

# 2-group Belyi maps

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# Outline

Motivation

Background

Computing permutation triples

A refined conjecture

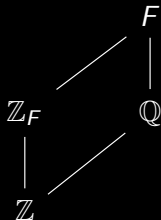
Computing equations

# Motivation



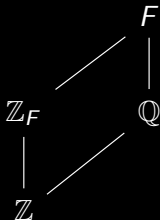
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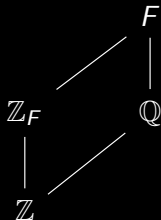
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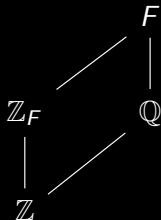


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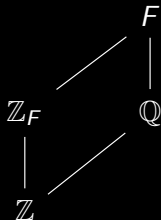
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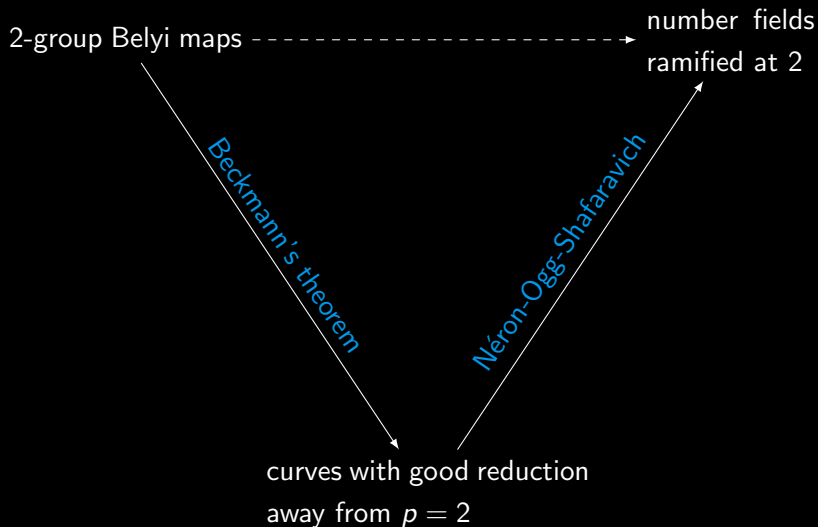
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Does there exist a number field where 2 is the *only* ramified prime?  
Nonsolvable?



## Why 2-group Belyi maps?



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$p = 7$  : existence (Dieulefait, Roberts)

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The hope is that an explicit nonsolvable field ramified only at 2 can be obtained as  $K(\text{Jac}(X)[2])$  where  $X$  is the domain of a **2-group Belyi map** (which we will define shortly).

## Main results

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- implementation of an algorithm to compute equations for 2-group Belyi maps over finite fields
- implementation of a *method* to compute equations for 2-group Belyi maps over number fields



# Main results

Motivated by the applications of 2-group Belyi maps to arithmetic geometry, we now state the main results.

- implementation of an algorithm to enumerate isomorphism classes of 2-group Belyi maps
- implementation of an algorithm to compute equations for 2-group Belyi maps over finite fields
- implementation of a *method* to compute equations for 2-group Belyi maps over number fields
- computational and theoretical evidence supporting a conjecture that every 2-group Belyi map is defined over an abelian extension of the rationals

# Background



## Belyi's theorem

A **Belyi map** is a morphism  $\phi: X \rightarrow \mathbb{P}^1$  of smooth projective algebraic curves over  $\mathbb{C}$  that is unramified outside of  $\{0, 1, \infty\}$ .

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## Theorem (Belyi 1979)

*An algebraic curve (smooth projective)  $X$  over  $\mathbb{C}$  can be defined over a number field if and only if  $X$  admits a Belyi map.*

## 2-group Belyi maps

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The **monodromy group** of  $\phi$ ,  $\text{Mon}(\phi)$ , is the image of the map

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A **2-group Belyi map** is a Galois Belyi map with monodromy group a 2-group.

# Beckmann's theorem

## Theorem (Beckmann 1989)

*Let  $\phi: X \rightarrow \mathbb{P}^1$  be a Galois Belyi map with monodromy group  $G$ .  
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*Then there exists a number field  $M$  satisfying the following properties.*

- *$p$  is unramified in  $M$*
- *$\phi$  is defined over  $M$*
- *$X$  is defined over  $M$*
- *$X$  has good reduction at all primes  $\mathfrak{p}$  of  $M$  above  $p$*

# Permutation Triples

A **transitive permutation triple of degree  $d$**  is a triple

$$\sigma = (\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3$$

such that

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The set of degree  $d$  Belyi maps up to isomorphism is in bijection with the set of degree  $d$  transitive permutation triples up to **simultaneous conjugation** and the group  $\langle \sigma \rangle$  is the monodromy group of  $\phi$ .



# Passports

A **passport**  $\mathcal{P}$  consists of the data  $(g, G, \lambda)$  where  $g \geq 0$  is an integer,  $G \leq S_d$  is a transitive subgroup, and  $\lambda = (\lambda_0, \lambda_1, \lambda_\infty)$  is a triple of partitions of  $d$ .

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The **passport of a Belyi map**  $\phi : X \rightarrow \mathbb{P}^1$  is  $(g(X), \text{Mon}(\phi), (\lambda_0, \lambda_1, \lambda_\infty))$  with  $g(X)$  the genus of  $X$ ,  $\text{Mon}(\phi)$  the monodromy group of  $\phi$ , and the partitions from ramification.

