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CALCULATING AND DRAWING BELYI PAIRS

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The paper contains a survey of the current state of a constructive part in dessin d'enfants theory. Namely, it is devoted to actual establishing the correspondence between the Belyi pairs and their combinatorial-topological representations. This correspondence is established in terms of equivalence of categories, and all relevant categories are introduced. Several connections with arithmetic are discussed. One of the sections presents a possible generalization of the theory, in which three branch points, allowed for the Belyi functions, are replaced by four. Several directions for further research are presented. Bibliography: 80 *titles.*

0. Introduction

We agree (more or less) in this conference on which kinds of embedded graphs deserve our attention. Some of us prefer to paint vertices, or edges, or both, some consider distinguished elements, some embed graphs into nonoriented or bordered surfaces, etc. All these classes of objects are rather close to each other and are very well classified and counted.

However, since the *dessins d'enfants* is one of the topics of the conference, I am going to concentrate on the embedded graphs as illustrations of other structures that, at the first glance, belong to totally different mathematics, that is, of the *Belyi pairs* in several versions.

According to Grothendieck, *...il y a une identité profonde entre la combinatoire des cartes finies d'une part, et la géometrie des courbes algébriques définies sur des corps de nombres, de l'autre. Ce résultat profond, joint à l'interprétation algébrico-géométrique des cartes finies, ouvre la porte sur un monde nouveau, inexploré – et à porté de main de tous qui passent sans le voir* [28]. This *monde nouveau* is several decades old now, and I give an incomplete overview of some of its inhabited parts, concentrating mostly on the *identité profonde* itself.

After [28] was generally accepted by the community as a *mathematical* text[1] (though of a highly non-standard form), the interested mathematicians split roughly in two groups. The representatives of the "abstract" one started to develope the general ideas from [28] (non-Abelian geometry, the Grothendieck-Teichmüller tower,...), and we do not discuss these subjects in the present paper. The other "down-to-Earth" group started the actual case-by-case (family-by-family...) realization of the *identité profonde*, i.e., studying the correspondence between the dessins d'enfants and Belyi pairs.

In the late 1980s and early 1990s, just several people around the globe devoted themselves to this activity; all of us seem to have known each other personally at that time. Nowadays it is a well-developed branch of mathematics with hundreds of active researchers (including physicists); see, e.g., survey [73] with 156 references therein.

The outline of the paper is as follows. In Sec. 1, several categories are introduced and equivalences between them are constructed; this section can be skipped by those who hate abstract nonsense (with just some general perspective lost). Section 2 is in a sense a central one: the problem of object-by-object correspondence is discussed there both formally and informally. Section 3 is devoted to a more special problem of studying the Galois orbits of dessins. Section 4 contains somewhat random examples of calculations that highlight certain problems of the general theory; an attempt was made to follow a chronological order, and the

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[1]The author became involved as a participant of I. M. Gelfand's Moscow seminar, where [28] was analyzed line by line.

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genus-0 case is discussed in some detail. In Sec. 5, some catalogs of dessins, Belyi pairs, and related objects are listed. Section 6 contains a discussion of a further possible development of the theory, where the Belyi restriction on the number of branch points (≤ 3) is replaced by weaker conditions. We complete the paper by brief concluding remarks in Sec. 7.

I am deeply grateful to the participants of my MSU seminar for many years of cooperation and understanding, both professional and human. I am also grateful to N. Adrianov, Yu. Kochetkov, and S. Natanzon for consultations on various mathematical subjects treated in this paper. I am indebted to N. Adrianov, S. Dawydiak, A. Zvonkin, and the referee for critical comments on preliminary versions of the text.

1. Categories of dessins d'enfants and Belyi pairs

We introduce two versions of categories of both types.

1.1. The category $\mathcal{DESS}$. The objects of $\mathcal{DESS}$ are the *dessins d'enfant* in the sense of [28], i.e., the triples of topological spaces

$$\mathrm{X}_0 \subset \mathrm{X}_1 \subset \mathrm{X}_2$$

such that X_0 is a nonempty finite set, whose elements are called *vertices*, X_2 is a compact connected oriented surface, and X_1 is an *embedded graph*, which means that the complement $\mathrm{X}_1 \setminus \mathrm{X}_0$ is homeomorphic to a disjoint union of real intervals called *edges*. We also require that the complement $\mathrm{X}_2 \setminus \mathrm{X}_1$ is homeomorphic to a disjoint union of open disks called *faces*. The difference between dessins and *two-dimensional cell complexes* lies in the concepts of *morphisms*.

In order to give a short definition of morphisms in $\mathcal{DESS}$, we add the condition $\mathrm{X}_{-1} = \varnothing$ to each above triple, and call a continuous mapping of surfaces *admissible* if it respects the orientation, is *open*,[2] and respects the differences, i.e., such a mapping of triples $f : (\mathrm{X}_2, \mathrm{X}_1, \mathrm{X}_0) \to (\mathrm{Y}_2, \mathrm{Y}_1, \mathrm{Y}_0)$ should satisfy the condition

$$f(\mathrm{X}_i \setminus \mathrm{X}_j) \subseteq \mathrm{Y}_i \setminus \mathrm{Y}_j$$

for $-1 \leq j < i \leq 3$. Two admissible mappings are called *admissibly equivalent* if they are homotopic in the class of admissible mappings; the morphisms in $\mathcal{DESS}$ are defined as the classes of admissible equivalence of admissible mappings.

1.2. The category $\mathcal{DESS}_3$. The objects of $\mathcal{DESS}_3$ are the *tricolored* dessins, i.e., the dessins $\mathrm{X}_0 \subset \mathrm{X}_1 \subset \mathrm{X}_2$ endowed with *coloring mapping*

$$\mathrm{col}_3 : X_1 \longrightarrow \{blue, green, red\}$$

constant on the edges. It is required that

(0) any vertex is incident with edges of only <u>two</u> colors;

(1) any edge has <u>two</u> vertices in its closure;

(2) any face has <u>three</u> edges in its closure that are colored pairwise differently.

Taking into account assumption **(0)**, we color every vertex by the (only remaining) color different from the colors of incident edges. In view of assumption **(2)**, the connected components of $\mathrm{X}_2 \setminus \mathrm{X}_1$ are called (topological) *triangles*. From the *orientability* of X_2, one can deduce that these triangles can also be colored, now in *black* and *white*, in such a way that the *neighboring* triangles, i.e., those having a common edge, are colored differently.

Thus the coloring mapping col_3 can be extended to

$$\mathrm{col}_5 : X_2 \longrightarrow \{black, blue, green, red, white\}$$

[2] According to the somewhat forgotten theory, developed by S. Stoilow, any open mapping of Riemann surfaces is locally topologically conjugated to a holomorphic one, see [75].

with exactly two choices of black/white coloring that correspond to the orientations of X_2. We agree that the positive-counter-clockwise orientation of the white triangles corresponds to the *blue-green-red-blue* cyclic order of the colors of edges in its closure; this choice will be motivated below.

The objects of $\mathcal{DESS}_3$ are called *colored triangulations*; we note, however, that there is exactly one object of this category that is not a triangulation of a surface in the usual sense; this object is formed by a pair of black and white triangles with colored edges after identifying edges with the same color.

The morphisms in $\mathcal{DESS}_3$ are defined in the same way as in $\mathcal{DESS}$ with additional assumption of *color-respecting*[3].

1.3. The category $\mathcal{BELP}(\daleth)$ over a field $\daleth$. We assume that $\daleth$ is *algebraically closed*. Then the objects of $\mathcal{BELP}(\daleth)$ are the *Belyi pairs* $(\mathbf{X}, \beta)$, where $\mathbf{X}$ is a complete irreducible smooth curve over $\daleth$ and β is a (normalized) *Belyi function*, i.e., a non-constant *separable* morphism $\beta : \mathbf{X} \to \mathbf{P}_1(\daleth)$ with at most three-element set of branch points

$$\operatorname{bran}(\beta) \subseteq \{0, 1, \infty\}.$$

In the main case of our concern, that is, under the assumption $\operatorname{char}(\daleth) = 0$, it means simply[4] that $\#\beta^{-1\circ}(c) = \deg \beta$ for all $c \in \mathbf{P}_1(\daleth) \setminus \{0, 1, \infty\}$.

A morphism in $\mathcal{BELP}(\daleth)$ from $(\mathbf{X}, \beta)$ to $(\mathbf{X}', \beta')$ is defined as a morphism $f : \mathbf{X} \to \mathbf{X}'$ of curves such that the diagram

$$\begin{tikzcd} \mathbf{X} \arrow[rr, "f"] \arrow[dr, "\beta"'] & & \mathbf{X}' \arrow[dl, "\beta'"] \\ & \mathbf{P}_1(\daleth) & \end{tikzcd}$$

is commutative.

1.4. The category $\mathcal{BELP}_2(\daleth)$ over a field $\daleth$. It is a full subcategory[5] of $\mathcal{BELP}(\daleth)$. A Belyi pair $(\mathbf{X}, \beta)$ is an object of $\mathcal{BELP}_2(\daleth)$ if and only if β is *clean*, i.e., all the ramification indices over 1 are precisely 2. In other words, for any $P \in \mathbf{X}$ the equality $\beta(P) = 1$ implies $\beta - 1 \in \mathfrak{m}_P^2 \setminus \mathfrak{m}_P^3$, where $\mathfrak{m}_P$ is the (only) maximal ideal of the local ring $\mathcal{O}_P$ of the rational functions on $\mathbf{X}$ that are regular in P.

1.5. Functors. The obvious functors are

- the inclusion of a full subcategory

$$\mathcal{BELP}_2(\daleth) \hookrightarrow \mathcal{BELP}(\daleth);$$

- the color-forgetting functor

$$\mathcal{DESS}_3 \longrightarrow \mathcal{DESS}.$$

The most important are

$$\mathbf{draw} : \mathcal{BELP}_2(\mathbb{C}) \longrightarrow \mathcal{DESS}$$

[3]The "same" category was considered in [38] under the name *oriented hypermaps*; our vertices of three colors were called *hypervertices*, *hyperedges*, and *hyperfaces*.

[4]I use the notation $^{-1\circ}$ for the *compositional* inverse in order to distinguish it from the *algebraic* inverse, e.g., $\tan^{-1\circ} = \arctan$ while $\tan^{-1} = \cot$.

