SOLVABLE NOTES

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Contents

I. Belyĭ Maps

TODO: [transitive permutation triples] TODO: [function fields] TODO: [the solvable story and the analogy with number fields] TODO: [iterated quadratic extensions of function fields] TODO: [naming convention so we can refer to specific examples]

II. Galois Groups and Coverings

TODO: [describe how the iterative procedure leads us to a group theory question] TODO: [covering spaces via Rotman]

III. An Algorithm to Produce Permutation Triples

III.1. **Intro.** Let σ be a transitive permutation triple TODO: [link to previous section where this is defined]. Recall that this means

$$\sigma = (\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3, \quad \sigma_\infty \sigma_1 \sigma_0 = 1,$$

and σ generates a transitive subgroup of S_d. As is described in TODO: [link], σ corresponds to a Belyĭ map $\phi : X \to \mathbb{P}^1$ of degree d. Given σ, we would like to describe all (Galois) Belyĭ maps $(\widetilde{X}, \widetilde{\phi})$ that are degree m covers of the given Belyĭ map (X, ϕ). We will only be concerned with the particular case $m = 2$, but for now we remain in the more general setting. Put another way, $(\widetilde{X}, \widetilde{\phi})$ makes the diagrams

$$\begin{tikzcd} \widetilde{X} \arrow[dd, "\widetilde{\phi}"'] \arrow[dr, "\psi"] & \\ & X \arrow[dl, "\phi"] \\ \mathbb{P}^1 & \end{tikzcd} \qquad \begin{tikzcd} K(\widetilde{X}) \arrow[dd, dash] \arrow[dr, dash, "m"] & \\ & K(X) \arrow[dl, dash, "d"] \\ K(\mathbb{P}^1) & \end{tikzcd}$$

commute. According to Galois theory we have the following (upside down) diagram.

$$\begin{tikzcd} \mathrm{Gal}(K(\widetilde{X})/K(\widetilde{X})) \arrow[dd, dash] \arrow[dr, dash, "m"] & \\ & \mathrm{Gal}(K(\widetilde{X})/K(X)) \arrow[dl, dash, "d"] \\ \mathrm{Gal}(K(\widetilde{X})/K(\mathbb{P}^1)) & \end{tikzcd}$$

Let $G, \widetilde{G}, H$ denote the Galois groups

$$G = \mathrm{Gal}(K(X)/K(\mathbb{P}^1))$$
$$\widetilde{G} = \mathrm{Gal}(K(\widetilde{X})/K(\mathbb{P}^1))$$
$$H = \mathrm{Gal}(K(\widetilde{X})/K(X)).$$

The above situation can be organized into an exact sequence of groups

$$1 \longrightarrow H \xrightarrow{\iota} \widetilde{G} \xrightarrow{f} G \longrightarrow 1.$$

Now recall from section TODO: [link] that the groups $\widetilde{G}$ and G are the monodromy groups for $\widetilde{X}$ and X respectively. Thus, there is an action of $\widetilde{G}$ on the md sheets of the covering $\widetilde{X} \to \mathbb{P}^1$ and there is an action of G on the d sheets of the covering $X \to \mathbb{P}^1$. Identifying H with its image $\iota(H)$ in $\widetilde{G}$, we also have an action of H on the md sheets of the covering $\widetilde{X} \to \mathbb{P}^1$. To summarize, we can view $H \trianglelefteq \widetilde{G} \leq S_{md}$ and $G \leq S_d$. The groups $\widetilde{G}$ and G are transitive since we only consider connected covers.

