FINITELY MAXIMAL FUCHSIAN GROUPS

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1. *Introduction*

By a Fuchsian group Γ we shall mean a finitely generated discrete subgroup of PSL(2, $\mathbb{R}$), the group of all conformal homeomorphisms of the upper half-plane U.

Greenberg [3] defines a Fuchsian group to be *finitely maximal* if there does not exist another Fuchsian group containing it with finite index. Let $R(\Gamma)$ denote the set of all isomorphisms $r : \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ with the property that $r(\Gamma)$ is discrete and r maps parabolic (hyperbolic boundary) elements to parabolic (hyperbolic boundary) elements. Greenberg showed that for some Fuchsian groups Γ there exists a Fuchsian group Γ_0 containing it with finite index, such that every $r \in R(\Gamma)$ is just the restriction to Γ of an $r_0 \in R(\Gamma_0)$. In this case $r(\Gamma)$ is never finitely maximal. (He also showed that for other groups Γ, $r(\Gamma)$ was " usually " finitely maximal). Greenberg determined some of these pairs Γ, Γ_0. In this paper we determine all such pairs.

2. *Signatures of Fuchsian groups*

It is known that every Fuchsian group has a presentation of the following form.

Generators:	$a_1, b_1, \ldots, a_g, b_g$.	(Hyperbolic)
	$x_1, x_2, \ldots, x_r$	(Elliptic)
	$p_1, \ldots, p_s$	(Parabolic)
	$h_1, \ldots, h_t$	(Hyperbolic Boundary elements)

Relations:

$$x_1^{m_1} = x_2^{m_2} = \ldots = x_r^{m_r} = \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \prod_{j=1}^{r} x_j \prod_{k=1}^{s} p_k \prod_{l=1}^{t} h_l = 1.$$

We then say that Γ has *signature*

$$(g;\ m_1, \ldots, m_r;\ s;\ t); \tag{1}$$

$m_1, \ldots m_r$ are integers ≥ 2 and are called the *periods* of Γ.

It is sometimes convenient to think of parabolic elements as being elliptic elements of infinite period. When this is the case we will write the signature (1) as

$$[g;\ m_1, \ldots, m_u;\ t], \tag{1'}$$

where $u = r+s$ and

$$m_{r+1} = \ldots = m_u = \infty .$$

For any Fuchsian group Γ we can define $L(\Gamma)$, the set of limit points of Γ [4; p. 86]. $L(\Gamma)$ is a subset of the real line of one of the following three types.

(a) $L(\Gamma)$ has at most two points;

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(b) $L(\Gamma) = \mathbb{R}$;

(c) $L(\Gamma)$ is a perfect nowhere-dense subset of $\mathbb{R}$.

Groups of type (a) will not interest us. They are cyclic or have signature

$$(0;\ 2, 2;\ 0;\ 1)$$

and, as Greenberg points out, are never finitely maximal. Groups of type (b) are called groups of the first kind and groups of type (c) of the second kind.

We now describe the Riemann–Hurwitz formula. For a group Γ with signature (1), define

$$M(\Gamma) = 2g-2+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)+s+t.$$

If $M(\Gamma) > 0$, there exists a Fuchsian group with signature (1) and if Γ is of the first kind $M(\Gamma) > 0$. If $\Gamma_1 \subseteq \Gamma$ is a subgroup of finite index, then

$$|\Gamma : \Gamma_1| = \frac{M(\Gamma_1)}{M(\Gamma)}.$$

(If $t = 0$ this follows from the fact that $2\pi M(\Gamma)$ is the hyperbolic measure of a fundamental region for Γ. If $t > 0$ it is a result of Maclachlan [7]. See Proposition 6.)

3. *The space* $T^{\#}(\Gamma)$

Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be a set of generators for Γ. We topologize $R(\Gamma)$ as a subset of $(\mathrm{PSL}(2, \mathbb{R}))^n$ by associating with each $r \in R(\Gamma)$ the point $(r(\gamma_1), r(\gamma_2), \ldots, r(\gamma_n))$.