[5]See, e.g., [52] for the standard categorical concepts.

and

$$\mathbf{paint} : \mathcal{BELP}(\mathbb{C}) \longrightarrow \mathcal{DESS}_3.$$

In both cases to a Belyi pair $(\mathbf{X}, \beta)$ a dessin d'enfant with

$$X_2 := \mathbf{top}(\mathbf{X})$$

is assigned; here **top** means the forgetting functor that assigns to a complex algebraic curve (= Riemann surface) the underlying topological oriented surface.

For $(\mathbf{X}, \beta) \in \mathcal{BELP}_2(\mathbb{C})$, we set

$$X_1 := \beta^{-1\circ}([0,1]) \text{ and } X_0 := \beta^{-1\circ}(\{0\}).$$

The condition on the ramification of β over 1 implies that while $P \in X_2$ moves along some edge (a connected component of $X_1 \setminus X_0$) from one vertex (an element of X_0) to another, the point $\beta(P)$ moves from 0 to 1 and back, the edge being *folded* in the point of $\beta^{-1\circ}(1)$; the local coordinate z centered at this point can be chosen so that $\beta = 1 + z^2$ in its domain.

In order to define the functor **paint**, we introduce the *Belyi sphere* $\mathbf{P}_1(\mathbb{C})^{\mathsf{Bel}}$, which is the *colored Riemann sphere* $\mathbf{P}_1(\mathbb{C})$. Decomposing

$$\mathbf{P}_1(\mathbb{C}) = \mathbb{C} \coprod \{\infty\},$$

we define this coloring as

$$\mathrm{col}_5^{\mathsf{Bel}} : \mathbf{P}_1(\mathbb{C}) \longrightarrow \{\mathit{black}, \mathit{blue}, \mathit{green}, \mathit{red}, \mathit{white}\} :$$

$$z \mapsto \begin{cases} \mathit{black} \text{ if } z \in \mathbb{C} \setminus \mathbb{R} \text{ and } \operatorname{Im} z < 0, \\ \mathit{white} \text{ if } z \in \mathbb{C} \setminus \mathbb{R} \text{ and } \operatorname{Im} z > 0, \\ \mathit{blue} \text{ if } z \in \mathbb{R}_{<0} \text{ or } z = 1, \\ \mathit{green} \text{ if } z \in (0,1) \text{ or } z = \infty, \\ \mathit{red} \text{ if } z \in \mathbb{R}_{>1} \text{ or } z = 0. \end{cases}$$

The choice of colors is motivated as follows. The *black* and *white* for the lower and the upper parts is quite traditional (hell and heaven...), while the real line is colored in such a way that *blue* (symbolizing *cold*) corresponds to negative numbers, while *red* (symbolizing *hot*) corresponds to positive ones. The *green* is just in between and is assigned no meaningful association. The vertices of the colored topological "triangle" $\mathbf{P}_1(\mathbb{R})$ have the same color as the opposite side.

Furthermore, the colors of the pieces of the real line occur in the *alphabetical* order. The above-promised motivation of the choice of "colored' orientation can be given now: the traditional counter-clockwise detour around the white triangle corresponds to moving along the real line form $-\infty$ to ∞.

The Belyi pair

$$(\mathbf{P}_1(\mathbb{C})^{\mathsf{Bel}}, \text{ identity})$$

can be considered as the (colored) *final* object of the category $\mathcal{BELP}(\mathbb{C})$.

Now we can complete the definition of the functor **paint**: for a Belyi pair $(\mathbf{X}, \beta)$, the surface $X_2 := \mathbf{top}(\mathbf{X})$ is colored by $\mathrm{col}_5 := \beta^* \mathrm{col}_5^{\mathsf{Bel}}$, i.e., the points of the surface are colored according to the colors of their images under the Belyi mapping: for any $P \in X_2$,

$$\mathrm{col}_5(P) := \mathrm{col}_5^{\mathsf{Bel}}(\beta(P)).$$

Obviously, the sets X_1 and X_0 turn out to be the closure of the union of the green edges and the set of isolated red points, respectively.

1.6. Intermediate category equivalences. The following result is a step towards *l'identité profonde.*

Theorem. *The functors*

$$\mathbf{draw} : \mathcal{BELP}_2(\mathbb{C}) \longrightarrow \mathcal{DESS}$$

and

$$\mathbf{paint} : \mathcal{BELP}(\mathbb{C}) \longrightarrow \mathcal{DESS}_3$$

define the equivalences of categories.

Sketch of the proof. The detailed proof (straightforward but tedious) is written up in [67]; some elements of it can be found in [29]. Similar formulations are contained in many papers, see, e.g., [73]. We just present some necessary constructions.

The functor $\mathcal{DESS}_3 \longrightarrow \mathcal{BELP}(\mathbb{C})$, that is the inverse to **paint**, is constructed in the following way.

Given a surface X_2 with a coloring function col_5 on it, one creates the sets B and W of black and white open triangles, respectively. Since each triangle has exactly three neighboring triangles of the opposite color and since each neighbor is defined by the color of the common edge in the closures, we have three involutions

$$\mathrm{b}, \mathrm{g}, \mathrm{r} : B \coprod W \xrightarrow{\sim} B \coprod W,$$

each of which is defined by "crossing" the edge of the corresponding color. Thus, the set $B \coprod W$ is acted upon by the group

$$\langle \mathrm{b} \rangle * \langle \mathrm{g} \rangle * \langle \mathrm{r} \rangle \simeq \mathsf{C}_2 * \mathsf{C}_2 * \mathsf{C}_2,$$

where $*$ means the amalgamated product and C_2 is a cyclic group of order 2. The elements of the product are the words in the alphabet {b,g,r} without repeated letters, and should be thought of as itineraries of vertex-avoiding trips around X_2, where each letter shows the color of the current edge to be crossed.

The introduced action of $\mathsf{C}_2 * \mathsf{C}_2 * \mathsf{C}_2$ on $(B \coprod W)$ is *transitive*, because of the *connectedness* of X_2.

Fixing an arbitrary "base" triangle $t_0 \in W$, we introduce a discrete analog of the *fundamental group* of a dessin. By definition, it is a *stationary group*

$$\Pi_1(X_2, \mathrm{col}_5; t_0) := (\mathsf{C}_2 * \mathsf{C}_2 * \mathsf{C}_2)_{t_0};$$

in the above terms it corresponds to *round trips* starting and ending at t_0.

This group is isomorphic to a true fundamental group

$$\Pi_1(X_2, \mathrm{col}_5; t_0) \simeq \pi_1(X_2 \setminus X_0, \star),$$

where $\star \in t_0$ is an arbitrary point.

The index $(\mathsf{C}_2 * \mathsf{C}_2 * \mathsf{C}_2 : \Pi_1(X_2, \mathrm{col}_5; t_0))$ is finite and is equal to the number of triangles $\#(B \coprod W)$. Moreover, in view of the orientability of X_2, resulting in the black and white coloring of the triangles, the "fundamental" group $\Pi_1(X_2, \mathrm{col}_5; t_0))$ consists of words of *even* length. The subgroup of such words in $\mathsf{C}_2 * \mathsf{C}_2 * \mathsf{C}_2$ has index 2 and is isomorphic to a free group with two generators. Fix

$$\mathbb{Z} * \mathbb{Z} \simeq \mathsf{Free}_2 = \langle \mathrm{gb}, \mathrm{rb} \rangle \hookrightarrow \mathsf{C}_2 * \mathsf{C}_2 * \mathsf{C}_2;$$

the reason for this choice will be explained soon.

Now define the *universal* colored triangulation as a tessellation of the hyperbolic plane $\mathcal{H} := \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$ by the *ideal* triangles formed by iterations of reflections across the sides starting, say, from the ideal triangle

$$\mathbf{t}_0 := \left\{\tau \in \mathcal{H} \mid 0 \le \operatorname{Re}\tau \le 1, \operatorname{Im}\tau \ge \sqrt{\frac{1}{4} - \left(\operatorname{Re}\tau - \frac{1}{2}\right)^2}\right\}.$$

Choose $\mathbf{t}_0$ to be *white*. Consider the only conformal equivalence of its interior with upper half-plane

$$\operatorname{Int}(\mathbf{t}_0) \xrightarrow{\simeq} \mathcal{H}$$

that sends 0 to 0, 1 to 1, and ∞ to ∞. Identifying the just introduced $\mathcal{H}$ (*different* from the plane containing $\mathbf{t}_0$) with the upper Belyi hemisphere

$$\mathcal{H} = \mathbf{P}_1(\mathbb{C})^{\mathsf{Bel}}_{white} \hookrightarrow \mathbf{P}_1(\mathbb{C})^{\mathsf{Bel}},$$

we color the boundary of $\mathbf{t}_0$ according to this identification; the sides $\operatorname{Re}\tau = 0$, $(\operatorname{Re}\tau - \frac{1}{2})^2 + (\operatorname{Im}\tau)^2 = \frac{1}{4}$, and $\operatorname{Re}\tau = 1$ turn out to be *blue*, green, and red, respectively. The coloring of the whole tessellated $\mathcal{H} \supset \mathbf{t}_0$ is defined by the following rule: any reflection *preserves the colors of sides and changes the colors of triangles.*

The reflections against the sides of $\mathbf{t}_0$ are given by the antiholomorphic involutions

$$\begin{aligned} &\mathrm{b}\colon \tau \mapsto -\overline{\tau}, \\ &\mathrm{g}\colon \tau \mapsto \frac{\overline{\tau}}{2\overline{\tau} - 1}, \\ &\mathrm{r}\colon \tau \mapsto 2 - \overline{\tau}. \end{aligned}$$

Therefore their compositions are the fractional-linear transformations

$$\begin{aligned} &\mathrm{gb}\colon \tau \mapsto \frac{\tau}{2\tau + 1}, \\ &\mathrm{rb}\colon \tau \mapsto \tau + 2, \end{aligned}$$

or, using the correspondence between fractional-linear transformations and matrices,

$$\mathrm{gb} \longleftrightarrow \pm\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \ \mathrm{rb} \longleftrightarrow \pm\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

where for a matrix $M \in \mathsf{SL}_2(\mathbb{Z})$ we denote by $\pm M$ its image in

$$\mathsf{PSL}_2(\mathbb{Z}) := \frac{\mathsf{SL}_2(\mathbb{Z})}{\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right\}}.$$