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The **passport of a permutation triple**  $\sigma$  is  $(g(\sigma), \langle \sigma \rangle, \lambda(\sigma))$  where

$$g(\sigma) = 1 - d + (e(\sigma_0) - e(\sigma_1) - e(\sigma_\infty))/2$$

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We now discuss the importance of organizing triples by passport.

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The situation improves, however, in the Galois setting.

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Then

- $\phi$  and  $X$  are defined over  $M(\phi)$ ,
- $\#G = d$ ,
- all cycles of  $\sigma_s$  have the same length for  $s \in \{0, 1, \infty\}$ ,
- and if we let  $a, b, c$  be the orders of  $\sigma_0, \sigma_1, \sigma_\infty$  respectively, we have

$$g(X) = 1 + \frac{\#G}{2} \left( 1 - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} \right).$$

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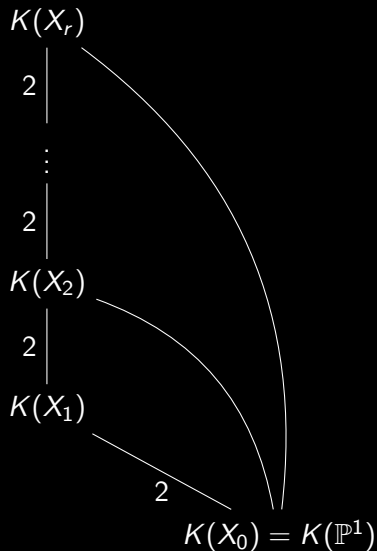
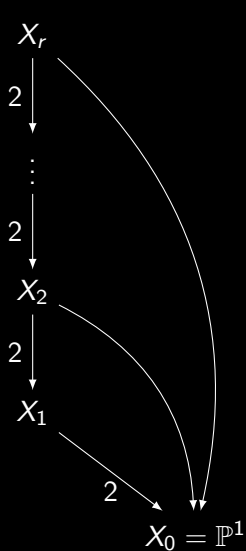
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The monodromy group in this setting corresponds to field automorphisms of the Galois closure of  $K(X)$  fixing  $K(x)$ .

## 2-group Belyi maps as iterated quadratic extensions



# Computing permutation triples



## Setup

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A **2-group permutation triple** of degree  $d \in \mathbb{Z}_{\geq 1}$  is a triple of permutations  $\sigma := (\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3$  satisfying

- $\sigma_\infty \sigma_1 \sigma_0 = \text{id}$ ;
- $G := \langle \sigma_0, \sigma_1 \rangle$  is a transitive subgroup of  $S_d$ ; and
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We say two degree  $d$  2-group permutation triples  $\sigma, \sigma'$  are **simultaneously conjugate** if there exists  $\tau \in S_d$  such that

$$\sigma^\tau := (\tau^{-1} \sigma_0 \tau, \tau^{-1} \sigma_1 \tau, \tau^{-1} \sigma_\infty \tau) = \sigma'$$

## Lifting permutation triples

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A **lift** of  $\sigma$  is a 2-group permutation triple  $\tilde{\sigma} \in S_{2d}^3$  such that  $\langle \tilde{\sigma} \rangle$  is isomorphic to some extension  $\tilde{G}$  of  $\mathbb{Z}/2\mathbb{Z}$  by  $G$  as in the exact sequence below.

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} \tilde{G} \xrightarrow{\pi} \langle \sigma \rangle \longrightarrow 1$$

## Lifting permutation triples

Let  $\sigma$  be a 2-group permutation triple.

A **lift** of  $\sigma$  is a 2-group permutation triple  $\tilde{\sigma} \in S_{2d}^3$  such that  $\langle \tilde{\sigma} \rangle$  is isomorphic to some extension  $\tilde{G}$  of  $\mathbb{Z}/2\mathbb{Z}$  by  $G$  as in the exact sequence below.

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} \tilde{G} \xrightarrow{\pi} \langle \sigma \rangle \longrightarrow 1$$

For a 2-group permutation triple  $\sigma$ , we denote the set of lifts of  $\sigma$  by  $\text{Lifts}(\sigma)$  and  $\text{Lifts}(\sigma)/\sim$  denotes the set of lifts up to simultaneous conjugation.

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**Input:**  $\sigma$  a 2-group permutation triple of degree  $d$

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5. Quotient  $\text{Lifts}(\sigma)$  by simultaneous conjugation

## Results : number of triples and passports

### Theorem (M.)

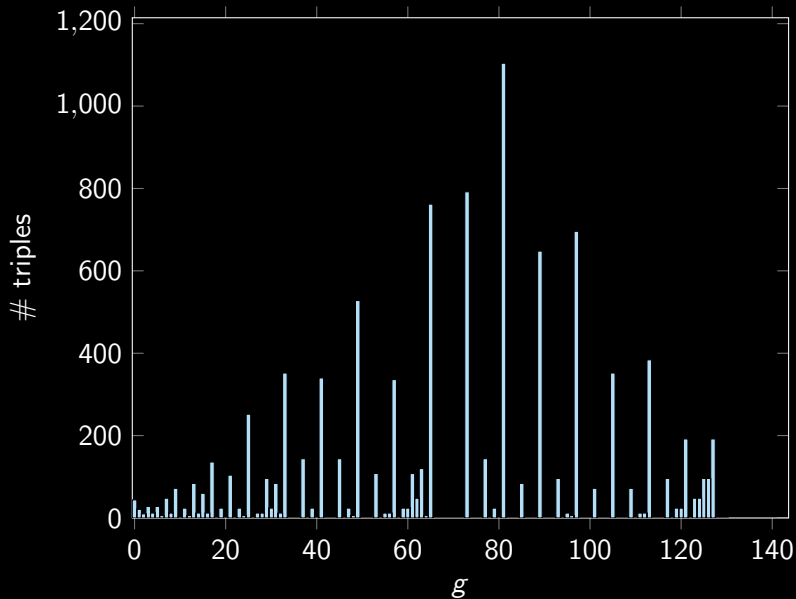
*The following tables list the number of isomorphism classes of 2-group Belyi maps, the number of passports, and number of lax passports respectively up to degree 256.*