Moreover, the action of $\widetilde{G}$ on the md sheets is required to factor through the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[dd, "\widetilde{\phi}"'] \arrow[dr, "\psi"] & \\ & X \arrow[dl, "\phi"] \\ \mathbb{P}^1 & \end{tikzcd} .$$

To explain, the map $\psi : \widetilde{X} \to X$ is a m-sheeted cover and identifies the md sheets of $\widetilde{X}$ into d blocks. From the exact sequence above, we have the diagram

$$\begin{tikzcd} \widetilde{G} \arrow[rr, "f"] \arrow[dr, "\pi"'] & & G \\ & G/H \arrow[ur, "\hat{f}"'] & \end{tikzcd}$$

with $\hat{f}$ an isomorphism. Let $\widetilde{g} \in f^{-1}(g)$ for some $g \in G$. According to the above diagrams, $\widetilde{G}$ must have a well-defined action on the d blocks realizing the element $\widetilde{g}$ as a permutation in S_d. MM: [more explanation?] Moreover, the action of $\widetilde{g}$ on blocks must agree with the action of g on $\{1, 2, \dots, d\}$. That is, we must have

$$\hat{f}(\pi(\widetilde{g})) = f(\widetilde{g}) = g.$$

MM: [do we need to prove anything here? G-equivariant bijection?]

III.2. **Degree** 4.

Example III.2.1. TODO: [$m = 2$, $d = 4$, the Galois story] Up to isomorphism, there is only one degree 2 Belyĭ map $\phi : X \to \mathbb{P}^1$ corresponding to the permutation triple

$$\sigma = ((1\,2), (1\,2), (1)(2)) .$$

From σ (and the above discussion) we would like to find all $\widetilde{\sigma}$ corresponding to degree $m = 2$ covers of X that are Galois Belyĭ maps.

Notes

that is, $\widetilde{X} \to \mathbb{P}^1$ is degree 4 with monodromy $\widetilde{G}$ having order 4.

That is, $\widetilde{\sigma}$ correspond to Belyĭ maps $(\widetilde{X}, \widetilde{\phi})$ and monodromy groups $\widetilde{G}$ making the diagrams

$$\begin{tikzcd} \widetilde{X} \arrow[dd, "\widetilde{\phi}"'] \arrow[dr, "\psi"] & \\ & X \arrow[dl, "\phi"] \\ \mathbb{P}^1 & \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[dd, dash, "4"'] \arrow[dr, dash, "2"] & \\ & H \arrow[dl, dash, "2"] \\ \widetilde{G} & \end{tikzcd}$$

commute. Since $m = 2$, H is an order 2 subgroup of S_4 that identifies 4 sheets to 2 sheets. Thus $H = \langle\tau\rangle$ with $\tau = (1\,2)(3\,4)$, $\tau = (1\,3)(2\,4)$, or $\tau = (1\,4)(2\,3)$. We choose $\tau = (1\,3)(2\,4)$ so that the identification of sheets corresponds to reducing the sheets modulo 2 (labeling them $1, 2, 3, 4$). This identifies the 4 sheets into 2 blocks $\left\{\boxed{1\,3}, \boxed{2\,4}\right\}$.

Moreover, by assuming $\widetilde{X}$ Galois, we get $H \trianglelefteq \widetilde{G}$. Also, $G = \langle\sigma\rangle = S_2 \cong \mathbb{Z}/2\mathbb{Z}$. Thus, we are looking for all permutation triples $\widetilde{\sigma} \in S_4^3$ that generate a (transitive) order 4 monodromy group $\widetilde{G} \leq S_4$ sitting in the exact sequence

$$\begin{tikzcd} 1 \arrow[r] & \langle\tau\rangle \arrow[r, "\iota"] & \widetilde{G} \arrow[r, "f"] \arrow[d, "\pi"'] & \langle\sigma\rangle \arrow[r] & 1 \\ & & \widetilde{G}/\langle\tau\rangle \arrow[ur, "\hat{f}"'] & & \end{tikzcd}.$$

Now we have the combinatorial task of pulling back σ under the map f.

Notes

One thing that makes me nervous is that we do not know the group $\widetilde{G}$ a priori. If we have $\widetilde{G}$ in hand (from a transitive group database or by some other means) with $\langle\tau\rangle \trianglelefteq \widetilde{G}$, then pulling back σ under f is nothing more than finding all $\widetilde{\sigma}$ that map to σ in the quotient (once this quotient is identified with $\langle\sigma\rangle$). MM: [When we don't have $\widetilde{G}$, how do we prove that identifying sheets via τ yields all possible $\widetilde{\sigma}$?]

