3 Fluid Dynamics Review

3.1 Mass conservation

We consider an Eulerian volume (static volume that does not move with the fluid) and conserve mass of the fluid inside it.

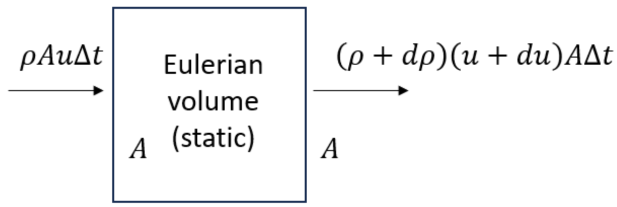

Figure 1: A fixed infinitesimal volume for mass conservation analysis

Balancing mass in time dt with $dt \to 0$, we obtain

$$dm = dm_{\text{in}} - dm_{\text{out}} = \rho(Audt) - (\rho + d\rho)(A(u + du)dt) = (\rho du + ud\rho + dud\rho)dt \tag{26}$$

$$V\frac{d\rho}{dt} = -Ad(\rho u) \implies \frac{d\rho}{dt} = -\frac{d(\rho u)}{dx} \implies \frac{d\rho}{dt} + \frac{d(\rho u)}{dx} = 0 \tag{27}$$

where $V = Adx$. The density of the fluid in the Eulerian volume is only a function of time, therefore the derivative with respect to time is the partial derivative. In 3D, the mass conservation is

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_1)}{\partial x_1} + \frac{\partial(\rho u_2)}{\partial x_2} + \frac{\partial(\rho u_3)}{\partial x_3} = 0 \tag{28}$$

$$\text{Index notation } \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j} = 0 \tag{29}$$

$$\text{Shorthand derivative notation } \rho_{,t} + (\rho u_j)_{,j} = 0 \tag{30}$$

3.2 Momentum conservation from Newton's second law

As laws are applied to Lagrangian fluid elements, we will come across the concept of a total derivative. Consider the Consider a scalar value c associated with the fluid element. This can be a velocity component or internal energy of the fluid element.

$$\begin{aligned} dc &= c(t + dt; a) - c(t; a) = c(x + dx, t + dt) - c(x, t) && (31)\\ &= c(x + dx, t + dt) - c(x + dx, t) + c(x + dx, t) - c(x, t) && (32)\\ &= \frac{\partial c(x + dx, t)}{\partial t}dt + \frac{\partial c(x, t)}{\partial x}dx && (33)\\ &= \frac{\partial c(x, t)}{\partial t}dt + \frac{\partial^2 c(x, t)}{\partial t \partial x}dxdt + \frac{\partial c(x, t)}{\partial x}dx && (34) \end{aligned}$$

Note that the distance traveled by the fluid element in time dt is $dx = u(x, t)dt$. The total derivative is then

$$\frac{dc}{dt} = \frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} \tag{35}$$

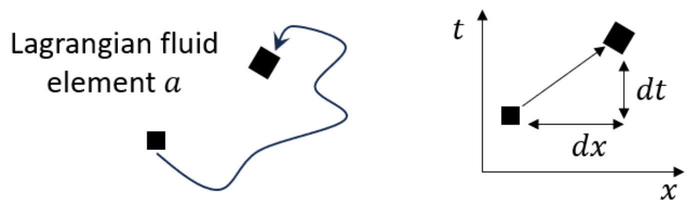

Figure 2: An infinitesimal Lagrangian fluid element, that can deform and has associated scalars. The total derivative analysis applies to changes of the scalar associated with the fluid element during an infinitesimal time change dt

where the mixed derivative term vanishes as $dx \to 0$. When the velocity vector is three-dimensional

$$\frac{dc}{dt} = \frac{\partial c}{\partial t} + u_1 \frac{\partial c}{\partial x_1} + u_2 \frac{\partial c}{\partial x_2} + u_3 \frac{\partial c}{\partial x_3} \tag{36}$$

$$\text{Index notation} = \frac{\partial c}{\partial t} + u_j \frac{\partial c}{\partial x_j} \tag{37}$$

$$\text{Shorthand derivative notation} = c_{,t} + u_j c_{,j} \tag{38}$$

Newton's second law states that

$$F = ma \tag{39}$$

$$-Adp = \rho V \frac{du}{dt} \tag{40}$$

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{dp}{dx} \tag{41}$$

In three-dimensional flows, the equation is

$$\frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{dp}{dx_i} \tag{42}$$

$$= u_{,i} + u_j u_{i,j} = -\upsilon p_{,i} \quad \text{Shorthand derivative notation} \tag{43}$$

