JACOBIANS WITH GOOD REDUCTION AWAY FROM 2 ARISING FROM 2-GROUP BELYI MAPS

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Contents

1. Introduction

TODO: [Motivation, then define Belyĭ maps, passports, Galois orbits, and the specific setting: 2-solvable permutation triples and everything Galois.]

Let X be a nice curve over $\mathbb{C}$. A *Belyĭ map* on X is a nonconstant map $\phi\colon X \to \mathbb{P}^1$ that is unramified away from $\{0, 1, \infty\}$.

A *passport* is the data (g, G, λ) consisting of a nonnegative integer $g \in \mathbb{Z}_{\geq 0}$, a transitive permutation group $G \leq S_d$, and three partitions $\lambda = (\lambda_0, \lambda_1, \lambda_\infty)$ of d. The *passport* of a Belyĭ map is given by its genus, its monodromy group, and the ramification degrees of the points above $0, 1, \infty$.

Throughout, let $K \subseteq \mathbb{C}$ be a field. A *(nice) curve* over K is a smooth, projective, connected (irreducible) scheme of finite type over K that is pure of dimension 1. After extension to $\mathbb{C}$, a curve may be thought of as a compact, connected Riemann surface. A *Belyĭ map* over K is a finite morphism $\phi\colon X \to \mathbb{P}^1$ over K that is unramified outside $\{0, 1, \infty\}$; we will sometimes write (X, ϕ) when we want to pay special attention to the source curve X. Two Belyĭ maps ϕ, ϕ' are *isomorphic* if there is an isomorphism $\iota\colon X \xrightarrow{\sim} X'$ of curves such that $\phi'\iota = \phi$.

Let $\phi\colon X \to \mathbb{P}^1$ be a Belyĭ map over $\mathbb{Q}^{\mathrm{al}}$ of degree $d \in \mathbb{Z}_{\geq 1}$. The *monodromy group* of ϕ is the Galois group $\mathrm{Mon}(\phi) := \mathrm{Gal}(\mathbb{C}(X) \,|\, \mathbb{C}(\mathbb{P}^1)) \leq S_d$ of the corresponding extension of function fields (understood as the action of the automorphism group of the normal closure); the group $\mathrm{Mon}(\phi)$ may also be obtained by lifting paths around $0, 1, \infty$ to X.

A *permutation triple* of degree $d \in \mathbb{Z}_{\geq 1}$ is a tuple $\sigma = (\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3$ such that $\sigma_\infty\sigma_1\sigma_0 = 1$. A permutation triple is *transitive* if the subgroup $\langle\sigma\rangle \leq S_d$ generated by σ is transitive. We say that two permutation triples σ, σ' are *simultaneously conjugate* if there exists $\tau \in S_d$ such that

$$\sigma^\tau := (\tau^{-1}\sigma_0\tau, \tau^{-1}\sigma_1\tau, \tau^{-1}\sigma_\infty\tau) = (\sigma_0', \sigma_1', \sigma_\infty') = \sigma'. \tag{1}$$

An automorphism of a permutation triple σ is an element of S_d that simultaneously conjugates σ to itself, i.e., $\mathrm{Aut}(\sigma) = Z_{S_d}(\langle\sigma\rangle)$, the centralizer inside S_d.

Lemma 2. *The set of transitive permutation triples of degree d up to simultaneous conjugation is in bijection with the set of Belyĭ maps of degree d up to isomorphism.*

Proof. The correspondence is via monodromy [6, Lemma 1.1]; in particular, the monodromy group of a Belyĭ map is (conjugate in S_d to) the group generated by σ. □

The group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}} \,|\, \mathbb{Q})$ acts on Belyĭ maps by acting on the coefficients of a set of defining equations; under the bijection of Lemma 2, it thereby acts on the set of transitive permutation triples, but this action is rather mysterious.

We can cut this action down to size by identifying some basic invariants, as follows. A *passport* consists of the data $\mathcal{P} = (g, G, \lambda)$ where $g \geq 0$ is an integer, $G \leq S_d$ is a transitive subgroup, and $\lambda = (\lambda_0, \lambda_1, \lambda_\infty)$ is a tuple of partitions λ_s of S_d for $s = 0, 1, \infty$. These partitions will be also be thought of as a tuple of conjugacy classes $C = (C_0, C_1, C_\infty)$ by cycle type, so we will also write passports as (g, G, C). The *passport* of a Belyĭ map $\phi\colon X \to \mathbb{P}^1$ is $(g(X), \mathrm{Mon}(\phi), (\lambda_0, \lambda_1, \lambda_\infty))$, where $g(X)$ is the genus of X and λ_s is the partition of d obtained by the ramification degrees above $s = 0, 1, \infty$, respectively. Accordingly, the *passport* of a transitive permutation triple σ is $(g(\sigma), \langle\sigma\rangle, \lambda(\sigma))$, where (by Riemann–Hurwitz)

$$g(\sigma) := 1 - d + (e(\sigma_0) - e(\sigma_1) - e(\sigma_\infty))/2 \tag{3}$$

and e is the index of a permutation (d minus the number of orbits), and $\lambda(\sigma)$ is the cycle type of σ_s for $s = 0, 1, \infty$. The *size* of a passport $\mathcal{P}$ is the number of simultaneous conjugacy classes (as in 1) of (necessarily transitive) permutation triples σ with passport $\mathcal{P}$.

