# 2-**group Belyi maps**

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# Acknowledgements

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# Outline

# Motivation

# Galois representations

Let $X$ be an irreducible, smooth projective algebraic curve of genus $g \geq 1$ over a number field $K$. Let $G_K := \mathrm{Gal}(K^{\mathrm{al}} \,|\, K)$ be the absolute Galois group of $K$ and let $\ell \in \mathbb{Z}$ be prime.

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The geometry of $X$ and the arithmetic of $\rho$ are inimately related. For example, if $X$ has good reduction at a prime $\mathfrak{p}$ above $p \neq \ell$, then $\mathfrak{p}$ will be unramified in the $\ell$-**torsion field** $K(J[\ell])$.

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**Theorem (Belyi 1979)**

*An algebraic curve (smooth projective) $X$ over $\mathbb{C}$ can be defined over a number field if and only if $X$ admits a Belyi map.*

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We can now state Beckmann's theorem to relate Belyi maps to our original motivation.

# Beckmann's theorem

**Theorem (Beckmann 1989)**

*Let* $\phi\colon X \to \mathbb{P}^1$ *be a Galois Belyi map with monodromy group* $G$*.*
*Let* $p$ *be a prime not dividing* $\#G$*.*

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*Let $p$ be a prime not dividing $\#G$.*
*Then there exists a number field $M$ satisfying the following properties.*

- *$p$ is unramified in $M$*
- *$\phi$ is defined over $M$*
- *$X$ is defined over $M$*
- *$X$ has good reduction at all primes $\mathfrak{p}$ of $M$ above $p$*

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The hope is that an explicit nonsolvable field ramified only at 2 can be obtained as $K(\mathrm{Jac}(X)[2])$ where $X$ is the domain of a Galois Belyi map with monodromy group a 2-group.
We call these Belyi maps 2-**group Belyi maps**.

# Main results

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- implementation of an algorithm to enumerate isomorphism classes of 2-group Belyi maps
- implementation of an algorithm to compute equations for 2-group Belyi maps over finite fields
- implementation of a *method* to compute equations for 2-group Belyi maps over number fields
- computational and theoretical evidence supporting a conjecture that every 2-group Belyi map is defined over an abelian extension of the rationals

# 2-group Belyi maps as iterated quadratic extensions

$$\begin{tikzcd}
X_r \arrow[d, "2"'] \arrow[ddddr, bend left] & \\
\vdots \arrow[d, "2"'] & \\
X_2 \arrow[d, "2"'] \arrow[ddr, bend left] & \\
X_1 \arrow[dr, "2"'] & \\
& X_0 = \mathbb{P}^1
\end{tikzcd}
\qquad\qquad
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# Background

## Isomorphism of Belyi maps

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Let $\phi\colon X \to \mathbb{P}^1$ and $\phi'\colon X' \to \mathbb{P}^1$ be Belyi maps of degree $d$. $\phi$ and $\phi'$ are **isomorphic** (respectively **lax isomorphic**) if the diagrams

$$\begin{tikzcd} X \arrow[rr, "\sim"] \arrow[dr, "\phi"'] & & X' \arrow[dl, "\phi'"] \\ & \mathbb{P}^1 & \end{tikzcd}, \text{ respectively } \begin{tikzcd} X \arrow[r, "\sim"] \arrow[d, "\phi"'] & X' \arrow[d, "\phi'"] \\ \mathbb{P}^1 \arrow[r, "\sim", "\beta"'] & \mathbb{P}^1 \end{tikzcd}$$

commute where $\beta(\{0,1,\infty\}) = \{0,1,\infty\}$.

# Permutation Triples

A **transitive permutation triple of degree** $d$ is a triple

$$\sigma = (\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3$$

such that

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The set of degree $d$ Belyi maps up to isomorphism is in bijection with the set of degree $d$ transitive permutation triples up to **simultaneous conjugation** and the group $\langle \sigma \rangle$ is the monodromy group of $\phi$.

# Passports

A **passport** $\mathcal{P}$ consists of the data $(g, G, \lambda)$ where $g \geq 0$ is an integer, $G \leq S_d$ is a transitive subgroup, and $\lambda = (\lambda_0, \lambda_1, \lambda_\infty)$ is a triple of partitions of $d$.

