times 5^2 \times \sin 100^\circ \\ &= 12.310\dots \text{cm}^2\end{aligned}$$

$$\begin{aligned}\text{Area of segment} &= 21.816\dots - 12.310\dots \\ &= 9.51 \text{ cm}^2 \text{ to 3 significant figures}\end{aligned}$$

Note

Write down more figures than you need in the working and round the final answer.

Using the calculator memory means you do not have to re-key the figures.

Note

When dealing with problems where you first have to work out how to solve them, follow these steps.

- Read the question carefully and plan.
What do I know?
What do I have to find?
What methods can I apply?
- When you have finished, ask ‘Have I answered the question?’
There may be one last step you have forgotten to do.

Exercise 35.14

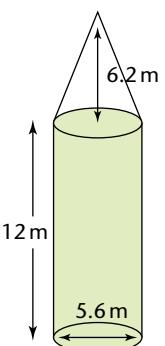
You may need to refer to the formulae given on pages 324, 327 and 330.

- 1 A tower of a castle has walls in the form of a cylinder and a roof in the shape of a cone.

The diameter of the tower is 5.6 m and the height of the wall is 12 m.

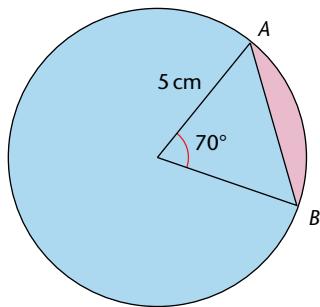
The roof has a height of 6.2 m.

Calculate the volume of the tower.



2 Calculate

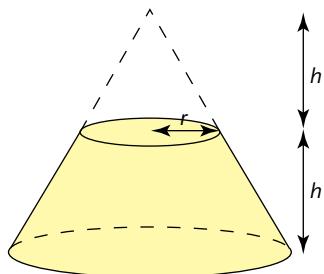
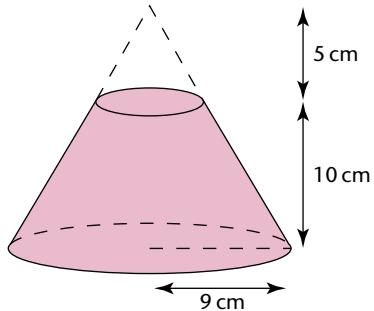
- a** the length of the chord AB
- b** the perimeter of the pink segment.



3 A cone of height 15 cm and base radius 9 cm has a cone of height 5 cm removed from its top as shown.

- a** What is the radius of the base of the top cone?
- b** Calculate the volume of the remaining frustum of the cone.

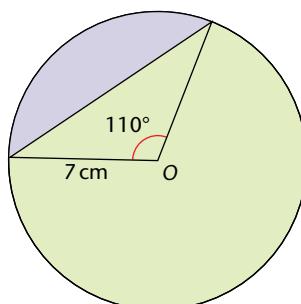
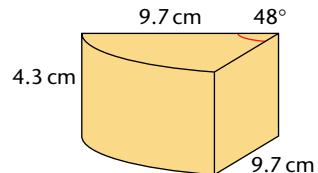
4 Show that the volume of this frustum is $\frac{1}{3} \pi r^2 h$.



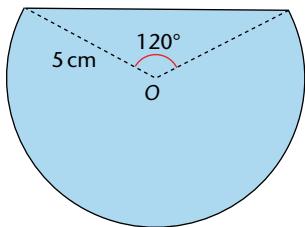
5 A piece of cheese is a prism whose cross-section is the sector of a circle with measurements as shown.

Calculate the volume of the piece of cheese.

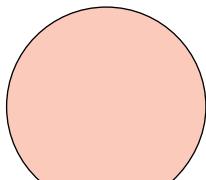
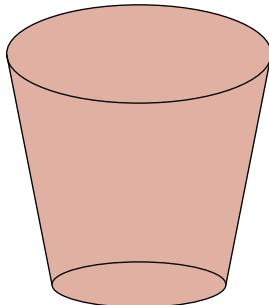
6 Calculate the area of the purple segment.



- 7 Calculate the perimeter of this segment of a circle of radius 5 cm.



- 8 The top of a flowerpot is a circle of radius 10 cm.
Its base is a circle of radius 8 cm.
The height of the flowerpot is 10 cm.
- Show that the flowerpot is a frustum of an inverted cone of complete height 50 cm and base radius 10 cm.
 - Calculate how many litres of soil the flowerpot can contain.
- 9 A train tunnel is a segment of a circle with a cross-section diameter of 7.6 m and a height of 6.5 m.
The tunnel is 5.3 km long.



Calculate the volume of soil removed when the tunnel was constructed.

36 TRIGONOMETRY

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- interpret and use three-figure bearings
- apply Pythagoras' Theorem and the sine, cosine and tangent ratios for acute angles to the calculation of a side or of an angle of a right-angled triangle
- solve trigonometrical problems in two dimensions involving angles of elevation and depression
- extend sine and cosine functions to angles between 90° and 180°
- solve problems using the sine and cosine rules for any triangle and the formula area of triangle = $\frac{1}{2} ab \sin C$
- solve simple trigonometrical problems in three dimensions.

CHECK YOU CAN:

- use a ruler and a protractor
- find the area of a triangle
- use your calculator to find squares and square roots
- solve simple equations.

Note

Make sure you use the clockwise scale.

Bearings

Bearings are used to describe direction.

They are measured clockwise from north.

Bearings are always given from some fixed point.

In the diagram, the bearing of B from A is 070° .

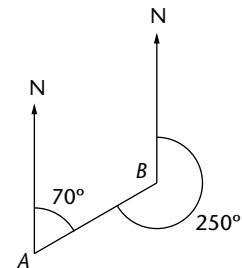
The bearing of A from B is called the reverse bearing.

In this case,

the bearing of A from B = the bearing of B from A + 180°

$$= 070^\circ + 180^\circ$$

$$= 250^\circ$$



Note

If the bearing is greater than 180° , the reverse bearing is found by subtracting 180° .

Note

Bearings must have three figures, so if the angle is less than 100° you must put a zero in front of the figures.

Example 36.1

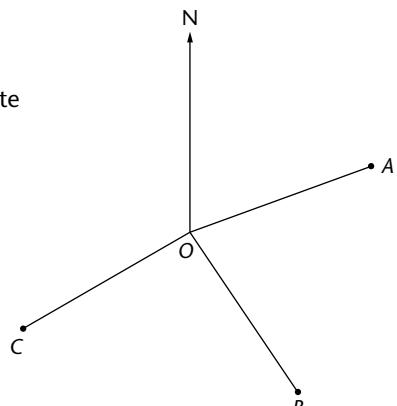
Question

Use a protractor to measure and write down each of these bearings.

- a A from O
- b B from O
- c C from O

Solution

- a 070°
- b 146°
- c 240°



If you have a circular protractor you can measure all of these directly.

If you only have a semi-circular protractor then for part c you need to measure the obtuse angle and subtract it from 360° .

Example 36.2**Question**

A , B and C are three towns.

B is 20 km from A , on a bearing of 085° from A .

C is 15 km from B , on a bearing of 150° from B .

- a Make a scale drawing showing the three towns.

Use a scale of 1 cm to 5 km.

- b i How far is A from C ?

ii What is the bearing of A from C ?

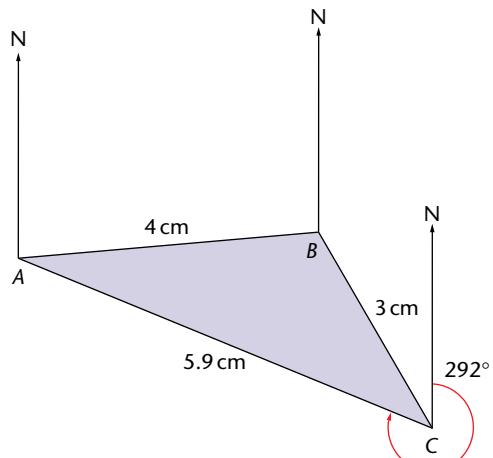
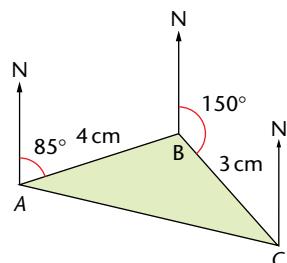
Solution

- a Make a sketch and label the angles and lengths for the final diagram.

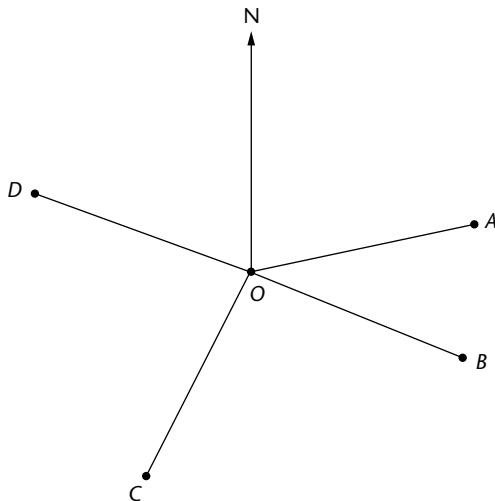
Then draw the diagram, starting far enough down the page to make sure it will all fit in.

- b i CA measures 5.9 cm on the scale drawing, so the real distance is $5.9 \times 5 = 29.5$ km.

ii The bearing is 292° .

**Exercise 36.1**

- 1 Measure the bearings of A , B , C and D from O .



- 2** Mark a point O . Draw a north line through O .
Mark points A , B , and C , all 5 cm from O on these bearings.

$A\ 055^\circ$ $B\ 189^\circ$ $C\ 295^\circ$

- 3** Point A is on a bearing of 124° from point B .
What is the bearing of point B from point A ?
4 Point C is on a bearing of 290° from point D .
What is the bearing of point D from point C ?
5 On this drawing, measure the bearing of
 a P from Q
 b Q from R .

- 6** A boat leaves port and sails for 10 km on a bearing of 285° . Make a sketch showing the direction in which the boat is sailing. Mark the distance and the angle clearly. Do not draw an accurate plan.

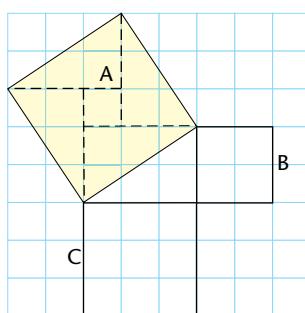
- 7** This sketch map shows the positions of three oil platforms – A , B and C .
 a Make an accurate drawing of the sketch.
 Use a scale of 1 cm to 5 km.
 b Measure the bearing of A from C .

- 8** This sketch map shows the positions of three hills – P , Q and R .
 Work out each of these bearings.

- a R from Q
 b R from P
 c P from R

Do not make a scale drawing.

Pythagoras' Theorem



Look at square A.

Area of square A = areas of the four triangles + small square

$$\begin{aligned} &= 4 \left(\frac{1}{2} \times 2 \times 3 \right) + 1 \\ &= 13 \end{aligned}$$

Area of square B + area of square C = $4 + 9 = 13$

This is an example of the rule linking the areas of squares around a right-angled triangle, known as Pythagoras' Theorem.

The largest square will always be on the longest side of the triangle – this is called the **hypotenuse** of the right-angled triangle.

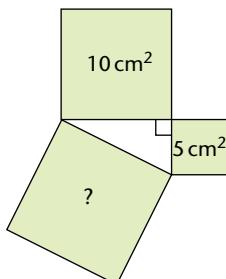
Pythagoras' Theorem can be stated like this.

The area of the square on the hypotenuse = the sum of the areas of the squares on the other two sides.

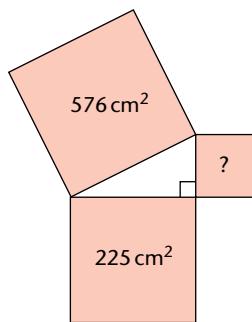
Exercise 36.2

Calculate the missing area in each of these diagrams.

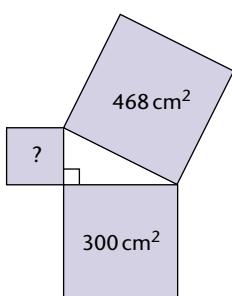
1



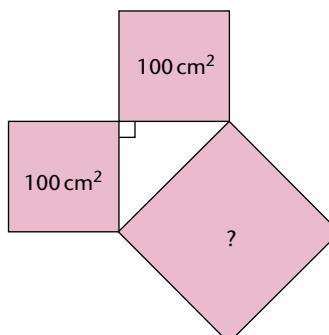
2



3



4



Using Pythagoras' Theorem

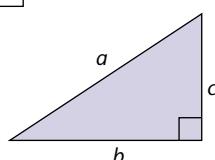
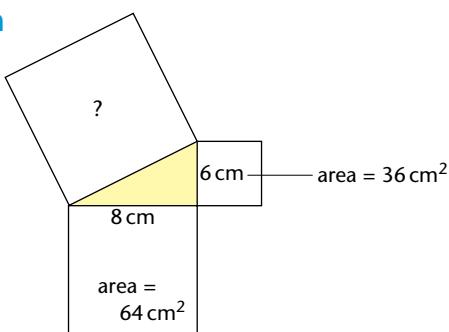
If you know the lengths of two sides of a right-angled triangle you can use Pythagoras' Theorem to find the length of the third side.

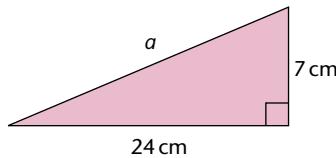
The unknown area = $64 + 36 = 100 \text{ cm}^2$.

This means that the sides of the unknown square have a length of $\sqrt{100} = 10 \text{ cm}$.

When using Pythagoras' Theorem, you don't need to draw the squares – you can simply use the rule.

$$a^2 = b^2 + c^2$$



Example 36.3**Question**Find the length a in the diagram.**Solution**

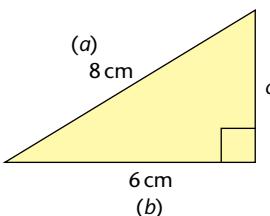
$$a^2 = 7^2 + 24^2$$

$$a^2 = 49 + 576$$

$$a^2 = 625$$

$$a = \sqrt{625}$$

$$a = 25 \text{ cm}$$

Example 36.4**Question**Find the length c in the diagram.**Solution**

$$a^2 = b^2 + c^2$$

$$8^2 = 6^2 + c^2$$

$$64 = 36 + c^2$$

$$c^2 = 64 - 36$$

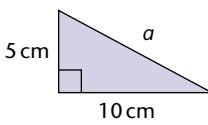
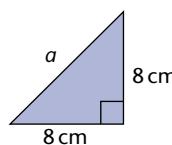
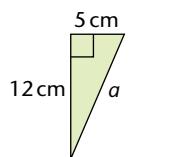
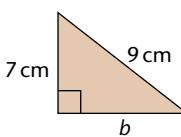
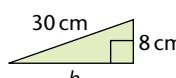
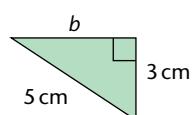
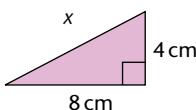
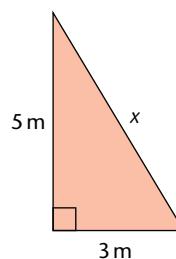
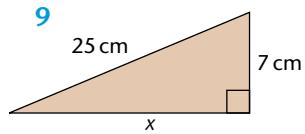
$$c = \sqrt{28}$$

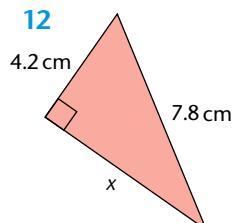
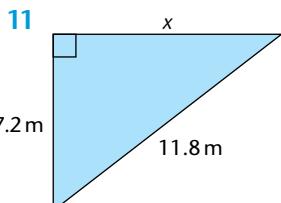
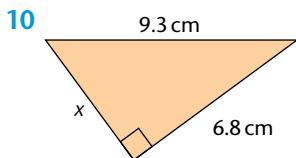
$$c = 5.29 \text{ cm (to 2 decimal places)}$$

Exercise 36.3

For each of the triangles in this exercise, find the length of the side marked with a letter.

Give your answers either exactly or correct to 2 decimal places.

1**2****3****4****5****6****7****8****9**



Using Pythagoras' Theorem to solve problems

You can use Pythagoras' Theorem to solve problems.

It is a good idea to draw a sketch if a diagram isn't given.

Try to draw it roughly to scale and mark on it any lengths you know.

Example 36.5

Question

Tao is standing 115 m from a vertical tower.

The tower is 20 m tall.

Work out the distance from Tao directly to the top of the tower.

Solution

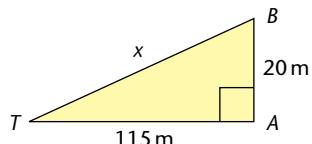
$$x^2 = 115^2 + 20^2$$

$$= 13\,625$$

$$x = \sqrt{13\,625}$$

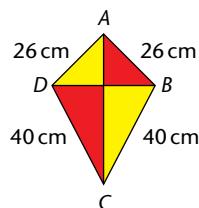
$$= 116.7 \text{ m (to 1 decimal place)}$$

Tao is 116.7 m (to 1 decimal place) from the top of the tower.



Exercise 36.4

- 1 A rectangular field is 225 m long and 110 m wide.
Find the length of the diagonal path across it.
- 2 A rectangular field is 25 m long.
A footpath 38.0 m long crosses the field diagonally.
Find the width of the field.
- 3 A ladder is 7 m long.
It is resting against a wall, with the top of the ladder 5 m above the ground.
How far from the wall is the base of the ladder?
- 4 Haleef is making a kite for his sister.
This is his diagram of the kite.
The kite is 30 cm wide.
Haleef needs to buy some cane to make the struts AC and DB .
What length of cane does he need to buy?



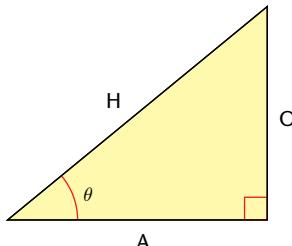
Note

Label the sides in the order hypotenuse, opposite, adjacent.

To identify the opposite side, go straight out from the middle of the angle. The side you hit is the opposite.

You can shorten the labels to 'H', 'O' and 'A'.

θ is the Greek letter 'theta'.

**Note**

Notice that the ratio of the lengths is written as a fraction, $\frac{\text{Opposite}}{\text{Hypotenuse}}$, rather than a ratio, Opposite : Hypotenuse.

Note

You need to learn the three ratios.

$$\sin \theta = \frac{O}{H}$$

$$\cos \theta = \frac{A}{H}$$

$$\tan \theta = \frac{O}{A}$$

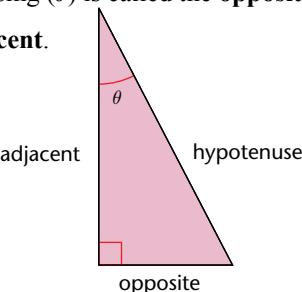
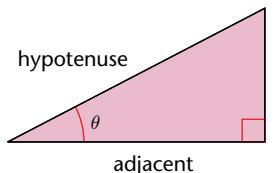
There are various ways of remembering these but one of the most popular is to learn the 'word' 'SOHCAHTOA'.

Trigonometry

You already know that the longest side of a right-angled triangle is called the **hypotenuse**.

The side opposite the angle you are using (θ) is called the **opposite**.

The remaining side is called the **adjacent**.



For a given angle θ° , all right-angled triangles with an angle θ° will be similar (angles in each triangle of 90° , θ° and $(90 - \theta^\circ)$). It follows that the ratios of the sides will be constant for that value of θ .

The ratio $\frac{\text{Opposite}}{\text{Hypotenuse}}$ is called the **sine** of the angle.

This is often shortened to 'sin'.

$$\sin \theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

The ratio $\frac{\text{Adjacent}}{\text{Hypotenuse}}$ is called the **cosine** of the angle.

This is often shortened to 'cos'.

$$\cos \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

The ratio $\frac{\text{Opposite}}{\text{Adjacent}}$ is called the **tangent** of the angle.

This is often shortened to 'tan'.

$$\tan \theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

Now find $\tan 70^\circ$.

Using the ratios 1

When you need to solve a problem using one of the ratios, you should follow these steps.

- Draw a clearly labelled diagram.
- Label the sides H, O and A.
- Decide which ratio you need to use.
- Solve the equation.

In one type of problem you will encounter you are required to find the numerator (top) of the fraction. This is demonstrated in the following examples.

Example 36.6

Question

Find the length marked x.

Solution

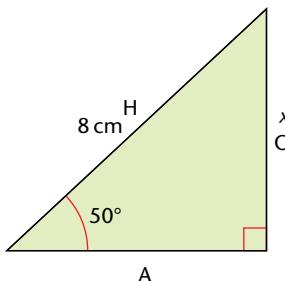
Since you know the hypotenuse (H) and want to find the opposite (O), you use the sine ratio.

$$\sin 50^\circ = \frac{O}{H}$$

$$\sin 50^\circ = \frac{x}{8}$$

$8 \times \sin 50^\circ = x$ Multiply both sides by 8.

$x = 6.12835\dots = 6.13$ cm correct to 3 significant figures



Note

Press these keys on your calculator to find x.

8 \times sin 5 0 =

Note

Make sure that your calculator is set to degrees. This is the default setting but, if you see 'rad' or 'R' or 'grad' or 'G' in the window, change the setting using the key labelled 'mode' or 'DRG' or 'set up'.

Example 36.7

Question

In triangle ABC, BC = 12 cm, angle B = 90° and angle C = 35° .

Find the length AB.

Solution

Draw the triangle and label the sides.

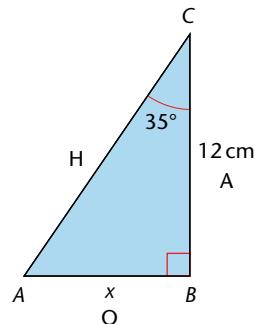
Since you know the adjacent (A) and want to find the opposite (O), you use the tangent ratio.

$$\tan 35^\circ = \frac{O}{A}$$

$$\tan 35^\circ = \frac{x}{12}$$

$12 \times \tan 35^\circ = x$ Multiply both sides by 12.

$x = 8.40249\dots = 8.40$ cm correct to 3 significant figures



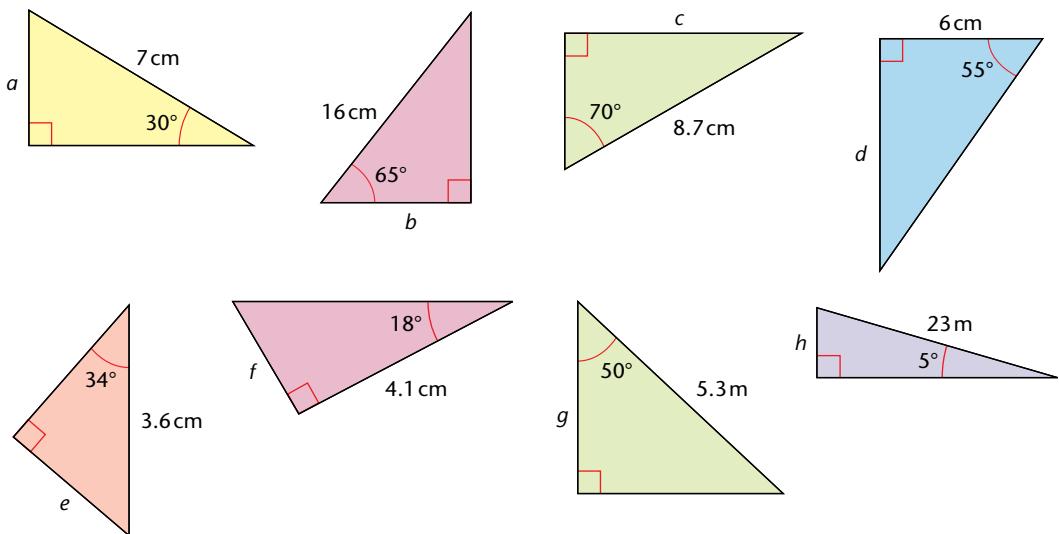
Note

Press these keys on your calculator to find x.

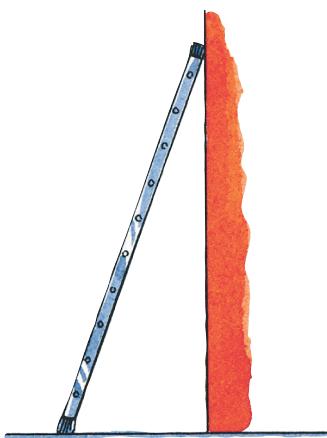
1 2 \times tan 3 5 =

Exercise 36.6

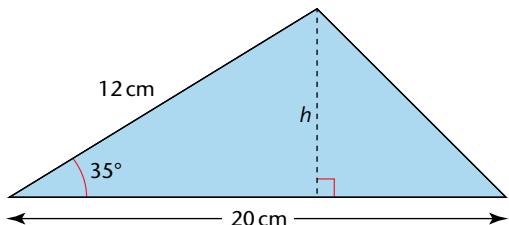
- 1 In these diagrams find the lengths marked a , b , c , d , e , f , g and h .



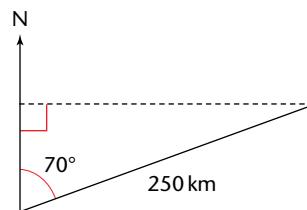
- 2 The ladder shown here is 6 metres long.
The angle between the ladder and the ground is 70° .
How far from the wall is the foot of the ladder?



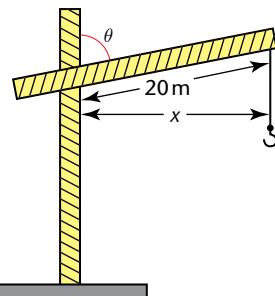
- 3 a Find the height, h , of the triangle.
b Use the height you found in part a to find the area of the triangle.



- 4 A ship sails on a bearing of 070° for 250 km.
- Find how far north the ship has travelled.
 - Find how far east the ship has travelled.



- 5 The diagram shows a crane.
The length of the crane's arm is 20 metres.
The crane can operate with the arm anywhere between 15° and 80° to the vertical.
Calculate the minimum and maximum values of x , the distance from the crane at which a load can be lowered.



Using the ratios 2

In the second type of problem you will encounter, you are required to find the denominator (bottom) of the fraction. This is demonstrated in the following example.

Example 36.8

Question

Find the length marked x .

Solution

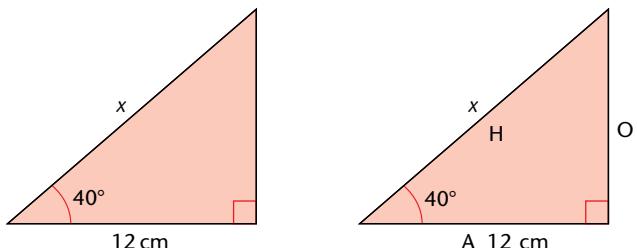
Since you know A and want to find H , you use the cosine ratio.

$$\cos 40^\circ = \frac{A}{H}$$

$$\cos 40^\circ = \frac{12}{x}$$

$$x = \frac{12}{\cos 40^\circ}$$

$$x = 15.66488\dots = 15.7 \text{ cm correct to 3 significant figures}$$



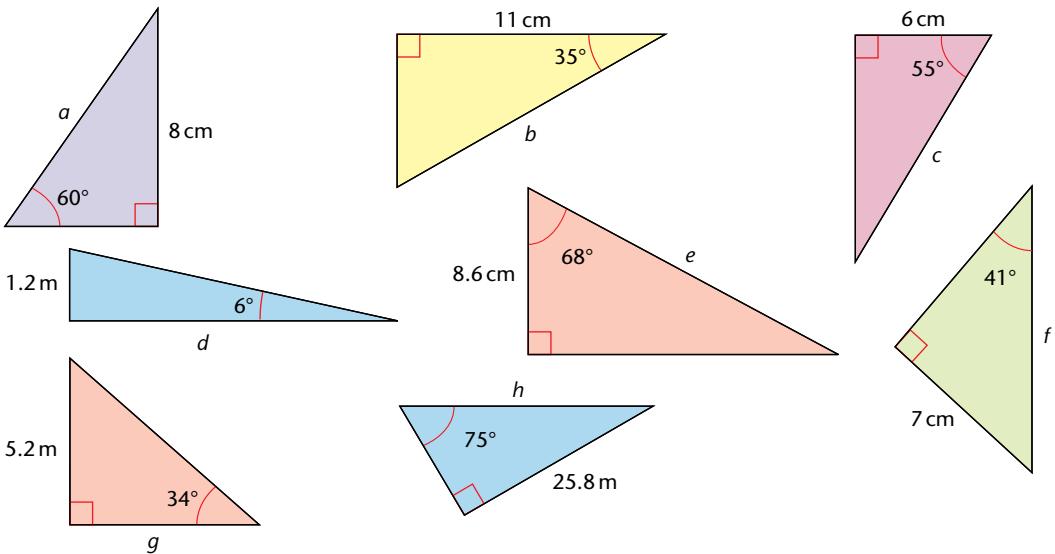
Note

Always look to see whether the length you are trying to find should be longer or shorter than the one you are given.

If your answer is obviously wrong, you have probably multiplied instead of divided.

Exercise 36.7

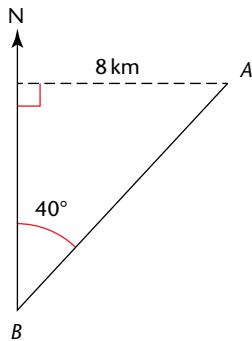
- 1 In these diagrams find the lengths marked a , b , c , d , e , f , g and h .



- 2 The bearing of A from B is 040° .

A is 8 kilometres east of B .

Calculate how far A is north of B .



- 3 The diagram shows a shed.

a Find the length d .

b The length of the shed is 2.5 m.
Find the area of the roof.

- 4 A ship sailed on a bearing of 140° .

It was then 90 km south of its original position.

a Draw a diagram to show the ship's journey.
b How far east of its original position is it?

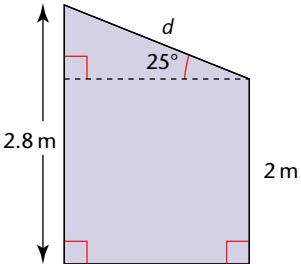
- 5 Mr Jones wants to buy a ladder.

His house is 5.3 metres high and he needs to reach the top.

The ladders are in two sections, each section being the same length.
When extended there must be an overlap of 1.5 metres between the two sections.

The safe operating angle is 76° .

Calculate the length of each of the sections of ladder he needs to buy.



Using the ratios 3

In the third type of problem you will encounter, you are given the value of two sides and are required to find the angle. This is demonstrated in the following examples.

Example 36.9

Question

Find the angle θ .

Solution

This time, look at the two sides you know.

Since they are O and H, you use the sine ratio.

$$\sin \theta = \frac{O}{H} \quad \sin \theta = \frac{5}{8}$$

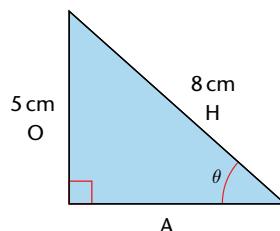
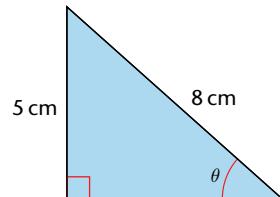
Work out $5 \div 8 = 0.625$ on your calculator and leave this number in the display.

Now use the \sin^{-1} function (the inverse of sine).

With 0.625 still in the display, press **SHIFT** **sin** **=**, or the equivalent on your calculator.

You should see 38.68218....

So $\theta = 38.7^\circ$ correct to 3 significant figures or 39° correct to the nearest degree.



Example 36.10

Question

Find the angle θ .

Solution

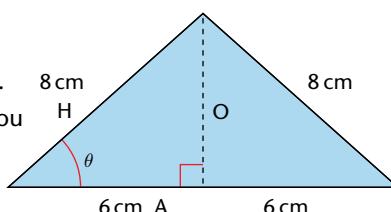
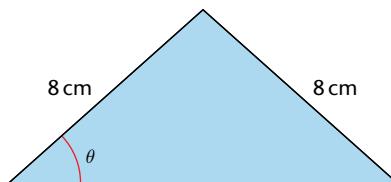
Since this is an isosceles triangle, not a right-angled triangle, you need to use the fact that the line of symmetry splits an isosceles triangle into two equal right-angled triangles.

The sides you know are A and H so you use the cosine ratio.

$$\cos \theta = \frac{A}{H} = \frac{6}{8}$$

$$\theta = \cos^{-1} \frac{6}{8}$$

$\theta = 41.4^\circ$ correct to 3 significant figures



Note

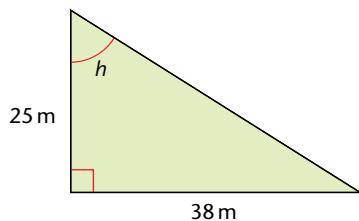
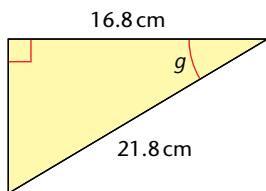
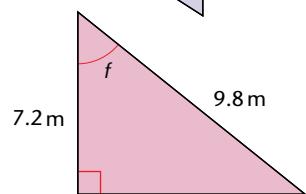
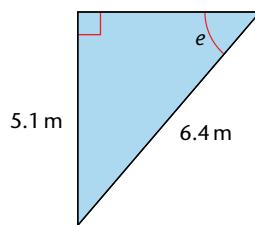
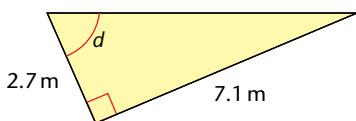
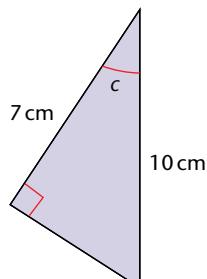
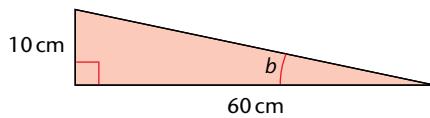
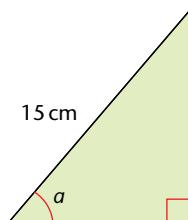
Example 36.10 shows how to deal with isosceles triangles.

You use the line of symmetry to split the triangle into two equal right-angled triangles.

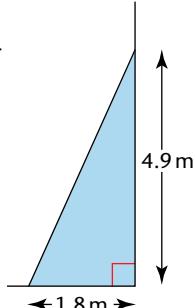
This works only with isosceles triangles because they have a line of symmetry.

Exercise 36.8

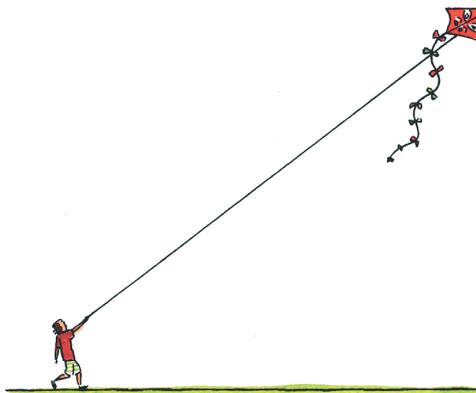
- 1** In these diagrams find the angles marked a, b, c, d, e, f, g and h .



- 2** The diagram represents a ladder leaning against a wall.
Find the angle the ladder makes with the horizontal.



- 3** In the picture the kite is 15 metres above the girl.
The string is 25 metres long.
Find the angle the string makes with the horizontal.



- 4 The diagram represents a pair of step ladders standing on a horizontal floor. Find the angle, θ , between the two parts of the step ladder.
- 5 An aircraft flies 180 km due east from A to B . It then flies 115 km due south from B to C .
- Draw a diagram to show the positions of A , B and C .
 - Calculate the bearing of C from A .

- 6 A television mast is 54 metres high and stands on horizontal ground. Six guy wires keep the mast upright.

Three of these are attached to the top and a point on the ground. These three make an angle of 16.5° with the vertical.

- Calculate the total length of these three wires.

The other three wires are attached $\frac{2}{3}$ of the way up the mast.

They are attached to the same points on the ground as the previous three.

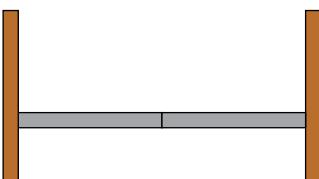
- Calculate the angle these make with the vertical.

- 7 A ship sails for 150 km on a bearing of 115° from A to B .

It then sails for 250 km on a bearing of 230° from B to C .

Calculate the distance AC and the bearing of C from A .

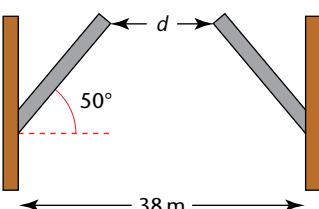
- 8 The diagrams show a bridge in the shut and open positions.



When opened, the bridge sections raise by 50° .

The distance between the supports is 38 metres.

Calculate the distance d .



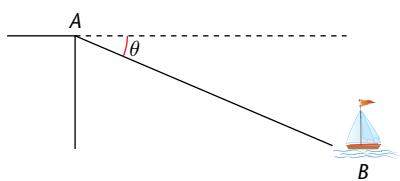
Angle of elevation and angle of depression

The angle a line makes above a horizontal plane is called the angle of elevation.

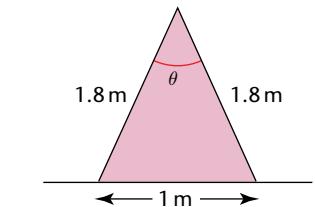
The angle of elevation of the top of this tree, A , from B is θ .



The angle a line makes below a horizontal plane is called the angle of depression.



The angle of depression of the boat, B , from A is θ .



Note

Since alternate angles are equal, the angle of elevation of A from B equals the angle of depression of B from A .

Example 36.11**Question**

The height of this tree is 24 m.