The group $\Gamma(2) := \left\langle \pm\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\rangle$, generated by the images of gb and rb, is called the *principal congruence subgroup*; it consists of the matrices with odd entries on the principal diagonal and even off-diagonal entries. It is known that there are no relations between the above generators and hence

$$\Gamma(2) \simeq \mathsf{Free}_2 \simeq \pi_1(\ddot{\mathbb{C}}),$$

where $\ddot{\mathbb{C}} := \mathbb{C} \setminus \{0, 1\}$. Moreover, the holomorphic mapping

$$\mathcal{H} \longrightarrow \frac{\mathcal{H}}{\Gamma(2)} \simeq \ddot{\mathbb{C}}$$

is the *universal cover*; it can be realized as the extension (by the *symmetry principle*) of the above conformal mapping $\mathrm{Int}(\mathbf{t}_0) \to \mathbf{P}_1(\mathbb{C})^{\mathsf{Bel}}_{white}$. For this cover, we introduce the nonstandard notation

$$\dddot{\beta}_\infty : \mathcal{H} \longrightarrow \ddot{\mathbb{C}}.$$

Note that it is a "world constant," i.e., does not depend on any arbitrary choices. Unfortunately, our (easily memorizable) normalizations are a bit inconsistent with the classical notations; e.g., according to [34],

$$\dddot{\beta}_\infty = 1 - \frac{1}{k^2},$$

where k^2 is defined by the beautiful formula

$$k(\tau)^2 \equiv 1 - \prod_{n=1}^{\infty} \Big[\frac{1 - \mathrm{e}^{(2n-1)\pi \mathrm{i}\tau}}{1 + \mathrm{e}^{(2n-1)\pi \mathrm{i}\tau}}\Big]^8.$$

Our notation $\dddot{\beta}_\infty$ looks more natural if we interpret the doubly punctured affine line as the triply punctured projective line

$$\ddot{\mathbb{C}} =: \dddot{\mathbb{P}}_1(\mathbb{C}) := \mathbb{P}_1(\mathbb{C}) \setminus \{0, 1, \infty\}$$

and for a Belyi pair $(\mathbf{X}, \beta)$ set $\dddot{\mathbf{X}} := \beta^{-1\circ}(\dddot{\mathbb{P}}_1(\mathbb{C}))$; then we consider the *nonramified* covering

$$\dddot{\beta} := \beta|_{\dddot{\mathbb{X}}} : \dddot{\mathbf{X}} \longrightarrow \dddot{\mathbb{P}}_1(\mathbb{C}).$$

We can call $(\dddot{\mathbf{X}}, \dddot{\beta})$ an *affine Belyi pair*. Then

$$\dddot{\beta}_\infty : \mathcal{H} \longrightarrow \dddot{\mathbb{P}}_1(\mathbb{C})$$

is the *universal affine Belyi pair*. According to the functorial properties of coverings, any affine Belyi pair $(\dddot{\mathbf{X}}, \dddot{\beta})$ can be included into the commutative diagram

$$\begin{tikzcd}
\mathcal{H} \arrow[rr, "/\pi_1(\dddot{\mathbf{X}})"] \arrow[dr, "\dddot{\beta}_\infty"'] & & \dddot{\mathbf{X}} \arrow[dl, "\dddot{\beta}"] \\
& \dddot{\mathbb{P}}_1(\mathbb{C}). &
\end{tikzcd}$$

The horizontal arrow means factorization over the group

$$\pi_1(\dddot{\mathbf{X}}) \hookrightarrow \mathsf{PSL}_2(\mathbb{Z}) \hookrightarrow \mathsf{PSL}_2(\mathbb{R}) = \mathrm{Aut}\mathcal{H}.$$

Now we are ready to complete the restoration of a Belyi pair $(\mathbf{X}, \beta)$ from the colored dessin (X_2, col_5) associated to it by the functor **paint**.

In view of the aforementioned isomorphism

$$\pi_1(X_2 \setminus X_0) \simeq \Pi_1(X_2, \mathrm{col}_5) \hookrightarrow \mathsf{Free}_2 \simeq \Gamma(2) \hookrightarrow \mathrm{Aut}\mathcal{H}$$

and after taking into account the equality $\mathbf{top}(\dddot{\mathbf{X}}) = X_2 \setminus X_0$, we obtain the inclusion $\pi_1(\dddot{\mathbf{X}}) \hookrightarrow \Gamma(2)$ and restore

$$\dddot{\mathbf{X}} := \frac{\mathcal{H}}{\Pi_1(X_2, \mathrm{col}_5)}$$

and

$$\dddot{\beta} : \frac{\mathcal{H}}{\Pi_1(X_2, \mathrm{col}_5)} \longrightarrow \frac{\mathcal{H}}{\Gamma(2)} \simeq \dddot{\mathbf{P}}_1(\mathbb{C}).$$

The Belyi pair $(\mathbf{X}, \beta)$ is restored as the compactification of the affine Belyi pair $(\dddot{\mathbf{X}}, \dddot{\beta})$.

Note that in addition to the equivalence of the categories $\mathcal{DESS}_3$ and $\mathcal{BELP}(\mathbb{C})$, we got the equivalence of both to the seemingly simpler category of finite homogeneous Free_2-sets (whose objects are usually understood as pairs of permutations generating a transitive group) perfectly suited for computer operations.

The equivalence between $\mathcal{DESS}$ and $\mathcal{BELP}_2(\mathbb{C})$ is established in the similar manner.

1.7. Arithmetic geometry enters. Though the preceding considerations seem to belong to combinatorial topology and complex analysis, arithmetic is very close.

Theorem. *The obvious category inclusions*

$$\mathcal{BELP}(\overline{\mathbb{Q}}) \hookrightarrow \mathcal{BELP}(\mathbb{C})$$

and

$$\mathcal{BELP}_2(\overline{\mathbb{Q}}) \hookrightarrow \mathcal{BELP}_2(\mathbb{C})$$

are category equivalences.

Only the *density* of the inclusion functors, *every complex Belyi pair is isomorphic to a Belyi pair defined over algebraic numbers*, deserves a discussion; but with the help of the constructions of the previous subsection, it can be easily deduced from the known results, see, e.g., [12].

However, Grothendieck was strongly impressed by the fact that *any* dessin is related to a curve over a field of algebraic numbers; his testimony [28] concerning the only comparable impact, hearing the definition of a circle at the age of 12 before which its *rotondité parfait* seemed *au delà des mots*, is widely quoted in the modern mathematical literature.

As for the *arithmetic geometry*, **Belyi's theorem** states that *all the curves over* $\overline{\mathbb{Q}}$ *are in the game*. This was proved in [7] and improved in [8]. At present, many detailed expositions are available, see, e.g., [50] or [26].

1.8. Ultimate category equivalences. Collecting the above constructions and results together, we get the following category equivalences:

$$\boxed{\mathcal{DESS} \longleftrightarrow \mathcal{BELP}_2(\overline{\mathbb{Q}})},$$

$$\boxed{\mathcal{DESS}_3 \longleftrightarrow \mathcal{BELP}(\overline{\mathbb{Q}})}.$$

All the objects of these categories are defined by finite amounts of information; the rest of the paper is devoted to discussion of the explicit realization of the boxed equivalences.

Naturally, the left-to-right realization is a *calculation*, while the right-to-left one is a *drawing*; cf. the title of the paper.

2. Object-by-object correspondence: dreams and goals

In this section, we discuss several reasons for establishing explicit correspondences between the dessins d'enfants and Belyi pairs.

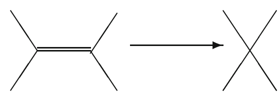
2.0. Fun. This reason is easily observable: lots of people from different countries are engaged in calculations of Belyi pairs and obviously enjoy it; the *fun* is often mentioned explicitly. Brian Birch called results of calculations *beautiful* "ballet of numbers" [10]. I devoted decades to these calculations involving a number of students who found them amazing.

However, I will try to present more serious reasons.

2.1. Transferring structures. The objects of the categories of dessins d'enfants and of Belyi pairs look quite dissimilar; the individual objects of each category, as well as the morphism sets and *moduli* (i.e., sets of classes of isomorphic objects), carry obvious additional structures that are hidden in the corresponding objects of the other category. Transferring these structures can be interesting and productive.

From $\mathcal{DESS}$ to $\mathcal{BELP}$. We give two examples related to objects and one related to moduli.

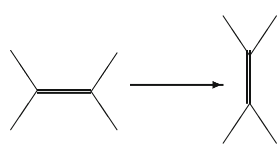
(a) Edge contraction. This operation is clear in terms of dessins



but is mysterious in terms of Belyi pairs.

It needs a *marked edge* that has no obvious meaning related to the corresponding Belyi pairs (unlike the *vertices* and *cells* corresponding to the zeros and poles of Belyi functions). However, according to empirical evidence and certain theoretical results, one can predict that the sets of *primes of bad reduction* (see below) of Belyi pairs do not change drastically under edge contractions of the corresponding dessins.

(b) Flip. It is another operation on dessins with marked edge



that has no clear meaning in terms of Belyi pairs.

However, when dessins are used as the *labels* of cells of the *moduli spaces* $\mathcal{M}_{g,n}(\mathbb{C})$ of pointed complex curves[6], the flips correspond to jumping to the neighboring cells.

(c) Enumeration. A remarkable activity in the enumeration of various combinatorial objects, including dessins d'enfants, can be observed during several last years; the corresponding results are well represented in our conference.

From the viewpoint of my talk, the decisive step was made in Zograf's paper [77]; the main recursion there is perfectly suited for the enumeration of (classes of isomorphic) objects of $\mathcal{DESS}_3$.

No corresponding techniques is known for $\mathcal{BELP}$. If it appears, then it will probably be related to Shafarevich's finiteness conjecture [72] proved by Faltings, see [22]. For the time being the arithmetic geometrical finiteness results, unlike their combinatorial-topological part, are very far from being constructive.

From $\mathcal{BELP}$ to $\mathcal{DESS}$. The general dream is to *visualize* the objects of arithmetic geometry. Here is a minimal wish list.

• **Primes of bad reduction**. A prime p is *good* for a Belyi pair $(\mathbf{X}, \beta)$ if $(\mathbf{X}, \beta)$ can be defined over the ring $\mathcal{O}$ of integers of some number field in such a way that its reduction $(\mathbf{X}_\mathfrak{p}, \beta_\mathfrak{p})$ over some prime ideal $\mathfrak{p} \lhd \mathcal{O}$ with $\frac{\mathcal{O}}{\mathfrak{p}} \supseteq \mathbb{F}_p$ is a Belyi pair over $\overline{\mathbb{F}_p}$, the genus of $\mathbf{X}_\mathfrak{p}$ equals that of $\mathbf{X}$, and $\deg \beta_\mathfrak{p} = \deg \beta$.