We define two isomorphisms $r_1, r_2 \in R(\Gamma)$ to be equivalent (and write $r_1 \sim r_2$) if there exists an angle-preserving (i.e. conformal or anti-conformal) homeomorphism $\lambda : U \to U$ such that

$$\lambda r_1(\gamma)\,\lambda^{-1} = r_2(\gamma) \qquad \text{for all } \gamma \in \Gamma.$$

Define

$$T^{\#}(\Gamma) = R(\Gamma)/\sim$$

and give it the quotient topology.

It is known that if Γ is a group with signature (1) then $T^{\#}(\Gamma)$ is a cell of dimension $d(\Gamma)$, where

$$d(\Gamma) = 6g-6+2r+2s+3t$$

[1, 2].

$T^{\#}(\Gamma)$ is known as the reduced Teichmüller space and it coincides with the usual Teichmüller space if and only if Γ is of the first kind **[2]**.

In **[3]** Greenberg gives the following result. (Also see **[6]**).

PROPOSITION 1. *Let Γ_1, Γ_2 be two Fuchsian groups with $\alpha : \Gamma_1 \to \Gamma_2$ an injective homomorphism such that $|\Gamma_2 : \alpha(\Gamma_1)|$ is finite. Then α induces a map*

$$[r] \to [r \circ \alpha]$$

which is an embedding of $T^{\#}(\Gamma_2)$ into $T^{\#}(\Gamma_1)$ whose image is a closed subset of $T^{\#}(\Gamma_1)$.

(By $[r]$ we mean the equivalence class of r.)

We are interested in finding all pairs of groups Γ, Γ_0 such that $\Gamma \subseteq \Gamma_0$ with finite index and every $r \in R(\Gamma)$ is just the restriction to Γ_0 of an $r_0 \in R(\Gamma_0)$, (so that the embedding of Theorem 1 is a surjection). Clearly this property only depends on the signatures of Γ, Γ_0, as the isomorphisms in $R(\Gamma)$ preserve the signature. If σ is the signature of Γ and σ_0 is the signature of Γ_0, we shall write $\sigma \subseteq \sigma_0$ and if $\Gamma \lhd \Gamma_0$ we shall write $\sigma \lhd \sigma_0$. Hence our problem is to find all pairs of signatures such that $\sigma \subseteq \sigma_0$.

PROPOSITION 2. *$\sigma \subseteq \sigma_0$ if and only if there exist Fuchsian groups Γ, Γ_0 such that $\Gamma \subseteq \Gamma_0$ with finite index and $d(\Gamma) = d(\Gamma_0)$.*

Proof. This follows from Proposition 1.

4. *Subgroups of Fuchsian groups*

The following theorem was proved by the author in [8].

PROPOSITION 3. *Let Γ_2 be a Fuchsian group with signature* (1). *Then Γ_2 contains a subgroup Γ_1 of index N with signature*

$$(g';\ n_{11}, n_{12}, \ldots, n_{1\rho_1}, \ldots, n_{r1}, n_{r2}, \ldots n_{r\rho_r};\ s';\ t')$$

if and only if

(a) *there exists a finite permutation group G transitive on N points, and an epimorphism $\theta : \Gamma \to G$ satisfying the following conditions*:

(i) *the permutation $\theta(x_j)$ has precisely ρ_j cycles of lengths less than m_j, the length of these cycles being*

$$m_j/n_{j1}, \ldots, m_j/n_{j\rho_j};$$

(ii) *if we denote the number of cycles in the permutation $\theta(\gamma)$ by $\delta(\gamma)$, then*

$$s' = \sum_{k=1}^{s} \delta(p_k), \quad t' = \sum_{l=1}^{t} \delta(h_l);$$

(b) $M(\Gamma_1)/M(\Gamma_2) = N$.

Notes A. The periods n_{ji} in Γ_1 are powers of conjugates of x_j. The parabolic (hyperbolic boundary) generators of Γ_1 are powers of conjugates of the parabolic (hyperbolic boundary) generators of Γ_2.

B. We can still interpret the result if Γ_2 is written with signature (1′). In this case, a parabolic element is represented by an elliptic element x of period ∞. If $\theta(x)$ has μ cycles then this means, by (i), that there are μ parabolic generators conjugate to powers of x in Γ_1. The number of parabolic generators of Γ_1 is then found by (ii).

