PRESCRIBED RAMIFICATION IN FUNCTION FIELDS

MICHAEL MUSTY

Contents

I. Introduction

I.1. **Motivating Question.** Let $\varphi : \widetilde{X} \to X$ be a degree 2 map of algebraic curves over $K = \overline{\mathbb{Q}}$. Then ϕ corresponds to a quadratic function field extension $K(\widetilde{X})/K(X)$. Consider an effective divisor $D \in \operatorname{Div}(X)$ of degree d with d even.

Question

How do we construct φ in such a way that we get simple ramification exactly at the points in the support of D?

MM: [maybe just start with "the picture"]

I.2. **Background: Curves and Function Fields.** Let X be a curve over K and let $K(X)$ denote its function field.

Proposition I.2.1. *There is a bijection*

$$K(X) \cup \{\infty\} \longleftrightarrow \{\textit{maps } X \to \mathbb{P}^1 \textit{ defined over } K\}.$$

Proof. Recall that $f \in K(X)$ defines a map $X \to \mathbb{P}^1$ defined by

$$p \mapsto \begin{cases} [f(p) : 1] & \text{if } f \text{ is regular at } p \\ [1 : 0] & \text{if } f \text{ has a pole at } p \end{cases}$$

On the other hand, given a map $\varphi : X \to \mathbb{P}^1$ where $\varphi = [f : g]$. If $g = 0$, then φ is the constant map $p \mapsto [1 : 0]$ (call this map ∞). Otherwise, φ is the map

$$p \mapsto [f/g : 1].$$

$\square$

Proposition I.2.2. *Let $\varphi : \widetilde{X} \to X$ be a map of curves. Then φ induces an injection*

$$\varphi^* : K(X) \to K(\widetilde{X}).$$

Proof. Let $f \in K(X)$. Then by the previous prop, f corresponds to a map $X \to \mathbb{P}^1$. Precompose with φ and use the previous proposition again. $\square$

In fact, even more is true.

Theorem I.2.3. [2, Thm 2.4]

(1) Let $\varphi : \widetilde{X} \to X$. Then $K(\widetilde{X})/\varphi^* K(X)$ is a finite extension.
(2) Let $\iota : K(X) \to K(\widetilde{X})$ be an injection fixing K. Then there exists a unique nonconstant map φ such that $\varphi^* = \iota$.
(3) Let F be a finite index subfield of $K(\widetilde{X})$ containing K. Then there exists a smooth curve X (unique up to K-iso) and a nonconstant map $\varphi : \widetilde{X} \to X$ such that $K(X) = F$.

Moreover, these categories are equivalent.

I.3. **Background: Ramification and Riemann-Hurwitz.**

Definition I.3.1. Let $\varphi : \widetilde{X} \to X$ be a nonconstant map of curves. Let $\widetilde{p} \in \widetilde{X}$ be smooth. The **ramification index** of φ at $\widetilde{p}$ is the quantity:

$$e_\varphi(\widetilde{p}) = \mathrm{ord}_{\widetilde{p}}(\varphi^* t_{\varphi(\widetilde{p})})$$

where $t_{\varphi(\widetilde{p})}$ is a uniformizer at $\varphi(\widetilde{p})$. We say φ is **unramified at** $\widetilde{p}$ if $e_\varphi(\widetilde{p}) = 1$.

Example I.3.2. Consider the map $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ given by $x \mapsto x^2 = y$ i.e. with projective coordinates

$$[z_0 : z_1] \mapsto [z_0^2 : z_1^2].$$

Let $\widetilde{p} = [1 : 1]$ in the domain. Then

$$\begin{aligned} e_\varphi([1:1]) &= \mathrm{ord}_{[1:1]}(\varphi^* t_{\varphi([1:1])}) \\ &= \mathrm{ord}_{[1:1]}(\varphi^* t_{[1:1]}) \\ &= \mathrm{ord}_{[1:1]}(\varphi^*(y-1)) \\ &= \mathrm{ord}_{[1:1]}(x^2 - 1) \\ &= 1. \end{aligned}$$

If instead $\widetilde{p} = [0 : 1]$, then $\varphi(\widetilde{p}) = [0 : 1] =: p$, and $t_p = y$, $\varphi^* t_p = x^2$, and $\mathrm{ord}_{\widetilde{p}}(x^2) = 2$.
MM: [also just think about preimages]