| $d$       | 1 | 2 | 4 | 8  | 16 | 32  | 64  | 128  | 256  |
|-----------|---|---|---|----|----|-----|-----|------|------|
| # triples | 1 | 3 | 7 | 19 | 55 | 151 | 503 | 1799 | 7175 |

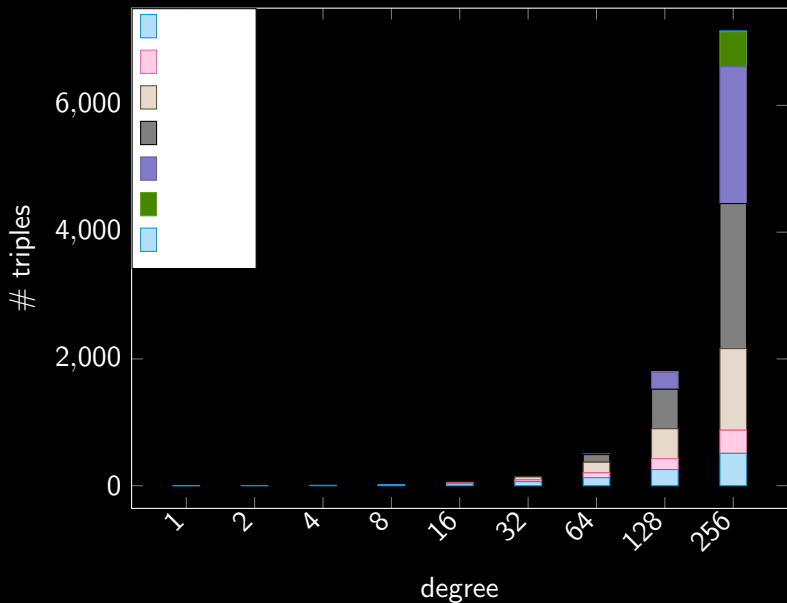
| $d$         | 1 | 2 | 4 | 8  | 16 | 32 | 64  | 128 | 256  |
|-------------|---|---|---|----|----|----|-----|-----|------|
| # passports | 1 | 3 | 7 | 16 | 41 | 96 | 267 | 834 | 2893 |

| $d$             | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 |
|-----------------|---|---|---|---|----|----|----|-----|-----|
| # lax passports | 1 | 1 | 3 | 6 | 14 | 31 | 85 | 257 | 882 |

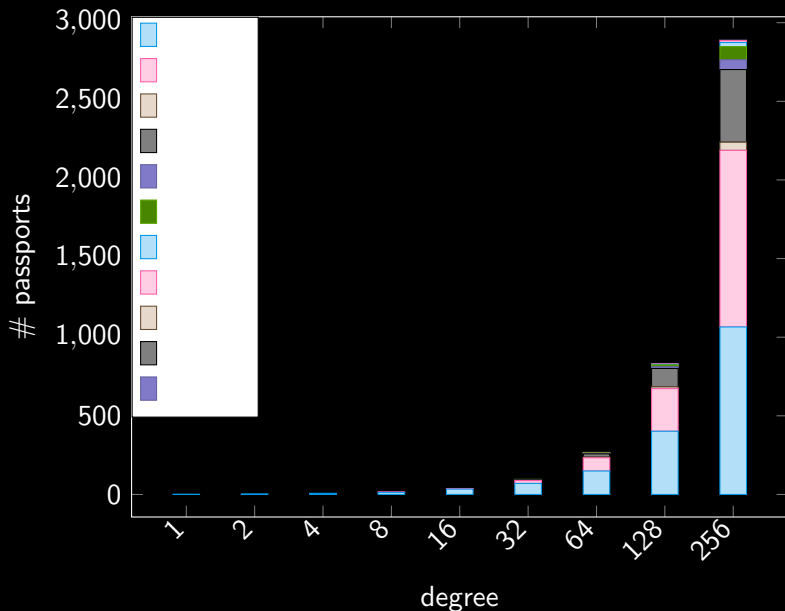
## Results : distribution of genera



## Results : groups by nilpotency class



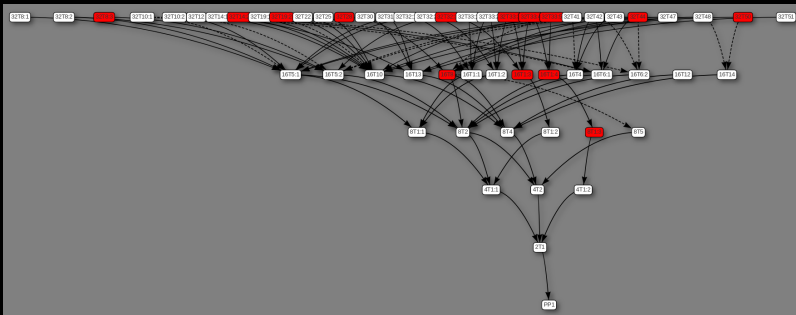
## Results : passport sizes



# The graph of 2-group Belyi maps

<https://math.dartmouth.edu/~mjmusty/32.html>

<https://math.dartmouth.edu/~mjmusty/32nh.html>



## A refined conjecture





# Passports

Recall that a passport  $\mathcal{P}$  consists of the data  $(g, G, \lambda)$  where  $g \in \mathbb{Z}_{\geq 0}$ ,  $G$  is a transitive subgroup of  $S_d$  and  $\lambda = (\lambda_0, \lambda_1, \lambda_\infty)$  is a triple of partitions of  $d$  corresponding to conjugacy classes  $(C_0, C_1, C_\infty)$  of  $S_d$ .

