

2-group Belyi maps

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Motivation



Galois representations

Let X be an irreducible, smooth projective algebraic curve of genus $g \geq 1$ over a number field K . Let $G_K := \text{Gal}(K^{\text{al}} | K)$ be the absolute Galois group of K and let $\ell \in \mathbb{Z}$ be prime.

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The geometry of X and the arithmetic of ρ are intimately related. For example, if X has good reduction at a prime \mathfrak{p} above $p \neq \ell$, then \mathfrak{p} will be unramified in the **ℓ -torsion field** $K(J[\ell])$.

Belyi's theorem

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Theorem (Belyi 1979)

An algebraic curve (smooth projective) X over \mathbb{C} can be defined over a number field if and only if X admits a Belyi map.

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We can now state Beckmann's theorem to relate Belyi maps to our original motivation.

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Theorem (Beckmann 1989)

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Then there exists a number field M satisfying the following properties.

- *p is unramified in M*
- *ϕ is defined over M*
- *X is defined over M*
- *X has good reduction at all primes \mathfrak{p} of M above p*

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Conjecture (Gross 1998)

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The hope is that an explicit nonsolvable field ramified only at 2 can be obtained as $K(\text{Jac}(X)[2])$ where X is the domain of a Galois Belyi map with monodromy group a 2-group.

We call these Belyi maps **2-group Belyi maps**.

Main results

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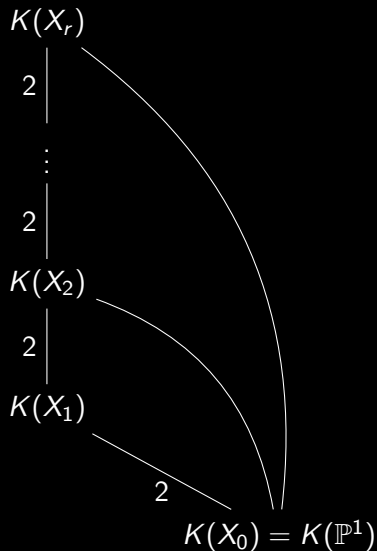
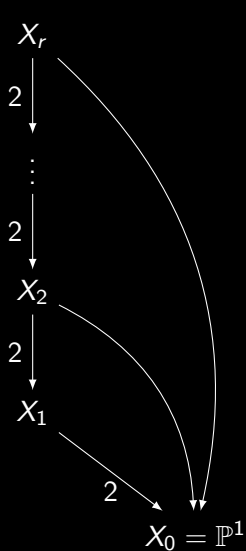
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Motivated by the applications of 2-group Belyi maps to arithmetic geometry, we now state the main results.

- implementation of an algorithm to enumerate isomorphism classes of 2-group Belyi maps
- implementation of an algorithm to compute equations for 2-group Belyi maps over finite fields
- implementation of a *method* to compute equations for 2-group Belyi maps over number fields
- computational and theoretical evidence supporting a conjecture that every 2-group Belyi map is defined over an abelian extension of the rationals

2-group Belyi maps as iterated quadratic extensions



Background



Isomorphism of Belyi maps

Let $\phi: X \rightarrow \mathbb{P}^1$ and $\phi': X' \rightarrow \mathbb{P}^1$ be Belyi maps of degree d .

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Let $\phi: X \rightarrow \mathbb{P}^1$ and $\phi': X' \rightarrow \mathbb{P}^1$ be Belyi maps of degree d . ϕ and ϕ' are **isomorphic** (respectively **lax isomorphic**) if the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\sim} & X' \\ & \searrow \phi & \swarrow \phi' \\ & \mathbb{P}^1 & \end{array}, \text{ respectively } \begin{array}{ccc} X & \xrightarrow{\sim} & X' \\ \phi \downarrow & & \downarrow \phi' \\ \mathbb{P}^1 & \xrightarrow[\beta]{\sim} & \mathbb{P}^1 \end{array}$$

commute where $\beta(\{0, 1, \infty\}) = \{0, 1, \infty\}$.

