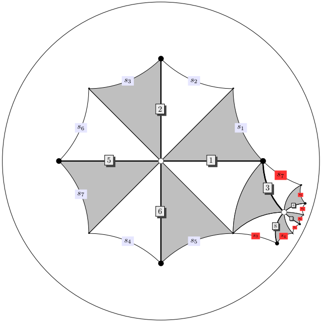
2-solvable Belyĭ maps



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Algebra and Number Theory Seminar
Dartmouth College

May 8, 2018

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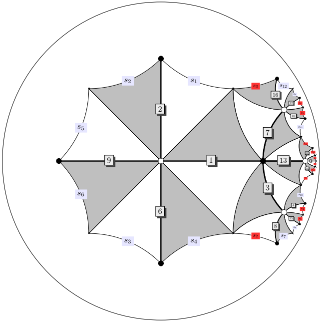

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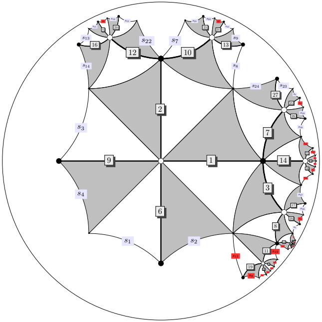

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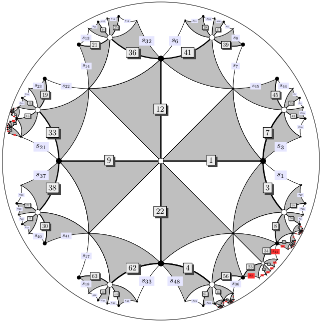

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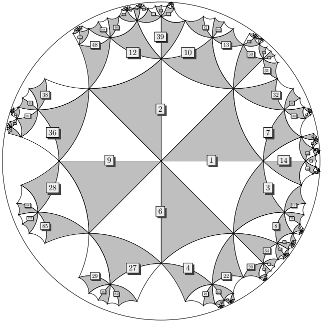

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1. What is a 2-solvable Belyĭ map?
2. Motivation: Beckmann's Theorem
3. An algorithm to compute 2-solvable Belyĭ maps
   (a) Computing permutation triples
   (b) Computing equations
4. Examples
5. Application: Number fields obtained from 2-torsion points

Theorem (G.V. Belyĭ 1979)

A smooth projective curve X over $\mathbb{C}$ can be defined over $\overline{\mathbb{Q}}$ if and only if there exists a branched covering of compact connected Riemann surfaces $\phi : X \to \mathbb{P}^1$ unramified (unbranched) above $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

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Two Belyĭ maps $\phi : X \to \mathbb{P}^1$ and $\phi' : X' \to \mathbb{P}^1$ are **isomorphic** if there is an isomorphism $\iota : X \to X'$ such that $\phi' \iota = \phi$.

Passports of Belyĭ maps

A **passport** $\mathcal{P}$ consists of the data (g, G, λ) where $g \geq 0$ is an integer, $G \leq S_d$ is a transitive subgroup, and $\lambda = (\lambda_0, \lambda_1, \lambda_\infty)$ is a triple of partitions of d.

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The passport of a Belyĭ map $\phi : X \to \mathbb{P}^1$ is $(g(X), \mathrm{Mon}(\phi), (\lambda_0, \lambda_1, \lambda_\infty))$ with $g(X)$ the genus of X, $\mathrm{Mon}(\phi)$ the monodromy group of ϕ, and the partitions specified by ramification.

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Passports of permutation triples

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Two such triples σ and σ' are **simultaneously conjugate** if there exists $\tau \in S_d$ with

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The **size** of a passport $\mathcal{P}$ is the number of simultaneous conjugacy classes of transitive permutation triples with passport $\mathcal{P}$.

A group-theoretic description of Belyĭ maps

Lemma

The set of transitive permutation triples of degree d up to simultaneous conjugation is in bijection with the set of Belyĭ maps of degree d up to isomorphism.

