

# **2-GROUP BELYI MAPS**

A Thesis

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by

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# Abstract

Write your abstract here.

# Preface

Preface and Acknowledgments go here!

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## Chapter 1

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# Introduction

### Section 1.1

#### **Belyi maps from a historical perspective**

In [2], G.V. Belyi proved that a Riemann surface  $X$  can be defined over a number field (when viewed as an algebraic curve over  $\mathbb{C}$ ) if and only if there exists a non-constant meromorphic function  $\phi : X \rightarrow \mathbb{P}_{\mathbb{C}}^1$  unramified outside the set  $\{0, 1, \infty\}$ . This result came to be known as Belyi's Theorem and the maps  $\phi$  came to be known as Belyi maps (or Belyi functions). Although Belyi's Theorem has an elementary proof, it was a starting point for a great deal of modern research in the area. This work was largely spurred on by Grothendieck's *Esquisse d'un programme* [3] where he was impressed enough to write

*jamais sans doute un résultat profond et déroutant ne fut démontré en si peu de lignes!*

never, without a doubt, was such a deep and disconcerting result proved in so few lines!

An intriguing aspect of the theory of Belyi maps that arose from Grothendieck's work in the 1980s is the reformulation of these objects in a purely topological way. The preimage  $\phi^{-1}([0, 1])$  is a graph embedded on  $X$ , and Grothendieck developed axioms for embedded graphs in such a way that they coincided exactly with the category of Belyi maps. He called these graphs *dessins d'enfants* or children's drawings.

Even as a standalone theorem, Belyi's Theorem is a remarkable result in the mysterious way that it allows us to distinguish between algebraic and transcendental objects. However, the main interest in Belyi maps arises from Galois theory. The absolute Galois group of  $\mathbb{Q}$  acts on the set of Belyi maps via the defining equations. The induced action on the set of dessins

### 1.1.1. Inverse Galois theory, Hurwitz families, and fields with few ramified primes

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*Inverse Galois theory.*

*Hurwitz families.*

### 1.1.2. Grothendieck's theory of dessins d'enfants

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## Chapter 2

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# Background

### Section 2.1

## Belyi maps

### 2.1.1. Algebraic curves and their function fields

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### 2.1.2. Riemann's existence theorem and covers of $\mathbb{P}^1$

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### 2.1.3. Belyi's theorem

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**Proposition 2.1.1.** [MM: \[Galois action on Belyi maps\]](#)

### 2.1.4. Belyi maps and $G$ -Belyi maps

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**Definition 2.1.2.** The geometry type of a Belyi map [MM: \[todo\]](#)

(degenerate)

(spherical)

(Euclidean)

## 2.2 GROUP THEORY

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(hyperbolic)

**Proposition 2.1.3.** *Galois correspondence of Belyi maps*

*Proof.*

□

### 2.1.5. Permutation triples and passports

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**Definition 2.1.4.** MM: [passports and such]

## Section 2.2

# Group theory

### 2.2.1. Central group extensions and $H^2(G, A)$

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**Definition 2.2.1.**

### 2.2.2. Holt's algorithm and Magma implementation

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### 2.2.3. Results on 2-groups

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**Lemma 2.2.2.** MM: [todo]

Section 2.3

## Jacobians of curves

### 2.3.1. Abel-Jacobi and the construction over $\mathbb{C}$

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### 2.3.2. Algebraic construction

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### 2.3.3. Riemann-Roch

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### 2.3.4. Torsion points and torsion fields

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Section 2.4

## Galois representations

### 2.4.1. Representations of Galois groups of number fields

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### 2.4.2. Representations coming from geometry

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## Chapter 3

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# A database of 2-group Belyi maps

In this chapter we describe an algorithm to generate 2-group Belyi maps of a given degree. We begin by defining this particular family of Belyi maps in Section 3.1. The algorithm is inductive in the degree. The base case in degree 1 is discussed in Section 3.2. We then move on to describe the inductive step of the algorithm which we describe in two parts. First we discuss the algorithm to enumerate the isomorphism classes using permutation triples in Section 3.3. For a discussion on the relationship between permutation triples and Belyi maps see Section 2.1. Next we discuss the inductive step to produce Belyi curves and maps in Section 3.4. In Section 3.5 we give a detailed description of the running time of the algorithm. Lastly, in Section 3.6, we discuss the implementation and computations that we have carried out explicitly.

### Section 3.1

## 2-group Belyi maps

Recall the definition of a Belyi map in Section 2.1. In this section we define a narrow our focus to a more specific family of Belyi maps which we now describe.

**Definition 3.1.1.** A degree  $d$  Belyi map  $\phi$  with monodromy group  $G$  is said to be Galois if  $\#G = d$ .

**Definition 3.1.2.** A 2-group Belyi map is a Galois Belyi map with monodromy group a 2-group.

MM: [\[some exposition\]](#)

Section 3.2

### Degree 1 Belyi maps

Section 3.3

### An algorithm to enumerate isomorphism classes of 2-group Belyi maps

The algorithm we describe here is iterative. The degree 1 case is discussed in Section 3.2. We now set up some notation for the iteration.

**Notation 3.3.1.** First we suppose that we are given  $\sigma$  a permutation triple corresponding to a 2-group Belyi map  $\phi : X \rightarrow \mathbb{P}^1$ .

