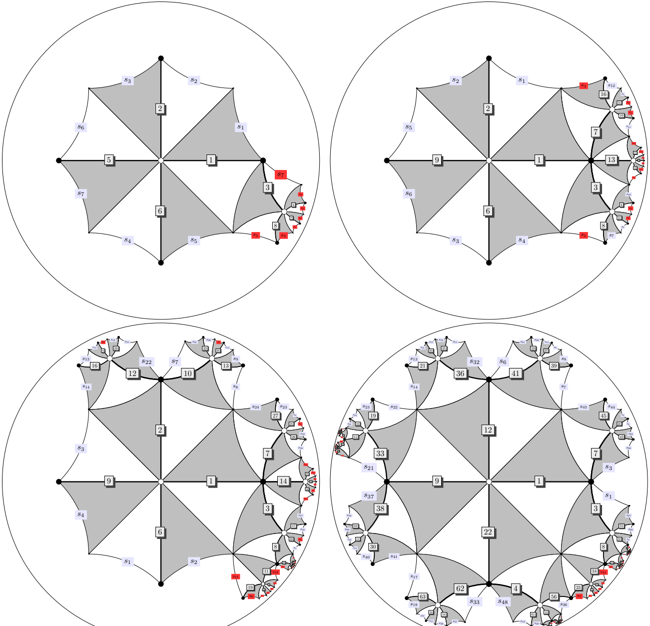
2-Group Belyi Maps



Why $p = 2$?

Conjecture (Gross 1998)

For every prime p, there exists a nonsolvable Galois number field ramified only at p.

$p \geq 11$: existence (Serre), explicit (Edixhoven, Mascot)

$p = 7$: existence (Dieulefait)

$p = 5$: existence (Dembélé, Greenberg, Voight), explicit (Roberts)

$p = 3$: existence (Dembélé, Greenberg, Voight)

$p = 2$: existence (Dembélé)

The hope is that an explicit nonsolvable field ramified only at 2 can be obtained from a 2-group Belyi curve.

Belyi's Theorem

Theorem (G.V. Belyi 1979)

A smooth projective curve X over $\mathbb{C}$ can be defined over $\overline{\mathbb{Q}}$ if and only if there exists a branched covering of compact connected Riemann surfaces $\phi : X \to \mathbb{P}^1$ unramified (unbranched) above $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Such a map is called a **Belyi map**. We will denote the **monodromy group** of a Belyi map ϕ by $\mathrm{Mon}(\phi)$.

Permutation Triples

A **transitive permutation triple of degree** d is a triple

$$\sigma = (\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3$$

such that

- $\sigma_\infty \sigma_1 \sigma_0 = 1$
- σ generates a transitive subgroup of S_d

The set of degree d Belyi maps up to isomorphism is in bijection with the set of degree d transitive permutation triples up to **simultaneous conjugation** and the group $\langle \sigma \rangle$ is the monodromy group of ϕ.

Passports

A **passport** $\mathcal{P}$ consists of the data (g, G, λ) where $g \geq 0$ is an integer, $G \leq S_d$ is a transitive subgroup, and $\lambda = (\lambda_0, \lambda_1, \lambda_\infty)$ is a triple of partitions of d.

The **passport of a Belyi map** $\phi : X \to \mathbb{P}^1$ is $(g(X), \mathrm{Mon}(\phi), (\lambda_0, \lambda_1, \lambda_\infty))$ with $g(X)$ the genus of X, $\mathrm{Mon}(\phi)$ the monodromy group of ϕ, and the partitions specified by ramification.

The **passport of a permutation triple** σ is $(g(\sigma), \langle\sigma\rangle, \lambda(\sigma))$ where

$$g(\sigma) = 1 - d + (e(\sigma_0) - e(\sigma_1) - e(\sigma_\infty))/2$$

with

$$e(\tau) = d - \#\text{cycles of } \tau,$$

and $\lambda(\sigma)$ is specified by cycle structures.

2-Group Belyi maps and Beckmann's Theorem

A **Galois Belyi map** is a degree d Belyi map ϕ with $\#\,\mathrm{Mon}(\phi) = d$.

