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The full automorphism groups of hyperelliptic Riemann surfaces

by

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For every integer $g \geq 2$ we obtain the complete list of groups acting as the full automorphisms groups on hyperelliptic Riemann surfaces of genus g.

Introduction. Hurwitz [12] and Wiman [20] initiated the study of automorphism groups of hyperelliptic Riemann surfaces of low genus at the end of the last century, and much later Brandt and Stichtenoth [4] determined all groups occurring as subgroups of the full automorphism group of hyperelliptic Riemann surfaces of fixed genus $g \geq 2$, containing the hyperellipticity automorphism. This is the starting point of our work, in which we obtain the complete list of groups acting as the full automorphism group on such surfaces and determine the genera of such surfaces as well (Theorem 3.1). The case $g = 2$ was settled in 1888 by Bolza [3] (see also Geyer [9]). The cases

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g = 3 and 4 are handled, and not only for hyperelliptic surfaces, by Henn [11], Kuribayashi and Komiya [14], Duma-Radtke [7] and Kato [13]. Finally it is worth to mention that Broughton [5] has recently classified all actions on Riemann surfaces of genera 2 and 3. The proofs involve Fuchsian groups and the theory of Teichmüller spaces. The same problem, in the category of hyperelliptic Klein surfaces, was solved by Etayo and the authors [6]. Along the paper, all Riemann surfaces are assumed to be compact.

1. Preliminaries. The approach mentioned in the introduction involve the theory of Fuchsian groups. By a *Fuchsian group* we mean a discrete subgroup Γ of the group Ω of Möbius transformations, which is known to act as the group of isometries of the hyperbolic plane H, with compact quotient space H/Γ. Such a group Γ has a presentation of the form

$$<x_1,\ldots,x_r,a_1,b_1,\ldots,a_g,b_g \mid x_i^{m_i},\ \prod_{i=1}^{r} x_i \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}> \tag{1.1}$$

which can be concentrated in the so called *signature of* Γ given by

$$\sigma(\Gamma)=(g;m_1,\ldots,m_r) \tag{1.2}$$

The nonnegative integer g is *the orbit genus and* $m_1,\ldots,m_r$ are *the periods* of Γ. We omit in the signature those periods which are equal to 1 and if there are no periods in σ then we write $\sigma(\Gamma)=(g;-)$ and we say that Γ is a *surface group*. Finally, if the orbit genus of Γ is zero, we shall write the signature shortly as $(m_1,\ldots,m_r)$. The hyperbolic area of a fundamental region for any Fuchsian group Γ with signature (1.2) is given by the positive rational

$$\mu(\Gamma) = 2\pi\ [2g-2+\sum_{i=1}^{r}(1-1/m_i)] \tag{1.3}$$

and by [21], a symbol $\sigma=(g;m_1,\ldots,m_r)$ is a *Fuchsian signature, i.e.* there exists a Fuchsian group Γ with $\sigma(\Gamma)=\sigma$, if and only if the right hand side of (1.3) is greater than zero. Moreover if Γ_1 is a subgroup of a Fuchsian group Γ

of finite index then it is a Fuchsian group itself and the classical Hurwitz Riemann formula can be read as

$$\mu(\Gamma_1)/\mu(\Gamma)=[\Gamma:\Gamma_1]. \tag{1.4}$$

Every Fuchsian group Γ acts properly discontinuously on H and so the quotient H/Γ admits a natural structure of a Riemann surface. Conversely, from the Riemann uniformization theorem (see *e.g.* [1]) every compact Riemann surface of genus $g\geq 2$ is isomorphic to H/Γ, for some surface group Γ of orbit genus g. The relevance of Fuchsian groups in the study of automorphism groups of Riemann surfaces of genus $g\geq 2$ comes from Macbeath's theorem [15] which says that having a surface X so represented a finite group G is (isomorphic to) a group of its automorphisms if and only if ($G\cong\Lambda/\Gamma$) $G=\Lambda/\Gamma$ for some Fuchsian group Λ containing Γ as a normal subgroup. In the proof of Theorem 3.1 we shall use the following result due to Maclachlan [16]:

Proposition 1.1. *Let* Γ' *be a Fuchsian group with signature* $(g';m_1,\ldots,m_r)$ *and canonical generators* $a_1,b_1,\ldots,a_{g'},b_{g'},x_1,\ldots,x_r$. *Let* Λ *be a normal subgroup of* Γ' *of finite index* M, *and denote by* p_i *the order of the image of* x_i *in the quotient group* Γ'/Λ. *Let finally* $I = \{1 \leq i \leq r : m_i \neq p_i\}$ *and* $\alpha_i = M/p_i$, $n_i = m_i/p_i$ *for every* $i\in I$. *Then* Λ *has the signature*

$$\sigma(\Lambda) = (g;\ n_i,\overset{\alpha_i}{\ldots},n_i\ :\ i\in I)$$

for some nonnegative integer g.