**Definition 3.3.2.** We say that a permutation triple  $\tilde{\sigma}$  is a **degree 2 lift** (or simply a **lift**) of a permutation triple  $\sigma$  if there exists a short exact sequence of groups as in Figure 3.3.1 with  $\iota(\mathbb{Z}/2\mathbb{Z})$  contained in the center of  $\langle \tilde{\sigma} \rangle$ .

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} \langle \tilde{\sigma} \rangle \xrightarrow{\pi} \langle \sigma \rangle \longrightarrow 1$$

Figure 3.3.1:  $\tilde{\sigma}$  a lift of  $\sigma$



### 3.3 AN ALGORITHM TO ENUMERATE ISOMORPHISM CLASSES OF 2-GROUP BELYI MAPS

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In Algorithm 3.3.5 below we describe how to determine all lifts  $\tilde{\sigma}$  (up to isomorphism) of a given permutation triple  $\sigma$ .

**Lemma 3.3.3.** *Let  $\sigma$  be a permutation triple corresponding to a 2-group Belyi map  $\phi : X \rightarrow \mathbb{P}^1$  and  $\tilde{\sigma}$  a lift of  $\sigma$  corresponding to a 2-group Belyi map  $\tilde{\phi} : \tilde{X} \rightarrow \mathbb{P}^1$ . Then there exists a permutation triple  $\tilde{\sigma}'$  that is simultaneously conjugate to  $\tilde{\sigma}$  with  $\iota(\langle \tilde{\sigma}' \rangle)$  contained in the center of  $\langle \sigma \rangle$ .*

*Proof.* □

*Remark 3.3.4.* In light of Lemma 3.3.3, we can restrict our attention to central extensions of  $\langle \sigma \rangle$  in Definition 3.3.2.

**Algorithm 3.3.5.** Let the notation be as described above in 3.3.1.

**Input:**  $\sigma = (\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3$  a permutation triple corresponding to a 2-group Belyi map

**Output:** all lifts  $\tilde{\sigma}$  of  $\sigma$  up to simultaneous conjugation in  $S_{2d}$  sorted by passport

1. Let  $G = \langle \sigma \rangle$  and compute all central extensions  $\tilde{G}$  sitting in the exact sequence in Figure 3.3.2 up to isomorphism (see Definition 2.2.1). For more information about the algorithms to do this see Section 2.2.2.

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} \tilde{G} \xrightarrow{\pi} G \longrightarrow 1$$

Figure 3.3.2:  $\tilde{G}$  a (central) extension of  $G$

2. For each extension  $\tilde{G}$  as in Figure 3.3.2 from the previous step we perform the following:

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- (a) Consider the set of triples

$$\{\tilde{\sigma} := (\tilde{\sigma}_0, \tilde{\sigma}_1, \tilde{\sigma}_\infty) : \tilde{\sigma}_s \in \pi^{-1}(\sigma_s) \text{ for } s \in \{0, 1, \infty\}\} \quad (3.3.1)$$

and let  $\text{Lifts}(\sigma)$  denote the set of such  $\tilde{\sigma}$  with the property that  $\tilde{\sigma}_\infty \tilde{\sigma}_1 \tilde{\sigma}_0 = 1$  and  $\langle \tilde{\sigma} \rangle = \tilde{G}$ .

- (b) For each  $\tilde{\sigma} \in \text{Lifts}(\sigma)$  compute  $\text{order}(\tilde{\sigma}) := (\text{order}(\tilde{\sigma}_0), \text{order}(\tilde{\sigma}_1), \text{order}(\tilde{\sigma}_\infty)) \in \mathbb{Z}^3$  and sort  $\text{Lifts}(\sigma)$  according to  $\text{order}(\tilde{\sigma})$ . Let

$$\text{Lifts}(\sigma, (a, b, c)) := \{\tilde{\sigma} \in \text{Lifts}(\sigma) : \text{order}(\tilde{\sigma}) = (a, b, c)\}. \quad (3.3.2)$$

- (c) For each set of triples  $\text{Lifts}(\sigma, (a, b, c))$  remove simultaneously conjugate triples so that  $\text{Lifts}(\sigma, (a, b, c))$  has exactly one representative from each simultaneous conjugacy class. **MM:** [TODO: reword]

3. Return the union of the sets  $\text{Lifts}(\sigma, (a, b, c))$  ranging over all extensions as in Figure 3.3.2 and for each extension ranging over all orders  $(a, b, c)$ .