Use $u_i\times$ Eq. 30 + $\rho\times$ Eq. 41 and chain rule to obtain the conservation form of the momentum equation,

$$(\rho u_i)_{,t} + (\rho u_i u_j)_{,j} = -p_{,i} \tag{44}$$

3.3 Energy conservation from first law of thermodynamics

The first law of thermodynamics states

$$\frac{de}{dt} = \frac{\delta q}{dt} - p \frac{d\upsilon}{dt} \tag{45}$$

$$= \frac{\delta q}{dt} + \frac{p}{\rho^2} \frac{d\rho}{dt} \tag{46}$$

$$= \frac{\delta q}{dt} - \frac{p}{\rho} \frac{\partial u}{\partial x} \tag{47}$$

Use $u_i\times$ Eq. 41 + Eq. 47 to obtain an equation for specific energy

$$\frac{d(e+u^2/2)}{dt}=\frac{\partial(e+u^2/2)}{\partial t}+u\frac{\partial(e+u^2/2)}{\partial x}=\frac{\delta q}{dt}-\frac{1}{\rho}\frac{\partial(pu)}{\partial x} \tag{48}$$

Use $\rho\times$ Eq. 48 + $(e+u^2/2)\times$ Eq. 30 to obtain the conservation form of total energy

$$\frac{\partial(\rho e+\rho u^2/2)}{\partial t}+\frac{\partial(\rho e+\rho u^2/2+p)u}{\partial x}=\rho\frac{\delta q}{dt} \tag{49}$$

For three-dimensional flows, the equation is

$$\frac{\partial(E+K)}{\partial t}+\frac{\partial(H+K)u_1}{\partial x_1}+\frac{\partial(H+K)u_2}{\partial x_2}+\frac{\partial(H+K)u_3}{\partial x_3}=\rho\frac{\delta q}{dt} \tag{50}$$

$$\text{where } K=\rho(u_1^2+u_2^2+u_3^2)/2,\quad E=\rho e,\quad H=\rho h=\rho(e+p/\rho)=E+p \tag{51}$$

$$\text{or } (E+K)_{,t}+((H+K)u_j)_{,j}=\rho\frac{\delta q}{dt}\qquad\text{Shorthand derivative notation} \tag{52}$$

Therefore, the governing equations for inviscid compressible flow are

$$\rho_{,t}+(\rho u_j)_{,j}=0 \tag{53}$$

$$(\rho u_i)_{,t}+(\rho u_i u_j)_{,j}=-p_{,i} \tag{54}$$

$$(E+K)_{,t}+((H+K)u_j)_{,j}=\rho\frac{\delta q}{dt} \tag{55}$$

In one-dimension, the equations reduce to

$$\rho_{,t}+(\rho u_1)_{,1}=0 \tag{56}$$

$$(\rho u_1)_{,t}+(\rho u_1 u_1)_{,1}=-p_{,1} \tag{57}$$

$$(E+K)_{,t}+((H+K)u_1)_{,1}=\rho\frac{\delta q}{dt} \tag{58}$$

or when written in the fully expanded form without any shorthand notations,

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0 \tag{59}$$

$$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}(\rho u^2+p)=0 \tag{60}$$

$$\frac{\partial}{\partial t}(\rho e+\rho u^2/2)+\frac{\partial}{\partial x}(\rho u^3/2+\rho eu+pu)=\rho\frac{\delta q}{dt} \tag{61}$$

3.4 Steady equations and control volume analysis

If the flow is steady, then $()_{,t}$ terms drop out, and we have

$$(\rho u_j)_{,j} = 0 \tag{62}$$

$$(\rho u_i u_j)_{,j} = -p_{,i} \tag{63}$$

$$(\rho(h+k)u_j)_{,j} = \rho u_j \frac{\delta q}{dx_j} \tag{64}$$

where $\delta q/dt = q_{,t} + u_j q_{,j}$ and $k = K/\rho$. The equations can be written in the divergence form

$$(\rho u_j)_{,j} = 0 \tag{65}$$

$$(\rho u_i u_j + p\delta_{ij})_{,j} = 0 \tag{66}$$

$$(\rho(h+k)u_j)_{,j} = \rho u_j \frac{\delta q}{dx_j} \tag{67}$$

We can use the Gauss divergence theorem to express these equations in the integral over a control volume. Note that $()_{,j} = \nabla \cdot ()$. The theorem states that for a vector $\mathbf{F}$

$$\iiint_V (\nabla \cdot \mathbf{F})\, \mathrm{d}V = \oiint_S (\mathbf{F} \cdot \hat{\mathbf{n}})\, \mathrm{d}S. \tag{68}$$

Integrating the steady conservation equations in the control volume give us

$$\oiint_S \rho u_n \, dS = 0 \tag{69}$$

$$\oiint_S \rho u_n u_i + p_i \, dS = 0 \tag{70}$$

$$\oiint_S \rho u_n (h+k)\, dS = \iiint_V \rho u_j \frac{\delta q}{dx_j} dV \tag{71}$$

For a constant cross-sectional control volume, we can write $\iiint_V \rho u \frac{\delta q}{dx} dV \approx \dot{m} \int \frac{\delta q}{dx} dx = \dot{m}\Delta q$, where $\dot{m} = (\rho u_n A)$ is the mass flow rate, which is a constant as will be shown below. Δq is the total heat transferred into the fluid in the control volume.