Permutation Triples

A **transitive permutation triple of degree d** is a triple

$$\sigma = (\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3$$

such that

- $\sigma_\infty \sigma_1 \sigma_0 = 1$
- σ generates a transitive subgroup of S_d

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The set of degree d Belyi maps up to isomorphism is in bijection with the set of degree d transitive permutation triples up to **simultaneous conjugation** and the group $\langle \sigma \rangle$ is the monodromy group of ϕ .

Passports

A **passport** \mathcal{P} consists of the data (g, G, λ) where $g \geq 0$ is an integer, $G \leq S_d$ is a transitive subgroup, and $\lambda = (\lambda_0, \lambda_1, \lambda_\infty)$ is a triple of partitions of d .

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The **passport of a Belyi map** $\phi : X \rightarrow \mathbb{P}^1$ is $(g(X), \text{Mon}(\phi), (\lambda_0, \lambda_1, \lambda_\infty))$ with $g(X)$ the genus of X , $\text{Mon}(\phi)$ the monodromy group of ϕ , and the partitions from ramification.

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The **passport of a permutation triple** σ is $(g(\sigma), \langle \sigma \rangle, \lambda(\sigma))$ where

$$g(\sigma) = 1 - d + (e(\sigma_0) - e(\sigma_1) - e(\sigma_\infty))/2$$

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$$e(\tau) = d - \#\text{cycles of } \tau,$$

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We now discuss the importance of organizing triples by passport. 12/48

Fields of moduli, fields of definition, and passports

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The situation improves, however, in the Galois setting...

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Then

- ϕ and X are defined over $M(\phi)$,
- $\#G = d$,
- all cycles of σ_s have the same length for $s \in \{0, 1, \infty\}$,
- and if we let a, b, c be the orders of $\sigma_0, \sigma_1, \sigma_\infty$ respectively, we have

$$g(X) = 1 + \frac{\#G}{2} \left(1 - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} \right).$$

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The monodromy group in this setting corresponds to field automorphisms of the Galois closure of $K(X)$ fixing $K(x)$.

Computing permutation triples



Setup

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- $\sigma_\infty \sigma_1 \sigma_0 = \text{id}$;
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We say two degree d 2-group permutation triples σ, σ' are **simultaneously conjugate** if there exists $\tau \in S_d$ such that

$$\sigma^\tau := (\tau^{-1} \sigma_0 \tau, \tau^{-1} \sigma_1 \tau, \tau^{-1} \sigma_\infty \tau) = \sigma'$$

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For a 2-group permutation triple σ , we denote the set of lifts of σ by $\text{Lifts}(\sigma)$ and $\text{Lifts}(\sigma)/\sim$ denotes the set of lifts up to simultaneous conjugation.

Algorithm to compute $\text{Lifts}(\sigma)/\sim$

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2. For each $f \in H^2(G, A)$ compute the corresponding extension

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2. For each $f \in H^2(G, A)$ compute the corresponding extension

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Choosing

$$\alpha := (1\ d+1)(2\ d+2) \dots (d-1\ 2d-1)(d\ 2d)$$

allows us to label blocks by reducing modulo d .

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Lastly, we quotient by simultaneous conjugation to obtain

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Bipartite graphs of permutation triples

Now that we can lift permutation triples, we now describe some notation for the bipartite graphs that organize these triples.

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For $i \in \mathbb{Z}_{\geq 1}$ we define the bipartite graph denoted \mathcal{G}_{2^i} with the following node sets.

- $\mathcal{G}_{2^i}^{\text{above}}$: the set of isomorphism classes of 2-group Belyi maps of degree 2^i indexed by 2-group permutation triples $\tilde{\sigma}$ up to simultaneous conjugation in S_{2^i}
- $\mathcal{G}_{2^i}^{\text{below}}$: the set of isomorphism classes of 2-group Belyi maps of degree 2^{i-1} indexed by 2-group permutation triples σ up to simultaneous conjugation in $S_{2^{i-1}}$

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For every pair of nodes $(\tilde{\sigma}, \sigma) \in \mathcal{G}_{2^i}^{\text{above}} \times \mathcal{G}_{2^i}^{\text{below}}$ there is an edge between σ and $\tilde{\sigma}$ if and only if $\tilde{\sigma}$ is simultaneously conjugate to a lift of σ .