*Proof of correctness.* The algorithms in Step 1 are addressed in Section 2.2.2. Let  $\phi : X \rightarrow \mathbb{P}^1$  be the 2-group Belyi map corresponding to  $\sigma$ . By Proposition 2.1.3, the groups obtained from Step 1 are precisely the groups that can occur as monodromy groups of degree 2 covers of  $X$ . **MM:** [ lemma in section about extensions (or in background about Belyi maps) to prove that two isomorphic extensions cannot produce nonisomorphic Belyi maps and that two nonisomorphic extensions cannot produce isomorphic Belyi maps ] In Step 2 we restrict our attention to a single extension of  $G$  as in Figure 3.3.2. When we pullback a triple  $\sigma$  under the map  $\pi$ , there are  $2^3 = 8$

### 3.3 AN ALGORITHM TO ENUMERATE ISOMORPHISM CLASSES OF 2-GROUP BELYI MAPS

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preimages  $\tilde{\sigma}$ . Of these 8 preimages, exactly 4 have the property that  $\tilde{\sigma}_\infty \tilde{\sigma}_1 \tilde{\sigma}_0 = 1$ . Of these 4 triples, we only take those that generate  $\tilde{G}$  and this makes up the set  $\text{Lifts}(\sigma)$ . In Step 2(b), we are sorting  $\text{Lifts}(\sigma)$  by passport. Since 2-group Belyi maps are Galois, the cycle structure of each  $\tilde{\sigma}_s \in \tilde{\sigma}$  is determined by the order of  $\tilde{\sigma}_s$  so that sorting by order is the same as sorting by cycle structure.

*Remark 3.3.6.* In fact, even though we do not need this for the algorithm, there are at most 2 different passports that can occur in  $\text{Lifts}(\sigma)$ . 2 different passports occur when one of  $\sigma_s \in \sigma$  is the identity. If  $\sigma$  does not contain an identity element, then all triples in  $\text{Lifts}(\sigma)$  have the same passport.

At this point, we have constructed the sets  $\text{Lifts}(\sigma, (a, b, c))$ . In light of Remark 3.3.6, there are only 2 possibilities:

- There is only one such set  $\text{Lifts}(\sigma, (a, b, c))$  consisting of at most 4 triples.
- There are 2 sets  $\text{Lifts}(\sigma, (a, b, c))$  and  $\text{Lifts}(\sigma, (a', b', c'))$  each consisting of at most 2 triples.

Step 2(c) is to eliminate simultaneous conjugation in each set  $\text{Lifts}(\sigma, (a, b, c))$ . After Step 2(c) is complete, the sets  $\text{Lifts}(\sigma, (a, b, c))$  contain exactly one permutation triple for each isomorphism class of 2-group Belyi map with passport determined by  $(a, b, c)$  and monodromy group  $\tilde{G}$  such that the diagram in Figure 3.3.3 commutes. In Step 3 we collect together all sets  $\text{Lifts}(\sigma, (a, b, c))$  as we range over all possible extensions in Step 1, and by the discussion for Step 2 yields the desired output.  $\square$

We now illustrate Algorithm 3.3.5 with the following example.

*Example 3.3.7.* In this example we carry out Algorithm 3.3.5 for the degree 2 permutation triple  $\sigma = ((12), (1)(2), (12))$ . Here  $G = \langle \sigma \rangle \cong \mathbb{Z}/2\mathbb{Z}$ . In Step 1, we obtain

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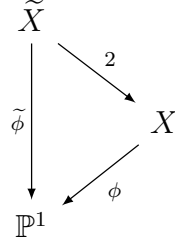


Figure 3.3.3: The permutation triples  $\tilde{\sigma}$  constructed in Algorithm 3.3.5 correspond to Belyi maps  $\tilde{\phi} : \tilde{X} \rightarrow \mathbb{P}^1$  in the above diagram.

two group extensions  $\tilde{G}_1 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  and  $\tilde{G}_2 \cong \mathbb{Z}/4\mathbb{Z}$ : We will consider the two

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\iota_1} & \tilde{G}_1 & \xrightarrow{\pi_1} & G \longrightarrow 1 \\ 1 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\iota_2} & \tilde{G}_2 & \xrightarrow{\pi_2} & G \longrightarrow 1 \end{array}$$

Figure 3.3.4: Two extensions of  $G$  in Example 3.3.7

extensions separately:

- For  $\tilde{G}_1$ , we have

$$\begin{aligned} \text{Lifts}(\sigma) = \big\{ & ((1\ 2)(3\ 4), (1)(2)(3)(4), (1\ 2)(3\ 4)), ((1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)), \\ & ((1\ 4)(2\ 3), (1)(2)(3)(4), (1\ 4)(2\ 3)), ((1\ 4)(2\ 3), (1\ 3)(2\ 4), (1\ 2)(3\ 4)) \big\} \end{aligned}$$

Before we continue with the algorithm, let us take a moment to explain this more closely in the following remark.

*Remark 3.3.8.* First, note that the image of  $\iota_1$  is an order 2 subgroup of  $\tilde{G}_1$ . Let  $\tau \in \tilde{G}_1$  denote the generator of this image. From the perspective of branched covers,  $\tau$  is identifying 4 sheets in a degree 4 cover down to 2 sheets in a degree 2 cover. Elements  $\tilde{\sigma}$  of  $\text{Lifts}(\sigma)$  must induce a well-defined action on the identified

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sheets and this action must be compatible with  $\sigma$ . In this example  $\tau = (1\,3)(2\,4)$  meaning that  $\tau$  identifies the sheets labeled 1 and 3 into a single sheet and  $\tau$  identifies the sheets labeled 2 and 4 into a single sheet. Another way of saying that  $\tilde{\sigma}$  induces a well-defined action is that  $\tilde{\sigma}$  acts on the blocks  $\{\boxed{1\,3}, \boxed{2\,4}\}$ . Saying that this action is compatible with  $\sigma$  means that for each  $s \in \{0, 1, \infty\}$  the induced action of  $\tilde{\sigma}_s$  on blocks is the same as  $\sigma_s$ . For

$$\tilde{\sigma} = ((1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,4))$$

we have  $\tilde{\sigma}_0 \boxed{1\,3} = \boxed{2\,4}$  and  $\tilde{\sigma}_0 \boxed{2\,4} = \boxed{1\,3}$  so that the induced permutation of blocks is

$$\left( \boxed{1\,3}, \boxed{2\,4} \right)$$

which is the same as the permutation  $\sigma_0 = (1\,2)$  (as long as we identify  $\boxed{1\,3}$  with 1 and  $\boxed{2\,4}$  with 2).