3.5 Examples of control volumes

Some common examples of control volumes are shown below. Please note that subscripts on the velocity, in this subsection, denote stations not vector components, since the analysis is 1D.

1. 1D Streamtube: In a streamtube, the velocity is parallel to the tube. Therefore, mass and energy conservation in a streamtube volume is

$$-\rho_1 u_1 A + \rho_2 u_2 A = 0 \implies \rho_1 u_1 = \rho_2 u_2 = \frac{\dot{m}}{A} \tag{72}$$

$$-\rho_1 u_1 A(h_1 + k_1) + \rho_2 u_2 A(h_2 + k_2) = \dot{m}\Delta q \implies h_1 + k_1 + \Delta q = h_2 + k_2 \tag{73}$$

   where $\dot{m}$ is the mass flow rate and Δq is the heat transferred into the fluid in the control volume.

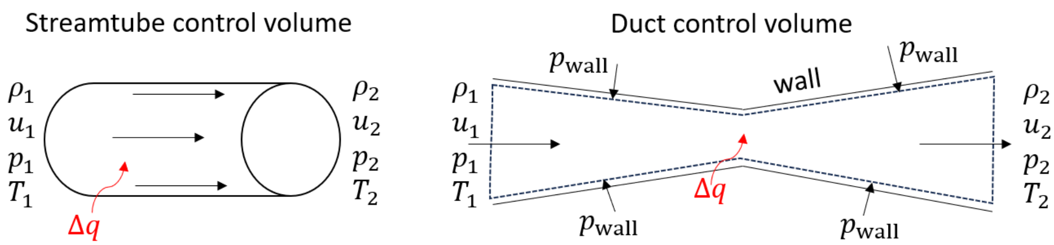

Figure 3: Examples of control volumes

2. Similarly, control volume analysis in a duct shows

$$\text{Mass: } -\rho_1 u_1 A + \rho_2 u_2 A = 0 \implies \rho_1 u_1 A_1 = \rho_2 u_2 A_2 = \dot{m} \tag{74}$$

$$\text{Momentum: } F_{\text{fluid to wall}} = -F_{\text{wall to fluid}} = -\int_{S3 \,\cup\, S4} p dS \cdot \hat{n} \tag{75}$$

$$= (p_2 + \rho_2 u_2^2)A_2 - (p_1 + \rho_1 u_1^2)A_1 = p_2 A_2 - p_1 A_1 + \dot{m}(u_2 - u_1) \tag{76}$$

$$\text{Energy: } h_1 + k_1 + \Delta q = h_2 + k_2 \tag{77}$$

3.6 Total/Stagnation quantities

The total/stagnation quantity is defined as the thermodynamic state of the fluid element if it were isentropically stagnated using a hypothetical process. Consider the hypothetical streamline along which this process takes place. Note that isentropic process also implies adiabatic process. Then, using a control volume analysis, the energy equation gives us

$$h_1 + k_1 = h_{01} + k_{01} = h_{01} \implies \frac{T_{01}}{T} = 1 + \frac{k_1}{c_p T} \tag{78}$$

$$s_1 = s_{01} \tag{79}$$

$$\frac{p_{01}}{p_1} = \left(\frac{T_{01}}{T}\right)^{\frac{\gamma}{\gamma-1}} \tag{80}$$

$$\frac{\rho_{01}}{\rho} = \left(\frac{T_{01}}{T_1}\right)^{\frac{1}{\gamma-1}} \tag{81}$$

3.6.1 Relations between total/Stagnation quantities for different fluid elements in adiabatic flow

Now we compare fluid elements in different streamlines/streamtubes. For any two fluid elements in a flow, if the flow is adiabatic (has no heat transfer), then

$$h_{01} = h_1 + k_1 = h_\infty + k_\infty, \; h_{02} = h_2 + k_2 = h_\infty + k_\infty \tag{82}$$

$$\implies h_{01} = h_{02} \implies T_{01} = T_{02} \tag{83}$$

Therefore, any points in an adiabatic flow have the same stagnation enthalpy (or equivalently stagnation temperature). Further, the relationship between the stagnation pressures is given by Eq.