The action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}} \,|\, \mathbb{Q})$ on Belyĭ maps preserves passports. Therefore, after computing equations for all Belyĭ maps with a given passport, we can try to identify the Galois orbits of this action. We say a passport is *irreducible* if it has one $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}} \,|\, \mathbb{Q})$-orbit and *reducible* otherwise.

2. Enumerating Triples

Definition 4. Let $m \in \mathbb{Z}_{\geq 2}$. Let σ be a permutation triple corresponding to a Belyĭ map $\phi : X \to \mathbb{P}^1$. We say that a permutation triple $\widetilde{\sigma}$ with corresponding Belyĭ map $\widetilde{\phi} : \widetilde{X} \to \mathbb{P}^1$ is an *m-cover* of σ if there exists a degree m map of curves $\psi : \widetilde{X} \to X$ such that $\widetilde{\phi} = \phi \circ \psi$. Two m-covers are *isomorphic* if they correspond to isomorphic Belyĭ maps.

Algorithm 5. Let σ be a permutation triple corresponding to a Belyĭ map $\phi : X \to \mathbb{P}^1$ of degree d and monodromy group G. This algorithm produces all m-covers of σ.

1. Compute representatives for all distinct isomorphism classes of (central) group extensions of the form
$$1 \longrightarrow \mathbb{Z}/m\mathbb{Z} \xrightarrow{\iota} E \xrightarrow{\pi} G \longrightarrow 1.$$
2. For each extension in the previous step, compute the lifts $\pi^{-1}(\sigma)$.
3. For each $\widetilde{\sigma} \in \pi^{-1}(\sigma)$, keep only those satisfying
   - $\widetilde{\sigma}_\infty \widetilde{\sigma}_1 \widetilde{\sigma}_0 = 1$

- and $E \cong \langle \widetilde{\sigma} \rangle \leq S_{md}$.

4. Take one triple from each simultaneous conjugacy class. The remaining $\widetilde{\sigma}$ are the m-covers of σ for the chosen extension.

Proof of correctness. Step 1 is carried out using [2]. In particular, the group cohomology algorithms are based on [4, 5]. For each extension in the previous step, there are m^3 possible triples $\widetilde{\sigma} := \pi^{-1}(\sigma)$. MM: [Let $\tau := \iota(\overline{1})$ in the case $m = 2$. 4 triples multiple to τ and 4 multiply to id. At the end of Step 3 there might not be any. If $\sigma_s = 1$ for some s, then each extension can produce 0,1, or 2 passports. Otherwise, every $\widetilde{\sigma}$ above σ is in the same passport.] □

TODO: [example]

Proposition 6. *Let σ be a permutation triple. If $\widetilde{\sigma}$ and $\widetilde{\sigma}'$ are m-covers of σ, then $\widetilde{\sigma}$ and $\widetilde{\sigma}'$ are in the same Galois orbit.* MM: [what is true about the converse?]

Proof. □

TODO: [m-covers -> passports]

Let $\mathcal{P}_d$ denote the set of solvable passports of degree d. For $\mathcal{P} \in \mathcal{P}_d$, let $\operatorname{Reps}(\mathcal{P})$ denote a set of passport representatives of $\mathcal{P}$, and let $\operatorname{Reps}(\mathcal{P}_d) := \cup_{\mathcal{P} \in \mathcal{P}_d} \operatorname{Reps}(\mathcal{P})$.

Algorithm 7. Given passport representatives $\operatorname{Reps}(\mathcal{P}_{2^{\ell-1}})$, this algorithm computes all solvable passports for a given degree $d = 2^{\ell}$. Moreover, it produces passport representatives $\operatorname{Reps}(\mathcal{P}_{2^{\ell}})$ that are 2-covers of triples in $\operatorname{Reps}(\mathcal{P}_{2^{\ell-1}})$.

1. Let $\sigma \in \operatorname{Reps}(\mathcal{P}_{2^{\ell-1}})$. Using Algorithm 5, compute all 2-covers of σ.
2. Organize these 2-covers of σ by cycle structure. Call these sets of covering triples $\{C_k\}$.
3. Eliminate redundant (simultaneously conjugate) triples in each C_k.
4. MM: [What if the C_k are redundant?]

Proof of correctness. By Proposition 6, all 2-covers of σ are in the same Galois orbit. MM: [Is each C_k a full Galois orbit in the passport?] □

3. Computing Maps

4. Finding Nice Models

5. Computing Galois Representations

6. Interesting Features

Lemma 8. *Let $\phi : X \to \mathbb{P}^1$ be a hyperelliptic Belyĭ map with monodromy G a 2-group. Let ι denote the hyperelliptic involution in $\operatorname{Aut}(X)$. Then $G \leq \operatorname{Aut}(X)$ and ι is central in G.*

Proof. MM: [follows from canonical map...] □

Theorem 9. *Let $\phi : X \to \mathbb{P}^1$ be a hyperelliptic 2-solvable Belyĭ map with monodromy G. Let ι denote the hyperelliptic involution in $\operatorname{Aut}(X)$. Then*

$$\begin{tikzcd} X \arrow[d, "G"'] \arrow[dr, "\langle\iota\rangle"] & \\ & \mathbb{P}^1 \arrow[dl, "G/\langle\iota\rangle"] \\ \mathbb{P}^1 & \end{tikzcd}$$

so that $\overline{G} := G/\langle\iota\rangle \leq \mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}_2(\mathbb{C})$*. In particular,* $\overline{G}$ *is either cyclic, dihedral, or exceptional.*

Proof. □

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