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The **passport of a Belyi map** $\phi : X \to \mathbb{P}^1$ is $(g(X), \mathrm{Mon}(\phi), (\lambda_0, \lambda_1, \lambda_\infty))$ with $g(X)$ the genus of $X$, $\mathrm{Mon}(\phi)$ the monodromy group of $\phi$, and the partitions from ramification.

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The **passport of a permutation triple** $\sigma$ is $(g(\sigma), \langle\sigma\rangle, \lambda(\sigma))$ where

$$g(\sigma) = 1 - d + (e(\sigma_0) - e(\sigma_1) - e(\sigma_\infty))/2$$

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We now discuss the importance of organizing triples by passport.

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The situation improves, however, in the Galois setting...

# The Galois setting

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Then

- $\phi$ and $X$ are defined over $M(\phi)$,
- $\#G = d$,
- all cycles of $\sigma_s$ have the same length for $s \in \{0, 1, \infty\}$,
- and if we let $a, b, c$ be the orders of $\sigma_0, \sigma_1, \sigma_\infty$ respectively, we have

$$g(X) = 1 + \frac{\#G}{2}\left(1 - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}\right).$$

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The monodromy group in this setting corresponds to field automorphisms of the Galois closure of $K(X)$ fixing $K(x)$.

# Computing permutation triples

## Setup

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- $\sigma_\infty \sigma_1 \sigma_0 = \mathrm{id}$;
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- $G$ is a 2-group of order $d$ embedded in $S_d$ via its left regular representation.

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We say two degree $d$ 2-group permutation triples $\sigma, \sigma'$ are **simultaneously conjugate** if there exists $\tau \in S_d$ such that

$$\sigma^\tau := (\tau^{-1}\sigma_0\tau, \tau^{-1}\sigma_1\tau, \tau^{-1}\sigma_\infty\tau, ) = \sigma'$$

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A **lift** of $\sigma$ is a 2-group permutation triple $\widetilde{\sigma} \in S_{2d}^3$ such that $\langle \widetilde{\sigma} \rangle$ is isomorphic to some extension $\widetilde{G}$ of $\mathbb{Z}/2\mathbb{Z}$ by $G$ as in the exact sequence below.

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} \widetilde{G} \xrightarrow{\pi} \langle \sigma \rangle \longrightarrow 1$$

## Lifting permutation triples

Let $\sigma$ be a 2-group permutation triple.

A **lift** of $\sigma$ is a 2-group permutation triple $\widetilde{\sigma} \in S_{2d}^3$ such that $\langle \widetilde{\sigma} \rangle$ is isomorphic to some extension $\widetilde{G}$ of $\mathbb{Z}/2\mathbb{Z}$ by $G$ as in the exact sequence below.

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} \widetilde{G} \xrightarrow{\pi} \langle \sigma \rangle \longrightarrow 1$$

For a 2-group permutation triple $\sigma$, we denote the set of lifts of $\sigma$ by $\mathrm{Lifts}(\sigma)$ and $\mathrm{Lifts}(\sigma)/\sim$ denotes the set of lifts up to simultaneous conjugation.

## **Algorithm to compute** $\mathrm{Lifts}(\sigma)/\!\sim$

**Input**: $\sigma$ a 2-group permutation triple of degree $d$

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# Example computing $\mathsf{Lifts}(\sigma)/\!\sim$ : setup

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$T_1 = \Big\{((1\,2)(3\,4), \mathrm{id}, (1\,2)(3\,4)), ((1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)),$
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Choosing

$$\alpha := (1\ d+1)(2\ d+2)\dots(d-1\ 2d-1)(d\ 2d)$$

allows us to label blocks by reducing modulo $d$.

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Lastly, we quotient by simultaneous conjugation to obtain
$\mathrm{Lifts}(\sigma)/\sim\; = \Big\{((1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)),$
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## Bipartite graphs of permutation triples

Now that we can lift permutation triples, we now describe some notation for the bipartite graphs that organize these triples.

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For $i \in \mathbb{Z}_{\geq 1}$ we define the bipartite graph denoted $\mathscr{G}_{2^i}$ with the following node sets.

