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Modular Forms (mod p) and Galois Representations

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This is a continuation of the paper *Algebraic Modular Forms* ([G]); we will continue to use the same notation and conventions. Let G be a reductive algebraic group over $\mathbf{Q}$, with the property that every arithmetic subgroup Γ of $G(\mathbf{Q})$ is finite (equivalent conditions are given in [G, Prop. 1.4]). For simplicity, we will assume that G is an inner form of a split group over $\mathbf{Q}$, so that the L-group of G defined in [G, §13] is just $\hat{G}$, the split dual group over $\mathbf{Z}$. We will also assume that there is a cocharacter $\eta\colon\ \mathbf{G}_m \to \hat{T}$ satisfying the condition [G, (13.5)]:

$$\langle \eta, \alpha \rangle = 1$$

for all simple roots α of $\hat{T}$ with respect to $\hat{B}$. We fix η once and for all.

Our aim in this paper is to review the theory of modular forms (mod p) on G described in [G, §9]. We will then discuss the action of the Hecke algebra on the finite-dimensional space M over $\mathbf{Z}/p\mathbf{Z}$ of modular forms of a fixed weight and level. Unlike the case of characteristic zero (see [G, Prop. 6.11]), M need not be a semisimple Hecke module. However, to each simple submodule $N \subset M$, we hope to associate a Galois representation

$$\rho_N\colon\ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \hat{G}(k),$$

where k is an algebraic closure of $\mathbf{Z}/p\mathbf{Z}$. The representation ρ_N should satisfy certain local properties, which are specified in Chapter 1. In Chapter 2, we discuss a particular example, with $\hat{G} = G_2$ and $p = 5$, which was studied by D. Pollack and J. Lansky.

In Chapter 3, we establish some general results on local Galois representations

$$\rho\colon\ \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to \hat{G}(k).$$

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For example, we show that the inertia group always maps to a Borel subgroup, and that the niveau of the characters of the tame inertia group (cf. [S2, §1.7]) divide the order of an element in the Weyl group $W(\hat{T}, \hat{G})$ of $\hat{G}$. In Chapter 4, we apply these results to the Galois representations associated to ordinary modular forms (mod p).

1 Galois representations associated to modular forms (mod p)

Let G be a reductive group over $\mathbf{Q}$, with the hypotheses made in the introduction. Let p be a prime where G is split over $\mathbf{Q}_p$, and let K_p be a hyperspecial maximal compact subgroup of $G(\mathbf{Q}_p)$. Let

$$K_p \to G(p)$$

be the reduction homomorphism, and let W be an irreducible representation of the finite group $G(p)$ over $\mathbf{Z}/p\mathbf{Z}$. Let

$$K = \prod_{\ell \neq p} K_\ell \times K_p$$

be an open compact subgroup of $G(\hat{\mathbf{Q}})$.

The group K acts on the vector space W via the maps

$$K \to K_p \to G(p) \to GL(W).$$

We define the vector space $M = M(K, W)$ of modular forms of level K and weight W (mod p) by

$$M = \{f\colon\ G(\mathbf{Q})\backslash G(\hat{\mathbf{Q}}) \to W\colon\ f(gk) = k^{-1}f(g) \quad \text{for all } \ k \in K\}.$$

Some motivation for this definition is given in [G, §8–9], where $G(p)$ is denoted $\overline{G}(p)$ and W is denoted $\overline{W}$. Since the double coset space

$$G(\mathbf{Q})\backslash G(\hat{\mathbf{Q}})/K$$

is finite [G, Prop. 4.3], and the representation W is finite-dimensional, M is a finite-dimensional vector space over $\mathbf{Z}/p\mathbf{Z}$.

