2-**group Belyi maps**

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Outline

Motivation

Galois representations

Let X be an irreducible, smooth projective algebraic curve of genus $g \geq 1$ over a number field K. Let $G_K := \mathrm{Gal}(K^{\mathrm{al}} \mid K)$ be the absolute Galois group of K and let $\ell \in \mathbb{Z}$ be prime.

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The geometry of X and the arithmetic of ρ are inimately related.

For example, if X has good reduction at a prime $\mathfrak{p}$ above $p \neq \ell$, then $\mathfrak{p}$ will be unramified in the ℓ-**torsion field** $K(J[\ell])$.

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Theorem (Belyi 1979)

An algebraic curve (smooth projective) X over $\mathbb{C}$ can be defined over a number field if and only if X admits a Belyi map.

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We can now state Beckmann's theorem to relate Belyi maps to our original motivation.

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Let $\phi\colon X \to \mathbb{P}^1$ be a Galois Belyi map with monodromy group G. Let p be a prime not dividing $\#G$.

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Let p be a prime not dividing $\#G$.
Then there exists a number field M satisfying the following properties.

- *p is unramified in M*
- *ϕ is defined over M*
- *X is defined over M*
- *X has good reduction at all primes $\mathfrak{p}$ of M above p*

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Conjecture (Gross 1998)

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$p = 7$: existence (Dieulefait)

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The hope is that an explicit nonsolvable field ramified only at 2 can be obtained as $K(\mathrm{Jac}(X)[2])$ where X is the domain of a Galois Belyi map with monodromy group a 2-group.
We call these Belyi maps 2-**group Belyi maps**.

Main results

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Motivated by the applications of 2-group Belyi maps to arithmetic geometry, we now state the main results.

- implementation of an algorithm to enumerate isomorphism classes of 2-group Belyi maps
- implementation of an algorithm to compute equations for 2-group Belyi maps over finite fields
- implementation of a *method* to compute equations for 2-group Belyi maps over number fields
- computational and theoretical evidence supporting a conjecture that every 2-group Belyi map is defined over an abelian extension of the rationals

2-group Belyi maps as iterated quadratic extensions

$$\begin{tikzcd}
X_r \arrow[d, "2"'] \arrow[ddddr, bend left] & \\
\vdots \arrow[d, "2"'] & \\
X_2 \arrow[d, "2"'] \arrow[ddr, bend left] & \\
X_1 \arrow[dr, "2"'] & \\
& X_0 = \mathbb{P}^1
\end{tikzcd}
\qquad
\begin{tikzcd}
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Background

Isomorphism of Belyi maps

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Let $\phi\colon X \to \mathbb{P}^1$ and $\phi'\colon X' \to \mathbb{P}^1$ be Belyi maps of degree d. ϕ and ϕ' are **isomorphic** (respectively **lax isomorphic**) if the diagrams

$$\begin{tikzcd} X \arrow[rr, "\sim"] \arrow[dr, "\phi"'] & & X' \arrow[dl, "\phi'"] \\ & \mathbb{P}^1 & \end{tikzcd}, \text{ respectively } \begin{tikzcd} X \arrow[r, "\sim"] \arrow[d, "\phi"'] & X' \arrow[d, "\phi'"] \\ \mathbb{P}^1 \arrow[r, "\sim", "\beta"'] & \mathbb{P}^1 \end{tikzcd}$$

commute where $\beta(\{0, 1, \infty\}) = \{0, 1, \infty\}$.

Permutation Triples

A **transitive permutation triple of degree** d is a triple

$$\sigma = (\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3$$

such that

- $\sigma_\infty \sigma_1 \sigma_0 = 1$
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The set of degree d Belyi maps up to isomorphism is in bijection with the set of degree d transitive permutation triples up to **simultaneous conjugation** and the group $\langle \sigma \rangle$ is the monodromy group of ϕ.

Passports

A **passport** $\mathcal{P}$ consists of the data (g, G, λ) where $g \geq 0$ is an integer, $G \leq S_d$ is a transitive subgroup, and $\lambda = (\lambda_0, \lambda_1, \lambda_\infty)$ is a triple of partitions of d.

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The **passport of a Belyi map** $\phi : X \to \mathbb{P}^1$ is $(g(X), \mathrm{Mon}(\phi), (\lambda_0, \lambda_1, \lambda_\infty))$ with $g(X)$ the genus of X, $\mathrm{Mon}(\phi)$ the monodromy group of ϕ, and the partitions from ramification.