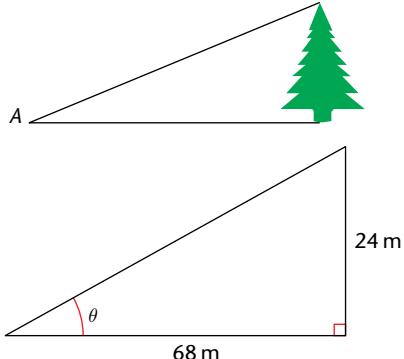
A is 68 m from the foot of the tree.

Find the angle of elevation of the top of the tree from A .

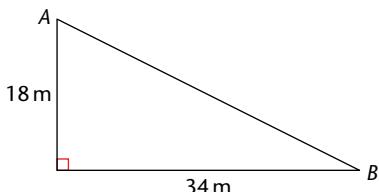
Solution

$$\tan \theta = \frac{24}{68}$$

$$\theta = 19.4^\circ$$

**Exercise 36.9**

- 1 Find the angle of elevation of A from B .



- 2 The angle of elevation of the top of a tree from A is 25° .

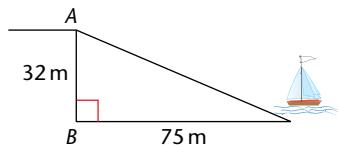
A is 38 m from the base of the tree.

Calculate the height of the tree.

- 3 AB is a vertical cliff 32 m high.

The boat is on the sea, 75 metres from the bottom of the cliff.

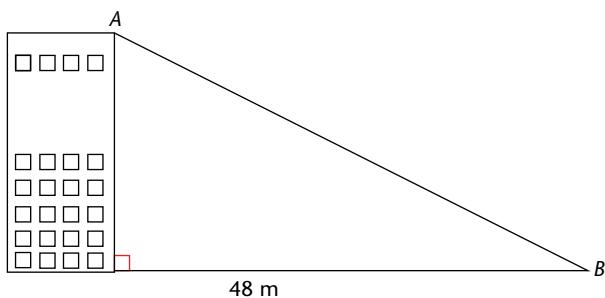
Calculate the angle of depression of the boat from the top of the cliff, A .



- 4 The diagram shows a tall building. B is a point on the ground 48 m from the foot of the building.

The angle of depression of B from a point A , on the top of the building, is 28° .

Calculate the height of the building.



- 5 Adil is standing on horizontal ground looking at the top of a building 18 m high.

The angle of elevation of the top of the building from Adil's eyes is 24° .

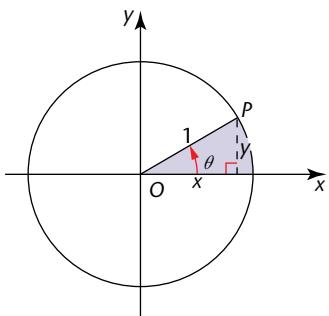
Adil's eyes are 1.8 m above the ground.

Find how far away from the building Adil is standing.

The sine and cosine functions for obtuse angles

So far you have only dealt with the sine and cosine functions for right-angled triangles so the angles have all been acute.

However, your calculator will give you the sine and cosine of any angle.



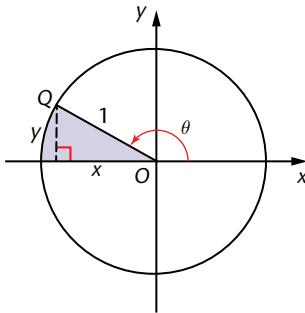
In this diagram you can see that for an acute angle

$$\cos \theta = \frac{x}{1} \text{ so } x = \cos \theta$$

$$\sin \theta = \frac{y}{1} \text{ so } y = \sin \theta$$

So P has coordinates $(\cos \theta, \sin \theta)$.

For other angles, the trigonometric functions are defined in a similar way, where the angle is measured anticlockwise from the x axis.



By symmetry, Q has coordinates $(-\cos \theta, \sin \theta)$.

So

$$\sin \theta = \sin (180^\circ - \theta)$$

$$\cos \theta = -\cos (180^\circ - \theta)$$

By symmetry $\sin \theta = y = \sin (180^\circ - \theta)$

and $\cos \theta = x = -\cos (180^\circ - \theta)$.

So for an obtuse angle θ , $\sin \theta$ is equal to $\sin (180^\circ - \theta)$

and $\cos \theta$ takes the same value as $\cos (180^\circ - \theta)$ but is negative.

Example 36.12**Question**

You are given that $\sin 40^\circ = 0.6428$ and $\cos 40^\circ = 0.7660$.

Without using your calculator, write down the values of

- a $\sin 140^\circ$
- b $\cos 140^\circ$.

Solution

a $\sin 140^\circ = \sin (180^\circ - 40^\circ) = \sin 40^\circ = 0.6428$

b $\cos 140^\circ = \cos (180^\circ - 40^\circ) = -\cos 40^\circ = -0.7660$

Check these values by finding $\sin 140^\circ$ and $\cos 140^\circ$ directly on your calculator.

Example 36.13**Question**

Solve the equation $\sin x = 0.8$ for $0^\circ \leq x \leq 180^\circ$.

Solution

Using a calculator gives $x = 53.13^\circ$ so this is one solution.

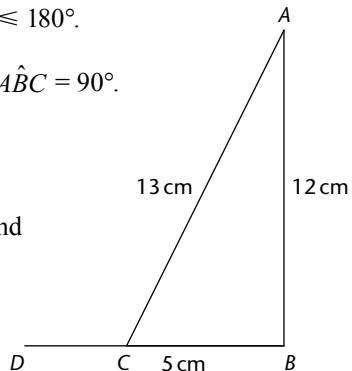
But $\sin x$ is also positive for values of x between 90° and 180° .

So another solution is $180^\circ - 53.13^\circ = 126.87^\circ$.

So $x = 53.13^\circ$ or 126.87°

Exercise 36.10

-  1 You are given that $\sin 60^\circ = 0.8660$ and $\cos 60^\circ = 0.5$. Without using your calculator, write down the values of
- a $\sin 120^\circ$
 - b $\cos 120^\circ$.
- 2 Solve the equation $\sin x = 0.3$ for $0^\circ \leq x \leq 180^\circ$.
- 3 Solve the equation $\cos x = 0.7$ for $0^\circ \leq x \leq 180^\circ$.
- 4 In the diagram, BCD is a straight line. $BC = 5\text{ cm}$, $AB = 12\text{ cm}$, $AC = 13\text{ cm}$ and $A\hat{B}C = 90^\circ$. Find
- a $\tan B\hat{A}C$
 - b $\cos A\hat{C}D$.
- 5 A is the point $(1, 2)$, B is the point $(4, 6)$ and C is the point $(-5, 2)$.
- a Calculate the length of AB .
 - b Find the value of $\sin C\hat{A}B$.
 - c Find the value of $\cos C\hat{A}B$.

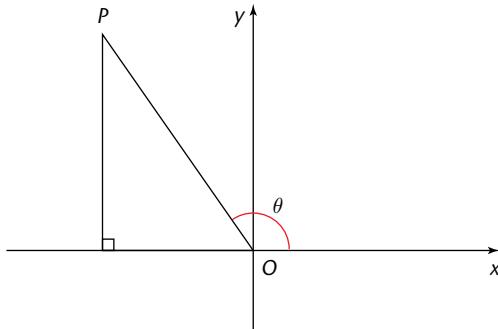


- 6 P is the point $(-6, 8)$.

Find the value of

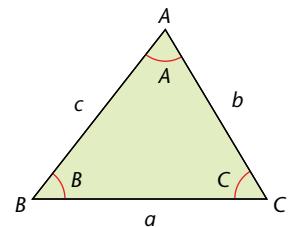
a $\sin \theta$

b $\cos \theta$.



Non-right-angled triangles

All the trigonometry that you have learned so far has been based on finding lengths and angles in right-angled triangles. However, many triangles are not right-angled. You need a method to find lengths and angles in these other triangles.



For simplicity we shall use a single letter to represent each side and each angle of the triangle, a capital letter for an angle and a lower case letter for a side. The side opposite an angle takes the same, lower case letter, as shown in the diagram.

There are two rules for dealing with non-right-angled triangles; they are called the **sine rule** and the **cosine rule**.

You will need to memorise the formulae.

The sine rule

The sine rule is derived from the trigonometry in right-angled triangles that you have already met.

Use triangle BCP to find an expression for h in terms of a and C .

$$\frac{h}{a} = \sin C \quad \text{so} \quad h = a \sin C$$

Use triangle ABP to find an expression for h in terms of c and A .

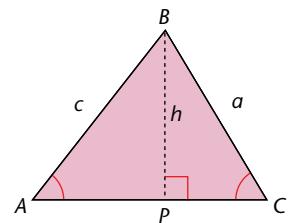
$$\frac{h}{c} = \sin A \quad \text{so} \quad h = c \sin A$$

Equating the expressions for h gives

$$h = a \sin C = c \sin A$$

Dividing both sides by $(\sin A \sin C)$ gives

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$



Note

You do not need to be able to prove the sine rule.

Similarly by drawing a perpendicular from point C to side AB we can show that

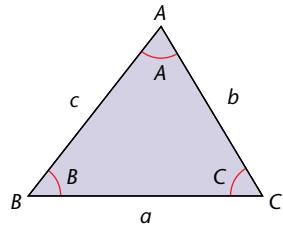
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

So the full sine rule is

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



To use the sine rule you must know one side and the angle opposite it. You also need to know one other angle or one other side.

Example 36.14**Question**

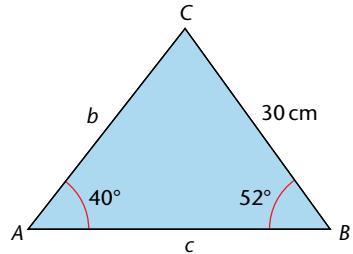
Find these.

- a Length b b Length c

Solution

- a Since you are finding a length, use the formula with lengths on top.

Choose pairs of angles and opposite sides where three of the four values are known and substitute into the formula.



$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$\frac{b}{\sin 52^\circ} = \frac{30}{\sin 40^\circ}$$

$$b = \frac{30}{\sin 40^\circ} \times \sin 52^\circ$$

$b = 36.8$ cm to 1 decimal place

- b Before you can find c , you need to find angle C .

$$C = 180 - (40 + 52)$$

$$C = 88^\circ$$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\frac{c}{\sin 88^\circ} = \frac{30}{\sin 40^\circ}$$

$$c = \frac{30}{\sin 40^\circ} \times \sin 88^\circ$$

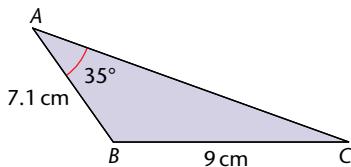
$c = 46.6$ cm to 1 decimal place

Note

Although you could use the pair b and B instead of a and A , you should whenever possible use values that are given rather than values that have been calculated.

Example 36.15**Question**

Find C.

**Solution**

Since you are finding an angle, use the formula with angles on top.

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\frac{\sin C}{7.1} = \frac{\sin 35^\circ}{9}$$

$$\sin C = \frac{\sin 35^\circ}{9} \times 7.1$$

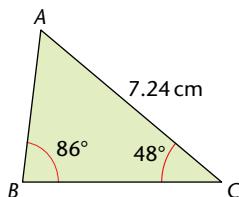
$$\sin C = 0.4524\dots$$

$$C = \sin^{-1}(0.4524\dots)$$

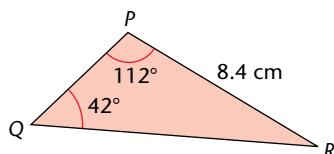
$$C = 26.9^\circ \text{ to 1 decimal place}$$

Exercise 36.11

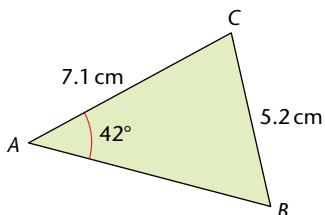
- 1 Find c, A and a.



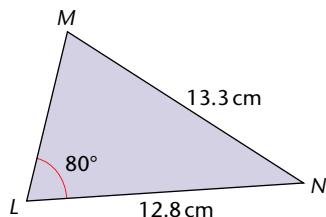
- 2 Find p, R and r.



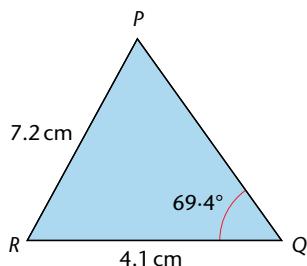
- 3 Find B, C and c.



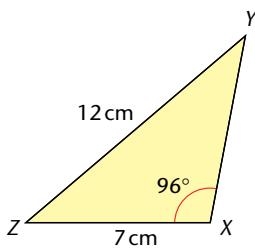
- 4 Find M, N and n.



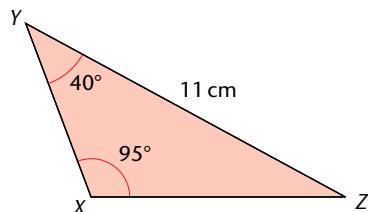
- 5 Find P, R and r.



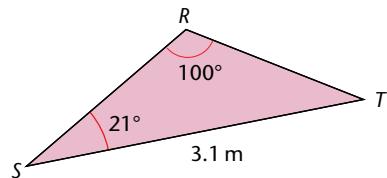
- 6 Find Y, Z and z.



- 7 Find y , Z and z .



- 8 Find s , T and t .

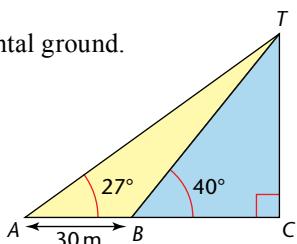


- 9 In triangle ABC , $c = 23$ cm, $b = 19.4$ cm and angle C is 54° .
Find the length of BC .

- 10 Calculate the largest angle of the triangle ABC given that $A = 35^\circ$, $a = 8.9$ cm and $c = 12$ cm.

- 11 Two points, A and B , are 30 m apart on horizontal ground.
The angles of elevation of the top, T , of a vertical tower, TC , from A and B
are 27° and 40° respectively.

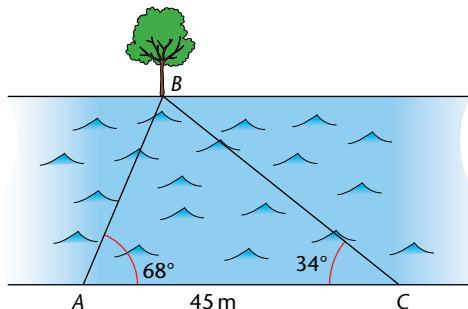
- a Find AT and BT .
b Find the height of the tower, TC .



- 12 A river has two parallel banks.

Points A and C are on one side of the river, 45 m apart.

The angles from these points to a tree, B , on the opposite bank are 68° and 34° , as shown in the diagram.

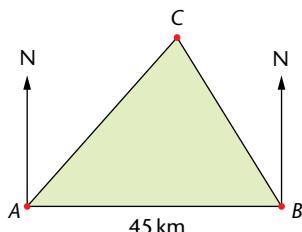


- a Find AB and BC .
b Find the width of the river.

- 13 Town B is 45 km due east of town A .

Town C is on a bearing of 057° from town A and a bearing of 341° from town B .

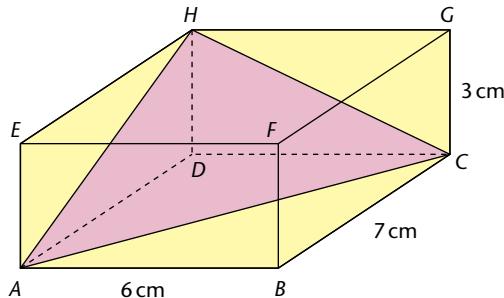
Find how far town C is from towns A and B .



14 ABCDEFGH is a cuboid.

ACH is a triangle contained within the cuboid.

$A\hat{H}C$ is 80° .



Calculate the size of these angles, giving your answers to the nearest degree.

a $A\hat{C}H$

b $C\hat{A}H$

The cosine rule

To derive the cosine rule, you combine right-angled trigonometry with Pythagoras' Theorem.

Use Pythagoras' Theorem for triangle BPA.

$$h^2 + x^2 = c^2 \quad \text{so} \quad h^2 = c^2 - x^2.$$

Use Pythagoras' Theorem for triangle BPC.

$$h^2 + (b-x)^2 = a^2 \quad \text{so} \quad h^2 = a^2 - (b-x)^2.$$

Equate the two expressions for h^2

$$h^2 = a^2 - (b-x)^2 = c^2 - x^2$$

Expand the bracket and simplify $a^2 - (b^2 - 2bx + x^2) = c^2 - x^2$

$$a^2 - b^2 + 2bx - x^2 = c^2 - x^2$$

$$a^2 - b^2 + 2bx = c^2$$

Rearrange $a^2 = c^2 + b^2 - 2bx$

But $\frac{x}{c} = \cos A$ so $x = c \cos A$ giving $a^2 = c^2 + b^2 - 2bc \cos A$

This is the cosine rule.

Changing the letters and rearranging gives the six formulae for the cosine rule.

$$a^2 = c^2 + b^2 - 2bc \cos A$$

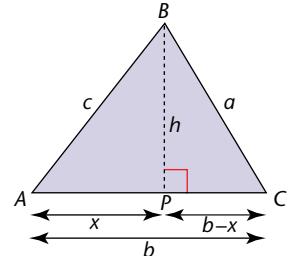
$$b^2 = c^2 + a^2 - 2ca \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

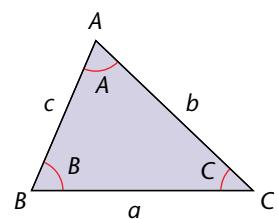
$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



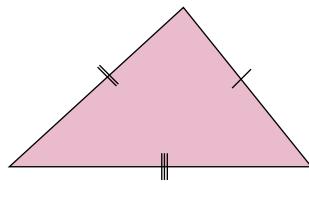
Note

You do not need to be able to prove the cosine rule.

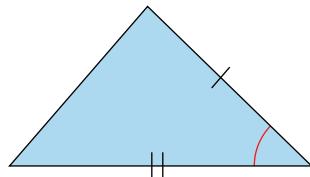


Use the cosine rule

- when you know all the sides



- when you know two sides and the included angle.



Example 36.16

Question

Find c .

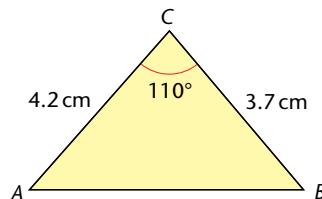
Solution

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$c^2 = 3.7^2 + 4.2^2 - (2 \times 3.7 \times 4.2 \times \cos 110^\circ)$$

$$c^2 = 41.959\dots$$

$c = 6.48\text{ cm}$ to 2 decimal places



Example 36.17

Question

Find R .

Solution

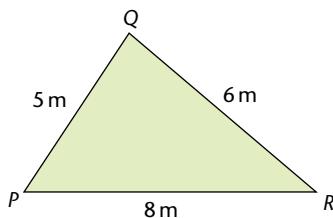
$$\cos R = \frac{p^2 + q^2 - r^2}{2pq}$$

$$\cos R = \frac{6^2 + 8^2 - 5^2}{2 \times 6 \times 8}$$

$$\cos R = 0.78125$$

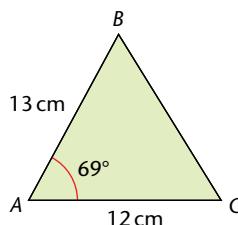
$$R = \cos^{-1} (0.78125)$$

$R = 38.6^\circ$ to 1 decimal place

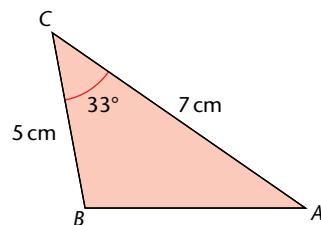


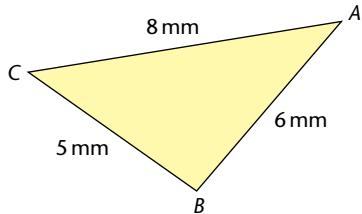
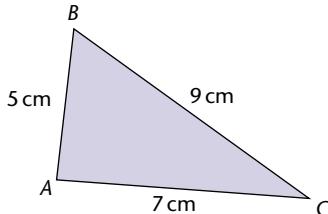
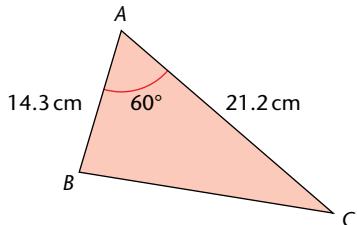
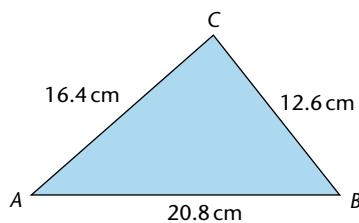
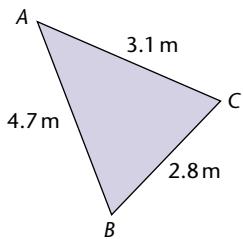
Exercise 36.12

- 1 Find a .

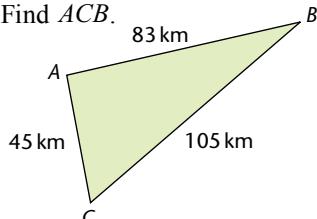
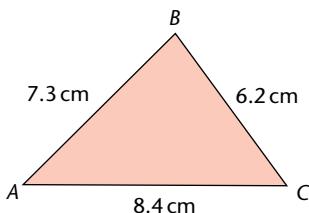


- 2 Find c .



3 Find C .**4** Find B .**5** Find a .**6** Find B .**7** Find B .

8 Three towns, A , B and C , are shown in the diagram.
Find \hat{ACB} .

**9** Find the smallest angle in this triangle.

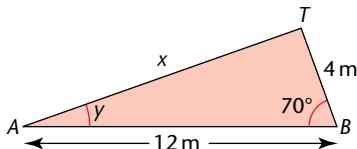
10 A cross-country runner runs 4 km due north and then 6.7 km in a south-east direction. How far is she from her starting point?

11 From a boat C , A is 9 km away on a bearing of 058° and B is 12 km away on a bearing of 110° . Calculate the distance AB .

12 The diagram shows the cross-section of a roof.

The house is 12 m wide.

Calculate the length of the roof, x , and the angle of the slope, y .

**Note**

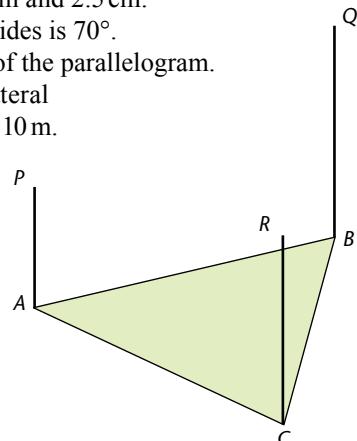
Always draw a sketch if you are not given a diagram.

Label the sides and angles with the information given.

- 13** A parallelogram has sides of length 5.1 cm and 2.5 cm.
The angle between one pair of adjacent sides is 70° .
Find the length of each of the diagonals of the parallelogram.

- 14** Three points, A , B and C , form an equilateral triangle on horizontal ground with sides 10 m.
Vertical posts PA , QB and RC are placed in the ground.
 $PA = 4$ m, $QB = 10$ m and $RC = 6$ m.

- a**
- i Find PQ .
 - ii Find PR .
 - iii Find QR .
- b** Hence find the size of
- i $\hat{R}PQ$
 - ii $P\hat{R}Q$.



The general formula for the area of any triangle

To find the area of a triangle when you do not know the perpendicular height, you can derive a formula from the trigonometry of right-angled triangles.

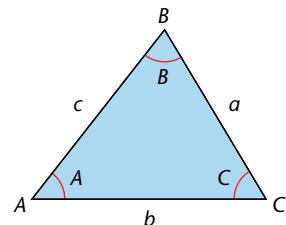
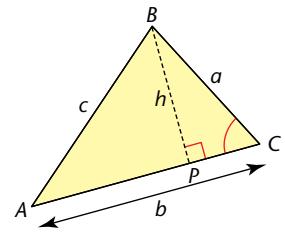
$$\text{Area of triangle } ABC = \frac{1}{2} \times b \times h$$

$$\text{But } \frac{h}{a} = \sin C \quad \text{so} \quad h = a \sin C$$

Substituting this expression in the formula for the area of a triangle gives

$$\text{Area of triangle } ABC = \frac{1}{2} ab \sin C$$

$$\begin{aligned}\text{Area of triangle } ABC &= \frac{1}{2} ab \sin C \\ &= \frac{1}{2} bc \sin A \\ &= \frac{1}{2} ca \sin B\end{aligned}$$



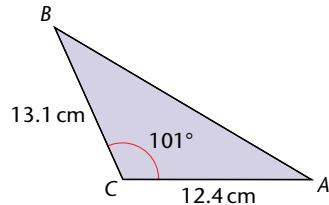
Example 36.18

Question

Find the area of the triangle shown.

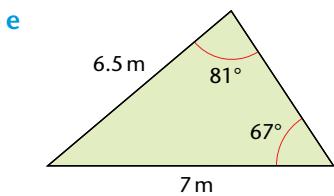
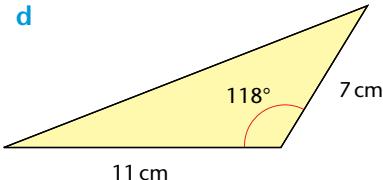
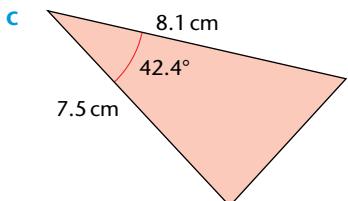
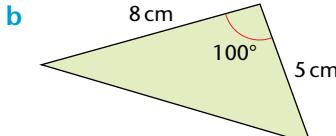
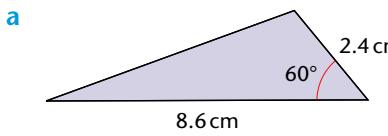
Solution

$$\begin{aligned}\text{Area} &= \frac{1}{2} ab \sin C \\ &= \frac{1}{2} \times 13.1 \times 12.4 \times \sin 101^\circ \\ &= 79.7 \text{ cm}^2 \text{ to 1 decimal place}\end{aligned}$$

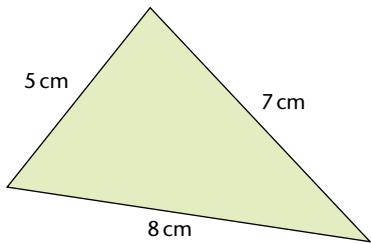


Exercise 36.13

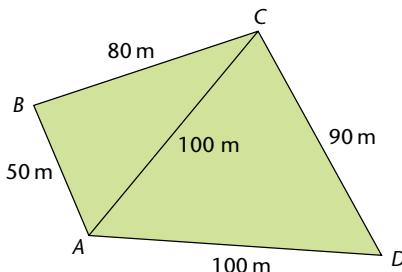
- 1** Find the area of each of these triangles.



- 2** In triangle ABC , $a = 10 \text{ cm}$, $c = 6 \text{ cm}$ and $B = 150^\circ$.
Find the area of the triangle.
- 3** In triangle ABC , $a = 4 \text{ cm}$, $c = 7 \text{ cm}$ and its area is 13.4 cm^2 .
Find the size of angle B .
- 4** The area of triangle PQR is 273 cm^2 .
Given that $PQ = 12.8 \text{ cm}$ and $\hat{P}QR = 107^\circ$, find QR .
- 5** Calculate the area of parallelogram $ABCD$ in which
 $AB = 6 \text{ cm}$, $BC = 9 \text{ cm}$ and $\hat{A}BC = 41.4^\circ$.
- 6** Calculate the area of this triangle.



- 7** $ABCD$ is a field with dimensions as shown in the diagram.
Calculate the area of the field.

**Note**

Remember to draw a diagram and label it with the information you know.

Finding lengths and angles in three dimensions

You can find lengths and angles of three-dimensional objects by identifying right-angled triangles within the object and using Pythagoras' Theorem or trigonometry.

Example 36.19

Question

A rectangular box measures 4 cm by 3 cm by 6 cm.

Calculate these.

- a AC b AR c Angle RBC d Angle ARC

Solution

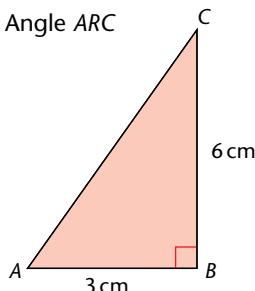
- a Identify the right-angled triangle required and draw a sketch.

$$AC^2 = AB^2 + BC^2$$

$$AC^2 = 3^2 + 6^2$$

$$AC^2 = 45$$

$$AC = 6.71 \text{ cm to 2 decimal places}$$

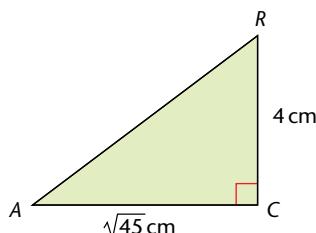


- b $AR^2 = AC^2 + RC^2$

$$AR^2 = 45 + 42$$

$$AR^2 = 61$$

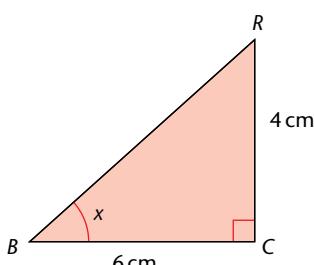
$$AR = 7.81 \text{ cm to 2 decimal places}$$



c $\tan x = \frac{4}{6}$

$$x = \tan^{-1} \frac{4}{6}$$

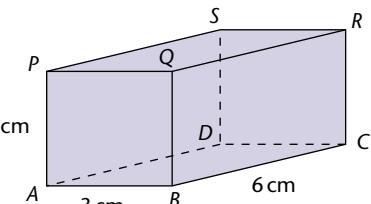
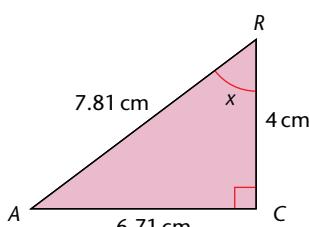
$$x = 33.7^\circ \text{ to 1 decimal place}$$



d $\tan x = \frac{\sqrt{45}}{4}$

$$x = \tan^{-1} \frac{\sqrt{45}}{4}$$

$$x = 59.2^\circ \text{ to 1 decimal place}$$



Note

Don't use 6.71^2 as this has been rounded.

You need AC^2 and this is 45.

Note

When there is a choice of formulae to use, use the one which contains as many given values as possible. You may have calculated a value incorrectly.

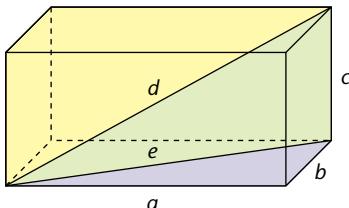
Note

You can't avoid using a calculated value here.

Using 6.71 will give an answer correct to 3 significant figures but it is good practice to use the more accurate value, $\sqrt{45}$, and round at the end.

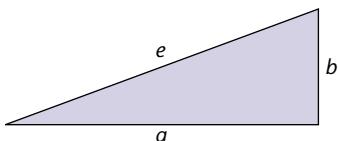
In Example 36.19, notice how you used the result of part **a** to work out the result of part **b**.

You can use this method to derive a general formula for the length of the diagonal of a cuboid measuring a by b by c .

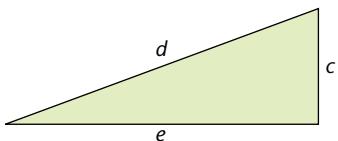


The diagonal of the rectangular base is labelled e .

$$e^2 = a^2 + b^2$$



The diagonal of the cuboid is labelled d .



$$d^2 = e^2 + c^2$$

$$\text{But } e^2 = a^2 + b^2$$

so

$$d^2 = a^2 + b^2 + c^2$$

$$d = \sqrt{a^2 + b^2 + c^2}$$

Example 36.20

Question

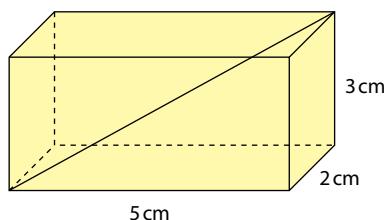
Find the length of the diagonal of this cuboid.

Solution

The length of the diagonal of a cuboid $= \sqrt{a^2 + b^2 + c^2}$.

In this cuboid $a = 5\text{ cm}$, $b = 2\text{ cm}$ and $c = 3\text{ cm}$.

$$\begin{aligned} \text{Length of diagonal} &= \sqrt{5^2 + 2^2 + 3^2} \\ &= \sqrt{38} \\ &= 6.2\text{ cm to 1 decimal place} \end{aligned}$$



Example 36.21**Question**

A tree, TC , is 20 m north of point A .

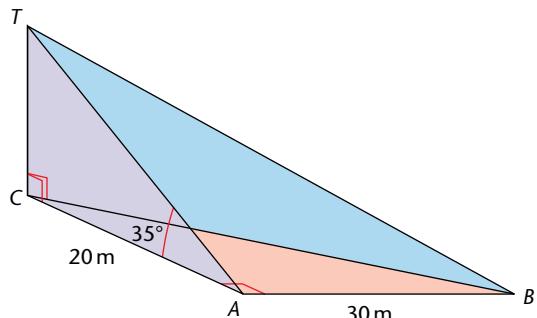
The angle of elevation of the top of the tree, T , from A , is 35° .

Point B is 30 m east of point A .

A , B and C are on horizontal ground.

Calculate

- the height of the tree, TC
- the length BC
- the angle of elevation of T from B .

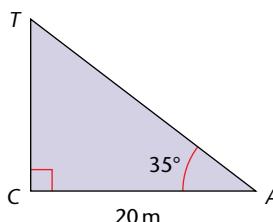
**Solution**

First, draw a diagram.

a $\tan 35^\circ = \frac{TC}{20}$

$$TC = 20 \tan 35^\circ$$

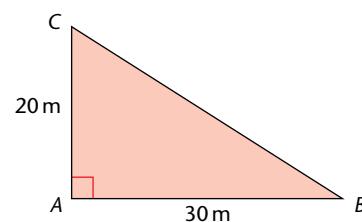
$TC = 14.0$ m to 1 decimal place



b $BC^2 = 20^2 + 30^2$

$$BC^2 = 1300$$

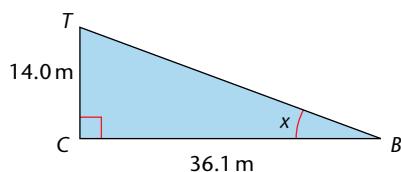
$BC = 36.1$ m to 1 decimal place



c $\tan x = \frac{14.0...}{36.1...}$

$$x = \tan^{-1} \left(\frac{14.0...}{36.1...} \right)$$

$x = 21.2^\circ$ to 1 decimal place

**Example 36.22****Question**

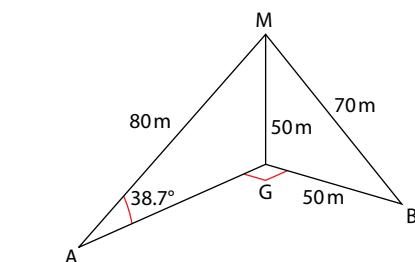
A mast, MG , is 50 m high.

It is supported by two ropes, AM and BM , as shown.

ABG is horizontal.

Other measurements are shown on the diagram.

Is the mast vertical?



Solution

If $M\hat{G}A$ is a right angle then $\sin 38.7^\circ = \frac{50}{80}$.

$$\sin^{-1}\left(\frac{50}{80}\right) = 38.68\dots$$

So $M\hat{G}A$ is a right angle.

If $M\hat{G}B$ is a right angle then the side lengths of triangle MGB will fit Pythagoras' Theorem.

$$50^2 + 50^2 = 5000$$

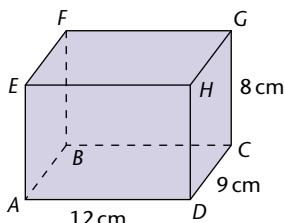
$$\sqrt{5000} = 70.7\dots$$

MB is shorter than this, so $M\hat{G}B$ is not a right angle.

The mast leans towards B ($M\hat{G}B < 90^\circ$).

Exercise 36.14

- 1 $ABCDEFGH$ is a cuboid with dimensions as shown.

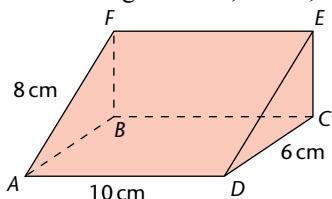


Calculate these.

- a $G\hat{D}C$ b EG c EC d $G\hat{E}C$

- 2 A wedge has a rectangular base, $ABCD$, on horizontal ground.

The rectangular face, $BCEF$, is vertical.



Calculate these.

- a FC b $D\hat{F}C$ c FD d $E\hat{D}F$

- 3 Three points, A , B and C , are on horizontal ground with B due west of C and A due south of C .

A chimney, CT , at C , is 50 m high.

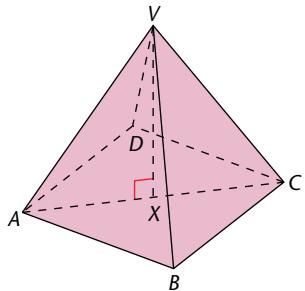
The angle of elevation of the top, T , of the chimney is 26° from A and 38° from B .

Calculate these.

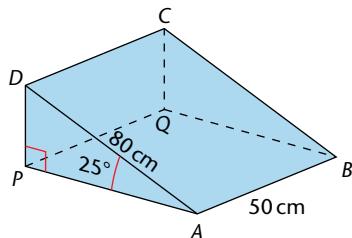
- a How far A and B are from C
 b The distance between A and B
 c The bearing of B from A

- 4** The diagram shows a square-based pyramid with V vertically above the centre, X , of the square $ABCD$.
 $AB = 8\text{ cm}$ and $AV = 14\text{ cm}$.
Calculate these.

- a** $C\hat{A}B$ **b** $V\hat{C}B$
c AC **d** AX
e VX **f** $V\hat{A}X$



- 5** $ABCDPQ$ is a triangular prism with $ABPQ$ horizontal and ADP vertical.