- $\mathscr{G}_{2^i}^{\text{above}}$ : the set of isomorphism classes of 2-group Belyi maps of degree $2^i$ indexed by 2-group permutation triples $\widetilde{\sigma}$ up to simultaneous conjugation in $S_{2^i}$
- $\mathscr{G}_{2^i}^{\text{below}}$ : the set of isomorphism classes of 2-group Belyi maps of degree $2^{i-1}$ indexed by 2-group permutation triples $\sigma$ up to simultaneous conjugation in $S_{2^{i-1}}$

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For every pair of nodes $(\widetilde{\sigma}, \sigma) \in \mathscr{G}_{2^i}^{\text{above}} \times \mathscr{G}_{2^i}^{\text{below}}$ there is an edge between $\sigma$ and $\widetilde{\sigma}$ if and only if $\widetilde{\sigma}$ is simultaneously conjugate to a lift of $\sigma$.

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**Input**: The bipartite graph $\mathscr{G}_{2^{i-1}}$

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4. For every pair $(\widetilde{\sigma}, \sigma) \in \mathscr{G}_{2i}^{\text{above}} \times \mathscr{G}_{2i}^{\text{below}}$ place an edge between $\widetilde{\sigma}$ and $\sigma$ if and only if there is a triple in the equivalence class $[\widetilde{\sigma}] \in \mathrm{Lifts}(\mathscr{G}_{2i-1})/\sim$ that is a lift of $\sigma$

**Theorem (M.)**

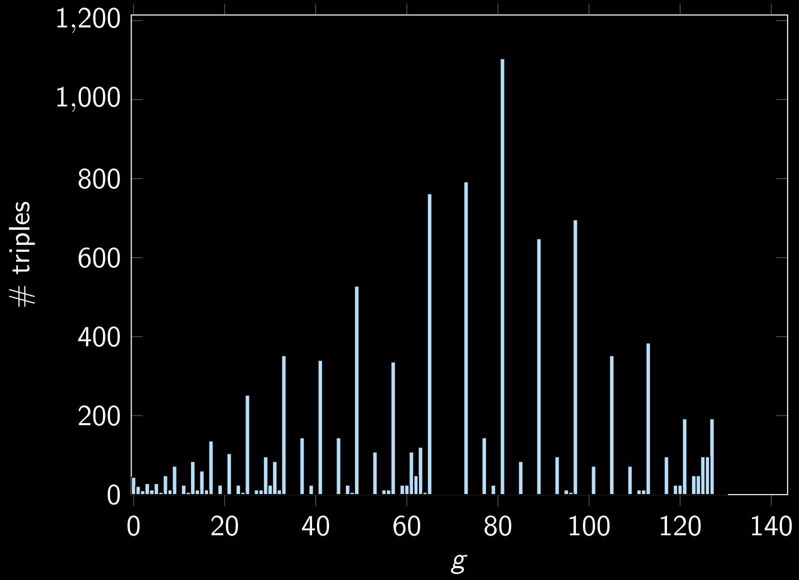
*The following tables list the number of isomorphism classes of 2-group Belyi maps, the number of passports, and number of lax passports respectively up to degree 256.*

| $d$ | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 |
|---|---|---|---|---|---|---|---|---|---|
| # *triples* | 1 | 3 | 7 | 19 | 55 | 151 | 503 | 1799 | 7175 |

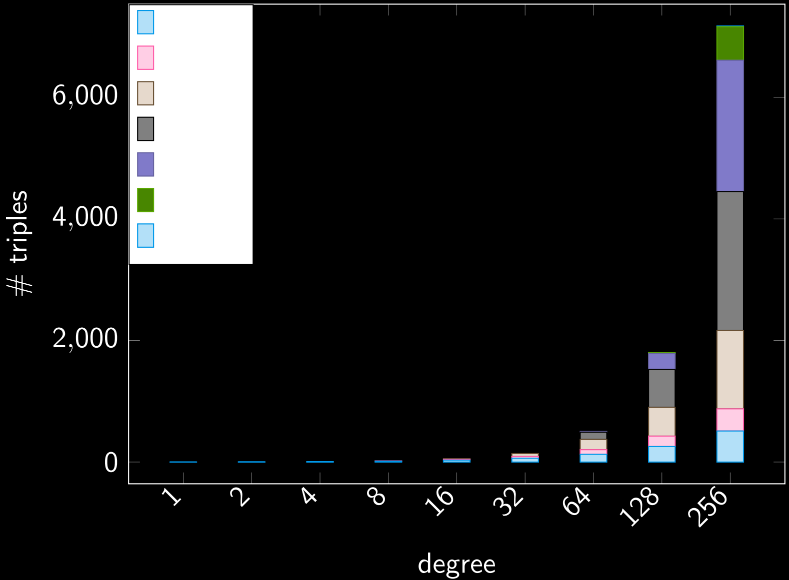
| $d$ | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 |
|---|---|---|---|---|---|---|---|---|---|
| # *passports* | 1 | 3 | 7 | 16 | 41 | 96 | 267 | 834 | 2893 |

| $d$ | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 |
|---|---|---|---|---|---|---|---|---|---|
| # *lax passports* | 1 | 1 | 3 | 6 | 14 | 31 | 85 | 257 | 882 |