• Finite sets of dessins corresponding to **GALOIS ORBITS** of Belyi pairs, see the next section (the bold capitals are used, because initially the problem of describing these orbits was one of the main motivations to develop the Grothendieck program).

[6]It is a long story not to be discussed here; see [47] and [62] for original constructions and, say, [19] for a detailed exposition.

• **Fields of definition**. As soon as the Galois orbits of dessins are defined, every dessin acquires a finite-index stabilizer in the absolute Galois group that corresponds by the Galois theory to a certain number field that I call the *field of definition*[7] of the dessin.
• **Discriminants** of the fields of definition. Since the number fields of large degree are not described easily, their discriminants constitute the natural observable quantities.

2.2. Defining and comparing complexities. Both classes of objects are definable by finite amounts of information; informally, the *complexity* of such an object means the (logarithm of the) length of the shortest description of an object. This idea is formalized in the theory of *Kolmogorov complexity*; see [80] for the original introduction and [54] for a modern discussion in the broad scientific context.

The general question is: are the complexities of the corresponding objects *related*? Put plainly, is it true that if one has a short enough description of a dessin, then it is possible to write equations of the corresponding Belyi pair in terms of small enough coefficients of reasonable size, and vice versa?

The naive answer seems to be negative; in Sec. **6**, we present a 4-edged dessin of genus 1, corresponding to a Belyi pair, supported on the elliptic curve with terribly long j-invariant.

However, certain theoretical results exist. On the dessins side, the natural measure of complexity is the *number of edges* of a graph; the known methods of describing dessins, say, in terms of the finite-index subgroups of Free_2, provide descriptions of the length uniformly bounded in terms of the number of edges. The above mentioned recent progress in the enumeration of dessins gives beautiful expressions for the (weighted by the orders of symmetry groups that are generically trivial) numbers of dessins with a prescribed genus and number of edges, the asymptotic of these numbers, etc.

Many years ago my teacher Yu. I. Manin drew my attention to the similarity between Kolmogorov complexity and *heights* in arithmetic geometry [53]. These functions (both defined up to bounded ones) cannot coincide, since the heights are algorithmically computable while the Kolmogorov complexity is not, but the Kolmogorov complexity of objects of arithmetic geometry can very well be bounded by heights from above.

It took decades of my mathematical life to find an appropriate context for developing this idea: it is exactly the idea we are discussing. The list of classical heights is naturally extended by the *Belyi* height, for the time being defined only on the moduli spaces of the curves

$$\mathrm{h}_{\mathsf{Bel}} : \mathcal{M}_g(\overline{\mathbb{Q}}) \longrightarrow \mathbb{N} : \mathbf{X} \mapsto \min\{\deg\beta \mid (\mathbf{X}, \beta) \in \mathcal{BELP}(\overline{\mathbb{Q}})\}.$$

This function is well-defined in view of Belyi's theorem; we do not assume the *cleanness* of β's, because of the implication

$$(\mathbf{X}, \beta) \in \mathcal{BELP}(\overline{\mathbb{Q}}) \Longrightarrow (\mathbf{X}, 4\beta(1-\beta)) \in \mathcal{BELP}_2(\overline{\mathbb{Q}}).$$

The relation between the Belyi height and the other known heights has been studied in the recent literature. According to [35], the Faltings height is polynomially bounded from above by the Belyi height,

$$-\log(2\pi)g \leq \mathrm{h}_{\mathsf{Fal}}(\mathbf{X}) \leq 13 \cdot 10^6 g \cdot \mathrm{h}_{\mathsf{Bel}}(\mathbf{X})^5.$$

The problem of finding upper bounds for the Belyi height (given an algebraic curve over $\overline{\mathbb{Q}}$, how do we *practically* find a Belyi function on it?) is more delicate. The estimate in [36] is

[7] Many authors call this field the *field of moduli*; I avoid this term, because, as was mentioned above, the dessins can be understood as labels of cells in the *moduli* spaces of curves, and confusion is possible.

given in terms of the first move in the *Belyi game*,[8]

$$\mathrm{h}_{\mathsf{Bel}}(\mathbf{X}) \leq (4m\mathrm{H}_\Lambda)^{9m^3 2^{m-2} m!} \deg(\phi),$$

where $\mathbf{X}$ is a curve of genus g defined over $\mathbb{K}$, $\phi \in \mathbb{K}(\mathbf{X})$ is an arbitrary rational nonconstant function, Λ is the set of finite critical values of Φ, H_Λ is the maximal value of the *Weil* height of an element of Λ, and $m = 4\mathrm{H}_\Lambda \cdot (\mathbb{K} : \mathbb{Q})(\deg\phi + g - 1)^2$.

Perhaps, better estimates can be based on the methods of [8]; see the discussion in [50].

3. Galois orbits of dessins

As was mentioned, the action of the absolute Galois group on dessins is one of the oldest and the most exciting objects of the theory.

3.0. The group. Denote the *absolute Galois group*[9] by

$$\lvertneqq := \mathrm{Aut}(\overline{\mathbb{Q}}).$$

This group is quite mysterious. It is *profinite*, i.e., best observable by means of its finite factors. However, nobody knows whether *every* finite group appears among the factors of $\lvertneqq$; this question is called the *inverse Galois problem*, see, e.g., [37]. It should be noted that Belyi first proved his theorem in [7] as a tool for solving the inverse Galois problem for certain series of Chevalley groups.

Considering $\overline{\mathbb{Q}}$ as a subfield of $\mathbb{C}$, we could ask which elements of $\lvertneqq$ act on $\overline{\mathbb{Q}}$ *continuously*. It turns out that there are only two: the identity and the complex conjugation. It is quite difficult to specify any other element of $\lvertneqq$; usually only their images in the factors of $\lvertneqq$ are discussed.

3.1. The action. We introduce four countable sets of classes of isomorphic objects:

$$\mathbf{DESS} := \frac{\mathcal{DESS}}{\approx}, \qquad \mathbf{DESS}_3 := \frac{\mathcal{DESS}_3}{\approx},$$
$$\mathbf{BELP} := \frac{\mathcal{BELP}(\overline{\mathbb{Q}})}{\approx}, \qquad \mathbf{BELP}_2 := \frac{\mathcal{BELP}_2(\overline{\mathbb{Q}})}{\approx}.$$

The above-discussed category equivalences define the bijections

$$\mathbf{DESS} \leftrightarrow \mathbf{BELP}_2$$

and

$$\mathbf{DESS}_3 \leftrightarrow \mathbf{BELP}.$$

The group $\lvertneqq$ acts on $\mathbf{BELP}$ and hence on $\mathbf{BELP}_2$ in an obvious manner: we take any realization of a pair $(\mathbf{X}, \beta)$ over $\overline{\mathbb{Q}}$ and act by $\lvertneqq$ on the coefficients of defining equations of $\mathbf{X}$ and on the coefficients of β and then check immediately that the result does not depend on the realization. Hence the introduced bijections provide the actions

$$\boxed{\lvertneqq : \mathbf{DESS}}$$

and $\lvertneqq : \mathbf{DESS}_3$; the latter is less popular.

[8]This term has been coined, because of the original method of demonstration of the Belyi theorem in [7]: starting with an arbitrary $\phi \in \overline{\mathbb{Q}}(\mathbf{X}) \backslash \overline{\mathbb{Q}}$, Belyi reduced step by step its set of finite critical values $\Lambda := \mathrm{CritVal}(\phi)$. The *move* in this game is defined by a polynomial $P \in \mathbb{Q}[x]$; it replaces ϕ by $P \circ \phi$ and Λ by $P(\Lambda) \bigcup \mathrm{CritVal}(P)$.

[9]There are lots of other notations: $G_{\mathbb{Q}}$, $\mathrm{Gal}(\mathbb{Q})$, etc. I use Grothendieck's notation.

3.2. Properties. We are considering the action of a profinite group (of cardinality *continuum*) on a *countable* set; the orbits of this action are obviously *finite* (all the algebraic numbers defining a certain Belyi pair lie in some number field, and there is a finite-index subgroup of $\lneqq$, leaving fixed all the elements of this field).

The sizes of $\lneqq$-orbits of dessins have obvious combinatorial upper bounds. Since the valencies of vertices are the orders of zeros of a Belyi function and the valencies of faces are the orders of poles, the vertices and faces are $\lneqq$-invariant. Therefore the $\lneqq$-orbit of any dessin belongs to the set of dessins with the same *valency lists* called *passports*. Unfortunately, the recent progress in the dessins enumeration seems not to provide explicit expressions for the *numbers of dessins with a given passport*.

The striking feature of the action of $\lneqq$ on **Dess** is its *faithfulness*. A simple proof can be found in [64], where it is shown that this action is faithful already on the *plane trees*.

Thus, theoretically dessins can give us the ability to *see* the whole of $\lneqq$; unfortunately, for the time being our vision is quite limited. We know for sure how the *complex conjugation* looks like; some images related to the $\lneqq$-orbits of the *quasiplatonic* dessins, defined over the cyclotomic fields, were clarified in [39]. Hopefully, considering further examples (as well as developing new concepts) will improve our arithmetic visual acuity.

4. Examples of calculations

Lots of Belyi pairs have been calculated since Grothendieck's *Esquisse* was generally accepted by mathematical and physical communities; some of them turned out to be calculated *before* it appeared. An impressively complete recent survey can be found in [73]; lots of other sources are available. The choice of examples in this section is rather random and corresponds to author's interests and the problems discussed in the present paper.

4.0. Pre-Grothendieck era. The *Platonic solids* are, of course, the most classical dessins d'enfants. The corresponding Belyi pairs $(\mathbf{P}_1(\mathbb{C}), \beta)$, or rather the multi-valued functions $\beta^{-1\circ}$ studied in terms of *Schwartzian* differential equations, are thoroughly discussed in the famous Klein's "Icosahedron" [40].

Highly symmetric curves of positive genera were intensively studied in the nineteenth century. Such a curve $\mathbf{X}$ usually defines a Belyi function

$$\beta : \mathbf{X} \longrightarrow \frac{\mathbf{X}}{\mathrm{Aut}\mathbf{X}} \simeq \mathbf{P}_1(\mathbb{C}).$$

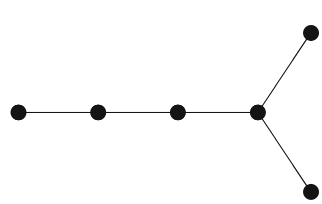
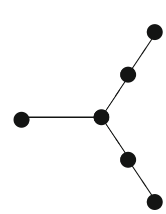
Among these curves we find, e.g., the *Klein quartic* (see [20] for a comprehensive exposition), the *Bring curve* (see paper [79] containing an interesting discussion of certain variations), the *Fricke-Macbeath curve* (see [32] for a dessin-theoretic discussion).