Proposition I.3.3. *Let $\varphi : \widetilde{X} \to X$ be a nonconstant map of smooth curves. Then for every $p \in X$,*

$$\sum_{\widetilde{p} \in \varphi^{-1}(p)} e_\varphi(\widetilde{p}) = \deg(\varphi).$$

Theorem I.3.4 (Riemann-Hurwitz)**.** *Let $\varphi : \widetilde{X} \to X$ be a nonconstant map of smooth curves. Then,*

$$2g(\widetilde{X}) - 2 = (2g(X) - 2)\deg(\varphi) + \sum_{\widetilde{p} \in \widetilde{X}} (e_\varphi(\widetilde{p}) - 1).$$

MM: [more analogies with number fields]

I.4. **Background: Divisors and Riemann-Roch.** Recall that for a curve X we have an exact sequence

$$0 \longrightarrow \mathrm{Div}^0(X) \longrightarrow \mathrm{Div}(X) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0.$$

For $f \in K(X)^\times$, recall that the map

$$\mathrm{div}(f) = \sum_{p \in X} \mathrm{ord}_p(f)(p) \in \mathrm{Div}(X)$$

yields the exact sequence

$$1 \longrightarrow K^\times \longrightarrow K(X)^\times \xrightarrow{\mathrm{div}} \mathrm{Div}^0(X) \longrightarrow \mathrm{Pic}^0(X) \longrightarrow 0.$$

Example I.4.1. Let $X_0 = \mathbb{P}^1$ with coordinate t. Let $p \in X$ and $a = t(p)$. $K(X_0) = K(x)$ i.e. rational functions in the indeterminate x. Let $f = x - a \in K(X_0)$. Then f corresponds to a map φ. Then $\mathrm{div}(f) = \varphi^*((0) - (\infty)) = (p) - (\infty)$.

Definition I.4.2. We say a divisor is **effective** if none of the coefficients are negative. Let $D \in \mathrm{Div}(X)$ and define

$$\mathscr{L}(D) := \{f \in K(X) : \mathrm{div}(f) + D \text{ effective}\} \cup \{0\}.$$

MM: [functions with poles no worse than those of D]

Theorem I.4.3 (Riemann-Roch)**.** [1] *There is an integer $g \geq 0$ and a divisor class $\mathscr{C}$ such that for all $C \in \mathscr{C}$ and $D \in \mathrm{Div}(X)$, we have*

$$\ell(D) - \ell(C - D) = \deg(D) - g + 1.$$

II. Examples

II.1. **General Setting.**

Question

Given a curve X and $D \in \mathrm{Div}(X)$ (D effective and even degree d), find $f \in K(X)^\times$ such that the function field

$$K(Y) = \frac{K(X)[y]}{(y^2 - f)}$$

corresponding to the curve Y has ramification exactly at D.

Notes

- $f \in K(X)^\times / K(X)^{\times 2}$.
- For an arbitrary D we can pick $D' \in \operatorname{Pic}(X)$ such that $D + 2D'$ has degree zero. Then

$$\begin{aligned} \operatorname{div}(f) &= D + 2D' \\ \operatorname{div}(fg^2) &= D + 2(D' + \operatorname{div}(g)). \end{aligned}$$

MM: [think of d distinct points not ∞]
Now we have to solve the following equation in $\operatorname{Pic}(X)$:

$$0 = [D] + 2[D'] \iff 2[D'] = -[D] = -d[\infty].$$

II.2. **Degree 2 to Degree 1.** Let $X_0 = \mathbb{P}^1$. Then $K(X_0) = K(x_0)$. Let $D = (0) - (\infty)$. Then if we let $f = x_0$, we have $\operatorname{div}(f) = D$. We can define a quadratic extension of $K(X_0)$ as follows:

$$K(X_1) = \frac{K(X_0)[x_1]}{(x_1^2 - f)}.$$

In the curves setting we have the curve $X_1 : x_1^2 - x_0$ and the degree 2 map

$$\varphi : X_1 \to X_0$$

defined by $x_0 \mapsto x_0^2$. Now where is φ ramified?