We can now deduce a known result in the case where $\Gamma_1 \lhd \Gamma_2$. We will assume Γ_2 to be of the first kind, so that $t = 0$, and write Γ_2 with signature (1′), omitting any reference to the t. Here we think of parabolic elements as being elliptic elements of infinite period, so that we might as well think of $x_1, x_2, \ldots, x_u$ as being the elliptic generators with $m_{r+1} = \ldots = m_u = \infty$. Let $\Gamma_1 \lhd \Gamma_2$ with index N. The *exponent* of x_i modulo Γ_1 is the least integer v_i such that $x_i^{v_i} \in \Gamma_1$. Clearly $v_i < \infty$ and $v_i | m_i$ if

$m_i < \infty$. Rearrange the periods so that x_i has exponent m_i modulo Γ_1 only for $0 \leqslant i \leqslant p$ and x_{i+p} has exponent $n_i < m_{i+p}$ otherwise. Then Γ_2 has signature

$$(g;\ m_1, \ldots, m_p, n_1\, t_1, \ldots, n_q\, t_q), \tag{2}$$

where $p+q = u$ and $1 < t_i \leqslant \infty$.

We now have

PROPOSITION 4. *Γ_1 has signature*

$$[g_1;\ t_1^{N/n_1}, \ldots, t_q^{N/n_q}] \tag{3}$$

(t_1^{N/n_1} means that the period t_1 occurs N/n_1 times).

Proof. Let the finite permutation group G of Proposition 3 be Γ_2/Γ_1 acting in its right regular representation and $\theta : \Gamma_2 \to G$ be the natural homomorphism. If $i \leqslant p$ then $\theta(x_i)$ has order m_i and so $\theta(x_i)$ is a product of N/m_i m_i cycles. $\theta(x_{i+p})$ has order n_i and so is a product of N/n_i n_i cycles. The result follows from Proposition 3 by observing that $n_i = m_{i+p}/t_i$ for $i \leqslant q$. (If a generator is parabolic the proof still applies by note B above).

We can now find all pairs of signatures σ, σ_0, representing groups of the first kind such that $\sigma \lhd \sigma_0$ with the finite index.

Let Γ, Γ_0 be groups with signatures σ, σ_0, respectively and such that $\Gamma \lhd \Gamma_0$ with index N. We may suppose that Γ_0 has signature (2) and Γ has signature (3). Then the Riemann–Hurwitz formula gives

$$2g_1 - 2 + \sum_{i=1}^{q} \frac{N}{n_i}\left(1 - \frac{1}{t_i}\right) = N\left(2g - 2 + \sum_{i=1}^{p}\left(1 - \frac{1}{m_i}\right) + \sum_{i=1}^{q}\left(1 - \frac{1}{t_i n_i}\right)\right).$$

Let

$$P = \sum_{i=1}^{p} \frac{1}{m_i},\ Q = \sum_{i=1}^{q} \frac{1}{n_i}.$$

Then the above equation reduces to

$$2g_1 - 2 = N(2g - 2 + p + q - P - Q). \tag{4}$$

The equation $d(\Gamma) = d(\Gamma_0)$ implies that

$$3g_1 + NQ = 3g + p + q. \tag{5}$$

Eliminating g_1 from (4) and (5) gives

$$(N-1)(6g-6) + (3N-2)(p+q) - 3NP - NQ = 0.$$

Now $P \leqslant p/2$, $Q \leqslant q$,
so that

$$(N-1)(6g-6) + (3N-2)(p+q) - \tfrac{3}{2}Np - Nq \leqslant 0. \tag{6}$$

As $N \geqslant 2$, $(N-1)(6g-6)$ is negative and so $g = 0$. (6) now implies that

$$(N-1)(3p+4q-12) \leqslant p$$

and so

$$3p + 4q - 12 \leqslant p.$$

Therefore

$$p+2q \leqslant 6.$$

Also, as $g = 0$ and Γ_0, being of the first kind, is non-cyclic, we must have

$$p+q \geqslant 3.$$

It is now a simple matter to determine all pairs of signatures σ, σ_0 such that $\sigma \lhd \sigma_0$. There are only ten possible cases which obey the restrictions on p and q. We will discuss two of these. First let us try $p = 4$, $q = 0$. Then $Q = 0$ and equation (5) implies that $3g_1 = 4$, an impossibility. Now take $p = 3, q = 1$. Equation (4) gives

$$2g_1-2 = N(2-P-Q).$$

Equation (5) gives

$$3g_1+NQ = 4$$

and so there are two possiblities:

(i) $g_1 = 1, NQ = 1$,

(ii) $g_1 = 0, NQ = 4$.