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The **passport of a permutation triple** σ is $(g(\sigma), \langle\sigma\rangle, \lambda(\sigma))$ where

$$g(\sigma) = 1 - d + (e(\sigma_0) - e(\sigma_1) - e(\sigma_\infty))/2$$

with

$$e(\tau) = d - \#\text{cycles of } \tau,$$

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We now discuss the importance of organizing triples by passport.

Fields of moduli, fields of definition, and passports

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The situation improves, however, in the Galois setting...

The Galois setting

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Then

- ϕ and X are defined over $M(\phi)$,
- $\#G = d$,
- all cycles of σ_s have the same length for $s \in \{0, 1, \infty\}$,
- and if we let a, b, c be the orders of $\sigma_0, \sigma_1, \sigma_\infty$ respectively, we have

$$g(X) = 1 + \frac{\#G}{2}\left(1 - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}\right).$$

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The monodromy group in this setting corresponds to field automorphisms of the Galois closure of $K(X)$ fixing $K(x)$.

Computing permutation triples

Setup

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- $\sigma_\infty \sigma_1 \sigma_0 = \mathrm{id}$;
- $G := \langle \sigma_0, \sigma_1 \rangle$ is a transitive subgroup of S_d; and
- G is a 2-group of order d embedded in S_d via its left regular representation.

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We say two degree d 2-group permutation triples σ, σ' are **simultaneously conjugate** if there exists $\tau \in S_d$ such that

$$\sigma^\tau := (\tau^{-1}\sigma_0\tau, \tau^{-1}\sigma_1\tau, \tau^{-1}\sigma_\infty\tau,) = \sigma'$$

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A **lift** of σ is a 2-group permutation triple $\widetilde{\sigma} \in S_{2d}^3$ such that $\langle \widetilde{\sigma} \rangle$ is isomorphic to some extension $\widetilde{G}$ of $\mathbb{Z}/2\mathbb{Z}$ by G as in the exact sequence below.

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For a 2-group permutation triple σ, we denote the set of lifts of σ by $\mathrm{Lifts}(\sigma)$ and $\mathrm{Lifts}(\sigma)/\sim$ denotes the set of lifts up to simultaneous conjugation.

Algorithm to compute Lifts(σ)/$\sim$

Input: σ a 2-group permutation triple of degree *d*
Output: Lifts(σ)/$\sim$

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$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "\iota_1"] & \widetilde{G}_1 \arrow[r, "\pi_1"] & G \arrow[r] & 1 \\ 1 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "\iota_2"] & \widetilde{G}_2 \arrow[r, "\pi_2"] & G \arrow[r] & 1 \end{tikzcd}$$

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Choosing

$$\alpha := (1\ d+1)(2\ d+2)\ldots(d-1\ 2d-1)(d\ 2d)$$

allows us to label blocks by reducing modulo d.

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Lastly, we quotient by simultaneous conjugation to obtain
$\mathrm{Lifts}(\sigma)/\!\sim\, = \Big\{((1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)),$
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Bipartite graphs of permutation triples

Now that we can lift permutation triples, we now describe some notation for the bipartite graphs that organize these triples.

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For $i \in \mathbb{Z}_{\geq 1}$ we define the bipartite graph denoted $\mathscr{G}_{2^i}$ with the following node sets.

- $\mathscr{G}_{2^i}^{\text{above}}$: the set of isomorphism classes of 2-group Belyi maps of degree 2^i indexed by 2-group permutation triples $\widetilde{\sigma}$ up to simultaneous conjugation in S_{2^i}
- $\mathscr{G}_{2^i}^{\text{below}}$: the set of isomorphism classes of 2-group Belyi maps of degree 2^{i-1} indexed by 2-group permutation triples σ up to simultaneous conjugation in $S_{2^{i-1}}$

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For every pair of nodes $(\widetilde{\sigma}, \sigma) \in \mathscr{G}_{2^i}^{\text{above}} \times \mathscr{G}_{2^i}^{\text{below}}$ there is an edge between σ and $\widetilde{\sigma}$ if and only if $\widetilde{\sigma}$ is simultaneously conjugate to a lift of σ.