To finish Step 2(a) we only take triples in  $\text{Lifts}(\sigma)$  that generate  $\tilde{G}_1$ , so at the end of Step 2(a) for this extension we have

$$\text{Lifts}(\sigma) = \left\{ ((1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)), ((1\,4)(2\,3), (1\,3)(2\,4), (1\,2)(3\,4)) \right\}.$$

In Step 2(b) we sort  $\text{Lifts}(\sigma)$  into passports as determined by orders of elements. Here, all  $\tilde{\sigma} \in \text{Lifts}(\sigma)$  have the same orders (and hence belong to the same passport). Thus we get a single set  $\text{Lifts}(\sigma, (2, 2, 2)) = \text{Lifts}(\sigma)$ . Lastly, in Step 2(c) we see that the two triples in  $\text{Lifts}(\sigma, (2, 2, 2))$  are simultaneously conjugate (by the permutation  $(2\,4)$ ) and hence we remove one of the triples

### 3.3 AN ALGORITHM TO ENUMERATE ISOMORPHISM CLASSES OF 2-GROUP BELYI MAPS

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from  $\text{Lifts}(\sigma, (2, 2, 2))$ .

- For  $\tilde{G}_2$ , we have

$$\begin{aligned} \text{Lifts}(\sigma) = \big\{ & ((1\ 4\ 3\ 2), (1)(2)(3)(4), (1\ 2\ 3\ 4)), ((1\ 2\ 3\ 4), (1\ 3)(2\ 4), (1\ 2\ 3\ 4)), \\ & ((1\ 2\ 3\ 4), (1)(2)(3)(4), (1\ 4\ 3\ 2)), ((1\ 4\ 3\ 2), (1\ 3)(2\ 4), (1\ 4\ 3\ 2)) \big\} \end{aligned}$$

All 4 of the above triples in  $\text{Lifts}(\sigma)$  generate  $\tilde{G}_2$ , so we continue to Step 2(b) with  $\# \text{Lifts}(\sigma) = 4$ . In Step 2(b), we sort  $\text{Lifts}(\sigma)$  into two sets  $\text{Lifts}(\sigma, (4, 1, 4))$  and  $\text{Lifts}(\sigma, (4, 2, 4))$  each containing 2 triples. In Step 2(c), we find that the 2 triples in  $\text{Lifts}(\sigma, (4, 1, 4))$  are simultaneously conjugate (by the permutation  $(2\ 4)$ ) and the 2 triples in  $\text{Lifts}(\sigma, (4, 2, 4))$  are simultaneously conjugate (also by the permutation  $(2\ 4)$ ), so we remove one permutation triple from each of these sets so that  $\text{Lifts}(\sigma, (4, 1, 4))$  and  $\text{Lifts}(\sigma, (4, 2, 4))$  both have cardinality 1.

In Step 3, we return

$$\text{Lifts}(\sigma, (2, 2, 2)) \cup \text{Lifts}(\sigma, (4, 1, 4)) \cup \text{Lifts}(\sigma, (4, 2, 4))$$

which is a set of 3 permutation triples each corresponding to an isomorphism class of 2-group Belyi map as in Figure 3.3.3.

Now that we have an algorithm to find all lifts of a single permutation triple, the next step is to describe how to use this to organize all isomorphism classes of 2-group Belyi maps of a given degree.

**Algorithm 3.3.9.** Let the notation be as described above in 3.3.1.

**Input:**  $d = 2^m$  for some positive integer  $m$

### 3.3 AN ALGORITHM TO ENUMERATE ISOMORPHISM CLASSES OF 2-GROUP BELYI MAPS

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**Output:** a sequence of bipartite graphs  $\mathcal{G}_2, \mathcal{G}_4, \dots, \mathcal{G}_{2^m}$  where the two sets of nodes of  $\mathcal{G}_{2^i}$  are

- $\mathcal{G}_{2^i}^{\text{above}}$  : the set of isomorphism classes of 2-group Belyi maps of degree  $2^i$  indexed by permutation triples  $\tilde{\sigma}$
- $\mathcal{G}_{2^i}^{\text{below}}$  : the set of isomorphism classes of 2-group Belyi maps of degree  $2^{i-1}$  indexed by permutation triples  $\sigma$

and there is an edge between  $\tilde{\sigma}$  and  $\sigma$  if and only if  $\tilde{\sigma}$  is a lift (as in Definition 3.3.2) of  $\sigma$ . This algorithm is iterative. For each  $i = 1, \dots, m$ , we use  $\mathcal{G}_{2^i}^{\text{below}}$  to compute  $\mathcal{G}_{2^i}^{\text{above}}$  and then we define

$$\mathcal{G}_{2^{i+1}}^{\text{below}} := \mathcal{G}_{2^i}^{\text{above}}$$

and continue the process.