(i) implies that $P+Q = 2$ and as $Q \leqslant \frac{1}{2}$ we must have $m_1 = m_2 = m_3 = n_1 = 2$ and so $\sigma_0 = (0;\ 2, 2, 2, 2t)$, $\sigma = (1;\ t)$. The index is two.

(ii) implies that $N = 2/2-P \leqslant 4$. But as $NQ = 4$, $N = 4$, $Q = 1, P = 3/2$ and so

$$\sigma_0 = (0;\ 2, 2, 2, t),\ \sigma = (0;\ t, t, t, t)$$

and the index is 4.

It is easy to check that the subgroups constructed *are* normal subgroups just by constructing the appropriate homomorphism. It is by these means that the following result is obtained.

THEOREM 1. *The following is the complete list of signatures σ, σ_0, representing Fuchsian groups of the first kind for which $\sigma \lhd \sigma_0$.*

σ	σ_0	*index*
$(2;\ -)$	$(0;\ 2, 2, 2, 2, 2, 2)$	2
$[1;\ t, t]$	$[0;\ 2, 2, 2, 2, t]$	2
$[1;\ t]$	$[0;\ 2, 2, 2, 2t]$	2
$[0;\ t, t, t, t]$	$[0;\ 2, 2, 2, t]$	4
$[0;\ t_1, t_1, t_2, t_2]$	$[0;\ 2, 2, t_1, t_2]$	2
$[0;\ t, t, t]$	$[0;\ 3, 3, t]$	3
$[0;\ t, t, t]$	$[0;\ 2, 3, 2t]$	6
$[0;\ t_1, t_1, t_2]$	$[0;\ 2, t_1, 2t_2]$	2

(The above list of groups also appears in [6]. The signatures σ represent signatures of groups Γ for which the action of the Modular group on $T^{\#}(\Gamma)$ is not effective). The parameters t_1, t_2 are restricted so that $M(\Gamma) > 0$.

5. *Non-normal subgroups*

We now look for all signatures σ, σ_0 representing groups of the first kind for which $\sigma \subseteq \sigma_0$ with finite index and the inclusion is not normal.

PROPOSITION 5. *The signatures σ, σ_0 represent triangle groups, i.e. groups with signature* $[0;\ m_1, m_2, m_3]$.

The proof is straightforward, though tedious; the calculation uses Propositions 2 and 3. Note the following simple consequence of Proposition 3. If Γ_2 is a Fuchsian group with k 2's amongst its periods and if Γ_1 is a subgroup of odd index, then Γ_1 must have at least k 2's amongst its periods. This, and other similar consequences of Proposition 3 will be used below in the proof of Proposition 4. First we prove

LEMMA. *If* $\sigma_0 = [0;\ m_1, m_2, m_3, m_4]$ *and* $\sigma \subseteq \sigma_0$ *then* $\sigma \lhd \sigma_0$.

(Hence these groups appear in the list given in Theorem 1.)

Proof. As $d(\Gamma) = d(\Gamma_0) = 1$, σ is a signature of one of the two types

(a) $[0;\ n_1, n_2, n_3, n_4]$,

(b) $[1;\ n]$.

We just discuss (a), (b) following in a similar way. If we suppose that the inclusion is not normal then the index $N > 2$. If Γ is of type (a) then $1/6 \leqslant M(\Gamma) \leqslant 2$, the bottom inequality holding only for $[0;\ 2, 2, 2, 3]$ the top one only for $[0;\ \infty, \infty, \infty, \infty]$. As we are interested only in the case when $N \geqslant 3$, we may suppose that $M(\Gamma_0) < 2/3$. Then it is easy to check that Γ_0 must have at least one period 2. Now suppose Γ is of type (a) and that $\Gamma \subseteq \Gamma_0$ with index 3. Then by the argument above there must be at least one period 2 in Γ. Therefore $M(\Gamma) \leqslant 3/2$ and so $M(\Gamma_0) < \frac{1}{2}$. It now follows that there must be at least two periods 2 in Γ_0 and hence two periods 2 in Γ. Therefore $M(\Gamma) \leqslant 1$ and so $M(\Gamma_0) < \frac{1}{3}$ and it follows that Γ_0 has three periods 2 and so Γ has three periods 2. Therefore $M(\Gamma) \leqslant \frac{1}{2}$ and $M(\Gamma_0) < \frac{1}{6}$, a contradiction.