Algorithm to compute $\mathscr{G}_{2^i}$

Input: The bipartite graph $\mathscr{G}_{2^{i-1}}$
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4. For every pair $(\widetilde{\sigma}, \sigma) \in \mathscr{G}_{2^i}^{\mathrm{above}} \times \mathscr{G}_{2^i}^{\mathrm{below}}$ place an edge between $\widetilde{\sigma}$ and σ if and only if there is a triple in the equivalence class $[\widetilde{\sigma}] \in \mathrm{Lifts}(\mathscr{G}_{2^{i-1}})/\sim$ that is a lift of σ

Results : number of triples and passports

Theorem (M.)

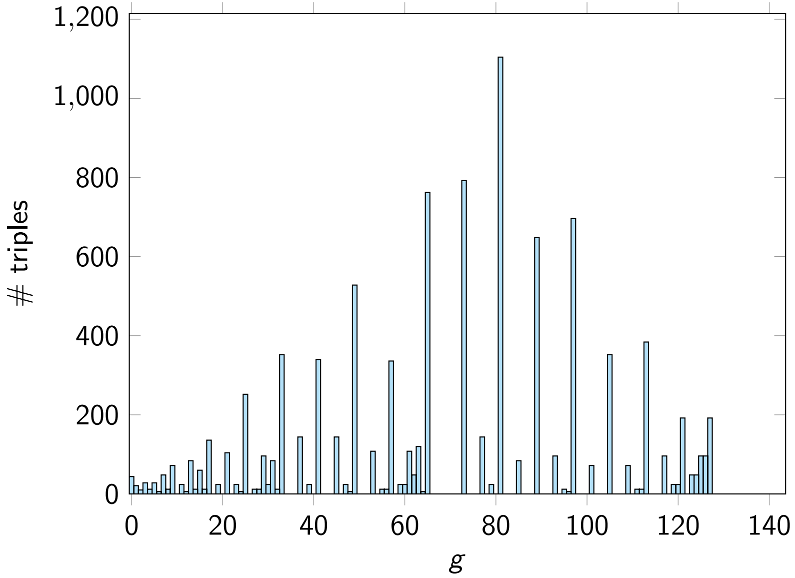
The following tables list the number of isomorphism classes of 2-group Belyi maps, the number of passports, and number of lax passports respectively up to degree 256*.*

d	1	2	4	8	16	32	64	128	256
# *triples*	1	3	7	19	55	151	503	1799	7175

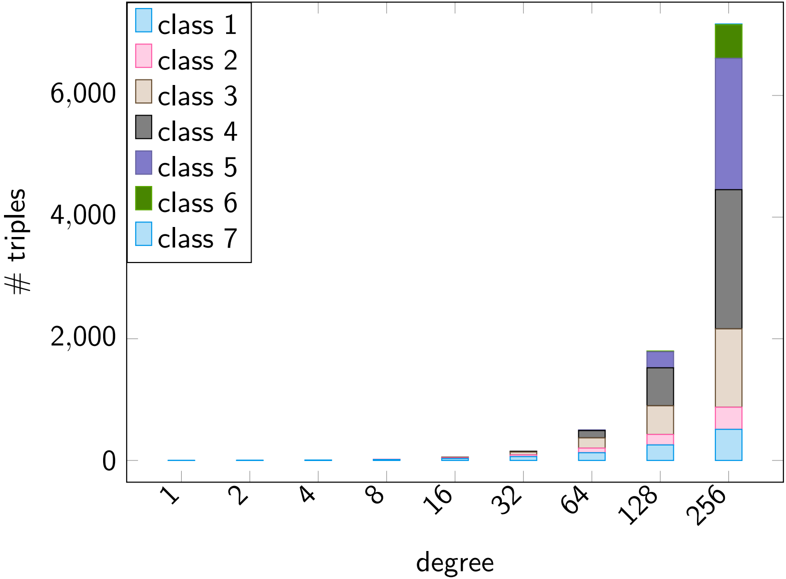
d	1	2	4	8	16	32	64	128	256
# *passports*	1	3	7	16	41	96	267	834	2893