Now G=Aut(X) if and only if $\Lambda=N_\Omega(\Gamma)$, the normalizer of Γ in Ω. Although the last equality is rather very difficult to decide, it is always the case i Λ is a *maximal Fuchsian* group *i.e.* it is not properly contained in any other Fuchsian group. Following Greenberg [10] (see also [6]) a Fuchsian signature σ is said to *be maximal* if for every Fuchsian group Λ with signature σ and for every Fuchsian group Λ' containing Λ as a proper subgroup, we have $d(\Lambda)\neq d(\Lambda')$, where $d(\Lambda)$ is the dimension of the Teichmüller space of Λ, whose value for a group with signature (1.2) is $6(g-1)+2r$, see **[2]**, **[8]**. Almost all Fuchsian signatures turns out to be maximal and a complete list of those which fail to be was found by Singerman in **[19]** and is given in Table 1 below.

The signatures σ_1 in the first column are the nonmaximal ones and for each one of them, the corresponding signature σ_2 is the signature of a Fuchsian group Λ' properly containing a group Λ with signature σ_1 and $d(\Lambda')=d(\Lambda)$. We say that σ_2 *extends* σ_1. The group Λ' can be chosen containing Λ as a normal subgroup just in the first eight cases and then we say that the pair (σ_1,σ_2) is *normal.*

Table 1

σ_1	σ_2	$[\sigma_2 : \sigma_1]$
(2;-)	(0;2,2,2,2,2,2)	2
(1;t,t)	(0;2,2,2,2,t)	2
(1;t)	(0;2,2,2,2t)	2
(0;t,t,t,t) ; $t\geq3$	(0;2,2,2,t)	4
$(0;t_1,t_1,t_2,t_2)$; $t_1+t_2\geq5$	$(0;2,2,t_1,t_2)$	2
(0;t,t,t); $t\geq4$	(0;3,3,t)	3
(0;t,t,t); $t\geq4$	(0;2,3,2t)	6
$(0;t_1,t_1,t_2)$; $t_1\geq3$, $t_1+t_2\geq7$	$(0;2,t_1,2t_2)$	2
(7,7,7)	(2,3,7)	24
(2,7,7)	(2,3,7)	9
(3,3,7)	(2,3,7)	8
(4,8,8)	(2,3,8)	12
(3,8,8)	(2,3,8)	10
(9,9,9)	(2,3,9)	12
(4,4,5)	(2,4,5)	6
(n,4n,4n) ; $n\geq2$	(2,3,4n)	6
(n,2n,2n) ; $n\geq3$	(2,4,2n)	4
(3,n,3n) ; $n\geq3$	(2,3,3n)	4
(2,n,2n) ; $n\geq4$	(2,3,2n)	3

Not all Fuchsian groups with maximal signature are maximal. However this is the case for triangle groups Λ as $d(\Lambda)=0$. Moreover from **[10]** (see also Theorem 5.1.2 in **[6]** for an explicit statement and proof in a more general setting) we have

Proposition 1.2. *Given a maximal Fuchsian signature σ there exists a maximal Fuchsian group Λ with $\sigma(\Lambda)=\sigma$.*

A Riemann surface X is said to be *hyperelliptic* if it has a central involution ϕ_h for which X/ϕ_h is a ramified sphere *i.e.* has genus 0. Such involution is said to be a *hyperellipticity automorphism* and it turns out to be unique. In terms of Fuchsian groups, Maclachlan's theorem [17] states that a Riemann surface $X=H/\Gamma$ of genus $g\geq 2$ is hyperelliptic if and only if there exists a unique Fuchsian group Γ_h, with signature $\sigma_h=(2,\overset{2g+2}{\ldots},2)$ called the *group of hyperellipticity*, containing Γ as a subgroup of index 2. The notations ϕ_h, Γ_h and σ_h are fixed throughout all the paper. Finally for a given integer $g\geq 2$, a group G is said to be a *g-hyperelliptic subgroup* (resp. a *g-hyperelliptic group*) if there exists a hyperelliptic surface X of genus g such that $\phi_h\in G\subseteq \mathrm{Aut}(X)$ (resp. $G=\mathrm{Aut}(X)$).