Algorithm to compute \mathcal{G}_{2^i}

Input: The bipartite graph $\mathcal{G}_{2^{i-1}}$

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4. For every pair $(\tilde{\sigma}, \sigma) \in \mathcal{G}_{2^i}^{\text{above}} \times \mathcal{G}_{2^i}^{\text{below}}$ place an edge between $\tilde{\sigma}$ and σ if and only if there is a triple in the equivalence class $[\tilde{\sigma}] \in \text{Lifts}(\mathcal{G}_{2^{i-1}})/\sim$ that is a lift of σ

Results : number of triples and passports

Theorem (M.)

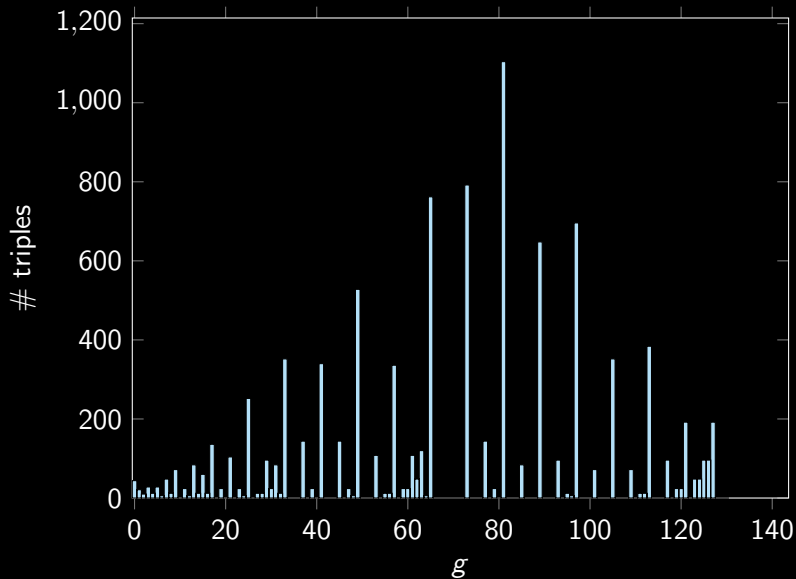
The following tables list the number of isomorphism classes of 2-group Belyi maps, the number of passports, and number of lax passports respectively up to degree 256.

d	1	2	4	8	16	32	64	128	256
# triples	1	3	7	19	55	151	503	1799	7175

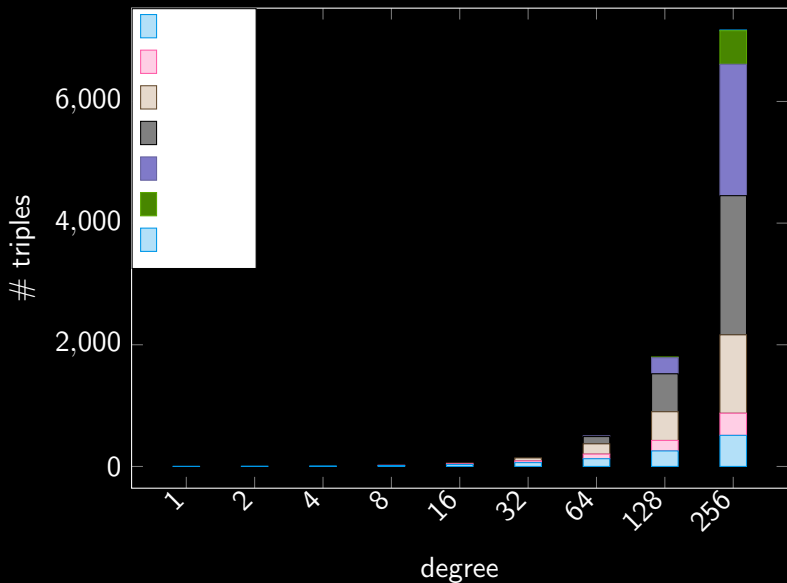
d	1	2	4	8	16	32	64	128	256
# passports	1	3	7	16	41	96	267	834	2893