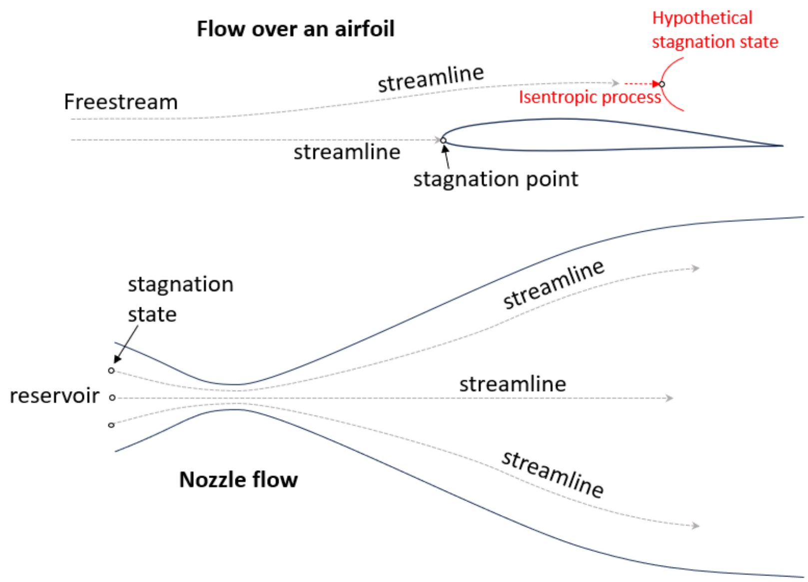

Figure 4: Encountering stagnation state in flow-fields.

84.

$$\frac{p_{02}}{p_{01}} = \exp\left(\frac{s_{01} - s_{02}}{c_p}\right) = \exp\left(\frac{s_1 - s_2}{c_p}\right) \tag{84}$$

where we have used $T_{01} = T_{02}$.

3.6.2 Relations between total/Stagnation quantities for different fluid elements in isentropic flow

Isentropic flow is also adiabatic. Therefore, we have $T_{01} = T_{02}$. Furthermore, $s_2 = s_1$, therefore $p_{02} = p_{01}$.

3.7 Adiabatic 1D flow with local entropy jump

Consider a fluid element along a steady streamline/pathline. We can use 1D conservation of energy using Navier-Stokes equations to obtain

$$\frac{\partial}{\partial x}(\rho u h_0) = \frac{\partial}{\partial x}(q_x + \tau_{xx} u) \tag{85}$$

Here q_x is conduction/heat diffusion, and τ_{xx} is viscous stress. If the fluid element encounters a shock, the entropy increases due to molecular dissipative processes. Control volume analysis using Eq. 85 will tell us that the total enthalpy change only depends on the outer faces of the volume, and not what occurs locally inside. This is because of the conservative nature of the diffusion processes and a consequence of total energy being conserved irrespective of molecular dissipation losses.

$$(\rho u h_0)_2 - (\rho u h_0)_1 = (q_x + \tau_{xx} u)_2 - (q_x + \tau_{xx} u)_1 \tag{86}$$

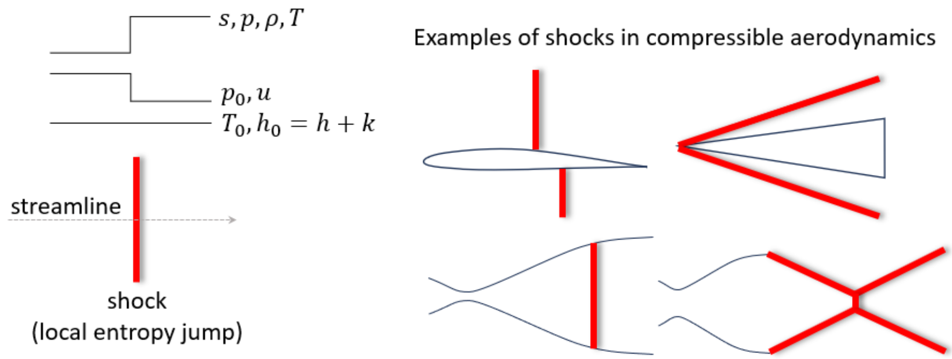

Figure 5: Example of a local entropy jump: a shock wave

Therefore, as long as there are no dissipative process at stations 1 and 2, or equivalently, $q_x = 0, \tau_{xx} = 0$

$$\rho_\infty u_\infty (h_{02} - h_{01}) = 0 \implies h_{02} = h_{01} \implies T_{02} = T_{01} \tag{87}$$

For example, this would correspond to stations before and after the shock. Therefore, the stagnation enthalpy remains constant even if the fluid element encounters isolated dissipative processes, such as shocks. Note that the total enthalpy is not constant *inside* the shock, because the molecular dissipation terms on the RHS need not be zero.