The first requirement for any element in a preimage of f is that it has a well-defined action on the set of blocks $\left\{\boxed{1\,3}, \boxed{2\,4}\right\}$ obtained from τ. For instance, $(1\,2\,3\,4)$ does not have a well-defined action on these blocks. The only other requirement is that the action on blocks agrees with the permutation we are pulling back. MM: [more explanation?]

In our example, $\sigma_\infty = (1)(2)$. Then $f^{-1}(\sigma_\infty)$ is precisely the $\widetilde{\sigma_\infty} \in S_4$ with a well-defined action on $B = \left\{\boxed{1\,3}, \boxed{2\,4}\right\}$ such that the action of $\widetilde{\sigma_\infty}$ on B agrees with the action of σ_∞ on $\{1, 2\}$. Thus

$$f^{-1}(\sigma_\infty) = \{(1)(3)(2)(4), (1\,3)(2\,4)\} = \{(1)(2)(3)(4), (1\,3)(2\,4)\}.$$

For each $\widetilde{\sigma_\infty} \in f^{-1}(\sigma_\infty)$ we can indeed check that the action of $\widetilde{\sigma_\infty}$ on blocks is given by $\left(\boxed{1\,3}\right)\left(\boxed{2\,4}\right)$ (i.e. $\widetilde{\sigma_\infty}$ fixes each block). Similarly, for $\sigma_0 = \sigma_1 = (1\,2)$ we have that

$$f^{-1}(\sigma_0) = f^{-1}(\sigma_1) = \{(1\,2\,3\,4), (1\,2)(3\,4), (1\,4\,2\,3), (1\,4)(2\,3)\}.$$

That is, each $\widetilde{\sigma_i} \in f^{-1}(\sigma_i)$ for $i \in \{0, 1\}$ acts on blocks by $\left(\boxed{1\,3}\,\boxed{2\,4}\right)$.

Thus we have the following possibilities for the preimages of $\sigma_0 = \sigma_1 = (1\,2)$ and $\sigma_\infty = (1)(2)$ under f:

$$\begin{array}{c} (1\,2\,3\,4), (1\,2)(3\,4), \\ (1\,4\,2\,3), (1\,4)(2\,3) \end{array} \mapsto (1\,2)$$

$$(1\,3)(2\,4), (1)(2)(3)(4) \mapsto (1)(2)$$

and we can sort through these possibilities to see which ones correspond to degree 4 (Galois) Belyĭ maps according to TODO: [link]. If $\widetilde{\sigma_\infty} = (1)(2)(3)(4)$, then the condition that

$$\widetilde{\sigma_\infty}\widetilde{\sigma_1}\widetilde{\sigma_0} = (1)(2)(3)(4)$$

allows for only the single possibility $\widetilde{\sigma_0} = (1\,2\,3\,4)$ and $\widetilde{\sigma_1} = (1\,4\,2\,3)$ (up to simultaneous conjugation). MM: [should we prove that simultaneous conjugation does not mess up our convention of identifying sheets mod 2?] In this case $\widetilde{G} \cong \mathbb{Z}/4\mathbb{Z}$. If $\widetilde{\sigma_\infty} = \tau$, then there are two possibilities up to simultaneous conjugation. One possibility is

$$\widetilde{\sigma} = ((1\,2\,3\,4), (1\,2\,3\,4), (1\,3)(2\,4))$$

with $\widetilde{G} \cong \mathbb{Z}/4\mathbb{Z}$. The other possibility is

$$\widetilde{\sigma} = ((1\,2)(3\,4), (1\,4)(3\,2), (1\,3)(2\,4))$$

with $\widetilde{G} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

In summary, there are 3 permutation triples $\widetilde{\sigma}$ (up to isomorphism) corresponding to degree 4 Galois Belyĭ maps that factor through the degree 2 Belyĭ map corresponding to σ. They are as follows: TODO: [make a table?]

