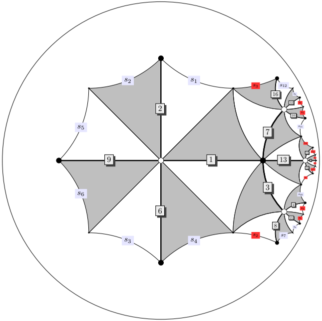
2-solvable Belyĭ maps



Michael Musty
Algebra and Number Theory Seminar
May 8, 2018

Outline

1. What is a 2-solvable Belyĭ map?
2. Motivation
3. Algorithm to compute explicitly
    3.1 Find permutation triples
    3.2 Compute equations
4. Explicit examples

Theorem (G.V. Belyĭ 1979)

A smooth projective curve X over $\mathbb{C}$ can be defined over $\overline{\mathbb{Q}}$ if and only if there exists a branched covering of compact connected Riemann surfaces $\varphi : X \to \mathbb{P}^1$ unramified (unbranched) above $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

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In the 1980s, Grothendieck described a bijection between Belyĭ maps and *dessins d'enfants*. Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on these sets.

A Zoo of Bijections

$$
\begin{tikzcd}
\left\{\begin{array}{l}\text{Transitive permutation}\\ \text{Triples } (\sigma_0, \sigma_1, \sigma_\infty) \in \\ S_d^3 \text{ with } \sigma_\infty\sigma_1\sigma_0 = 1\end{array}\right\} \arrow[dr, leftrightarrow, "\sim"] & & \left\{\begin{array}{l}\text{Index } d \text{ triangle sub-}\\ \text{groups } \Gamma \leq \Delta(a, b, c)\end{array}\right\} \arrow[dl, leftrightarrow, "\sim"'] \\
& \left\{\begin{array}{l}\text{dessins with}\\ d \text{ edges}\end{array}\right\} \arrow[dl, leftrightarrow, "\sim"] \arrow[dr, leftrightarrow, "\sim"] & \\
\left\{\begin{array}{l}\text{Degree } d \text{ function field}\\ \text{extensions } K \supseteq \mathbb{C}(z)\\ \text{with discriminant sup-}\\ \text{ported above } 0, 1, \infty\end{array}\right\} & & \left\{\text{Belyĭ maps of degree } d\right\}
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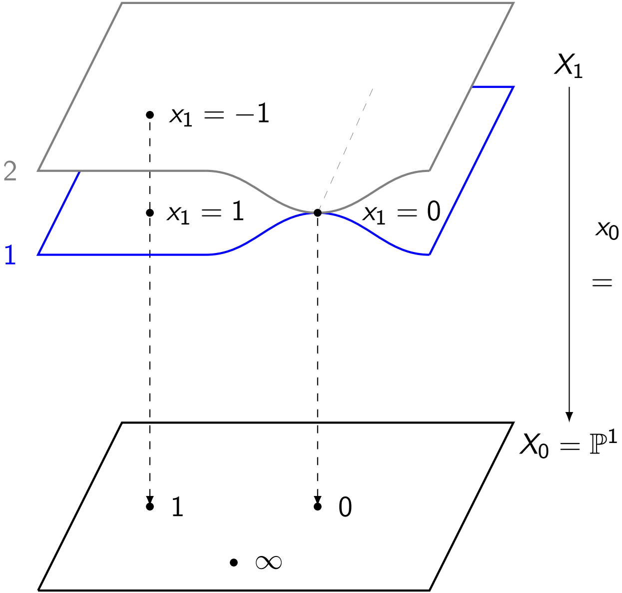
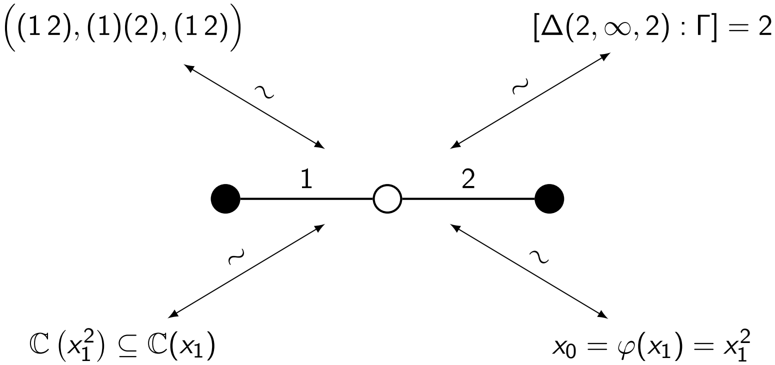
All up to the appropriate version of equivalence in each category.

Example ☕



$$x_0 = \varphi(x_1) = x_1^2$$
$$= (x_1 + 1)(x_1 - 1) + 1$$

Example ☕



$\big((1\,2),(1)(2),(1\,2)\big)$
$[\Delta(2,\infty,2):\Gamma]=2$
$\sim$
$\sim$
1
2
$\sim$
$\sim$
$\mathbb{C}\left(x_1^2\right)\subseteq\mathbb{C}(x_1)$
$x_0=\varphi(x_1)=x_1^2$

$$\begin{tikzcd}
X_r \arrow[d, "2"'] \arrow[ddddr, bend left] & \\
\vdots \arrow[d, "2"'] & \\
X_2 \arrow[d, "2"'] \arrow[ddr, bend left] & \\
X_1 \arrow[dr, "2"'] & \\
& X_0 = \mathbb{P}^1
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
K(X_r) \arrow[d, dash, "2"'] \arrow[ddddr, dash, bend left] & \\
\vdots \arrow[d, dash, "2"'] & \\
K(X_2) \arrow[d, dash, "2"'] \arrow[ddr, dash, bend left] & \\
K(X_1) \arrow[dr, dash, "2"'] & \\
& K(X_0) = K(\mathbb{P}^1)
\end{tikzcd}$$

Beckmann's Theorem

Theorem (Beckmann-Kazez 1989)

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Upshot:

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Upshot: Every 2-solvable Belyĭ curve we write down has good reduction away from $p = 2$.