1. To begin the iteration we let  $\mathcal{G}_2^{\text{below}} = \{\sigma\}$  where  $\sigma = ((1), (1), (1)) \in S_1^3$  corresponds to the degree 1 Belyi map.
2. Now suppose we have computed  $\mathcal{G}_{2^i}^{\text{below}}$ . We compute  $\mathcal{G}_{2^i}^{\text{above}}$  as follows:
  - (a) Apply Algorithm 3.3.5 to every  $\sigma \in \mathcal{G}_{2^i}^{\text{below}}$  to obtain  $\#\mathcal{G}_{2^i}^{\text{below}}$  sets  $\text{Lifts}(\sigma)$ . As a word of caution, the notation  $\text{Lifts}(\sigma)$  has a different meaning here than in Algorithm 3.3.5. Here  $\text{Lifts}(\sigma)$  is the set of lifts of  $\sigma$  up to simultaneous conjugation. Let

$$\mathcal{G}_{2^i}^{\text{above}} := \bigcup_{\sigma \in \mathcal{G}_{2^i}^{\text{below}}} \text{Lifts}(\sigma)$$

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and place an edge of  $\mathcal{G}_{2^i}$  between  $\tilde{\sigma} \in \mathcal{G}_{2^i}^{\text{above}}$  and  $\sigma \in \mathcal{G}_{2^i}^{\text{below}}$  if and only if  $\tilde{\sigma} \in \text{Lifts}(\sigma)$ .

- (b) Consider all pairs  $(\tilde{\sigma}, \tilde{\sigma}') \in \mathcal{G}_{2^i}^{\text{above}}$  and for each pair test if  $\tilde{\sigma}$  is simultaneously conjugate to  $\tilde{\sigma}'$  in  $S_{2^i}$ . If the pair is simultaneously conjugate, then combine the nodes  $\tilde{\sigma}$  and  $\tilde{\sigma}'$  into a single node (take either triple) and combine the edge sets of  $\tilde{\sigma}$  and  $\tilde{\sigma}'$  to be the edge set of the new node.
- (c) Return the resulting bipartite graph as  $\mathcal{G}_{2^i}$ .
- (d) If  $i < m$ , then let  $\mathcal{G}_{2^{i+1}}^{\text{below}} := \mathcal{G}_{2^i}^{\text{above}}$  and repeat Step 2 with  $i + 1$ . If  $i = m$ , then return the sequence of bipartite graphs  $\mathcal{G}_2, \mathcal{G}_4, \dots, \mathcal{G}_{2^m}$ .

*Proof of correctness.* We first address the claim that every 2-group Belyi map  $\phi : X \rightarrow \mathbb{P}^1$  of degree  $2^i$  is represented by a permutation triple in  $\mathcal{G}_{2^i}^{\text{above}}$ . Let  $G$  be the monodromy group of  $\phi$ . Since  $\#G = 2^i$ , by Lemma 2.2.2, there exists a normal tower of groups

$$G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_i \tag{3.3.3}$$

where  $G_0 = \{1\}$ ,  $G_i = G$ , and each consecutive quotient is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ . By the Galois correspondence, Proposition 2.1.3, this normal tower of groups corresponds to the diagram in Figure 3.3.5. Let  $\sigma_j$  be the permutation triple corresponding to  $\phi_j$  in Figure 3.3.5. Applying Algorithm 3.3.5 to  $\sigma_j$  we obtain  $\sigma_{j+1}$  as a lift of  $\sigma_j$  so that the permutation triple corresponding to  $\phi$  appears in  $\mathcal{G}_{2^i}^{\text{above}}$ . This shows that every 2-group Belyi map of degree  $2^i$  is represented by at least one node in  $\mathcal{G}_{2^i}$ . We now claim that every 2-group Belyi map of degree  $2^i$  is represented by exactly one node in  $\mathcal{G}_{2^i}$ . Since we are applying Algorithm 3.3.5 to every permutation triple in  $\mathcal{G}_{2^i}^{\text{below}}$ , it is possible that in Step 2(a), the set of permutation triples in  $\mathcal{G}_{2^i}^{\text{above}}$  has



### 3.3 AN ALGORITHM TO ENUMERATE ISOMORPHISM CLASSES OF 2-GROUP BELYI MAPS

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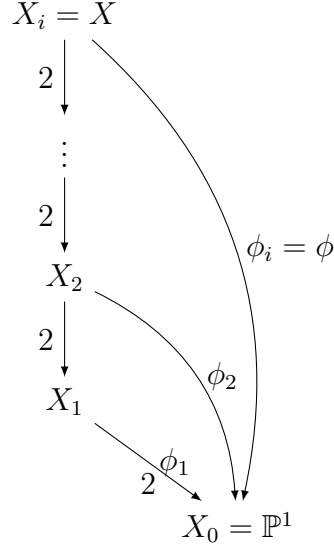


Figure 3.3.5: A 2-group Belyi map  $\phi$  as a sequence of degree 2 covers. For  $j \in \{1, \dots, i\}$ ,  $\phi$  factors through a degree  $2^j$  Belyi map denoted  $\phi_j$ .