$$\Big((1\,2\,3\,4),(1\,4\,2\,3),(1)(2)(3)(4)\Big),\Big((1\,2\,3\,4),(1\,2\,3\,4),(1\,3)(2\,4)\Big),\Big((1\,2)(3\,4),(1\,4)(3\,2),(1\,3)(2\,4)\Big).$$

III.3. **Degree 8 Galois.**

Example III.3.1. TODO: [Galois degree 2 covers of 4T1-441] Let

$$\sigma=(\sigma_0,\sigma_1,\sigma_\infty)=\Big((1\,2\,3\,4),(1\,4\,2\,3),(1)(2)(3)(4)\Big).$$

We organize $f^{-1}(\sigma)$ in the following table.

$f^{-1}\Big((1\,2\,3\,4)\Big)$	$f^{-1}\Big((1\,4\,2\,3)\Big)$	$f^{-1}\Big((1)(2)(3)(4)\Big)$
$(1\,2\,3\,4)(5\,6\,7\,8)$	$(1\,8\,3\,2)(4\,7\,6\,5)$	$(1)(2)(3)(4)(5)(6)(7)(8)$
$(1\,6\,7\,4)(2\,3\,8\,5)$	$(1\,4\,3\,6)(2\,5\,8\,7)$	$(1\,5)(2\,6)(3\,7)(4\,8)$
$(1\,6\,3\,4)(2\,7\,8\,5)$	$(1\,4\,7\,2)(3\,6\,5\,8)$	
$(1\,6\,7\,8)(2\,3\,4\,5)$	$(1\,4\,7\,6)(2\,5\,8\,3)$	
$(1\,2\,7\,4)(3\,8\,5\,6)$	$(1\,8\,3\,6)(2\,5\,4\,7)$	
$(1\,2\,3\,8)(4\,5\,6\,7)$	$(1\,4\,3\,2)(5\,8\,7\,6)$	
$(1\,2\,7\,8)(3\,4\,5\,6)$	$(1\,8\,7\,2)(3\,6\,5\,4)$	
$(1\,6\,3\,8)(2\,7\,4\,5)$	$(1\,8\,7\,6)(2\,5\,4\,3)$	
$(1\,6\,7\,8\,5\,2\,3\,4)$	$(1\,8\,7\,2\,5\,4\,3\,6)$	
$(1\,2\,3\,8\,5\,6\,7\,4)$	$(1\,4\,7\,6\,5\,8\,3\,2)$	
$(1\,2\,7\,8\,5\,6\,3\,4)$	$(1\,8\,3\,6\,5\,4\,7\,2)$	
$(1\,6\,3\,8\,5\,2\,7\,4)$	$(1\,8\,3\,2\,5\,4\,7\,6)$	
$(1\,2\,3\,4\,5\,6\,7\,8)$	$(1\,4\,7\,2\,5\,8\,3\,6)$	
$(1\,6\,7\,4\,5\,2\,3\,8)$	$(1\,8\,7\,6\,5\,4\,3\,2)$	
$(1\,6\,3\,4\,5\,2\,7\,8)$	$(1\,4\,3\,6\,5\,8\,7\,2)$	
$(1\,2\,7\,4\,5\,6\,3\,8)$	$(1\,4\,3\,2\,5\,8\,7\,6)$	

These possibilities combine to yield $512=16^2\cdot 2$ possible $\widetilde{\sigma}$. Of these 512 possibilities, only 32 have $\widetilde{\sigma_\infty}\widetilde{\sigma_1}\widetilde{\sigma_0}=\mathrm{id}_{S_8}$. Of these 32, only 24 generate a transitive order 8 subgroup of S_8. MM: [include some of these lists?] Finally, if we identify triples that are simultaneously conjugate or in the same orbit under the S_3 action on triples, we get the following list of 3 possibilities for $\widetilde{\sigma}$ organized by $\widetilde{G}$:

