PRESCRIBED RAMIFICATION IN FUNCTION FIELDS

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Contents

I.	Introduction	1
I.1.	Motivating Question	1
I.2.	Background: Curves and Function Fields	1
I.3.	Background: Ramification and Riemann-Hurwitz	2
I.4.	Background: Divisors and Riemann-Roch	3
II.	Examples	3
II.1	. General Setting	3
II.2	2. Degree 2 to Degree 1	4
II.3	3. Degree 4 to Degree 2	5
II.4	Degree 8 to Degree 4	6
References		6

I. Introduction

I.1. **Motivating Question.** Let $\varphi: \widetilde{X} \to X$ be a degree 2 map of algebraic curves over $K = \overline{\mathbb{Q}}$. Then ϕ corresponds to a quadratic function field extension $K(\widetilde{X})/K(X)$. Consider an effective divisor $D \in \text{Div}(X)$ of degree d with d even.

Question

How do we construct φ in such a way that we get simple ramification exactly at the points in the support of D?

MM: [maybe just start with "the picture"]

I.2. Background: Curves and Function Fields. Let X be a curve over K and let K(X) denote its function field.

Proposition I.2.1. There is a bijection

$$K(X) \cup \{\infty\} \longleftrightarrow \{maps \ X \to \mathbb{P}^1 \ defined \ over \ K\}.$$

Proof. Recall that $f \in K(X)$ defines a map $X \to \mathbb{P}^1$ defined by

$$p \mapsto \begin{cases} [f(p):1] & \text{if } f \text{ is regular at } p\\ [1:0] & \text{if } f \text{ has a pole at } p \end{cases}$$

On the other hand, given a map $\varphi: X \to \mathbb{P}^1$ where $\varphi = [f:g]$. If g = 0, then φ is the constant map $p \mapsto [1:0]$ (call this map ∞). Otherwise, φ is the map

$$p \mapsto [f/g:1].$$

Proposition I.2.2. Let $\varphi: \widetilde{X} \to X$ be a map of curves. Then φ induces an injection

$$\varphi^*: K(X) \to K(\widetilde{X}).$$

Proof. Let $f \in K(X)$. Then by the previous prop, f corresponds to a map $X \to \mathbb{P}^1$. Precompose with φ and use the previous proposition again.

In fact, even more is true.

Theorem I.2.3. [2, Thm 2.4]

- (1) Let $\varphi: \widetilde{X} \to X$. Then $K(\widetilde{X})/\varphi^*K(X)$ is a finite extension.
- (2) Let $\iota: K(X) \to K(\widetilde{X})$ be an injection fixing K. Then there exists a unique nonconstant map φ such that $\varphi^* = \iota$.
- (3) Let F be a finite index subfield of $K(\widetilde{X})$ containing K. Then there exists a smooth curve X (unique up to K-iso) and a nonconstant map $\varphi: \widetilde{X} \to X$ such that K(X) = F.

Moreover, these categories are equivalent.

I.3. Background: Ramification and Riemann-Hurwitz.

Definition I.3.1. Let $\varphi: \widetilde{X} \to X$ be a nonconstant map of curves. Let $\widetilde{p} \in \widetilde{X}$ be smooth. The **ramification index** of φ at \widetilde{p} is the quantity:

$$e_{\varphi}(\widetilde{p}) = \operatorname{ord}_{\widetilde{p}}(\varphi^* t_{\varphi(\widetilde{p})})$$

where $t_{\varphi(\widetilde{p})}$ is a uniformizer at $\varphi(\widetilde{p})$. We say φ is **unramified at** \widetilde{p} if $e_{\varphi}(\widetilde{p}) = 1$.