d	1	2	4	8	16	32	64	128	256
# lax passports	1	1	3	6	14	31	85	257	882

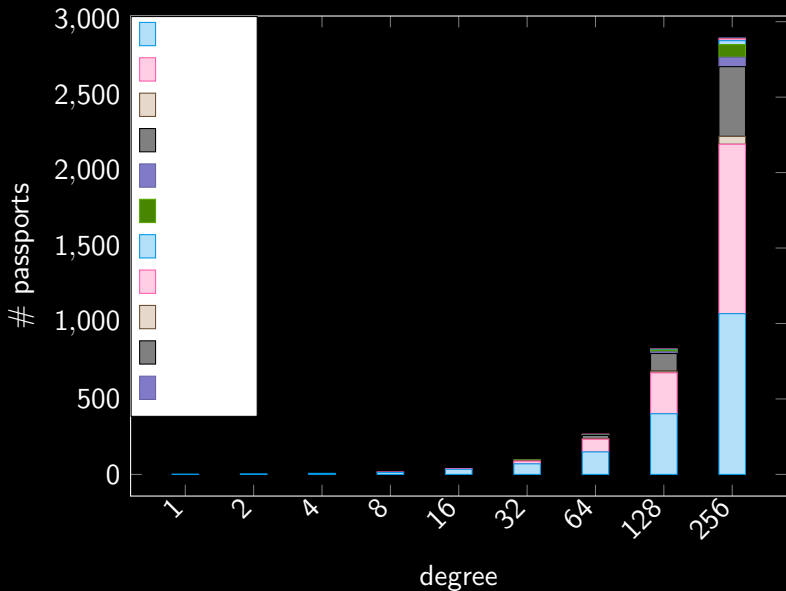
Results : distribution of genera



Results : groups by nilpotency class



Results : passport sizes



A refined conjecture



Passports

Recall that a passport \mathcal{P} consists of the data (g, G, λ) where $g \in \mathbb{Z}_{\geq 0}$, G is a transitive subgroup of S_d and $\lambda = (\lambda_0, \lambda_1, \lambda_\infty)$ is a triple of partitions of d corresponding to conjugacy classes (C_0, C_1, C_∞) of S_d .

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The size of \mathcal{P} is the cardinality of the set $\Sigma_{\mathcal{P}}$ defined by

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To instead analyze $\text{Gal}(\mathbb{Q}^{\text{al}} | \mathbb{Q}^{\text{ab}})$ we *refine* the notion of a passport.

Refined passports

A **refined passport** \mathcal{P} consists of the data (g, G, c) where $g \in \mathbb{Z}_{\geq 0}$, G is a transitive subgroup of S_d and $c = (c_0, c_1, c_\infty)$ is a triple of conjugacy classes of G .

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As was the case with passport, every permutation triple σ determines a refined passport $\mathcal{P}(\sigma)$.

A refined conjecture

Theorem (M.)

The size of $\mathcal{P}(\sigma)$ is equal to 1 for every 2-group permutation triple σ with degree ≤ 256 .

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Theorem (M.)

ARC is true for 2-group permutation triples σ with $\langle \sigma \rangle$ dihedral.

Computing equations



A motivating example : setup

Let F be a number field with integers \mathbb{Z}_F . Let $\text{Pl}(F)$ denote the places of F and S_∞ the archimedean places. For $v \in \text{Pl}(F) \setminus S_\infty$ let \mathfrak{p}_v denote the prime ideal of \mathbb{Z}_F corresponding to v .

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Question

How do we construct a quadratic extension of F with ramification prescribed by \mathfrak{a} ?