2. On g-hyperelliptic subgroups and ramification indices. The result of Brandt and Stichtenoth mentioned in the introduction can be understood as the determination of all finite groups G that can stand as g-hyperelliptic subgroups for some integer $g\geq 2$.

Let X be a hyperelliptic Riemann surface of genus $g\geq 2$ and let G be a subgroup of Aut(X) containing ϕ_h. Then $X=H/\Gamma$, where $\sigma(\Gamma)=(g;-)$ and $G=\Lambda/\Gamma$ for some Fuchsian group Λ with signature $\sigma(\Lambda)$. Then Proposition 2.1 and Hauptsatz (§5) in the paper [4] of Brandt-Stichtenoth can be stated in the following way:

Theorem 2.1. *Let* $g\geq 2$ *be an integer. For every* $N\geq 1$, *a group* G *of order* 2N *is a* g-*hyperelliptic subgroup if and only if it appears in the Table* 2 *given below, the corresponding value* $t=t(g,N)$ *is a non-negative integer and the signature* $\sigma(\Lambda)$ *is a Fuchsian signature. -i.e. it has positive area-. Moreover, in cases 3.a - 3.f, N must be even*

Table 2

Case	$\sigma=\sigma(\Lambda)$	$t = t(g,N)$	$G=\Lambda/\Gamma$
1	$(2,\overset{t}{\ldots},2)$	$2g + 2$	$\mathbb{Z}_2$
2.a	$(2,\overset{t}{\ldots},2,N,N)$	$(2g+2)/N$	$\mathbb{Z}_2\oplus\mathbb{Z}_N = \langle z\rangle\oplus\langle x\rangle$
2.b	$(2,\overset{t}{\ldots},2,N,2N)$	$(2g+1)/N$	$\mathbb{Z}_{2N} = \langle x\rangle$
2.c	$(2,\overset{t}{\ldots},2,2N,2N)$	$2g/N$	$\mathbb{Z}_{2N}$
3.a	$(2,\overset{t}{\ldots},2,2,2,N/2)$	$(2g+2)/N$	$\mathbb{Z}_2\oplus D_{N/2}=\langle z\rangle\oplus\langle x,y \mid x^2, y^2,(xy)^{N/2}\rangle$
3.b	$(2,\overset{t}{\ldots},2,2,4,N/2)$	$(2g+2)/N-1/2$	$V_{N/2}=\langle x,y \mid x^4,y^{N/2},(xy)^2,(x^{-1}y)^2\rangle$
3.c	$(2,\overset{t}{\ldots},2,4,4,N/2)$	$(2g+2)/N-1$	$H_{N/2} = \langle x,y \mid x^4,y^2x^2,(xy)^{N/2}\rangle$
3.d	$(2,\overset{t}{\ldots},2,2,2,N)$	$2g/N$	$D_N = \langle x,y \mid x^2,y^2,(xy)^N \rangle$
3.e	$(2,\overset{t}{\ldots},2,2,4,N)$	$2g/N-1/2$	$U_{N/2} = \langle x,y \mid x^2,y^N,xyxy^{N/2+1}\rangle$
3.f	$(2,\overset{t}{\ldots},2,4,4,N)$	$2g/N-1$	$G_{N/2} = \langle x,y \mid x^2y^{N/2},y^N,x^{-1}yxy\rangle$
4.a	$(2,\overset{t}{\ldots},2,2,3,3)$	$(g+1)/6$	$\mathbb{Z}_2\oplus A_4= \langle z\rangle\oplus\langle x,y \mid x^2,y^3,(xy)^3\rangle$
4.b	$(2,\overset{t}{\ldots},2,2,3,6)$	$(g-1)/6$	$\mathbb{Z}_2\oplus A_4$
4.c	$(2,\overset{t}{\ldots},2,2,6,6)$	$(g-3)/6$	$\mathbb{Z}_2\oplus A_4$
4.d	$(2,\overset{t}{\ldots},2,4,3,3)$	$(g-2)/6$	$SL(2,3)=\langle x,y \mid x^4,y^3,(xy)^3,yx^2y^{-1}x^2\rangle$
4.e	$(2,\overset{t}{\ldots},2,4,3,6)$	$(g-4)/6$	$SL(2,3)$
4.f	$(2,\overset{t}{\ldots},2,4,6,6)$	$(g-6)/6$	$SL(2,3)$
5.a	$(2,\overset{t}{\ldots},2,2,3,4)$	$(g+1)/12$	$\mathbb{Z}_2\oplus S_4=\langle z\rangle\oplus\langle x,y \mid x^2,y^3,(xy)^4\rangle$
5.b	$(2,\overset{t}{\ldots},2,2,6,4)$	$(g-3)/12$	$\mathbb{Z}_2\oplus S_4$
5.c	$(2,\overset{t}{\ldots},2,2,3,8)$	$(g-2)/12$	$W_1=\langle x,y \mid x^2,y^3,(xy)^4(yx)^4,(xy)^8\rangle$
5.d	$(2,\overset{t}{\ldots},2,2,6,8)$	$(g-6)/12$	W_1
5.e	$(2,\overset{t}{\ldots},2,4,3,4)$	$(g-5)/12$	$W_2=\langle x,y \mid x^4,y^3, yx^2y^{-1}x^2,(xy)^4\rangle$
5.f	$(2,\overset{t}{\ldots},2,4,6,4)$	$(g-9)/12$	W_2
5.g	$(2,\overset{t}{\ldots},2,4,3,8)$	$(g-8)/12$	$W_3=\langle x,y \mid x^4,y^3,(xy)^8,x^2(xy)^4\rangle$
5.h	$(2,\overset{t}{\ldots},2,4,6,8)$	$(g-12)/12$	W_3
6.a	$(2,\overset{t}{\ldots},2,2,3,5)$	$(g+1)/30$	$\mathbb{Z}_2\oplus A_5=\langle z\rangle\oplus\langle x,y \mid x^2,y^3,(xy)^5\rangle$
6.b	$(2,\overset{t}{\ldots},2,2,3,10)$	$(g-5)/30$	$\mathbb{Z}_2\oplus A_5$
6.c	$(2,\overset{t}{\ldots},2,2,6,10)$	$(g-15)/30$	$\mathbb{Z}_2\oplus A_5$
6.d	$(2,\overset{t}{\ldots},2,2,6,5)$	$(g-9)/30$	$\mathbb{Z}_2\oplus A_5$
6.e	$(2,\overset{t}{\ldots},2,4,3,5)$	$(g-14)/30$	$SL(2,5)=\langle x,y \mid x^4,y^3,(xy)^5,yx^2y^{-1}x^2\rangle$
6.f	$(2,\overset{t}{\ldots},2,4,3,10)$	$(g-20)/30$	$SL(2,5)$
6.g	$(2,\overset{t}{\ldots},2,4,6,5)$	$(g-24)/30$	$SL(2,5)$
6.h	$(2,\overset{t}{\ldots},2,4,6,10)$	$(g-30)/30$	$SL(2,5)$