Let $\mathcal{H}$ denote the Hecke algebra of functions

$$f\colon \prod_{\ell \neq p} G(\mathbf{Q}_\ell) \to \mathbf{Z}/p\mathbf{Z}$$

which are

$$\text{bi} - \prod_{\ell \neq p} K_\ell\text{-invariant.}$$

This acts linearly on the space M, by the formula of [G, (6.6)]. Let $N \subset M$ denote a nonzero simple submodule, which exists whenever $M \neq 0$ (as M is finite-dimensional). Let

$$E = \mathrm{End}_{\mathcal{H}}(N),$$

which is a finite field of characteristic p. We fix an embedding of E into k, an algebraic closure of $\mathbf{Z}/p\mathbf{Z}$.

If ℓ is a prime, where K_ℓ is hyperspecial, the local Hecke algebra $\mathcal{H}_\ell$ of K_ℓ-bi-invariant functions on $G(\mathbf{Q}_\ell)$ is abelian, and is contained in the center of $\mathcal{H}$. Hence the simple submodule N gives rise to a homomorphism of $\mathbf{Z}/p\mathbf{Z}$-algebras:

$$\varphi_\ell\colon\ \mathcal{H}_\ell \to E \to k.$$

Since $p \neq \ell$ and we have fixed the cocharacter η of $\hat{T}$, the homomorphism φ_ℓ has Satake parameter (see [G, §16])

$$s_\ell(N) \quad \text{in} \quad C\ell(\hat{G})(k).$$

Here $C\ell(\hat{G})$ is the affine scheme (over $\mathbf{Z}$) of semisimple conjugacy classes in $\hat{G}$, which is isomorphic to the quotient of $\hat{T}$ by the finite Weyl group $W(\hat{T}, \hat{G})$.

We say, after Serre, that a representation

$$\rho\colon\ \Gamma \to \hat{G}(k)$$

is completely reducible provided the following condition holds: Whenever the image $\rho(\Gamma)$ is contained in a parabolic subgroup $\hat{P}$ of $\hat{G}$, it is contained in a Levi factor $\hat{L}$ of $\hat{P}$. When $\hat{G} = GL_n$, completely reducible representations correspond to semisimple Γ-modules of rank n over k.

Conjecture 1.1. Associated to a simple submodule $N \subset M$, there is a continuous, completely reducible Galois representation

$$\rho_N\colon\ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \hat{G}(k),$$

which satisfies the following properties.

(a) For every prime $p \neq \ell$, where K_ℓ is hyperspecial, the representation ρ is unramified at ℓ, and any Frobenius Fr_ℓ maps to an element whose semisimple part $\rho_N(Fr_\ell)_{ss}$ has class $s_\ell(N)$ in $C\ell(\hat{G})(k)$.

(b) The semisimple part $\rho_N(Fr_\infty)_{ss}$ of the image of any complex conjugation maps to the class of the involution $\eta(-1)$ in $Cl(\hat{G})(k)$.

Condition (b) asserts, when $p \neq 2$ and the group G is semisimple, that complex conjugation maps to the most negative involution in $\hat{G}$. By our hypothesis that G is an inner form of a split group, the opposition involution of the Dynkin diagram of G is trivial. This implies that $\eta(-1)$ has Brauer trace equal to $-\mathrm{rank}(\hat{G})$ on the adjoint representation $\mathrm{Lie}(\hat{G})$ of $\hat{G}$.

For inner forms G of $GL(2)$, Conjecture 1.1 can be proved using methods of Serre [S]. In this case, the representation ρ is uniquely determined. In general, we can ask: How unique is the representation ρ, conjectured to exist in 1.1?

Proposition 1.2. The simple submodule N determines the kernel of the representation ρ_N, and so determines the fixed field $L \subset \overline{\mathbf{Q}}$ of the kernel. The field L is a finite, normal extension of $\mathbf{Q}$ with Galois group a subgroup of $\hat{G}(k)$. □

Proof (following a letter of Serre). Let ρ_1 and ρ_2 be two representations associated to N, with fixed fields L_1 and L_2. To show that $L_1 = L_2$, we consider the composite extension $L = L_1 L_2$, which is fixed by the kernel of

$$\rho_1 \times \rho_2\colon \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \hat{G}(k) \times \hat{G}(k).$$

Consider the representation

$$\rho_1\colon \mathrm{Gal}(L/L_2) \to \hat{G}(k).$$

The Frobenius elements at unramified primes in $\mathrm{Gal}(L/L_2)$ map to unipotent classes in $\hat{G}(k)$. Indeed, they have the same semisimple parts as for ρ_2, and ρ_2 is the trivial representation of $\mathrm{Gal}(L/L_2)$. Hence the image of ρ_1 is a finite, unipotent subgroup $\Gamma \subset \hat{G}(k)$.