Calculate these.

- a** DP
b AC
c $C\hat{A}Q$
d The volume of the prism

- 6** A pyramid has a rectangular base, $ABCD$, with $AB = 15\text{ cm}$ and $BC = 8\text{ cm}$. The vertex, V , of the pyramid is directly above the centre, X , of $ABCD$ with $VX = 10\text{ cm}$.

- a** Calculate these.

i AC **ii** AV **iii** $A\hat{V}B$

- b** M is the midpoint of AB .

Calculate these.

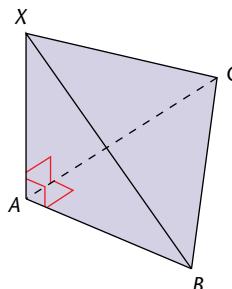
i VM **ii** $V\hat{M}X$

- 7** Triangle ABC is horizontal.

X is vertically above A .

$AC = AX = 15\text{ cm}$.

$A\hat{C}B = 27^\circ$ and $B\hat{A}C = 90^\circ$.



Calculate these.

- a** XC **b** BC **c** BX

- 8 A field, $ABCD$, is a quadrilateral with opposite sides equal.

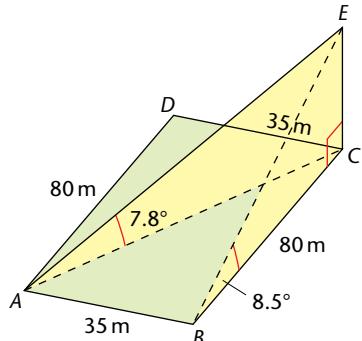
$$AD = BC = 80 \text{ m}$$

$$DC = AB = 35 \text{ m}$$

A vertical post, CE , is at one corner of the field.

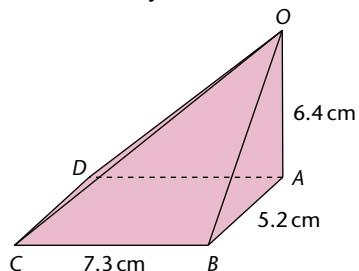
The angles of elevation of the top of the post are 7.8° from A and

$$8.5^\circ \text{ from } B.$$



Is the field a rectangle? Show how you decide.

- 9 The pyramid $OABCD$ has a horizontal rectangular base $ABCD$ as shown. O is vertically above A .



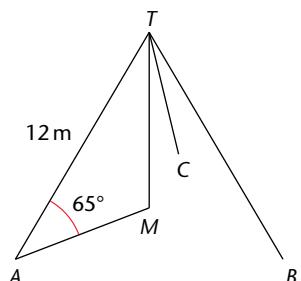
Calculate these.

- a The length of OC b \hat{OCA}

- 10 A vertical mast, MT , has its base, M , on horizontal ground.

It is supported by a wire, AT , which makes an angle of 65° with the horizontal and is of length 12 m, and two more wires, BT and CT , where A, B and C are on the ground.

$$BM = 4.2 \text{ m.}$$



Calculate these.

- a The height of the mast
 b The angle which BT makes with the ground
 c The length of wire BT

37 VECTORS IN TWO DIMENSIONS

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- describe a translation by using a vector represented by $\begin{pmatrix} x \\ y \end{pmatrix}$, \overrightarrow{AB} or \mathbf{a}
- add and subtract vectors
- multiply a vector by a scalar
- calculate the magnitude of a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ as $\sqrt{x^2 + y^2}$
- represent vectors by directed line segments
- use the sum and difference of two vectors to express given vectors in terms of two coplanar vectors
- use position vectors.

CHECK YOU CAN:

- add, subtract, multiply and divide numerical fractions
- simplify and factorise algebraic expressions
- rearrange formulae.

Vectors and translations

A transformation that moves points in a given direction for a given distance is called a **translation**.

The distance and direction that a shape moves in a translation can be written as a **column vector**.

The top number tells you how far the shape moves across, or in the x -direction.

The bottom number tells you how far the shape moves up or down, or in the y -direction.

A positive top number is a move to the right.

A negative top number is a move to the left.

A positive bottom number is a move up.

A negative bottom number is a move down.

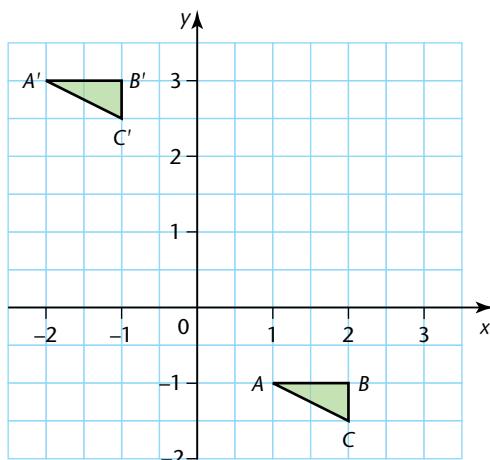
A translation of 3 to the right and 2 down is written as $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

Example 37.1

Question

Translate the triangle ABC by $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

Solution



$\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ means move 3 units

left and 4 units up.

Point A moves from $(1, -1)$ to $(-2, 3)$.

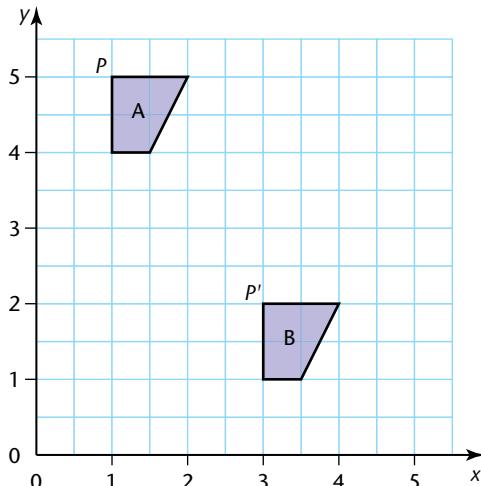
Point B moves from $(2, -1)$ to $(-1, 3)$.

Point C moves from $(2, -1.5)$ to $(-1, 2.5)$.

Example 37.2

Question

Describe fully the single transformation that maps shape A on to shape B.



Note

Try not to confuse the words *transformation* and *translation*.

Transformation is the general name for all changes made to shapes.

Translation is the particular transformation where all points of an object move the same distance in the same direction.

Solution

It is clearly a translation as the shape stays the same way up and the same size.

To find the movement, choose one point on the original shape and the image and count the units moved.

For example, P moves from (1, 5) to (3, 2). This is a movement of 2 to the right and 3 down.

The transformation is a translation of $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

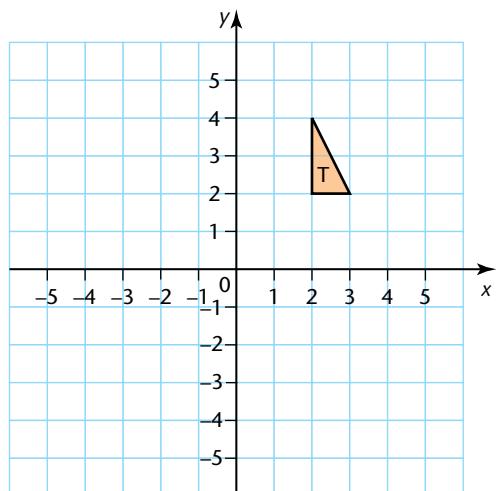
Note

You must state that the transformation is a translation and give the column vector.

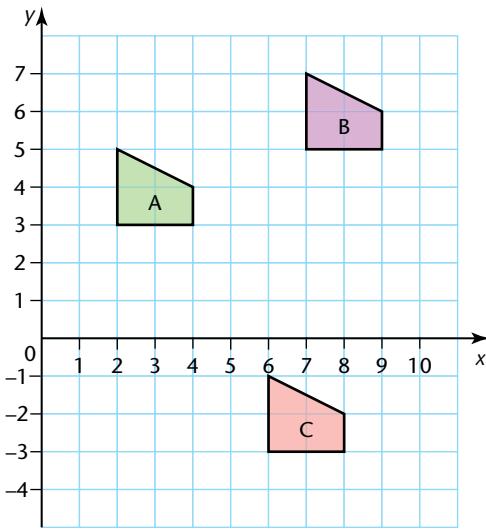
Exercise 37.1

- 1 Copy the diagram.

- Translate triangle T 2 units to the right and 1 up.
Label the image A.
- Translate triangle T 1 unit to the right and 3 down.
Label the image B.
- Translate triangle T by $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$.
Label the image C.
- Translate triangle T by $\begin{pmatrix} -6 \\ -6 \end{pmatrix}$.
Label the image D.
- Translate triangle T by $\begin{pmatrix} -2 \\ -5 \end{pmatrix}$.
Label the image E.



2 Look at the diagram.



Describe the translation that maps

- a** shape A on to shape B
- b** shape A on to shape C
- c** shape C on to shape B
- d** shape C on to shape A.

Vector notation

You have already met directed numbers, these are numbers which go in either a positive or a negative direction. Vectors, however, can go in any direction. They have two parts, a **magnitude** (length) and a **direction**. They are very useful for the study of motion.

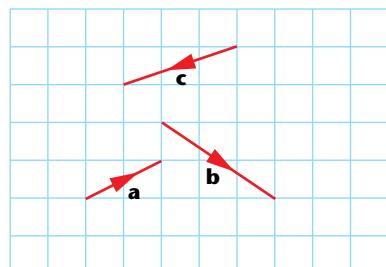
Vectors are usually written in one of three ways.

- The most common notation for a vector is a column vector, for example $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, meaning 3 in the positive x -direction and 2 in the positive y -direction.
- These are similar to coordinates, except written in a column, and they can start anywhere (not just at the origin).
- \overrightarrow{AB} meaning the vector starts at A and finishes at B .
- **a**. This is the algebraic form of a vector.

Column vectors

If a vector is drawn on a grid then it can be described by a column vector $\begin{pmatrix} x \\ y \end{pmatrix}$, where x is the length across to the right and y is the length upwards.

In the diagram $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$.

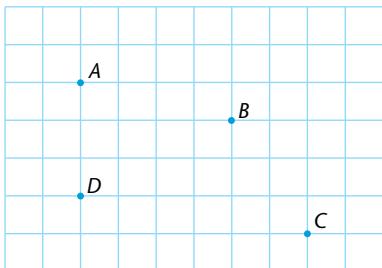


Example 37.3

Question

Write down these column vectors.

- a The vector that maps point A to point B.
- b The vector \overrightarrow{CD} .
- c The vector \overrightarrow{DC} .



Note

Students often make an error of 1 when working out the values for the vector.

Take care with the counting.

Solution

a $\overrightarrow{AB} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

b $\overrightarrow{CD} = \begin{pmatrix} -6 \\ 1 \end{pmatrix}$

c $\overrightarrow{DC} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$

Example 37.4

Question

Find the column vector that maps $(-1, 6)$ on to $(5, 2)$.

Solution

The x-coordinate has changed from -1 to 5 so the increase is 6 .

The y-coordinate has changed from 6 to 2 so the decrease is 4 .

The vector is $\begin{pmatrix} 6 \\ -4 \end{pmatrix}$.

Note

As you can see, it is not necessary to plot the points to do this question. However, you may prefer to do so.

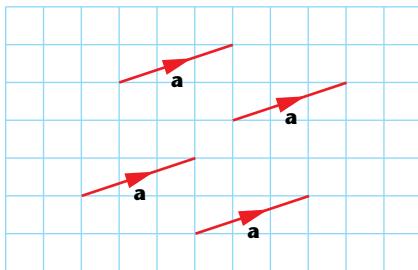
General vectors

A vector has both length and direction but can be in any position. The vector going from A to B can be labelled \overrightarrow{AB} or it can be given a letter \mathbf{a} , in bold type.

When you write a vector, you must indicate that the quantity is a vector in some definite way. You can use an arrow, for example, \overrightarrow{AB} to show the vector that goes from A to B or you can use underlining, for example, $\underline{\mathbf{a}}$, to show a vector that in print is bold, \mathbf{a} .

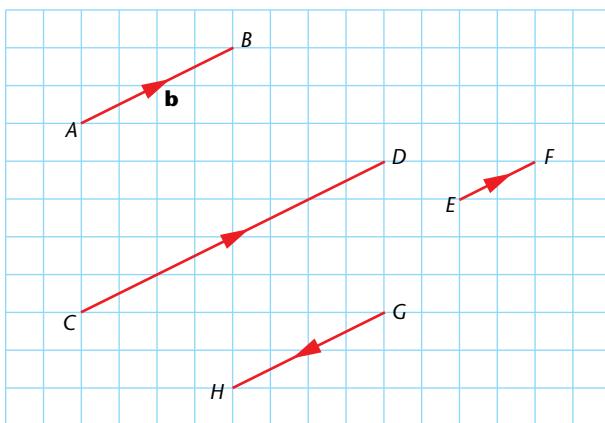


All the four lines drawn below are of equal length and go in the same direction and they can all be called \mathbf{a} .



Look at this diagram.

$$\overrightarrow{AB} = \mathbf{b}.$$



The line CD is parallel to AB and twice as long so $\overrightarrow{CD} = 2\mathbf{b}$.

EF is parallel to AB and half the length so $\overrightarrow{EF} = \frac{1}{2}\mathbf{b}$.

GH is parallel and equal in length to AB but in the opposite direction so $\overrightarrow{GH} = -\mathbf{b}$.

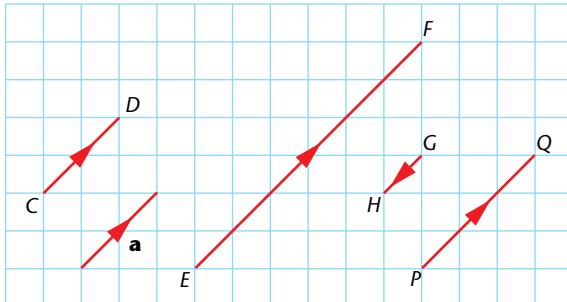
Example 37.5

Question

For the diagram, write down the vectors \overrightarrow{CD} , \overrightarrow{EF} , \overrightarrow{GH} and \overrightarrow{PQ} in terms of \mathbf{a} .

Solution

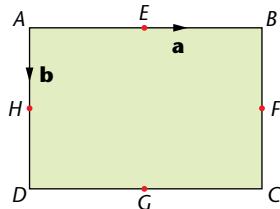
$$\overrightarrow{CD} = \mathbf{a}, \quad \overrightarrow{EF} = 3\mathbf{a}, \quad \overrightarrow{GH} = -\frac{1}{2}\mathbf{a}, \quad \overrightarrow{PQ} = \frac{3}{2}\mathbf{a}$$



Example 37.6**Question**

$ABCD$ is a rectangle and E, F, G, H are the midpoints of the sides.

$\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$.



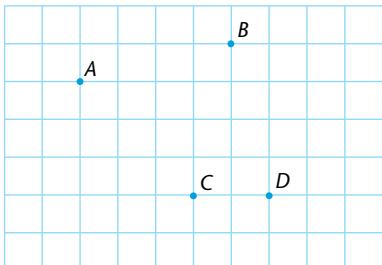
Write the vectors \overrightarrow{BC} , \overrightarrow{CD} , \overrightarrow{AE} , \overrightarrow{AH} , \overrightarrow{EG} , \overrightarrow{CF} and \overrightarrow{FH} in terms of \mathbf{a} or \mathbf{b} .

Solution

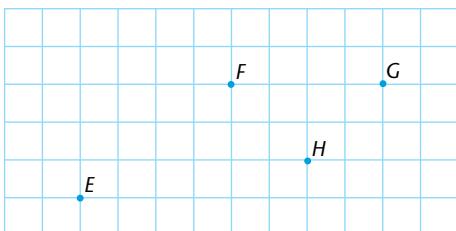
$$\overrightarrow{BC} = \mathbf{b}, \quad \overrightarrow{CD} = -\mathbf{a}, \quad \overrightarrow{AE} = \frac{1}{2}\mathbf{a}, \quad \overrightarrow{AH} = \frac{1}{2}\mathbf{b}, \quad \overrightarrow{EG} = \mathbf{b}, \quad \overrightarrow{CF} = -\frac{1}{2}\mathbf{b}, \quad \overrightarrow{FH} = -\mathbf{a}$$

Exercise 37.2

- 1 Write down the column vectors for \overrightarrow{AB} , \overrightarrow{CD} , \overrightarrow{CB} , \overrightarrow{AD} and \overrightarrow{CA} .



- 2 Write down the column vectors for \overrightarrow{EF} , \overrightarrow{GH} , \overrightarrow{EH} , \overrightarrow{GF} and \overrightarrow{FH} .



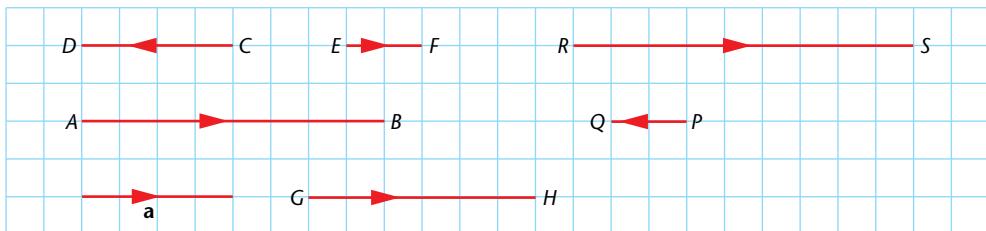
- 3 Find the column vector that describes these translations.

- | | | |
|---------------------------|-----------------------------|----------------------------|
| a (1, 2) to (1, 4) | b (2, 3) to (-2, 3) | c (1, 0) to (-1, 3) |
| d (4, 2) to (5, 9) | e (-3, 2) to (5, -4) | f (6, 1) to (0, 5) |

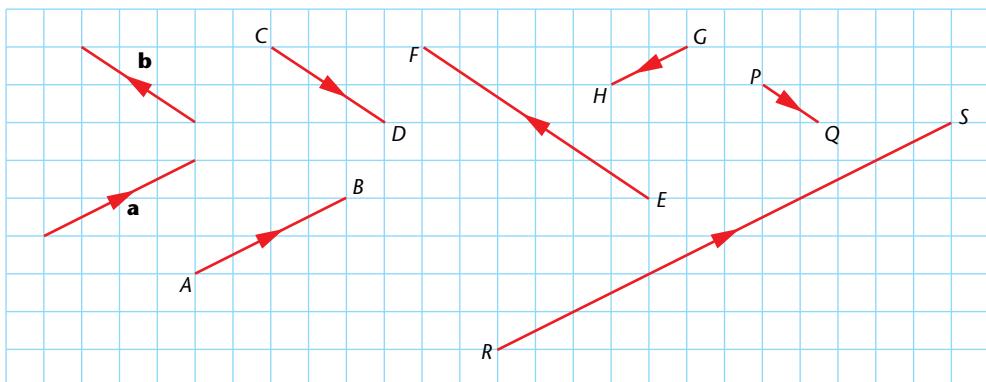
4 Copy and complete this table.

	Original point	Vector	New point
a	(1, 2)	$\begin{pmatrix} 3 \\ 2 \end{pmatrix}$	
b	(2, 3)	$\begin{pmatrix} 4 \\ 1 \end{pmatrix}$	
c	(1, 0)	$\begin{pmatrix} -3 \\ 2 \end{pmatrix}$	
d	(4, 2)	$\begin{pmatrix} 0 \\ -3 \end{pmatrix}$	
e	(-3, 2)	$\begin{pmatrix} -5 \\ -2 \end{pmatrix}$	
f	(6, 1)	$\begin{pmatrix} -6 \\ -1 \end{pmatrix}$	

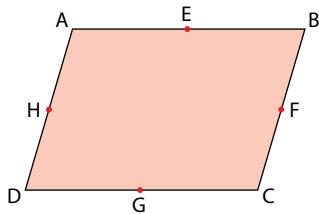
5 For the diagram below, write down the vectors \overrightarrow{AB} , \overrightarrow{CD} , \overrightarrow{EF} , \overrightarrow{GH} , \overrightarrow{PQ} and \overrightarrow{RS} in terms of \mathbf{a} .



6 For the diagram below, write down the vectors \overrightarrow{AB} , \overrightarrow{CD} , \overrightarrow{EF} , \overrightarrow{GH} , \overrightarrow{PQ} and \overrightarrow{RS} in terms of \mathbf{a} or \mathbf{b} .



- 7 $ABCD$ is a parallelogram. E, F, G, H are the midpoints of the sides.



$$\overrightarrow{AE} = \mathbf{a} \text{ and } \overrightarrow{AH} = \mathbf{b}.$$

Write down the vectors for $\overrightarrow{AB}, \overrightarrow{CD}, \overrightarrow{EB}, \overrightarrow{GD}, \overrightarrow{HF}$, and \overrightarrow{FC} in terms of \mathbf{a} or \mathbf{b} .

- 8 $ABCD$ is a square. E, F, G, H are the midpoints of sides AB, BC, CD, DA , respectively.

$$\overrightarrow{AB} = \mathbf{a} \text{ and } \overrightarrow{AD} = \mathbf{b}.$$

Write down the vectors $\overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{EB}, \overrightarrow{GD}, \overrightarrow{HF}$, and \overrightarrow{FC} in terms of \mathbf{a} or \mathbf{b} .

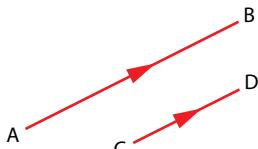
Multiplying a vector by a scalar

A quantity that has magnitude but not direction is called a **scalar**.

Multiplying a vector by a scalar produces a vector in the same direction but longer by a factor equal to the scalar.

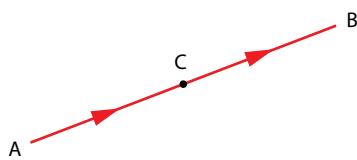
If you know that $\overrightarrow{AB} = k\overrightarrow{CD}$, you can conclude that

- \overrightarrow{AB} is parallel to \overrightarrow{CD}
- \overrightarrow{AB} is k times the length of \overrightarrow{CD} .



If you know that $\overrightarrow{AB} = k\overrightarrow{AC}$ and there is a common point A , you can conclude that

- A, B and C are in a straight line
- \overrightarrow{AB} is k times the length of \overrightarrow{AC} .



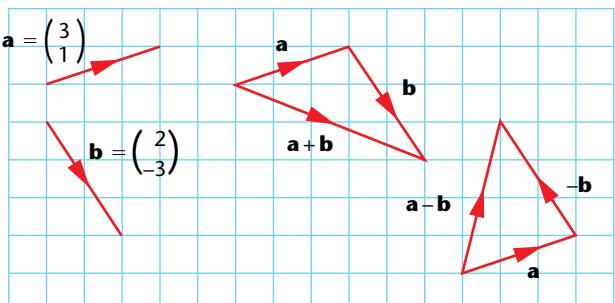
Addition and subtraction of column vectors

In the diagram, $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

You can see that

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

$$\text{and } \mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$



So, to add or subtract column vectors, you add or subtract the components.

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix} \text{ and } \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a-c \\ b-d \end{pmatrix}$$

Example 37.7

Question

Given $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, work out these.

a $2\mathbf{a}$

b $\mathbf{a} + 2\mathbf{b}$

c $\mathbf{a} - \mathbf{b} + \mathbf{c}$

d $2\mathbf{a} + \mathbf{b} - \mathbf{c}$

e $\frac{1}{2}\mathbf{a}$

Solution

a $2\mathbf{a} = 2 \times \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

b $\mathbf{a} + 2\mathbf{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 2 \times \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$

c $\mathbf{a} - \mathbf{b} + \mathbf{c} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

d $2\mathbf{a} + \mathbf{b} - \mathbf{c} = 2 \times \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$

Note

When adding and subtracting column vectors, be very careful with the signs as most errors are made in that way.

Exercise 37.3

1 Work out these.

a $2 \times \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

b $\begin{pmatrix} 6 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

c $\frac{1}{2} \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

d $\begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

e $\begin{pmatrix} 3 \\ 4 \end{pmatrix} + 2 \times \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

2 Work out these.

a $2 \times \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

b $\begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

c $\frac{1}{2} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

d $\begin{pmatrix} 2 \\ -1 \end{pmatrix} + 2 \times \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

e $\frac{1}{2} \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

3 Work out these.

a $3 \times \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

b $\begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 8 \end{pmatrix}$

c $\frac{1}{2} \begin{pmatrix} 8 \\ 10 \end{pmatrix}$

d $2 \times \begin{pmatrix} 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

e $2 \times \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 5 \times \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

4 Work out these.

a $2 \times \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

b $\begin{pmatrix} 1 \\ 6 \end{pmatrix} - \begin{pmatrix} 7 \\ 3 \end{pmatrix}$

c $\frac{1}{2} \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

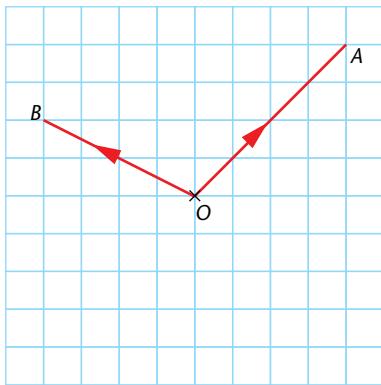
d $\begin{pmatrix} 1 \\ -4 \end{pmatrix} - 2 \times \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

e $\frac{1}{2} \begin{pmatrix} 2 \\ 6 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 3 \\ -5 \end{pmatrix}$

Position vectors

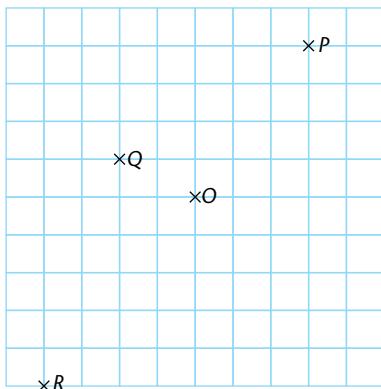
The vector \overrightarrow{OA} is the **position vector** of A in relation to O .

Similarly \overrightarrow{OB} is the position vector of B in relation to O .



Exercise 37.4

1 Find as column vectors the position vectors relative to O of P , Q and R .



2 A is the point $(-2, 1)$, B is the point $(4, 3)$ and C is the point $(7, 4)$.

a What are the position vectors \overrightarrow{OA} , \overrightarrow{OB} and \overrightarrow{OC} ?

b Work out these column vectors.

i \overrightarrow{AB}

ii \overrightarrow{BC}

c What can you say about A , B and C ?

- 3 A is the point (2, 1), B is the point (4, 4), C is the point (7, 4) and D is the point (3, -2).
- What are the position vectors \overrightarrow{OA} , \overrightarrow{OB} and \overrightarrow{OC} ?
 - Work out these column vectors.
 i \overrightarrow{AB} ii \overrightarrow{CD}
 - What can you say about AB and CD ?

The magnitude of a vector

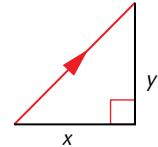
The magnitude of a vector is its length.

The magnitude of the vector \overrightarrow{AB} is written as $|\overrightarrow{AB}|$.

The magnitude of the vector \mathbf{a} is written as $|\mathbf{a}|$.

Pythagoras' Theorem can be used to calculate the magnitude of a vector.

The **magnitude** of the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is $\sqrt{x^2 + y^2}$.



Example 37.8

Question

$$\overrightarrow{CD} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

Find $|\overrightarrow{CD}|$.

Solution

$$\begin{aligned} |\overrightarrow{CD}| &= \sqrt{5^2 + (-2)^2} \\ &= \sqrt{25 + 4} \\ &= \sqrt{29} \\ &= 5.39 \text{ to 3 significant figures} \end{aligned}$$

Exercise 37.5

- Given that $\mathbf{a} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$, work out these.

a $2\mathbf{a}$	b $-\mathbf{a}$	c $4\mathbf{a}$
d $\frac{1}{2}\mathbf{a}$	e $-\frac{1}{3}\mathbf{a}$	f $ \mathbf{a} $
- Given that $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, work out these.

a $3\mathbf{a}$	b $\mathbf{a} + \mathbf{b}$	c $\mathbf{b} - \mathbf{a}$
d $2\mathbf{a} + \mathbf{b}$	e $3\mathbf{a} - 2\mathbf{b}$	f $ \mathbf{a} $
g $ \mathbf{b} $		
- Given that $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$, work out these.

a $3\mathbf{c}$	b $4\mathbf{c} - 2\mathbf{b}$	c $\mathbf{a} - \mathbf{b} + \mathbf{c}$
d $2\mathbf{a} + 3\mathbf{b} + 2\mathbf{c}$	e $\frac{1}{2}\mathbf{a} - \mathbf{b} - \frac{1}{2}\mathbf{c}$	f $ \mathbf{b} $
g $ \mathbf{c} $		

4 Given that $\mathbf{p} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$, work out these.

a $4\mathbf{p}$

b $-2\mathbf{p}$

c $\frac{1}{2}\mathbf{p}$

d $9\mathbf{p}$

e $0.4\mathbf{p}$

f $|\mathbf{p}|$

5 Given that $\mathbf{p} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, work out these.

a $2\mathbf{p}$

b $\mathbf{p} + \mathbf{q}$

c $\mathbf{q} - \mathbf{p}$

d $2\mathbf{p} + \mathbf{q}$

e $3\mathbf{q} - 2\mathbf{p}$

f $|\mathbf{p} + \mathbf{q}|$

6 Given that $\mathbf{a} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$, work out these.

a $3\mathbf{c}$

b $3\mathbf{c} + 2\mathbf{b}$

c $\mathbf{a} - \mathbf{b} + \mathbf{c}$

d $\mathbf{a} + 4\mathbf{b} - 2\mathbf{c}$

e $\frac{1}{2}\mathbf{a} + \mathbf{b} - \frac{1}{2}\mathbf{c}$

f $|\mathbf{a} - \mathbf{c}|$

Vector geometry

In the diagram, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $OACB$ is a parallelogram.

AC is parallel and equal to OB so $\overrightarrow{AC} = \mathbf{b}$.

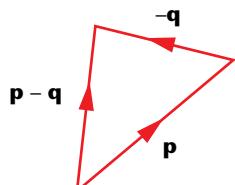
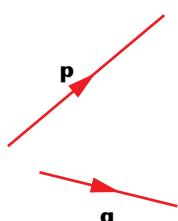
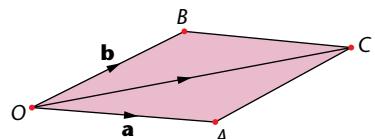
$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \mathbf{a} + \mathbf{b}.$$

You can also see in the diagram that $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = \mathbf{b} + \mathbf{a}$.

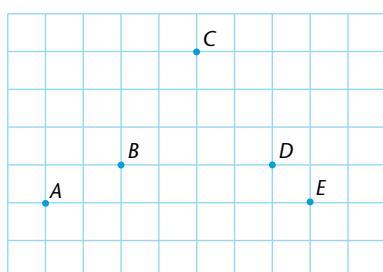
This shows that $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.

The vectors can be added in either order.

To subtract vectors you use the fact that $\mathbf{p} - \mathbf{q} = \mathbf{p} + (-\mathbf{q})$.



There are different routes you can use to get from A to E on this diagram.



For instance $\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CE} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$.

Whichever route you use, adding the column vectors gives the same result,

$$\overrightarrow{AE} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$$

The fact that the result is the same is a very important rule which, together with your knowledge of how to multiply a vector by a scalar, is used to find vectors in geometrical diagrams.

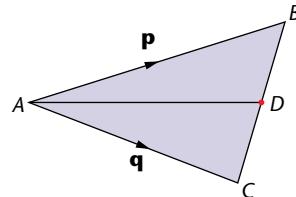
Example 37.9

Question

In the triangle ABC , $\overrightarrow{AB} = \mathbf{p}$, $\overrightarrow{AC} = \mathbf{q}$ and D is the midpoint of BC .

Write these vectors in terms of \mathbf{p} and \mathbf{q} .

- a \overrightarrow{BC} b \overrightarrow{BD} c \overrightarrow{AD}



Solution

$$\begin{aligned} \text{a } \overrightarrow{BC} &= \overrightarrow{BA} + \overrightarrow{AC} & \text{c } \overrightarrow{AD} &= \overrightarrow{AB} + \overrightarrow{BD} \\ &= -\mathbf{p} + \mathbf{q} & &= \mathbf{p} + \frac{1}{2}(\mathbf{q} - \mathbf{p}) \\ &= \mathbf{q} - \mathbf{p} & &= \mathbf{p} + \frac{1}{2}\mathbf{q} - \frac{1}{2}\mathbf{p} \\ \text{b } \overrightarrow{BD} &= \frac{1}{2}\overrightarrow{BC} & &= \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q} \\ &= \frac{1}{2}(\mathbf{q} - \mathbf{p}) & &= \frac{1}{2}(\mathbf{p} + \mathbf{q}) \end{aligned}$$

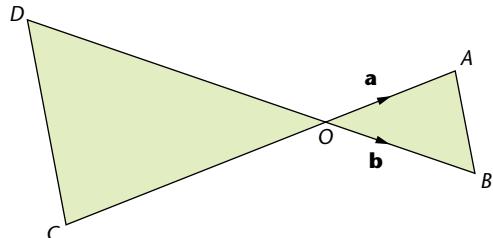
Example 37.10

Question

In this diagram, $OC = 2 \times OA$ and $OD = 2 \times OB$.

$\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

- a Write these vectors in terms of \mathbf{a} and \mathbf{b} .
- i \overrightarrow{OC}
 - ii \overrightarrow{OD}
 - iii \overrightarrow{AB}
 - iv \overrightarrow{DC}
- b What does this show about the lines AB and DC ?



Solution

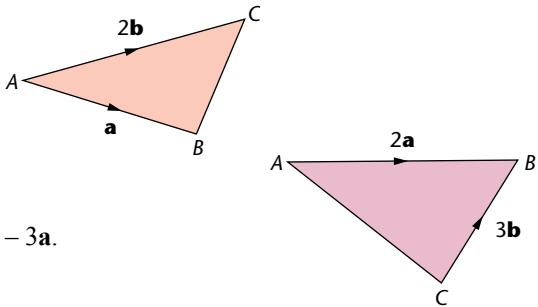
- a i OC is on the same line as OA , in the opposite direction and twice as long.
 $\overrightarrow{OC} = -2 \times \overrightarrow{OA} = -2\mathbf{a}$
- ii By the same reasoning as above, $\overrightarrow{OD} = -2 \times \overrightarrow{OB} = -2\mathbf{b}$.
- iii $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$
- iv $\overrightarrow{DC} = \overrightarrow{DO} + \overrightarrow{OC} = 2\mathbf{b} - 2\mathbf{a} = 2(\mathbf{b} - \mathbf{a})$

- b The vector for DC is twice the vector for AB .

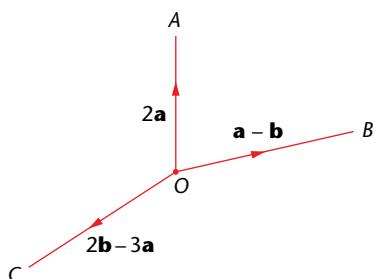
So AB and DC are parallel and DC is twice as long as AB .

Exercise 37.6

- 1** In the triangle, $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AC} = 2\mathbf{b}$.
Find the vector \overrightarrow{BC} in terms of \mathbf{a} and \mathbf{b} .
- 2** In the triangle ABC , $\overrightarrow{AB} = 2\mathbf{a}$ and $\overrightarrow{CB} = 3\mathbf{b}$.
Work out the vector \overrightarrow{AC} .

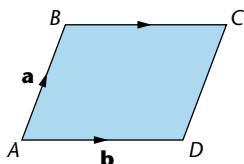


- 3** In this diagram, $\overrightarrow{OA} = 2\mathbf{a}$, $\overrightarrow{OB} = \mathbf{a} - \mathbf{b}$, and $\overrightarrow{OC} = 2\mathbf{b} - 3\mathbf{a}$.



Write these vectors in terms of \mathbf{a} and/or \mathbf{b} , as simply as possible.

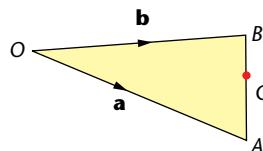
- a** \overrightarrow{AB} **b** \overrightarrow{BC} **c** \overrightarrow{AC}
- 4** ABCD is a parallelogram. $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$.



Work out the vectors \overrightarrow{BC} , \overrightarrow{CD} , \overrightarrow{BD} and \overrightarrow{AC} in terms of \mathbf{a} and/or \mathbf{b} .

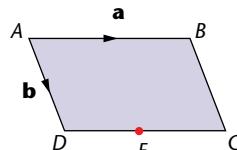
- 5** In the triangle OAB , C is a point on AB so that $AC = 2 \times CB$.

$\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.
Work out the vectors \overrightarrow{AB} , \overrightarrow{CB} and \overrightarrow{OC} in terms of \mathbf{a} and/or \mathbf{b} .



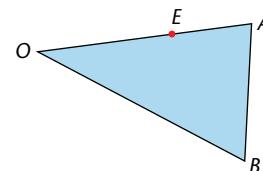
- 6** ABCD is a parallelogram.
 E is the midpoint of the DC line.

$\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$.
Write down the vector \overrightarrow{EB} in terms of \mathbf{a} and/or \mathbf{b} .



- 7** OAB is a triangle with E a point on OA so that $OE = 2 \times EA$.
 $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

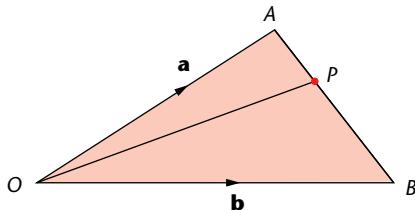
Work out the vector \overrightarrow{EB} in terms of \mathbf{a} and/or \mathbf{b} .