$\widetilde{G} \cong \mathbb{Z}/8\mathbb{Z}$
$\Big((1\,6\,7\,8\,5\,2\,3\,4), (1\,4\,3\,2\,5\,8\,7\,6), \mathrm{id}_{S_8}\Big)$
$\Big((1\,6\,7\,8\,5\,2\,3\,4), (1\,8\,3\,6\,5\,4\,7\,2), (1\,5)(2\,6)(3\,7)(4\,8)\Big)$.
$\widetilde{G} \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$\Big((1\,2\,3\,4)(5\,6\,7\,8), (1\,8\,3\,6)(2\,5\,4\,7), (1\,5)(2\,6)(3\,7)(4\,8)\Big)$

Example III.3.2. TODO: [Galois degree 2 covers of 4T1-442] Let

$$\sigma = (\sigma_0, \sigma_1, \sigma_\infty) = \Big((1\,2\,3\,4), (1\,2\,3\,4), (1\,3)(2\,4)\Big).$$

We organize $f^{-1}(\sigma)$ in the following table.

$f^{-1}(\sigma_0)$	$f^{-1}(\sigma_1)$	$f^{-1}(\sigma_\infty)$
$(1,2,3,4)(5,6,7,8)$	$(1,2,3,4)(5,6,7,8)$	$(1,3)(2,4)(5,7)(6,8)$
$(1,6,7,4)(2,3,8,5)$	$(1,6,7,4)(2,3,8,5)$	$(1,3)(2,8)(4,6)(5,7)$
$(1,6,3,4)(2,7,8,5)$	$(1,6,3,4)(2,7,8,5)$	$(1,7)(2,4)(3,5)(6,8)$
$(1,6,7,8)(2,3,4,5)$	$(1,6,7,8)(2,3,4,5)$	$(1,7)(2,8)(3,5)(4,6)$
$(1,2,7,4)(3,8,5,6)$	$(1,2,7,4)(3,8,5,6)$	$(1,3,5,7)(2,4,6,8)$
$(1,2,3,8)(4,5,6,7)$	$(1,2,3,8)(4,5,6,7)$	$(1,3,5,7)(2,8,6,4)$
$(1,2,7,8)(3,4,5,6)$	$(1,2,7,8)(3,4,5,6)$	$(1,7,5,3)(2,4,6,8)$
$(1,6,3,8)(2,7,4,5)$	$(1,6,3,8)(2,7,4,5)$	$(1,7,5,3)(2,8,6,4)$
$(1,6,7,8,5,2,3,4)$	$(1,6,7,8,5,2,3,4)$	
$(1,2,3,8,5,6,7,4)$	$(1,2,3,8,5,6,7,4)$	
$(1,2,7,8,5,6,3,4)$	$(1,2,7,8,5,6,3,4)$	
$(1,6,3,8,5,2,7,4)$	$(1,6,3,8,5,2,7,4)$	
$(1,2,3,4,5,6,7,8)$	$(1,2,3,4,5,6,7,8)$	
$(1,6,7,4,5,2,3,8)$	$(1,6,7,4,5,2,3,8)$	
$(1,6,3,4,5,2,7,8)$	$(1,6,3,4,5,2,7,8)$	
$(1,2,7,4,5,6,3,8)$	$(1,2,7,4,5,6,3,8)$	

These possibilities combine to yield 2048 possible $\widetilde{\sigma}$. Of these, 128 have $\widetilde{\sigma_\infty}\widetilde{\sigma}_1\widetilde{\sigma}_0 = \mathrm{id}_{S_8}$. Of these, 120 generate a transitive subgroup of S_8. Of these 120 transitive groups, 24 have order equal to 8. Finally, after identifying triples up to simultaneous conjugation in S_8 and the S_3 action on triples, we get the following list of 3 possibilities for $\widetilde{\sigma}$ organized by $\widetilde{G}$:

$\widetilde{G} \cong \mathbb{Z}/8\mathbb{Z}$
$\Big((1\,6\,7\,8\,5\,2\,3\,4), (1\,6\,7\,8\,5\,2\,3\,4), (1\,3\,5\,7)(2\,8\,6\,4)\Big)$ $\Big((1\,6\,7\,8\,5\,2\,3\,4), (1\,2\,7\,4\,5\,6\,3\,8), (1\,7\,5\,3)(2\,4\,6\,8)\Big)$.
$\widetilde{G} \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$\Big((1\,2\,3\,4)(5\,6\,7\,8), (1\,6\,3\,8)(2\,7\,4\,5), (1\,7)(2\,8)(3\,5)(4\,6)\Big)$

MM: [3 things to mention here...