We now invoke a result of Borel and Tits [BT], which associates to Γ a canonical parabolic subgroup $\hat{P} \subset \hat{G}$ with the properties

$$N_{\hat{G}(k)}(\Gamma) \subset \hat{P}(k),$$
$$\Gamma \subset R_u(\hat{P})(k).$$

Since $\mathrm{Gal}(L/\mathbf{Q})$ normalizes $\mathrm{Gal}(L/L_2)$, the image of ρ_1 is contained in $\hat{P}(k)$. By complete reducibility, the image of ρ_1 is contained in a Levi factor $\hat{L}(k)$ of $\hat{P}(k)$.

But $\hat{L} \cap R_u(\hat{P}) = 1$, so $\Gamma = 1$, and ρ_1 is trivial on $\mathrm{Gal}(L/L_2)$. This shows that $L_2 \supset L_1$. Reversing the roles and considering the representation

$$\rho_2\colon \mathrm{Gal}(L/L_1) \to \hat{G}(k),$$

shows that $L_1 \supset L_2$. Hence $L_1 = L_2$, as claimed.

We note that there may be more than one $\hat{G}(k)$-conjugacy class of homomorphisms $\rho\colon \mathrm{Gal}(L/\mathbf{Q}) \hookrightarrow \hat{G}(k)$, which have the same local data in $C\ell(\hat{G})(k)$. When there is a single class, we predict that the image is contained in a conjugate of the finite group $\hat{G}(E)$.

2 An example

In this section, we assume that G is the anisotropic form of the exceptional group of type G_2 over $\mathbf{Q}$. Then G may be given explicitly as the automorphism group of the $\mathbf{Q}$-algebra of Cayley's octonions.

The dual group $\hat{G}$ is the split group G_2 over $\mathbf{Z}$, and the cocharacter η is uniquely determined by its inner product with the simple roots for $\hat{T}$ with respect to $\hat{B}$.

The group G is split over $\mathbf{Q}_p$ for all primes p, and we fix a hyperspecial subgroup K_p so that $K = \prod K_p = \underline{G}(\hat{\mathbf{Z}}) \subset G(\hat{\mathbf{Q}})$, where $\underline{G}$ is the model over $\mathbf{Z}$ discussed in [G2, §4]. In the example, we take $p = 5$ and let W be the irreducible representation of $G_2(5)$ of dimension $5^6 = 15,625$, which is the reduction of the Steinberg representation (mod 5), and has highest weight $\mu = 4\omega_1 + 4\omega_2$.

By computations of Lansky and Pollack, the space $M(K, W)$ defined in §1 has dimension 1 over $\mathbf{Z}/5\mathbf{Z}$. Hence $M = N$ is a simple module for the unramified Hecke algebra

$$\widehat{\bigotimes_{\ell \neq 5}} \mathcal{H}_\ell = \mathcal{H}$$

with endomorphism ring $E = \mathbf{Z}/5\mathbf{Z}$. Lansky and Pollack computed the action of $\mathcal{H}_2$ and $\mathcal{H}_3$ on N, and found the characteristic polynomials of $s_2(N)$ and $s_3(N)$ in $C\ell(G_2)(5)$ acting on the 7-dimensional representation of $G_2(5)$ over $\mathbf{Z}/5\mathbf{Z}$:

$$x^7 - x^6 - x^5 + 2x^4 - 2x^3 + x^2 + x - 1 = 0 \quad \text{for } s_2,$$

$$x^7 - 2x^6 - x^4 + x^3 + 2x - 1 = 0 \quad \text{for } s_3.$$

Conjecture 1.1 predicts the existence of an extension L of $\mathbf{Q}$, which is Galois and totally complex, with

$$\rho_N\colon \operatorname{Gal}(L/\mathbf{Q}) \hookrightarrow G_2(5).$$

The extension L should be unramified outside of 5, and the primes 2 and 3 should have behavior predicted by the polynomials for s_2 and s_3. Presumably, the representation ρ_N is surjective; we will return to this question in §5.