- 8** $ABCDEF$ is a regular hexagon.
 O is the centre of the hexagon.
 $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.
 Find each of these vectors in terms of \mathbf{a} and/or \mathbf{b} .

a \overrightarrow{FA} b \overrightarrow{BD} c \overrightarrow{AB} d \overrightarrow{AC}

- 9** In the diagram, P is one-third of the way along AB .
 The position vectors of A and B in relation to O are \mathbf{a} and \mathbf{b} , respectively.

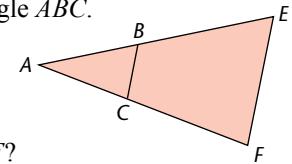


Find each of these vectors, as simply as possible, in terms of \mathbf{a} and \mathbf{b} .

a \overrightarrow{AB} b \overrightarrow{AP} c \overrightarrow{OP}

- 10** Triangle AEF is a 3 times enlargement of triangle ABC .

$\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{b}$.

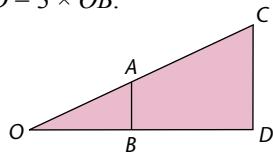


a Write down the vectors \overrightarrow{AE} , \overrightarrow{AF} , \overrightarrow{BC} and \overrightarrow{EF} in terms of \mathbf{a} and/or \mathbf{b} .

- b What do the vectors show about BC and EF ?

- 11** In the triangle OCD , $AC = 3 \times OA$ and $BD = 3 \times OB$.

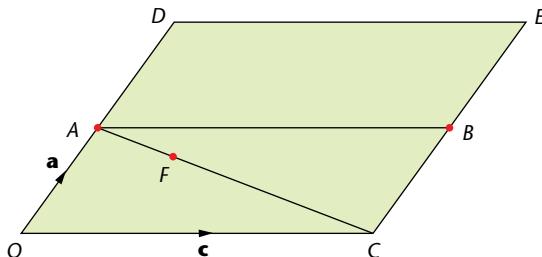
$\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.



- a Use vectors to show that AB is parallel to CD .
 b What is the ratio of the lengths of AB and CD ?

- 12** In the diagram, $\overrightarrow{OA} = \overrightarrow{AD} = \overrightarrow{CB} = \overrightarrow{BE} = \mathbf{a}$ and $\overrightarrow{OC} = \overrightarrow{AB} = \overrightarrow{DE} = \mathbf{c}$.

F is one third of the way along AC .



- a Find each of these vectors in terms of \mathbf{a} and \mathbf{c} . Simplify your answers where possible.

i \overrightarrow{OE} ii \overrightarrow{AC} iii \overrightarrow{OF}

- b What two facts can you conclude about the points O , F and E ?

38 MATRICES

Displaying information as a matrix

A **matrix** is a way to display information.

The information is displayed in a rectangular array as in matrix **A**.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Matrix **A** is a 2×3 matrix because it has 2 rows and 3 columns.

The **order**, or size, of matrix **A** is 2×3 .

Each individual piece of information in the matrix is an **element**.

Matrix **A** has six elements.

The **zero matrix** is a matrix in which all the elements are zero.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
 is the 2×2 zero matrix.

A shopkeeper sells bags of apples and bananas. The bags are small and large.

One day he sells 8 small bags of apples, 5 large bags of apples, 15 small bags of bananas and 9 large bags of bananas.

This information can be displayed as the 2×2 matrix below.

$$\begin{pmatrix} 8 & 5 \\ 15 & 9 \end{pmatrix}$$

Multiplying a matrix by a scalar quantity

A **scalar** is a number.

When a matrix is multiplied by a scalar quantity, each element in the matrix is multiplied by that number.

The shopkeeper sells the same number of bags of fruit for two days.

This calculation can be used to calculate the total number of bags sold.

$$2 \begin{pmatrix} 8 & 5 \\ 15 & 9 \end{pmatrix} = \begin{pmatrix} 16 & 10 \\ 30 & 18 \end{pmatrix}$$

The elements of the matrix represent the number of each type of bag sold over the two days.

The matrix shows that the shopkeeper sold 16 small bags of apples, 10 large bags of apples, 30 small bags of bananas and 18 large bags of bananas.

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- display information in the form of a matrix of any order
- solve problems involving the calculation of the sum and product (where appropriate) of two matrices, and interpret the results
- calculate the product of a matrix and a scalar quantity
- use the algebra of 2×2 matrices including the zero and identity 2×2 matrices
- calculate the determinant $|A|$ and inverse A^{-1} of a non-singular matrix A .

CHECK YOU CAN:

- calculate with negative numbers.

Note

The plural of *matrix* is *matrices*.

Example 38.1**Question**

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Work out $3\mathbf{A}$.

Solution

$$3\mathbf{A} = 3 \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 3 \times 1 & 3 \times 2 & 3 \times 3 \\ 3 \times 4 & 3 \times 5 & 3 \times 6 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{pmatrix}$$

When a matrix is multiplied by a scalar the result is a matrix of the same order as the original matrix.

 **Exercise 38.1**

- 1** Write down the order of each of these matrices.

a $\begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$

b $\begin{pmatrix} -1 & 6 \\ 8 & -2 \end{pmatrix}$

c $\begin{pmatrix} 5 & 3 \\ 1 & 0 \\ -1 & 2 \end{pmatrix}$

d $(4 \ 3)$

- 2** Work out these.

a $4 \begin{pmatrix} 2 & 6 \\ 0 & 1 \\ 3 & 2 \end{pmatrix}$

b $2 \begin{pmatrix} 3 & 4 \\ -1 & 7 \end{pmatrix}$

c $3 \begin{pmatrix} 5 & -2 & 0 \\ 3 & 1 & -1 \end{pmatrix}$

d $5 \begin{pmatrix} 7 & -3 \\ 0 & -1 \end{pmatrix}$

- 3** A baker makes small and large loaves of bread.

Every day he makes 15 small white loaves, 10 small brown loaves, 20 large white loaves and 25 large brown loaves.

This information is displayed in the matrix \mathbf{L} .

$$\mathbf{L} = \begin{pmatrix} 15 & 10 \\ 20 & 25 \end{pmatrix}$$

- a** Work out $7\mathbf{L}$.

- b** Explain what the numbers in your answer to part **a** represent.

Adding and subtracting matrices

Matrices can only be added or subtracted if they are of the same order.

When two matrices are added, the corresponding elements in each matrix are added.

When two matrices are subtracted, the corresponding elements in each matrix are subtracted.

The result is a matrix of the same order as the original matrices.

Example 38.2

Question

Work out these.

a $\begin{pmatrix} 4 & 1 \\ -2 & 0 \end{pmatrix} + \begin{pmatrix} -3 & 2 \\ -1 & 5 \end{pmatrix}$

b $\begin{pmatrix} 3 & 0 & 1 \\ -1 & 4 & 2 \end{pmatrix} - \begin{pmatrix} -2 & 1 & 4 \\ -3 & 2 & 1 \end{pmatrix}$

c $2\begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ 2 & 4 \end{pmatrix}$

Solution

a $\begin{pmatrix} 4 & 1 \\ -2 & 0 \end{pmatrix} + \begin{pmatrix} -3 & 2 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 4 + -3 & 1 + 2 \\ -2 + -1 & 0 + 5 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -3 & 5 \end{pmatrix}$

b $\begin{pmatrix} 3 & 0 & 1 \\ -1 & 4 & 2 \end{pmatrix} - \begin{pmatrix} -2 & 1 & 4 \\ -3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 - -2 & 0 - 1 & 1 - 4 \\ -1 - -3 & 4 - 2 & 2 - 1 \end{pmatrix}$
 $= \begin{pmatrix} 5 & -1 & -3 \\ 2 & 2 & 1 \end{pmatrix}$

c $2\begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 2 & 6 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 6 + 0 & 4 + -2 \\ 2 + 2 & 6 + 4 \end{pmatrix}$
 $= \begin{pmatrix} 6 & 2 \\ 4 & 10 \end{pmatrix}$

Note

If a calculation involves multiplying by a scalar quantity as well as addition or subtraction, do the multiplication first, then add or subtract.

Exercise 38.2

1 Work out these.

a $\begin{pmatrix} 3 & 5 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 4 & 7 \\ 1 & 5 \end{pmatrix}$

b $\begin{pmatrix} 1 & 8 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} -4 & 6 \\ -2 & 5 \end{pmatrix}$

c $\begin{pmatrix} 10 & 6 \\ 4 & -1 \end{pmatrix} + \begin{pmatrix} -3 & 9 \\ -8 & 6 \end{pmatrix}$

d $\begin{pmatrix} -7 & 0 \\ 6 & -3 \end{pmatrix} + \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}$

2 Work out these.

a $\begin{pmatrix} 4 & 7 \\ 9 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 6 \\ 8 & 1 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 0 \\ -2 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$

c $\begin{pmatrix} -7 & 6 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} -5 & 3 \\ -2 & 6 \end{pmatrix}$ **d** $\begin{pmatrix} 0 & -5 \\ 10 & -7 \end{pmatrix} - \begin{pmatrix} 3 & -5 \\ 4 & -3 \end{pmatrix}$

3 $\mathbf{A} = \begin{pmatrix} 3 & 0 & -1 \\ 4 & -2 & 5 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 6 & 2 & 0 \\ -5 & 7 & 8 \end{pmatrix}$

Work out these.

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{A} - \mathbf{B}$ **c** $3\mathbf{A}$ **d** $2\mathbf{B}$
e $2\mathbf{A} - \mathbf{B}$ **f** $\mathbf{B} - 3\mathbf{A}$ **g** $2\mathbf{A} + 2\mathbf{B}$ **h** $3\mathbf{B} - 2\mathbf{A}$

4 A library records the number of visitors it has in one day.

Matrix **M** shows the visitors in the morning.

Matrix **N** shows the visitors in the afternoon.

	male	female	
M	children	$\begin{pmatrix} 25 & 40 \end{pmatrix}$	
	adults	$\begin{pmatrix} 59 & 65 \end{pmatrix}$	
			male
			female
			N
	children	$\begin{pmatrix} 38 & 52 \end{pmatrix}$	
	adults	$\begin{pmatrix} 75 & 68 \end{pmatrix}$	

- a** **i** How many female adults visited the library in the morning?
ii How many male children visited the library in the afternoon?
- b** **i** Work out $\mathbf{M} + \mathbf{N}$.
ii What do the elements of $\mathbf{M} + \mathbf{N}$ represent?
- c** **i** Work out $\mathbf{N} - \mathbf{M}$.
ii What do the elements of $\mathbf{N} - \mathbf{M}$ represent?

Multiplying two matrices

Two matrices can be multiplied together only if the number of columns in the first matrix is the same as the number of rows in the second matrix.

The elements in the rows of the first matrix are multiplied by the elements in the columns of the second matrix.

Multiplying an $m \times n$ matrix by an $n \times p$ matrix results in an $m \times p$ matrix.

So, for example, multiplying a 3×2 matrix by a 2×1 matrix will result in a 3×1 matrix.

The method for multiplying two matrices is shown in the following example.

Example 38.3**Question**

Work out $\begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 4 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 3 & 4 \\ 0 & 1 \end{pmatrix}$.

Solution

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 4 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 3 & 4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \times 2 + 3 \times 3 + 2 \times 0 & \dots \\ \dots & \dots \end{pmatrix}$$

$$= \begin{pmatrix} 11 & \dots \\ \dots & \dots \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 4 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 3 & 4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \times 2 + 2 \times 3 + 4 \times 0 & \dots \end{pmatrix}$$

$$= \begin{pmatrix} 11 & \dots \\ 6 & \dots \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 4 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 3 & 4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 1 \times 1 + 3 \times 4 + 2 \times 1 \\ 6 & 0 \times 1 + 2 \times 4 + 4 \times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 15 \\ 6 & 12 \end{pmatrix}$$

First multiply the elements in the first row by the elements in the first column and add the products.

Write the result in the first row of the first column in the answer matrix.

Next multiply the elements in the second row by the elements in the first column and add the products.

Write the result in the second row of the first column in the answer matrix.

Complete the answer matrix by multiplying the first row and the second column together and the second row and the second column together.

The result is a 2×2 matrix.

Reversing the order of the multiplication does not give the same result.

$$\begin{pmatrix} 2 & 1 \\ 3 & 4 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 4 \end{pmatrix}$$
 is a 3×2 matrix multiplied by a 2×3 matrix.

The result will be a 3×3 matrix.

In matrix multiplication reversing the order may give a different result or may be impossible.

Exercise 38.3

1 $P = \begin{pmatrix} 1 & 3 & 1 \end{pmatrix}$ $Q = \begin{pmatrix} 4 & 0 & 3 \\ 2 & -1 & 5 \end{pmatrix}$ $R = \begin{pmatrix} 1 & 0 \\ -3 & -2 \end{pmatrix}$ $S = \begin{pmatrix} 2 & 3 \end{pmatrix}$

State which of these multiplications are possible.

If the multiplication is possible, state the order of the result.

- a PQ
- b QP
- c RQ
- d QR
- e RS
- f SR

2 Work out these.

a $\begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$

b $\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 0 & -3 \end{pmatrix}$

c $\begin{pmatrix} 2 & -3 & 5 \\ -1 & 0 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$

d $\begin{pmatrix} 4 & 0 \\ 1 & -1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 2 & -1 \end{pmatrix}$

3 $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 4 & -3 \\ 1 & 5 \end{pmatrix}$

Work out these.

a \mathbf{AB}

b \mathbf{BA}

c \mathbf{A}^2

d \mathbf{B}^2

4 $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\mathbf{Z} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

a Work out

i \mathbf{MZ} **ii** \mathbf{ZM} .

b What do these results show you about the zero matrix?

5 A sports club has morning and afternoon classes.

The matrix, \mathbf{M} , shows the number of children attending the club one day.

a.m. p.m.

$$\mathbf{M} \quad \begin{array}{l} \text{boys} \\ \text{girls} \end{array} \quad \begin{pmatrix} 25 & 40 \\ 50 & 55 \end{pmatrix}$$

The fee for the morning class is \$5 and the fee for the afternoon class is \$4.

a Represent the fees in a 2×1 matrix, \mathbf{N} .

b Work out $\mathbf{P} = \mathbf{MN}$.

c Explain what the elements in \mathbf{P} represent.

The determinant and the inverse of a matrix

The inverse of a matrix \mathbf{A} is represented by \mathbf{A}^{-1} .

The product $\mathbf{A} \times \mathbf{A}^{-1}$ gives the identity matrix.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ is the } 2 \times 2 \text{ identity matrix.}$$

The first step in finding the inverse of a matrix is to find the **determinant** of the matrix.

$|\mathbf{A}|$ represents the determinant of \mathbf{A} .

If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the **determinant** of the matrix, $|\mathbf{A}| = ad - bc$.

The inverse of matrix \mathbf{A} is then found using this formula.

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If the determinant is zero then the matrix has no inverse.

Example 38.4

Question

Given that $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$, find \mathbf{A}^{-1} .

Solution

$$|\mathbf{A}| = (2 \times 4) - (3 \times -1) = 8 + 3 = 11 \quad \text{First find the determinant of } \mathbf{A}.$$

$$\mathbf{A}^{-1} = \frac{1}{11} \begin{pmatrix} 4 & -3 \\ 1 & 2 \end{pmatrix}$$

The inverse can be left in this form.

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{4}{11} & -\frac{3}{11} \\ \frac{1}{11} & \frac{2}{11} \end{pmatrix}$$

Alternatively, the elements of the matrix can be multiplied by the fraction.

Exercise 38.4

- 1** Find the inverse of each of these matrices.

a $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 4 & 8 \\ 2 & 5 \end{pmatrix}$ **c** $\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}$ **d** $\begin{pmatrix} 5 & 6 \\ 1 & 2 \end{pmatrix}$

e $\begin{pmatrix} 0 & -1 \\ 2 & 4 \end{pmatrix}$ **f** $\begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix}$ **g** $\begin{pmatrix} 1 & 2 \\ -3 & -5 \end{pmatrix}$ **h** $\begin{pmatrix} -4 & 1 \\ -3 & 2 \end{pmatrix}$

2 $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 6 & 3 \end{pmatrix}$

- a** Find $|\mathbf{A}|$.

- b** Explain how you know that \mathbf{A} has no inverse.

3 $\mathbf{B} = \begin{pmatrix} -6 & 3 \\ 4 & -1 \end{pmatrix}$

- a** Find \mathbf{B}^{-1} .

- b** Work out

i $\mathbf{B}\mathbf{B}^{-1}$

ii $\mathbf{B}^{-1}\mathbf{B}$.

- c** Given that $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, work out

i \mathbf{BI}

ii \mathbf{IB} .

Matrix algebra

When solving problems involving matrices you need to remember these facts.

- \mathbf{AB} means matrix \mathbf{A} multiplied by matrix \mathbf{B} .
- $\mathbf{AB} \neq \mathbf{BA}$.
- $\mathbf{AA}^{-1} = \mathbf{I}$, where \mathbf{I} is the identity matrix.
- Multiplying an $m \times n$ matrix by an $n \times p$ matrix results in an $m \times p$ matrix.

Example 38.5

Question

$$\mathbf{M} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 9 \end{pmatrix}$$

Find the matrix \mathbf{M} .

Solution

$m \times n$ multiplied by a 2×2 matrix gives a 1×2 matrix.

So \mathbf{M} is a 1×2 matrix.

$$\text{Let } \mathbf{M} = \begin{pmatrix} a & b \end{pmatrix}.$$

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 9 \end{pmatrix}$$

This can be solved using simultaneous equations or by using inverse matrices.

Using simultaneous equations

$$\begin{pmatrix} 2a+0b & a+2b \end{pmatrix} = \begin{pmatrix} 8 & 9 \end{pmatrix}$$

Multiplying the matrices on the left-hand side leads to simultaneous equations.

$$2a = 8$$

$$a = 4$$

$$a + 2b = 9$$

$$2b = 5$$

$$b = 2.5$$

$$\mathbf{M} = \begin{pmatrix} 4 & 2.5 \end{pmatrix}$$

The first equation can be solved to find a .

Substitute into the second equation to find b .

The solutions give the elements of \mathbf{M} .

Using inverse matrices

determinant = $(2 \times 2) - (1 \times 0) = 4 - 0 = 4$ First find the determinant of the second matrix.

$$\text{inverse} = \frac{1}{4} \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix}$$

Then find the inverse of the second matrix.

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 9 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix}$$

Multiply both sides of the equation by the inverse of the second matrix.

$$\begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} 4 & 2.5 \end{pmatrix}$$

The result is the matrix **M**.

Exercise 38.5

1 $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 3 & -1 \end{pmatrix}$

Find **B** when

- a** $2\mathbf{I} = \mathbf{B}$ **b** $\mathbf{I} + \mathbf{B} = \mathbf{A}$ **c** $\mathbf{B} - 2\mathbf{I} = \mathbf{A}$
d $\mathbf{IB} = \mathbf{A}$ **e** $\mathbf{AB} = \mathbf{I}$.

2 $\mathbf{P} = \begin{pmatrix} 2 & -4 \\ 5 & 0 \end{pmatrix}$ $\mathbf{Q} = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}$

Find matrix **Y**, when

- a** $\mathbf{P} - \mathbf{Y} = \mathbf{Q}$
b $\mathbf{Y} = \mathbf{PQ}$.

3 $\mathbf{X} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 5 \end{pmatrix}$

Find matrix **X**.

4 $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \mathbf{Y} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

Find matrix **Y**.

5 $\mathbf{Z} \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 5 & 6 \end{pmatrix}$

Find matrix **Z**.

39 TRANSFORMATIONS

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- use the following transformations of the plane: reflection (M), rotation (R), translation (T), enlargement (E) and their combinations
- identify and give precise descriptions of transformations connecting given figures
- describe transformations using coordinates and matrices.

CHECK YOU CAN:

- plot points in all four quadrants
- draw and recognise simple straight lines on a graph such as $x = k$, $y = k$ and $y = \pm x$
- use a protractor
- multiply matrices.

The language of transformations

When a shape is transformed, the original shape is called the **object** and the new shape is called the **image**.

A transformation **maps** the object on to the image.

If the vertices of an object are labelled A, B, C, \dots , the corresponding vertices of the image can be labelled $A', B', C' \dots$

Anything that stays the same when a transformation is performed is **invariant**.

Reflection

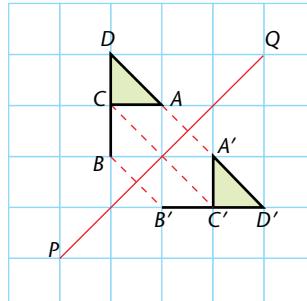
Reflections are carried out in **mirror lines**.

In a reflection corresponding points are the same distance from the mirror line but on the opposite side.

The object and the image are **congruent** but the image is reversed.

Points on the mirror lines are invariant points under a reflection.

In this diagram the shape has been reflected in the line PQ .

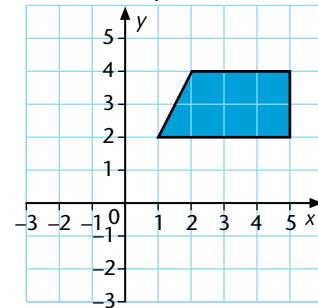


Instead of counting squares you could trace the shape and the mirror line, then turn the tracing paper over and line up the mirror line and your tracing of it.

Example 39.1

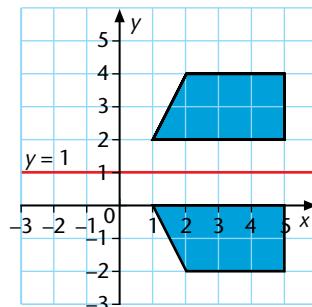
Question

Reflect the trapezium in the line $y = 1$. The diagram shows the reflection.



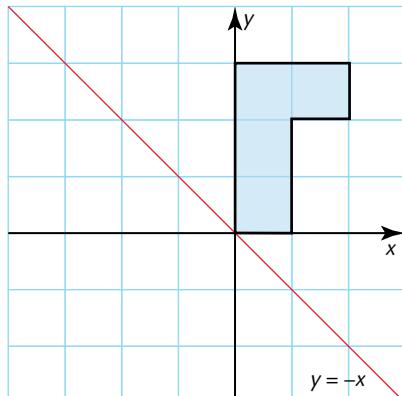
Solution

Instead of drawing the trapezium and its reflection, we can draw the line of reflection and then draw the trapezium below it. The trapezium is reflected across the line $y = 1$ to form a second trapezium with vertices at (1, 0), (4, 0), (4, 2), and (1, 2).

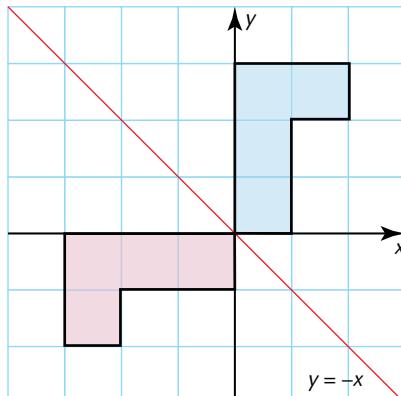


Example 39.2**Question**

Reflect this shape in the line $y = -x$.

**Solution**

The diagram shows the reflection.

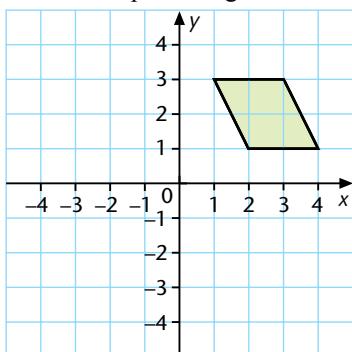
**Note**

When you have drawn a reflection in a sloping line, check it by turning the page so the mirror line is vertical. Then you can easily see if it has been reflected correctly.

Exercise 39.1

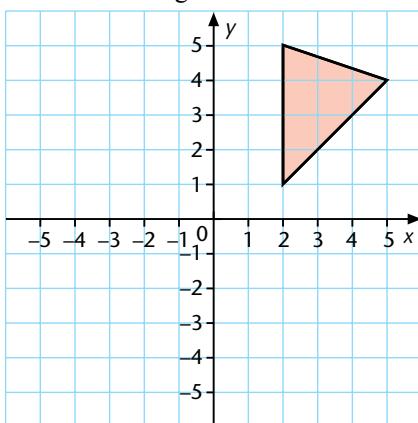
In each of questions 1 to 6, copy the diagram on to squared paper first.

- 1 Reflect the parallelogram in the y axis.

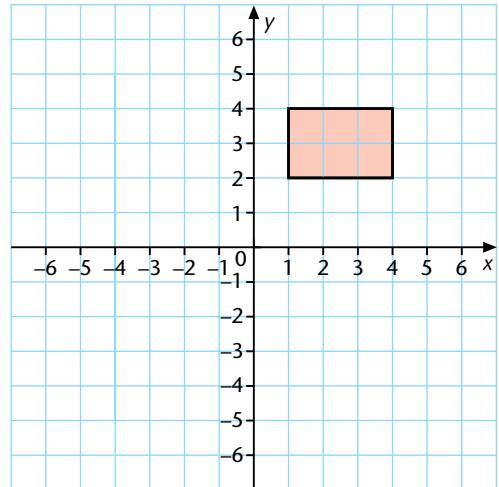
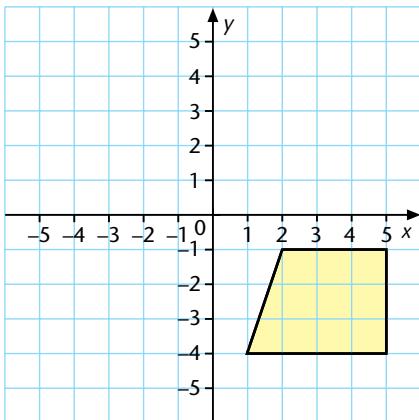
**Note**

You can count the squares or use tracing paper to help you draw the image.

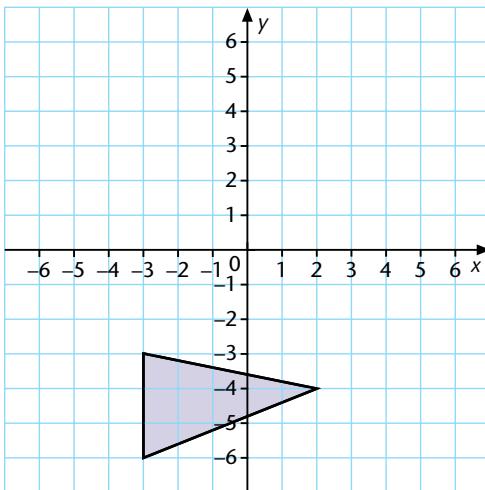
- 2 Reflect the triangle in the x axis.



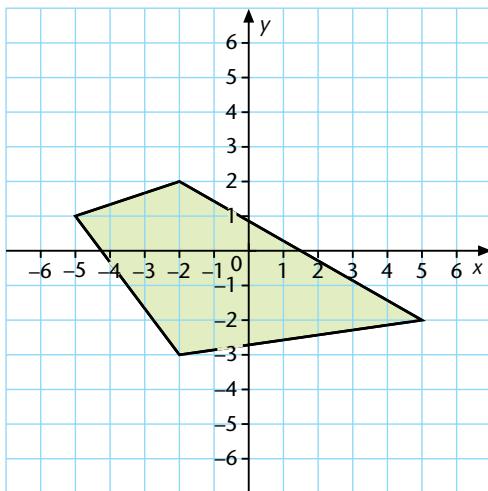
- 3** Reflect the trapezium in the x axis. **4** Reflect the rectangle in the line $x = -1$.



- 5** Reflect the triangle in the line $y = -2$.



- 6** Reflect the quadrilateral in the line $y = 2$.



- 7 Draw a pair of axes and label them -4 to 4 for x and y .
- Draw a triangle with vertices at $(1, 0)$, $(1, -2)$ and $(2, -2)$. Label it A.
 - Reflect triangle A in the line $y = 1$. Label it B.
 - Reflect triangle B in the line $y = x$. Label it C.
- 8 Draw a pair of axes and label them -4 to 4 for x and y .
- Draw a triangle with vertices at $(1, 1)$, $(2, 3)$ and $(3, 3)$. Label it A.
 - Reflect triangle A in the line $y = 2$. Label it B.
 - Reflect triangle A in the line $y = -x$. Label it C.

Rotation

In a **rotation** you turn an object about a point.

The point is called the **centre of rotation**.

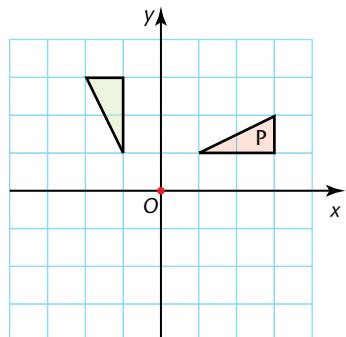
When an object is rotated, the object and its image are congruent.

Unlike reflections, when an object is rotated, the object and image are the same way round.

In this diagram, shape P has been rotated 90° anticlockwise about O .

To rotate a shape you need to know three things

- the angle of rotation
- the direction of rotation
- the centre of rotation.



Example 39.3

Question

Rotate this shape 270° anticlockwise about O .

Solution

You can think of 270° anticlockwise as the same as 90° clockwise.

You can draw the rotation by counting squares.

A is 2 squares above O so its image, A' , is 2 squares to the right.

B is 3 squares above O so its image, B' , is 3 squares to the right.

C is 2 squares above and 2 to the right of O so its image, C' , is 2 to the right and 2 below O .

D is 3 squares above and 2 to the right of O so its image, D' , is 3 to the right and 2 below O .

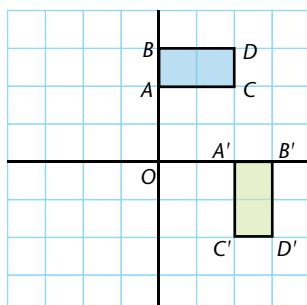
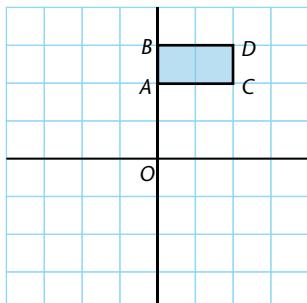
Alternatively, you could do the rotation using tracing paper.

Trace the shape.

Hold the centre of rotation still with the point of a pin or a pencil and rotate the paper through a quarter-turn clockwise.

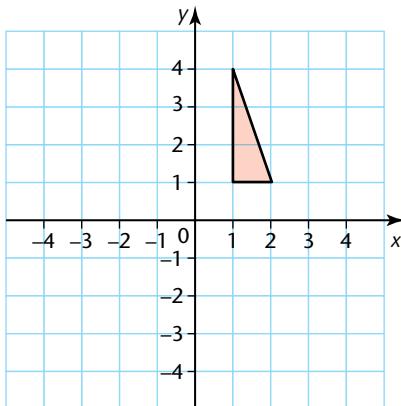
Use another pin or the point of your compasses to prick through the corners.

Join the pin holes to form the image.



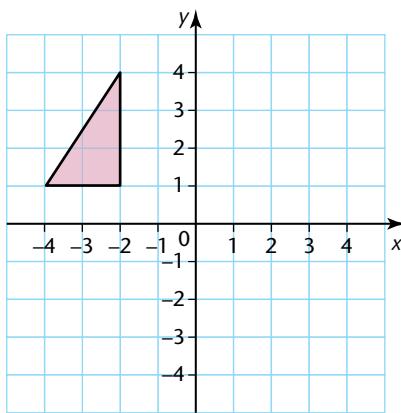
Exercise 39.2

- 1** Make three copies of this diagram and answer each part of the question on a separate diagram.



- a** Rotate the shape through 180° about the origin.
- b** Rotate the shape through 90° clockwise about the origin.
- c** Rotate the shape through 90° anticlockwise about the origin.

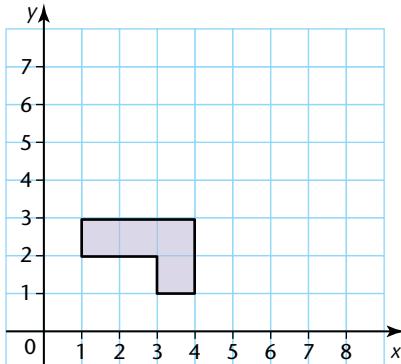
- 2** Make three copies of this diagram and answer each part of the question on a separate diagram.



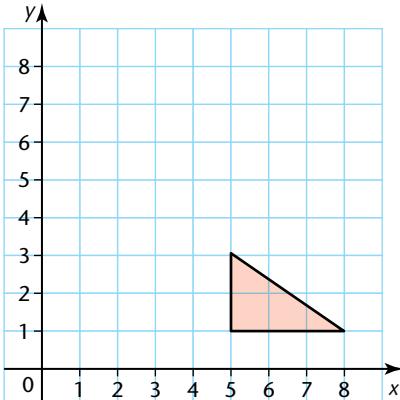
- a** Rotate the shape through 90° anticlockwise about the origin.
- b** Rotate the shape through 90° clockwise about the origin.
- c** Rotate the shape through 180° about the origin.

In each of questions **3** to **5**, copy the diagram on to squared paper first.

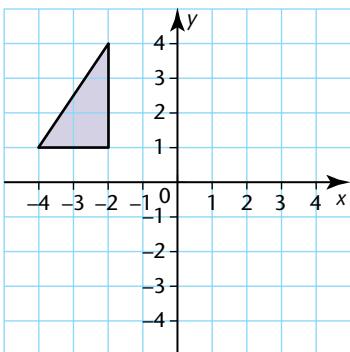
- 3** Rotate the shape through 180° about the point $(4, 3)$.



- 4 Rotate the shape through 180° about the point $(5, 3)$.



- 5 Rotate the shape through 90° clockwise about the point $(-1, -2)$.



Translation

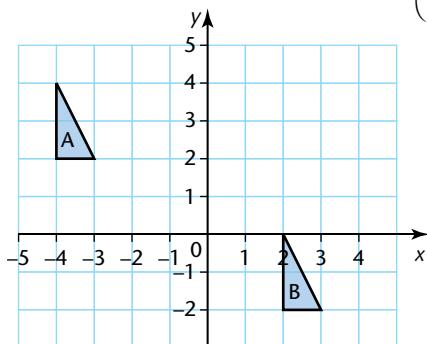
In a **translation**, every point on the object moves the same distance in the same direction as every other point.

The object and the image are **congruent**.

In this diagram triangle A maps on to triangle B.

Each vertex of the triangle moves 6 squares to the right and 4 squares down.

The translation is given by the vector $\begin{pmatrix} 6 \\ -4 \end{pmatrix}$.



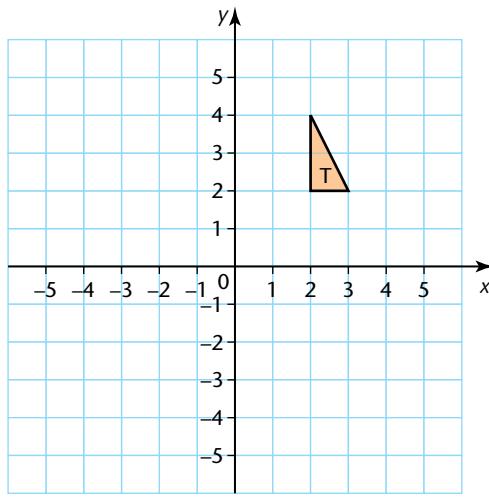
Note

Take care with the counting.

Choose a point on both the object and the image and count the units from one to the other.

Exercise 39.3

In each of questions 1 to 3, copy the diagram on to squared paper first.

1

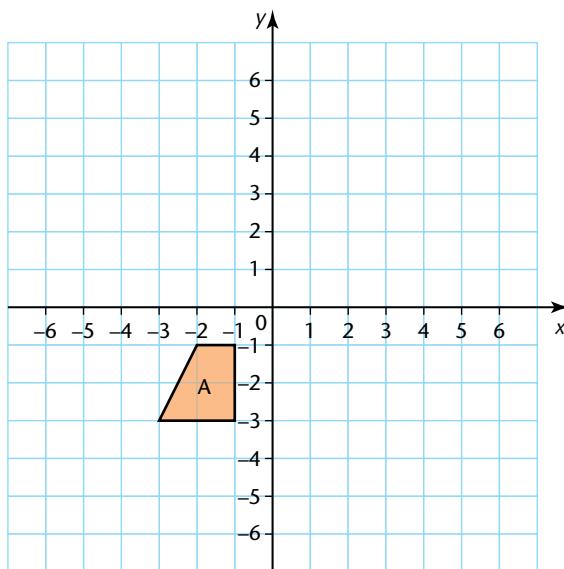
- a Translate triangle T by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
Label the image A.

- b Translate triangle T by $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.
Label the image B.

- c Translate triangle T by $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$.
Label the image C.

- d Translate triangle T by $\begin{pmatrix} -6 \\ -6 \end{pmatrix}$.
Label the image D.

- e Translate triangle T by $\begin{pmatrix} -2 \\ -5 \end{pmatrix}$.
Label the image E.

2

- a Translate shape A by $\begin{pmatrix} -3 \\ 6 \end{pmatrix}$.
Label the image B.

- b Translate shape B by $\begin{pmatrix} 7 \\ -8 \end{pmatrix}$.
Label the image C.

- c Translate shape C by $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$.
Label the image D.

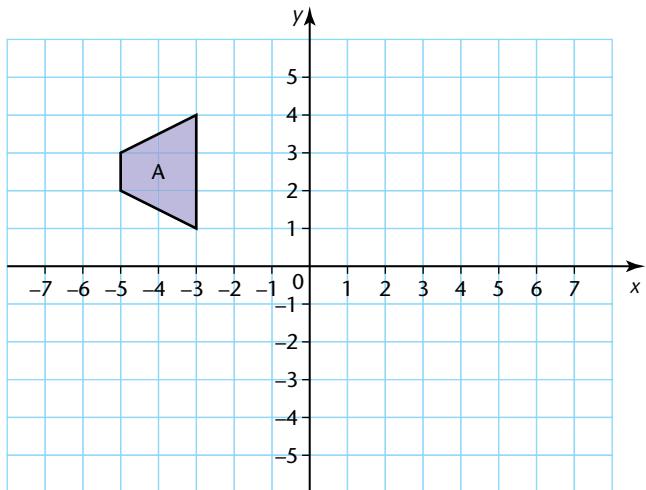
- d Translate shape D by $\begin{pmatrix} -5 \\ -3 \end{pmatrix}$.
What do you notice?