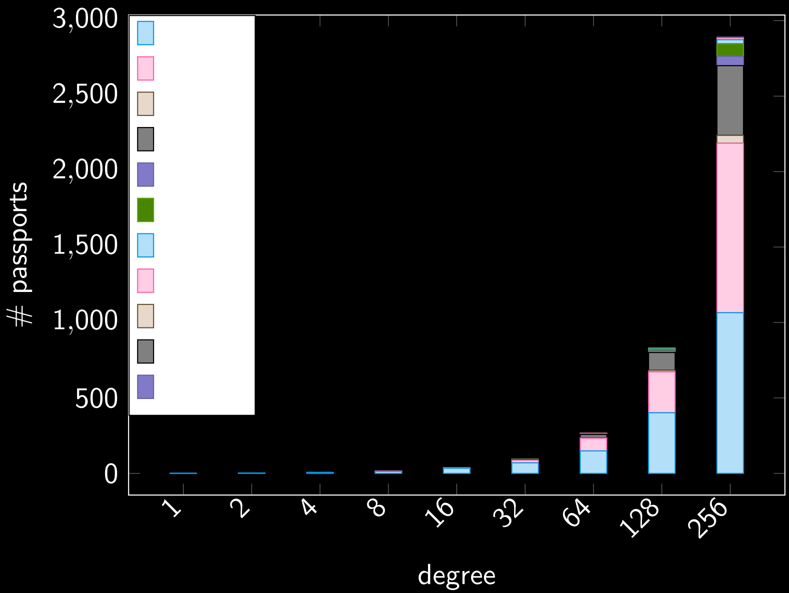
## Results : distribution of genera



## Results : groups by nilpotency class



# Results : passport sizes



# A refined conjecture

## Passports

Recall that a passport $\mathcal{P}$ consists of the data $(g, G, \lambda)$ where $g \in \mathbb{Z}_{\geq 0}$, $G$ is a transitive subgroup of $S_d$ and $\lambda = (\lambda_0, \lambda_1, \lambda_\infty)$ is a triple of partitions of $d$ corresponding to conjugacy classes $(C_0, C_1, C_\infty)$ of $S_d$.

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As a result of the action of $G_{\mathbb{Q}}$ on $\mathcal{P}$, the size of $\mathcal{P}$ bounds the degree of the field of moduli of any Belyi map with passport $\mathcal{P}$.

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To instead analyze $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}} \,|\, \mathbb{Q}^{\mathrm{ab}})$ we *refine* the notion of a passport.

# Refined passports

A **refined passport** $\mathscr{P}$ consists of the data $(g, G, c)$ where $g \in \mathbb{Z}_{\geq 0}$, $G$ is a transitive subgroup of $S_d$ and $c = (c_0, c_1, c_\infty)$ is a triple of conjugacy classes of $G$.

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where $(\sigma_0, \sigma_1, \sigma_\infty) \sim (\sigma_0', \sigma_1', \sigma_\infty')$ if there exists $\alpha \in \mathrm{Aut}(G)$ with $\alpha(\sigma_s) = \sigma_s'$ for every $s \in \{0, 1, \infty\}$.

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The size of $\mathscr{P}$ is the cardinality of the set $\Sigma_{\mathscr{P}}$ defined by

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where $(\sigma_0, \sigma_1, \sigma_\infty) \sim (\sigma_0', \sigma_1', \sigma_\infty')$ if there exists $\alpha \in \mathrm{Aut}(G)$ with $\alpha(\sigma_s) = \sigma_s'$ for every $s \in \{0, 1, \infty\}$.

As was the case with passport, every permutation triple $\sigma$ determines a refined passport $\mathscr{P}(\sigma)$.

