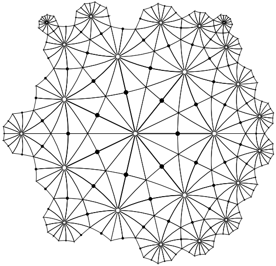
2-solvable Belyĭ maps



Michael Musty, Dartmouth College (grad)
math.dartmouth.edu/~mjmusty
Maine Québec Number Theory Conference
October 14, 2017

Acknowledgements

- Sam Schiavone
- Jeroen Sijsling
- John Voight

Outline

Theorem (G.V. Belyĭ 1979)

A curve X over $\mathbb{C}$ can be defined over $\overline{\mathbb{Q}}$ if and only if there exists a branched covering of compact connected Riemann surfaces $\varphi : X \to \mathbb{P}^1$ unramified (unbranched) above $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Theorem (G.V. Belyĭ 1979)

A curve X over $\mathbb{C}$ can be defined over $\overline{\mathbb{Q}}$ if and only if there exists a branched covering of compact connected Riemann surfaces $\varphi : X \to \mathbb{P}^1$ unramified (unbranched) above $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Such a map is called a **Belyĭ map**.

Theorem (G.V. Belyĭ 1979)

A curve X over $\mathbb{C}$ can be defined over $\overline{\mathbb{Q}}$ if and only if there exists a branched covering of compact connected Riemann surfaces $\varphi : X \to \mathbb{P}^1$ unramified (unbranched) above $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Such a map is called a **Belyĭ map**. The **degree** of a Belyĭ map is the number of sheets in the covering.

Theorem (G.V. Belyĭ 1979)

A curve X over $\mathbb{C}$ can be defined over $\overline{\mathbb{Q}}$ if and only if there exists a branched covering of compact connected Riemann surfaces $\varphi : X \to \mathbb{P}^1$ unramified (unbranched) above $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Such a map is called a **Belyĭ map**. The **degree** of a Belyĭ map is the number of sheets in the covering.

In the 1980s, Grothendieck described a bijection between Belyĭ maps and *dessins d'enfants*.

Theorem (G.V. Belyĭ 1979)

A curve X *over* $\mathbb{C}$ *can be defined over* $\overline{\mathbb{Q}}$ *if and only if there exists a branched covering of compact connected Riemann surfaces* $\varphi : X \to \mathbb{P}^1$ *unramified (unbranched) above* $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Such a map is called a **Belyĭ map**. The **degree** of a Belyĭ map is the number of sheets in the covering.

In the 1980s, Grothendieck described a bijection between Belyĭ maps and *dessins d'enfants*. $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on these sets.

A Zoo of Bijections

$\left\{\begin{array}{l}\text{Transitive permutation} \\ \text{Triples } (\sigma_0, \sigma_1, \sigma_\infty) \in \\ S_d^3 \text{ with } \sigma_\infty\sigma_1\sigma_0 = 1\end{array}\right\}$

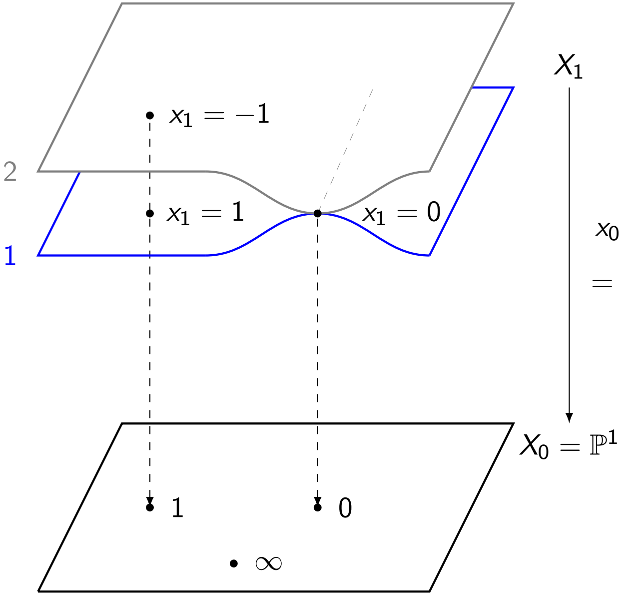
$\left\{\begin{array}{l}\text{Index } d \text{ triangle sub-} \\ \text{groups } \Gamma \leq \Delta(a, b, c)\end{array}\right\}$