3 Local Galois representations

The material in this section is independent of the theory of modular forms (mod p), and we will change the notation slightly.

Let p be a prime, and let k be an algebraically closed field of characteristic p. Let $\underline{\hat{G}}$ be a connected, reductive group over k; since the field k will be fixed throughout the section, we will write $\hat{G}$ for the discrete group $\underline{\hat{G}}(k)$. Our aim is to study continuous, local Galois representations

$$\rho\colon \operatorname{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to \hat{G}. \tag{3.1}$$

We first recall some of the structure of the local Galois group $D = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Let I be the inertia subgroup of D, and let I_w be the wild inertia subgroup of I. Both are normal subgroups of D, and we have isomorphism (see [S3, Ch. IV])

$$D/I \simeq \hat{\mathbf{Z}}, \quad \text{generated by } \mathrm{Fr}_p,$$

$$I_t = I/I_w \simeq \prod_{\ell \neq p} \mathbf{Z}_\ell(1),$$

where I_t is the tame quotient, and Fr_p acts on I_t by multiplication by p.

Proposition 3.2. The image $\rho(I)$ of the inertia subgroup is contained in a Borel subgroup $\hat{B}$ of $\hat{G}$. □

Proof. Since I_w is a pro-p-group, $\rho(I_w)$ is a finite p-group. Hence $\rho(I_w) = \Gamma$ is a unipotent subgroup of $\hat{G}$.

Let $\hat{P}$ be the canonical parabolic subgroup which Borel and Tits associate to Γ [BT]. Since I_w is normal in D, we have

$$\begin{array}{cccc} \rho: & D & \to & \hat{P} \\ & \cup & & \cup \\ & I_w & \to & R_u(\hat{P}). \end{array} \tag{3.3}$$

Let $\hat{L}$ be a Levi factor of $\hat{P}$. Taking the quotient of (3.3) by $R_u(\hat{P})$, we obtain a partial semisimplification of ρ:

$$\rho_s\colon\ D/I_w \to \hat{L}. \tag{3.4}$$

The image $\rho_s(I_t)$ is cyclic, of order prime to p. Hence it is generated by a semisimple element a in $\hat{L}$, and is contained in a maximal torus $\hat{T}$ of $\hat{L}$. Let $\hat{B}_L$ be a Borel subgroup of $\hat{L}$ containing $\hat{T}$, and let $\hat{B}$ be the inverse image of $\hat{B}_L$ under the map

$$\hat{P} \to \hat{P}/R_u(\hat{P}) = \hat{L}.$$

Then $\hat{B}$ is a Borel subgroup of $\hat{G}$ containing the image $\rho(I)$.

We now consider the partial semisimplification ρ_s of ρ constructed in (3.4):

$$\begin{array}{cccc} \rho_s: & D/I_w & \to & \hat{L} \\ & \cup & & \cup \\ & I_t & \to & \hat{T}. \end{array}$$

Let a be a generator of the cyclic group $\rho_s(I_t)$ in $\hat{T}$.

Proposition 3.5. There is an element n in the normalizer of $\hat{T}$ in $\hat{L}$, which satisfies

$$nan^{-1} = a^p. \qquad \square$$

Proof. The cyclic group $\langle a \rangle$ in $\hat{T}$ is normalized by $g = \rho_s(\mathrm{Fr}_p)$, and $gag^{-1} = a^p$. Hence the tori $\hat{T}$ and $g^{-1}\hat{T}g$ both centralize a, and are maximal in the connected reductive group $Z_{\hat{L}}(a)^0 \subset \hat{L}$. They are therefore conjugate by an element $z \in Z_{\hat{L}}(a)^0$: $z\hat{T}z^{-1} = g^{-1}\hat{T}g$. The element $n = gz$ in $\hat{L}$ lies in the normalizer of $\hat{T}$, and satisfies $nan^{-1} = a^p$.