It is a very beautiful task, to draw the corresponding dessins: all of us *see* the Fricke–Macbeath curve at our conference poster.

Another gem of nineteenth century mathematics came not from algebraic geometry, but from group theory and combinatorics. The *Cayley graphs* are closely related to dessins d'enfants, see [74] and [27].

4.1. Grothendieck era. I just recall few examples that promoted some understanding. The first ones seem very simple now. We are considering only the objects of $\mathcal{DESS}$ and clean Belyi pairs.

(a) 5- and 6-edged trees. There are two plane trees with vertex valency list (3,2,2,1,1,1):

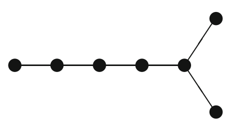
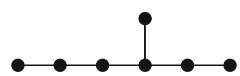
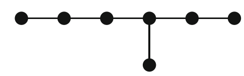
 and  .

When in the early 1990's Voevodsky and me started to calculate the corresponding Belyi functions[10], we first supposed naively that these trees constitute a Galois orbit over a quadratic field (a real one since the trees are not mutually mirror-symmetric). So we were surprised to find out that they are defined over $\mathbb{Q}$; the corresponding (non-normalized) polynomials are (see [71])

$$z^3(z-1)^2 \text{ and } z^3(9z^2-15z+40).$$

It is assumed that the Belyi functions β are expressed in terms of the normalized (with critical values ± 1) tree polynomials P by the formula $\beta = 1 - P^2$.

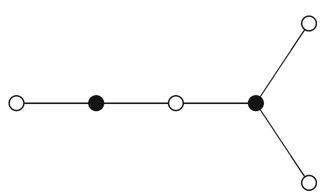
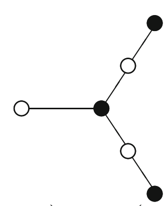
A similar guess concerning 6-edged trees was confirmed. There are three of them with valency list (3,2,2,2,1,1,1):

 ,  , and  .

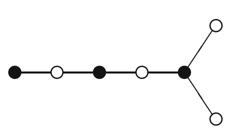
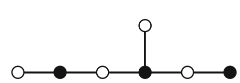
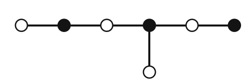
This triple is really defined over a cubic field: the corresponding polynomials are $z^3(z{+}1)^2(z{+}a)$, where a runs over the roots of the polynomial[11]

$$25a^3 - 12a^2 - 24a - 16 = 0.$$

The explanation of the definability of the above 5-edged trees over $\mathbb{Q}$ is given by a *hidden*[12] $\lneqq$-invariant: the *bicolored* structure of a plane tree. The trees

 and 

have different bicolored valency lists $(3,2 \mid 2,1,1,1)$ and $(3,1,1 \mid 2,2,1)$, therefore they are the only elements in their $\lneqq$-orbits, while the trees

 ,  , and 

have the same bicolored valency list $(3,2,1 \mid 2,2,1,1)$ and constitute a 3-element $\lneqq$-orbit.

(b) Leila's flower. A much more enigmatic case was found soon by Leila Schneps, see [64]. Denote by $\mathrm{IV}_{p_1p_2\ldots p_k}$ the plane tree of *diameter* 4, i.e., an abstract tree with bicolored valency list $(k,1,\ldots,1 \mid p_1,p_2,\ldots,p_k)$, embedded in the plane in such a way that the *paracentral* (white) vertices of valencies $p_1,p_2,\ldots,p_k$ go counterclockwise around the (black) *center* of valency k. If all the white valencies are different, then there are $(k-1)!$ *cyclic orders* on the set of paracentral vertexes, and the resulting $(k-1)!$ plane trees constitute a good candidate for a Galois orbit. However, when Leila considered the simplest nontrivial example IV_{23456}, it

[10] Now they are called the *Shabat polynomials.*

[11] As was noted by Drinfel'd and his student Pushnya, the substitution $a = -\frac{2b}{b-3}$ turns the minimal polynomial for a into b^3-2.

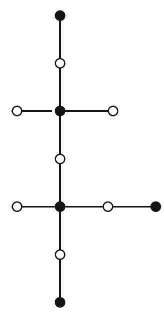
[12] It was hidden in the early 1990's, and is very well known now.

turned out that the 24-element set of the corresponding plane trees is split by $\lvertneqq$-action into the two 12-element orbits (corresponding to the parity of permutations).

Later, Kochetkov (see [41]) found other examples with the same kind of splitting. The "explanation" of Leila's phenomenon emerged: the *sum times product* of paracentral valencies $2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot (2 + 3 + 4 + 5 + 6)$ turned out to be a square. The theoretical explanation was provided soon by Zapponi, see [76]; further generalizations can be found in [43].

(c) Mathieu trees. The *edge rotation group* of a bicolored plane tree acts transitively on its edges and is generated by two elements: the rotation around black vertices, and the rotation around the white vertices. It is a highly nontrivial Galois invariant, see, e.g., [25].

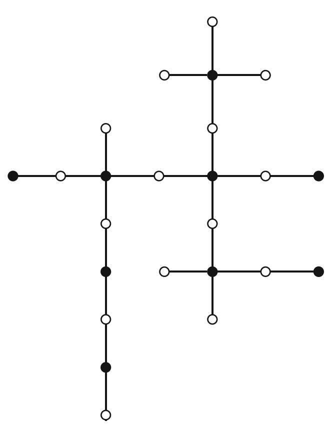
There are ten plane trees with bicolored valency list $(4, 4, 1, 1, 1 \mid 2, 2, 2, 2, 1, 1, 1)$, and the plane tree



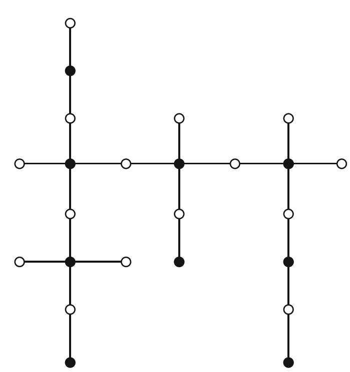
is the only one of them (together with its mirror reflection) whose edge rotation group is the Mathieu group M_{11}, see [3]. Those who do not remember (rather cumbersome) standard definitions of this group can *define* it as the edge rotation group of the above tree; its order is 7920.

So this tree and its mirror image are supposed to constitute the Galois orbit, and the calculations show (see [55, 5]) that they are really defined over $\mathbb{Q}(\sqrt{-11})$.

A similar phenomenon can be observed for the trees



and



related to the Mathieu group M_{23} that can again be *defined* as the edge rotation group of these trees; its order is 10 200 960.

These trees and their mirror images constitute just 4 out of 60 060 trees with the same bicolored valency list (see, e.g., [1]), whose edge rotation group is the Mathieu group M_{23}; the edge rotation group of *all* the remaining 60056 trees is the alternating group A_{23} (see, e.g., [21]).

So there is a chance to write down the Shabat polynomial explicitly only for the 4 exceptional trees among this huge amount. It is a difficult problem, first solved by Matiyasevich in [56], where it was shown that all the four trees are defined over $\mathbb{Q}(\sqrt{-\frac{23}{2}-\frac{5}{2}\sqrt{-23}})$. The detailed modern treatment can be found in [21].

It was shown in [3] that no other Mathieu group can be realized as the edge rotation group of a plane tree; however, they can be realized as the monodromy groups of more general Belyi mappings, see [78].

(d) Fields of realizations vs field of definition. We would like to have all Belyi pairs of genus 0 "written down" as explicitly as possible; in any case, they are parametrized over $\mathbb{C}$ just by spherical graphs. However, on the way to the desired explicitness we encounter several obstacles, one of which is discussed below.

Return to the arbitrary algebraically closed field $\daleth$ and consider the *set*

$$\mathbf{Bel}(\daleth) \subset \daleth(z)$$

of all *clean* Belyi functions. According to the above main theorems, this set is an infinite union of three-dimensional quasi-projective varieties acted upon by the group $\mathsf{PSL}_2(\daleth)$ of fractional linear transformations of the argument z. The countable set

$$\frac{\mathbf{Bel}(\daleth)}{\mathsf{PSL}_2(\daleth)} = \coprod_{\mathcal{P}\in\mathbf{Pass}_0^{\mathsf{clean}}} \mathcal{B}_{\mathcal{P}}$$

is the union of finite sets $\mathcal{B}_{\mathcal{P}}$ of $\mathsf{PSL}_2(\daleth)$-orbits of pure Belyi functions. These sets are labeled by *passports*, i.e., the lists

$$\mathcal{P} = \begin{pmatrix} \alpha_1 & 2 & \gamma_1 \\ \alpha_2 & 2 & \gamma_2 \\ \dots & \dots & \dots \\ \alpha_v & 2 & \gamma_f \end{pmatrix}$$

of multiplicities of the corresponding Belyi function over $(0,1,\infty)$; in the case of a function of degree $2n$, they satisfy the conditions

$$\alpha_1 + \cdots + \alpha_v = 2 + \cdots + 2 = \gamma_1 + \cdots + \gamma_f = 2n \qquad (\mathbf{deg})$$

and[13]

$$v - n + f = 2. \qquad (\mathbf{Euler})$$

Any orbit has the form $\mathcal{B}_{\mathcal{P}} = [\beta]_{\mathsf{PSL}_2(\daleth)} := \{\beta \circ \mathrm{T}^{-1\circ} \mid \mathrm{T} \in \mathsf{PSL}_2(\daleth)\}$, where $\beta \in \daleth(z)$ is a rational function such that

$$\beta = k_0 \frac{(z-A_1)^{\alpha_1} \dots (z-A_v)^{\alpha_v}}{(z-C_1)^{\gamma_1} \dots (z-C_{\mathfrak{c}})^{\gamma_{\mathfrak{c}}}}$$

and

$$\beta - 1 = k_1 \frac{(z-B_1)^2 \dots (z-B_n)^2}{(z-C_1)^{\gamma_1} \dots (z-C_f)^{\gamma_f}}.$$

Taking the difference, we obtain a polynomial equation

$$\begin{aligned} &k_0 \cdot (z-A_1)^{\alpha_1} \dots (z-A_v)^{\alpha_v} - k_1 \cdot (z-B_1)^2 \dots (z-B_n)^2 \\ &\quad = (z-C_1)^{\gamma_1} \dots (z-C_f)^{\gamma_f} \end{aligned} \qquad (\bigstar)$$

in the unknowns

$$k_0, k_1 \in \daleth; A_1, \dots, A_v; B_1, \dots, B_n; C_1, \dots, C_f \in \mathbf{P}_1(\daleth);$$

[13] If the passport originated from the true spherical dessin, just use the Euler formula; if it is a table of multiplicities of an algebraically defined β, then consider the degree of the divisor $\mathrm{div}(\mathrm{d}\beta)$.

this equation basically comprises the whole theory of Belyi pairs in genus 0 over an arbitrary field.