$\left\{\begin{array}{l}\text{dessins with} \\ d \text{ edges}\end{array}\right\}$

$\left\{\begin{array}{l}\text{Degree } d \text{ function field} \\ \text{extensions } K \supseteq \mathbb{C}(z) \\ \text{with discriminant sup-} \\ \text{ported above } 0, 1, \infty\end{array}\right\}$

$\{\text{Belyĭ maps of degree } d\}$

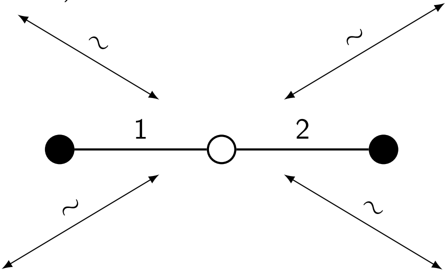
Example ☕



$$x_0 = \varphi(x_1) = x_1^2$$
$$= (x_1 + 1)(x_1 - 1) + 1$$

$$\big((1\,2),(1)(2),(1\,2)\big)$$

$$[\Delta(2,1,2):\Gamma]=2$$



$$\mathbb{C}\left(x_1^2\right)\subseteq\mathbb{C}(x_1)$$

$$x_0=\varphi(x_1)=x_1^2$$

2-solvable (Galois) Belyĭ maps

$$\begin{tikzcd}
X_r \arrow[d, "2"'] \arrow[ddddr, bend left] & \\
\vdots \arrow[d, "2"'] & \\
X_2 \arrow[d, "2"'] \arrow[ddr, bend left] & \\
X_1 \arrow[dr, "2"'] & \\
& X_0 = \mathbb{P}^1
\end{tikzcd}
\qquad
\begin{tikzcd}
K(X_r) \arrow[d, dash, "2"'] \arrow[ddddr, dash, bend left] & \\
\vdots \arrow[d, dash, "2"'] & \\
K(X_2) \arrow[d, dash, "2"'] \arrow[ddr, dash, bend left] & \\
K(X_1) \arrow[dr, dash, "2"'] & \\
& K(X_0) = K(\mathbb{P}^1)
\end{tikzcd}$$

Beckmann's Theorem

Beckmann's Theorem

The **field of moduli of a curve** X (defined over $\overline{\mathbb{Q}}$) is the fixed field of the group $\{\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : X^\tau \cong X\}$.

The **field of moduli of a curve** X (defined over $\overline{\mathbb{Q}}$) is the fixed field of the group $\{\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : X^\tau \cong X\}$.

Similarly, the **field of moduli of a Belyĭ map** φ is the fixed field of $\{\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : \varphi^\tau \cong \varphi\}$.

Beckmann's Theorem

The **field of moduli of a curve** X (defined over $\overline{\mathbb{Q}}$) is the fixed field of the group $\{\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : X^\tau \cong X\}$.

Similarly, the **field of moduli of a Belyĭ map** φ is the fixed field of $\{\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : \varphi^\tau \cong \varphi\}$.

Theorem (Sybilla Beckmann 1989)

Assume (X, φ) is a pair consisting of a curve X and a Belyĭ map φ for X. Let M be the field of moduli of the pair (X, φ). Let G be the Galois group of the Galois closure of the cover $\varphi : X \to \mathbb{P}^1$. For every prime $p \in \mathbb{Z}$, if p does not divide $\#G$, then X has good reduction at primes of M above p. Moreover, p is unramified in M.

