2-group Belyi maps

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July 9, 2019

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Outline

Motivation

Factoring in number fields

Let F be a number field with ring of integers $\mathbb{Z}_F$.

$$\begin{tikzcd}[column sep=small] & F \arrow[dl, dash] \arrow[d, dash] \\ \mathbb{Z}_F \arrow[d, dash] & \mathbb{Q} \arrow[dl, dash] \\ \mathbb{Z} & \end{tikzcd}$$

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Nonsolvable?

Why 2-group Belyi maps?

$$
\begin{tikzcd}
\text{2-group Belyi maps} \arrow[rr, dashed] \arrow[dr, "\text{Beckmann's theorem}"] & & \text{number fields ramified at 2} \\
& \text{curves with good reduction away from } p = 2 \arrow[ur, "\text{Néron-Ogg-Shafaravich}"] &
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The hope is that an explicit nonsolvable field ramified only at 2 can be obtained as $K(\mathrm{Jac}(X)[2])$ where X is the domain of a 2-**group Belyi map** (which we will define shortly).

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- implementation of a *method* to compute equations for 2-group Belyi maps over number fields

Main results

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- implementation of an algorithm to enumerate isomorphism classes of 2-group Belyi maps
- implementation of an algorithm to compute equations for 2-group Belyi maps over finite fields
- implementation of a *method* to compute equations for 2-group Belyi maps over number fields
- computational and theoretical evidence supporting a conjecture that every 2-group Belyi map is defined over an abelian extension of the rationals

Background

Belyi's theorem

A **Belyi map** is a morphism $\phi\colon X \to \mathbb{P}^1$ of smooth projective algebraic curves over $\mathbb{C}$ that is unramified outside of $\{0, 1, \infty\}$.

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Theorem (Belyi 1979)

An algebraic curve (smooth projective) X over $\mathbb{C}$ can be defined over a number field if and only if X admits a Belyi map.

2-**group Belyi maps**

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A 2-**group Belyi map** is a Galois Belyi map with monodromy group a 2-group.

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Let $\phi\colon X \to \mathbb{P}^1$ be a Galois Belyi map with monodromy group G. Let p be a prime not dividing $\#G$.

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Then there exists a number field M satisfying the following properties.

- *p is unramified in M*
- *ϕ is defined over M*
- *X is defined over M*
- *X has good reduction at all primes $\mathfrak{p}$ of M above p*

Permutation Triples

A **transitive permutation triple of degree** d is a triple

$$\sigma = (\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3$$

such that

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The set of degree d Belyi maps up to isomorphism is in bijection with the set of degree d transitive permutation triples up to **simultaneous conjugation** and the group $\langle \sigma \rangle$ is the monodromy group of ϕ.

Passports

A **passport** $\mathcal{P}$ consists of the data (g, G, λ) where $g \geq 0$ is an integer, $G \leq S_d$ is a transitive subgroup, and $\lambda = (\lambda_0, \lambda_1, \lambda_\infty)$ is a triple of partitions of d.

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The **passport of a Belyi map** $\phi : X \to \mathbb{P}^1$ is $(g(X), \mathrm{Mon}(\phi), (\lambda_0, \lambda_1, \lambda_\infty))$ with $g(X)$ the genus of X, $\mathrm{Mon}(\phi)$ the monodromy group of ϕ, and the partitions from ramification.

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The **passport of a permutation triple** σ is $(g(\sigma), \langle\sigma\rangle, \lambda(\sigma))$ where

$$g(\sigma) = 1 - d + (e(\sigma_0) - e(\sigma_1) - e(\sigma_\infty))/2$$

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$$e(\tau) = d - \#\text{cycles of } \tau,$$

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We now discuss the importance of organizing triples by passport.

Fields of moduli, fields of definition, and passports

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The situation improves, however, in the Galois setting.

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Let $\phi\colon X \to \mathbb{P}^1$ be a *Galois* Belyi map of degree d with monodromy group G corresponding to the permutation triple σ.

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Then

- ϕ and X are defined over $M(\phi)$,
- $\#G = d$,
- all cycles of σ_s have the same length for $s \in \{0, 1, \infty\}$,
- and if we let a, b, c be the orders of $\sigma_0, \sigma_1, \sigma_\infty$ respectively, we have

$$g(X) = 1 + \frac{\#G}{2}\left(1 - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}\right).$$

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The monodromy group in this setting corresponds to field automorphisms of the Galois closure of $K(X)$ fixing $K(x)$.