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To instead analyze  $\text{Gal}(\mathbb{Q}^{\text{al}} | \mathbb{Q}^{\text{ab}})$  we *refine* the notion of a passport.

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A **refined passport**  $\mathcal{P}$  consists of the data  $(g, G, c)$  where  $g \in \mathbb{Z}_{\geq 0}$ ,  $G$  is a transitive subgroup of  $S_d$  and  $c = (c_0, c_1, c_\infty)$  is a triple of conjugacy classes of  $G$ .

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As was the case with passport, every permutation triple  $\sigma$  determines a refined passport  $\mathcal{P}(\sigma)$ .

## A refined conjecture

### Theorem (M.)

*The size of  $\mathcal{P}(\sigma)$  is equal to 1 for every 2-group permutation triple  $\sigma$  with degree  $\leq 256$ .*



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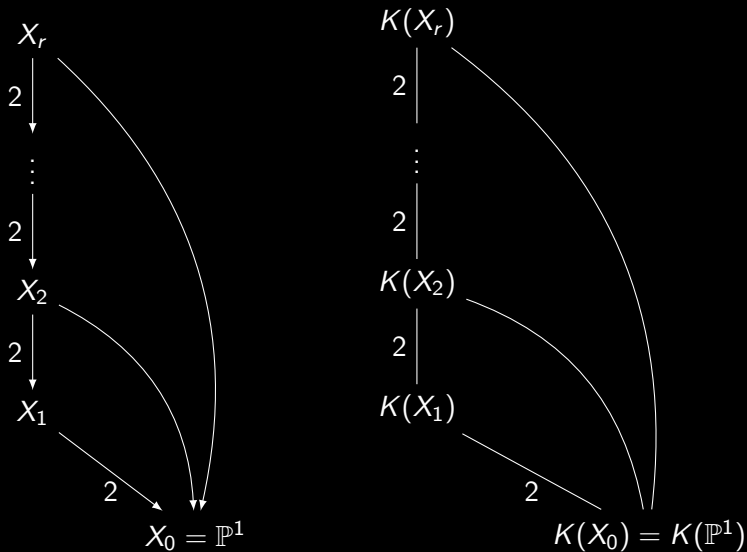
### **Theorem (M.)**

*ARC is true for 2-group permutation triples  $\sigma$  with  $\langle \sigma \rangle$  dihedral.*

# Computing equations



## 2-group Belyi maps as iterated quadratic extensions



## A motivating example : setup

Let  $F$  be a number field with integers  $\mathbb{Z}_F$ . Let  $\text{Pl}(F)$  denote the places of  $F$  and  $S_\infty$  the archimedean places. For  $v \in \text{Pl}(F) \setminus S_\infty$  let  $\mathfrak{p}_v$  denote the prime ideal of  $\mathbb{Z}_F$  corresponding to  $v$ .

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If  $\mathfrak{a}$  is not principal, then the question requires more care.

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The algorithms in this section rely on transporting this technique to the function field setting.



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A **2-group Belyi map modulo  $q$**  is a Galois extension of function fields  $\mathbb{F}_q(x) \hookrightarrow F$  with  $[F : \mathbb{F}_q(x)]$  a power of 2 unramified outside  $\{0, 1, \infty\}$  with  $p \geq 3$ .

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1. **Input:**  $F$  a 2-group Belyi map modulo  $q$  with passport  $(g, G, (a, b, c))$ ,  $G$  explicitly identified with  $\text{Gal}(F | \mathbb{F}_q(x))$ , and a triple  $(r_0, r_1, r_\infty) \in \{0, 1\}^3$ .
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$$f = \frac{\alpha^6 + 2x\alpha^4 + (2x^3 + 2x)\alpha^2 + (2x^3 + 2x^2)\alpha + x^4 + 2x^3}{x^3(x+1)}$$

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$$16T4-4,4,4-g3 : F(\sqrt{f})$$

## Algorithm in characteristic $p \geq 3$ : example

$F(\sqrt{f}) = \mathbb{F}_3(x)(\alpha)$  where  $\alpha$  satisfies

$$\begin{aligned} & y^{16} + (x^7 + 2x^6 + x^5)y^{14} + (x^{16} + 2x^{14} + x^{13} + 2x^{11})y^{12} \\ & + (x^{23} + x^{21} + 2x^{20} + 2x^{18} + x^{17} + x^{15})y^{10} \\ & + (x^{31} + 2x^{30} + 2x^{29} + 2x^{28} + x^{27} + 2x^{26} + x^{25} + 2x^{24} + x^{23} + x^{20})y^8 \\ & + (2x^{38} + 2x^{36} + x^{35} + 2x^{29} + 2x^{27} + x^{26})y^6 \\ & + (x^{48} + 2x^{47} + 2x^{46} + 2x^{44} + 2x^{43} + 2x^{42} \\ & \quad + x^{39} + 2x^{38} + 2x^{37} + 2x^{35} + 2x^{34} + 2x^{33})y^4 \\ & + (x^{54} + x^{52} + x^{51} + x^{50} + x^{49} + x^{47} \\ & \quad + x^{45} + x^{43} + x^{42} + x^{41} + x^{40} + x^{38})y^2 \\ & + x^{64} + 2x^{62} + x^{61} + 2x^{59} + 2x^{58} + x^{56} \\ & + x^{52} + 2x^{50} + 2x^{49} + x^{47} + 2x^{46} + x^{44} \end{aligned}$$