Lifting permutation triples

Lifting permutation triples

$$
\begin{tikzcd}
\widetilde{X} \arrow[dr, "\psi"] \arrow[dd, "\widetilde{\varphi}"'] & \\
 & X \arrow[dl, "\varphi"] \\
\mathbb{P}^1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
K(\widetilde{X}) \arrow[dr, dash, "m"] \arrow[dd, dash] & \\
 & K(X) \arrow[dl, dash, "d"] \\
K(\mathbb{P}^1) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\{1\} \arrow[dr, dash, "m"] \arrow[dd, dash] & \\
 & H \arrow[dl, dash, "d"] \\
\widetilde{G} &
\end{tikzcd}
$$

Lifting permutation triples

$$\begin{tikzcd}
\widetilde{X} \arrow[dd, "\widetilde{\varphi}"'] \arrow[dr, "\psi"] & \\
& X \arrow[dl, "\varphi"] \\
\mathbb{P}^1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
K(\widetilde{X}) \arrow[dd, dash] \arrow[dr, dash, "m"] & \\
& K(X) \arrow[dl, dash, "d"] \\
K(\mathbb{P}^1) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\{1\} \arrow[dd, dash] \arrow[dr, dash, "m"] & \\
& H \arrow[dl, dash, "d"] \\
\widetilde{G} &
\end{tikzcd}$$

$$G = \mathrm{Gal}(K(X)/K(\mathbb{P}^1)) \qquad G \cong \langle \sigma \rangle \leq S_d$$

$$\widetilde{G} = \mathrm{Gal}(K(\widetilde{X})/K(\mathbb{P}^1)) \qquad \widetilde{G} \cong \langle \widetilde{\sigma} \rangle \leq S_{md}$$

$$H = \mathrm{Gal}(K(\widetilde{X})/K(X)) \qquad H \cong \langle \tau \rangle \leq S_{md}$$

Lifting permutation triples

$$
\begin{tikzcd}
\widetilde{X} \arrow[dd, "\widetilde{\varphi}"'] \arrow[dr, "\psi"] & \\
& X \arrow[dl, "\varphi"] \\
\mathbb{P}^1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
K(\widetilde{X}) \arrow[dd, dash] \arrow[dr, dash, "m"] & \\
& K(X) \arrow[dl, dash, "d"] \\
K(\mathbb{P}^1) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\{1\} \arrow[dd, dash] \arrow[dr, dash, "m"] & \\
& H \arrow[dl, dash, "d"] \\
\widetilde{G} &
\end{tikzcd}
$$

$$G = \mathrm{Gal}(K(X)/K(\mathbb{P}^1)) \qquad G \cong \langle \sigma \rangle \leq S_d$$

$$\widetilde{G} = \mathrm{Gal}(K(\widetilde{X})/K(\mathbb{P}^1)) \qquad \widetilde{G} \cong \langle \widetilde{\sigma} \rangle \leq S_{md}$$

$$H = \mathrm{Gal}(K(\widetilde{X})/K(X)) \qquad H \cong \langle \tau \rangle \leq S_{md}$$

$$1 \longrightarrow H \xrightarrow{\iota} \widetilde{G} \xrightarrow{f} G \longrightarrow 1$$

$$\widetilde{\sigma} \xrightarrow{?} \sigma$$

Lifting example ☕

Lifting example ☕

$$\begin{tikzcd} \widetilde{X} \arrow[dd, "\widetilde{\varphi}"'] \arrow[dr, "\psi"] & \\ & X \arrow[dl, "\varphi"] \\ \mathbb{P}^1 & \end{tikzcd} \qquad \begin{tikzcd} K(\widetilde{X}) \arrow[dd, dash] \arrow[dr, dash, "2"] & \\ & K(X) \arrow[dl, dash, "2"] \\ K(\mathbb{P}^1) & \end{tikzcd} \qquad \begin{tikzcd} \{1\} \arrow[dd, dash] \arrow[dr, dash, "2"] & \\ & H \arrow[dl, dash, "2"] \\ \widetilde{G} & \end{tikzcd}$$

$$
\begin{tikzcd}
\widetilde{X} \arrow[dd, "\widetilde{\varphi}"'] \arrow[dr, "\psi"] & \\
& X \arrow[dl, "\varphi"] \\
\mathbb{P}^1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
K(\widetilde{X}) \arrow[dd, dash] \arrow[dr, dash, "2"] & \\
& K(X) \arrow[dl, dash, "2"] \\
K(\mathbb{P}^1) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\{1\} \arrow[dd, dash] \arrow[dr, dash, "2"] & \\
& H \arrow[dl, dash, "2"] \\
\widetilde{G} &
\end{tikzcd}
$$

$$
\begin{aligned}
G &= \mathrm{Gal}(K(X)/K(\mathbb{P}^1)) & \qquad G &\cong \Big\langle \Big((1\,2), (1)(2), (1\,2) \Big) \Big\rangle \leq S_2 \\
\widetilde{G} &= \mathrm{Gal}(K(\widetilde{X})/K(\mathbb{P}^1)) & \qquad \widetilde{G} &\cong \langle \widetilde{\sigma} \rangle \leq S_4 \\
H &= \mathrm{Gal}(K(\widetilde{X})/K(X)) & \qquad H &\cong \langle (1\,3)(2\,4) \rangle \leq S_4
\end{aligned}
$$