d	1	2	4	8	16	32	64	128	256
# *lax passports*	1	1	3	6	14	31	85	257	882

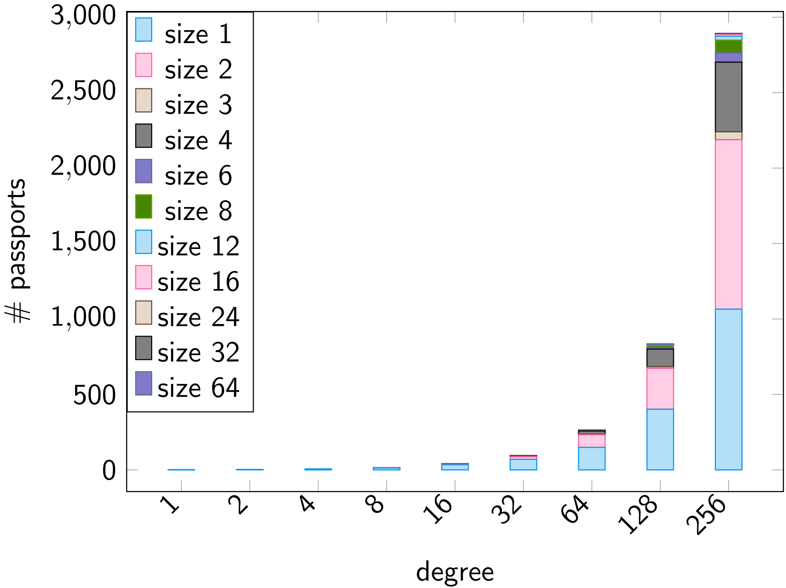
Results : distribution of genera



Results : groups by nilpotency class



Results : passport sizes



A refined conjecture

Passports

Recall that a passport $\mathcal{P}$ consists of the data (g, G, λ) where $g \in \mathbb{Z}_{\geq 0}$, G is a transitive subgroup of S_d and $\lambda = (\lambda_0, \lambda_1, \lambda_\infty)$ is a triple of partitions of d corresponding to conjugacy classes (C_0, C_1, C_∞) of S_d.

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The size of $\mathcal{P}$ is the cardinality of the set $\Sigma_{\mathcal{P}}$ defined by

$$\Big\{(\sigma_0, \sigma_1, \sigma_\infty) \in C_0 \times C_1 \times C_\infty : \sigma_\infty \sigma_1 \sigma_0 = 1 \text{ and } \langle \sigma_0, \sigma_1 \rangle = G\Big\}/\sim$$

where $\sim$ denotes simultaneous conjugation in S_d.

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As a result of the action of $G_{\mathbb{Q}}$ on $\mathcal{P}$, the size of $\mathcal{P}$ bounds the degree of the field of moduli of any Belyi map with passport $\mathcal{P}$.

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To instead analyze $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}} \,|\, \mathbb{Q}^{\mathrm{ab}})$ we *refine* the notion of a passport.

Refined passports

A **refined passport** $\mathscr{P}$ consists of the data (g, G, c) where $g \in \mathbb{Z}_{\geq 0}$, G is a transitive subgroup of S_d and $c = (c_0, c_1, c_\infty)$ is a triple of conjugacy classes of G.

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where $(\sigma_0, \sigma_1, \sigma_\infty) \sim (\sigma_0', \sigma_1', \sigma_\infty')$ if there exists $\alpha \in \mathrm{Aut}(G)$ with $\alpha(\sigma_s) = \sigma_s'$ for every $s \in \{0, 1, \infty\}$.

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As was the case with passport, every permutation triple σ determines a refined passport $\mathscr{P}(\sigma)$.

A refined conjecture

Theorem (M.)

The size of $\mathscr{P}(\sigma)$ is equal to 1 for every 2-group permutation triple σ with degree ≤ 256.

A refined conjecture

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The size of $\mathscr{P}(\sigma)$ *is equal to* 1 *for every 2-group permutation triple* σ *with degree* ≤ 256.

Conjecture (ARC)

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The size of $\mathscr{P}(\sigma)$ *is equal to* 1 *for every* 2*-group permutation triple* σ *with degree* ≤ 256*.*

Conjecture (ARC)

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Theorem (M.)

ARC is true for 2*-group permutation triples* σ *with* $\langle\sigma\rangle$ *dihedral.*

Computing equations

A motivating example : setup

Let F be a number field with integers $\mathbb{Z}_F$. Let $\mathrm{Pl}(F)$ denote the places of F and S_∞ the archimedean places. For $v \in \mathrm{Pl}(F) \setminus S_\infty$ let $\mathfrak{p}_v$ denote the prime ideal of $\mathbb{Z}_F$ corresponding to v.

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Let $S \subset \mathrm{Pl}(F) \setminus S_\infty$ and let $\mathfrak{a} := \prod_{v \in S} \mathfrak{p}_v$.

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Let $S \subset \mathrm{Pl}(F) \setminus S_\infty$ and let $\mathfrak{a} := \prod_{v \in S} \mathfrak{p}_v$.