We say the local representation ρ is *regular* if a is a regular, semisimple element in $\hat{L}$: $Z_{\hat{L}}(a)^0 = \hat{T}$. In this case, the element n constructed in the proof of Proposition 3.5 is uniquely determined in the quotient group $N_{\hat{L}}(\hat{T})/\hat{T} = W(\hat{T}, \hat{L})$, and ρ determines a unique element (up to conjugacy) in the Weyl group of $\hat{L}$. Note that this Weyl group is a subgroup of the Weyl group of $\hat{G}$.

Corollary 3.6. If the element n of Proposition 3.5 has order f in the quotient group $W(\hat{L}, \hat{T}) = N_{\hat{L}}(\hat{T})/\hat{T}$, then

$$a^{p^f-1} = a \quad \text{in} \quad \hat{T}. \qquad \square$$

We now study, in more detail, the case when $f = 1$, so n can be chosen to lie in $\hat{T}$.

Proposition 3.7. The following conditions are all equivalent.

(a) The element n in Proposition 3.5 can be chosen to lie in $\hat{T}$.

(b) $a^{p-1} = 1$ in $\hat{T}$.

(c) The homomorphism ρ_s: $I_t \to \hat{T}$ factors as $\rho_s = \overline{\lambda} \circ \omega$, where ω: $I_t \to (\mathbf{Z}/p\mathbf{Z})^*$ is the fundamental character giving the Galois action on the p-th roots of unity in $\overline{\mathbf{Q}}_p$, and the map $(\mathbf{Z}/p\mathbf{Z})^* \to \hat{T}$ is given by a class $\overline{\lambda}$ in $X_\bullet(\hat{T})/(p-1)X_\bullet(\hat{T})$. $\square$

Proof. The first two conditions are clearly equivalent, and show $\rho_s(I_t)$ is cyclic of order dividing $p - 1$. Hence ρ_s, restricted to I_t, factors through the quotient $I_t \underset{\omega}{\to} (\mathbf{Z}/p\mathbf{Z})^*$, and a homomorphism $\overline{\lambda}$: $(\mathbf{Z}/p\mathbf{Z})^* \to \hat{T}$ is given by a cocharacter (mod $(p-1)$).

When the equivalent conditions of Proposition 3.7 hold, we say ρ has niveau 1. One simple situation in which ρ is regular, of niveau 1, is the following proposition.

Proposition 3.8. Assume that the finite unipotent group $\Gamma = \rho(I_w)$ contains a regular unipotent element of $\hat{G}$. Then ρ is regular, of niveau 1. $\square$

Proof. When Γ contains a regular element, the canonical parabolic subgroup which Borel and Tits associate to Γ is a Borel subgroup $\hat{B}$ of $\hat{G}$. Hence ρ: $D \to \hat{B}$ and ρ_s: $D/I_w \to \hat{T}$. In this case, a is clearly regular, and n lies in $\hat{T}$.

4 Ordinary modular forms (mod p)

In this section, we use the general results on local Galois representations to predict the restriction of the representation ρ_N to a decomposition group at p, when G is simply connected and the simple module $N \subset M$ is ordinary at p.

The prediction involves the irreducible representation W of $G(p)$, used to define the space $M = M(K, W)$. We first recall the parameters of irreducible representations of $G(p)$ when G is simply connected, which are due to Chevalley [C] and Steinberg [St]. A good exposition of this theory can be found in [B].