## Algorithm in characteristic $p \geq 3$ : compute entire passport

One issue with our technique to compute 2-group Belyi maps is that it only guarantees that the resulting Belyi map has the desired *passport* and does not allow us to control which *isomorphism class* we get.

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To recover from this we use isomorphism testing of function fields to determine if we have redundant Belyi maps with a given passport.

Since we know the sizes of passports from our work with permutation triples, we know that we have representatives from every isomorphism class even if we cannot match the Belyi maps to their corresponding permutation triples.

## Implementation in characteristic zero

Our inability to compute  $\text{Pic}(F)$  for  $F$  over a number field requires us to resort to adhoc techniques in this setting.



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However, we do have access to the ramification points of the Belyi maps and instead use combinations of these points to try to build a candidate function.

Although this implementation does not allow us to compute all 2-group Belyi maps for a given degree, it does work well in practice.

# Results

<https://github.com/michaelmusty/2GroupDessins>

- *all* 2-group Belyi maps modulo 3 up to degree 32
- hundreds of 2-group Belyi maps up to degree 256

## Future work

- higher degree over  $\mathbb{F}_3$
- non-Galois setting
- an algorithm in characteristic zero
- prove ARC for other families of 2-groups
- $p$ -group Belyi maps for  $p$  odd
- compute torsion fields

# Acknowledgements

- Dave, Tom, Carl, and John
- Sam, Jeroen, Edgar, Florian, and Richard
- Mary, Jim, Matt, and Nicole

## Acknowledgements

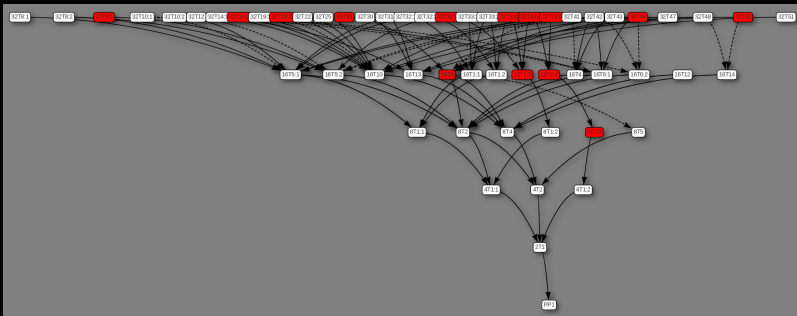
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Thanks for listening!

# The graph of 2-group Belyi maps

<https://math.dartmouth.edu/~mjmusty/32.html>

<https://math.dartmouth.edu/~mjmusty/32nh.html>





## Galois representations

Let  $X$  be an irreducible, smooth projective algebraic curve of genus  $g \geq 1$  over a number field  $K$ . Let  $G_K := \text{Gal}(K^{\text{al}} | K)$  be the absolute Galois group of  $K$  and let  $\ell \in \mathbb{Z}$  be prime.

Let  $J := \text{Jac}(X)$  be the **Jacobian variety** of  $X$ .  $J$  is an abelian variety of dimension  $g$ .

$G_K$  acts on the  $\ell$ -torsion points  $J[\ell](K^{\text{al}}) \cong (\mathbb{Z}/\ell\mathbb{Z})^{2g}$  of  $X$ .

This action determines a **mod- $\ell$  Galois representation**

$$\rho: G_K \rightarrow \text{Aut}(J[\ell]) \cong \text{GL}_{2g}(\mathbb{Z}/\ell\mathbb{Z}).$$

The geometry of  $X$  and the arithmetic of  $\rho$  are intimately related. For example, if  $X$  has good reduction at a prime  $\mathfrak{p}$  above  $p \neq \ell$ , then  $\mathfrak{p}$  will be unramified in the  $\ell$ -**torsion field**  $K(J[\ell])$ .

# Isomorphism of Belyi maps

Let  $\phi: X \rightarrow \mathbb{P}^1$  and  $\phi': X' \rightarrow \mathbb{P}^1$  be Belyi maps of degree  $d$ .  $\phi$  and  $\phi'$  are **isomorphic** (respectively **lax isomorphic**) if the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\sim} & X' \\ & \searrow \phi & \swarrow \phi' \\ & \mathbb{P}^1 & \end{array}, \text{ respectively } \begin{array}{ccc} X & \xrightarrow{\sim} & X' \\ \phi \downarrow & & \downarrow \phi' \\ \mathbb{P}^1 & \xrightarrow[\beta]{\sim} & \mathbb{P}^1 \end{array}$$

commute where  $\beta(\{0, 1, \infty\}) = \{0, 1, \infty\}$ .

## Example computing $\text{Lifts}(\sigma)/\sim$ : setup

Let  $\sigma = ((1\ 2), \text{id}, (1\ 2))$ . Then  $G = \langle \sigma \rangle = \mathbb{Z}/2\mathbb{Z}$ .