Question

How do we construct a quadratic extension of F with ramification prescribed by $\mathfrak{a}$?

A motivating example : setup

Let F be a number field with integers $\mathbb{Z}_F$. Let $\mathrm{Pl}(F)$ denote the places of F and S_∞ the archimedean places. For $v \in \mathrm{Pl}(F) \setminus S_\infty$ let $\mathfrak{p}_v$ denote the prime ideal of $\mathbb{Z}_F$ corresponding to v.

Let $S \subset \mathrm{Pl}(F) \setminus S_\infty$ and let $\mathfrak{a} := \prod_{v \in S} \mathfrak{p}_v$.

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If $\mathfrak{a}$ is not principal, then the question requires more care.

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To summarize, in the case where $\mathfrak{a}$ is not principal but there exists $\mathfrak{b}$ with $\mathfrak{a}\mathfrak{b}^2$ principal we have $[\mathfrak{a}] \in \mathrm{Cl}_F^2$ and $[\mathfrak{b}]$ is unique up to multiplication by $[\mathfrak{c}] \in \mathrm{Cl}_F[2]$.

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Given $\mathfrak{a}$ encoding ramification data, we want to find $\mathfrak{b}^2$ and d such that $\mathfrak{a}\mathfrak{b}^2 = (d)$.

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The algorithms in this section rely on transporting this technique to the function field setting.

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As an example, let X be an irreducible affine plane curve (possibly singular) defined by the equation $f(x, y) = 0$ with $f \in K[x, y]$. Then the **function field of** X, denoted $K(X)$ is the field of fractions of the coordinate ring $K[x, y]/(f(x, y))$ of X.

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The set of places of F is denoted $\mathrm{Pl}(F)$ and the **degree** of P is the index $[\mathcal{O}_P/P : K]$ of the **residue class field**.

The **divisor class group** $\mathrm{Div}(F)$ of F is the free abelian group generated by the places of F. A **divisor** $D \in \mathrm{Div}(F)$ is represented by a sum of places $\sum_P a_P P$ and the **degree** of D is $\sum_P a_P \deg(P)$. The set of **degree zero divisors** is denoted $\mathrm{Div}^0(F)$.

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The image of $\mathrm{div}\colon F^\times \to \mathrm{Div}(F)$ defined by $\mathrm{div}(f) = \sum_P \mathrm{ord}_P(f)P$ is the subgroup of **principal divisors** of F denoted $\mathrm{Princ}(F)$.

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The **Riemann-Roch space** of a divisor $D \in \mathrm{Div}(F)$ is defined by $\mathscr{L}(D) := \{f \in F : \mathrm{div}(f) + D \geq 0\} \cup \{0\}$.

Lemma

Let $aF^{\times 2}$ be a nontrivial coset of $F^{\times}/F^{\times 2}$ and consider the extension $L := F(\sqrt{a})$. Then a prime P of F is ramified in L if and only if $\operatorname{ord}_P(a)$ is odd.

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The simple cases are when no such a exists or when R is principal. The last case occurs when there exists D such that $R - 2D = \operatorname{div}(a)$ for some $a \in F$.

As in the number field setting, this implies $R \in 2\operatorname{Pic}(F)$ and D is unique up to addition by $T \in \operatorname{Pic}^0(F)[2]$.

Algorithm in characteristic $p \geq 3$: setup

Let F be a function field with field of constants $\mathbb{F}_q$ with $q = p^r$ and p an odd prime. Let $\mathbb{F}_q(x)$ denote the rational function field in the variable x.

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A 2-**group Belyi map modulo** q is a Galois extension of function fields $\mathbb{F}_q(x) \hookrightarrow F$ with $[F : \mathbb{F}_q(x)]$ a power of 2 unramified outside $\{0, 1, \infty\}$.

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Picard groups are implemented in the tame case.

Algorithm in characteristic $p \geq 3$: Galois test

Input:

- F a Galois extension of $\mathbb{F}_q(x)$
- $\mathrm{Gal}(F \,|\, \mathbb{F}_q(x))$ explicitly given as automorphisms of F
- $a \in F$

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- For each generator $\sigma \in \mathrm{Gal}(F \mid \mathbb{F}_q(x))$ test if $\sigma(a)/a$ is a square in F
- Return True if $\sigma(a)/a$ is a square in F for all generators σ and otherwise return False

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Similarly, we can apply the same test after extending the constant field from $\mathbb{F}_q$ to $\mathbb{F}_{q^2}$.