- 3 a Translate shape A by $\begin{pmatrix} -2 \\ -6 \end{pmatrix}$.
Label the image B.

- b Translate shape B by $\begin{pmatrix} 9 \\ 5 \end{pmatrix}$.
Label the image C.

- c Translate shape C by $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$.
Label the image D.

- d What must you translate D by to get back to A?



Enlargement

In an **enlargement** of a shape, the image is the same shape as the object, but it is larger or smaller.

The object and image are not congruent, because they are different sizes, but they are **similar**.

An enlargement is drawn from a given point, the **centre of enlargement**.

For a '2 times enlargement', each point on the image must be twice as far from the centre as the corresponding point on the object.

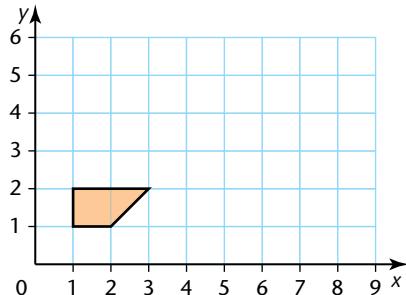
If the centre is on the object, that point will not move and the enlargement will overlap the original shape.

The number used to multiply the lengths for the enlargement is the **scale factor**.

Example 39.4

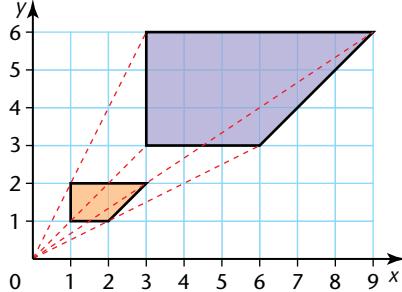
Question

Enlarge this shape by scale factor 3 with the origin as the centre of enlargement.



Solution

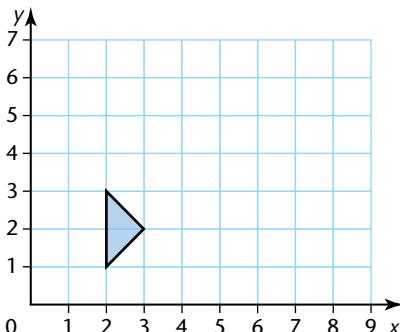
Scale factor 3 means this is a '3 times enlargement', so each point in the image is three times as far from the origin as the corresponding point in the original shape.



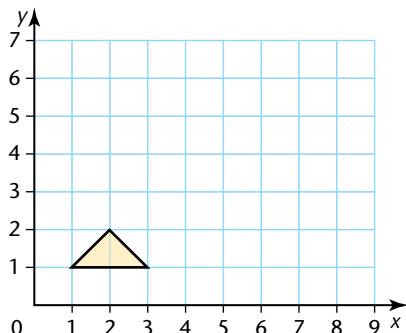
Exercise 39.4

In each of questions 1 to 3, copy the diagram on to squared paper first.

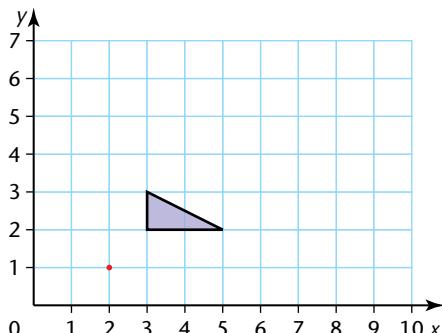
- 1 Enlarge the shape by scale factor 2, centre the origin.



- 2 Enlarge the shape by scale factor 3, centre the origin.



- 3 Enlarge the shape by scale factor 2, centre the point (2, 1).

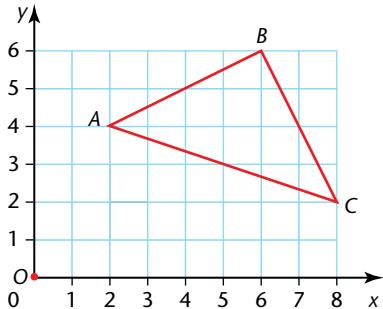


Enlargement with a fractional scale factor

The technique for drawing an enlargement with a fractional scale factor is the same as for a positive integral scale factor. You can see how this is done in the following example.

Example 39.5**Question**

Draw an enlargement of triangle ABC with centre O and scale factor $\frac{1}{2}$.

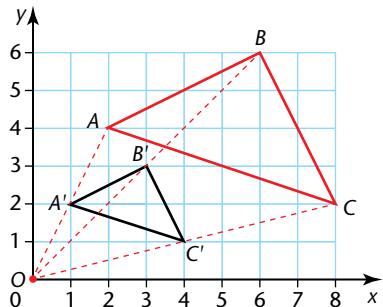
**Solution**

Draw lines from O to A , B and C .

By counting squares, mark A' on OA so that A' is the midpoint of OA .

Similarly mark B' and C' at the midpoints of OB and OC .

Join $A'B'C'$.



$A'B'C'$ is the required enlargement with scale factor $\frac{1}{2}$.

Check that the lengths of $A'B'$, $B'C'$ and $A'C'$ are $\frac{1}{2}$ the lengths of the corresponding lengths on the original triangle.

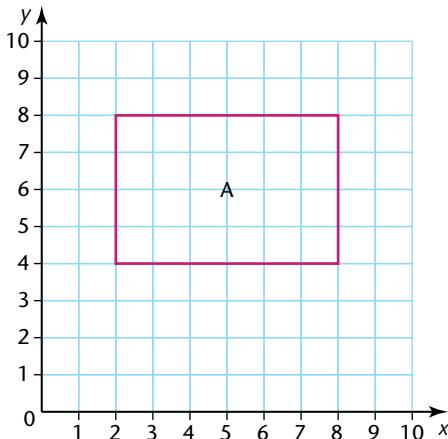
An enlargement with scale factor $\frac{1}{2}$ is the inverse of an enlargement with scale factor 2.

The transformation is still called an enlargement even though the image is smaller than the original.

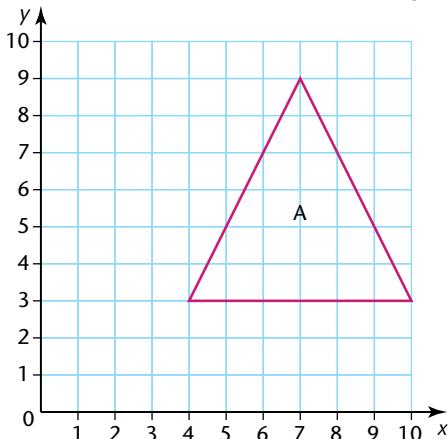
Exercise 39.5

In each of questions 1 to 4, copy the diagram on to squared paper first.

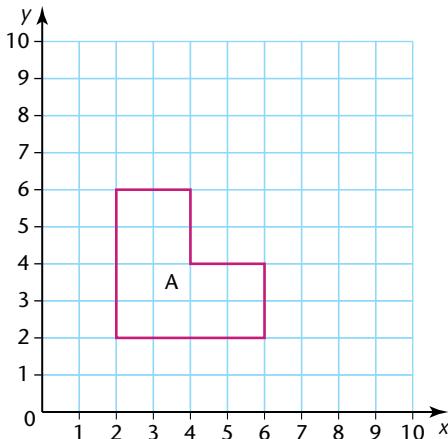
- 1 Enlarge the rectangle with scale factor $\frac{1}{2}$ and centre the origin.



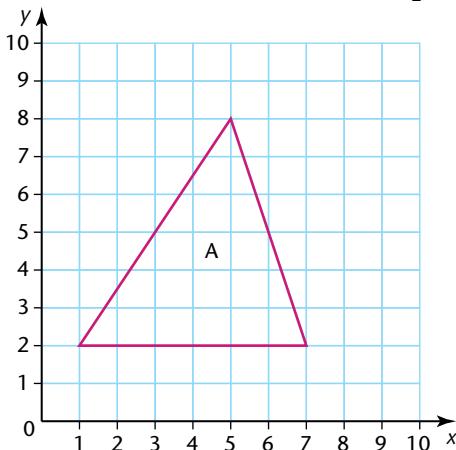
- 2 Enlarge the triangle with scale factor $\frac{1}{3}$ and centre (1, 6).



- 3 Enlarge the shape A with scale factor $1\frac{1}{2}$ and centre (0, 2).



- 4 Enlarge the shape A with scale factor $\frac{1}{2}$ and centre (5, 4).

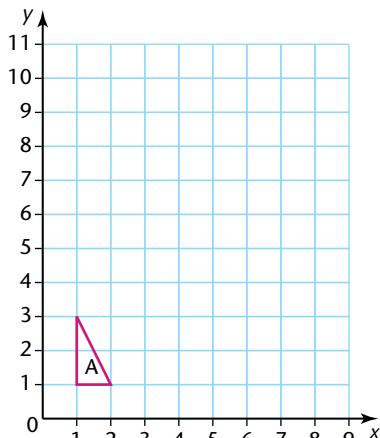


Enlargement with a negative scale factor

When the scale factor of an enlargement is negative, the image is on the opposite side of the centre of enlargement from the object and the image is inverted.

Example 39.6

Question



Enlarge triangle A with scale factor -2 and centre (3, 4).

Solution

Plot the centre of enlargement, O, at (3, 4).

Draw the line AO and extend it. Measure the length OA and mark the point A' so that $OA' = 2 \times OA$.

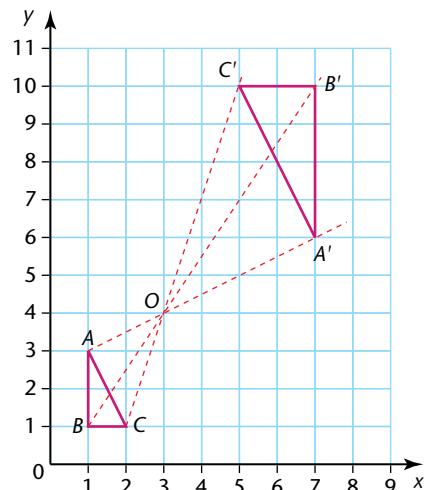
Note that A' is on the opposite side of O to A.

Or, by counting squares, O is 2 to the right and one above A, so A' is 4 to the right and 2 above O.

Do the same for BO to give point B' and CO to give point C'.

A'B'C' is the required image of ABC.

Check that the sides of triangle A'B'C' are twice as long as the corresponding sides of triangle ABC.



Note

It is easy to confuse negative and fractional scale factors.

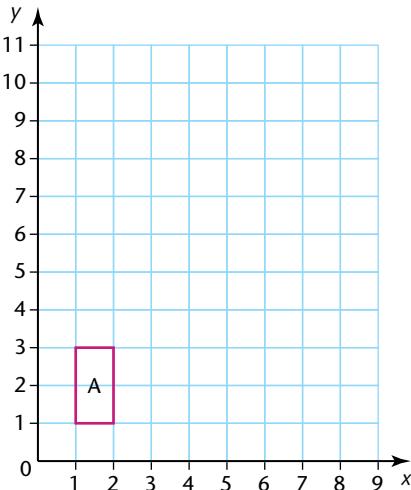
Remember that fractional scale factors make the image smaller than the object.

With negative scale factors, the image is inverted and on the opposite side of the centre of enlargement to the object.

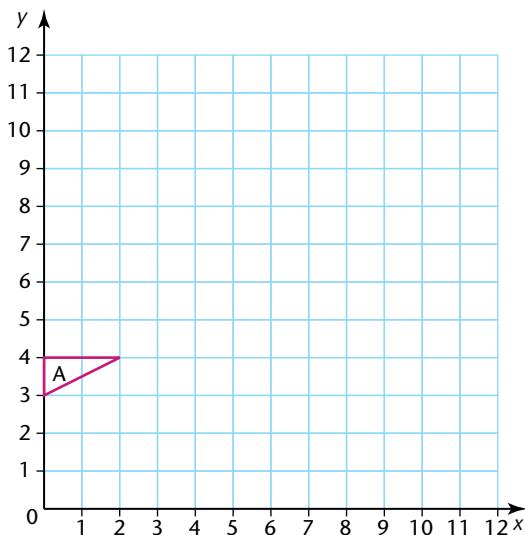
Exercise 39.6

In each of questions 1 to 4, copy the diagram on to squared paper first.

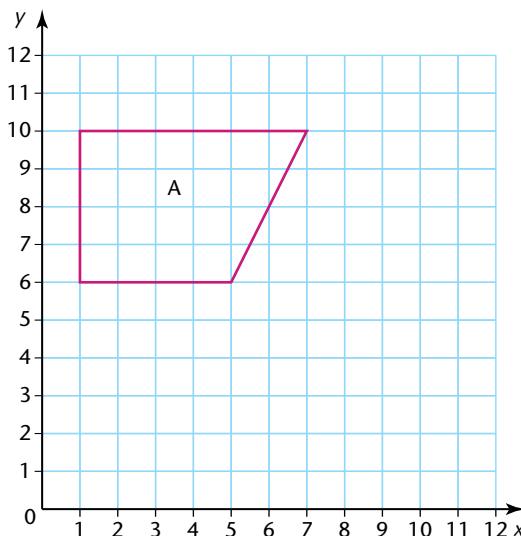
- 1 Enlarge rectangle A with scale factor -2 and centre $(3, 4)$.



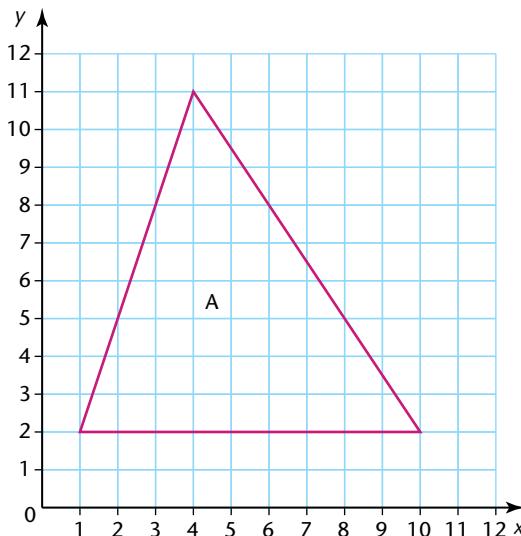
- 2 Enlarge triangle A with scale factor -3 and centre $(3, 5)$.



- 3 Enlarge shape A with scale factor $-\frac{1}{2}$ and centre (7, 6).



- 4 Enlarge triangle A with scale factor $-\frac{1}{3}$ and centre (4, 5).



Recognising and describing transformations

You may be given the object and the image and asked to describe fully the transformation.

You must first state which type of transformation it is.

Trace the object.

If you have to turn it over to fit on the image, the transformation is a reflection.

If you have to turn the tracing round but not turn it over, the transformation is a rotation.

If you just slide the tracing without turning over or turning round, the transformation is a translation.

If the object is not the same size as the image, the transformation is an enlargement.

You then need to complete the description.

The extra information required is shown in this table.

Note

If you are asked to describe fully a *single* transformation, do not give a combination of transformations.

Transformation	Extra information required
Reflection	The equation of the mirror line or a statement that it is the x axis or the y axis
Rotation	Angle Direction Centre
Translation	Column vector
Enlargement	Scale factor Centre

Reflection

Example 39.7

Question

Describe fully the single transformation that maps triangle ABC on to triangle $A'B'C'$.

Solution

You can check that the transformation is a reflection by using tracing paper.

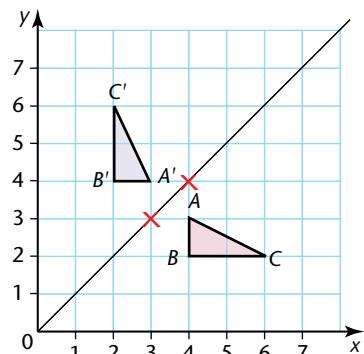
To find the mirror line, put a ruler between two corresponding points (B and B') and mark a point halfway between them, at $(3, 3)$.

Repeat this for two other corresponding points (C and C'). The midpoint is $(4, 4)$.

Join the two midpoints to find the mirror line. The mirror line is $y = x$.

The transformation is a reflection in the line $y = x$.

Again, the result can be checked using tracing paper.

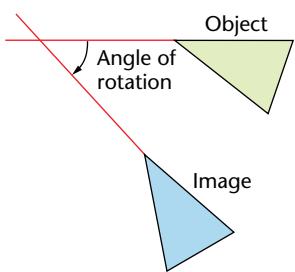


Rotation

To find the angle of rotation, find a pair of sides that correspond in the object and the image. Measure the angle between them. You may need to extend both of the sides to do this.

If the centre of rotation is not on the object, its position may not be obvious. However, you can usually find it, either by counting squares or using tracing paper.

Always remember to state whether the rotation is clockwise or anticlockwise.



Example 39.8

Question

Describe fully the single transformation that maps flag A on to flag B.

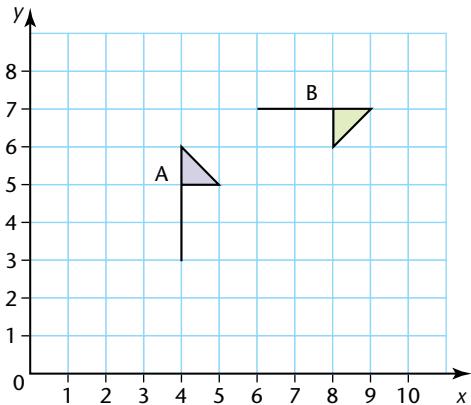
Solution

You have to turn the shape round but you don't turn it over so the transformation is a rotation.

The angle is 90° clockwise.

You may need to make a few trials, using tracing paper and a compass point centred on different points, to find that the centre of rotation is (7, 4).

So the transformation that maps flag A on to flag B is a rotation of 90° clockwise with centre of rotation (7, 4).



Translation

Example 39.9

Question

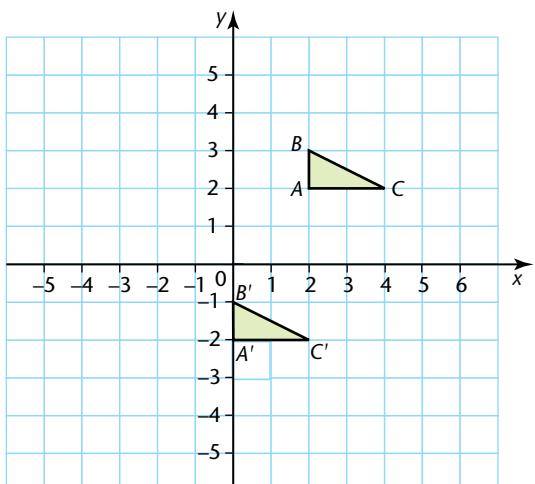
Describe fully the single transformation that maps shape ABC on to shape A'B'C'.

Solution

The shape stays the same way up and is the same size so the transformation is a translation.

A moves from (2, 2) to (0, -2). This is a movement of 2 to the left and 4 down.

The transformation is a translation by the vector $\begin{pmatrix} -2 \\ -4 \end{pmatrix}$.



Enlargement

Example 39.10

Question

Describe fully the transformation that maps

- a A on to B b B on to A.

Solution

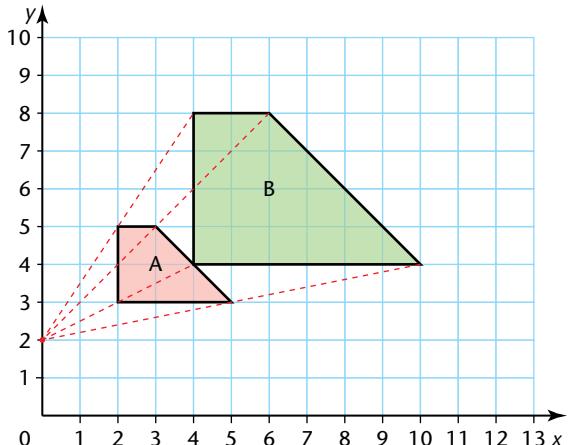
By measuring the sides you can see that shape B is twice as big as shape A.

To find the centre of enlargement, you join corresponding vertices of the two shapes and extend them until they cross.

The point where they cross is the centre of enlargement.

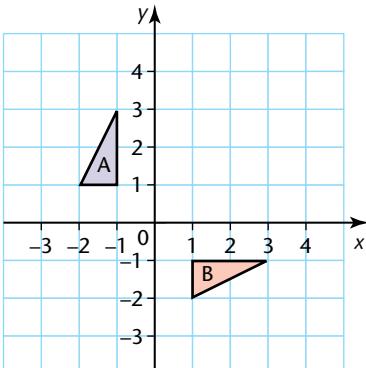
The centre of enlargement is (0, 2).

- a The transformation that maps shape A on to shape B is an enlargement, scale factor 2, centre (0, 2).
 b The transformation that maps shape B on to shape A is an enlargement, scale factor $\frac{1}{2}$, centre (0, 2).

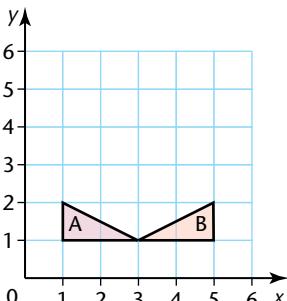


Exercise 39.7

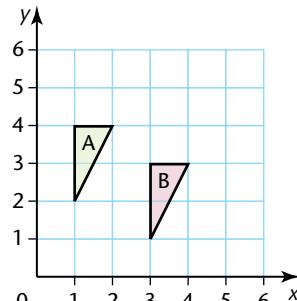
- 1 Describe fully the transformation that maps A on to B.



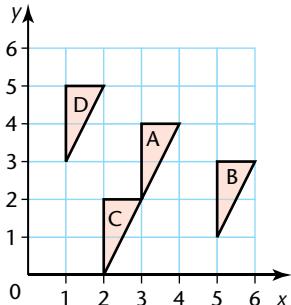
- 2 Describe fully the transformation that maps A on to B.



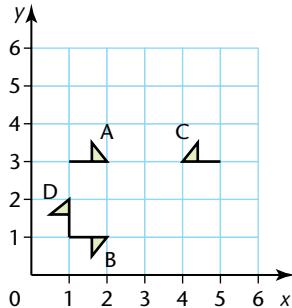
- 3 Describe fully the transformation that maps A on to B.



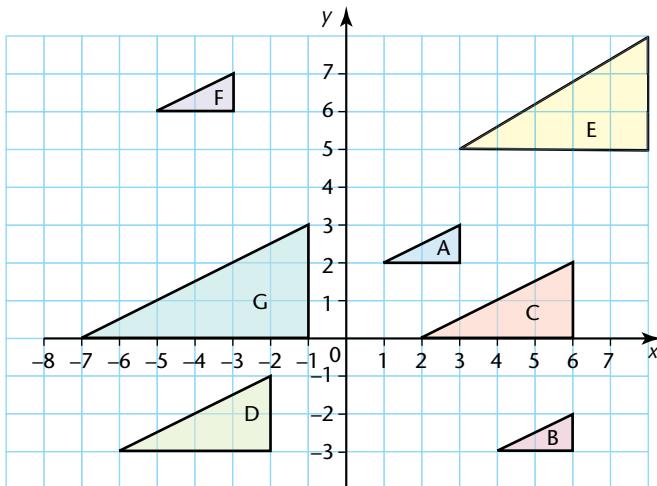
- 4** Describe fully the single transformation that maps
- triangle A on to triangle B
 - triangle A on to triangle C
 - triangle A on to triangle D
 - triangle B on to triangle D.



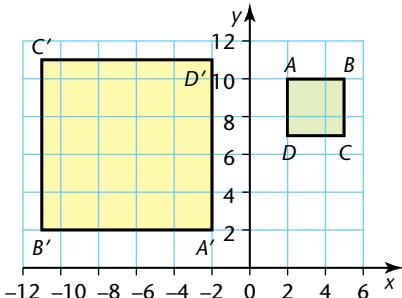
- 5** Describe fully the single transformation that maps
- flag A on to flag B
 - flag A on to flag C
 - flag B on to flag D.



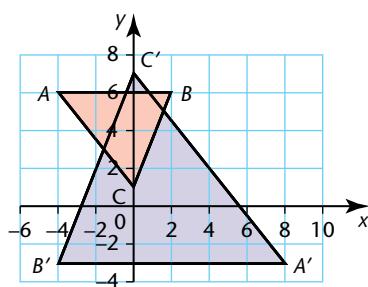
- 6** Describe fully the single transformation that maps
- triangle A on to triangle B
 - triangle A on to triangle C
 - triangle C on to triangle D
 - triangle A on to triangle E
 - triangle A on to triangle F
 - triangle G on to triangle A.



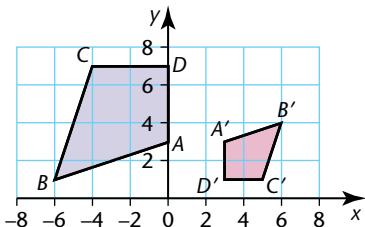
- 7** Describe fully the single transformation that maps shape ABCD on to shape A'B'C'D'.



- 8** Describe fully the single transformation that maps shape ABC on to shape A'B'C'.



- 9 Describe fully the single transformation that maps shape $ABCD$ on to shape $A'B'C'D'$.



Combining transformations

Note

It is very easy to assume PQ means P followed by Q .

It does not. It means Q followed by P .

This notation is like that used for functions.

If a transformation is followed by another transformation, the result is sometimes equivalent to a third transformation.

If shape A is transformed by transformation P , the image is $P(A)$.

If the image $P(A)$ is then transformed by transformation Q , the final image is $Q(P(A))$.

This is shortened to $QP(A)$.

So $QP(A)$ represents the result of doing P followed by Q .

So the transformation QP is the combined effect of P followed by Q .

Example 39.11

Question

Describe fully the single transformation equivalent to a rotation of 90° clockwise about the origin followed by a reflection in the y axis.

Solution

Choose a simple shape like triangle A to start with.

Rotating triangle A 90° clockwise about the origin gives triangle B .

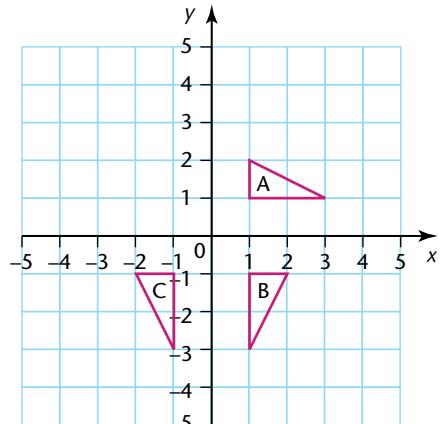
Reflecting triangle B in the y axis gives triangle C .

You need to find the single transformation that will map triangle A on to triangle C .

If you trace triangle A , you will need to turn the tracing over to fit it on to triangle C , so the transformation must be a reflection.

The mirror line is the $y = -x$ line.

So the single transformation is a reflection in the line $y = -x$.



Note

Make sure you carry out the transformations in the order stated in the question. It usually makes a difference.

If you are asked for a single transformation, do not give a combination of transformations.

Example 39.12

Question

E is the transformation: an enlargement, scale factor 3, centre (3, 3).

T is the transformation: a translation by the vector $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$.
Describe fully the single transformation TE.

Solution

Again start with a simple object like triangle A.

Enlarge triangle A with scale factor 3, centre (3, 3).

This maps triangle A on to triangle B. So $B = E(A)$.

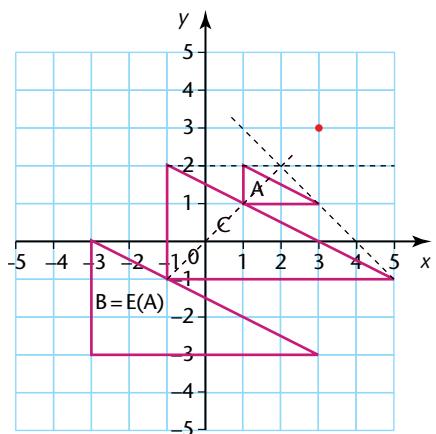
Then translate triangle B by the vector $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

This maps triangle B on to triangle C. So $C = T(E(A)) = TE(A)$.

Clearly the equivalent transformation is still an enlargement with scale factor 3. It just remains to find the centre of enlargement.

Draw lines through the corresponding vertices of triangle A and triangle C. These intersect at (2, 2).

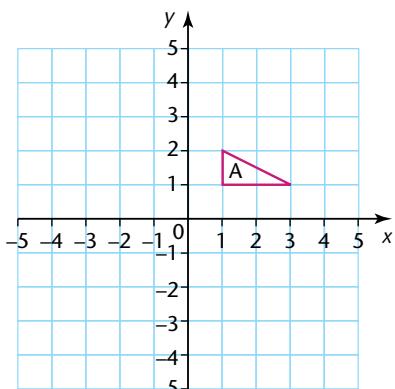
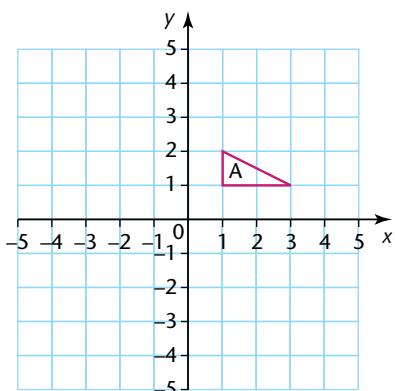
So the single transformation, TE, is an enlargement, scale factor 3, with centre (2, 2).



Exercise 39.8

- 1 a Rotate triangle A 180° about the origin.
Label the image B.
- b Reflect triangle B in the x axis.
Label the image C.
- c Describe fully the single transformation equivalent to a rotation of 180° about the origin followed by a reflection in the x axis.

- 2 a Translate triangle A by the vector $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$.
Label the image B.
- b Rotate triangle B 90° clockwise about the point $(-2, 3)$.
Label the image C.
- c Describe fully the single transformation equivalent to a translation by the vector $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ followed by a rotation, 90° clockwise, about the point $(-2, 3)$.



In questions 3 to 10, carry out the transformations on a simple shape of your choice.

- 3 Find the single transformation equivalent to a reflection in the y axis followed by a rotation of 90° clockwise about the origin.
- 4 Describe fully the single transformation equivalent to a reflection in the line $x = 3$ followed by a reflection in the line $y = -2$.
- 5 Describe fully the single transformation equivalent to a reflection in $y = 3$ followed by a reflection in $y = -1$.
- 6 R is a rotation of 90° anticlockwise about the point $(2, 3)$.
M is a reflection in the line $x = 2$.
Describe fully the single transformation equivalent to MR.
- 7 E is an enlargement with scale factor 2, centre the origin.
R is a rotation of 180° about $(0, 3)$.
Describe fully the single transformation equivalent to RE.
- 8 M_1 is a reflection in $y = x + 1$.
 M_2 is a reflection in $y = 2$.
 - a Describe the single transformation equivalent M_2M_1 .
 - b Describe the single transformation equivalent to M_1M_2 .
- 9 M is a reflection in $y = -x$.
R is a rotation of 90° anticlockwise about $(1, -1)$.
Describe fully the single transformation that is equivalent to RM.
- 10 T is a translation by the vector $\begin{pmatrix} -3 \\ 3 \end{pmatrix}$.
M is a reflection in the line $y = x$.
Describe fully the single transformation that is equivalent to MT.

Matrices of transformations

P is the point $(5, 3)$.

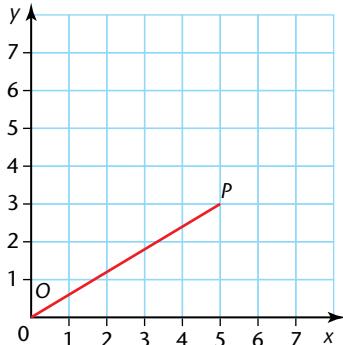
The column vector $OP = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ is called the **position vector** of P .

Some transformations can be represented by a 2×2 matrix.

Multiplying the position vector by the matrix gives the position vector of the image point.

$$\begin{pmatrix} 2 & 3 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 19 \\ -6 \end{pmatrix}$$

So the matrix $\begin{pmatrix} 2 & 3 \\ 0 & -2 \end{pmatrix}$ has transformed the point $(5, 3)$ to the point $(19, -6)$.



Example 39.13**Question**

- a Transform each of the points A , B and C in the

diagram using the matrix $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

- b Plot the image points A' , B' and C' .

- c What do you notice?

Solution

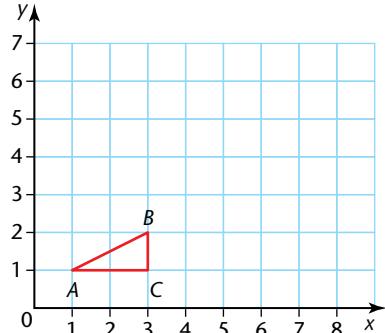
- a Multiply the position vectors of the points A , B and C

by the matrix $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

$$M(A) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$M(B) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

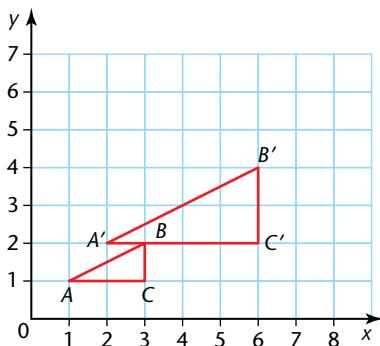
$$M(C) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

**Note**

The calculations in part a can be written in a shorter way.

$$\begin{pmatrix} A & B & C \\ 2 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} A' & B' & C' \\ 2 & 6 & 6 \\ 2 & 4 & 2 \end{pmatrix}$$

- b



- c The triangle ABC has been enlarged with scale factor 2 and centre the origin.

So the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ represents an enlargement with scale factor 2 and centre the origin.

Exercise 39.9

- 1 In each of questions 1 to 3, copy the diagram to squared paper first.

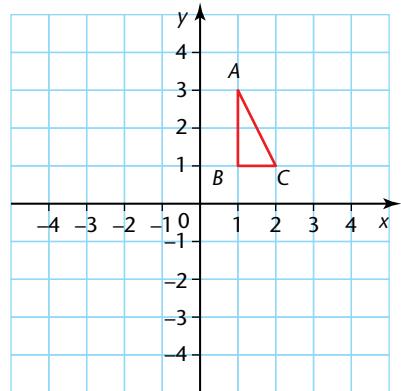
- a Transform each of the points A , B and C in the diagram using the

matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

- b Plot the image points A' , B' and C' .

- c Describe fully the transformation represented by

the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

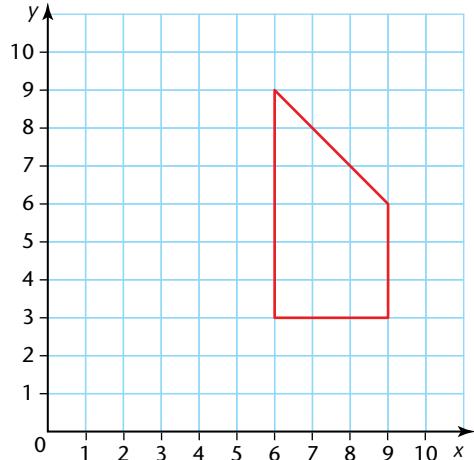


- 2 a** Transform the triangle in question 1 using the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
- b** Describe fully the transformation represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

- 3 a** Transform the trapezium

using the matrix $\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$.

- b** Describe fully the transformation represented by the matrix $\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$.



Combined transformations using matrices

If $\mathbf{M} = \begin{pmatrix} 2 & 3 \\ 0 & -2 \end{pmatrix}$, $\mathbf{N} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and P is the point $(5, 3)$,

then, as before,

$$\mathbf{M}(P) = \begin{pmatrix} 2 & 3 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 19 \\ -6 \end{pmatrix}$$

$$\text{and } \mathbf{N}(\mathbf{M}(P)) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 19 \\ -6 \end{pmatrix} = \begin{pmatrix} 13 \\ -6 \end{pmatrix}$$

So

$$\mathbf{NM} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix}$$

$$\text{and } \mathbf{NM}(P) = \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 13 \\ -6 \end{pmatrix}$$

So transforming a point using matrix \mathbf{M} followed by transforming the image using matrix \mathbf{N} is the same as transforming the original point by the matrix \mathbf{NM} .

This is the same convention as with the transformations themselves.

\mathbf{NM} means \mathbf{M} followed by \mathbf{N} . \mathbf{NM} is the matrix of the combined transformation.

Exercise 39.10

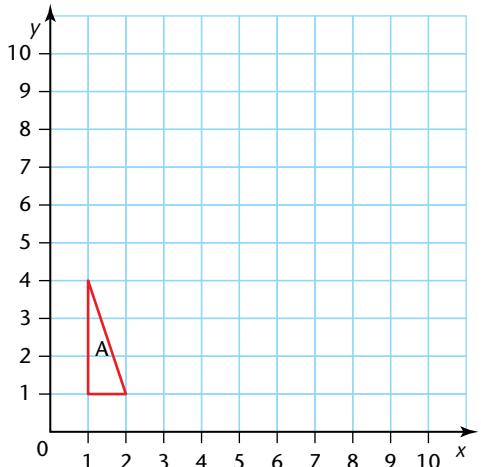
1 $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $Q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

- a Transform triangle A using matrix P.
Label the image B.
- b Transform triangle B using matrix Q.
Label the image C.
- c Find the matrix QP .
- d Transform triangle A using matrix QP .
Check that the coordinates of the image are the same as the coordinates of C.

2 $P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

A is the point $(-1, 3)$, B is the point $(-2, 1)$ and C is the point $(-1, 1)$.

- a Draw triangle ABC.
- b Transform triangle ABC on to triangle $A'B'C'$ using matrix P.
- c Transform triangle $A'B'C'$ on to triangle $A''B''C''$ using matrix Q.
- d Find matrix QP .
- e Describe fully the transformation that is represented by the matrix QP .

**Finding the matrix of a transformation**

If P is the point $(1, 0)$ then the position vector of $P = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

If Q is the point $(0, 1)$ then the position vector of $Q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

For a general transformation matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}.$$

The position vectors of the transformed points have components from the columns of \mathbf{M} .

The vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are called the **base vectors**.

This gives a method for finding the matrix of a transformation, which is shown in the following example.

Example 39.14**Question**

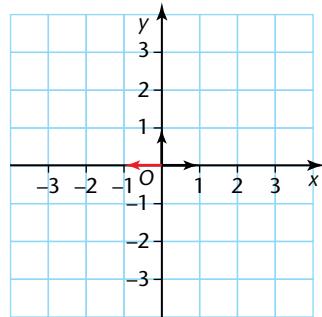
Find the matrix which represents a rotation of 90° anticlockwise about the origin, O .

Solution

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ maps to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ so the first column of the matrix is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ maps to $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ so the second column of the matrix is $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

So the matrix which represents a rotation of 90° anticlockwise about the origin O is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.



Notice that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so the origin must remain invariant whatever the matrix.

This means that you can only find the matrix for transformations where the origin remains invariant.

To use a 2×2 transformation matrix for

- reflections, the mirror line must go through the origin
- rotations, the centre of rotation must be the origin
- enlargements, the centre of enlargement must be the origin.

Translations cannot be represented by a 2×2 matrix because every point moves in a translation, including the origin.

Example 39.15**Question**

M is a reflection in the y axis.

E is an enlargement scale factor 2, centre O .

Find the single matrix which represents M followed by E .

Solution**Method 1**

With $M \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ maps to $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and with $E \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ maps to $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$.

With $M \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ stays as $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and then with E maps to $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

So the matrix for M followed by E is $\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$.

Method 2

With $M \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ maps to $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ stays as $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

So the matrix for M is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

With $E \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ maps to $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ maps to $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

So the matrix for E is $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

The matrix for M followed by E is $EM = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$.

Exercise 39.11

- 1** Find the matrix which represents a rotation of 180° about the origin.
- 2** Find the matrix which represents a reflection in the line $y = x$.
- 3** Find the matrix which represents a reflection in the x axis.
- 4** Find the matrix which represents an enlargement with scale factor -3 and centre the origin.
- 5** Find the matrix which represents a rotation 90° clockwise about the origin.
- 6** Find the matrix which represents a reflection in the line $y = -x$.
- 7** R is a rotation of 90° anticlockwise about the origin.
 M is a reflection in the line $y = 0$.

Find the single matrix which represents the transformation R followed by M .

- 8** E is an enlargement with scale factor 2 centre the origin.
 R is a rotation of 180° about $(0, 0)$.

Find the single matrix which represents the transformation E followed by R .

- 9** M_1 is a reflection in $y = x$.
 M_2 is a reflection in the x axis.
 - a** Find the single matrix which represents the transformation M_1 followed by M_2 .
 - b** Find the single matrix which represents the transformation M_2 followed by M_1 .

- 10** M is a reflection in $y = -x$.
 R is a rotation of 90° anticlockwise about $(0, 0)$.

Find the single matrix which represents the transformation M followed by R .

40 PROBABILITY

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- calculate the probability of a single event as either a fraction or a decimal
- understand that the probability of an event occurring = $1 -$ the probability of the event not occurring
- understand relative frequency as an estimate of probability
- calculate the probability of simple combined events using possibility diagrams and tree diagrams where appropriate.

CHECK YOU CAN:

- calculate with fractions
- use the probability scale from 0 to 1.

The probability of a single event

When you toss a coin, there are two possible **outcomes**, heads or tails.

If the coin is fair, these outcomes are equally likely.

The probability of getting a head is $\frac{1}{2}$ or 0.5.

The probability of getting a tail is $\frac{1}{2}$ or 0.5.

The sum of these two probabilities is 1 because there are no other possible outcomes.

If there is a set of equally likely outcomes then the probability of each **event** is given by this formula.

$$\text{Probability of an event} = \frac{\text{number of outcomes which give this event}}{\text{total number of outcomes}}$$

If it is **impossible** that the event will happen, the probability is 0.

If it is **certain** that the event will happen, the probability is 1.

Probabilities can be written as either fractions or decimals.

Example 40.1

Question

A bag contains six red counters, three blue counters and one green counter.

A counter is taken at random from the bag.

Find the probability that the counter is

- a red b green c black.

Solution

- a There are $6 + 3 + 1 = 10$ counters in the bag, so there are ten equally likely outcomes.

Six of the counters are red, so six outcomes give the required event.

The probability that the counter is red is $\frac{6}{10}$ which can be simplified to $\frac{3}{5}$.

- b One of the counters is green, so only one outcome gives the required event.

The probability that the counter is green is $\frac{1}{10}$. This fraction is in its simplest form.

- c There are no black counters in the bag, so this event is impossible.

The probability that the counter is black is 0.

Note

When you write a probability as a fraction, write your final answer as a fraction in its simplest form.

Exercise 40.1

In this exercise, give your answers as fractions in their simplest form.

- 1** A fair dice is rolled.

Find the probability of getting

- a** a 6 **b** an even number **c** a number less than 5.

- 2** There are six blue balls and four red balls in a bag.

A ball is taken at random from the bag.

Find the probability that the ball is

- a** red **b** blue.

- 3** There are seven blue balls, three red balls and ten yellow balls in a bag.

A ball is taken at random from the bag.

Find the probability that the ball is

- a** blue **b** red **c** yellow.

- 4** A letter is chosen at random from the word REPRESENT.

Find the probability that it is

- a** an E **b** an R.

- 5** The table shows information about a group of 40 children.

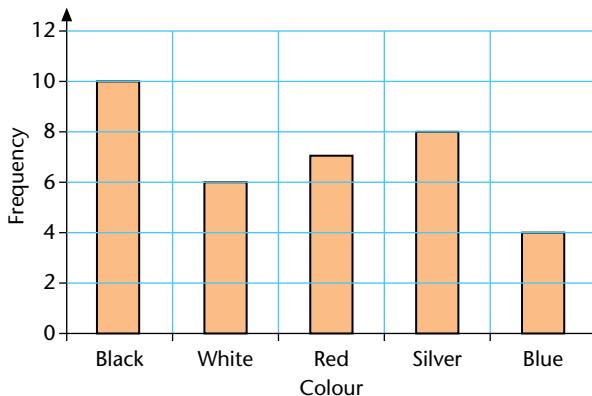
	Boy	Girl
Left-handed	6	8
Right-handed	11	15

One of the children is selected at random.

Find the probability that the child is

- a** a left-handed boy **b** a right-handed girl **c** a girl.

- 6** The bar chart shows the colours of cars parked in a car park.



One of the cars is selected at random.

Find the probability that it is

- a** black **b** silver.

The probability of an event not occurring

There are ten counters in a bag.

Seven of the counters are black.

A counter is taken at random from the bag.

The probability that the counter is black is $\frac{7}{10}$.

You can write this probability as $P(\text{black}) = \frac{7}{10}$.

To find the probability that the counter is not black, you need to think about how many outcomes there are where the counter is not black.

As there are 10 counters in the bag, there are three counters that are not black.

The probability that the counter is not black is $\frac{3}{10}$.

You can write this as $P(\text{not black}) = \frac{3}{10}$

Note that $P(\text{black}) + P(\text{not black}) = \frac{7}{10} + \frac{3}{10} = 1$.

This is because there are no other possible outcomes for this experiment.

This formula can be used to find out the probability that an event does not occur.

The probability of an event not occurring = $1 - \text{probability of the event occurring}$.

This rule can also be used when there are more than two possible events and you know the probability of all but one of them occurring.

Example 40.2

Question

- a The probability that Nayim's bus is late is $\frac{1}{8}$.
What is the probability that Nayim's bus is not late?

- b Salma can travel to school by bus, car or bike.
The probability that she travels by bus is 0.5.
The probability that she travels by car is 0.2.
What is the probability that she travels by bike?

Solution

a $P(\text{not late}) = 1 - P(\text{late})$
 $= 1 - \frac{1}{8}$
 $= \frac{7}{8}$

b $P(\text{bike}) = 1 - P(\text{bus}) - P(\text{car})$
 $= 1 - 0.5 - 0.2$
 $= 0.3$

The only possibilities are bus, car and bike, so these three probabilities sum to 1.

Exercise 40.2

- 1 The probability that Husna will do the washing up tonight is $\frac{3}{8}$.
What is the probability that she will not do the washing up?
- 2 The probability that Chris will get a motorcycle for his sixteenth birthday is 0.001.
What is the probability that he will not get a motorcycle?
- 3 In a game of tennis you can only win or lose. A draw is not possible.
Qasim plays Roshan at tennis.
The probability that Qasim wins is 0.7.
What is the probability that Roshan wins?
- 4 James has ten T-shirts.
Four of them have pictures on.
James takes one of the T-shirts at random.
What is the probability that it does not have a picture?
- 5 A bag contains red, white and blue balls.
I pick one ball out of the bag at random.
The probability that I pick a red ball is $\frac{1}{12}$.
The probability that I pick a white ball is $\frac{7}{12}$.
What is the probability that I pick a blue ball?
- 6 A shop has black, grey and blue dresses on a rail.
Mina picks one at random.
The probability that she picks a grey dress is 0.2.
The probability that she picks a black dress is 0.1.
What is the probability that she picks a blue dress?
- 7 When she goes to the gym, Yasmin runs, cycles or rows first.
The probability that she runs first is $\frac{7}{20}$.
The probability that she cycles first is $\frac{9}{20}$.
What is the probability that she rows first?
- 8 The table shows the probability of getting some of the scores when a biased six-sided dice is rolled.

Score	1	2	3	4	5	6
Probability	0.27	0.16	0.14		0.22	0.1

What is the probability of getting 4?

Note

A dice is **biased** if the outcomes are *not* all equally likely.

Estimating from a population

If you know the probability of an event happening, then you can use this probability to make an estimate of the number of times this event will occur.

Example 40.3**Question**

Saira goes to work by bus each morning.

The probability that the bus is late is 0.15.

Estimate the number of times out of 40 mornings that the bus is expected to be late.

Solution

$$\text{Number of times late} = 0.15 \times 40 = 6$$

The bus is expected to be late on approximately six out of the 40 mornings.

Note

This does not mean that the bus will definitely be late six times.

It is an estimate based on the probability that has been given.

 **Exercise 40.3**

- 1 A fair coin is tossed 250 times.
Estimate the number of times it will land on heads.
- 2 A fair six-sided dice is rolled 300 times.
Estimate the number of times it will land on 6.
- 3 The probability that a football team will win a match is 0.6.
How many of their next 30 matches are they expected to win?
- 4 A factory makes plates.
The probability that a plate is faulty is $\frac{1}{20}$.
In a batch of 1000 plates, how many would you expect to be faulty?
- 5 A biased spinner is numbered from 1 to 5.
The probability that it lands on 3 is 0.4.
It is spun 50 times.
Estimate the number of times it will land on 3.

Relative frequency and probability

It is not always possible to find probabilities by looking at equally likely outcomes.

For example, you may have to work out the probability of rolling a 6 with a dice that may be biased, the probability of a driver having a car accident or the probability that a person will visit a certain shop.

For this type of event you need to set up an experiment, carry out a survey or use past results.

Take the example of rolling a 6 with a dice that may be biased.

For a fair (unbiased) dice, the probability of getting a 6 is $\frac{1}{6} = 0.166\ldots = 0.17$ approximately.

You can record the number of 6s you get when you roll the dice a number of times and use the results to decide whether the dice may be biased.

The proportion of times a 6 occurs is known as the **relative frequency**.

Relative frequency = $\frac{\text{number of times an outcome occurs}}{\text{total number of trials}}$

The table shows the results of an experiment.

Number of trials	Number of 6s	Relative frequency
10	4	$\frac{4}{10} = 0.4$
50	18	$\frac{18}{50} = 0.36$
100	35	$\frac{35}{100} = 0.35$
500	180	$\frac{180}{500} = 0.36$

The relative frequency gives an estimate of the probability of rolling a 6.

The relative frequency changes depending on the number of trials.

The greater the number of trials, the better the estimate of the probability will be.

The results of this experiment suggest that the dice is biased.

An estimate for the probability of rolling a 6 with this dice is 0.36.

Example 40.4

Question

Farid carries out a survey on the colours of the cars passing his school.

The table shows his results.

Colour	Black	Red	Blue	White	Green	Other	Total
Number of cars	51	85	64	55	71	90	416

Use his results to estimate the probability that the next car that passes will be

- a red b not red.

Solution

Relative frequency can be used as an estimate of probability.

a Relative frequency of red cars = $\frac{\text{number of red cars}}{\text{total number of cars}} = \frac{85}{416}$

Estimate of probability = $\frac{85}{416} = 0.204$ to 3 decimal places

b $P(\text{not red}) = 1 - P(\text{red})$

Estimate of $P(\text{not red}) = 1 - \frac{85}{416} = \frac{331}{416} = 0.796$ to 3 decimal places

Note

The relative frequency can be written as either a fraction or a decimal.

Exercise 40.4

- 1** Use the figures from Farid's survey in Example 40.4 to estimate the probability that the next car will be
a blue **b** black or white.
 Give your answers correct to 3 decimal places.

- 2** Solomon has a spinner in the shape of a pentagon with sides labelled 1, 2, 3, 4, 5. Solomon spins the spinner 500 times. His results are shown in the table.

Number on spinner	Frequency
1	102
2	103
3	98
4	96
5	101

- a** What is the relative frequency of scoring
i 2 **ii** 4?
b Do the results suggest that Solomon's spinner is fair? Explain your answer.
- 3** In an experiment, a drawing pin is thrown. It can land either point up or point down. It lands point up 87 times in 210 throws. Use these figures to estimate the probability that, the next time it is thrown, it will land
a point up **b** point down.
 Give your answers correct to 2 decimal places.
- 4** The table shows the results of a survey on the type of detergent households use to do their washing.

Type of detergent	Number of households
Liquid	120
Powder	233
Tablets	85

- Use these figures to estimate, correct to 2 decimal places, the probability that the next household surveyed will use
a liquid **b** tablets.
- 5** Dina surveys 80 students from her school to find out how they travel to school. Of these students, 28 travel by bus.
a Estimate the probability that a student chosen at random travels to school by bus.
b There are 1200 students in the school. Estimate the number of students in the school who travel by bus.

- 6 Pavan tested a four-sided spinner.
He spun it 120 times and it landed on red 24 times.
- Estimate the probability that the spinner lands on red.
 - How many times would he expect it to land on red if he spun it 300 times?

The probability of combined events

To find the probability of more than one event happening, you need to make sure that you consider all of the possible outcomes.

One way of finding all of the outcomes is to use a **possibility diagram**.

If two fair coins are tossed, the first coin can show either a head or a tail and the second coin can show either a head or a tail.

The possibility diagram below shows all of the possible outcomes.

	1st coin		
2nd coin	H	T	
	H	X	X
	T	X	X

This shows that there are four possible outcomes: HH, HT, TH, TT.

These are all equally likely.

The probability of each event can be found from the possibility diagram.

$$P(2 \text{ heads}) = \frac{1}{4}$$

$$P(1 \text{ head}, 1 \text{ tail}) = \frac{2}{4} = \frac{1}{2}$$

$$P(2 \text{ tails}) = \frac{1}{4}$$

When the events are numerical – such as when two dice are rolled – it can be helpful to show the numerical outcomes on the possibility diagram rather than using a cross.

The possibility diagram here shows the outcomes for rolling two fair dice when the two scores are added together.

This shows that there are 36 possible outcomes.

Each outcome is equally likely.

The diagram can be used to find the probability of different events.

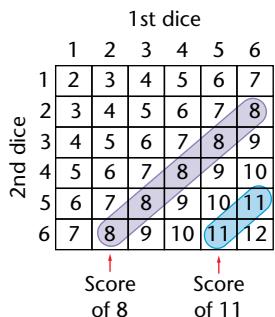
There are five different outcomes that give a score of 8, so $P(8) = \frac{5}{36}$.

There are two different outcomes that give a score of 11, so $P(11) = \frac{2}{36}$.

Suppose that you want to find the probability of scoring 8 **or** 11.

From the possibility diagram, you can see that there are a total of seven

outcomes that give a score of 8 or 11, so $P(8 \text{ or } 11) = \frac{7}{36}$.



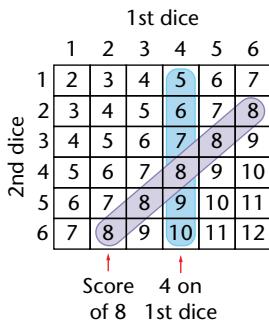
You can also see that $P(8) + P(11) = \frac{5}{36} + \frac{2}{36} = \frac{7}{36}$.

So $P(8 \text{ or } 11) = P(8) + P(11)$.

When two events cannot happen at the same time they are known as **mutually exclusive events**.

The events ‘scoring 8’ and ‘scoring 11’ are mutually exclusive because they cannot both happen at the same time.

If events A and B are mutually exclusive, then $P(A \text{ or } B) = P(A) + P(B)$.



This rule does not apply if the events can happen at the same time, for example ‘scoring 8’ and ‘scoring a 4 on the first dice’.

$$P(8) = \frac{5}{36}$$

$$P(\text{4 on first dice}) = \frac{6}{36}$$

From the possibility diagram, you can see that there are 10 outcomes that give a score of 8 **or** score a 4 on the first dice, so $P(8 \text{ or } 4 \text{ on first dice}) = \frac{10}{36}$.

$$P(8 \text{ or } 4 \text{ on first dice}) \neq P(8) + P(4 \text{ on first dice})$$

This is because scoring a double 4 fits both events.

These two events are not mutually exclusive so the probabilities cannot be added.

Example 40.5

Question

Ravi rolls two dice and adds the scores.

Find the probability that he scores

a 12

b less than 6

c 10 or 11.

Solution

The outcomes can be seen in the possibility diagram shown above.

a There is only one way to score 12, so $P(12) = \frac{1}{36}$.

b It can be seen from the diagram that there are 10 outcomes with a score of less than 6.

$$P(\text{less than } 6) = \frac{10}{36} = \frac{5}{18} \quad \text{Write the fraction in its simplest form.}$$

c It can be seen from the diagram that there are five outcomes with a score of 10 or 11.

$$P(10 \text{ or } 11) = \frac{5}{36}$$

Note that $P(10) = \frac{3}{36}$ and $P(11) = \frac{2}{36}$.

Scoring 10 and 11 are mutually exclusive,
so $P(10 \text{ or } 11) = P(10) + P(11)$

$$= \frac{3}{36} + \frac{2}{36} = \frac{5}{36}.$$

Exercise 40.5

- 1** **a** Copy and complete this table to show the total score when a fair dice is rolled and a spinner numbered from 1 to 4 is spun.
- b** What is the probability of getting a total of 9?
- c** What is the probability of getting a total of less than 6?
- d** What is the probability of getting a total that is an even number?
- 2** Fatima rolls an ordinary dice and tosses a coin.
- a** Draw a possibility diagram to show all the possible outcomes.
- b** Find the probability that Fatima scores
- i** a head and a 6
 - ii** a tail and an odd number.
- 3** A fair spinner is numbered from 1 to 4.
- The spinner is spun twice and the score is the **product** of the two numbers shown.
- a** Draw a possibility diagram to show all of the possible outcomes.
- b** Find the probability that the score is
- i** 4
 - ii** more than 8
 - iii** an odd number.
- 4** A fair spinner is numbered 2, 4, 6, 8, 10.
- The spinner is spun twice and the score is the sum of the two numbers shown.
- a** Draw a possibility diagram to show all the possible outcomes.
- b** What is the probability that the score is greater than 10?
- c** What is the probability that the score is 8 or 14?

Spinner	Dice					
	1	2	3	4	5	6
1	2	3				
2						
3						
4						

Independent events

When you toss a coin twice, the outcome of the second toss is not affected by the outcome of the first toss.

Two events are said to be **independent** when the outcome of the second event is not affected by the outcome of the first.

If events A and B are independent, $P(A \text{ and } B) = P(A) \times P(B)$.

The probability of getting a head when you toss a fair coin is $\frac{1}{2}$.

You have seen from the possibility diagram that the probability of getting two heads when you toss a coin twice is $\frac{1}{4}$.

$$P(\text{head and head}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

The result for independent events can be extended to more than two events. The probability of all of the events happening is found by multiplying together the probabilities of each individual event.

Example 40.6

Question

- a The probability that Imran's bus to school is late is 0.2.

Assuming that the events are independent, find the probability that the bus is late

- i two mornings in a row
- ii three mornings in a row.

- b There are four green balls and one blue ball in a bag.

Anna selects a ball, notes its colour and replaces it.

She then selects another ball.

What is the probability that the first ball is green and the second ball is blue?

Solution

- a The events are independent, so the probabilities are multiplied.

$$\text{i } P(\text{late two mornings}) = P(\text{late}) \times P(\text{late}) = 0.2 \times 0.2 = 0.04$$

$$\text{ii } P(\text{late three mornings}) = P(\text{late}) \times P(\text{late}) \times P(\text{late}) = 0.2 \times 0.2 \times 0.2 = 0.008$$

- b The first ball is replaced in the bag before the second is taken, so the events are independent.

$$P(\text{green}) = \frac{4}{5}$$

$$P(\text{blue}) = \frac{1}{5}$$

$$P(\text{first green and second blue}) = P(\text{green}) \times P(\text{blue}) = \frac{4}{5} \times \frac{1}{5} = \frac{4}{25}$$

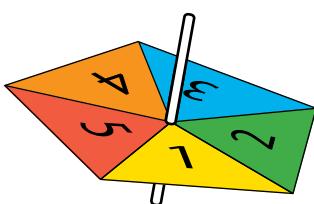
Note

The larger the number of repeats of the event, the smaller the probability that it occurs.



Exercise 40.6

- 1 There are five green balls, three red balls and two yellow balls in a bag. Jai chooses a ball at random, notes its colour and puts it back in the bag. He then does this a second time.
Find the probability that both Jai's choices are red.
- 2 The probability that I cycle to work is 0.4.
The probability that I take the bus to work is 0.6.
Assuming the events are independent, what is the probability that I cycle on Monday and take the bus on Tuesday?
- 3 a What is the probability that I get a multiple of 3 when I roll a single fair dice?
b If I roll the dice twice, what is the probability that both rolls give a multiple of 3?
- 4 There is an equal likelihood that someone is born on any day of the week.
What is the probability that Gary and Rushna were both born on a Monday?
- 5 The probability that Hira wins the 100-metre race is 0.4.
The probability that Rabia wins the 400-metre race is 0.3.
What is the probability that
 - a both girls win their race
 - b neither of them wins their race?
- 6 Salma spins this five-sided spinner three times.
What is the probability that all Salma's spins land on 1?



Tree diagrams for combined events

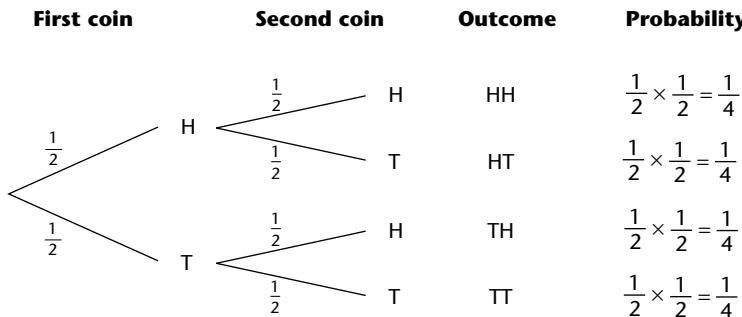
You have seen that possibility diagrams are one way of showing all of the outcomes for two combined events.

A tree diagram can be used to show the outcomes, and their probabilities, for two or more events.

The tree diagram for tossing two coins looks like this.

		1st coin
	H	T
H	X	X
T	X	X

		2nd coin
	H	T
H	X	X
T	X	X



To find the probability of one outcome, **multiply** the probabilities on the branches.

To find the probability of more than one outcome, **add** the probabilities of each outcome.

Notice that on each pair of branches the probabilities add up to 1.

Also the probabilities of all the four possible outcomes add up to 1.

If there are more than two combined events, further sets of branches can be added to the tree.

Example 40.7

Question

There are six red balls and four black balls in a bag.

Gita selects a ball, notes its colour and replaces it.

She then selects another ball.

What is the probability that Gita selects

- a two red balls
- b one of each colour?



Solution

A tree diagram can be drawn to show this information.

First ball	Second ball	Outcome	Probability
	R	RR	$0.6 \times 0.6 = 0.36$
0.6	B	RB	$0.6 \times 0.4 = 0.24$
	R	BR	$0.4 \times 0.6 = 0.24$
0.4	B	BB	$0.4 \times 0.4 = 0.16$

a $P(RR) = 0.6 \times 0.6 = 0.36$

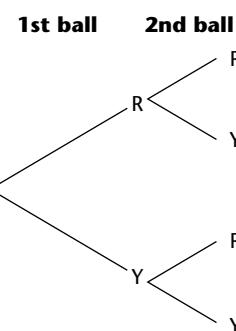
b For one of each colour, the outcome is either RB or BR.

$$\begin{aligned} P(\text{one of each colour}) &= P(RB) + P(BR) \\ &= (0.6 \times 0.4) + (0.4 \times 0.6) \\ &= 0.24 + 0.24 = 0.48 \end{aligned}$$

Note

Some questions may not ask you to draw a tree diagram. However, drawing a tree diagram will help you to make sure that you have considered all of the possible outcomes.

You may be able to answer some probability questions without a tree diagram.

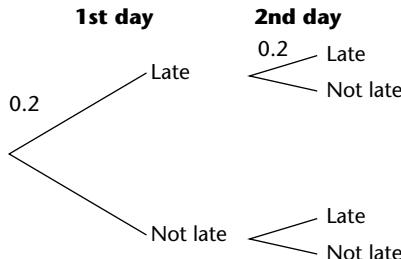
 **Exercise 40.7**

- 1 There are seven red balls and three yellow balls in a bag. Lee chooses a ball at random, notes its colour and replaces it. He then chooses another.

Copy and complete the tree diagram to show Lee's choices. What is the probability that Lee chooses

- a two red balls
- b a red ball and then a yellow ball
- c a yellow ball and then a red ball
- d a red ball and a yellow ball in either order?

- 2 On any day the probability that Sarah's bus is late is 0.2. a Copy the tree diagram and complete it for two days.



- b Calculate the probability that Sarah's bus is
- i late on both days
 - ii late on one of the two days.

- 3** In an experiment a drawing pin falls point up 300 times in 500 throws.
- Write down, as a fraction in its lowest terms, the probability of the pin landing point up.
 - Draw a tree diagram to show the result of two throws, and the pin landing point up or point down.
 - Find the probability that the pin lands point up on
 - both throws
 - one of the two throws.
- 4** Extend the tree diagram you drew for question 3 to show the results of three throws.
Find the probability that the pin lands point up on
- all three throws
 - one of the three throws.
- 5** There are ten red balls, three blue balls and seven yellow balls in a bag.
Waseem chooses a ball at random, notes its colour and replaces it.
He then chooses another.
- Draw a tree diagram to show the results of Waseem's choices.
 - Calculate the probability that Waseem chooses
 - two blue balls
 - two balls of the same colour
 - two balls of different colours. (Look for the quick way of doing it.)
- 6** The probability that a train arrives on time is 0.7.
Find the probability that on three consecutive days
- the train arrives on time
 - the train is late on the first two days
 - the train is on time for two out of the three days.

The probability of dependent events

In the last section, all of the events were independent.

The outcome of the second event was not affected by the outcome of the first.

In some situations, the outcome of the second event *is* affected by the first event.

In this situation, the probability of the second event depends on what happened in the first event.

In this case the events are called **dependent** events.

You can solve problems involving dependent events using tree diagrams.

When events are dependent, the probabilities on the second sets of branches are not the same as the probabilities on the first set.

Example 40.8

Question

There are seven blue balls and three red balls in a bag.

A ball is selected at random and **not** replaced.

A second ball is then selected.

Find the probability that

- a two blue balls are chosen
- b two balls of the same colour are chosen.

Solution

A tree diagram can be used to find the probabilities of the outcomes.

The probabilities for the second ball depend on the colour of the first ball.

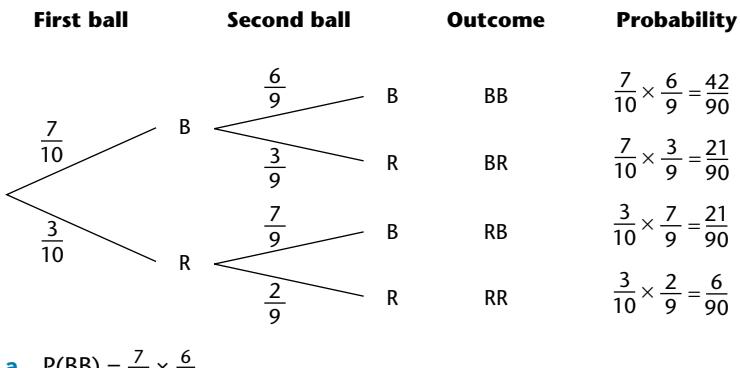
If the first ball was blue there are now six blue balls and three red balls left in the bag.

The probabilities are therefore $\frac{6}{9}$ and $\frac{3}{9}$, respectively.

If the first ball was red there are now seven blue balls and two red balls left in the bag.

The probabilities are therefore $\frac{7}{9}$ and $\frac{2}{9}$, respectively.

The tree diagram looks like this.



$$\begin{aligned} \text{a } P(BB) &= \frac{7}{10} \times \frac{6}{9} \\ &= \frac{42}{90} \\ &= \frac{7}{15} \end{aligned}$$

$$\begin{aligned} \text{b } P(BB \text{ or } RR) &= \frac{7}{10} \times \frac{6}{9} + \frac{3}{10} \times \frac{2}{9} \\ &= \frac{42}{90} + \frac{6}{90} \\ &= \frac{48}{90} \\ &= \frac{8}{15} \end{aligned}$$

Note

You should give your final answer as a fraction in its simplest form, but it is usually better not to simplify the fractions in the tree diagram. This is because you may need to add the probabilities and you will need them with a common denominator.

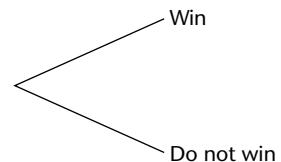
You use the tree diagram in the same way as you would for independent events.

The multiplication and addition rules still apply.

It is the probabilities that are different from the independent case.

 **Exercise 40.8**

- 1** There are five blue balls in a bag and four white balls.
If a white ball is selected first, what is the probability that the second ball is blue?
- 2** The probability that a school will win the first hockey match of the season is 0.6.
If they win the first match of the season the probability that they win the second is 0.7, otherwise it is 0.4.
 - a** Copy and complete the tree diagram.
 - b** Find the probability that they win one of the first two matches only.
 - c** Find the probability that they win at least one of the first two matches.
- 3** There are eight green balls in a bag and two white balls.
Two balls are selected at random without replacement.
 - a** Draw a tree diagram to show the probabilities of the possible outcomes.
 - b** Find the probability of selecting two white balls.
 - c** Find the probability of selecting two balls of different colours.
- 4** Salima walks to school, cycles or goes by bus.
The probability that she walks is 0.5.
The probability that she cycles is 0.2.
If she walks, the probability that she is late is 0.2. If she cycles, it is 0.1 and if she goes by bus, it is 0.4.
Find the probability that, on any given day, she is on time for school.
- 5** There are three red pens and five blue pens in a pencil case.
Basma picks two pens at random.
Find the probability that Basma picks at least one blue pen.

1st match

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- collect, classify and tabulate statistical data
- read, interpret and draw simple inferences from tables and statistical diagrams
- calculate the mean, median, mode and range for individual and discrete data and distinguish between the purposes for which they are used
- calculate an estimate of the mean for grouped and continuous data
- identify the modal class from a grouped frequency distribution.

CHECK YOU CAN:

- make tally charts
- add, subtract, multiply and divide numbers.

Collecting and grouping data

You may have seen headlines like these in newspapers or on television.

ONE DIRECTION IS THE MOST POPULAR BAND

or

TEENAGERS VOTE DIET COKE AS THE BEST SOFT DRINK

You may not agree with either headline, but how is such information found out?

One way is to ask people questions – that is, to carry out a survey using a questionnaire and collect the answers on a data collection sheet. The answers are then analysed to give the results.

Example 41.1

Question

Jamila is asked to find out how many people there are in cars that pass the school gates each day between 8.30 a.m. and 9.00 a.m.

She writes down the number of people in the first 40 cars that pass on the first Monday.

1	2	4	1	1	3	1	2	2	3
4	2	1	1	1	2	2	3	2	1
1	1	2	2	1	3	4	1	1	1
2	1	3	3	1	1	4	2	1	2

Make a frequency table to show her results.

Solution

You make a tally chart in the usual way.

Remember to group the tally marks into fives, which are shown like this , for easier counting.

A frequency table is a tally chart with an extra column for the totals.

The frequencies are the total of the tallies.

Number of people	Tally	Frequency
1		18
2		12
3		6
4		4

Surveys

When you are trying to find out some statistical information, it is often impossible to find this out from everyone concerned. Instead, you ask a smaller section of the population. This selection is called a **sample**.

Choosing a sample

The size of your sample is important. If your sample is too small, the results may not be very reliable. In general, the sample size needs to be at least 30. If your sample is too large, the data may take a long time to collect and analyse. You need to decide what represents a reasonable sample size for the hypothesis you are investigating.

You also need to eliminate **bias**. A biased sample is unreliable because it means that some results are more likely than others.

It is often a good idea to use a **random sample**, where every person or piece of data has an equal chance of being selected. You may want, however, to make sure that your sample has certain characteristics.

For example, when investigating the hypothesis ‘boys are taller than girls’, random sampling within the whole school could mean that all the boys selected happen to be in Year 7 and all the girls in Year 11: this would be a biased sample, as older children tend to be taller. So you may instead want to use random sampling to select five girls and five boys from within each year group.

Example 41.2

Question

Candace is doing a survey about school meals.

She asks every tenth person going into lunch.

Why may this not be a good method of sampling?

Solution

She will not get the opinions of those who dislike school meals and have stopped having them.

Data collection sheets

When you collect large amounts of data, you may need to group it in order to analyse it or to present it clearly. It is usually best to use equal class widths for this.

Tally charts are a good way of obtaining a frequency table, or you can use a spreadsheet or other statistics program to help you. Before you collect your data, make sure you design a suitable data collection sheet or spreadsheet.

Think first how you will show the results as this may influence the way you collect the data.

You could quite easily collect data from every student in your class on the number of children in their family. One way to collect this data would be to ask each person individually and make a list like the following one.

Class 10G

1	2	1	1	2	3	2	1	2	1	1	2	4	2	1	1	2
5	2	3	1	1	4	10	3	2	5	1	2	1	1	1	2	

The data for Class 10G is shown in the completed frequency table.

Using the same method to collect data for all the students in your year could get very messy. One way to make the collection of data easier is to use a data collection sheet. Designing a table like the one here can make collecting the data easy and quick.

Number of children	Tally	Frequency
1		12
2		10
3		3
4		2
5		2
6		0
7		0
8		0
9		0
10		1

Before you design a data collection sheet it is useful to know what the answers might be, although this is not always possible. For example, what would happen if you were using the tally chart above and someone said 13?

One way to deal with this problem is to have an extra line at the end of the table to record all other responses. The table here shows how this might look.

Number of children	Tally	Frequency
1		12
2		10
3		3
4		2
5		2
More than 5		1

Adding the ‘More than 5’ line allows all possible responses to be recorded and gets rid of some of the lines where there are no responses.

Designing a questionnaire

A **questionnaire** is often a good way of collecting data.

You need to think carefully about what information you need and how you will analyse the answers to each question. This will help you get the data in the form you need.

Here are some points to bear in mind when you design a questionnaire.

- Make the questions short, clear and relevant to your task.
- Only ask one thing at a time.
- Make sure your questions are not ‘leading’ questions. Leading questions show bias. They ‘lead’ the person answering them towards a particular answer. For example: ‘Because of global warming, don’t you think that families should be banned from having more than one car?’
- If you give a choice of answers, make sure there are neither too few nor too many.

Example 41.3

Question

Suggest a sensible way of asking an adult their age.

Solution

Please tick your age group:

- 18–25 years 26–30 years 31–40 years
 41–50 years 51–60 years Over 60 years

This means that the person does not have to tell you their exact age.

Note

When you are using groups, make sure that all the possibilities are covered and that there are no overlaps.

When you have written your questionnaire, test it out on a few people.

This is called doing a **pilot survey**.

Try also to analyse the data from the pilot survey, so that you can check whether it is possible. You may then wish to reword one or two questions, regroup your data or change your method of sampling before you do the proper survey.

Any practical problems encountered in collecting the data would be described in a report of the survey.

Two-way tables

Sometimes the data collected involve two factors.

For example, data can be collected about the colour of a car and where it was made.

You can show both of these factors in a two-way table.

Example 41.4**Question**

Peter has collected data about cars in a car park.

He has recorded the colour of each car and where it was made.

The two-way table below shows some of his data.

Complete the table.

	Made in Europe	Made in Asia	Made in the USA	Total
Red	15	4	2	
Not red	83			154
Total		73		

Solution

	Made in Europe	Made in Asia	Made in the USA	Total
Red	15	4	2	21
Not red	83	69	2	154
Total	98	73	4	175

All the totals can now be completed by adding across the rows or down the columns.

Note

A useful check is to calculate the grand total (the number in the bottom right corner of the table) twice.

The number you get by adding down the last column should be the same as the number you get by adding across the bottom row.

73 cars are made in Asia so $73 - 4$ are not red.

154 cars are not red so $154 - 83 - 69$ are not red and are made in the USA.

Exercise 41.1

- 1 These are the number of letters delivered one morning to 80 houses.

0	2	5	3	1	2	0	3	4	7
2	1	0	0	1	4	6	3	2	1
5	3	2	1	1	2	0	3	2	7
1	2	0	3	5	2	0	1	1	2
2	0	1	2	0	2	1	0	0	3
8	1	6	2	0	1	1	2	3	2
0	1	2	6	2	4	1	0	3	2
2	1	4	0	1	2	3	1	4	0

- a Make a frequency table.
b How many houses received three letters?