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How do we construct a quadratic extension of F with ramification prescribed by \mathfrak{a} ?

First, it is possible that no such extension exists.

A motivating example : setup

Let F be a number field with integers \mathbb{Z}_F . Let $\text{Pl}(F)$ denote the places of F and S_∞ the archimedean places. For $v \in \text{Pl}(F) \setminus S_\infty$ let \mathfrak{p}_v denote the prime ideal of \mathbb{Z}_F corresponding to v .

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If \mathfrak{a} is not principal, then the question requires more care.

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Let Cl_F denote the class group of F and suppose $\mathfrak{ab}^2 = (d)$.

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Let Cl_F denote the class group of F and suppose $ab^2 = (d)$.

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To summarize, in the case where \mathfrak{a} is not principal but there exists \mathfrak{b} with $\mathfrak{a}\mathfrak{b}^2$ principal we have $[\mathfrak{a}] \in \text{Cl}_F^2$ and $[\mathfrak{b}]$ is unique up to multiplication by $[\mathfrak{c}] \in \text{Cl}_F[2]$.

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The algorithms in this section rely on transporting this technique to the function field setting.

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The set of places of F is denoted $\text{Pl}(F)$ and the **degree** of P is the index $[\mathcal{O}_P/P : K]$ of the **residue class field**.

Algebraic function fields : Picard group and $\mathcal{L}(D)$

The **divisor class group** $\text{Div}(F)$ of F is the free abelian group generated by the places of F . A **divisor** $D \in \text{Div}(F)$ is represented by a sum of places $\sum_P a_P P$ and the **degree** of D is $\sum_P a_P \deg(P)$. The set of **degree zero divisors** is denoted $\text{Div}^0(F)$.

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The image of $\text{div}: F^\times \rightarrow \text{Div}(F)$ defined by $\text{div}(f) = \sum_P \text{ord}_P(f) P$ is the subgroup of **principal divisors** of F denoted $\text{Princ}(F)$.

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The **Riemann-Roch space** of a divisor $D \in \text{Div}(F)$ is defined by $\mathcal{L}(D) := \{f \in F : \text{div}(f) + D \geq 0\} \cup \{0\}$.

Algebraic function fields : quadratic extensions

Lemma

Let $aF^{\times 2}$ be a nontrivial coset of $F^{\times}/F^{\times 2}$ and consider the extension $L := F(\sqrt{a})$. Then a prime P of F is ramified in L if and only if $\text{ord}_P(a)$ is odd.

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As in the number field setting, this implies $R \in 2\text{Pic}(F)$ and D is unique up to addition by $T \in \text{Pic}^0(F)[2]$.

Algorithm in characteristic $p \geq 3$: setup

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Picard groups are implemented in the tame case.

Algorithm in characteristic $p \geq 3$: Galois test

Input:

- F a Galois extension of $\mathbb{F}_q(x)$
- $\text{Gal}(F | \mathbb{F}_q(x))$ explicitly given as automorphisms of F
- $a \in F$

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- For each generator $\sigma \in \text{Gal}(F | \mathbb{F}_q(x))$ test if $\sigma(a)/a$ is a square in F
- Return True if $\sigma(a)/a$ is a square in F for all generators σ and otherwise return False

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Similarly, we can apply the same test after extending the constant field from \mathbb{F}_q to \mathbb{F}_{q^2} .

Algorithm in characteristic $p \geq 3$: get candidates

Input:

- F a 2-group Belyi map modulo q of degree $d = 2^m$ corresponding to a 2-group permutation triple σ
- A passport $\mathcal{P} = (\tilde{G}, (a, b, c))$ with \tilde{G} a 2-group of order $2d$ such that there exists a 2-group permutation triple $\tilde{\sigma}$ with passport \mathcal{P} that is a lift of σ
- $\text{Gal}(F | \mathbb{F}_q(x)) \cong \langle \sigma \rangle$ explicitly given as automorphisms of F

Output: A list of candidate functions $\{f_i\}$ with each $f_i \in F$ such that $F(\sqrt{f_i})$ is a 2-group Belyi map modulo q with passport \mathcal{P} .