2-group Belyi maps as iterated quadratic extensions

$$
\begin{tikzcd}
X_r \arrow[d, "2"'] \arrow[ddddr, bend left] & \\
\vdots \arrow[d, "2"'] & \\
X_2 \arrow[d, "2"'] \arrow[ddr, bend left] & \\
X_1 \arrow[dr, "2"'] & \\
& X_0 = \mathbb{P}^1
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
K(X_r) \arrow[d, dash, "2"'] \arrow[ddddr, dash, bend left] & \\
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Computing permutation triples

Setup

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- $\sigma_\infty \sigma_1 \sigma_0 = \mathrm{id}$;
- $G := \langle \sigma_0, \sigma_1 \rangle$ is a transitive subgroup of S_d; and
- G is a 2-group of order d embedded in S_d via its left regular representation.

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We say two degree d 2-group permutation triples σ, σ' are **simultaneously conjugate** if there exists $\tau \in S_d$ such that

$$\sigma^\tau := (\tau^{-1}\sigma_0\tau, \tau^{-1}\sigma_1\tau, \tau^{-1}\sigma_\infty\tau,) = \sigma'$$

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A **lift** of σ is a 2-group permutation triple $\widetilde{\sigma} \in S_{2d}^3$ such that $\langle \widetilde{\sigma} \rangle$ is isomorphic to some extension $\widetilde{G}$ of $\mathbb{Z}/2\mathbb{Z}$ by G as in the exact sequence below.

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} \widetilde{G} \xrightarrow{\pi} \langle \sigma \rangle \longrightarrow 1$$

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For a 2-group permutation triple σ, we denote the set of lifts of σ by $\mathrm{Lifts}(\sigma)$ and $\mathrm{Lifts}(\sigma)/\sim$ denotes the set of lifts up to simultaneous conjugation.

Algorithm to compute $\mathsf{Lifts}(\sigma)/\sim$

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4.
$$\text{Lifts}(\sigma) := \bigcup_{f \in H^2(G,A)} \text{Lifts}(\sigma, f)$$

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$$\left\{ \widetilde{\sigma} : \widetilde{\sigma}_s \in \pi_f^{-1}(\sigma_s) \text{ for } s \in \{0, 1, \infty\},\ \widetilde{\sigma}_\infty \widetilde{\sigma}_1 \widetilde{\sigma}_0 = 1,\ \langle \widetilde{\sigma} \rangle = \widetilde{G}_f \right\}$$
4. $$\mathrm{Lifts}(\sigma) := \bigcup_{f \in H^2(G,A)} \mathrm{Lifts}(\sigma, f)$$
5. Quotient Lifts(σ) by simultaneous conjugation

Example computing $\mathsf{Lifts}(\sigma)/\!\sim$: setup

Let $\sigma = \big((1\,2), \mathrm{id}, (1\,2)\big)$. Then $G = \langle \sigma \rangle = \mathbb{Z}/2\mathbb{Z}$.

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Choosing

$$\alpha := (1\ d+1)(2\ d+2)\ldots(d-1\ 2d-1)(d\ 2d)$$

allows us to label blocks by reducing modulo d.

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Lastly, we quotient by simultaneous conjugation to obtain
$$\mathrm{Lifts}(\sigma)/\sim = \Big\{((1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)), ((1\,4\,3\,2), \mathrm{id}, (1\,2\,3\,4)), ((1\,2\,3\,4), (1\,3)(2\,4), (1\,2\,3\,4))\Big\}$$

Bipartite graphs of permutation triples

Now that we can lift permutation triples, we now describe some notation for the bipartite graphs that organize these triples.

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For $i \in \mathbb{Z}_{\geq 1}$ we define the bipartite graph denoted $\mathscr{G}_{2^i}$ with the following node sets.

- $\mathscr{G}_{2^i}^{\text{above}}$: the set of isomorphism classes of 2-group Belyi maps of degree 2^i indexed by 2-group permutation triples $\widetilde{\sigma}$ up to simultaneous conjugation in S_{2^i}
- $\mathscr{G}_{2^i}^{\text{below}}$: the set of isomorphism classes of 2-group Belyi maps of degree 2^{i-1} indexed by 2-group permutation triples σ up to simultaneous conjugation in $S_{2^{i-1}}$

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For every pair of nodes $(\widetilde{\sigma}, \sigma) \in \mathscr{G}_{2^i}^{\text{above}} \times \mathscr{G}_{2^i}^{\text{below}}$ there is an edge between σ and $\widetilde{\sigma}$ if and only if $\widetilde{\sigma}$ is simultaneously conjugate to a lift of σ.