$\tilde{G}_1 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  and  $\tilde{G}_2 \cong \mathbb{Z}/4\mathbb{Z}$  with

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota_1} \tilde{G}_1 \xrightarrow{\pi_1} G \longrightarrow 1$$

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota_2} \tilde{G}_2 \xrightarrow{\pi_2} G \longrightarrow 1$$

Each map  $\pi_1, \pi_2$  pulls back to 4 triples that multiply to id:

$$T_1 = \left\{ ((1\ 2)(3\ 4), \text{id}, (1\ 2)(3\ 4)), ((1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)), \right. \\ \left. ((1\ 4)(2\ 3), \text{id}, (1\ 4)(2\ 3)), ((1\ 4)(2\ 3), (1\ 3)(2\ 4), (1\ 2)(3\ 4)) \right\}$$

$$T_2 = \left\{ ((1\ 4\ 3\ 2), \text{id}, (1\ 2\ 3\ 4)), ((1\ 2\ 3\ 4), (1\ 3)(2\ 4), (1\ 2\ 3\ 4)), \right. \\ \left. ((1\ 2\ 3\ 4), \text{id}, (1\ 4\ 3\ 2)), ((1\ 4\ 3\ 2), (1\ 3)(2\ 4), (1\ 4\ 3\ 2)) \right\}$$

## Example computing $\text{Lifts}(\sigma)/\sim$ : action on blocks

Choose  $\alpha = (1\ 3)(2\ 4)$  to be the generator of  $\iota_1(\mathbb{Z}/2\mathbb{Z})$  in  $\tilde{G}_1$ .

Each triple in  $T_1$  must act on the *blocks*  $\{\boxed{1\ 3}, \boxed{2\ 4}\}$  corresponding to the permutations in  $\sigma$ .

Let  $(\tilde{\sigma}_0, \tilde{\sigma}_1, \tilde{\sigma}_\infty) = ((1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3))$ .

Note that  $\tilde{\sigma}_0(\boxed{1\ 3}) = \boxed{2\ 4}$  and  $\tilde{\sigma}_0(\boxed{2\ 4}) = \boxed{1\ 3}$ .

The induced permutation of  $\tilde{\sigma}_0$  on blocks is  $(\boxed{1\ 3}, \boxed{2\ 4})$  which is the same as the permutation  $\sigma_0 = (1\ 2)$ .

Similarly,  $\tilde{\sigma}_1$  acts as id on blocks and  $\tilde{\sigma}_\infty$  acts as  $(1\ 2)$  on blocks.

Choosing

$$\alpha := (1\ d+1)(2\ d+2) \dots (d-1\ 2d-1)(d\ 2d)$$

allows us to label blocks by reducing modulo  $d$ .

## Example computing $\text{Lifts}(\sigma)/\sim$ : conclude

We currently have triples that multiply to id and have the correct action on blocks, but we only want triples that generate the correct group.

$$\text{Lifts}(\sigma, \tilde{G}_1 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) = \\ \left\{ ((1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)), ((1\ 4)(2\ 3), (1\ 3)(2\ 4), (1\ 2)(3\ 4)) \right\}$$

$$\text{Lifts}(\sigma, \tilde{G}_2 \cong \mathbb{Z}/4\mathbb{Z}) = T_2 = \\ \left\{ ((1\ 4\ 3\ 2), \text{id}, (1\ 2\ 3\ 4)), ((1\ 2\ 3\ 4), (1\ 3)(2\ 4), (1\ 2\ 3\ 4)), \right. \\ \left. ((1\ 2\ 3\ 4), \text{id}, (1\ 4\ 3\ 2)), ((1\ 4\ 3\ 2), (1\ 3)(2\ 4), (1\ 4\ 3\ 2)) \right\}$$

Lastly, we quotient by simultaneous conjugation to obtain

$$\text{Lifts}(\sigma)/\sim = \left\{ ((1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)), \right. \\ \left. ((1\ 4\ 3\ 2), \text{id}, (1\ 2\ 3\ 4)), ((1\ 2\ 3\ 4), (1\ 3)(2\ 4), (1\ 2\ 3\ 4)) \right\}$$

## Bipartite graphs of permutation triples

Now that we can lift permutation triples, we now describe some notation for the bipartite graphs that organize these triples.

For  $i \in \mathbb{Z}_{\geq 1}$  we define the bipartite graph denoted  $\mathcal{G}_{2^i}$  with the following node sets.

- $\mathcal{G}_{2^i}^{\text{above}}$  : the set of isomorphism classes of 2-group Belyi maps of degree  $2^i$  indexed by 2-group permutation triples  $\tilde{\sigma}$  up to simultaneous conjugation in  $S_{2^i}$
- $\mathcal{G}_{2^i}^{\text{below}}$  : the set of isomorphism classes of 2-group Belyi maps of degree  $2^{i-1}$  indexed by 2-group permutation triples  $\sigma$  up to simultaneous conjugation in  $S_{2^{i-1}}$

For every pair of nodes  $(\tilde{\sigma}, \sigma) \in \mathcal{G}_{2^i}^{\text{above}} \times \mathcal{G}_{2^i}^{\text{below}}$  there is an edge between  $\sigma$  and  $\tilde{\sigma}$  if and only if  $\tilde{\sigma}$  is simultaneously conjugate to a lift of  $\sigma$ .