Algorithm in characteristic $p \geq 3$: get candidates (steps 1-4)

1. For $s \in \{0, 1, \infty\}$ compute

$$r_s := \begin{cases} 0 & \text{if } \text{order}(\sigma_s) = \text{order}(\tilde{\sigma}_s) \\ 1 & \text{if } \text{order}(\sigma_s) < \text{order}(\tilde{\sigma}_s) \end{cases}$$

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2. Compute

$$R := \sum_{s \in \{0, 1, \infty\}} r_s R_s \in \text{Div}(F)$$

where R_0, R_1, R_∞ are defined to be the supports of $\text{div}(x)$, $\text{div}(x - 1)$, and $\text{div}(1/x)$ respectively.

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4. Compute $[R] := \psi^{-1}(R)$.

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 - (c) If $\mathcal{L}(R - 2D_a)$ has dimension 1, then compute $f_a \in F$ with $\text{div}(f_a)$ generating $\mathcal{L}(R - 2D_a)$ and go to Step 5d Otherwise go to the next $a \in \text{Pic}(F)[2]$.

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 - (d) Apply Galois test to F , $\text{Gal}(F | \mathbb{F}_q(x))$, and f_a from Step 5c to see if $F(\sqrt{f_a})$ generates a Galois extension. If $F(\sqrt{f_a})$ is Galois over $\mathbb{F}_q(x)$ then save f_a and go to the next $a \in \text{Pic}(F)[2]$. If $F(\sqrt{f_a})$ is not Galois over $\mathbb{F}_q(x)$, then go to Step 5e.

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 - (e) Let F' be the function field F after extending the field of constants \mathbb{F}_q to \mathbb{F}_{q^2} . Apply Galois test to F' , $\text{Gal}(F' | \mathbb{F}_{q^2}(x))$, and f_a (viewed as an element of F') from Step 5c to see if $F'(\sqrt{f_a})$ generates a Galois extension. If $F'(\sqrt{f_a})$ is Galois over $\mathbb{F}_{q^2}(x)$ then save f_a . Go to the next $a \in \text{Pic}(F)[2]$.

Algorithm in characteristic $p \geq 3$: get candidates (steps 6-8)

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8. Return the list S''

Algorithm in characteristic $p \geq 3$: compute entire passport

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However, one issue with this technique is that it only guarantees that the resulting Belyi map has the desired *passport* and does not allow us to control which *isomorphism class* we get.

Algorithm in characteristic $p \geq 3$: compute entire passport

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Since we know the sizes of passports from our work with permutation triples, we know that we have representatives from every isomorphism class even if we cannot match the Belyi maps to their corresponding permutation triples.

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Although this implementation does not allow us to compute all 2-group Belyi maps for a given degree, it does work well in practice.

<https://github.com/michaelmusty/2GroupDessins>

- *all* 2-group Belyi maps modulo 3 up to degree 32
- hundreds of 2-group Belyi maps up to degree 256

Examples



Notation

DNG-a, b, c-gE-H

D : degree in $\{2, 4, 8, 16, 32, 64, 128, 256\}$

N : either T or S identifying group database

G : a positive integer identifying the group

a : ramification index of 0 in $\{2, 4, 8, 16, 32, 64, 128, 256\}$

b : ramification index of 1 in $\{2, 4, 8, 16, 32, 64, 128, 256\}$

c : ramification index of ∞ in $\{2, 4, 8, 16, 32, 64, 128, 256\}$

g : just the letter g

E : the genus in $\mathbb{Z}_{\geq 0}$

H : the hash of the 2-group permutation triple a positive integer

An interesting example

<https://michaelmusty.github.io/d3ssins>

Future work

- higher degree over \mathbb{F}_3
- an algorithm in characteristic zero
- prove ARC for other families of 2-groups
- p -group Belyi maps for p odd
- compute torsion fields

Backup slides