simultaneously conjugate triples which arise when a degree  $2^i$  Belyi map is a degree 2 cover of more than one nonisomorphic Belyi map of degree  $2^{i-1}$ . In Step 2(b), we combine permutation triples in  $\mathcal{G}_{2^i}^{\text{above}}$  that are simultaneously conjugate by taking a single permutation triple to represent this isomorphism class of 2-group Belyi map. Note that in Step 2(b) we never remove any edges in the graph  $\mathcal{G}_{2^i}$ . It follows from Step 2(b) that  $\mathcal{G}_{2^i}^{\text{above}}$  has at most one node for each 2-group Belyi map isomorphism class of degree  $2^i$ .  $\square$

**Theorem 3.3.10.** *The following table lists the number of isomorphism classes of 2-group Belyi maps of degree  $d$  for  $d$  up to 256.*

$d$	2	4	8	16	32	64	128	256
#								

### 3.4 AN ALGORITHM TO COMPUTE 2-GROUP BELYI CURVES AND MAPS

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*Proof.* Apply Algorithm 3.3.9. □

**Algorithm 3.3.11.** We use Algorithm 3.3.9 to count the number of Passports of 2-group Belyi maps of a given degree. [MM: \[todo\]](#)

**Theorem 3.3.12.** *The following table lists the number of passports of 2-group Belyi maps of degree  $d$  for  $d$  up to 256.*

$d$	2	4	8	16	32	64	128	256
# passports	3	7	16	41	96	267	834	2893

*Proof.* Apply Algorithm 3.3.11. □

#### Section 3.4

## An algorithm to compute 2-group Belyi curves and maps

The algorithm we describe here is iterative. The degree 1 case is discussed in Section 3.2. We now set up some notation for the iteration.

**Notation 3.4.1.** First we suppose we are given the following data:

- $X \subset \mathbb{P}_K^n$  defined over a number field  $K$  with coordinates  $x_0, \dots, x_n$  cut out by the equations  $\{h_i = 0\}_i$  with  $h_i \in K[x_0, \dots, x_n]$
- $\phi : X \rightarrow \mathbb{P}^1$  a 2-group Belyi map of degree  $d = 2^n$  given by  $\phi([x_0 : \dots : x_n]) = [x_0 : x_1]$  with monodromy group  $G = \langle \sigma \rangle$  (necessarily a 2-group) with  $\sigma$  a permutation triple corresponding to  $\phi$

### 3.4 AN ALGORITHM TO COMPUTE 2-GROUP BELYI CURVES AND MAPS

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- For  $s \in \{0, 1, \infty\}$  and  $\tau$  a cycle of  $\sigma_s \in \sigma$ , denote the ramification point above  $s$  corresponding to  $\tau$  by  $Q_{s,\tau}$
- $Y \subset \mathbb{A}_K^n$  the affine patch of  $X$  with  $x_0 \neq 0$  with coordinates  $(y_1, \dots, y_n)$  where  $y_i = x_i/x_0$  cut out by the equations  $\{g_i = 0\}_i$  with  $g_i \in K[y_1, \dots, y_n]$  so that  $\phi : Y \rightarrow \mathbb{A}^1$  is given by  $\phi(y_1, \dots, y_n) = y_1$
- $\tilde{\sigma}$  as in the output of Algorithm 3.3.5 applied to the input  $\sigma$

Algorithm 3.4.3 below describes how to lift the degree  $d$  Belyi map  $\phi$  to a degree  $2d$  Belyi map  $\tilde{\phi}$  with ramification prescribed by  $\tilde{\sigma}$  (also see Figure 3.4).

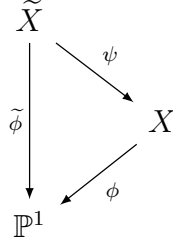


Figure 3.4.1: Algorithm 3.4.3 describes how to construct  $\tilde{\phi}$  corresponding to a permutation triple  $\tilde{\sigma}$  from a given 2-group Belyi map  $\phi$ .

**Lemma 3.4.2.** *Let  $D$  be a degree 0 divisor on  $X$ . Then  $\dim \mathcal{L}(D) \leq 1$ .*

*Proof.* Suppose  $\deg D = 0$ , and Let  $f, g \in \mathcal{L}(D) \setminus \{0\}$ . Write  $D = D_0 - D_\infty$  with  $D_0, D_\infty \geq 0$ . Since  $f, g \in \mathcal{L}(D)$ , we have  $\operatorname{div} f, \operatorname{div} g \geq D_0 - D_\infty$ . In particular,  $f/g \in K^\times$ . □

**Algorithm 3.4.3.** Let the notation be as described above in 3.4.1.

**Input:** A 2-group Belyi map  $\phi : X \rightarrow \mathbb{P}_K^1$  and a permutation triple  $\tilde{\sigma}$

**Output:** A model (over  $\mathbb{Q}^{\text{al}}$ ) for the Belyi map  $\tilde{\phi} : \tilde{X} \rightarrow \mathbb{P}_K^1$  with monodromy  $\tilde{\sigma}$