As a system of scalar equations, ($\bigstar$) is underdetermined:

$$\#\text{unknowns} - \#\text{equations} =_{(\mathbf{deg})} 2 + v + n + f - (2n+1) =_{(\mathbf{Euler})} 3.$$

This number is in perfect agreement with the existence of the *rational* action of $\mathsf{PSL}_2(\daleth)$ on the set of solutions of ($\bigstar$); in the case of $\daleth = \mathbb{C}$, this action corresponds to moving the vertices, "midpoints" of the edges, and "centers" of the faces by the common conformal transformation of the Riemann sphere.

Some remarks concerning the system ($\bigstar$) are in order.

(I). We mean $\mathbf{P}_1(\daleth) \simeq \daleth \coprod \{\infty\}$, and ($\bigstar$) makes sense literally only under the additional assumption that all the points $A_1, \ldots, C_f$ are *finite* and hence considered as *numbers*, i.e., elements of $\daleth$. However, it is often convenient to put some of these points to ∞ (traditionally it is C_1 with the maximal multiplicity), and in this case the system ($\bigstar$) should be modified by crossing out the corresponding factor (say, $(z - C_1)^{\gamma_1}$). This is obvious if we rewrite the system in a more careful way using *homogeneous* coordinates on $\mathbf{P}_1(\daleth)$. Then the number of unknowns reduces by one and the group $\mathsf{PSL}_2(\daleth)$ of fractional-linear transformations, acting on the set of solutions, is replaced by the 2-dimensional group $\mathsf{Aff}_1(\daleth) \simeq \daleth^+ \rtimes \daleth^\times$ of the affine transformations $z \mapsto pz + q$.

(II). The above rational $\mathsf{PSL}_2(\daleth)$-action on the set of solutions of ($\bigstar$) can be seen directly: applying the transformation $z = \frac{pz'+q}{rz'+s}, A_1 = \frac{pA_1'+q}{rA_1'+s}, \ldots$ with $ps - qr = 1$, and using the identities like $z - A_1 = \frac{1}{rA_1'+s}\frac{z'-A_1'}{rz'+s}$, we find that all the three terms of ($\bigstar$) are multiplied by

$$s\mathrm{Const}_0 \prod_{i=1}^{v} \frac{1}{(rz'+s)^{a_i}} = \frac{\mathrm{Const}_0}{(rz'+s)^{2n}}$$

and alike.

(III). Generically the irreducible components of the set of solutions of ($\bigstar$) are *principal* homogeneous $\mathsf{PSL}_2(\daleth)$-spaces, i.e., are birationally isomorphic to $\mathsf{PSL}_2(\daleth)$. However, in special cases of existence of nontrivial *symmetries*, when the group

$$\mathrm{Aut}(\mathbf{P}_1(\daleth), \beta) := \{\mathrm{T} \in \mathsf{PSL}_2(\daleth) \mid \beta \circ \mathrm{T}^{-1\circ} = \beta\}$$

is nontrivial, the components are the *orbifolds* $\mathsf{PSL}_2(\daleth)/\mathrm{Aut}(\mathbf{P}_1(\daleth), \beta)$. Of course, in the latter case the groups belong to a well-known restricted list.

(IV). The points $A_1, \ldots, C_f$ should be all *different*, otherwise a solution of ($\bigstar$) is called *parasitic*, see [49, 48]. The number of non-parasitic components of the set of solutions of ($\bigstar$) has a clear combinatorial meaning (at least in the case of $\mathrm{char}(\daleth) = 0$): it is the number of dessins d'enfants with prescribed passport. However, the author is unaware of the complete study of this direct relation between polynomial algebra and combinatorial topology.

Now we turn to the point of our discussion. Suppose that for a given passport we are interested not only in the general picture of orbits but want to see an explicit representative of each orbit. Informally, we would like to choose this representative as concise as possible.

From now on, let $\daleth = \overline{\mathbb{Q}}$. In order to formulate the precise problem, we recall the definition of a *field of realization*: for any solution $\{k_0, k_1, A_1, ..., C_f\} \subset \overline{\mathbb{Q}}$, it is the field generated by the coefficients of the polynomials

$$k_0 \prod_{i=1}^{v} (z - A_i)^{a_i}, \quad k_1 \prod_{i=1}^{n} (z - B_i)^2, \text{ and } \prod_{i=1}^{f} (z - C_i)^{c_i}.$$

The problem is to find representatives of the orbits that <u>minimize</u> the degree of their field of realization. Here the absolute Galois group $\lneqq$ enters: this time not as an object of study but as a tool.

For a Belyi function $\beta \in \overline{\mathbb{Q}}(z)$ and $\gamma \in \lneqq$, denote by ${}^{\gamma}\beta$ the result of coefficient-wise application of an automorphism. Denote by $\lneqq_\beta$ the stationary group of the Belyi pair $(\mathbb{P}_1(\overline{\mathbb{Q}}), \beta)$ with respect to the above defined action of $\lneqq$ on Belyi pairs; denote by

$$\mathbb{D}_\beta \leftrightarrow \lneqq_\beta$$

the field of algebraic numbers corresponding to the subgroup according to Galois theory. Note that we call it the *field of definition* of the dessin $(\mathbb{P}_1(\overline{\mathbb{Q}}), \beta)$ (unlike most writers who call it the *field of moduli* of the dessin).

By definition, for any $\gamma \in \lneqq_\beta$ there exists an isomorphism

$$(\mathbb{P}_1(\overline{\mathbb{Q}}), \ {}^{\gamma}\beta) \simeq (\mathbb{P}_1(\overline{\mathbb{Q}}), \beta),$$

which means that there exists a transformation $\mathrm{T}_\gamma \in \mathsf{PSL}_2(\overline{\mathbb{Q}})$ such that

$${}^{\gamma}\beta = \beta \circ \mathrm{T}_\gamma.$$

This transformation is uniquely defined if $(\mathbb{P}_1(\overline{\mathbb{Q}}), \beta)$ has no nontrivial automorphisms, and we assume this from now on.

One checks that for any $\gamma, \delta \in \lneqq_\beta$,

$${}^{\gamma\delta}\beta = \beta \circ \mathrm{T}_{\gamma\delta} = \ {}^{\gamma}({}^{\delta}\beta) = \ {}^{\gamma}(\beta \circ \mathrm{T}_\delta) = \ {}^{\gamma}\beta \circ \ {}^{\gamma}\mathrm{T}_\delta = \beta \circ \mathrm{T}_\gamma \circ \ {}^{\gamma}\mathrm{T}_\delta,$$

from which by our no-automorphism assumption we deduce that

$$\forall \gamma, \delta \in \lneqq_\beta [\mathrm{T}_{\gamma\delta} = \mathrm{T}_\gamma \circ \ {}^{\gamma}\mathrm{T}_\delta],$$

which means that we have just associated the *noncommutative* $\mathsf{PSL}_2(\overline{\mathbb{Q}})$*-valued* 1*-cocycle*

$$(\gamma \mapsto \mathrm{T}_\gamma) \in \mathrm{Z}^1(\lneqq_\beta, \mathsf{PSL}_2(\overline{\mathbb{Q}}))$$

of the stationary group $\lneqq_\beta$ of any Belyi[14] function $\beta \in \overline{\mathbb{Q}}(z)$, see [65].

Note that actually, any Belyi function $\beta \in \overline{\mathbb{Q}}(z)$ belongs to a smaller field $\beta \in \mathbb{K}(z)$; we called any such $\mathbb{K}$ a *field of realization* of β and consider only the cases $(\mathbb{K} : \mathbb{Q}) < \infty$. Our goal is to choose $\mathbb{K}$ as small as possible, and by definition we have a lower bound

$$\mathbb{K} \supseteq \mathbb{D}_\beta.$$

In the fortunate cases $\mathbb{K} = \mathbb{D}_\beta$, there is no need to correct the transformations T_β, since for all $\gamma \in \lneqq_\beta$ the functions ${}^{\gamma}\beta$ and β not only define the *isomorphic* Belyi pairs but are *equal*. Moreover, if it is possible to find a γ-independent correction $\mathrm{t} \in \mathsf{PSL}_2(\overline{\mathbb{Q}})$ such that

$${}^{\gamma}(\beta \circ \mathrm{t}) = \beta \circ \mathrm{t}$$

for all $\gamma \in \lneqq_\beta$, then $\beta \circ \mathrm{t} \in \mathbb{D}_\beta$, and in this case the problem of minimization of the field of realization is solved. Now we translate this condition to the cohomological language (see [65]).