3. The family of g-hyperelliptic groups. Here we prove the result of the paper.

Theorem 3.1. *Let* $N\geq 1$, $g\geq 2$ *be integers. A group* G *of order* 2N *is a g-hyperelliptic group if and only if it is a g-hyperelliptic subgroup and the corresponding triple* (G,g,N) *from Table 2 is not in the following list:*

Table 3

Case	G	Relation between g and N	Table 2
3.3.1	$\mathbb{Z}_2\oplus \mathbb{Z}_N$	g=N/2-1	2a, t=1
3.3.2	$\mathbb{Z}_2\oplus \mathbb{Z}_N$	g=N-1	2a, t=2
3.3.3	$\mathbb{Z}_{2N}$	g=N/2	2c, t=1
3.3.4	$\mathbb{Z}_{2N}$	g=N	2c, t=2
3.3.5	$H_{N/2}$	g=N/2-1	3c, t=0
3.3.6	$G_{N/2}$	g=N/2	3f, t=0
3.3.7	$\mathbb{Z}_2\oplus A_4$	g=3	4c, t=0
3.3.8	SL(2,3)	g=2	4d, t=0
3.3.9	SL(2,3)	g=6	4f, t=0
3.3.10	U_4	g=2	3e, t=0, N=8

Proof. Signature σ in Table 2 does not appear, in most cases, in the fir column of Table 1, and so it is a maximal Fuchsian signature. Hence, the group in Theorem 2.1 can be chosen to be a maximal Fuchsian group, by virtue Proposition 1.2. Consequently, whenever σ does not appear in the first column Table 1, the group G occurring in the corresponding row of Table 2 is g-hyperelliptic group. Thus, we are just concerned with cases in which σ is n a maximal Fuchsian signature. By inspection of the first columns of Tables 1 a 2, these signatures, with their correspondences with Tables 2 and 3, are t following:

Table 4

Case	Table 2	σ	Data	G	Table 3
1	2a	(2,N,N)	t=1,g=N/2-1	$\mathbb{Z}_2 \oplus \mathbb{Z}_N$	3.3.1
2	2a	(2,2,N,N)	t=2,g=N-1	$\mathbb{Z}_2 \oplus \mathbb{Z}_N$	3.3.2
3	2b	(2,N,2N)	t=1,g=(N-1)/2	$\mathbb{Z}_{2N}$	
4	2c	(2,2N,2N)	t=1,g=N/2	$\mathbb{Z}_{2N}$	3.3.3
5	2c	(2,2,2N,2N)	t=2,g=N	$\mathbb{Z}_{2N}$	3.3.4
6	3b	(2,4,8)	t=0,N=16,g=3	V_8	
7	3b	(2,2,4,4)	t=1,N=8,g=5	V_4	
8	3c	(4,4,N/2)	t=0,g=N/2-1	$H_{N/2}$	3.3.5
9	3e	(2,4,8)	t=0,N=8,g=2	U_4	3.3.10
10	3f	(4,4,N)	t=0,g=N/2	$G_{N/2}$	3.3.6
11	4a	(2,2,3,3)	t=1,g=5	$\mathbb{Z}_2 \oplus A_4$	
12	4c	(2,6,6)	t=0,g=3	$\mathbb{Z}_2 \oplus A_4$	3.3.7
13	4c	(2,2,6,6)	t=1,g=9	$\mathbb{Z}_2 \oplus A_4$	
14	4d	(3,3,4)	t=0,g=2	SL(2,3)	3.3.8
15	4f	(4,6,6)	t=0,g=6	SL(2,3)	3.3.9
16	5c	(3,4,4)	t=0,g=5	W_2	
17	5f	(4,4,6)	t=0,g=9	W_2	

The attentive reader should realize that also signature (2,2,4,4) occurring in Table 2 in

3.c with the group H_2 for $t = 1$, $N = 4$, $g = 3$,

3.e with the group U_2 for the same values of t, N, g,

3.f with the group G_1 for $t = 1$, $N = g = 2$

is not maximal, since it appears in Table 1. However, $H_2 = U_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_4$, whilst $G_1 = \mathbb{Z}_4$ and so these three possibilities are covered by case 2 with $N = 4$ and case 5 for $N = 2$. Hence we do not need to discuss them separately.

Note that some cases in Table 4 do not correspond to anyone in Table 3. This means that they provide us g-hyperelliptic groups though the signature σ is not maximal, and this is what we must do, in order to prove the "if" part of the theorem. Let us explain the general strategy we follow. Consider a group G and the corresponding signature σ in one of the cases 3,6,7,11,13,16, 17 in Table 4 and assume, by the way of contradiction, that G is not a g-hyperelliptic group. Since the pair (σ,G) appears in Table 2 we know that G is a g-hyperelliptic subgroup, say represented as Λ/Γ. So there exists a Fuchsian group $\tilde{\Lambda}$ containing properly Λ as a subgroup and Γ as a normal subgroup. Observe that repeating the arguments in [6, Theorem 5.1.2] Λ can be so chosen that $d(\tilde{\Lambda})=d(\Lambda)$. So the signature $\tilde{\sigma}$ of $\tilde{\Lambda}$ should appear in the second column of Table 1 in the same row as σ. Moreover as $\tilde{G}$ acts on H/Γ it should appear in Table 2 for the genus g. But inspecting this table we realize that whenever $\tilde{\sigma}$ appears the corresponding genus $\tilde{g}$ is different from g. This is summarized in the following

Table 5

Case	Genus g	Signature $\tilde{\sigma}$	Position of $\tilde{\sigma}$ in Table 2	Genus $\tilde{g}$
3	(N-1)/2	(2,3,2N)	6b, t=0, N=5	N
6	3	(2,3,8)	5c, t=0	2
7	5	(2,2,2,4)	3a, t=1, N=8 3b, t=1, N=4 3d, t=1, N=4	3 2 2
11	5	(2,2,2,3)	3a, t=1, N=6	2
13	9	(2,2,2,6)	3a, t=1, N=12 3d, t=1, N=6	5 3
16	5	(2,4,6)	3b, t=0, N=12 5b, t=0,	2 3
17	9	(2,4,12)	3b, t=0, N=24 3e, t=0, N=12	5 3

Let us now explain how to prove the "only if" part. Take a pair (σ,G) appearing in Table 4 in one of the cases 1,2,4,5,8,9,10,12,14 or 15. It is enough to prove the following:

Claim 3.2. For every Fuchsian group Λ with signature σ and every epimorphism $\theta\colon \Lambda \longrightarrow G$ whose kernel Γ has signature $\sigma = (g;-)$ and H/Γ is hyperelliptic, there exist a Fuchsian group $\tilde{\Lambda}$, a group $\tilde{G}$, group embeddings $i\colon\Lambda \hookrightarrow \tilde{\Lambda}$ and $j\colon G \hookrightarrow \tilde{G}$ and an epimorphism $\tilde{\theta}\colon\tilde{\Gamma} \longrightarrow \tilde{G}$ such that $1 \neq [\tilde{\Gamma}:\Lambda] = [\tilde{G}:G]$ and $\tilde{\theta}\cdot i = j\cdot\theta$.