Lifting example ☕

$$\begin{tikzcd}
\widetilde{X} \arrow[dr, "\psi"] \arrow[dd, "\widetilde{\varphi}"'] & \\
& X \arrow[dl, "\varphi"] \\
\mathbb{P}^1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
K(\widetilde{X}) \arrow[dr, dash, "2"] \arrow[dd, dash] & \\
& K(X) \arrow[dl, dash, "2"] \\
K(\mathbb{P}^1) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\{1\} \arrow[dr, dash, "2"] \arrow[dd, dash] & \\
& H \arrow[dl, dash, "2"] \\
\widetilde{G} &
\end{tikzcd}$$

$$G = \mathrm{Gal}(K(X)/K(\mathbb{P}^1)) \qquad G \cong \Big\langle \Big((1\,2), (1)(2), (1\,2) \Big) \Big\rangle \leq S_2$$

$$\widetilde{G} = \mathrm{Gal}(K(\widetilde{X})/K(\mathbb{P}^1)) \qquad \widetilde{G} \cong \langle \widetilde{\sigma} \rangle \leq S_4$$

$$H = \mathrm{Gal}(K(\widetilde{X})/K(X)) \qquad H \cong \langle (1\,3)(2\,4) \rangle \leq S_4$$

$$1 \longrightarrow H \xrightarrow{\iota} \widetilde{G} \xrightarrow{f} G \longrightarrow 1$$

$$\widetilde{\sigma} \xrightarrow{?} \sigma$$

Lifting example ☕

$$\sigma = (\sigma_0, \sigma_1, \sigma_\infty) = \Big((1\,2), (1)(2), (1\,2)\Big) \in S_2^3$$

$$\tau = (1\,3)(2\,4) \in S_4$$

$$\widetilde{G} = \langle \widetilde{\sigma} \rangle \leq S_4$$

$$\sigma = (\sigma_0, \sigma_1, \sigma_\infty) = \Big((1\,2), (1)(2), (1\,2)\Big) \in S_2^3$$
$$\tau = (1\,3)(2\,4) \in S_4$$
$$\widetilde{G} = \langle \widetilde{\sigma} \rangle \leq S_4$$

$$1 \longrightarrow \langle \tau \rangle \xrightarrow{\iota} \widetilde{G} \xrightarrow{f} \langle \sigma \rangle \longrightarrow 1$$

$$\widetilde{\sigma} \xrightarrow{?} \sigma$$

Lifting example ☕

$$\sigma = (\sigma_0, \sigma_1, \sigma_\infty) = \big((1\,2), (1)(2), (1\,2)\big) \in S_2^3$$
$$\tau = (1\,3)(2\,4) \in S_4$$
$$\widetilde{G} = \langle \widetilde{\sigma} \rangle \leq S_4$$

$$1 \longrightarrow \langle \tau \rangle \xrightarrow{\iota} \widetilde{G} \xrightarrow{f} \langle \sigma \rangle \longrightarrow 1$$

$$\widetilde{\sigma} \xrightarrow{?} \sigma$$

$f^{-1}(\sigma_0)$	$f^{-1}(\sigma_1)$	$f^{-1}(\sigma_\infty)$
$(1\,2)(3\,4)$	$(1)(2)(3)(4)$	$(1\,2)(3\,4)$
$(1\,4)(2\,3)$	$(1\,3)(2\,4)$	$(1\,4)(2\,3)$
$(1\,4\,3\,2)$		$(1\,4\,3\,2)$
$(1\,2\,3\,4)$		$(1\,2\,3\,4)$

Lifting example ☕

Lifting example ☕

There are 32 possible σ from these lists.

There are 32 possible σ from these lists.