Example I.3.2. Consider the map $\varphi: \mathbb{P}^1 \to \mathbb{P}^1$ given by $x \mapsto x^2 = y$ i.e. with projective coordinates

$$[z_0:z_1] \mapsto [z_0^2:z_1^2].$$

Let $\widetilde{p} = [1:1]$ in the domain. Then

$$e_{\varphi}([1:1]) = \operatorname{ord}_{[1:1]}(\varphi^* t_{\varphi([1:1])})$$

$$= \operatorname{ord}_{[1:1]}(\varphi^* t_{[1:1]})$$

$$= \operatorname{ord}_{[1:1]}(\varphi^* (y-1))$$

$$= \operatorname{ord}_{[1:1]}(x^2 - 1)$$

$$= 1$$

If instead $\widetilde{p} = [0:1]$, then $\varphi(\widetilde{p}) = [0:1] =: p$, and $t_p = y$, $\varphi^* t_p = x^2$, and $\operatorname{ord}_{\widetilde{p}}(x^2) = 2$. MM: [also just think about preimages]

Proposition I.3.3. Let $\varphi : \widetilde{X} \to X$ be a nonconstant map of smooth curves. Then for every $p \in X$,

$$\sum_{\widetilde{p}\in\varphi^{-1}(p)}e_{\varphi}(\widetilde{p})=\deg(\varphi).$$

Theorem I.3.4 (Riemann-Hurwitz). Let $\varphi: \widetilde{X} \to X$ be a nonconstant map of smooth curves. Then,

$$2g(\widetilde{X}) - 2 = (2g(X) - 2)\deg(\varphi) + \sum_{\widetilde{p} \in \widetilde{X}} (e_{\varphi}(\widetilde{p}) - 1).$$

MM: [more analogies with number fields]

I.4. **Background: Divisors and Riemann-Roch.** Recall that for a curve X we have an exact sequence

$$0 \longrightarrow \operatorname{Div}^{0}(X) \longrightarrow \operatorname{Div}(X) \xrightarrow{\operatorname{deg}} \mathbb{Z} \longrightarrow 0.$$

For $f \in K(X)^{\times}$, recall that the map

$$\operatorname{div}(f) = \sum_{p \in X} \operatorname{ord}_p(f)(p) \in \operatorname{Div}(X)$$

yields the exact sequence

$$1 \longrightarrow K^{\times} \longrightarrow K(X)^{\times} \xrightarrow{\operatorname{div}} \operatorname{Div}^{0}(X) \longrightarrow \operatorname{Pic}^{0}(X) \longrightarrow 0.$$

Example I.4.1. Let $X_0 = \mathbb{P}^1$ with coordinate t. Let $p \in X$ and a = t(p). $K(X_0) = K(x)$ i.e. rational functions in the indeterminate x. Let $f = x - a \in K(X_0)$. Then f corresponds to a map φ . Then $\operatorname{div}(f) = \varphi^*((0) - (\infty)) = (p) - (\infty)$.

Definition I.4.2. We say a divisor is **effective** if none of the coefficients are negative. Let $D \in \text{Div}(X)$ and define

$$\mathscr{L}(D) := \{ f \in K(X) : \operatorname{div}(f) + D \text{ effective} \} \cup \{ 0 \}.$$

MM: [functions with poles no worse than those of D]

Theorem I.4.3 (Riemann-Roch). [1] There is an integer $g \ge 0$ and a divisor class \mathscr{C} such that for all $C \in \mathscr{C}$ and $D \in \operatorname{Div}(X)$, we have

$$\ell(D) - \ell(C - D) = \deg(D) - g + 1.$$

II. Examples

II.1. General Setting.

Question

Given a curve X and $D \in \text{Div}(X)$ (D effective and even degree d), find $f \in K(X)^{\times}$ such that the function field

$$K(Y) = \frac{K(X)[y]}{(y^2 - f)}$$

corresponding to the curve Y has ramification exactly at D.

Notes

- $f \in K(X)^{\times}/K(X)^{\times 2}$.
- For an arbitrary D we can pick $D' \in Pic(X)$ such that D + 2D' has degree zero. Then

$$\operatorname{div}(f) = D + 2D'$$
$$\operatorname{div}(fg^2) = D + 2(D' + \operatorname{div}(g)).$$

MM: [think of d distinct points not ∞]