$$\begin{tikzcd}
 & K(X_1) = \dfrac{K(x_0)[x_1]}{(x_1^2 - f)} \arrow[dl, dash] \arrow[dd, dash] \\
R_1 = \dfrac{K[x_0, x_1]}{(x_1^2 - f)} \arrow[dd, dash] & \\
 & K(X_0) = K(x_0) \arrow[dl, dash] \\
R_0 = K[x_0] &
\end{tikzcd}$$

We can factor the ideal $I = fR_1$ to see that $I = \mathfrak{p}^2$ where $\mathfrak{p} = (x_0, x_1)$. Similarly, we can see that the ramification index at ∞ is 2, but the infinite prime is $(1/x_0, x_1/x_0)$.
MM: [what if we change the divisor D?]
Suppose instead we take $D = (0) - (1)$? Then $f = x_0/(x_0 - 1)$ has $\operatorname{div}(f) = D$. The diagram stays the same, but now the ideals $I_1 = x_0 R_1$ and $I_2 = (x_0 - 1)R_1$ factor as

$$\begin{aligned} I_1 = \mathfrak{p}_1^2 &= (x_0 - 1, (x_0 - 1)x_1)^2 \\ I_2 = \mathfrak{p}_2^2 &= (x_0, (x_0 - 1)x_1)^2. \end{aligned}$$

In summary, when $D = (0) - (\infty)$, the discriminant of R_1 is $4x_0$ whereas for $D = (0) - (1)$, the discriminant of R_1 is $4x_0^2 - 4x_0$.

II.3. **Degree 4 to Degree 2.** Now suppose we have the previous diagram:

$$
\begin{tikzcd}
 & K(X_1) = \dfrac{K(x_0)[x_1]}{(x_1^2 - x_0)} \arrow[dd, dash] \\
R_1 = \dfrac{K[x_0, x_1]}{(x_1^2 - x_0)} \arrow[ur, dash] \arrow[dd, dash] & \\
 & K(X_0) = K(x_0) \\
R_0 = K[x_0] \arrow[ur, dash] &
\end{tikzcd}
$$

The curve X_1 is given by the affine equation $x_1^2 - x_0 = 0$. Let $p_1 = (1, 1)$ and $p_{-1} = (1, -1)$ be points on X_1 and let $D = (p_1) - (p_{-1})$. Then we need to pick $f \in K(X_1)$ to extract a sqrt. Let

$$f = \frac{x_1 - 1}{x_1 + 1}$$

so that $\mathrm{div}(f) = D$. Then we can continue our diagram:

$$
\begin{tikzcd}
 & K(X_2) = \dfrac{K(X_1)[x_2]}{(x_2^2 - f)} \arrow[dd, dash] \\
R_2 \arrow[ur, dash] \arrow[dd, dash] & \\
 & K(X_1) = \dfrac{K(x_0)[x_1]}{(x_1^2 - x_0)} \arrow[dd, dash] \\
R_1 = \dfrac{K[x_0, x_1]}{(x_1^2 - x_0)} \arrow[ur, dash] \arrow[dd, dash] & \\
 & K(X_0) = K(x_0) \\
R_0 = K[x_0] \arrow[ur, dash] &
\end{tikzcd}
$$

As before, we can check that we have the desired ramification.

$$(x_1-1)R_2=\mathfrak{P}_1^2=\left(x_0-1,\frac{1}{2}x_1x_2+3x_0x_2-\frac{5}{2}x_2-\frac{9}{2}x_1+\frac{9}{2}\right)^2$$

$$(x_1+1)R_2=\mathfrak{P}_2^2=\left(x_0-1,\left(\left(\frac{13}{2}x_0-\frac{17}{2}\right)x_1+\left(\frac{3}{2}x_0^2+\frac{3}{2}x_0-5\right)\right)x_2+\left(-6x_0+2\right)x_1+\frac{3}{2}x_0-\frac{11}{2}\right)^2.$$

II.4. **Degree 8 to Degree 4.** like...

References

1. Michael Rosen, *Number theory in function fields*, vol. 210, Springer Science & Business Media, 2013.
2. Joseph H Silverman, *The arithmetic of elliptic curves*, vol. 106, Springer Science & Business Media, 2009.