Algorithm to compute $\mathscr{G}_{2^i}$

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Theorem (M.)

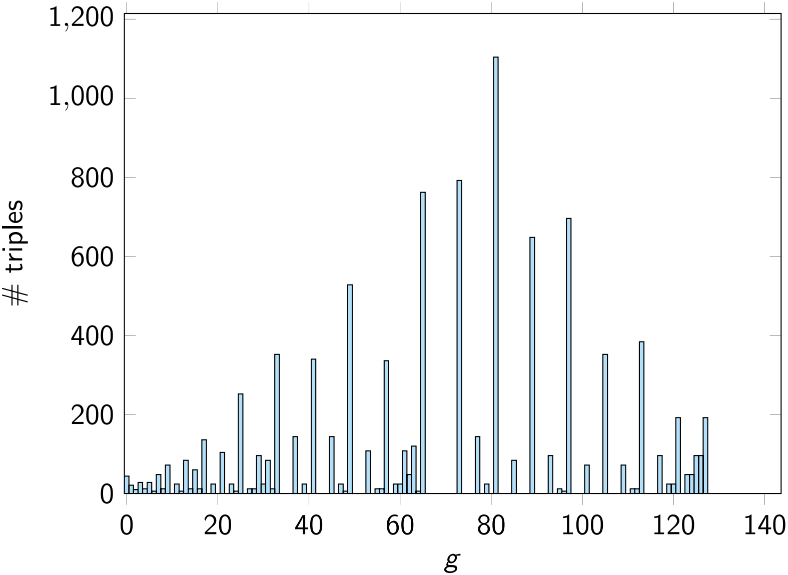
The following tables list the number of isomorphism classes of 2-group Belyi maps, the number of passports, and number of lax passports respectively up to degree 256.

d	1	2	4	8	16	32	64	128	256
# *triples*	1	3	7	19	55	151	503	1799	7175

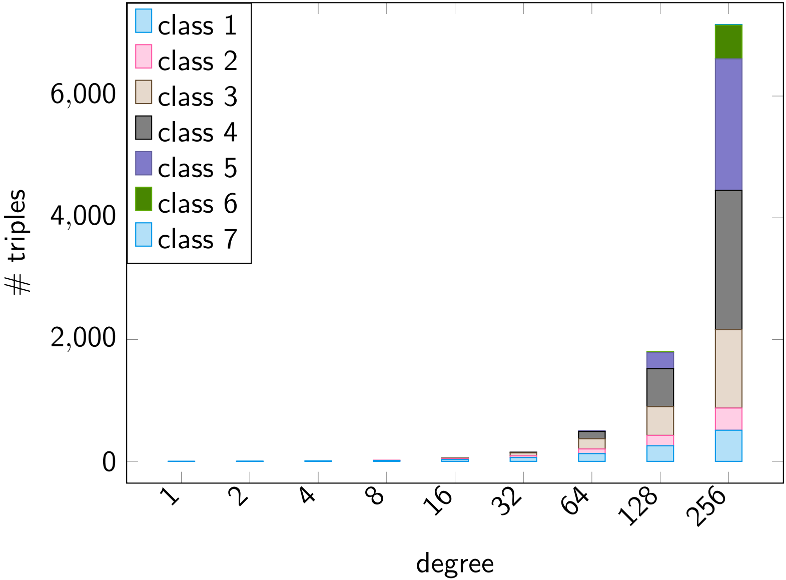
d	1	2	4	8	16	32	64	128	256
# *passports*	1	3	7	16	41	96	267	834	2893