### 3.4 AN ALGORITHM TO COMPUTE 2-GROUP BELYI CURVES AND MAPS

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1. Let  $R$  be the empty set of points on  $X$ . For each  $s \in \{0, 1, \infty\}$ , If the order of  $\sigma_s$  is strictly less than the order of  $\tilde{\sigma}_s$ , then append the ramification points  $\{Q_{s,\tau}\}_{\tau \in \sigma_s}$  (the ramification points on  $X$  above  $s$  corresponding to the cycles of  $\sigma_s$ ) to  $R$ .
2. Let  $D = \sum_P n_P P$  be a degree 0 divisor on  $X$  with  $n_P$  odd for every  $P \in R$  and  $n_P = 0$  for  $P \notin R$ . MM: [class group and base field]
3. Compute  $f \in \overline{K}(X)^\times$  corresponding to a generator of the Riemann-Roch space  $\mathcal{L}(D)$ .
4. Write  $f = a/b$  with  $a, b \in \overline{K}[y_1, \dots, y_n]$  and construct the ideal

$$\tilde{I} := \langle g_1, \dots, g_k, by_{n+1}^2 - a \rangle$$

in  $\overline{K}[y_1, \dots, y_n, y_{n+1}]$ .

5. Saturate  $\tilde{I}$  at  $\langle b \rangle$  and denote this ideal by  $\text{sat}(\tilde{I})$ .
6. Let  $\tilde{X}$  be the curve corresponding to  $\text{sat}(\tilde{I})$  and  $\tilde{\phi}$  the map  $(y_1, \dots, y_{n+1}) \mapsto y_1$ .

*Proof of correctness.* By Algorithm 3.3.5, there exists a 2-group Belyi map  $\tilde{\phi} : \tilde{X} \rightarrow \mathbb{P}^1$  with ramification according to  $\tilde{\sigma}$ . Since  $\tilde{\phi}$  is Galois, the ramification behavior above each  $s \in \{0, 1, \infty\}$  is constant (i.e. for a fixed  $s$ , all  $Q_{s,\tau}$  are either unramified or ramified to order 2). This ensures that the set  $R$  constructed in Step 1 is precisely the set of ramification values of  $\psi$  (in Figure 3.4). Now that we have the ramification values, we can construct the new Belyi map and curve. We do this by extracting a square root in the function field. More precisely, again by Algorithm 3.3.5, there

### 3.5 RUNNING TIME ANALYSIS

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exists  $\tilde{X}$  with  $\overline{K}(\tilde{X}) = \overline{K}(X, \sqrt{f})$  where  $f \in \overline{K}(X)^\times / \overline{K}(X)^{\times 2}$  and

$$\operatorname{div} f = \sum_{Q_{s,\tau} \in R} Q_{s,\tau} + 2D_\epsilon \in \frac{\operatorname{Div}^0(X)}{2\operatorname{Div}^0(X)} \quad (3.4.1)$$

□

*Example 3.4.4.*

Section 3.5

**Running time analysis**

Section 3.6

**Explicit computations**

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## Chapter 4

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# Results on 2-group Belyi maps

In this chapter we discuss some results on 2-group Belyi maps obtained, in part, from the data in Chapter 3. Section 4.1 details which types of Belyi maps can occur as 2-group Belyi maps. Section 4.2 describes the possible fields of definition and fields of moduli for 2-group Belyi maps.

### Section 4.1

## Classifying 2-group Belyi maps

The conditions that need to be satisfied for a general Belyi map to be a 2-group Belyi map are quite stringent. For example, the ramification indices can be encoded as a triple of integers  $(a, b, c) \in \{1, 2, 4, \dots\}^3$ . We start Section 4.1 by giving a complete description of the geometry type (see Definition 2.1.2) of a 2-group Belyi map in Proposition 4.1.1 and finish Section 4.1 with a classification of hyperelliptic 2-group Belyi maps in Theorem 4.1.3.

**Proposition 4.1.1.** *Let  $\phi : X \rightarrow \mathbb{P}^1$  be a 2-group Belyi map corresponding to a*

#### 4.1 CLASSIFYING 2-GROUP BELYI MAPS

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permutation triple  $\sigma$  with  $(a, b, c)$  the triple of orders of permutations in  $\sigma$ . Then the geometry type of  $\phi$  is determined by  $(a, b, c)$  and is one of the following:

(degenerate)  $\phi$  is degenerate if  $1 \in (a, b, c)$ .

(spherical)  $\phi$  is spherical if  $(a, b, c)$  is of the form  $(2, 2, c)$ ,  $(2, b, 2)$ , or  $(a, 2, 2)$ .

(Euclidean)  $\phi$  is Euclidean if  $(a, b, c)$  is of the form  $(2, 4, 4)$ ,  $(4, 2, 4)$ , or  $(4, 4, 2)$ .

(hyperbolic)  $\phi$  is hyperbolic (with genus at least 2) in all other cases.

*Proof.*

□

Before classifying the hyperelliptic 2-group Belyi maps, we prove the following Lemma.