## Algorithm to compute $\mathcal{G}_{2^i}$

**Input:** The bipartite graph  $\mathcal{G}_{2^{i-1}}$

**Output:** The bipartite graph  $\mathcal{G}_{2^i}$

1.

$$\text{Lifts}(\mathcal{G}_{2^{i-1}}) := \bigcup_{\sigma \in \mathcal{G}_{2^{i-1}}^{\text{above}}} \text{Lifts}(\sigma)/\sim$$

2. Quotient  $\text{Lifts}(\mathcal{G}_{2^{i-1}})$  by simultaneous conjugation in  $S_{2^i}$  to obtain  $\text{Lifts}(\mathcal{G}_{2^{i-1}})/\sim$

3. Define  $\mathcal{G}_{2^i}^{\text{below}} := \mathcal{G}_{2^{i-1}}^{\text{above}}$  and define  $\mathcal{G}_{2^i}^{\text{above}}$  by representatives of  $\text{Lifts}(\mathcal{G}_{2^{i-1}})/\sim$

4. For every pair  $(\tilde{\sigma}, \sigma) \in \mathcal{G}_{2^i}^{\text{above}} \times \mathcal{G}_{2^i}^{\text{below}}$  place an edge between  $\tilde{\sigma}$  and  $\sigma$  if and only if there is a triple in the equivalence class  $[\tilde{\sigma}] \in \text{Lifts}(\mathcal{G}_{2^{i-1}})/\sim$  that is a lift of  $\sigma$

## Algebraic function fields : setup

Let  $K$  be a perfect field. An **algebraic function field in one variable over  $K$**  is a field extension  $F$  over  $K$  such that there exists  $x \in F$  transcendental over  $K$  and  $[F : K(x)]$  is finite.

$K$  is the **constant field** of  $F$  and the **exact constant field** of  $F$  is the algebraic closure of  $K$  in  $F$ .

As an example, let  $X$  be an irreducible affine plane curve (possibly singular) defined by the equation  $f(x, y) = 0$  with  $f \in K[x, y]$ .

Then the **function field of  $X$** , denoted  $K(X)$  is the field of fractions of the coordinate ring  $K[x, y]/(f(x, y))$  of  $X$ .

A **place** of  $F$  is the maximal ideal of some DVR  $\mathcal{O}_P$  of  $F$ . The valuation on  $F$  corresponding to  $P$  is denoted  $\text{ord}_P$ .

The set of places of  $F$  is denoted  $\text{Pl}(F)$  and the **degree** of  $P$  is the index  $[\mathcal{O}_P/P : K]$  of the **residue class field**.



## Algebraic function fields : Picard group and $\mathcal{L}(D)$

The **divisor class group**  $\text{Div}(F)$  of  $F$  is the free abelian group generated by the places of  $F$ . A **divisor**  $D \in \text{Div}(F)$  is represented by a sum of places  $\sum_P a_P P$  and the **degree** of  $D$  is  $\sum_P a_P \deg(P)$ . The set of **degree zero divisors** is denoted  $\text{Div}^0(F)$ .

The image of  $\text{div}: F^\times \rightarrow \text{Div}(F)$  defined by  $\text{div}(f) = \sum_P \text{ord}_P(f) P$  is the subgroup of **principal divisors** of  $F$  denoted  $\text{Princ}(F)$ .

The **Picard group** of  $F$  is  $\text{Pic}(F) := \text{Div}(F) / \text{Princ}(F)$ .

The **Jacobian** of  $F$  is  $\text{Pic}^0(F) := \text{Div}^0(F) / \text{Princ}(F)$ .

There is a partial order on  $\text{Div}(F)$  defined by  $D_1 \geq D_2$  if  $\text{ord}_P(D_1) \geq \text{ord}_P(D_2)$  for every  $P \in \text{Pl}(F)$ .

The **Riemann-Roch space** of a divisor  $D \in \text{Div}(F)$  is defined by  $\mathcal{L}(D) := \{f \in F : \text{div}(f) + D \geq 0\} \cup \{0\}$ .

## Algebraic function fields : quadratic extensions

### Lemma

*Let  $aF^{\times 2}$  be a nontrivial coset of  $F^{\times}/F^{\times 2}$  and consider the extension  $L := F(\sqrt{a})$ . Then a prime  $P$  of  $F$  is ramified in  $L$  if and only if  $\text{ord}_P(a)$  is odd.*

We now revisit the question of finding a quadratic extension of  $F(\sqrt{a})$  of  $F$  with ramification prescribed by  $R \in \text{Div}(F)$ .

The simple cases are when no such  $a$  exists or when  $R$  is principal. The last case occurs when there exists  $D$  such that  $R - 2D = \text{div}(a)$  for some  $a \in F$ .

As in the number field setting, this implies  $R \in 2\text{Pic}(F)$  and  $D$  is unique up to addition by  $T \in \text{Pic}^0(F)[2]$ .

## Algorithm in characteristic $p \geq 3$ : Galois test

**Input:**

- $F$  a Galois extension of  $\mathbb{F}_q(x)$
- $\text{Gal}(F | \mathbb{F}_q(x))$  explicitly given as automorphisms of  $F$
- $a \in F$

**Output:** True if  $F(\sqrt{a})$  is Galois over  $\mathbb{F}_q(x)$  and False otherwise

- For each generator  $\sigma \in \text{Gal}(F | \mathbb{F}_q(x))$  test if  $\sigma(a)/a$  is a square in  $F$
- Return True if  $\sigma(a)/a$  is a square in  $F$  for all generators  $\sigma$  and otherwise return False

Similarly, we can apply the same test after extending the constant field from  $\mathbb{F}_q$  to  $\mathbb{F}_{q^2}$ .