d	1	2	4	8	16	32	64	128	256
# *lax passports*	1	1	3	6	14	31	85	257	882

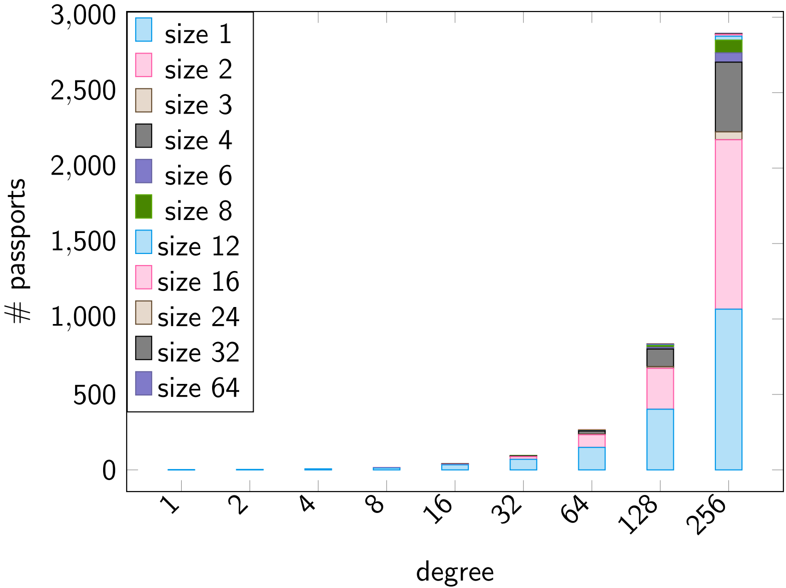
Results : distribution of genera



Results : groups by nilpotency class



Results : passport sizes



A refined conjecture

Passports

Recall that a passport $\mathcal{P}$ consists of the data (g, G, λ) where $g \in \mathbb{Z}_{\geq 0}$, G is a transitive subgroup of S_d and $\lambda = (\lambda_0, \lambda_1, \lambda_\infty)$ is a triple of partitions of d corresponding to conjugacy classes (C_0, C_1, C_∞) of S_d.

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The size of $\mathcal{P}$ is the cardinality of the set $\Sigma_{\mathcal{P}}$ defined by

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To instead analyze $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}} \mid \mathbb{Q}^{\mathrm{ab}})$ we *refine* the notion of a passport.

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A **refined passport** $\mathscr{P}$ consists of the data (g, G, c) where $g \in \mathbb{Z}_{\geq 0}$, G is a transitive subgroup of S_d and $c = (c_0, c_1, c_\infty)$ is a triple of conjugacy classes of G.

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As was the case with passport, every permutation triple σ determines a refined passport $\mathscr{P}(\sigma)$.

A refined conjecture

Theorem (M.)

The size of $\mathscr{P}(\sigma)$ *is equal to* 1 *for every* 2*-group permutation triple* σ *with degree* ≤ 256.

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ARC is true for 2*-group permutation triples* σ *with* $\langle\sigma\rangle$ *dihedral.*

Computing equations

A motivating example : setup

Let F be a number field with integers $\mathbb{Z}_F$. Let $\mathrm{Pl}(F)$ denote the places of F and S_∞ the archimedean places. For $v \in \mathrm{Pl}(F) \setminus S_\infty$ let $\mathfrak{p}_v$ denote the prime ideal of $\mathbb{Z}_F$ corresponding to v.

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First, it is possible that no such extension exists.

Second, if $\mathfrak{a} = (d)$ is principal, then $F(\sqrt{d})$ is ramified exactly at the $\mathfrak{p}_v$, and d is unique up to a unit.

A motivating example : setup

Let F be a number field with integers $\mathbb{Z}_F$. Let $\mathrm{Pl}(F)$ denote the places of F and S_∞ the archimedean places. For $v \in \mathrm{Pl}(F) \setminus S_\infty$ let $\mathfrak{p}_v$ denote the prime ideal of $\mathbb{Z}_F$ corresponding to v.

Let $S \subset \mathrm{Pl}(F) \setminus S_\infty$ and let $\mathfrak{a} := \prod_{v \in S} \mathfrak{p}_v$.

Question

How do we construct a quadratic extension of F with ramification prescribed by $\mathfrak{a}$?

First, it is possible that no such extension exists.

Second, if $\mathfrak{a} = (d)$ is principal, then $F(\sqrt{d})$ is ramified exactly at the $\mathfrak{p}_v$, and d is unique up to a unit.

If $\mathfrak{a}$ is not principal, then the question requires more care.

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To summarize, in the case where $\mathfrak{a}$ is not principal but there exists $\mathfrak{b}$ with $\mathfrak{a}\mathfrak{b}^2$ principal we have $[\mathfrak{a}] \in \mathrm{Cl}_F^2$ and $[\mathfrak{b}]$ is unique up to multiplication by $[\mathfrak{c}] \in \mathrm{Cl}_F[2]$.

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The algorithms in this section rely on transporting this technique to the function field setting.

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As an example, let X be an irreducible affine plane curve (possibly singular) defined by the equation $f(x, y) = 0$ with $f \in K[x, y]$. Then the **function field of** X, denoted $K(X)$ is the field of fractions of the coordinate ring $K[x, y]/(f(x, y))$ of X.

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The set of places of F is denoted $\mathrm{Pl}(F)$ and the **degree** of P is the index $[\mathcal{O}_P/P : K]$ of the **residue class field**.