Once this is done we have $G = \Lambda/\Gamma \subsetneqq \tilde{\Lambda}/\Gamma = \tilde{G} \subseteq \mathrm{Aut}(X)$, for all hyperelliptic surfaces X of genus g on which G acts as a group of automorphisms.

Begin for example with the case 3.1.5, *i.e.* $\sigma=(4,4,N/2)$ and the group is $H_{N/2}$. By Table 1 the signature $\tilde{\sigma}=(2,4,N)$ is the only signature extending σ. Let $\tilde{\Lambda}$ be a Fuchsian group with signature $\tilde{\sigma}$ containing Λ. By Proposition 1.1 the images of the canonical generators y_1, y_2 and y_3 of $\tilde{\Lambda}$ in $\tilde{\Lambda}/\Lambda$ have orders 2, 1 and 2, respectively. Thus $x_1 = y_2$, $x_2 = y_3y_2y_3^{-1}$ and $x_3 = y_3^2$ are in Λ, x_1 and x_2 have order 4, x_3 has order N/2 since N is even (see Table 2). Hence, in order to prove that x_1,x_2,x_3 form a canonical system of generators of Λ it is enough to check that $x_1x_2x_3 = 1$. Notice that $y_1y_2y_3 = 1$ and $y_1^2 = 1$, and so $x_1x_2x_3 = (y_2y_3)^2 = (y_1^{-1})^2 = 1$. Consequently, the embedding $i\colon\Lambda \hookrightarrow \tilde{\Lambda}$ is induced by the assignment:

$$i(x_1) = y_2,\ i(x_2) = y_3y_2y_3^{-1},\ i(x_3) = y_3^2.$$

On the other hand, let θ be the canonical epimorphism from Λ onto $G=\Lambda/\Gamma$. Since Γ is a surface group it follows from Prop 1.1 that $u = \theta(x_1)$ and $v = \theta(x_2)$ are two elements in G of order 4. Since $x_1x_2x_3 = 1$ we have $\theta(x_3) = (uv)^{-1}$ and so uv has order N/2. Clearly u and v generate G since θ is surjective. Moreover, let z be the central involution of G. Since we are dealing with Case 3.c of Table 2, the images of u and v in the quotient group

$G/<z>$ have order 2. Thus $u^2 = z = v^2$ and so G is generated by two elements u and v of order 4 whose product has order N/2 and in addition $u^2=v^2$. So $G = <u,v \mid u^4, u^2v^2, (uv)^{N/2}>$ since the last is isomorphic to $H_{N/2}$.

Inspecting Table 2 we find that $\tilde{\sigma}$ appears just in the row 3.b and so $\tilde{G} = \tilde{\Lambda}/\Gamma$ must be V_N. Now it is easy to check that the assignment $j(u) = x$, $j(v)=x^{-1}y^2$ induces a group monomorphism

$$j:H_{N/2} \longrightarrow V_N = <x,y \mid x^4, y^N, (xy)^2, (x^{-1}y)^2>$$

and without loss of generality we can assume that j is a group embedding (*i.e.* $H_{N/2} \subset V_N$) Now let $\tilde{\theta}: \tilde{\Lambda} \longrightarrow V_N$ be the group homomorphism induced by the assignment $\tilde{\theta}(y_1) = yx^{-1}$, $\tilde{\theta}(y_2) = x$, $\tilde{\theta}(y_3) = y^{-1}$. It is a homomorphism indeed since $\tilde{\theta}(y_1)\tilde{\theta}(y_2)\tilde{\theta}(y_3) = 1$ and it is obviously surjective. Finally it is straightforward to check the equality $\tilde{\theta} \cdot i = j \cdot \theta$.

For the remainder cases we denote by $\tilde{\sigma}$, $y_1,\ldots,y_r$ the signature and the canonical generators of $\tilde{\Lambda}$, by $x_1,\ldots,x_r$ the ones of Λ, and we give the definitions of homomorphisms $\theta, i, j, \tilde{\theta}$ and the presentations of G and $\tilde{G}$ in the Table 6 below.