Proposition 4.1. Assume that G is simply connected and split over $\mathbf{Z}/p\mathbf{Z}$, of rank r. Then there are p^r distinct, irreducible representations W of $G(p)$ over $\mathbf{Z}/p\mathbf{Z}$, parametrized by the cocharacters λ in $X_\bullet(\hat{T})$ which satisfy $1 \leq \langle\lambda, \alpha\rangle \leq p$ for all simple roots α of $\hat{T}$ with respect to $\hat{B}$. □

Proof. We fix $T \subset B \subset G$ over $\mathbf{Z}/p\mathbf{Z}$. Then Chevalley proved that for each dominant weight μ of T, there is an irreducible algebraic representation W_μ of G with highest weight μ for B. Steinberg showed that the restriction of W_μ to $G(p)$ is irreducible if and only if $\langle\mu, \alpha\rangle \leq p - 1$ for all simple coroots α of T. Our parameter λ is simply $\mu + \eta$, where η is half the sum of the positive roots for T with respect to B. (Note that η is equal to the cocharacter of $\hat{T}$ introduced earlier.)

Although it is not needed in what follows, we make a few remarks about the irreducible representations W_λ of $G(p)$. The trivial representation has parameter $\lambda = \eta$, and the Steinberg representation has parameter $\lambda = p\eta$. Very little is known about the character of W_λ, or even the dimension of W_λ, except in the case when λ lies in the small alcove:

$$\langle\lambda, \beta\rangle \leq p, \tag{4.2}$$

where β is the highest root of $\hat{T}$ with respect to $\hat{B}$. In this case, W_λ is the reduction (mod p) of the corresponding irreducible representation V_λ for G in characteristic zero, and the character of W_λ on T is given by Weyl's character formula:

$$\mathrm{char}(t \mid W_\lambda) = \sum \mathrm{sign}(\sigma) t^{\sigma\lambda} / \sum \mathrm{sign}(\sigma) t^{\sigma\eta}.$$

In general, if $\lambda = \mu + \eta$ is a weight of T, which satisfies $1 \leq \langle\lambda, \alpha\rangle \leq p$ for all simple roots α of $\hat{T}$ with respect to $\hat{B}$, let V_λ be the irreducible representation of G over $\mathbf{Q}_p$ with highest weight μ. Since K_p is compact, there are $\mathbf{Z}_p$-lattices in V_λ which are stable under K_p. One can choose a stable lattice V_λ^0 so that the $G(p)$-module V_λ^0/pV_λ^0 has a unique simple quotient, isomorphic to W_λ (see [B]). When λ is in the small alcove, or λ is the parameter of the trivial or Steinberg representation, V_λ^0/pV_λ^0 is irreducible and isomorphic to W_λ.

The choice of V_λ^0 gives a lattice inside the $\mathbf{Q}_p$-vector space $M(K, V_\lambda)$ of modular forms of level K and weight V_λ for G. This space has two equivalent definitions (see [G, §8–9]):

$$M(K, V_\lambda) = \left\{F\colon\ G(\hat{\mathbf{Q}})/K \to V_\lambda\colon\ F(\gamma g) = \gamma F(g) \text{ for } \gamma \text{ in } G(\mathbf{Q})\right\}$$
$$= \left\{ f : G(\hat{\mathbf{Q}})\backslash G(\hat{\mathbf{Q}})/\prod_{\ell \neq p} K_\ell \to V_\lambda : f(gk_p) = k_p^{-1} f(g) \text{ for } k_p \text{ in } K_p \right\}. \tag{4.3}$$

The first admits an action of the $\mathbf{Q}_p$-Hecke algebra of K in $G(\hat{\mathbf{Q}})$, and the second allows us to define the lattice $M(K, V_\lambda^0)$ consisting of those f with values in V_λ^0. The $\mathbf{Z}_p$-Hecke algebra $\mathcal{H}(\mathbf{Z}_p)$ of

$$\prod_{\ell \neq p} K_\ell \quad \text{in} \quad \prod_{\ell \neq p} G(\mathbf{Q}_\ell)$$

acts on $M(K, V_\lambda^0)$, but the Hecke algebra of K_p in $G(\mathbf{Q}_p)$ does not act on this lattice. The homomorphism of $G(p)$-modules