The **divisor class group** $\mathrm{Div}(F)$ of F is the free abelian group generated by the places of F. A **divisor** $D \in \mathrm{Div}(F)$ is represented by a sum of places $\sum_P a_P P$ and the **degree** of D is $\sum_P a_P \deg(P)$. The set of **degree zero divisors** is denoted $\mathrm{Div}^0(F)$.

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The image of $\mathrm{div}\colon F^\times \to \mathrm{Div}(F)$ defined by $\mathrm{div}(f) = \sum_P \mathrm{ord}_P(f) P$ is the subgroup of **principal divisors** of F denoted $\mathrm{Princ}(F)$.

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The **Riemann-Roch space** of a divisor $D \in \mathrm{Div}(F)$ is defined by $\mathscr{L}(D) := \{f \in F : \mathrm{div}(f) + D \geq 0\} \cup \{0\}$.

Lemma

Let $aF^{\times 2}$ be a nontrivial coset of $F^{\times}/F^{\times 2}$ and consider the extension $L := F(\sqrt{a})$. Then a prime P of F is ramified in L if and only if $\operatorname{ord}_P(a)$ is odd.

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As in the number field setting, this implies $R \in 2\,\mathrm{Pic}(F)$ and D is unique up to addition by $T \in \mathrm{Pic}^0(F)[2]$.

Algorithm in characteristic $p \geq 3$: setup

Let F be a function field with field of constants $\mathbb{F}_q$ with $q = p^r$ and p an odd prime. Let $\mathbb{F}_q(x)$ denote the rational function field in the variable x.

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Picard groups are implemented in the tame case.

Algorithm in characteristic $p \geq 3$: Galois test

Input:

- F a Galois extension of $\mathbb{F}_q(x)$
- $\mathrm{Gal}(F \,|\, \mathbb{F}_q(x))$ explicitly given as automorphisms of F
- $a \in F$

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- For each generator $\sigma \in \mathrm{Gal}(F \,|\, \mathbb{F}_q(x))$ test if $\sigma(a)/a$ is a square in F
- Return True if $\sigma(a)/a$ is a square in F for all generators σ and otherwise return False

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Similarly, we can apply the same test after extending the constant field from $\mathbb{F}_q$ to $\mathbb{F}_{q^2}$.

Algorithm in characteristic $p \geq 3$: get candidates

Input:

- F a 2-group Belyi map modulo q of degree $d = 2^m$ corresponding to a 2-group permutation triple σ
- A passport $\mathcal{P} = (\widetilde{G}, (a, b, c))$ with $\widetilde{G}$ a 2-group of order $2d$ such that there exists a 2-group permutation triple $\widetilde{\sigma}$ with passport $\mathcal{P}$ that is a lift of σ
- $\mathrm{Gal}(F \,|\, \mathbb{F}_q(x)) \cong \langle \sigma \rangle$ explicitly given as automorphisms of F

Output: A list of candidate functions $\{f_i\}$ with each $f_i \in F$ such that $F(\sqrt{f_i})$ is a 2-group Belyi map modulo q with passport $\mathcal{P}$.

Algorithm in characteristic $p \geq 3$: get candidates (steps 1-4)

1. For $s \in \{0, 1, \infty\}$ compute

$$r_s := \begin{cases} 0 & \text{if } \operatorname{order}(\sigma_s) = \operatorname{order}(\widetilde{\sigma}_s) \\ 1 & \text{if } \operatorname{order}(\sigma_s) < \operatorname{order}(\widetilde{\sigma}_s) \end{cases}$$

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2. Compute

$$R := \sum_{s \in \{0,1,\infty\}} r_s R_s \in \operatorname{Div}(F)$$

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4. Compute $[R] := \psi^{-1}(R)$.

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   (c) If $\mathscr{L}(R - 2D_a)$ has dimension 1, then compute $f_a \in F$ with $\mathrm{div}(f_a)$ generating $\mathscr{L}(R - 2D_a)$ and go to Step 5d Otherwise go to the next $a \in \mathrm{Pic}(F)[2]$.

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   (d) Apply Galois test to F, $\mathrm{Gal}(F \,|\, \mathbb{F}_q(x))$, and f_a from Step 5c to see if $F(\sqrt{f_a})$ generates a Galois extension. If $F(\sqrt{f_a})$ is Galois over $\mathbb{F}_q(x)$ then save f_a and go to the next $a \in \mathrm{Pic}(F)[2]$. If $F(\sqrt{f_a})$ is not Galois over $\mathbb{F}_q(x)$, then go to Step 5e.