If we consider the case $N = 4$ in a similar way, we find that we are left with only one possibility, i.e., $(0;\ t, t, t, t) \subseteq (0;\ 2, 2, 2, t)$. This occurred in Theorem 1. We can deduce that the inclusion must be normal because the only possible permutation group acting transitively on 4 points which $(0;\ 2, 2, 2, t)$ can be mapped onto homomorphically, is a group acting in its regular representation, (*viz* $Z_2 \times Z_2$). The cases where $N > 4$ are disposed of similarly.

LEMMA. *Let* Γ_0 *have signature* $\sigma_0 = [g;\ m_1, \ldots, m_r]$ *and* Γ *have signature* $\sigma = [g_1;\ n_1, \ldots, n_s]$. *If* $\sigma \subseteq \sigma_0$ *then* $\sigma \lhd \sigma_0$ *or* σ, σ_0 *represent triangle groups.*

Proof. We have two equations,

$$3g + r = 3g_1 + s \tag{7}$$

and

$$N\left(2g - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right) = 2g_1 - 2 + \sum_{i=1}^{s}\left(1 - \frac{1}{n_i}\right). \tag{8}$$

Eliminating g_1 from (7) and (8), we obtain

$$(N-1)(2g-2) = \tfrac{2}{3}r+\tfrac{1}{3}s - \sum_{i=1}^{s} \frac{1}{n_i} - N \sum_{i=1}^{r} \left(1-\frac{1}{m_i}\right). \tag{9}$$

Eliminating g from (7) and (8), we obtain

$$(N-1)(2g_1-2) = s - \sum_{i=1}^{s} \frac{1}{n_i} - \tfrac{2}{3}N(s-r) - N \sum_{i=1}^{r} \left(1-\frac{1}{m_i}\right). \tag{10}$$

Now suppose $s \leqslant r$. Then (9), together with the inequality

$$\sum_{i=1}^{r} \left(1-\frac{1}{m_i}\right) \geqslant \frac{r}{2},$$

implies that

$$(N-1)(2g-2) \leqslant r - \frac{Nr}{2}.$$

If $N = 2$ then the inclusion is normal; so suppose that $N > 2$. Then $g = 0$ and if we assume $r > 4$ then

$$2 < N \leqslant \frac{2r-4}{r-4}. \tag{11}$$

(If $r = 3$ then σ_0 and hence σ represent triangle groups. $r = 4$ has been covered by the previous lemma). (11) implies that $r = 5, 6, 7$ or 8 and these can be dismissed separately as it is impossible to satisfy (7) and (8) in these cases.

Now suppose $r < s$. Then (10) implies that

$$(N-1)(2g_1-2) < s - \frac{Ns}{2}.$$

Arguing as before we see that $g_1 = 0$, $s \leqslant 7$. Again, these are dismissed separately.

6. *Triangle groups*

Note that a non-cyclic Fuchsian group Γ is a triangle group if and only if $d(\Gamma) = 0$. It follows from Proposition 1 that the only Fuchsian groups which contain triangle groups are triangle groups and from Proposition 2 that all we need do is find all inclusion relationships amongst triangle groups. We shall denote the triangle group $[0;\ l, m, n]$ by $[l, m, n]$.

THEOREM 2. *The following is the complete list of signatures σ, σ_0, representing triangle groups for which $\sigma \subseteq \sigma_0$ with finite index and the inclusion is not normal, (the normal case being given in Theorem* 1).