**Lemma 4.1.2.** *Let  $\phi : X \rightarrow \mathbb{P}^1$  be a hyperelliptic 2-group Belyi map with monodromy group  $G$ . Let  $\iota$  denote the hyperelliptic involution in  $\text{Aut}(X)$ . Then  $G \leq \text{Aut}(X)$  and  $\iota$  is central in  $G$ .*

*Proof.*

□

**Theorem 4.1.3.** *Let  $\phi : X \rightarrow \mathbb{P}^1$  be a hyperelliptic 2-group Belyi map with monodromy group  $G$ . Let  $\iota$  denote the hyperelliptic involution in  $\text{Aut}(X)$ . Then  $\overline{G} := G/\langle \iota \rangle$  is a subgroup of  $\text{Aut}(\mathbb{P}^1) \cong \text{PGL}_2(\mathbb{C})$ . (see Figure 4.1.1). In particular,  $\overline{G}$  is either cyclic, dihedral, or exceptional.*

*Proof.*

□

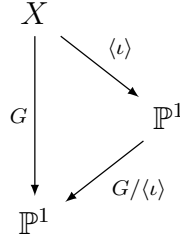


Figure 4.1.1: A hyperelliptic 2-group Belyi map with monodromy group  $G$  factors through a degree 2 map to  $\mathbb{P}^1$  and an automorphism of  $\mathbb{P}^1$ .

## Section 4.2

### Fields of definition of 2-group Belyi maps

In Section 4.2 we use the data from Chapter 3 to formulate a conjecture about the possible fields of definition of 2-group Belyi maps. Recall the action of  $G_{\mathbb{Q}}$  on the set of Belyi maps described in Proposition 2.1.1. For a fixed Belyi map, we can simplify matters as described in the following definition.

**Definition 4.2.1.** The field of moduli of a Belyi map  $\phi : X \rightarrow \mathbb{P}^1$  is the fixed field

$$\{\tau \in G_{\mathbb{Q}} : \phi^{\tau} \cong \phi\}.$$

Definition 4.2.1 allows us to study a more manageable finite extension. Moreover, passports (recall Definition 2.1.4) allow us to bound the degree of the field of moduli.

**Theorem 4.2.2.** *Let  $\phi : X \rightarrow \mathbb{P}^1$  be a Belyi map with passport  $\mathcal{P}$ . Then the degree of the field of moduli of  $\phi$  is bounded by the size of  $\mathcal{P}$ .*

*Proof.*

□



**Definition 4.2.3.** A refined passport  $\mathcal{P}$  consists of the data  $(g, G, C)$  where  $g \geq 0$  is an integer,  $G \leq S_d$  is a transitive subgroup, and  $C = (C_0, C_1, C_\infty)$  is a triple of conjugacy classes of  $G$ .

**MM:** [\[some exposition about refined passports\]](#) For a refined passport  $\mathcal{P}$  consider the set

$$\Sigma_{\mathcal{P}} = \{(\sigma_0, \sigma_1, \sigma_\infty) \in C_0 \times C_1 \times C_\infty : \sigma_\infty \sigma_1 \sigma_0 = 1, \text{ and } \langle \sigma \rangle = G\} / \sim$$

where  $(\sigma_0, \sigma_1, \sigma_\infty) \sim (\sigma'_0, \sigma'_1, \sigma'_\infty)$  if and only if there exists  $\alpha \in \text{Aut}(G)$  with  $\alpha(\sigma_s) = \sigma'_s$  for  $s \in \{0, 1, \infty\}$ .

**Conjecture 4.2.4.** Let  $\mathcal{P} = (g, G, C)$  be a refined passport with  $G = \text{Mon}(\phi)$  for some 2-group Belyi map  $\phi$ . Then  $\#\Sigma_{\mathcal{P}} = 0$  or 1.

*Proof.*

□

**Corollary 4.2.5.** Every 2-group Belyi map is defined over a cyclotomic field  $\mathbb{Q}(\zeta_{2^m})$  for some  $m$ .

*Proof.*

□

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## Chapter 5

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# Gross's conjecture for $p = 2$

We begin this chapter with Theorem 5.1.1 which provides the arithmetic motivation to study 2-group Belyi maps. We then detail past results on Gross's conjecture in Section 5.2 and finish with some discussion on 2-group Belyi maps in relation to the  $p = 2$  case of Gross's conjecture.

### Section 5.1

## Beckmann's theorem

In this Section we state Beckmann's theorem for Belyi maps over  $\mathbb{C}$  from 1989 which can be found in [1]. We then adapt Theorem 5.1.1 to our particular situation in Corollary 5.1.2.

**Theorem 5.1.1.** *Let  $\phi : X \rightarrow \mathbb{P}^1$  be a Belyi map with monodromy group  $G$  and suppose  $p$  does not divide  $\#G$ . Then there exists a number field  $M$  with the following properties:*

- *$p$  is unramified in  $M$*

## 5.2 PAST RESULTS ON GROSS'S CONJECTURE

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- the Belyi map  $\phi$  is defined over  $M$
- the Belyi curve  $X$  is defined over  $M$
- $X$  has good reduction at all primes  $\mathfrak{p}$  of  $M$  above  $p$

*Proof.* [1]

□

**Corollary 5.1.2.** *Let  $\phi : X \rightarrow \mathbb{P}^1$  be a 2-group Belyi map. Then there exists a smooth projective model for  $X$  with good reduction away from  $p = 2$ .*

*Proof.*

□

Section 5.2

**Past results on Gross's conjecture**

Section 5.3

**A nonsolvable Galois number field ramified only  
at 2**

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