Input:

- F a 2-group Belyi map modulo q of degree $d = 2^m$ corresponding to a 2-group permutation triple σ
- A passport $\mathcal{P} = (\widetilde{G}, (a, b, c))$ with $\widetilde{G}$ a 2-group of order $2d$ such that there exists a 2-group permutation triple $\widetilde{\sigma}$ with passport $\mathcal{P}$ that is a lift of σ
- $\mathrm{Gal}(F \,|\, \mathbb{F}_q(x)) \cong \langle \sigma \rangle$ explicitly given as automorphisms of F

Output: A list of candidate functions $\{f_i\}$ with each $f_i \in F$ such that $F(\sqrt{f_i})$ is a 2-group Belyi map modulo q with passport $\mathcal{P}$.

Algorithm in characteristic $p \geq 3$: get candidates (steps 1-4)

1. For $s \in \{0, 1, \infty\}$ compute

$$r_s := \begin{cases} 0 & \text{if } \operatorname{order}(\sigma_s) = \operatorname{order}(\widetilde{\sigma}_s) \\ 1 & \text{if } \operatorname{order}(\sigma_s) < \operatorname{order}(\widetilde{\sigma}_s) \end{cases}$$

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2. Compute

$$R := \sum_{s \in \{0,1,\infty\}} r_s R_s \in \operatorname{Div}(F)$$

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4. Compute $[R] := \psi^{-1}(R)$.

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   (d) Apply Galois test to F, $\mathrm{Gal}(F \mid \mathbb{F}_q(x))$, and f_a from Step 5c to see if $F(\sqrt{f_a})$ generates a Galois extension. If $F(\sqrt{f_a})$ is Galois over $\mathbb{F}_q(x)$ then save f_a and go to the next $a \in \mathrm{Pic}(F)[2]$. If $F(\sqrt{f_a})$ is not Galois over $\mathbb{F}_q(x)$, then go to Step 5e.

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   (e) Let F' be the function field F after extending the field of constants $\mathbb{F}_q$ to $\mathbb{F}_{q^2}$. Apply Galois test to F', $\mathrm{Gal}(F' \mid \mathbb{F}_{q^2}(x))$, and f_a (viewed as an element of F') from Step 5c to see if $F'(\sqrt{f_a})$ generates a Galois extension. If $F(\sqrt{f_a})$ is Galois over $\mathbb{F}_{q^2}(x)$ then save f_a. Go to the next $a \in \mathrm{Pic}(F)[2]$.

Algorithm in characteristic $p \geq 3$: get candidates (steps 6-8)

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$$G_a \cong \mathrm{Gal}(F(\sqrt{f_a}) \,|\, \mathbb{F}_q(x)),$$

   and let $S'' = \{f_a \in S : G_a \cong \widetilde{G}\}$.

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8. Return the list S''

Algorithm in characteristic $p \geq 3$: compute entire passport

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Since we know the sizes of passports from our work with permutation triples, we know that we have representatives from every isomorphism class even if we cannot match the Belyi maps to their corresponding permutation triples.

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However, we do have access to the ramification points of the Belyi maps and instead use combinations of these points to try to build a candidate function.

Although this implementation does not allow us to compute all 2-group Belyi maps for a given degree, it does work well in practice.

Results

https://github.com/michaelmusty/2GroupDessins

- *all* 2-group Belyi maps modulo 3 up to degree 32
- hundreds of 2-group Belyi maps up to degree 256

Examples

Notation

`DNG-a,b,c-gE-H`

- `D` : degree in $\{2, 4, 8, 16, 32, 64, 128, 256\}$
- `N` : either `T` or `S` identifying group database
- `G` : a positive integer identifying the group
- `a` : ramification index of 0 in $\{2, 4, 8, 16, 32, 64, 128, 256\}$
- `b` : ramification index of 1 in $\{2, 4, 8, 16, 32, 64, 128, 256\}$
- `c` : ramification index of ∞ in $\{2, 4, 8, 16, 32, 64, 128, 256\}$
- `g` : just the letter g
- `E` : the genus in $\mathbb{Z}_{\geq 0}$
- `H` : the hash of the 2-group permutation triple a positive integer

An interesting example

https://michaelmusty.github.io/d3ssins

Future work

- higher degree over $\mathbb{F}_3$
- an algorithm in characteristic zero
- prove ARC for other families of 2-groups
- p-group Belyi maps for p odd
- compute torsion fields

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Backup slides