	σ	σ_0	*Index*		σ	σ_0	*Index*
A.	$[7, 7, 7]$	$[2, 3, 7]$	24	G.	$[4, 4, 5]$	$[2, 4, 5]$	6
B.	$[2, 7, 7]$	$[2, 3, 7]$	9	H.	$[n, 4n, 4n]$	$[2, 3, 4n]$	6
C.	$[3, 3, 7]$	$[2, 3, 7]$	8	I.	$[n, 2n, 2n]$	$[2, 4, 2n]$	4
D.	$[4, 8, 8]$	$[2, 3, 8]$	12	J.	$[3, n, 3n]$	$[2, 3, 3n]$	4
E.	$[3, 8, 8]$	$[2, 3, 8]$	10	K.	$[2, n, 2n]$	$[2, 3, 2n]$	3
F.	$[9, 9, 9]$	$[2, 3, 9]$	12				

Outline of Proof. As may be expected, the proof involves a large number of calculations, so we just outline the method. If $\sigma_0 = [l, m, n]$, $\sigma = [l_1, m_1, n_1]$ then by the Riemann–Hurwitz formula [see § 2] the index N is given by

$$N = \frac{1 - \dfrac{1}{l_1} - \dfrac{1}{m_1} - \dfrac{1}{n_1}}{1 - \dfrac{1}{l} - \dfrac{1}{m} - \dfrac{1}{n}}. \tag{12}$$

Thus, one criterion is that the right-hand side of (12) should be an integer > 2. Another is that the periods of σ have to divide the periods of σ_0. If both these criteria are satisfied, we then have to decide whether the appropriate permutation group of Proposition 3 exists.

As can be seen from the statement of the theorem, the only candidates for σ_0 which turn up are groups of the type $[2, 3, m]$ or $[2, 4, m]$. It is easy to see why this is so. For suppose $\sigma_0 = [2, 5, m]$, $m \geqslant 5$. Then the triangle subgroup of largest index would have signature $[m, m, m]$ and then the index by (12) would be

$$N = \frac{10(m-3)}{3m-10} \leqslant 4.$$

In fact, the only possibility with integral m, N is $m = 5$, $N = 4$ (as $N > 2$). But if $[5, 5, 5] \subseteq [2, 5, 5]$ there would have to be a homomorphic image of $[2, 5, 5]$ acting transitively on 4 points (Proposition 3). An element of order 5 would then fix all 4 points and this would imply, by Proposition 3, that there would be 4 periods 5 in σ, a contradiction. (We should also consider the possibility of $[n, n, n] \subseteq [3, 3, n]$ with index 3, but this has already occurred in Theorem 1. Every such inclusion is normal, as the only permutation group possible is Z_3 in its regular representation).

Thus, we only need consider $\sigma_0 = [2, 3, m]$ or $[2, 4, m]$ and determine whether the above two criteria on subgroups are satisfied. Then we have to look for the appropriate permutation group. Some cases are quickly dismissed. For example, let $\sigma_1 = [2, 3, 8]$, $\sigma_2 = [8, 8, 8]$. If $\sigma_2 \subseteq \sigma_1$ then the index, by (12), would be 15 and so the element of order 2 in the permutation group would fix at least one point. But then, by Proposition 3, there would have to be a period 2 in the subgroup. Other cases are more difficult to dismiss. For example, $[3, 7, 7] \not\subseteq [2, 3, 7]$ with index 16. Trial and error shows that there is no homomorphic image of $[2, 3, 7]$ acting transitively on 16 points.

We now exhibit all the permutation groups in the cases A–K of the theorem. If Γ_0 has signature $[l, m, n]$ and presentation

$$\{x, y \mid x^l = y^m = (xy)^n = 1\},$$

we need only write down the permutations $\theta(x)$, $\theta(y)$ and $\theta(xy)$, where θ is the homomorphism of Proposition 3.