Lifting example ☕

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$$\begin{tikzcd}
\widetilde{X} \arrow[dd, "\widetilde{\varphi}"'] \arrow[dr, "\psi"] & \\
& X \arrow[dl, "\varphi"] \\
\mathbb{P}^1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
K(\widetilde{X}) \arrow[dd, dash] \arrow[dr, dash, "2"] & \\
& K(X) \arrow[dl, dash, "2"] \\
K(\mathbb{P}^1) &
\end{tikzcd}
\qquad
\begin{tikzcd}
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$$G = \mathrm{Gal}(K(X)/K(\mathbb{P}^1)) \qquad G \cong \Big\langle \Big((1\,2),(1)(2),(1\,2)\Big) \Big\rangle \leq S_2$$

$$\widetilde{G} = \mathrm{Gal}(K(\widetilde{X})/K(\mathbb{P}^1)) \qquad \widetilde{G} \cong \langle \widetilde{\sigma} \rangle \leq S_4$$

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$$1 \longrightarrow H \xrightarrow{\iota} \widetilde{G} \xrightarrow{f} G \longrightarrow 1$$

$$\widetilde{\sigma} \xrightarrow{?} \sigma$$

Lifting example ☕

$$\sigma = (\sigma_0, \sigma_1, \sigma_\infty) = \Big((1\,2), (1)(2), (1\,2)\Big) \in S_2^3$$

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$f^{-1}(\sigma_0)$	$f^{-1}(\sigma_1)$	$f^{-1}(\sigma_\infty)$
$(1\,2)(3\,4)$	$(1)(2)(3)(4)$	$(1\,2)(3\,4)$
$(1\,4)(2\,3)$	$(1\,3)(2\,4)$	$(1\,4)(2\,3)$
$(1\,4\,3\,2)$		$(1\,4\,3\,2)$
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$\widetilde{G} \cong \mathbb{Z}/4\mathbb{Z}$
$\big((1\,4\,3\,2), (1)(2)(3)(4), (1\,2\,3\,4)\big)$
$\big((1\,4\,3\,2), (1\,3)(2\,4), (1\,4\,3\,2)\big)$

$\widetilde{G} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$\big((1\,2)(3\,4), (1\,4)(2\,3), (1\,3)(2\,4)\big)$

4T1-[4,2,4]-4-22-4-g1

$$(\sigma_0, \sigma_1, \sigma_\infty) = \Big((1\,4\,3\,2), (1\,3)(2\,4), (1\,4\,3\,2)\Big)$$

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$$(\sigma_0, \sigma_1, \sigma_\infty) = \Big((1\,4\,3\,2), (1\,3)(2\,4), (1\,4\,3\,2)\Big)$$

$$\begin{tikzcd}
(0,0,0) \arrow[d, equal] & (1,1,0) \arrow[d, equal] & (1,-1,0) \arrow[d, equal] & \infty \arrow[d, equal] \\
(0,0) \arrow[d, equal] & (1,1) \arrow[dr, dash] & (1,-1) \arrow[d, dash] & \infty \arrow[d, equal] \\
0 & & 1 & \infty
\end{tikzcd}
\qquad
\begin{tikzcd}
X_2 : {\scriptstyle \substack{x_1^2 = x_0 \\ x_2^2 = x_1^3 - x_1}} \arrow[d] \arrow[ddr, bend left] & \\
X_1 : x_1^2 = x_0 \arrow[dr] & \\
& X_0
\end{tikzcd}$$

We now exhibit a genus 5 Belyĭ map $\varphi : X \to \mathbb{P}^1$ defined by $x_0 \in K(X)$ with monodromy group $C_8 : C_2$.

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X is cut out by the following equations in $\mathbb{A}^5$:

16T6-[8,8,4]-88-88-4444-g5

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X is cut out by the following equations in $\mathbb{A}^5$:

$$
\begin{aligned}
x_1^2 &= x_0 \\
x_3 x_4^2 &= x_1 x_2 + x_1 + x_3^2 \\
x_3 x_4^2 &= x_1 + x_1 x_3^2 \\
x_3 x_4^4 &= x_3^3 + 2 x_1 x_4^2 \\
2 x_3^2 x_4^4 &= x_2^2 + 2 x_3^3 x_4^2 + 2 x_3^2 - 2 x_3 x_4^2 + 2 x_4^4 + 1 \\
x_3^3 &= x_2 x_3 + x_3^2 x_4^2 - x_4^2 \\
x_3^2 x_4^2 &= x_2 x_4^2 + x_3^3 + x_3 \\
x_3^2 x_4^4 &= x_3^4 + x_3^2 + x_4^4
\end{aligned}
$$

Acknowledgements

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- Sam Schiavone
- Jeroen Sijsling
- John Voight

Thanks for listening!

https://math.dartmouth.edu/~mjmusty/32.html