- 4T1-442 to 8T2-442 unramified.
- Notice that 8T1-882 does not factor through 4T1-442 even though it seems like it should be possible at first.
- We have identified a size 2 passport... should probably prove something about these always showing up and make sure attempts to optimize algorithm don't eliminate these...

]

Example III.3.3. TODO: [Galois degree 2 covers of 4T2-222] Let

$$\sigma = (\sigma_0, \sigma_1, \sigma_\infty) = \Big((1\,2\,3\,4), (1\,2\,3\,4), (1\,3)(2\,4)\Big).$$

We organize $f^{-1}(\sigma)$ in the following table.

$f^{-1}(\sigma_0)$	$f^{-1}(\sigma_1)$	$f^{-1}(\sigma_\infty)$
$(1,2)(3,4)(5,6)(7,8)$	$(1,4)(2,3)(5,8)(6,7)$	$(1,3)(2,4)(5,7)(6,8)$
$(1,2)(3,8)(4,7)(5,6)$	$(1,4)(2,7)(3,6)(5,8)$	$(1,3)(2,8)(4,6)(5,7)$
$(1,6)(2,5)(3,4)(7,8)$	$(1,8)(2,3)(4,5)(6,7)$	$(1,7)(2,4)(3,5)(6,8)$
$(1,6)(2,5)(3,8)(4,7)$	$(1,8)(2,7)(3,6)(4,5)$	$(1,7)(2,8)(3,5)(4,6)$
$(1,2,5,6)(3,4,7,8)$	$(1,4,5,8)(2,3,6,7)$	$(1,3,5,7)(2,4,6,8)$
$(1,2,5,6)(3,8,7,4)$	$(1,4,5,8)(2,7,6,3)$	$(1,3,5,7)(2,8,6,4)$
$(1,6,5,2)(3,4,7,8)$	$(1,8,5,4)(2,3,6,7)$	$(1,7,5,3)(2,4,6,8)$
$(1,6,5,2)(3,8,7,4)$	$(1,8,5,4)(2,7,6,3)$	$(1,7,5,3)(2,8,6,4)$

These possibilities combine to yield 512 possible $\widetilde{\sigma}$. Of these, 64 have $\widetilde{\sigma_\infty}\widetilde{\sigma_1}\widetilde{\sigma_0} = \mathrm{id}_{S_8}$. Of these, 56 generate a transitive subgroup of S_8 and all 56 have order 8. Finally, after identifying triples up to simultaneous conjugation in S_8 and the S_3 action on triples, we get the following list of 3 possibilities for $\widetilde{\sigma}$ organized by $\widetilde{G}$:

$\widetilde{G} \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$\big((1\,2)(3\,4)(5\,6)(7\,8), (1\,4\,5\,8)(2\,3\,6\,7), (1\,7\,5\,3)(2\,8\,6\,4)\big)$
$\widetilde{G} \cong D_8$
$\big((1\,2)(3\,4)(5\,6)(7\,8), (1\,4)(2\,7)(3\,6)(5\,8), (1\,7\,5\,3)(2\,4\,6\,8)\big)$
$\widetilde{G} \cong Q_8$
$\big((1\,2\,5\,6)(3\,4\,7\,8), (1\,4\,5\,8)(2\,3\,6\,7), (1\,3\,5\,7)(2\,8\,6\,4)\big)$

III.4. **The Non-Galois Story.**

Example III.4.1. TODO: [$m = 2$, $d = 4$, the non-Galois story] MM: [maybe do non-Galois in a different section. . .]