Now we have to solve the following equation in Pic(X):

$$0 = [D] + 2[D'] \iff 2[D'] = -[D] = -d[\infty].$$

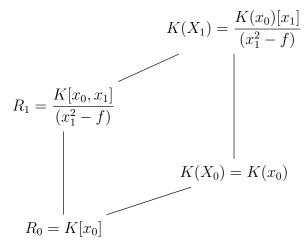
II.2. **Degree 2 to Degree 1.** Let $X_0 = \mathbb{P}^1$. Then $K(X_0) = K(x_0)$. Let $D = (0) - (\infty)$. Then if we let $f = x_0$, we have $\operatorname{div}(f) = D$. We can define a quadratic extension of $K(X_0)$ as follows:

$$K(X_1) = \frac{K(X_0)[x_1]}{(x_1^2 - f)}.$$

In the curves setting we have the curve $X_1: x_1^2 - x_0$ and the degree 2 map

$$\varphi: X_1 \to X_0$$

defined by $x_0 \mapsto x_0^2$. Now where is φ ramified?



We can factor the ideal $I = fR_1$ to see that $I = \mathfrak{p}^2$ where $\mathfrak{p} = (x_0, x_1)$. Similarly, we can see that the ramification index at ∞ is 2, but the infinite prime is $(1/x_0, x_1/x_0)$. MM: [what if we change the divisor D?]

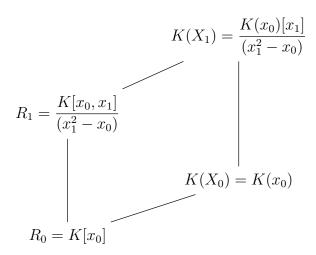
Suppose instead we take D = (0) - (1)? Then $f = x_0/(x_0 - 1)$ has $\operatorname{div}(f) = D$. The diagram stays the same, but now the ideals $I_1 = x_0 R_1$ and $I_2 = (x_0 - 1) R_1$ factor as

$$I_1 = \mathfrak{p}_1^2 = (x_0 - 1, (x_0 - 1)x_1)^2$$

 $I_2 = \mathfrak{p}_2^2 = (x_0, (x_0 - 1)x_1)^2$.

In summary, when $D = (0) - (\infty)$, the discriminant of R_1 is $4x_0$ whereas for D = (0) - (1), the discriminant of R_1 is $4x_0^2 - 4x_0$.

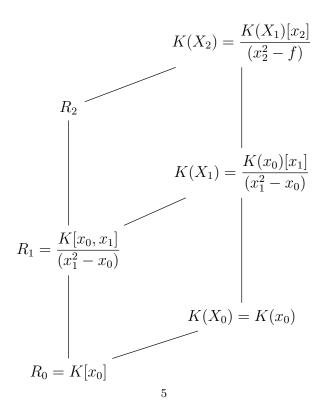
II.3. **Degree 4 to Degree 2.** Now suppose we have the previous diagram:



The curve X_1 is given by the affine equation $x_1^2 - x_0 = 0$. Let $p_1 = (1, 1)$ and $p_{-1} = (1, -1)$ be points on X_1 and let $D = (p_1) - (p_{-1})$. Then we need to pick $f \in K(X_1)$ to extract a sqrt. Let

$$f = \frac{x_1 - 1}{x_1 + 1}$$

so that div(f) = D. Then we can continue our diagram:



As before, we can check that we have the desired ramification.

$$(x_1 - 1)R_2 = \mathfrak{P}_1^2 = \left(x_0 - 1, \frac{1}{2}x_1x_2 + 3x_0x_2 - \frac{5}{2}x_2 - \frac{9}{2}x_1 + \frac{9}{2}\right)^2$$

$$(x_1 + 1)R_2 = \mathfrak{P}_2^2 = \left(x_0 - 1, \left(\left(\frac{13}{2}x_0 - \frac{17}{2}\right)x_1 + \left(\frac{3}{2}x_0^2 + \frac{3}{2}x_0 - 5\right)\right)x_2 + (-6x_0 + 2)x_1 + \frac{3}{2}x_0 - \frac{11}{2}\right)^2.$$

II.4. Degree 8 to Degree 4. like...

References

- 1. Michael Rosen, Number theory in function fields, vol. 210, Springer Science & Business Media, 2013.
- 2. Joseph H Silverman, The arithmetic of elliptic curves, vol. 106, Springer Science & Business Media, 2009.