Only 6 such triples correspond to Belyĭ maps.

There are 32 possible σ from these lists.

Only 6 such triples correspond to Belyĭ maps.

Eliminating redundancies we are left with 3 permutation triples corresponding to distinct Belyĭ maps.

Lifting example ☕

There are 32 possible σ from these lists.

Only 6 such triples correspond to Belyĭ maps.

Eliminating redundancies we are left with 3 permutation triples corresponding to distinct Belyĭ maps.

$\widetilde{G} \cong \mathbb{Z}/4\mathbb{Z}$
$\big((1\,4\,3\,2), (1)(2)(3)(4), (1\,2\,3\,4)\big)$
$\big((1\,4\,3\,2), (1\,3)(2\,4), (1\,4\,3\,2)\big)$

$\widetilde{G} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$\big((1\,2)(3\,4), (1\,4)(2\,3), (1\,3)(2\,4)\big)$

4T1-[4,2,4]-4-22-4-g1

$$(\sigma_0, \sigma_1, \sigma_\infty) = \Big((1\,4\,3\,2), (1\,3)(2\,4), (1\,4\,3\,2)\Big)$$

4T1-[4,2,4]-4-22-4-g1

$$(\sigma_0, \sigma_1, \sigma_\infty) = \Big((1\,4\,3\,2), (1\,3)(2\,4), (1\,4\,3\,2)\Big)$$

$$\begin{tikzcd}
(0,0,0) \arrow[d, equal] & (1,1,0) \arrow[d, equal] & (1,-1,0) \arrow[d, equal] & \infty \arrow[d, equal] \\
(0,0) \arrow[d, equal] & (1,1) \arrow[dr, dash] & (1,-1) \arrow[d, dash] & \infty \arrow[d, equal] \\
0 & & 1 & \infty
\end{tikzcd}
\qquad
\begin{tikzcd}
X_2 : \frac{x_1^2 - x_0}{x_2^2 - (x_1^3 - x_1)} \arrow[d] \arrow[dd, bend left] \\
X_1 : x_1^2 - x_0 \arrow[d] \\
X_0
\end{tikzcd}$$

16T6-[8,8,4]-88-88-4444-g5

Let X by the curve in $\mathbb{P}^5$ cut out by the following equations:

Let X by the curve in $\mathbb{P}^5$ cut out by the following equations:

$$
\begin{aligned}
&x_0 - x_1^2\\
&x_1x_2 + x_1 + x_3^2 - x_3x_4^2\\
&x_1x_3^2 + x_1 - x_3x_4^2\\
&x_1x_4^2 + (1/2)x_3^3 - (1/2)x_3x_4^4\\
&x_2^2 + 2x_3^3x_4^2 - 2x_3^2x_4^4 + 2x_3^2 - 2x_3x_4^2 + 2x_4^4 + 1\\
&x_2x_3 - x_3^3 + x_3^2x_4^2 - x_4^2\\
&x_2x_4^2 + x_3^3 - x_3^2x_4^2 + x_3\\
&x_3^4 - x_3^2x_4^4 + x_3^2 + x_4^4
\end{aligned}
$$

Let X by the curve in $\mathbb{P}^5$ cut out by the following equations:

$$
\begin{aligned}
&x_0 - x_1^2 \\
&x_1x_2 + x_1 + x_3^2 - x_3x_4^2 \\
&x_1x_3^2 + x_1 - x_3x_4^2 \\
&x_1x_4^2 + (1/2)x_3^3 - (1/2)x_3x_4^4 \\
&x_2^2 + 2x_3^3x_4^2 - 2x_3^2x_4^4 + 2x_3^2 - 2x_3x_4^2 + 2x_4^4 + 1 \\
&x_2x_3 - x_3^3 + x_3^2x_4^2 - x_4^2 \\
&x_2x_4^2 + x_3^3 - x_3^2x_4^2 + x_3 \\
&x_3^4 - x_3^2x_4^4 + x_3^2 + x_4^4
\end{aligned}
$$

The map to $\mathbb{P}^1$ defined by $x_0 \in K(X)$ is a genus 5 Belyĭ map with monodromy group $C_8 : C_2$.

Thanks for listening!

https://math.dartmouth.edu/~mjmusty/32.html