The action of $\mathsf{PSL}_2(\overline{\mathbb{Q}})$ on $\mathrm{Z}^1(\lneqq_\beta, \mathsf{PSL}_2(\overline{\mathbb{Q}}))$ is defined by

$$\mathsf{PSL}_2(\overline{\mathbb{Q}}) \times \mathrm{Z}^1(\lneqq_\beta, \mathsf{PSL}_2(\overline{\mathbb{Q}})) \longrightarrow \mathrm{Z}^1(\lneqq_\beta, \mathsf{PSL}_2(\overline{\mathbb{Q}})) : (\mathrm{t}, \mathrm{T}) \mapsto \mathrm{t} \cdot \mathrm{T},$$

[14]We did not use the assumption that the rational function β is a Belyi one.

where

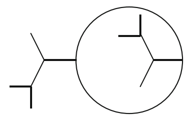
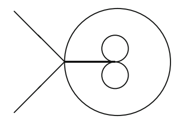
$$\mathrm{t}\cdot\mathrm{T} : \lneqq_\beta \longrightarrow \mathsf{PSL}_2(\overline{\mathbb{Q}}) : \gamma \mapsto (\mathrm{t}\cdot\mathrm{T})_\gamma := \mathrm{t}\circ\mathrm{T}\circ^\gamma \mathrm{t}^{-1\circ};$$

the cohomology *set* is defined as a set of orbits

$$\mathrm{H}^1(\lneqq_\beta, \mathsf{PSL}_2(\overline{\mathbb{Q}})) := \frac{\mathrm{Z}^1(\lneqq_\beta, \mathsf{PSL}_2(\overline{\mathbb{Q}}))}{\mathsf{PSL}_2(\overline{\mathbb{Q}})}.$$

It does not carry a structure of a group and only has a distinguished element, corresponding to *cohomologically trivial cocycles* of the form $\gamma \mapsto \mathrm{t}\circ^\gamma \mathrm{t}^{-1\circ}$ with a fixed $\mathrm{t} \in \mathsf{PSL}_2(\overline{\mathbb{Q}})$. The above formula ${}^\gamma(\beta\circ\mathrm{t}) = \beta\circ\mathrm{t}$ is equivalent in our notations to the statement that *the corresponding cocycle* $\mathrm{T}_\gamma = \mathrm{t}\circ^\gamma \mathrm{t}^{-1\circ}$ *is cohomologically trivial.* Hence *the obstruction to the realization of a Belyi function* β *over its field of definition* $\mathbb{D}_\beta$ *lies in the cohomology set* $\mathrm{H}^1(\lneqq_\beta, \mathsf{PSL}_2(\overline{\mathbb{Q}}))$.

The examples of spherical dessins, for which this obstruction is nontrivial, were constructed in 1990's:

 in [15] and  in [23].

The first of these dessins is defined over $\mathbb{Q}$ and can be realized over $\mathbb{Q}(\mathrm{i})$, while the second (together with its $\lneqq$-partner) is defined over $\mathbb{Q}(\sqrt{5})$ and is realizable over $\mathbb{Q}(\sqrt{5}, \sqrt{-2})$. It is proved in both papers [15] and [23] that for any spherical dessin, its field of realization can be chosen as no more than a quadratic extension of the field of definition.

A nice list of 14 new examples can be found in [33]; the example from [23] can be found there under the label F11. A simple example of similar (properly defined) phenomenon in the case of positive genus has been constructed in [16, Sec. 2.5].

Summarizing, we have some amount of beautiful examples, showing that the impossibility of realizing a Belyi pair over its field of definition is a well-hidden Galois invariant; however, to the best of my knowledge, we are far from a complete understanding of this phenomenon, e.g., of the *combinatorial* nature of the corresponding cohomological obstruction.

5. Catalogs

One of the most straightforward approaches to understanding the relation between dessins d'enfants and Belyi pairs is to create *complete* lists of objects of bounded complexity and establish the corresonding bijections. Calling these lists with bijections *catalogs*, I list[15] some of the ones I am aware of.

1991, Shabat: the dessins with ≤ 3 edges $\leftrightarrow$ the clean Belyi pairs $(\mathbf{X}, \beta)$ with $\deg\beta \leq 6$ [66].
1992, Bétréma, Péré, and Zvonkin: the plane trees with ≤ 8 edges $\leftrightarrow$ the Shabat polynomials of degree ≤ 8 [6].
1994, Birch: the Belyi pairs $(\mathbf{X}, \beta)$ with $\deg\beta \leq 5$ [10].
2008, Beukers and Montanus: the rational Belyi functions of degree 24 that are the j-invariants $t \mapsto \mathrm{j}(\mathbf{E}_t)$ of the families $\pi : \mathbf{E} \to \mathbf{P}_1$, where $\mathbf{E}$ is a K3-surface fibered into elliptic curves $\mathbf{E}_t := \pi^{-1\circ}(t)$; under certain assumptions,[16] there are 112 of them [9].
2009, Adrianov, Amburg, Dremov, Kochetkov, Kreines, Levitskaya, Nasretdinova, and Shabat: the dessins with ≤ 4 edges $\leftrightarrow$ the clean Belyi pairs $(\mathbf{X}, \beta)$ with $\deg\beta \leq 8$ [2].
2009, Kochetkov: the plane trees with 9 edges $\leftrightarrow$ the Shabat polynomials of degree 9 [45].

[15] The catalogs are ordered according to the date of publication, possibly in the preprint form; some (e.g., Birch's) were created long before the publication.

[16] Basically equivalent to the existence of precisely 6 singular fibers of Kodaira type I_n, see [58].

2012, Hoeij and Vidunas: the uniform[17]-with-4-exceptions spherical dessins (there are 366 $\lneqq$-orbits of them) $\leftrightarrow$ the corresponding rational Belyi functions of degree ≤ 60 [33].
2013, He, McKay, and Read: 33 torsion-free genus zero congruence subgroups of $\mathsf{PSL}_2(\mathbb{Z}) \leftrightarrow$ the corresponding dessins and 112 Beukers–Montanus families $\leftrightarrow$ the quintuples of generators of the corresponding (with the exception of 9, non-congruence) subgroups of $\mathsf{PSL}_2(\mathbb{Z})$ [31].
2014, Kochetkov: the plane trees with 10 edges $\leftrightarrow$ the Shabat polynomials of degree 10 in the cases of decomposable Galois orbits [46].

What did we learn from all these lists of (sometimes terribly long) formulas and complicated drawings? Here are some answers.

• The calculation skills has advanced considerably, both in terms of computer technologies and mathematically meaningful tricks.
• Certain new phenomena have been found and some of them were explained.
• We are close to the bounds of degrees d for which the calculation of *all* the Belyi pairs $(\mathbf{X}, \beta)$ of certain types with $\deg \beta \leq d$ is practically possible, and we know *what* bounds to use: it is the complexity of Belyi pairs (not the number of dessins/Belyi pairs and not the complexity of dessins). Indeed, the authors of the catalogs often have to omit the explicit expression for the Belyi pair (or refer to special sites) because of their length, but always draw the corresponding dessins.
• Moreover, compiling the catalogs requires clarifying the concept of *explicit calculation* of a Belyi pair. E.g., among Kochetkov's 9-trees we find a Galois orbit of 30 trees; they are labeled by the roots of the polynomial from $\mathbb{Z}[a]$ of degree 30 that occupies about a page. There seems to be no psychologically comfort and traditional way to specify these roots; however, they can be naturally labeled by plane trees that are actually drawable!
• Some evidence has been collected concerning the *generic* behavior of the Galois orbits of dessins: the orbit of a random one will probably consist of all the dessins with the same passport. The *special* behavior (i.e., the splitting of this set into smaller Galois orbits) is in most cases explained either in categorical terms (existence of automorphisms or morphisms onto the smaller dessins) or by a special invariant: cartographic $\approx$ monodromy $\approx$ edge rotation group. The rare remaining cases (like Leila's flower and other trees of diameter 4) are explained in more special ways.

Summarizing, if we consider the catalogs as dictionaries, then we see that the two basic languages related by them are highly asymmetric. The ”pictographic” language of dessins turns out to be considerably more compact and hence informative. Unfortunately, for the time being our vision is weak: we hardly see the most superficial structures, e.g., symmetries. The non-archimedean geometry of Belyi pairs, also encoded in dessins, remains in the dark.

6. Relax the branching?

This section is devoted to the last method of calculating Belyi pairs among the methods considered in this paper.

6.0. Hurwitz spaces. Any Belyi pair $(\mathbf{X}, \beta)$ over a field $\daleth$ corresponds to a point of the *Hurwitz space*

$$\mathcal{HUR}_{g,d}(\daleth) := \frac{\{(\mathbf{X}, f) \mid \mathbf{X} \in \mathcal{M}_g(\daleth), f \in \daleth(\mathbf{X}), \deg f = d\}}{\approx};$$

[17] A tricolored dessin d'enfant is called *uniform* if its valencies of a given color are constant.

unfortunately, there is no common notation for this space, and the authors often mean by $\mathcal{H}_{g,d}$ a set of pairs with the *simplest branching* of a function (see, e.g., [63]). This assumption implies the *maximal* possible number of the branch points, while we are interested in precisely the opposite case, when there are only 3 of them.

We are not going into the details of definitions of the Hurwitz spaces, since we will discuss just certain finite subsets of them and certain curves connecting the points of these subsets, so a serious foundational work is not needed for it. These finite subsets are Belyi pairs of a given genus and a given degree; the goal of this section is to introduce the curves obtained by a minimal relaxing of the branching assumptions.

So we suggest the following modification of the definition of Belyi pairs: replace

- 3 by 4;
- *Belyi* by *Fried* (see [24]);
- *curves* by *families.*

The latter modification is related to the fact that the covers $\mathbf{X} \to \mathbf{P}_1$ with 4 branch points acquire the continuous parameter: the cross-ratio of these points.

6.1. Fried families. The formal definition of a *Fried family* is as follows: it is a (smooth complete connected) surface $\mathbf{X}$ together with two morphisms:
1) $\pi : \mathbf{X} \to \mathbf{B}$ onto a (smooth complete connected) base curve $\mathbf{B}$ such that a *generic* fiber $\mathbf{X}_b := \pi^{-1\circ}(b)$ is a smooth connected curve;
2) $\Phi : \mathbf{X} \to \mathbf{P}_1$ such that for a *generic* $b \in \mathbf{B}$, the restriction

$$\Phi|_{\mathbf{X}_b} : \mathbf{X}_b \to \mathbf{P}_1$$

is a cover with 4 branch points. The Belyi pairs occur as restrictions of Φ to *special* fibers, where the branch points collide.

For a given Fried family defined by a quadruple $(\mathbf{X}, \mathbf{B}, \pi, \Phi)$, the set of *generic* pairs $\{(\mathbf{X}_b, \Phi|_{\mathbf{X}_b}) \mid b \in \mathbf{B}\}$ constitutes an algebraic curve in the appropriate Hurwitz space; for a fixed pair (d, g) the set of curves of this kind in $\mathcal{HUR}_{d,g}$ is finite (over $\mathbb{C}$ such a curve is defined locally by the conjugacy class of the monodromy $\pi_1(\mathbf{P}_1(\mathbb{C}) \setminus 4 \text{ branch points}) \to \mathsf{S}_d$ and a point on it via a cross-ratio of these points). The union of these curves can be called the *Fried net*; its projections to the moduli space $\mathcal{M}_g$ also deserve this name.