$\left\{ \begin{array}{l} \text{Transitive permutation} \\ \text{Triples } (\sigma_0, \sigma_1, \sigma_\infty) \in \\ S_d^3 \text{ with } \sigma_\infty \sigma_1 \sigma_0 = 1 \end{array} \right\}$

$\left\{ \begin{array}{l} \text{Index } d \text{ triangle sub-} \\ \text{groups } \Gamma \leq \Delta(a, b, c) \end{array} \right\}$

$\left\{ \begin{array}{l} \text{dessins} \\ \text{with } d \\ \text{edges} \end{array} \right\}$

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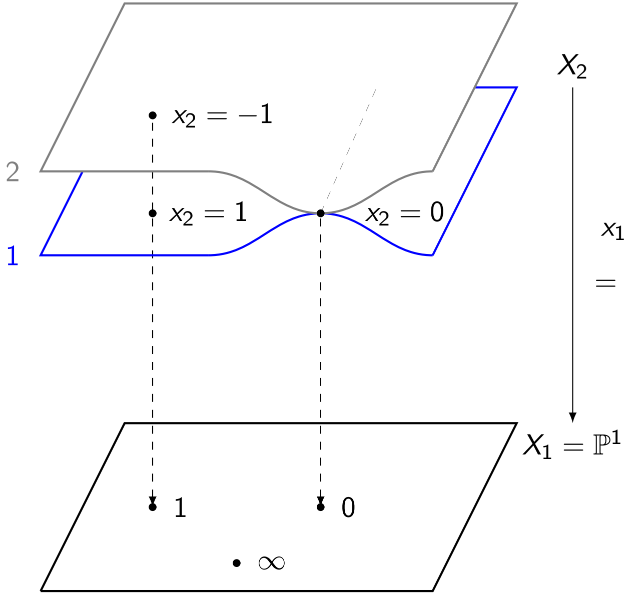
All up to the appropriate version of equivalence in each category.

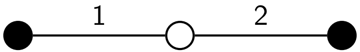


$$x_1 = \phi(x_2) = x_2^2$$
$$= (x_2+1)(x_2-1)+1$$

Example 2T1-2,1,2-g0

$\big((1\,2),(1)(2),(1\,2)\big)$

$[\Delta(2,\infty,2):\Gamma]=2$



$\mathbb{C}(x_2^2)\subseteq\mathbb{C}(x_2)$

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2-solvable (Galois) Belyĭ maps

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$$
\begin{tikzcd}
X_r \arrow[d, "2"'] \arrow[ddddr, bend left] & \\
\vdots \arrow[d, "2"'] & \\
X_3 \arrow[d, "2"'] \arrow[ddr, bend left] & \\
X_2 \arrow[dr, "2"'] & \\
& X_1 = \mathbb{P}^1
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
K(X_r) \arrow[d, dash, "2"'] \arrow[ddddr, dash, bend left] & \\
\vdots \arrow[d, dash, "2"'] & \\
K(X_3) \arrow[d, dash, "2"'] \arrow[ddr, dash, bend left] & \\
K(X_2) \arrow[dr, dash, "2"'] & \\
& K(X_1) = K(\mathbb{P}^1)
\end{tikzcd}
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Theorem (Beckmann-Kazez 1989)

Let $\phi : X \to \mathbb{P}^1$ be a Belyĭ map with monodromy group G.

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Upshot:

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Upshot: Every 2-solvable Belyĭ curve has a model with good reduction away from $p = 2$.

Let $\phi : X \to \mathbb{P}^1$ be a Belyĭ map of degree $d = 2^\ell$ corresponding to $\sigma \in S_d^3$.

Computing 2-solvable permutation triples

Let $\phi : X \to \mathbb{P}^1$ be a Belyĭ map of degree $d = 2^\ell$ corresponding to $\sigma \in S_d^3$. We want to find $\widetilde{\phi} : \widetilde{X} \to \mathbb{P}^1$ corresponding to $\widetilde{\sigma} \in S_{2d}^3$ such that

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For each extension, we get 8 possible $\widetilde{\sigma}$. We then check the necessary conditions and do some bookkeeping.