- 2** Kamil wants to survey public opinion about the local swimming pool.
Give one disadvantage of each of the following sampling situations.
- Selecting people to ring at random from the local phone directory.
 - Asking people who are shopping on Saturday morning.
- 3** Paul plans to ask 50 people at random how long they spent doing homework yesterday evening.

Here is the first draft of his data collection sheet.

Time spent	Tally	Frequency
Up to 1 hour		
1–2 hours		
2–3 hours		

Give two ways in which Paul could improve his collection sheet.

- 4** State what is wrong with each of these survey questions.
Then write a better version for each of them.
- What is your favourite sport: cricket, tennis or athletics?
 - Do you do lots of exercise each week?
 - Don't you think this government should encourage more people to recycle waste?
- 5** Design a questionnaire to investigate the use of a school library or resource centre.
You need to know
- which year group the student is in
 - how often they use the library
 - how many books they usually borrow on each visit.
- 6** Here is a two-way table showing the results of a car survey.
- Copy and complete the table.

	Japanese	Not Japanese	Total
Red	35	65	
Not red	72	438	
Total			

- How many cars were surveyed?
 - How many Japanese cars were in the survey?
 - How many of the Japanese cars were not red?
 - How many red cars were in the survey?
- 7** A drugs company compared a new type of drug for hay fever with an existing drug.

The two-way table shows the results of the trial.

- Copy and complete the table.

	Existing drug	New drug	Total
Symptoms eased	700	550	
No change in symptoms	350	250	
Total			

- How many people took part in the trial?
- How many people using the new drug had their symptoms eased?

- 8** At the indoor athletics championships, the USA, Germany and China won the most medals.
- a Copy and complete the table.

	Gold	Silver	Bronze	Total
USA	31		10	
Germany	18	16		43
China		9	11	42
Total		43		

- b Which country won the most gold medals?
 c Which country won the most bronze medals?

Averages and range

The mode and the median

The **mode** and the **median** are two different types of ‘average’.

The mode is the most common number in a set of numbers.

A set of data may have more than one mode or it may have no mode.

The median is the middle number of a set of numbers arranged in order.

In a list of n numbers, the middle number is $\frac{1}{2}(n + 1)$.

Example 41.5

Question

Here is a list of the weights of people in an exercise class.

73 kg, 58 kg, 61 kg, 43 kg, 81 kg, 53 kg, 73 kg, 70 kg, 73 kg, 62 kg, 60 kg, 85 kg

The instructor wants to know the average weight for the group.

Solution

One way is to find the most common weight, the mode.

The mode is 73 kg.

Another way is to put the weights in order and find the middle weight, the median.

There are 12 numbers.

$$\frac{1}{2}(n + 1) = \frac{1}{2}(12 + 1) = 6\frac{1}{2}$$

The middle numbers are the sixth and seventh.

The median is halfway between the two.

It is found by adding the values in the two middle positions and dividing by 2.

43 kg, 53 kg, 58 kg, 60 kg, 61 kg, 62 kg, 70 kg, 73 kg, 73 kg, 81 kg, 85 kg

$$\text{The median} = \frac{62 + 70}{2} = 66 \text{ kg.}$$

Exercise 41.2

- 1 Find the median mark for each of these tests.

Remember to write the marks in order, smallest to largest, first.

a 3 5 6 7 9

b 8 9 4 3 7 3 1 7

- 2 Twelve people have their handspan measured.

These are the results.

225 mm 216 mm 188 mm 212 mm 205 mm 198 mm

194 mm 180 mm 194 mm 198 mm 200 mm 194 mm

a How many of the group had a handspan greater than 200 mm?

b What are the mode and the median?

- 3 These are the earnings of ten workers in a small company.

\$10 000 \$10 000 \$10 000 \$10 000 \$13 000

\$13 000 \$15 000 \$21 000 \$23 000 \$70 000

a Find the mode and the median for the data.

b Which of these averages do you think gives the better impression of the average pay?

Explain your answer.

- 4 Here are the recent scores of two basketball players.

Harvey 24 0 0 32 0 17 29 19 23

Nick 12 9 3 16 8 6 9 0 11 13

Work out the mode and the median for each player.

Who would you pick for the next match?

Explain your answer.

Mean and range

The **mean** is another type of average.

$$\text{Mean} = \frac{\text{sum of data values}}{\text{number of data values}}$$

The **range** is a measure of how spread out the data is.

$$\text{Range} = \text{largest data value} - \text{smallest data value}$$

Example 41.6

Question

Madia and Parveen play cricket.

These are the number of runs they score in five innings.

Madia 7 8 5 4 7

Parveen 10 10 2 1 6

a Calculate the mean number of runs per innings for each player.

b Who is the more consistent batsman? →

Solution

a Mean = $\frac{\text{total number of runs}}{\text{number of innings}}$

Mean for Madia = $31 \div 5 = 6.2$

Mean for Parveen = $29 \div 5 = 5.8$

- b Use the range to determine who has more consistent scores.

Range = largest data value – smallest data value

Range for Madia = $8 - 4 = 4$

Range for Parveen = $10 - 1 = 9$

Madia has a smaller range so he is the more consistent batsman.

Exercise 41.3

- 1 Find the mean and range for each of these sets of data.

a 3 12 4 6 8 5 4

b 12 1 10 1 9 3 4 9 7 9

c 15 17 12 29 21 18 31 22

d 313 550 711 365 165 439 921 264

- 2 Copy and complete the table for each of these sets of data.

Data set A: 1, 1, 2, 2, 3, 3, 3, 4, 5, 6, 7

Data set B: 1, 1, 2, 2, 3, 3, 3, 4, 5, 6, 7, 1, 1, 2, 2, 3, 3, 3, 4, 5, 6, 7

Data set C: 2, 2, 4, 4, 6, 6, 6, 8, 10, 12, 14

	Data set A	Data set B	Data set C
Range			
Mean			

Write down anything that you notice about these results.

- 3 a Find the mean of this set of data.

1, 2, 2, 3, 7

- b Write down the mean and range of each of these sets of data.

i 1, 2, 2, 3, 7, 1, 2, 2, 3, 7

ii 10, 20, 20, 30, 70

iii 110, 120, 120, 130, 170

- 4 These are the salaries of ten workers in a small company.

\$10 000 \$10 000 \$10 000 \$10 000 \$13 000

\$13 000 \$15 000 \$21 000 \$23 000 \$70 000

Find the mean and range for the data.

- 5 These are marks scored in a test.

20 16 18 17 16 18 14 13

18 18 15 18 19 9 12 13

- a Find the mean.

- b Find the range.

- 6 Here are the weekly wages of five people who work in a shop.
\$157 \$185 \$189 \$177 \$171
- a i Find the range of these wages.
ii Find the mean of these wages.
- b A new employee starts who earns \$249.
i Find the new range of these wages.
ii Find the new mean of these wages.
- 7 A group of 12 people have each measured the length of their stride to the nearest centimetre.
These are the results.
87 78 91 73 84 98 77 82 83 73 89 81
- Work out
- a the mean length of stride.
b the range of these lengths.
- 8 a Nine students scored a mean of 7 marks for their last maths test.
What was their total mark?
b A tenth student scored 10.
What was the mean mark of all ten students?

Which average to use when comparing data

When you are comparing sets of data, you need to compare the size of the data values and how spread out the data values are.

You compare the size of the data values using an average. You have now met three types of average: the mode, the median and the mean. You can use any of the three to compare sets of data but in some circumstances one is better than the others.

The **mean** has the advantage that it uses every data value but it can be misleading if there are one or two very high or very low data values.

The **median** is not affected by a few high or low values.

The **mode** has the advantage that is often quick and easy to find but can be unreliable, particularly with a small sample of numbers.

Also, some sets of data will not have a mode and some will have more than one. However, as you will see in Example 41.8 below, in certain circumstances it can be very useful.

The range is a measure of how spread out the data values are.

The greater the range the more spread out the data values are. A greater range tells you that the data values are less consistent.

The smaller the range the less spread out the data values are. A smaller range tells you that the data values are more consistent.

You will meet another measure of spread in Chapter 42.

Note

When you are asked to compare two sets of data, state your answer in the context of the question.

So in Example 41.7, don't just state the potatoes from the market had a higher mean, state that this means that they are heavier on average.

Also don't just state that the range is smaller for the potatoes from the market, state that this means they are more consistently the same mass.

Example 41.7**Question**

These are the masses, in grams, of 12 potatoes in a bag bought from a supermarket.

200 410 300 250 280 290 420 380 310 280 210 320

These are the masses, in grams, of 12 potatoes in a bag bought from a local market.

230 400 350 360 270 390 410 370 360 380 410 380

Compare the masses.

Solution

Arrange the data in order and calculate the averages and range.

Supermarket potatoes

200 210 250 280 280 290 300 310 320 380 410 420

The mode is 280g.

The median is halfway between 290 and 300, so it is 295g.

Mean = $3650 \div 12 = 304$ g (to the nearest gram)

Range = $420 - 200 = 220$ g

Market potatoes

230 270 350 360 360 370 380 380 390 400 410 410

There are three modes, 360g, 380g and 410g.

The median is halfway between 370 and 380, so it is 375g.

Mean = $4310 \div 12 = 359$ g (to the nearest gram)

Range = $410 - 230 = 180$ g

The mean shows that the potatoes from the market are heavier on average.

The range shows that the potatoes from the market are more consistently the same mass than the ones from the supermarket.

In Example 41.7, the mode was not useful. Either the median or the mean could be used to compare the masses but there were no extreme values affecting the mean and the mean uses all the data. Also the total quantity of the potatoes is important. The bags of potatoes from the market weigh more in total.

Example 41.8**Question**

At the local shoe shop 15 pairs of ladies' shoes were sold.

These were the sizes sold. (They do not sell half sizes.)

3, 5, 6, 3, 3, 4, 4, 5, 5, 5, 5, 7, 3, 4 and 4

a Work out the mean, median and mode for these sizes.

b Which measure of average is the most useful in this case, and why? →

Solution

- a Arrange the data in order and calculate the averages and range.
- $$3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 7$$
- $$\text{Mean} = 66 \div 15 = 4.4$$
- The median is the 8th value.
- $$\text{Median} = 4$$
- $$\text{Mode} = 5$$
- b The most useful measure in this case is the mode, because it can be used to find out which size of shoe to order most of.

Example 41.9**Question**

The label on a matchbox states 'Average contents 50 matches'.

A survey of the number of matches in ten matchboxes produced the following data.

45 51 48 47 46 47 49 50 52 47

- a Find the mode, median, mean and range of the contents.
 b What do these results tell you about the statement on the label?

Solution

- a Arrange the data in order and calculate the averages and range.

45 46 47 47 47 48 49 50 51 52

Mode = 47

The median is halfway between the 5th and 6th values.

$$\text{Median} = \frac{47 + 48}{2} = 47.5$$

$$\text{Mean} = \frac{482}{10} = 48.2$$

$$\text{Range} = 52 - 45 = 7$$

- b The three measures of average – the mode, the median and the mean – are all below 50, so the statement on the label is untrue.
 The range shows that the number of matches varies greatly from one box to another.

Exercise 41.4

- 1 The number of points scored by two players, Carl and Adam, in a basketball team in eight games are shown in the table.

Carl	17	19	21	24	16	18	14	23
Adam	19	20	14	25	26	29	12	13

- a Work out the mean and range for each player.
 b Compare the number of points scored by the two players.

- 2** The number of hours of sunshine in June during the last eight years at two holiday resorts, A and B, are given in the table.

Resort A	180	170	109	171	165	190	162	173
Resort B	170	168	120	172	158	178	157	169

- a** Work out the mean, median and range for each resort.
b Compare the amount of sunshine at the two resorts.
- 3** A shopkeeper records the sizes of the pairs of jeans sold. These are the sizes sold.
 34, 28, 36, 30, 32, 30, 28, 32, 30, 36,
 34, 30, 26, 30, 32, 34, 36, 28, 34, 32
a Work out the mean, median and mode for these figures.
b Which measure of average is most useful when the shopkeeper wants to order more jeans?
- 4** At a gymnastics competition Olga was given these marks by the ten judges.
 8.7, 8.9, 9.3, 9.1, 8.4, 8.1, 8.9, 9.1, 9.2, 9.1
a Work out the mean, the median and the range for Olga.
b Janice had a mean of 9.1, a median of 8.8 and a range of 1.6. Which is the best average to use to compare the two girls? Give a reason for your answer.
- 5** In an ice-skating competition, Jamilla was given the following marks by the ten judges.
 9.1, 9.0, 8.7, 9.5, 8.8, 8.2, 7.9, 9.2, 8.9, 8.7
 The highest and lowest marks are ignored and the mean of the remaining marks is her score.
a Work out Jamilla's score.
b Is this better or worse than calculating the mean mark from all the judges?
- 6** At Prothero fabric factory, the salaries have a mean of \$25 000, a median of \$20 000 and a range of \$21 000.
 At Jaline fabric factory, the salaries have a mean of \$23 250, a median of \$20 000 and a range of \$10 000.
 Compare the salaries at the two factories.

Working with larger data sets

When working with larger data sets, it is often easier to see any patterns in the data if it is displayed in a frequency table.

For example, this is a list of goals scored in 20 matches.

1	1	3	2	0	0	1	4	0	2
2	0	6	3	4	1	1	3	2	1

This is the same data displayed in a frequency table.

Number of goals	Frequency
0	4
1	6
2	4
3	3
4	2
5	0
6	1

From the table, it is easy to identify the mode. It is the number of goals with the greatest frequency. Here the mode is 1 goal.

The median is the middle value. In this case, it is halfway between the 10th and 11th values.

Looking at the frequency column in the table, the 10th value is 1 and the 11th value is 2.

The median is, therefore, 1.5.

The table can also be used to calculate the mean.

There are:

four matches with 0 goals	$4 \times 0 = 0$ goals
six matches with 1 goal	$6 \times 1 = 6$ goals
four matches with 2 goals	$4 \times 2 = 8$ goals

and so on.

To find the total number of goals scored altogether, multiply each number of goals by its frequency and then add the results. You can add an extra column to your table to help you work out the values multiplied by their frequencies.

Number of goals	Frequency	Number of goals × frequency
0	4	0
1	6	6
2	4	8
3	3	9
4	2	8
5	0	0
6	1	6
Totals	20	37

Then, dividing by the total number of matches (20) gives the mean.

$$\text{Mean} = 37 \div 20 = 1.85 \text{ goals}$$

Example 41.10**Question**

Work out the mean, mode and range for the number of children in the houses in Berry Road, listed in this table.

Number of children (c)	Frequency (number of houses)	Total number of children (c × frequency)
0	6	0
1	4	4
2	5	10
3	7	21
4	1	4
5	2	10
Totals	25	

Solution

$$\text{Mean} = 49 \div 25 = 1.96 \text{ children}$$

Mode = 3 children

Range = 5 – 0 = 5 children

Exercise 41.5

- 1 A taxi company did a survey of the number of passengers carried in their taxis on each trip in one shift.
Calculate the mean number of passengers per taxi.

Number of passengers in a taxi	Frequency
1	84
2	63
3	34
4	15
5	4
Total	200

- 2 The table shows the number of minutes late that the students in a class arrived for registration one Monday morning.

Number of minutes late	Frequency
1	0
2	1
3	5
4	7
5	11
6	2
7	1
8	0
9	1
10	0

- a What was the modal number of minutes late?
b What is the mean number of minutes late?

- 3** A football team recorded the number of goals it scored each match one football season.

Number of goals	Frequency
0	7
1	10
2	9
3	4

- a** How many matches were played?
b What was the total number of goals scored?
c What was the mean number of goals scored?
d What was the median number of goals scored?
- 4** Bus tickets cost 50c, \$1.00, \$1.50 or \$2.00 depending on the length of the journey.

The frequency table shows the numbers of tickets sold on one Friday.
Calculate the mean fare paid on that Friday.

Price of ticket (\$)	Number of tickets
0.50	140
1.00	207
1.50	96
2.00	57

- 5** 800 people were asked how many cups of coffee they had bought one week.
Calculate the mean, median and mode of the number of cups of coffee bought.

Number of coffees	Frequency
0	20
1	24
2	35
3	26
4	28
5	49
6	97
7	126
8	106
9	54
10	83
11	38
12	67
13	21
14	26

Working with grouped and continuous data

So far in this chapter most of the data has been whole numbers only. The information is obtained through counting not measuring.

Data obtained by counting is called **discrete** data.

Data obtained by measuring is called **continuous** data.

This is because you have the whole range of measurement between any two values.

For example, a length L given as 18 cm to the nearest centimetre means $17.5 \leq L < 18.5$.

Any length between these values will be recorded as 18 cm.

Often, however, the groups are larger, to make handling the data easier.

Discrete data can also be grouped.

For example, if you are looking at examination results in an examination marked out of 100 it is difficult to draw any conclusions from individual scores so it is often better to group the marks.

For example, 1 to 10 marks, 11 to 20 marks, 21 to 30 marks, etc.

When you group data, some of the detail is lost.

For example, a frequency table may show that one student scored between 1 and 10 marks. You can no longer tell whether that student scored 1 mark or 10 marks or any of the scores in between.

For this reason, you cannot obtain exact values for the mode, median, mean or range.

The **modal class** may be found when data are given as a table, frequency polygon or bar graph. It is the class with the highest frequency.

Similarly, you cannot find the median from grouped data but you can find the class which contains the median.

You can use the midpoints of the largest and smallest classes to estimate the range.

Similarly, you can use the midpoint of the classes to calculate an estimate of the mean.

Example 41.11**Question**

The frequency table shows some information about the heights of a year group in a school.

Height h (cm)	Frequency
$155 \leq h < 160$	2
$160 \leq h < 165$	6
$165 \leq h < 170$	18
$170 \leq h < 175$	25
$175 \leq h < 180$	9
$180 \leq h < 185$	4
$185 \leq h < 190$	1

Note

Add two columns to the frequency table to help you work out the mean: one column for the midpoint of each class and one for the midpoint multiplied by the frequency of the class.

- a State the modal class.
- b State the class which contains the median.
- c Estimate the range.
- d Calculate an estimate of the mean.

Solution

a The highest frequency is 25 so the modal class is $170 \text{ cm} \leq h < 175 \text{ cm}$.

b The total frequency is 65 so the median is the 33rd data value.

Adding the frequencies $2 + 6 = 8$, $8 + 18 = 26$, so 26 are less than 170cm.

$26 + 25 = 51$ so all of the 27th to 51st heights are in the $170 < h \leq 175$ class.

So the 33rd height is in the $170 \text{ cm} < h \leq 175 \text{ cm}$ class.

c Using the midpoints of the first and last classes,
estimate of the range = $187.5 - 157.5 = 30$

Height h (cm)	Frequency	Midpoint	Midpoint \times frequency
$155 \leq h < 160$	2	157.5	315
$160 \leq h < 165$	6	162.5	975
$165 \leq h < 170$	18	167.5	3015
$170 \leq h < 175$	25	172.5	4312.5
$175 \leq h < 180$	9	177.5	1597.5
$180 \leq h < 185$	4	182.5	730
$185 \leq h < 190$	1	187.5	187.5
Total	65		11132.5

$$\begin{aligned}\text{Estimate of the mean} &= 11132.5 \div 65 \\ &= 171.3 \text{ cm (to 1 decimal place)}\end{aligned}$$

Exercise 41.6

- 1** The table shows a distribution of times.
- State the modal class.
 - State the class which contains the median.
 - Estimate the range.
 - Calculate an estimate of the mean.

Time t (seconds)	Frequency
$0 \leq t < 2$	4
$2 \leq t < 4$	6
$4 \leq t < 6$	3
$6 \leq t < 8$	2
$8 \leq t < 10$	7

- 2** The table shows a distribution of heights.
- State the modal class.
 - State the class which contains the median.
 - Estimate the range.
 - Calculate an estimate of the mean.

Height h (cm)	Frequency
$50 \leq h < 60$	15
$60 \leq h < 70$	23
$70 \leq h < 80$	38
$80 \leq h < 90$	17
$90 \leq h < 100$	7

- 3** The table shows a distribution of lengths.
- State the modal class.
 - State the class which contains the median.
 - Estimate the range.
 - Calculate an estimate of the mean.

Length l (metres)	Frequency
$1.0 \leq l < 1.2$	2
$1.2 \leq l < 1.4$	7
$1.4 \leq l < 1.6$	13
$1.6 \leq l < 1.8$	5
$1.8 \leq l < 2.0$	3

- 4 Calculate an estimate of the mean length for this distribution.

Length y (cm)	Frequency
$10 \leq y < 20$	2
$20 \leq y < 30$	6
$30 \leq y < 40$	9
$40 \leq y < 50$	5
$50 \leq y < 60$	3

- 5 Calculate an estimate of the mean mass of these tomatoes.

Mass of tomato t (g)	Frequency
$35 < t \leq 40$	7
$40 < t \leq 45$	13
$45 < t \leq 50$	20
$50 < t \leq 55$	16
$55 < t \leq 60$	4

- 6 Calculate an estimate of the mean of these times.

Time t (seconds)	Frequency
$0 \leq t < 20$	4
$20 \leq t < 40$	9
$40 \leq t < 60$	13
$60 \leq t < 80$	8
$80 \leq t < 100$	6

- 7 Calculate an estimate of the mean of these heights.

Height h (metres)	Frequency
$0 < h \leq 2$	12
$2 < h \leq 4$	26
$4 < h \leq 6$	34
$6 < h \leq 8$	23
$8 < h \leq 10$	5

- 8 Calculate an estimate of the mean of these lengths.

Length l (cm)	Frequency
$3.0 \leq l < 3.2$	3
$3.2 \leq l < 3.4$	8
$3.4 \leq l < 3.6$	11
$3.6 \leq l < 3.8$	5
$3.8 \leq l < 4.0$	3

Grouped discrete data

One method for finding the midpoint of a class is to add the end values of the class and divide by 2.

For example, if the class is $60 \leq s < 80$, then the midpoint is $\frac{60 + 80}{2} = 70$.

Strictly, this method is not correct. The class $60 \leq s < 80$ comprises the digits 60, 61, 62, 63, ..., 78, 79. The midpoint of these is $\frac{60 + 79}{2} = 69.5$.

However, using midpoints is an approximation and the error in using the simpler calculation is not significant.

Example 41.12

Question

The table shows the scores of 40 students in a history test.

Score (s)	Frequency
$0 \leq s < 20$	2
$20 \leq s < 40$	4
$40 \leq s < 60$	14
$60 \leq s < 80$	16
$80 \leq s < 100$	4
Total	40

Calculate an estimate of the mean score.

Solution

Score (s)	Frequency	Midpoint	Midpoint \times frequency
$0 \leq s < 20$	2	10	20
$20 \leq s < 40$	4	30	120
$40 \leq s < 60$	14	50	700
$60 \leq s < 80$	16	70	1120
$80 \leq s < 100$	4	90	360
Total	40		2320

$$\text{Mean} = 2320 \div 40 = 58$$

Note

Don't forget to divide by the *total frequency*, not by the number of groups.

Exercise 41.7

- The table shows a summary of the marks scored by students in a test.
 - State the modal class.
 - State the class which contains the median.
 - Estimate the range.
 - Calculate an estimate of the mean.

Marks (m)	Frequency
$0 \leq m < 20$	1
$20 \leq m < 40$	3
$40 \leq m < 60$	15
$60 \leq m < 80$	9
$80 \leq m < 100$	2

- 2** The table summarises the number of words in each sentence on one page of a book.
- State the modal class.
 - State the class which contains the median.
 - Estimate the range.
 - Calculate an estimate of the mean.

Number of words	Frequency
1–3	0
4–6	2
7–9	7
10–12	6
13–15	15

- 3** The table shows a summary of the attendance at the first 30 games of a football club.
- State the modal class.
 - State the class which contains the median.
 - Estimate the range.
 - Calculate an estimate of the mean.

Attendance (a)	Frequency
$0 \leq a < 4000$	0
$4000 \leq a < 8000$	2
$8000 \leq a < 12\,000$	7
$12\,000 \leq a < 16\,000$	6
$16\,000 \leq a < 20\,000$	15

- 4** The table shows a summary of the number of points scored by a school basketball player.
Calculate an estimate of the mean number of points scored.

Points (p)	Frequency
$0 \leq p < 5$	2
$5 \leq p < 10$	3
$10 \leq p < 15$	8
$15 \leq p < 20$	9
$20 \leq p < 25$	12
$25 \leq p < 30$	3

42 STATISTICAL DIAGRAMS

BY THE END OF THIS CHAPTER YOU WILL BE ABLE TO:

- construct and interpret bar charts, pie charts, pictograms, simple frequency distributions, frequency polygons, histograms with equal and unequal intervals and scatter diagrams
- construct and use cumulative frequency diagrams
- estimate and interpret the median, percentiles, quartiles and interquartile range for cumulative frequency diagrams
- calculate with frequency density
- understand what is meant by positive, negative and zero correlation with reference to a scatter diagram
- draw a straight line of best fit by eye.

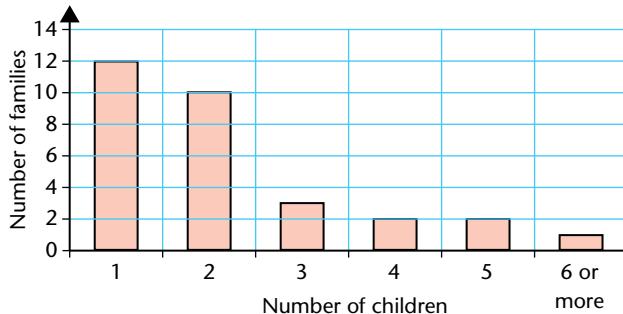
CHECK YOU CAN:

- calculate with fractions and percentages
- draw simple bar charts and pictograms
- measure angles
- understand grouped data
- find the median and the mode.

Bar charts and pictograms

Bar charts and pictograms can both be used to represent data that is divided into categories.

This bar chart shows the number of children in each of the 30 families of Class 11S.



You can find the number of families with different numbers of children from the bar chart.

There are 10 families with 2 children.

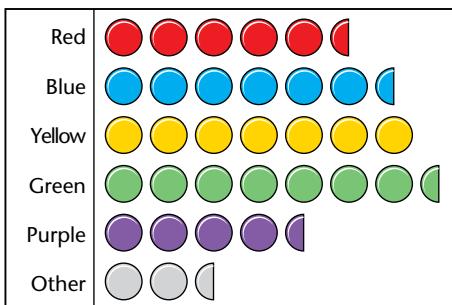
There is 1 family with 6 or more children.

Example 42.1

Question

This pictogram shows people's favourite colour for their mp3 player.

● represents 2 people.



- What does ● represent?
- How many people chose blue?
- How many people altogether were asked about their favourite colour of mp3 player?

Solution

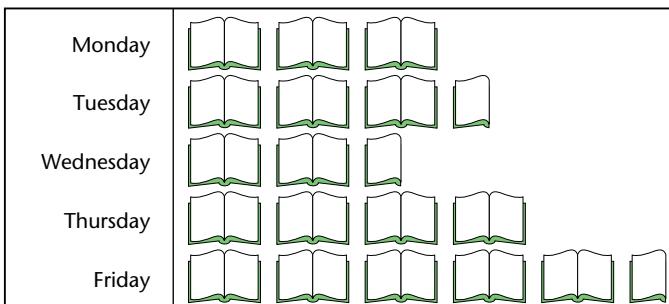
- a  represents 2 people.
- b  is half a symbol so it represents 1 person.
- c There are $6\frac{1}{2}$ symbols for blue.
Each symbol represents 2 people, so 13 people chose blue.
- c There are a total of $33\frac{1}{2}$ symbols in the pictogram, so 67 people were asked altogether.

Exercise 42.1

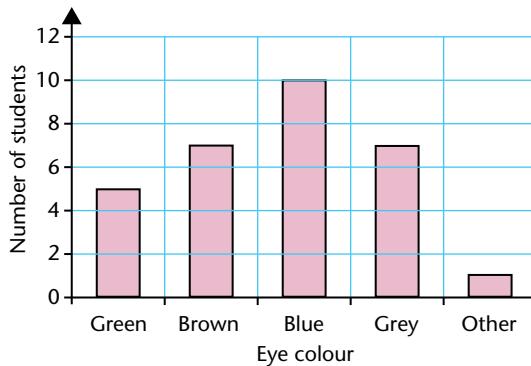
- 1 Ameera drew this pictogram to show the number of books borrowed from the school library one week.



 represents 10 books.



- a How many books were borrowed on each of the days?
b Which day was the most popular? Why do you think that might be?
- 2 The bar chart shows the eye colour of students in Class 10F.



- a How many students have brown eyes?
b How many more students have blue eyes than have grey eyes?
c How many students are there in the class?

- 3 The table shows the numbers of students absent from school during one week.

Day	Frequency
Monday	8
Tuesday	5
Wednesday	7
Thursday	10
Friday	6

Draw a bar chart to represent this information.

- 4 The table shows the numbers of bikes sold by a shop in four weeks.

Week	Frequency
Week 1	12
Week 2	10
Week 3	19
Week 4	13

Draw a pictogram to represent this information.

Use the symbol  to represent 8 bikes.

Pie charts

A **pie chart** is another way to display data from a frequency table.

The angle of the sector is proportional to the frequency.

The sectors show the proportions not the actual numbers.

Example 42.2

Question

The table shows data from a survey about which mathematics topic students preferred.

Draw a pie chart using this data.

Topic	Frequency
Number	8
Algebra	2
Geometry	4
Handling data	6



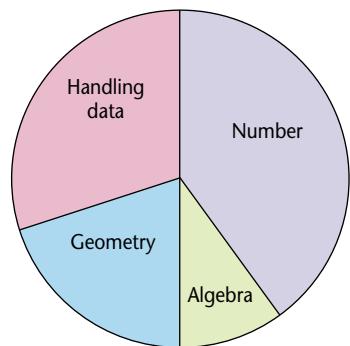
Solution

To draw a pie chart for this data, you need to work out the angle for each sector.

Topic	Frequency	Calculation	Angle
Number	8	$\frac{8}{20} \times 360$	144°
Algebra	2	$\frac{2}{20} \times 360$	36°
Geometry	4	$\frac{4}{20} \times 360$	72°
Handling data	6	$\frac{6}{20} \times 360$	108°
Total	20		360°

Note

You should check that the angles add up to 360° and measure all sectors accurately when you draw the pie chart.



A pie chart shows proportions.

You can write these as fractions or percentages.

To find out the number of data items in each group, you need to know the total number of data items.

Example 42.3**Question**

This pie chart shows the age groups of the people who entered a shop one morning.

- a What was the most common age group?
- b What fraction of the people were aged over 70?
- c A total of 180 people entered the shop.
How many people were aged 51–70?

Solution

- a The most common age group has the largest sector angle.
The most common age group was 20–30.

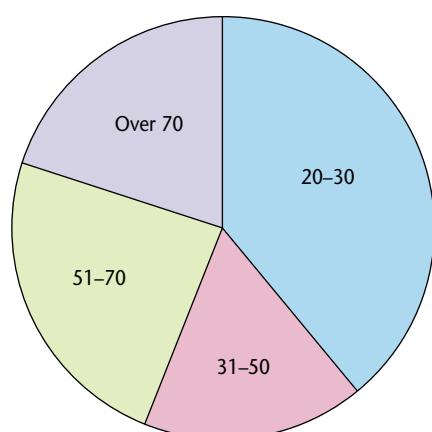
- b The sector angle for over 70 is 72° .

The fraction is $\frac{72}{360} = \frac{1}{5}$.

- c The sector angle for 51–70 is 86° .

$$\frac{86}{360} \times 180 = 43$$

There were 43 people aged 51–70.



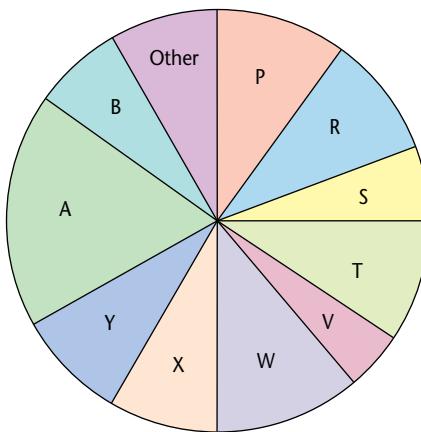
Exercise 42.2

- 1 The table shows the lengths of 60 pea pods.

Length (mm)	Frequency
50–59	6
60–69	20
70–79	14
80–89	12
90–99	8
Total	60

Draw a pie chart showing the lengths of the pea pods.

- 2 This pie chart shows the first letter of the names of 200 students.

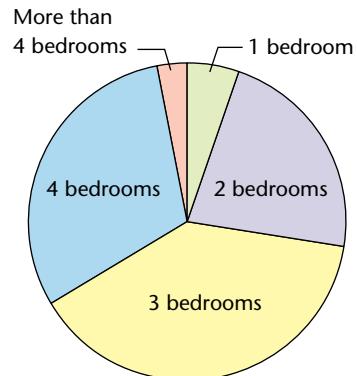


- a What was the most common letter?
 b How many names started with
 i P ii V?

- 3 Nesrin did a survey to find the number of bedrooms people had in their home.

The pie chart shows her results.

- a What fraction of people had three bedrooms?
 b 16 people had two bedrooms in their home.
 How many people did she ask altogether?



Representing grouped and continuous data

When data is grouped it can be represented in a frequency diagram.

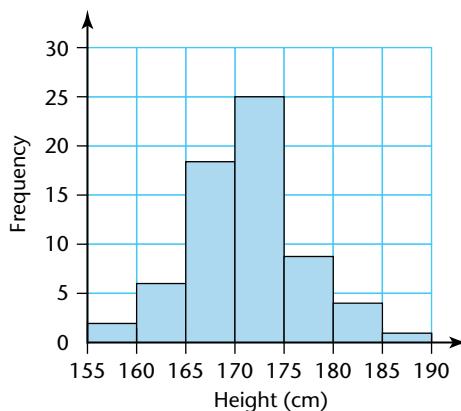
This is like a bar chart but the axes use a continuous scale.

The frequency table and diagram show information about the heights of a group of 16-year-old students.

The horizontal scale is labelled continuously and not with the groups.

The boundaries of the bars match the boundaries of the groups.

Height h (cm)	Frequency
$155 \leq h < 160$	2
$160 \leq h < 165$	6
$165 \leq h < 170$	18
$170 \leq h < 175$	25
$175 \leq h < 180$	9
$180 \leq h < 185$	4
$185 \leq h < 190$	1

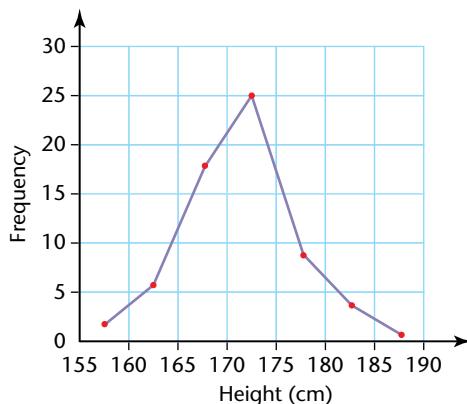


A **frequency polygon** is another way to represent grouped continuous data.

In a frequency polygon a single point is used to represent each group.

This point is plotted at the midpoint of the group and shows the frequency for the group.

This frequency polygon shows the data from the table above.



The midpoint of the group $155 \leq h < 160$ is found using the calculation

$$\frac{155 + 160}{2} = 157.5.$$

The midpoints of the groups are 157.5, 162.5, 167.5, 172.5, 177.5, 182.5, 187.5.

More than one frequency polygon can be drawn on a single grid. This makes it easier to make comparisons between the sets of data.

Example 42.4

Question

The table shows the number of words per sentence in the first 50 sentences of two books.

Number of words (w)	Frequency Book 1	Frequency Book 2
$0 < w \leq 10$	2	27
$10 < w \leq 20$	9	11
$20 < w \leq 30$	14	9
$30 < w \leq 40$	7	0
$40 < w \leq 50$	4	3
$50 < w \leq 60$	8	0
$60 < w \leq 70$	6	0

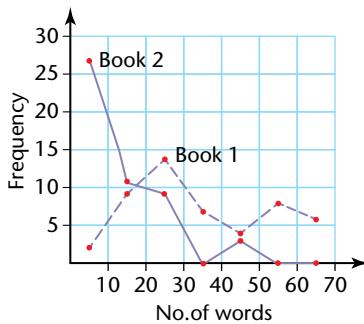
- a On the same grid, draw a frequency polygon for each book.
- b Use the frequency polygons to make two comparisons between the numbers of words per sentence in each book.

Solution

- a The midpoints of the groups are 5, 15, 25, 35, 45, 55, 65.

The two frequency polygons are plotted on the same axes.

Each line should be labelled.



- b Book 1 has a greater spread of sentence lengths than Book 2.

Book 2 has more short sentences than Book 1.

Note

When making comparisons between sets of data, you should refer back to the context of the data.

A sentence like 'the range is bigger for Book 1' is not enough.

Exercise 42.3

- 1 The manager of a sports centre recorded the ages of the women who used the swimming pool one morning.

Here are his results.

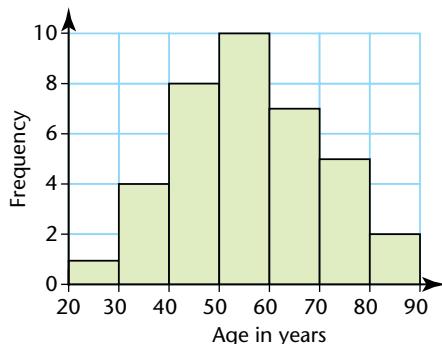
Draw a frequency diagram to show this data.

Age a (years)	Frequency
$15 \leq a < 20$	4
$20 \leq a < 25$	12
$25 \leq a < 30$	17
$30 \leq a < 35$	6
$35 \leq a < 40$	8
$40 \leq a < 45$	3
$45 \leq a < 50$	12

- 2 In a survey, the annual rainfall was measured at 100 different towns. Here are the results of the survey.

- a Draw a frequency diagram to show this data.
 b Write down the modal class.
 c Which interval contains the median value?