$$V_\lambda^0/pV_\lambda^0 \to W_\lambda$$

gives a linear map of $\mathcal{H}$-modules (see [G, §9]):

$$M(K, V_\lambda^0)/pM(K, V_\lambda^0) \to M(K, W_\lambda). \tag{4.4}$$

Now let $\tilde{N} \subset M(K, V_\lambda)$ be a simple Hecke submodule, and let $\tilde{E}$ be the center of the endomorphism ring of $\tilde{N}$. Then $\tilde{E}$ is a finite extension field of $\mathbf{Q}_p$, and the local Satake parameter $s_p(\tilde{N})$ lies in $\hat{T}/W(\tilde{E})$. We say $\tilde{N}$ is *ordinary* if there is a representative $\tilde{s}_p$ for $s_p(\tilde{N})$ in $\hat{T}(\tilde{E})$ such that the quotient

$$\tilde{s}_p/\lambda(p)$$

lies in the maximal compact subgroup of $\hat{T}(\tilde{E})$ (see [G, §19]). Let $\tilde{N}_0 = \tilde{N} \cap M(K, V_\lambda^0)$, and let N be the image of $\tilde{N}_0$ in $M(K, W_\lambda)$ under (4.4). We say N is ordinary at p if it is nonzero, and arises from the reduction of an ordinary module $\tilde{N}$ as above.

Conjecture 4.5. Assume that the simple $\mathcal{H}$-submodule $N \subset M(K, W_\lambda)$ is ordinary at p. Then the restriction of the global Galois representation ρ_N to a decomposition group at p has niveau 1, and the restriction of ρ_s to I_t factors as $\overline{\lambda} \circ \omega$, where $\overline{\lambda}$ is the image of λ in $X_\bullet(\hat{T})/(p-1)X_\bullet(\hat{T})$. □

We note the image of λ in $X_\bullet(\hat{T})/(p-1)X_\bullet(\hat{T})$ is almost enough to determine the restricted dominant weight λ, which satisfies $1 \leq \langle\lambda, \alpha\rangle \leq p$ for all simple α. Only the cases where $\langle\lambda, \alpha\rangle = 1$ and $\langle\lambda, \alpha\rangle = p$ need to be distinguished. Perhaps in the former case, the action of wild inertia on the simple root space U_α in U^{ab} is peu ramifiée, as a Kummer extension of $\mathbf{Q}_p(\mu_p)$, in the sense of [S4, p. 186].

5 An example, revisited

In the example of §2, $W = W_\lambda$ with $\lambda = 5\cdot\eta$ the parameter of the Steinberg representation. In this case, the map

$$M(K, V_\lambda^0)/5M(K, V_\lambda^0) \to M(K, W_\lambda)$$

is an isomorphism. Both spaces have dimension 1 over $\mathbf{Z}/5\mathbf{Z}$, as 5 is not a torsion prime for G_2, and

$$M(K, V_\lambda) \simeq V_\lambda^{G(\mathbf{Z})}$$

has dimension 1 over $\mathbf{Q}_5$. The simple module $N = M(K, W_\lambda)$ is ordinary at 5, so we predict that the restriction of

$$\rho_N : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to G_2(5)$$

to a decomposition group at 5, takes values in a Borel subgroup $\hat{B}$. Here we even predict that

$$\rho_s = \overline{\lambda} \circ \omega = \overline{\eta} \circ \omega$$

on all of D/I_w, so that the fixed field of the kernel of ρ_s is $\mathbf{Q}_5(\mu_5)$, and the characters of D/I_w on the representation

$$\rho_s : D/I_w \to G_2(5) \to GL_7(5)$$

are $\{\omega^3, \omega^2, \omega, 1, \omega^{-1}, \omega^{-2}, \omega^{-3}\}$.

Finally, since $\langle\lambda, \alpha\rangle = 5$ for both simple roots α, we predict that the finite group $\rho(I_w) \subset \hat{U}$ contains regular unipotent elements, and so has exponent 25. If this is the case, the global representation is surjective.

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