# A refined conjecture

**Theorem (M.)**

*The size of* $\mathscr{P}(\sigma)$ *is equal to* $1$ *for every 2-group permutation triple* $\sigma$ *with degree* $\leq 256$.

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*ARC is true for* $2$*-group permutation triples* $\sigma$ *with* $\langle\sigma\rangle$ *dihedral.*

# Computing equations

## A motivating example : setup

Let $F$ be a number field with integers $\mathbb{Z}_F$. Let $\mathrm{Pl}(F)$ denote the places of $F$ and $S_\infty$ the archimedean places. For $v \in \mathrm{Pl}(F) \setminus S_\infty$ let $\mathfrak{p}_v$ denote the prime ideal of $\mathbb{Z}_F$ corresponding to $v$.

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If $\mathfrak{a}$ is not principal, then the question requires more care.

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To summarize, in the case where $\mathfrak{a}$ is not principal but there exists $\mathfrak{b}$ with $\mathfrak{a}\mathfrak{b}^2$ principal we have $[\mathfrak{a}] \in \mathrm{Cl}_F^2$ and $[\mathfrak{b}]$ is unique up to multiplication by $[\mathfrak{c}] \in \mathrm{Cl}_F[2]$.

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The algorithms in this section rely on transporting this technique to the function field setting.

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As an example, let $X$ be an irreducible affine plane curve (possibly singular) defined by the equation $f(x, y) = 0$ with $f \in K[x, y]$. Then the **function field of** $X$, denoted $K(X)$ is the field of fractions of the coordinate ring $K[x, y]/(f(x, y))$ of $X$.

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The set of places of $F$ is denoted $\mathrm{Pl}(F)$ and the **degree** of $P$ is the index $[\mathcal{O}_P/P : K]$ of the **residue class field**.

The **divisor class group** $\mathrm{Div}(F)$ of $F$ is the free abelian group generated by the places of $F$. A **divisor** $D \in \mathrm{Div}(F)$ is represented by a sum of places $\sum_P a_P P$ and the **degree** of $D$ is $\sum_P a_P \deg(P)$.

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# Algebraic function fields : Picard group

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The image of $\mathrm{div}\colon F^\times \to \mathrm{Div}(F)$ defined by

$$\mathrm{div}(f) = \sum_P \mathrm{ord}_P(f) P$$

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The **Picard group** of $F$ is $\mathrm{Pic}(F) := \mathrm{Div}(F)/\mathrm{Princ}(F)$.
The **Jacobian** of $F$ is $\mathrm{Pic}^0(F) := \mathrm{Div}^0(F)/\mathrm{Princ}(F)$.

# Algebraic function fields : quadratic extensions

## Algorithm in characteristic $p \geq 3$ : setup

# Algorithm in characteristic $p \geq 3$ : Galois test

# Algorithm in characteristic $p \geq 3$ : get candidates

# Algorithm in characteristic $p \geq 3$ : lift Belyi map

# Implementation in characteristic zero

## Results

`https://github.com/michaelmusty/2GroupDessins`

- *all* 2-group Belyi maps modulo 3 up to degree 32
- hundreds of 2-group Belyi maps up to degree 256

# Examples

# Notation

$$\texttt{DNG-a,b,c-gE-H}$$

$\texttt{D}$ : degree in $\{2, 4, 8, 16, 32, 64, 128, 256\}$

$\texttt{N}$ : either $\texttt{T}$ or $\texttt{S}$ identifying group database

$\texttt{G}$ : a positive integer identifying the group

$\texttt{a}$ : ramification index of 0 in $\{2, 4, 8, 16, 32, 64, 128, 256\}$

$\texttt{b}$ : ramification index of 1 in $\{2, 4, 8, 16, 32, 64, 128, 256\}$

$\texttt{c}$ : ramification index of $\infty$ in $\{2, 4, 8, 16, 32, 64, 128, 256\}$

$\texttt{g}$ : just the letter $\texttt{g}$

$\texttt{E}$ : the genus in $\mathbb{Z}_{\geq 0}$

$\texttt{H}$ : the hash of the 2-group permutation triple a positive integer

# An interesting example

## d3ssins

https://michaelmusty.github.io/d3ssins

## Future work

- higher degree over $\mathbb{F}_3$
- an algorithm in characteristic zero
- prove ARC for other families of 2-groups
- $p$-group Belyi maps for $p$ odd
- compute torsion fields

# Backup slides