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   (e) Let F' be the function field F after extending the field of constants $\mathbb{F}_q$ to $\mathbb{F}_{q^2}$. Apply Galois test to F', $\mathrm{Gal}(F' \mid \mathbb{F}_{q^2}(x))$, and f_a (viewed as an element of F') from Step 5c to see if $F'(\sqrt{f_a})$ generates a Galois extension. If $F(\sqrt{f_a})$ is Galois over $\mathbb{F}_{q^2}(x)$ then save f_a. Go to the next $a \in \mathrm{Pic}(F)[2]$.

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$$G_a \cong \mathrm{Gal}(F(\sqrt{f_a}) \,|\, \mathbb{F}_q(x)),$$

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8. Return the list S''

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Since we know the sizes of passports from our work with permutation triples, we know that we have representatives from every isomorphism class even if we cannot match the Belyi maps to their corresponding permutation triples.

Implementation in characteristic zero

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Although this implementation does not allow us to compute all 2-group Belyi maps for a given degree, it does work well in practice.

Results

https://github.com/michaelmusty/2GroupDessins

- *all* 2-group Belyi maps modulo 3 up to degree 32
- hundreds of 2-group Belyi maps up to degree 256

Examples

Notation

`DNG-a,b,c-gE-H`

- `D` : degree in $\{2, 4, 8, 16, 32, 64, 128, 256\}$
- `N` : either `T` or `S` identifying group database
- `G` : a positive integer identifying the group
- `a` : ramification index of 0 in $\{2, 4, 8, 16, 32, 64, 128, 256\}$
- `b` : ramification index of 1 in $\{2, 4, 8, 16, 32, 64, 128, 256\}$
- `c` : ramification index of ∞ in $\{2, 4, 8, 16, 32, 64, 128, 256\}$
- `g` : just the letter g
- `E` : the genus in $\mathbb{Z}_{\geq 0}$
- `H` : the hash of the 2-group permutation triple a positive integer

An interesting example

d3ssins

https://michaelmusty.github.io/d3ssins

Future work

- higher degree over $\mathbb{F}_3$
- an algorithm in characteristic zero
- prove ARC for other families of 2-groups
- p-group Belyi maps for p odd
- compute torsion fields

Acknowledgements

- Dave, Tom, Carl, and John
- Sam, Anna, Jeroen, Edgar, Florian, and Richard
- Mary, Jim, Matt, and Nicole

Galois representations

Let X be an irreducible, smooth projective algebraic curve of genus $g \geq 1$ over a number field K. Let $G_K := \mathsf{Gal}(K^{\mathsf{al}} \mid K)$ be the absolute Galois group of K and let $\ell \in \mathbb{Z}$ be prime.

Let $J := \mathsf{Jac}(X)$ be the **Jacobian variety** of X. J is an abelian variety of dimension g.

G_K acts on the ℓ-torsion points $J[\ell](K^{\mathsf{al}}) \cong (\mathbb{Z}/\ell\mathbb{Z})^{2g}$ of X.

This action determines a **mod-ℓ Galois representation**

$$\rho\colon G_K \to \mathsf{Aut}(J[\ell]) \cong \mathsf{GL}_{2g}(\mathbb{Z}/\ell\mathbb{Z}).$$

The geometry of X and the arithmetic of ρ are inimately related. For example, if X has good reduction at a prime $\mathfrak{p}$ above $p \neq \ell$, then $\mathfrak{p}$ will be unramified in the ℓ-**torsion field** $K(J[\ell])$.

Isomorphism of Belyi maps

Let $\phi\colon X \to \mathbb{P}^1$ and $\phi'\colon X' \to \mathbb{P}^1$ be Belyi maps of degree d. ϕ and ϕ' are **isomorphic** (respectively **lax isomorphic**) if the diagrams

$$\begin{tikzcd} X \arrow[rr, "\sim"] \arrow[dr, "\phi"'] & & X' \arrow[dl, "\phi'"] \\ & \mathbb{P}^1 & \end{tikzcd}, \text{ respectively } \begin{tikzcd} X \arrow[r, "\sim"] \arrow[d, "\phi"'] & X' \arrow[d, "\phi'"] \\ \mathbb{P}^1 \arrow[r, "\sim", "\beta"'] & \mathbb{P}^1 \end{tikzcd}$$

commute where $\beta(\{0, 1, \infty\}) = \{0, 1, \infty\}$.