A 2-**group Belyi map** is a Galois Belyi map ϕ with $\mathrm{Mon}(\phi)$ a 2-group.

Theorem (Beckmann 1989)

Let $\phi : X \to \mathbb{P}^1$ be a Belyi map with monodromy group G. Suppose p does not divide $\#G$. Then there exists a number field M such that p is unramified in M and ϕ is defined over M with good reduction at all primes $\mathfrak{p}$ of M above p.

The upshot of Beckmann's theorem is that every 2-group Belyi curve has a model with good reduction away from $p = 2$.

2-Group Belyi maps

$$
\begin{tikzcd}
X_r \arrow[d, "2"'] \arrow[ddddr, bend left] & \\
\vdots \arrow[d, "2"'] & \\
X_3 \arrow[d, "2"'] \arrow[ddr, bend left] & \\
X_2 \arrow[dr, "2"'] & \\
 & X_1 = \mathbb{P}^1
\end{tikzcd}
$$

Computing 2-group permutation triples

Let $\phi : X \to \mathbb{P}^1$ be a Belyi map of degree $d = 2^\ell$ corresponding to $\sigma \in S_d^3$. We want to find $\widetilde{\phi} : \widetilde{X} \to \mathbb{P}^1$ corresponding to $\widetilde{\sigma} \in S_{2d}^3$ such that

$$\begin{tikzcd} \widetilde{X} \arrow[dd, "\widetilde{\phi}"'] \arrow[dr, "2"] & \\ & X \arrow[dl, "\phi"] \\ \mathbb{P}^1 & \end{tikzcd}$$

Such a $\widetilde{\sigma}$ sits in the following exact sequence of groups:

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} \langle \widetilde{\sigma} \rangle \xrightarrow{\pi} \langle \sigma \rangle \longrightarrow 1$$

Without loss of generality we can restrict our attention to *central* extensions.

Computing 2-group permutation triples

Let σ correspond to a 2-group Belyi map ϕ.

- Compute all equivalence classes of (central) extensions

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} E \xrightarrow{\pi} \langle \sigma \rangle \longrightarrow 1$$

  The main tool used here is Derek Holt's algorithm to compute the second cohomology group of a finite group.
- For each extension, we get 8 possible $\widetilde{\sigma}$. We then check the necessary conditions for $\widetilde{\sigma}$ to correspond to a Belyi map to obtain all possible lifts of σ.

Passport counts

Theorem

The following table lists the number of passports of 2-group Belyi maps of degree d for d up to 256.

d	2	4	8	16	32	64	128	256
# *passports*	3	7	16	41	96	267	834	2893

Computing 2-group Belyi maps

Let $\phi : X \to \mathbb{P}^1$ be a Belyi map of degree $d = 2^\ell$ corresponding to $\sigma \in S_d^3$. Given a permutation triple $\widetilde{\sigma}$ with

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} \langle \widetilde{\sigma} \rangle \xrightarrow{\pi} \langle \sigma \rangle \longrightarrow 1 \ ,$$

let us now consider the problem of finding the Belyi map corresponding to $\widetilde{\sigma}$. Let $X \subseteq \mathbb{A}_K^n$ with defining equations $\{g_i\}_{i=1}^s \subset K[x_1, \ldots, x_n]$. Our goal is to find $f \in K(X)^\times$ such that

$$\begin{tikzcd}
\widetilde{X} \arrow[dr, "\psi"] \arrow[dd, "\widetilde{\phi}"'] & \\
& X \arrow[dl, "\phi"] \\
\mathbb{P}^1 &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\frac{\overline{K}(X)[y]}{(y^2 - f)} \arrow[dr, dash, "2"] \arrow[dd, dash] & \\
& \overline{K}(X) \arrow[dl, dash, "d"] \\
\overline{K}(\mathbb{P}^1) &
\end{tikzcd}$$

with ψ (and hence $\widetilde{\phi}$) satisfying the ramification conditions imposed by $\widetilde{\sigma}$.