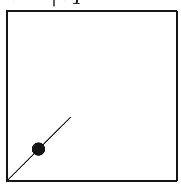
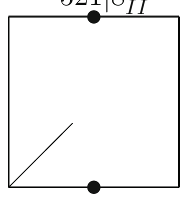
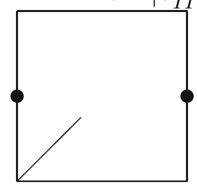
The author believes that geometry and arithmetic of the Fried nets need a thorough study, see [18]. A beautiful class of examples is delivered by the above-mentioned catalogs in [9] and [31], see also [60].

6.2. Five Belyi pairs on one curve. In the present paper, however, the Fried families are mentioned just as a tool for calculating Belyi pairs as special fibers in families with relaxed branching, and this tool is illustrated by just one example. It is related to the calculation of the Belyi pairs $(\mathbf{E}, \beta)$ corresponding to the 4-edged toric clean dessins with only one face, see catalog [2].

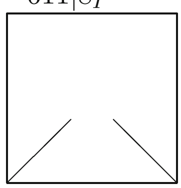
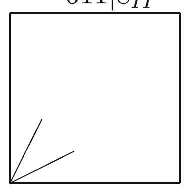
There are 11 of them, but six are *easy*, i.e., either centrally symmetric or *bicolorable*, and hence with the square $1 - \beta$ (this is the statement of the *Dremov lemma*, see [69]).

The remaining five dessins

$521|8_I$ $521|8^+_{II}$ $521|8^-_{II}$

$611|8_I$ $611|8_{II}$

(where the opposite sides of the squares are identified) can be called the *hard* ones. However, these five "live together": all of them are special fibers of <u>one</u> family! Here is a brief explanation.

By Dremov's lemma, $1-\beta$ is a square for none of them: all have loops and hence are not bicolorable. Now, since the dessins are clean,

$$\mathrm{div}(1-\beta) = 2(B_1+B_2+B_3+B_4) - 8O_{\mathbf{E}}$$

for some $B_1,\ldots,B_4 \in \mathbf{E}$ and the neutral element $O_{\mathbf{E}} \in \mathbf{E}$ chosen to be the pole of β. Therefore, $B_1+B_2+B_3+B_4-4O_{\mathbf{E}}$ is a point of order 2 in the Jacobian $\mathrm{Jac}(\mathbf{E}) \simeq \mathbf{E}$. But this point of order 2 is nontrivial, because otherwise $1-\beta$ would be a square (by the Abel and Riemann-Roch theorems). Hence we have a distinguished nontrivial point of order 2, so the appropriate form for a defining equation of all our five $\mathbf{E}$'s is

$$\mathbf{E}_{a,b} : y^2 = (x-1)(ax^2+bx-1),$$

where $(x=1, y=0)$ is the above distinguished point and $(x=0, y=1)$ is the zero of β of maximal valency. As was just explained, $(x-1)(1-\beta)$ is a square, so all the desired β are defined by the relation

$$(x-1)(1-\beta) = (P+Qy)^2,$$

where P and Q are polynomials in x of degrees 2 and 1. Considering $x=1$, we see that P is divisible by $x-1$, so the coefficients of β are polynomial in the coefficients of $\frac{P}{x-1}$ and Q. These coefficients are determined by the condition that β has a zero of order 4 in $(x=0, y=1)$, and we get a family of functions on $\mathbf{E}_{a,b}$ generically paramerized by the points of the affine (a,b)-planes. The condition of further colliding of the branch points defines the affine algebraic curve (rather messy, see [69]) which is the base of the desired family.

The dessins $611|8_I$ and $611|8_{II}$ constitute a Galois orbit over $\mathbb{Q}(\sqrt{2})$; the corresponding values of the parameters are

$$(a = -\frac{3}{64} \pm \frac{3}{32}\sqrt{2}, b = \frac{1}{4} \mp \frac{1}{4}\sqrt{2}).$$

The dessins $521|8_I$ and $521|8^{\pm}_{II}$ constitute a cubic Galois orbit. The corresponding parameters are the roots of the polynomials

$$65536\,a^3 - 238080\,a^2 + 216425\,a + 14000$$

and

$$64\,b^3 - 272\,b^2 + 1427\,b - 344.$$

6.3. Brief discussion. The minimal cubic polynomial, the roots of which are the j-invariants of all the three curves corresponding to the dessins of the orbit, is

$$
\begin{gathered}
56495049800000000000000 j^3 \\
-315629560922285350000000000 j^2 \\
+748295885321347996073297265625 j \\
-564055135320668135938721399828128.
\end{gathered}
$$

So the arithmetic height of these elliptic curves is far from being small, while the (clean) Belyi height is 4, one of the smallest possible.

The leading coefficient of the polynomial is, as usual, the product of big powers of small primes

$$56495049800000000000000 = 2^{15}5^{14}7^{10},$$

which indicates a *terribly* bad reduction over 2, 5, and 7=2+5. The author is unaware of the clear theoretical explanation of this phenomenon that is encountered very often.

This method can be applied in many other cases, see [18]. An infinite family of interesting (at least from the viewpoint of the Inverse Galois Problem) Fried families over $\mathbb{Q}$ was constructed in [30].

7. Concluding remarks

I briefly mention some issues that did not fit into present paper.

7.0. The advanced methods of calculation of Belyi pairs. Most of the known methods seem to be reviewed in [73]. They include

- reductions over *good* primes p (preferably over *very good* p, which means that if a Galois orbit of a dessin is parametrized by the roots of a polynomial $P \in \mathbb{Z}[a]$, then the polynomial $P \mod p \in \mathbb{F}_p[a]$ decomposes into linear factors[18]) with the subsequent lifting to p-adics and using $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$;
- direct computer algebra methods;
- approximate calculations (circle packing [13], multidimensional Newton method, ...);
- using modular forms.

Added to this is the use of the *Mulase–Penkava* operator $\beta \mapsto \frac{(\mathrm{d}\beta)^2}{\beta(1-\beta)}$ (see [59, 68, 76, 17]) and post-composing with the Jukovsky function $\beta \mapsto \frac{1}{2}(\beta + \frac{1}{\beta})$ that rises the degree but allows to reduce the genus, because of the acquired symmetry (see, e.g., [14]).

7.1. Hopes related to the discrete complex analysis. This domain on the border of pure and applied mathematics is actively developing during the last decades together with the discrete Riemannian and differential geometry, see [57, 11] and many other papers. In particular, the theory of *discrete period matrices* has been constructed. This theory can be directly applied to the *equilateral triangulations*, which by [71] is just a version of the dessins d'enfants theory.

Hopefully, somebody will use this powerful tool to calculate (at least approximately) the *Jacobians* of the curves carrying dessins, which suffices by the Torelli theorem to restore the curves.

[18]The "probability" that a random prime p has this property is (by the Frobenius theorem) a positive rational number whose denominator equals the order of the Galois group of P, see [51].

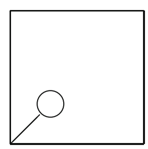
7.2. Drawing. Nothing has been said in the main body of this paper (but in the title...) about practical realizations of the functors **draw** and **paint**. At least by typographical reasons, we mention only **draw**ing.

The problem can be understood in two ways: combinatorially and metrically. As for the combinatorial drawing, the powerful computer tools have been developed, see, e.g., [31] and [4].

The problem of drawing a dessin in its *true shape* can be posed in two cases: *plane trees* and *toric dessins*; in the latter case, the dessin is to be drawn as a doubly-periodic infinite graph on the universal covering of an elliptic curve. I give just a couple of remarks, trying to express my feeling that these true shapes constitute a huge domain of unexplored geometry.

The collection of true shapes of trees was first presented in [6]. One of the arising questions was whether the edges can have *inflections*? The direct inspection of the computer-produced pictures did not give an immediate answer. However, an inflection was found, see [44]. This paper, as well as the earlier one [42], contains a number of interesting observations, questions, and conjectures.

As for the toric dessins, we mention just one case. The publication of catalog [2] was delayed for a couple of years, because nobody was able to calculate a Belyi pair corresponding to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Finally, Volodya Dremov managed it, see [17]; since then it was called the *Dremov pan*. However, it turned out that this dessin can <u>not</u> be drawn in its true shape realistically: Dremov has found that "his" pan is about a hundred times smaller than the ambient parallellogram.

7.3. Open problems. Some problems were formulated in the main text. I will add three more.

•**Families**. We perceive certain countable sets of dessins as *families*. The simplest examples among the plane trees are *chains*, *stars*, and *double stars* corresponding to the polynomials $\mathrm{T}_n(z) = \cos(n \arccos z)$, z^n, and $z^m(1-z)^n$, respectively. There are much more interesting ones like the *propellers* Y_{abc} and *crosses* X_{abcd}, see [61].

Can anybody give a precise <u>mathematical definition</u> of a *family*?

•**Cohomology of critical strata**. For $g, d, b \in \mathbb{N}$, we introduce the set of (isomorphism classes of) the curves of genus g over $\daleth$, on which the rational functions of degree d with no more than b critical values exist,

$$\mathcal{M}_{g;d,b}(\daleth) := \{\mathbf{X} \in \mathcal{M}_g(\Bbbk) \mid \exists f \in \daleth(\mathbf{X}) : \deg f = d, \#\mathrm{CritVal}(f) \leq b\}.$$

These sets are (usually reducible) quasi-projective subvarieties of the moduli spaces $\mathcal{M}_g(\daleth)$. It is known that for $d \geq 2g+1$,

$$\mathcal{M}_g = \mathcal{M}_{g;d,2(d+g)-2} \supseteq \mathcal{M}_{g;d,2(d+g)-3} \supseteq \cdots \supset \mathcal{M}_{g;d,4} \supset \mathcal{M}_{g;d,3},$$

where the last two strata correspond to the finite set of curves carrying the Belyi functions of degree d and the above defined Fried nets.

What can we say about the (appropriately defined) cohomology of $\mathcal{M}_{g;d,b}(\daleth)$, their components and the closures of components in the Deligne-Mumford compactification? What is the intersection behavior of the components? How do these structures depend on $\daleth$?

The last question is related to the following one.

• **The categories** $\mathcal{BELP}(\overline{\mathbb{F}_p})$. Can we visualize Belyi pairs in positive characteristic? How “close” are the categories $\mathcal{BELP}(\overline{\mathbb{F}_p})$ and $\mathcal{BELP}(\overline{\mathbb{Q}})$? Do they become “closer” after adding *stable* and *wild* Belyi pairs?

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