- 3 This frequency diagram shows the ages of people on a train one morning.



- a Make a grouped frequency table for this data.
 b Write down the modal group.
 c Explain why the youngest person may not be 20 years old.

Rainfall r (cm)	Frequency
$50 \leq r < 70$	14
$70 \leq r < 90$	33
$90 \leq r < 110$	27
$110 \leq r < 130$	8
$130 \leq r < 150$	16
$150 \leq r < 170$	2

- 4 The ages of all the people under 70 in a small village were recorded in 1995 and 2015.

The results are given in the table.

Age a (years)	Frequency in 1995	Frequency in 2015
$0 \leq a < 10$	85	50
$10 \leq a < 20$	78	51
$20 \leq a < 30$	70	78
$30 \leq a < 40$	53	76
$40 \leq a < 50$	40	62
$50 \leq a < 60$	28	64
$60 \leq a < 70$	18	56

- a On the same grid, draw a frequency polygon for each year.
 b Use the frequency polygons to compare the distribution of the ages in the two years.

Scatter diagrams

A **scatter diagram** is used to find out whether there is a **correlation**, or relationship, between two sets of data.

The data is presented as pairs of values, each of which is plotted as a coordinate point on a graph.

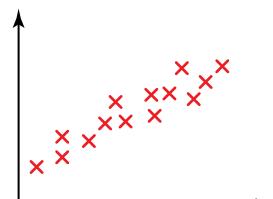
Here are some examples of what a scatter diagram could look like and how it might be interpreted.

Strong positive correlation

Here, one quantity increases as the other increases.

This is called **positive correlation**.

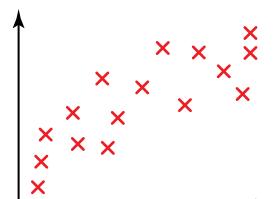
When the points are closely in line, the correlation is strong.



Weak positive correlation

These points also display positive correlation.

The points are more scattered so the correlation is weak.

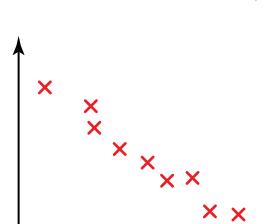


Strong negative correlation

Here, one quantity decreases as the other increases.

This is called **negative correlation**.

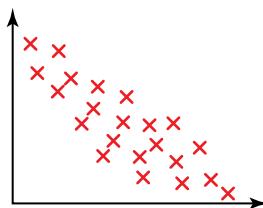
Again, the points are closely in line so the correlation is strong.



Weak negative correlation

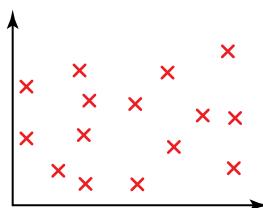
These points also display negative correlation.

The points are more scattered so the correlation is weak.



No correlation

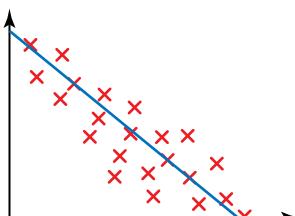
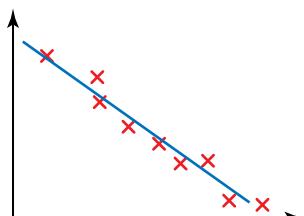
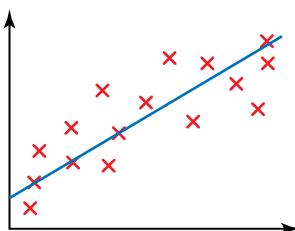
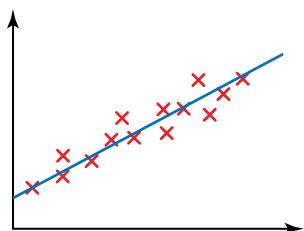
When the points are totally scattered and there is no clear pattern, there is no correlation between the two quantities.



Line of best fit

If a scatter diagram shows correlation, you can draw a **line of best fit** on it.

This line goes in the same general direction as the points and has approximately the same number of points on each side of the line.



You cannot draw a line of best fit on a scatter diagram with no correlation.

You can use a line of best fit to estimate a value when only one of the pair of quantities is known.

An estimate should only be made if the given value is within the range of the data used in the scatter diagram.

If it is outside this range, the estimate will be unreliable.

Example 42.5**Question**

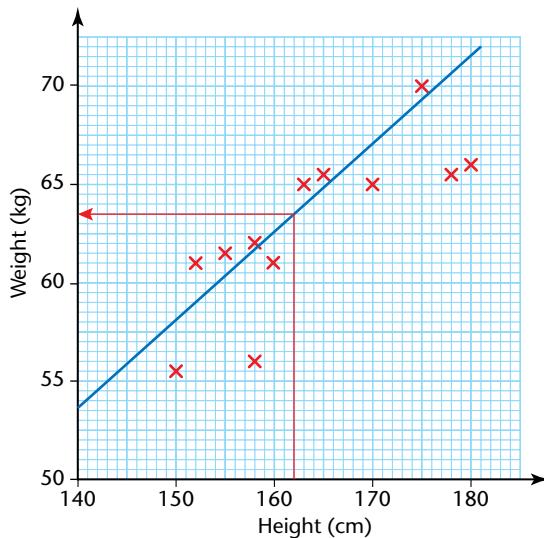
The table shows the heights and weights of 12 people.

Height (cm)	150	152	155	158	158	160	163	165	170	175	178	180
Weight (kg)	56	62	63	57	64	62	65	66	65	70	66	67

- a Draw a scatter diagram to show this data.
- b Comment on the strength and type of correlation between these heights and weights.
- c Draw a line of best fit on your scatter diagram.
- d Nimal is 162 cm tall. Use your line of best fit to estimate his weight.

Solution

a, c



b Weak positive correlation.

- d Draw a line up from 162 cm on the height axis to meet your line of best fit.

Now draw a horizontal line and read off the value where it meets the weight axis.

Nimal's probable weight is about 63 kg.

Exercise 42.4

- 1 The table shows the maths and science marks of eight students in their last examination.

Student	Maths mark	Science mark
A	10	30
B	20	28
C	96	80
D	50	55
E	80	62
F	70	70
G	26	38
H	58	48

- a** Draw a scatter diagram to show this information, with the maths mark on the horizontal axis.
- b** Describe the correlation shown in the scatter diagram.
- c** Draw a line of best fit.
- d** Use your line of best fit to estimate
- the science mark of a student who scored 40 in maths
 - the maths mark of a student who scored 75 in science.
- 2** The table shows the amount of fuel left in the tank of a car after different numbers of kilometres travelled.

Number of kilometres	Number of litres
20	7
40	5.2
60	4.2
80	2.6
100	1.2
120	0.4

- a** Draw a scatter diagram to show this information, with the number of kilometres on the horizontal axis.
- b** Describe the correlation shown in the scatter diagram.
- c** Draw a line of best fit.
- d** Use your line of best fit to estimate the number of litres left after 70 kilometres.
- 3** The table shows the marks of 15 students taking Paper 1 and Paper 2 of a maths examination. Both papers were marked out of 40.

Paper 1	Paper 2
36	39
34	36
23	27
24	20
30	33
40	35
25	27
35	32
20	28
15	20
35	37
34	35
23	25
35	33
27	30

- a** Draw a scatter diagram to show this information.
- b** Describe the correlation shown in the scatter diagram.
- c** Draw a line of best fit on your scatter diagram.
- d** Joe scored 32 on Paper 1 but was absent for Paper 2.
Use your line of best fit to estimate his score on Paper 2.

- 4 The table shows the engine size and fuel consumption of nine cars.

Engine size (litres)	Fuel consumption (km/l)
1.9	17
1.1	21
4.0	11
3.2	14
5.0	9
1.4	21
3.9	13
1.1	24
2.4	17

- a Draw a scatter diagram to show this information.
- b Describe the correlation shown in the scatter diagram.
- c Draw a line of best fit on your scatter diagram.
- d A tenth car has an engine size of 2.8 litres.

Use your line of best fit to estimate the fuel consumption of this car.

Cumulative frequency diagrams

A **cumulative frequency diagram** is a way to display a large set of grouped continuous data.

It can be used to estimate the median of the data and to give information about the spread of the data.

This table shows the distribution of the heights of 60 plants.

Height h (cm)	Frequency
$0 < h \leq 10$	1
$10 < h \leq 20$	3
$20 < h \leq 30$	9
$30 < h \leq 40$	25
$40 < h \leq 50$	11
$50 < h \leq 60$	6
$60 < h \leq 70$	3
$70 < h \leq 80$	1
$80 < h \leq 90$	1

The data has been combined in the cumulative frequency table.

The cumulative frequency is the total of all the frequencies up to and including the selected group.

Height h (cm)	Cumulative frequency
$h \leq 0$	0
$h \leq 10$	1
$h \leq 20$	4
$h \leq 30$	13
$h \leq 40$	38
$h \leq 50$	49
$h \leq 60$	55
$h \leq 70$	58
$h \leq 80$	59
$h \leq 90$	60

This row is included in the table to show that there are no plants with a height of 0 cm.

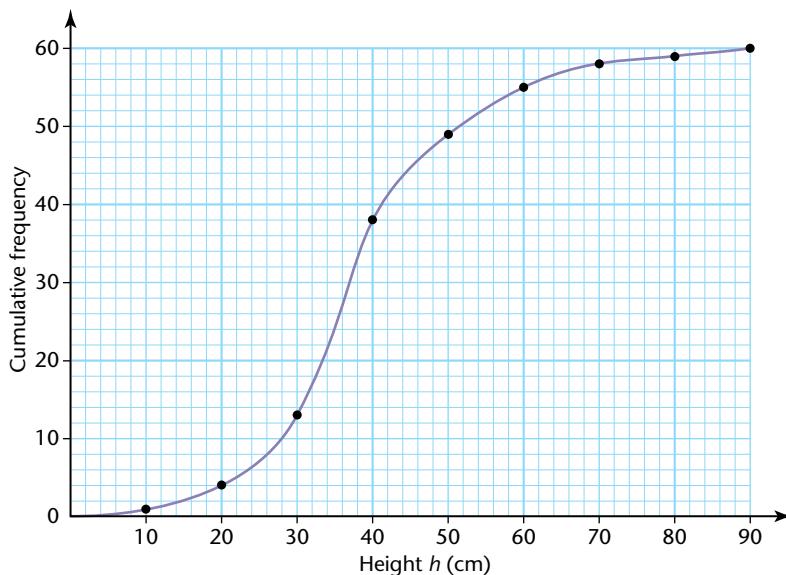
This tells you that there are 38 plants that have a height of 40 cm or less.

It is the sum of all the frequencies of the groups up to and including this one.

$$1 + 3 + 9 + 25 = 38$$

The cumulative frequency values can be used to plot a cumulative frequency diagram.

The cumulative frequency values are plotted at the upper value of each interval, so the points plotted are $(0, 0), (10, 1), (20, 4)$ and so on.



Example 42.6**Question**

The table shows the distribution of the masses of 100 apples.

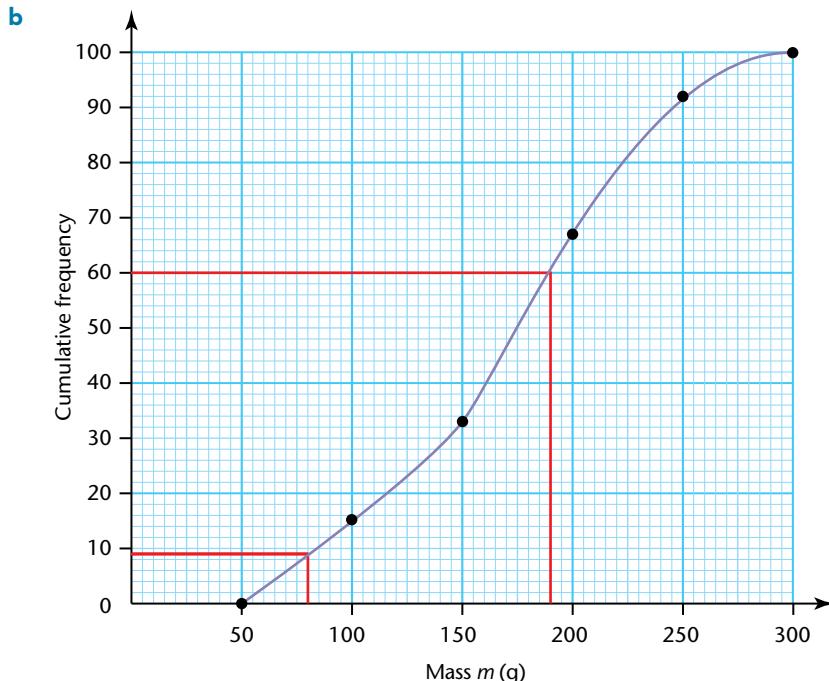
Mass m (g)	$50 < m \leq 100$	$100 < m \leq 150$	$150 < m \leq 200$	$200 < m \leq 250$	$250 < m \leq 300$
Frequency	15	18	34	25	8

- a Make a cumulative frequency table for the data.
- b Draw a cumulative frequency diagram for the data.
- c Estimate the number of apples with a mass of
 - i less than 80 g
 - ii greater than 190 g.

Solution

a	Mass m (g)	$m \leq 50$	$m \leq 100$	$m \leq 150$	$m \leq 200$	$m \leq 250$	$m \leq 300$
	Cumulative frequency	0	15	33	67	92	100

There are no apples with a mass less than or equal to 50 g, so this column is added to the cumulative frequency table.



- c i To find an estimate of the number of apples with a mass less than 80 g, read from the graph at this point. There are 9 apples with a mass less than 80 g.
- ii To find an estimate of the number of apples with a mass greater than 190 g, read from the graph at this point. There are 60 apples with a mass less than 190 g.
So there are $100 - 60 = 40$ apples with a mass greater than 190 g.

Exercise 42.5

- 1 The table shows the times taken for the 200 runners in a race.

Time in minutes (t)	Number of runners
$0 < t \leq 60$	0
$60 < t \leq 80$	10
$80 < t \leq 100$	37
$100 < t \leq 120$	72
$120 < t \leq 140$	55
$140 < t \leq 160$	18
$160 < t \leq 180$	7
$180 < t \leq 200$	1

- a Make a cumulative frequency table for the data.
 - b Draw a cumulative frequency diagram for the data.
 - c Estimate the number of runners who took
 - i less than 90 minutes
 - ii more than 150 minutes.
- 2 The table shows the lengths, in hours, of the lives of 90 lamp bulbs.

Life of lamp bulbs in hours (t)	Number of lamps
$0 < t \leq 600$	0
$600 < t \leq 625$	3
$625 < t \leq 650$	18
$650 < t \leq 675$	29
$675 < t \leq 700$	25
$700 < t \leq 725$	13
$725 < t \leq 750$	2

- a Make a cumulative frequency table for the data.
- b Draw a cumulative frequency diagram for the data.
- c Estimate the number of lamp bulbs that lasted for more than 680 hours.

Median, quartiles and percentiles

A cumulative frequency diagram can be used to find the median value of the distribution.

Look back at the cumulative frequency table and diagram on page 467.

There were 60 plants so the median value is the height of the 30th plant.

The median can also be called the 50th **percentile** because 50% of the data has a value less than or equal to that value.

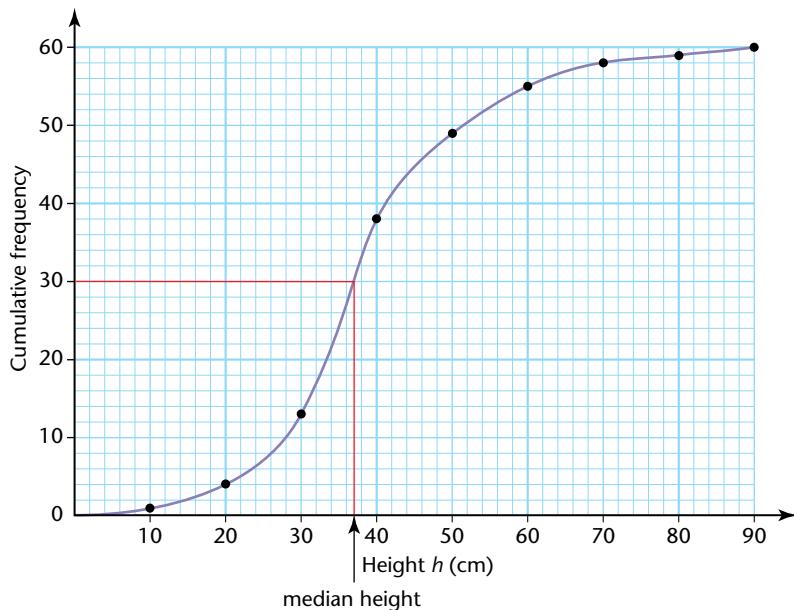
The median height is approximately 37 cm.

Note

If you were using a list of values, then for an even number of values the median would be halfway between the middle two values.

However, when using a cumulative frequency diagram for grouped data, the values of the individual items are not known.

Any results are therefore approximate and it is reasonable to use the 50th percentile.



The **lower quartile** (LQ) of the data is the 25th percentile.

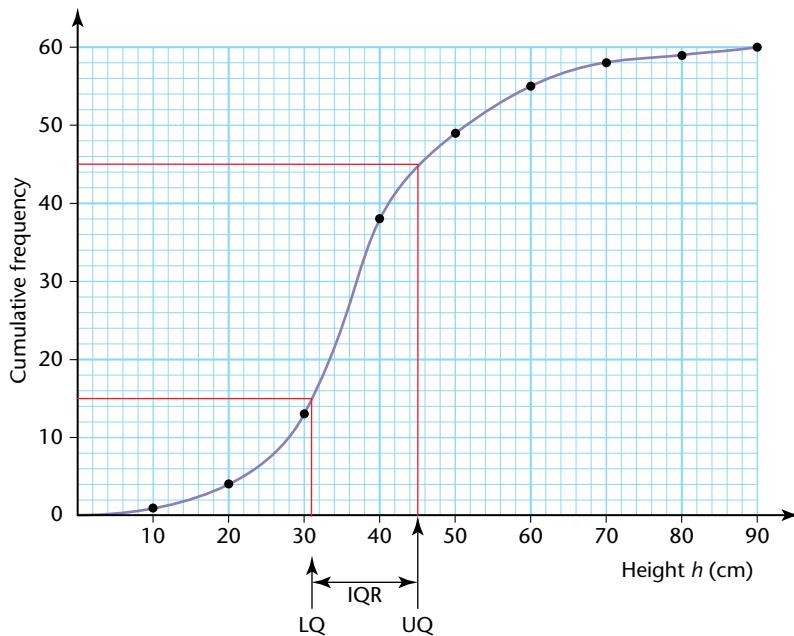
25% of the data has a value less than or equal to this value.

The lower quartile is the height of the 15th plant.

The **upper quartile** (UQ) of the data is the 75th percentile.

75% of the data has a value less than or equal to this value.

The upper quartile is the height of the 45th plant.



The **interquartile range** (IQR) is a measure of the spread of the middle 50% of the data.

The interquartile range is the difference between the upper quartile and the lower quartile.

Lower quartile = 31 cm

Upper quartile = 45 cm

Interquartile range = $45 - 31 = 14$ cm

Other percentiles can be found from the data.

For example, 10% of the data has a value less than or equal to the 10th percentile.

For this data, the 10th percentile is the height of the 6th plant and is 24 cm.

Example 42.7

Question

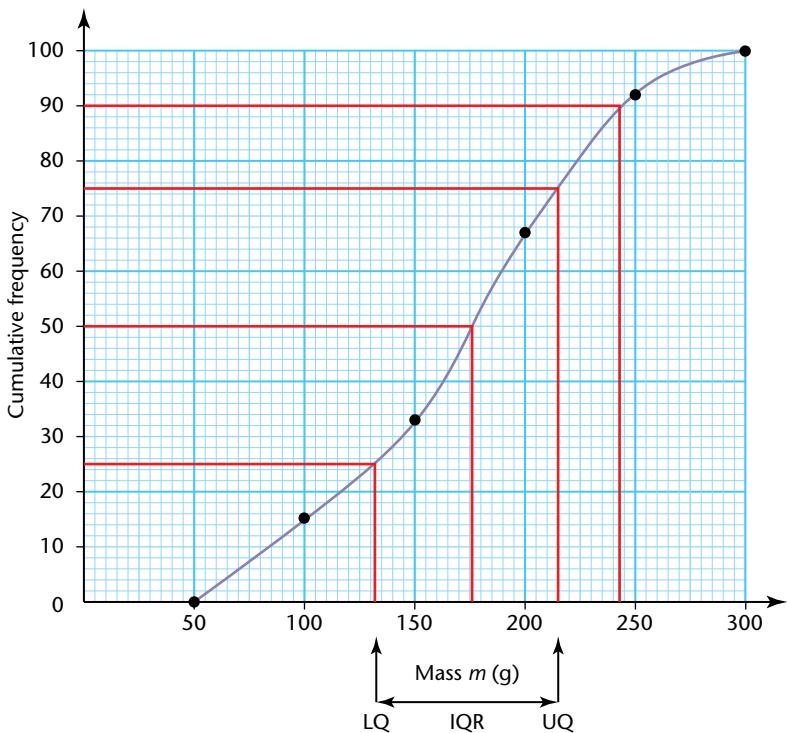
For the data in Example 42.6, estimate

- the median
- i the lower quartile
ii the upper quartile
iii the interquartile range
- the 90th percentile.

Solution

The required values can be read from the cumulative frequency diagram.

- There are 100 values, so the median is the 50th value.
Median = 175 g
- i The LQ is the 25th value.
LQ = 132 g
ii The UQ is the 75th value.
UQ = 215 g
iii $IQR = 215 - 132 = 83$ g
- 90th percentile = 243 g



Exercise 42.6

- 1 The table shows the distribution of the heights of 120 people.

Height h (cm)	Number of people
$0 < h \leq 130$	0
$130 < h \leq 140$	5
$140 < h \leq 150$	12
$150 < h \leq 160$	26
$160 < h \leq 170$	35
$170 < h \leq 180$	23
$180 < h \leq 190$	15
$190 < h \leq 200$	4

- a Make a cumulative frequency table for the data and use it to draw a cumulative frequency diagram.
- b Use your cumulative frequency diagram to estimate
- i the median height
 - ii the lower quartile
 - iii the upper quartile
 - iv the interquartile range.
- 2 The cumulative frequency table shows information about the masses of 200 potatoes.

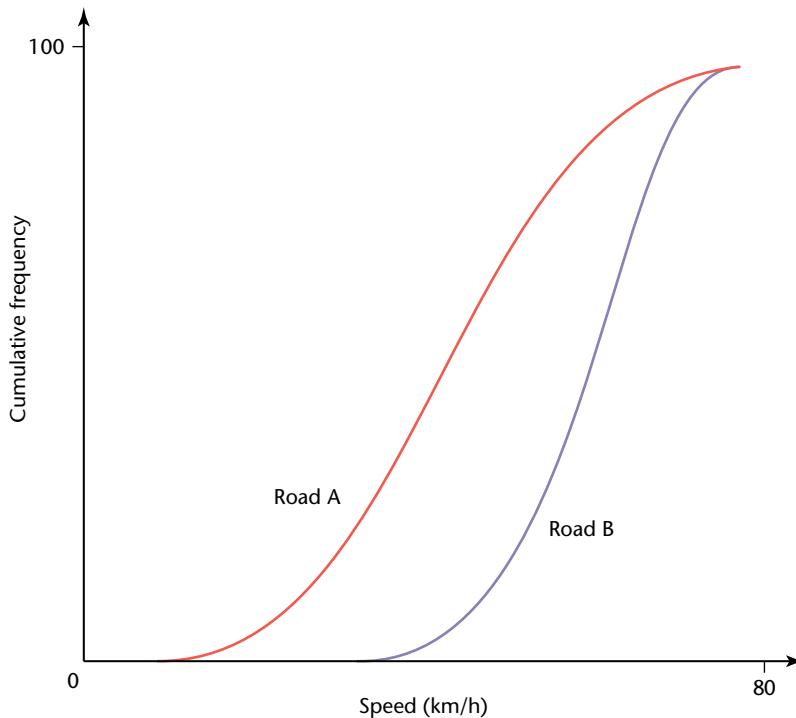
Mass m (grams)	$m \leq 0$	$m \leq 50$	$m \leq 100$	$m \leq 150$	$m \leq 200$	$m \leq 250$	$m \leq 300$	$m \leq 350$
Cumulative frequency	0	16	38	81	143	183	196	200

- a Draw a cumulative frequency diagram for this data.
- b Use your diagram to estimate the median and interquartile range of these masses.
- c Estimate the 80th percentile.
- 3 The cumulative frequency table shows information about the yield, in litres of milk, from 120 cows.

Yield x (litres)	$x \leq 5$	$x \leq 10$	$x \leq 15$	$x \leq 20$	$x \leq 25$	$x \leq 30$	$x \leq 35$
Cumulative frequency	0	10	33	68	94	111	120

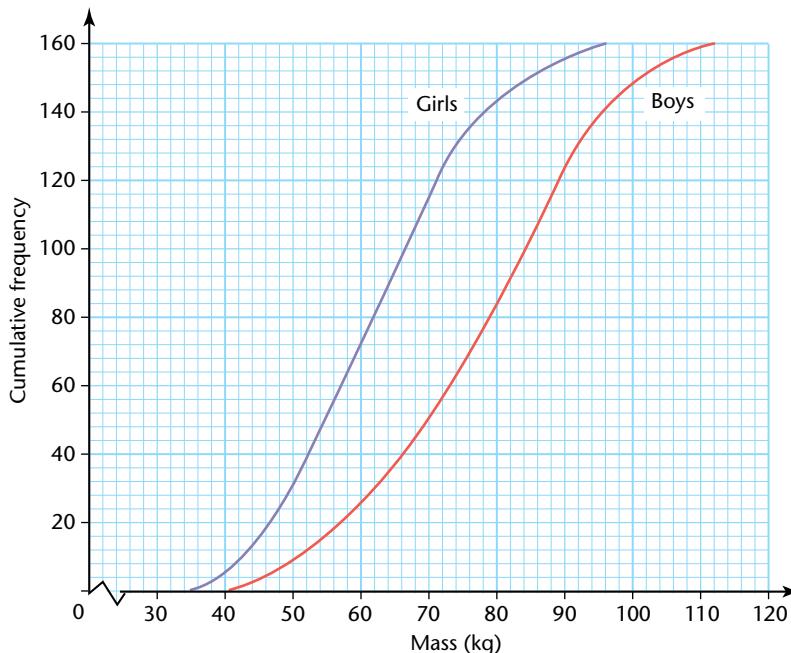
- a Draw a cumulative frequency diagram for this data.
- b Use your diagram to estimate the median and interquartile range of these amounts.
- c Estimate the number of cows that produced more than 28 litres of milk.
- d Estimate the 20th percentile.

- 4 The cumulative frequency diagram shows the distribution of the speeds of 100 cars on each of two roads.



- a On which road are the cars travelling faster on average?
State your reasons.
b On which road are the cars' speeds more consistent?
State your reasons.

- 5 The cumulative frequency diagram shows the distributions of the masses of 160 boys and 160 girls in a college.



- a Estimate the median mass of
 i the girls
 ii the boys.
 b Estimate the interquartile range for
 i the girls
 ii the boys.
 c Make two comments comparing the masses of the girls and the masses of the boys.

Histograms

A **histogram** is a frequency diagram that is used to represent continuous data.

The **area** of each bar is used to represent the frequency of the group.

The width of the bar is proportional to the class width.

The vertical scale on a histogram is the **frequency density**.

$$\text{Frequency density} = \frac{\text{Frequency}}{\text{Class width}}$$

The units of frequency density depend on the units of the data.

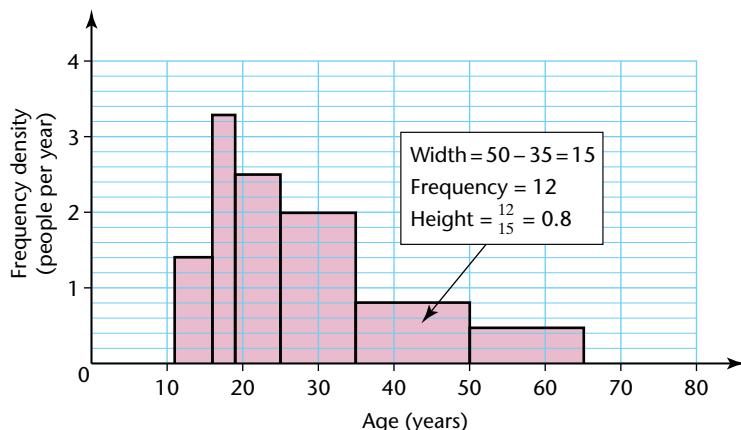
The table shows the distribution of the ages of people in a sports club.

Age in years	Frequency
11–15	7
16–18	10
19–24	15
25–34	20
35–49	12
50–64	7

The histogram below shows this distribution.

The upper boundary for the 11–15 age group is the 16th birthday. So the boundaries of this group are 11 and 16.

This applies similarly to the other groups, so the boundaries of the bars in the histogram are at 11, 16, 19, 25, 35, 50 and 65.



The calculation of the frequency density for the 35–49 group has been added to the histogram.

A similar calculation is done for each group.

For this histogram, the units of frequency density are people per year.

Note that the bars are not equal in width because the class widths are not equal.

Example 42.8

Question

An airline investigated the ages of passengers flying between London and Johannesburg.

The table shows the findings.

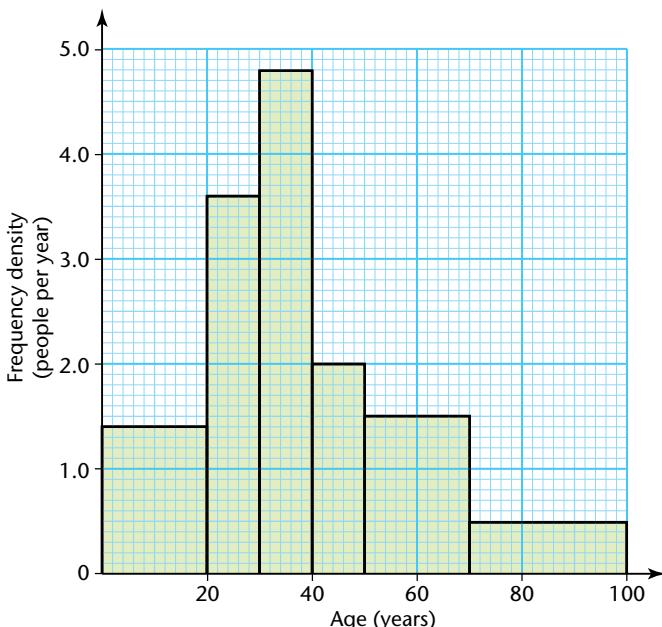
- Draw a histogram to represent this data.
- Estimate the number of passengers aged between 60 and 80.

Age A (years)	Frequency
$0 \leq A < 20$	28
$20 \leq A < 30$	36
$30 \leq A < 40$	48
$40 \leq A < 50$	20
$50 \leq A < 70$	30
$70 \leq A < 100$	15

Solution

- To draw a histogram you must first calculate the frequency density.

Age A (years)	Frequency	Class width	Frequency density (people per year)
$0 \leq A < 20$	28	20	$28 \div 20 = 1.4$
$20 \leq A < 30$	36	10	$36 \div 10 = 3.6$
$30 \leq A < 40$	48	10	$48 \div 10 = 4.8$
$40 \leq A < 50$	20	10	$20 \div 10 = 2$
$50 \leq A < 70$	30	20	$30 \div 20 = 1.5$
$70 \leq A < 100$	15	30	$15 \div 30 = 0.5$



Note

If a range includes parts of a class, then the calculation only gives an estimate as we do not know how the data is distributed within that group.

- An estimate of the number of people can be found using frequency = frequency density × class width. The range 60 to 80 contains 10 years from the 50 to 70 group and 10 years from the 70 to 100 group, so add the areas of those two sections of the histogram.

$$\text{Number of people} = 1.5 \times 10 + 0.5 \times 10 = 20$$

There are approximately 20 people aged between 60 and 80.

Exercise 42.7

- 1 200 people were surveyed to find the distances they travelled to work. The table shows the results.

Distance d (km)	Frequency
$0 \leq d < 5$	3
$5 \leq d < 10$	9
$10 \leq d < 15$	34
$15 \leq d < 20$	49
$20 \leq d < 30$	17
$30 \leq d < 40$	41
$40 \leq d < 60$	27
$60 \leq d < 100$	20

Draw a histogram to show this information.

- 2 The table shows the results of a survey to find the areas, to the nearest hectare, of 160 farms.

Area A (hectares)	Frequency
$1 \leq A < 4$	29
$4 \leq A < 8$	18
$8 \leq A < 12$	34
$12 \leq A < 16$	26
$16 \leq A < 24$	28
$24 \leq A < 30$	11
$30 \leq A < 34$	8
$34 \leq A < 40$	6

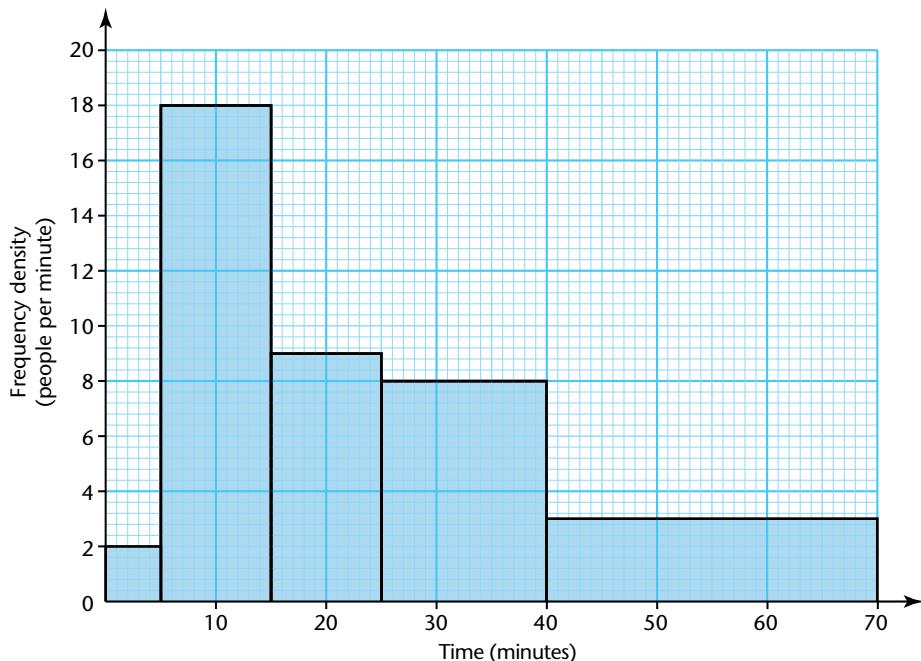
Draw a histogram to show this information.

- 3 The age of each person on a bus was recorded. The table shows the results.

Age A (years)	Frequency
$0 \leq A < 10$	0
$10 \leq A < 20$	2
$20 \leq A < 30$	3
$30 \leq A < 45$	8
$45 \leq A < 50$	5
$50 \leq A < 70$	18
$70 \leq A < 100$	12

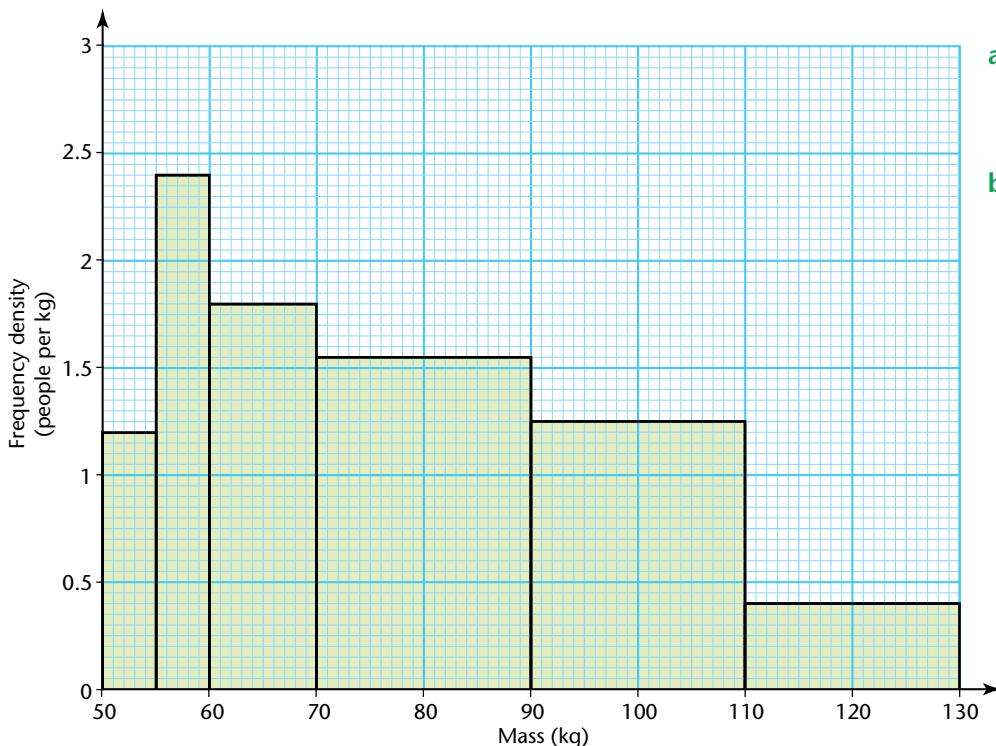
Draw a histogram to show this information.

- 4 A group of students were asked how long it took them to travel from home to school one day. The results are shown in the histogram.



- a How many students took between 15 and 25 minutes to travel from home to school?
- b Work out the total number of students asked.
- c Estimate the number of students who took between 30 and 60 minutes to travel from home to school.

- 5 The members of a gym club were weighed. The histogram represents their masses.



- a How many members of the gym were weighed?
- b Calculate an estimate of their mean mass.

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