Computing 2-group Belyi maps

The procedure to find $f \in K(X)$ is as follows:

1. Let $\{Q_i\}$ be the points on X that we want to be ramification values of ψ. These are determined by $\widetilde{\sigma}$.
2. Build a degree 0 divisor $D = \sum_P n_P P$ with n_{Q_i} odd for every i.
3. Try to find f in the (computable) Riemann-Roch space $L(D)$.

There are (at least) two remarks to make about this process:

- Extending the base field K may be necessary to determine D.
- Class group obstruction.

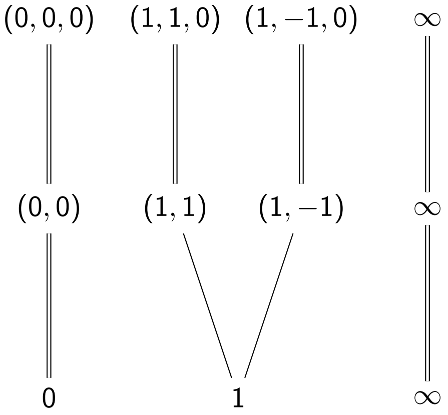
Degree 4 example

$$\widetilde{\sigma} = \Big((1\,4\,3\,2), (1\,3)(2\,4), (1\,4\,3\,2)\Big), \quad \sigma = \Big((1\,2), (1)(2), (1\,2)\Big)$$

$$\begin{tikzcd}
(0,0) \arrow[d, equal] & (1,1) \arrow[dr, dash] & & (1,-1) \arrow[dl, dash] & \infty \arrow[d, equal] & X_2 : x_2^2 = x_1 \arrow[dr] & \\
0 & & 1 & & \infty & & X_1
\end{tikzcd}$$

Degree 4 example

$$\widetilde{\sigma} = \Big((1\,4\,3\,2), (1\,3)(2\,4), (1\,4\,3\,2)\Big), \quad \sigma = \Big((1\,2), (1)(2), (1\,2)\Big)$$



$$\begin{tikzcd}
X_3 : \substack{x_2^2 = x_1 \\ x_3^2 = x_2^3 - x_2} \arrow[d] \arrow[dr, bend left] & \\
X_2 : x_2^2 = x_1 \arrow[r] & X_1
\end{tikzcd}$$

A refined conjecture (by day)

A **refined passport** $\mathcal{P}$ consists of the data (g, G, C) where $g \geq 0$ is an integer, $G \leq S_d$ is a transitive subgroup, and $C = (C_0, C_1, C_\infty)$ is a triple of conjugacy classes of G.

For a refined passport $\mathcal{P}$ consider the set

$$\Sigma_{\mathcal{P}} = \{(\sigma_0, \sigma_1, \sigma_\infty) \in C_0 \times C_1 \times C_\infty : \sigma_\infty \sigma_1 \sigma_0 = 1, \text{ and } \langle \sigma \rangle = G\} / \sim$$

where $(\sigma_0, \sigma_1, \sigma_\infty) \sim (\sigma_0', \sigma_1', \sigma_\infty')$ if and only if there exists $\alpha \in \mathrm{Aut}(G)$ with $\alpha(\sigma_s) = \sigma_s'$ for $s \in \{0, 1, \infty\}$.

Conjecture

Let $\mathcal{P} = (g, G, C)$ be a refined passport with $G = \mathrm{Mon}(\phi)$ for some 2-group Belyi map ϕ. Then $\#\Sigma_{\mathcal{P}} = 0$ or 1.

Corollary

Every 2-group Belyi map is defined over a cyclotomic field $\mathbb{Q}(\zeta_{2^m})$ for some m.

Searching for a nonsolvable field (by night)

Let X be a 2-group Belyi curve defined over a field K with Jacobian A. The hope is that $K(A[2])$ will be nonsolvable.

How to compute $K(A[2])$ given X?

- Sage (Bruin, Sijsling)
- Sage/Magma (Costa, Mascot, Sijsling, Voight)
- Magma (Neurohr)
- Pari/gp (Mascot)

Which curve?

- Coarse factorizations of A (Paulhus)
- Compute $\mathrm{Aut}(X)$ exploiting 2-group structure
- Consider Belyi maps that are not Galois?

Thanks for listening!