Passport counts

degree	2	4	8	16	32	64	128
# genus 0 passports	3	4	6	6	6	6	6
# genus 1 passports		3	3	3	3	3	3
# genus 2 passports			4	6	0	0	0
# genus 3 passports			3	8	12	0	0
# genus 4 passports				6	6	0	0
# genus 5 passports				6	8	12	0
# genus 6 passports				3	0	0	0
# genus 7 passports				3	18	12	0
# genus 8 passports					6	6	0
# genus 9 passports					15	18	24
# genus 11 passports					7	12	0
# genus 12 passports					3	0	0
# genus 13 passports					6	30	12
# genus 14 passports					3	0	0
# genus 15 passports					3	18	12
# genus 16 passports						6	6

Passport counts

degree	2	4	8	16	32	64	128
# genus 17 passports						39	25
# genus 19 passports						18	0
# genus 21 passports						30	48
# genus 23 passports						9	12
# genus 24 passports						3	0
# genus 25 passports						24	78
# genus 27 passports						6	0
# genus 28 passports						3	0
# genus 29 passports						6	30
# genus 30 passports						3	0
# genus 31 passports						3	18
# genus 32 passports							6
# genus 33 passports							117
# genus 37 passports							114
# genus 39 passports							18
# genus 41 passports							93

Passport counts

degree	2	4	8	16	32	64	128
# genus 45 passports							48
# genus 47 passports							9
# genus 48 passports							3
# genus 49 passports							72
# genus 53 passports							26
# genus 55 passports							6
# genus 56 passports							3
# genus 57 passports							24
# genus 59 passports							6
# genus 60 passports							3
# genus 61 passports							6
# genus 62 passports							3
# genus 63 passports							3
total passports	3	7	16	41	96	267	834

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$$\begin{tikzcd}
\widetilde{X} \arrow[dd, "\widetilde{\phi}"'] \arrow[dr, "\psi"] & \\
& X \arrow[dl, "\phi"] \\
\mathbb{P}^1 &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\frac{\overline{K}(X)[y]}{(y^2 - f)} \arrow[dd, dash] \arrow[dr, dash, "2"] & \\
& \overline{K}(X) \arrow[dl, dash, "d"] \\
\overline{K}(\mathbb{P}^1) &
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with ψ (and hence $\widetilde{\phi}$) satisfying the ramification conditions imposed by $\widetilde{\sigma}$.

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4T1-4,2,4-g1

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(0,0) \arrow[d, no head, Rightarrow] & (1,1) \arrow[dr, no head] & & (1,-1) \arrow[dl, no head] & \infty \arrow[d, no head, Rightarrow] & X_2 : x_2^2 = x_1 \arrow[dr] & \\
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X_3 : \substack{x_2^2=x_1 \\ x_3^2=x_2^3-x_2} \arrow[d] \arrow[ddr, bend left] & \\
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Passport: `8T1-8,4,8-g3`, size 2
Belyĭ curve: $X : y^2 + (x^4 + 1)y = -2x^4$
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Nonhyperelliptic Belyĭ maps

`128S1-128,32,128-g62` $\to$ `64S1-64,16,64-g30` $\to$ `32S1-32,8,32-g14` $\to$ `16T1-16,4,16-g6` $\to$ `8T1-8,2,8-g2` $\to$ `4T1-4,1,4-g0` $\to$ `2T1-2,1,2-g0`

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$$
\begin{aligned}
X \subset \mathbb{A}^6 : \; & x_1^5 - x_1 - x_2^2 \\
& x_1 - x_1^3 + x_2 x_4^4 \\
& x_1^3 x_3 - x_1 x_3 - x_2 x_4^2 \\
& x_1^2 x_4^2 - x_2 x_3 + x_4^2 \\
& x_2 x_3 - x_1^2 - 1 \\
& x_3 x_4^2 - 1 \\
& x_5^2 - x_4 \\
& x_6^2 - x_5 \\
\phi : \; & x_3^4 x_2^2 - 2 x_3^2 x_2 + 1
\end{aligned}
$$

`128S69-8,16,16-g49`: size 4

`64S7-4,8,8-g17`

`32S10-4,8,4-g7`

`16T12-4,8,2-g2`

`8T4-2,4,2-g0`

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`2T1-2,2,1-g0`

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The coordinates of the Q_j generate the field $K(J[2])$.