	$\theta(x)$	$\theta(y)$	$\theta(xy)$
A	(1, 22), (2, 7), (3, 11), (4, 17), (5, 20), (6, 12), (9, 14), (13, 19), (10, 18), (8, 23), (15, 24), (16, 21).	(1, 22, 2), (3, 12, 7), (4, 18, 11), (8, 23, 9), (5, 21, 17), (6, 13, 20), (10, 19, 14), (15, 24, 16).	(1, 2, ..., 7), (8, 9, ..., 14), (15, ..., 21), (22), (23), (24).
B	(1, 3), (2, 8), (7, 5), (6, 9), (4).	(1, 4, 5), (2, 8, 3), (7, 6, 9).	(1, 2, ..., 7), (8), (9).
C	(1, 8), (2, 7), (3, 4), (5, 6).	(1, 8, 2), (3, 5, 7), (4), (6).	(1, ..., 7), (8).
D	(12, 5), (4, 6), (3, 9), (2, 8), (1, 11), (10, 7).	(12, 6, 5), (11, 2, 1). (9, 4, 7), (10, 8, 3).	(1, ..., 8), (9, 10), (11), (12).
E	(7, 9), (10, 4), (8, 6), (3, 5), (1, 2).	(8, 7, 9), (5, 4, 10), (1, 3, 6), (2).	(1, ..., 8), (9), (10).
F	(1, 10), (6, 8), (11, 4) (3, 5), (12, 7), (2, 9).	(1, 10, 2), (4, 11, 5), (7, 12, 8), (9, 3, 6).	(1, ..., 9), (10), (11), (12).
G	(1, 6), (2, 3), (4, 5).	(1, 6, 2, 4), (3), (5).	(1, ..., 5), (6).
H	(1, 3), (4, 5), (2, 6).	(1, 4, 5), (2, 6, 3).	(1, ..., 4), (5), (6).
I	(1, 3), (2, 4).	(1, 3, 2, 4).	(1, 2), (3), (4).
J	(1, 4), (2, 3).	(1, 4, 2), (3).	(1, 2, 3), (4).
K	(1, 3), (2).	(1, 3, 2).	(1, 2), (3).

7. *Groups of the second kind*

To deal with groups of the second kind we use the following result of Maclachlan [7].

PROPOSITION 6. *For each group Γ with signature*

$$(g;\ m_1,\ m_2,\ \ldots,\ m_r;\ s;\ t) \qquad (t > 0)$$

there exists a group $\tilde{\Gamma}$ of the first kind with signature

$$(2g+t-1;\ m_1, m_1, \ldots, m_r, m_r;\ 2s;\ 0)$$

such that $\Gamma_1 \subseteq \Gamma_2$ with index N implies $\tilde{\Gamma}_1 \subseteq \tilde{\Gamma}_2$ with index N.

Note that $d(\tilde{\Gamma}) = 2d(\Gamma)$, so that $d(\Gamma) = d(\Gamma_0)$ if and only if $d(\tilde{\Gamma}) = d(\tilde{\Gamma}_0)$. Therefore, if σ, σ_0, $\tilde{\sigma}$, $\tilde{\sigma}_0$ are the signatures of Γ, Γ_0, $\tilde{\Gamma}$, $\tilde{\Gamma}_0$, then $\sigma \subseteq \sigma_0$ implies $\tilde{\sigma} \subseteq \tilde{\sigma}_0$ and hence $\tilde{\sigma}$, $\tilde{\sigma}_0$ will appear in Theorems 1 and 2. The periods in $\tilde{\sigma}$, $\tilde{\sigma}_0$ will appear an even number of times, so that the only possibilities are

$$\tilde{\sigma} = (2;\ -;\ -),\ \tilde{\sigma}_0 = (0;\ 2, 2, 2, 2, 2, 2;\ -;\ -)$$

or

$$\tilde{\sigma} = [0;\ t_1, t_1, t_1, t_1, -],\ \tilde{\sigma}_0 = [0;\ 2, 2, t_1, t_1;\ -].$$

We therefore deduce

THEOREM 3. *The following are the only signatures σ, σ_0 defining groups of the second kind for which $\sigma \subseteq \sigma_0$*

σ	σ_0	*Index*
$(1;\ -;\ -;\ 1)$	$(0;\ 2, 2, 2;\ -;\ 1)$	2
$[0;\ t_1, t_1;\ 1]$	$[0;\ 2, t_1;\ 1]$	2

(The fact that these subgroups actually occur is a simple consequence of Proposition 3.)

The list of pairs σ, σ_0 which appear in Theorems 1, 2, 3, together with certain pairs of groups which have two limit points or less, form the complete list of pairs σ, σ_0 for which $\sigma \subseteq \sigma_0$.

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