In all the cases above but 3.1.10, we have found a system of canonical generators of the group Λ in terms of the canonical generators of a group $\tilde{\Lambda}$ containing Λ using Proposition 1.1, as we explained in detail for 3.1.5. The last case 3.1.10 requires a different approach, since from Table 1 we see that there is no signature $\tilde{\sigma}$ extending σ such that the pair $(\sigma,\tilde{\sigma})$ is normal. In fact, the only signature extending σ is $\tilde{\sigma}=(2,3,8)$ and so there exists a Fuchsian group $\tilde{\Lambda}$ with this signature containing Λ as a non-normal subgroup of index 3. The main difference now is that Proposition 1.1 does not work in the non-normal case. Let $\tilde{y}_1,\tilde{y}_2,\tilde{y}_3$ be the system of canonical generators for $\tilde{\Lambda}$. As Λ has a period 8 and the only elements of finite order in $\tilde{\Lambda}$ are those that are conjugated to powers of its canonical generators [15], we see that conjugation in $\tilde{\Lambda}$ of $\tilde{y}_3$, say by element w, belongs to Λ. Now $y_i=w\tilde{y}_iw^{-1}$ is still a system of canonical generators of $\tilde{\Lambda}$ and we have $y_3 \in \Lambda$. Clearly y_2 does not belong to Λ since Λ has not a period equal to 3. So $1=\Lambda$, $2=y_2\Lambda$, $3=y_2^2\Lambda$ is a system of cosets of Λ in $\tilde{\Lambda}$. Now $\tilde{\Lambda}$ acts on $\{1,2,3\}$ as a group of permutations and since $y_3 \in \Lambda$ we have $y_3(1)=1$ and so by a theorem of Singerman **[18]** $y_3=(1)(2,3)$ and thus in particular $y_3^2 y_2 \Lambda = y_2 \Lambda$ and therefore $y_2^2 y_3^2 y_2 \in \Lambda$. Now taking

$$x_1 = y_2 y_1 y_2^2 \; ; \; x_2 = y_2^2 y_3^2 y_2 \text{ and } x_3 = y_3$$

we have that $x_1x_2x_3=1$ and so $x_1 \in \Lambda$. Furthermore x_1,x_2 and x_3 are elements of order 2, 4 and 8 respectively and therefore they form a system of canonical generators for Λ.

Table 6

Case	$\tilde{\tau}$	$\tilde{G}$	Generators of Λ	Presentation of G relative to θ	Homomorphisms j and $\tilde{\theta}$
3.1.1	(2,4,N)	V_N	$x_1=y_2^2$	$\theta(x_1)=z$	$j(z)=x^2$
			$x_2=y_3$	$\theta(x_2)=u$	$j(u)=y$
			$x_3=y_1y_3y_1^{-1}$	$\theta(x_3)=(zu)^{-1}$	$\tilde{\theta}(y_1)=(xy)^{-1}$
				$\mathbb{Z}_2\oplus\mathbb{Z}_N=\langle z\rangle\oplus\langle u\rangle$	$\tilde{\theta}(y_2)=x$
					$\tilde{\theta}(y_3)=y$
3.1.2	(2,2,2,N)	$\mathbb{Z}_2\oplus D_N$	$x_1=y_1$	$\theta(x_1)=\theta(x_2)=z$	$j(z)=z$
			$x_2=y_2y_1y_2$	$\theta(x_3)=u$	$j(u)=xy$
			$x_3=y_3y_4y_3$	$\theta(x_4)=u^{-1}$	$\tilde{\theta}(y_1)=z$
			$x_4=y_4$	$\mathbb{Z}_2\oplus\mathbb{Z}_N=\langle z\rangle\oplus\langle u\rangle$	$\tilde{\theta}(y_2)=zx$
					$\tilde{\theta}(y_3)=y$
					$\tilde{\theta}(y_4)=(xy)^{-1}$