Example III.4.2. TODO: [$m = 3\ldots$]

III.5. **The Algorithm.** We now formally describe an algorithm to compute all Galois degree 2 covers of a given Belyĭ map.

Algorithm III.5.1 (Brute Force)**.**

Proof. □

III.6. **Graph of Examples.**

Example III.6.1. TODO: [graph of examples]

IV. Orders in Function Fields

IV.1. **Intro.** Following [5], we will start with the general setting with data (A, K, L) where A is a commutative domain, $K = \mathrm{Frac}(A)$, and L a finite extension of K. Given such a triple (A, K, L), we associate a subring $B \subset L$ such that $L = \mathrm{Frac}(B)$. TODO: [define order] TODO: [define maximal order] TODO: [define ring of integers]

V. Computing Belyĭ Maps

V.1. **Intro.** We begin by explaining a bit about the task of computing Belyĭ maps in general. TODO: [reference the various methods for computing maps in general and in specific cases too....] For a (much) more detailed account see [7].

TODO: [Explain the special setting we are in]

V.2. **Explicit Example.** We now explain an explicit example. Let $K = \overline{\mathbb{Q}}$. Let $X_0 = \mathbb{P}^1$ over K. Up to isomorphism, there is a unique degree 2 Belyĭ map $\varphi : X_1 \to X_0$. Let $K(X_0)$ be the function field of X_0 which is $K(x_0)$ (the rational function field). To write down an equation for X_1, we must specify the ramification values (branch points) on X_0. We can do this by recording the ramification values in a divisor $D^0 \in \mathrm{Div}(X_0)$. For the degree 2 Belyĭ map, we get to choose 2 branch points on X_0, say 0 and ∞ so that $D^0 = 0 - \infty$.

MM: [mention how this comes to us from previous section]

Next, we find $f_0 \in K(X_0)$ such that the quadratic function field extension

$$K(X_1) := \frac{K(X_0)[x_1]}{(x_1^2 - f_0)}$$

corresponds to φ.

MM: [write down explicitly the correspondence?]

MM: [explain choices involved in choosing f... i.e. up to square and such that $\mathrm{div}(f)$ has property...]

We get that $f_0 = x_0$ and X_1 is the projective closure of the affine curve defined by the equation $x_1^2 - x_0$. On the function field side, we have the following diagram.

$$\begin{tikzcd}
 & K(X_1) = \dfrac{K(x_0)[x_1]}{(x_1^2 - x_0)} \arrow[dl, dash] \arrow[dd, dash] \\
R_1 = \dfrac{K[x_0, x_1]}{(x_1^2 - x_0)} \arrow[dd, dash] & \\
 & K(X_0) = K(x_0) \arrow[dl, dash] \\
R_0 = K[x_0] &
\end{tikzcd}$$

Let $\mathfrak{p}_0 = (x_0, x_1)$. We can see that we have the right ramification since

$$(x_0) = \mathfrak{p}_0^2$$
$$(1/x_0) = \mathfrak{p}_0^{-2}$$

and the discriminant of R_1 is $4x_0$. What are the primes above (x_0-1)? Well, $(x_0-1) = \mathfrak{p}_1\mathfrak{p}_2$ where

$$\begin{aligned}\mathfrak{p}_1 &= (x_0-1, x_1-1)\\ \mathfrak{p}_2 &= (x_0-1, x_1+1).\end{aligned}$$

Now...

MM: [In summary,]

$$\begin{tikzcd}[column sep=small]
 & K(X_2) = \dfrac{K(X_1)[x_2]}{(x_2^2 - f)} \arrow[dl, dash] \arrow[dd, dash] \\
R_2 \arrow[dd, dash] & \\
 & K(X_1) = \dfrac{K(x_0)[x_1]}{(x_1^2 - x_0)} \arrow[dl, dash] \arrow[dd, dash] \\
R_1 = \dfrac{K[x_0, x_1]}{(x_1^2 - x_0)} \arrow[dd, dash] & \\
 & K(X_0) = K(x_0) \arrow[dl, dash] \\
R_0 = K[x_0] &
\end{tikzcd}$$

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