## Algorithm in characteristic $p \geq 3$ : get candidates

### Input:

- $F$  a 2-group Belyi map modulo  $q$  of degree  $d = 2^m$  corresponding to a 2-group permutation triple  $\sigma$
- A passport  $\mathcal{P} = (\tilde{G}, (a, b, c))$  with  $\tilde{G}$  a 2-group of order  $2d$  such that there exists a 2-group permutation triple  $\tilde{\sigma}$  with passport  $\mathcal{P}$  that is a lift of  $\sigma$
- $\text{Gal}(F | \mathbb{F}_q(x)) \cong \langle \sigma \rangle$  explicitly given as automorphisms of  $F$

**Output:** A list of candidate functions  $\{f_i\}$  with each  $f_i \in F$  such that  $F(\sqrt{f_i})$  is a 2-group Belyi map modulo  $q$  with passport  $\mathcal{P}$ .

## Algorithm in characteristic $p \geq 3$ : get candidates (steps 1-4)

1. For  $s \in \{0, 1, \infty\}$  compute

$$r_s := \begin{cases} 0 & \text{if } \text{order}(\sigma_s) = \text{order}(\tilde{\sigma}_s) \\ 1 & \text{if } \text{order}(\sigma_s) < \text{order}(\tilde{\sigma}_s) \end{cases}$$

2. Compute

$$R := \sum_{s \in \{0, 1, \infty\}} r_s R_s \in \text{Div}(F)$$

where  $R_0, R_1, R_\infty$  are defined to be the supports of  $\text{div}(x)$ ,  $\text{div}(x - 1)$ , and  $\text{div}(1/x)$  respectively.

3. Compute the abelian group  $\text{Pic}(F) = T \oplus \mathbb{Z}$  (with  $T$  a finite abelian group) along with a map  $\psi: \text{Pic}(F) \rightarrow \text{Div}(F)$ .
4. Compute  $[R] := \psi^{-1}(R)$ .

## Algorithm in characteristic $p \geq 3$ : get candidates (step 5)

5. For each  $a \in \text{Pic}(F)[2]$  compute the following:
  - (a) Let  $D_a := \psi(a + [R]/2) \in \text{Div}(F)$ .
  - (b) Compute  $\mathcal{L}(R - 2D_a)$ .
  - (c) If  $\mathcal{L}(R - 2D_a)$  has dimension 1, then compute  $f_a \in F$  with  $\text{div}(f_a)$  generating  $\mathcal{L}(R - 2D_a)$  and go to Step 5d Otherwise go to the next  $a \in \text{Pic}(F)[2]$ .
  - (d) Apply Galois test to  $F$ ,  $\text{Gal}(F | \mathbb{F}_q(x))$ , and  $f_a$  from Step 5c to see if  $F(\sqrt{f_a})$  generates a Galois extension. If  $F(\sqrt{f_a})$  is Galois over  $\mathbb{F}_q(x)$  then save  $f_a$  and go to the next  $a \in \text{Pic}(F)[2]$ . If  $F(\sqrt{f_a})$  is not Galois over  $\mathbb{F}_q(x)$ , then go to Step 5e.
  - (e) Let  $F'$  be the function field  $F$  after extending the field of constants  $\mathbb{F}_q$  to  $\mathbb{F}_{q^2}$ . Apply Galois test to  $F'$ ,  $\text{Gal}(F' | \mathbb{F}_{q^2}(x))$ , and  $f_a$  (viewed as an element of  $F'$ ) from Step 5c to see if  $F'(\sqrt{f_a})$  generates a Galois extension. If  $F'(\sqrt{f_a})$  is Galois over  $\mathbb{F}_{q^2}(x)$  then save  $f_a$ . Go to the next  $a \in \text{Pic}(F)[2]$ .

## Algorithm in characteristic $p \geq 3$ : get candidates (steps 6-8)

6. Let  $S$  be the set of  $f_a$  saved in Step 5d. Let  $S'$  be the set of  $f_a$  saved in Step 5e.

7.     • If  $S$  is nonempty, then for each  $f_a \in S$  compute  $F(\sqrt{f_a})$ ,

$$G_a \cong \text{Gal}(F(\sqrt{f_a}) \mid \mathbb{F}_q(x)),$$

and let  $S'' = \{f_a \in S : G_a \cong \tilde{G}\}$ .

• If  $S$  is empty, then for each  $f_a \in S'$  compute  $F'(\sqrt{f_a})$ ,

$$G_a \cong \text{Gal}(F'(\sqrt{f_a}) \mid \mathbb{F}_{q^2}(x)),$$

and let  $S'' = \{f_a \in S' : G_a \cong \tilde{G}\}$ .

8. Return the list  $S''$

## Notation for examples

DNG-a,b,c-gE-H

D : degree in  $\{2, 4, 8, 16, 32, 64, 128, 256\}$

N : either T or S identifying group database

G : a positive integer identifying the group

a : ramification index of 0 in  $\{2, 4, 8, 16, 32, 64, 128, 256\}$

b : ramification index of 1 in  $\{2, 4, 8, 16, 32, 64, 128, 256\}$

c : ramification index of  $\infty$  in  $\{2, 4, 8, 16, 32, 64, 128, 256\}$

g : just the letter g

E : the genus in  $\mathbb{Z}_{\geq 0}$

H : the hash of the 2-group permutation triple a positive integer

To identify a passport we omit the hash.