3.1.3	(2,4,2N)	U_N	$x_1=y_2^2$	$\theta(x_1)=u^N$	$j(u)=y$
			$x_2=y_3$	$\theta(x_2)=u$	$\tilde\theta(y_1)=x$
			$x_3=y_1y_3y_1^{-1}$	$\theta(x_3)=u^{N-1}$	$\tilde\theta(y_2)=xy^{-1}$
				$\mathbb{Z}_{2N}=\langle u\rangle$	$\tilde\theta(y_3)=y$
3.1.4	(2,2,2N,2N)	D_{2N}	$x_1=y_1$	$\theta(x_1)=\theta(x_2)=u^N$	$j(u)=xy$
			$x_2=y_2y_1y_2$	$\theta(x_3)=u$	$\tilde\theta(y_1)=(xy)^N$
			$x_3=y_3y_4y_3$	$\theta(x_4)=u^{-1}$	$\tilde\theta(y_2)=x(xy)^{N-1}$
			$x_4=y_4$	$\mathbb{Z}_{2N}=\langle u\rangle$	$\tilde\theta(y_3)=x$
					$\tilde\theta(y_4)=yx$
3.1.5	(2,4,N)	V_N	$x_1=y_2$	$\theta(x_1)=u$	$j(u)=x$
			$x_2=y_3y_2y_3^{-1}$	$\theta(x_2)=v$	$j(v)=x^{-1}y^2$
			$x_3=y_3^2$	$\theta(x_3)=(uv)^{-1}$	$\tilde\theta(y_1)=yx^{-1}$
				$H_{N/2}=$	$\tilde\theta(y_2)=x$
				$\langle u,v \mid u^4, u^2v^2, (uv)^{N/2}\rangle$	$\tilde\theta(y_3)=y^{-1}$
3.1.6	(2,4,2N)	U_N	$x_1=y_2$	$\theta(x_1)=u$	$j(u)=xy$
			$x_2=y_1y_2y_1$	$\theta(x_2)=v$	$j(v)=yx$
			$x_3=(y_2y_1)^{-2}$	$\theta(x_3)=(uv)^{-1}$	$\tilde\theta(y_1)=x$
				$G_{N/2}=$	$\tilde\theta(y_2)=xy$
				$\langle u,v \mid u^4, u^2v^2, u^2(uv)^{N/2}\rangle$	$\tilde\theta(y_3)=y^{-1}$
3.1.7	(2,4,6)	$\mathbb{Z}_2\oplus S_4$	$x_1=y_2^2$	$\theta(x_1)=u$	$j(u)=(xy)^2$
			$x_2=y_3$	$\theta(x_2)=v$	$j(v)=zy$
			$x_3=y_2y_3y_2^{-1}$	$\theta(x_3)=(uv)^{-1}$	$\tilde\theta(y_1)=xz$
				$\mathbb{Z}_2\oplus A_4=$	$\tilde\theta(y_2)=xy$
				$\langle u,v \mid u^2, v^6, v^3(uv)^3\rangle$	$\tilde\theta(y_3)=y^{-1}z$

3.1.8	(2,3,8)	W_1	$x_1=y_2$	$\theta(x_1)=u$	$j(u)=xyx$
			$x_2=y_3y_2y_3^{-1}$	$\theta(x_2)=v$	$j(v)=(xy)(yx)$
			$x_3=y_3^2$	$\theta(x_3)=(uv)^{-1}$	$\tilde{\theta}(y_1)=x$
				$SL(2,3)=$	$\tilde{\theta}(y_2)=xyx$
				$\langle u,v \mid u^3,v^3,(uv)^4,(uv)^2(vu)^2\rangle$	$\tilde{\theta}(y_3)=(yx)^{-1}$
3.1.9	(2,6,8)	W_1	$x_1=y_3^2$	$\theta(x_1)=a$	$j(a)=(xy)^6$
			$x_2=y_3^{-1}y_2y_3$	$\theta(x_2)=b$	$j(b)=(xy)^5x$
			$x_3=y_2$	$\theta(x_3)=(ab)^{-1}$	$\tilde{\theta}(y_1)=(xy)^4x$
				$SL(2,3)=$	$\tilde{\theta}(y_2)=(xy)^4y$
				$\langle a,b \mid a^4,b^6,a^2b^3,a^2(ab)^3\rangle$	$\tilde{\theta}(y_3)=(xy)^7$
3.1.10	(2,3,8)	W_1	$x_1=y_2y_1y_2^2$	$\theta(x_1)=a$	$j(a)=yxy^2$
			$x_2=y_2^2y_3^2y_2$	$\theta(x_2)=b$	$j(b)=yxy^2xy$
			$x_3=y_3$	$U_4=$	$\tilde{\theta}(y_1)=x$
				$\langle a,b \mid a^2,b^4,b^2(ab)^4\rangle$	$\tilde{\theta}(y_2)=y$
					$\tilde{\theta}(y_3)=(xy)^{-1}$

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After the typing of this paper we have heard about a Ph.D. Thesis by Britta Krapp on questions related to the problem studied here.

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