# Number fields unramified away from 2

John W. Jones

*School of Mathematical and Statistical Sciences, Arizona State University, Box 871804, Tempe, AZ, United States*

**A B S T R A C T**

Consider the set of number fields unramified away from 2, i.e., unramified outside $\{2, \infty\}$. We show that there do not exist any such fields of degrees 9 through 15. As a consequence, the following simple groups are ruled out for being the Galois group of an extension which is unramified away from 2: Mathieu groups $M_{11}$ and $M_{12}$, $PSL(3, 3)$, and alternating groups $A_j$ for $8 < j < 16$ (values $j \leqslant 8$ were previously known).

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Let $\mathcal{K}_2$ be the set of number fields $K \subset \mathbf{C}$ which are unramified outside of the set $\{2, \infty\}$, i.e., fields with discriminant $\pm 2^a$. We say that such a field is unramified away from 2. A field is in $\mathcal{K}_2$ if and only if its Galois closure is in $\mathcal{K}_2$. Accordingly, we let $\mathcal{G}_2$ be the set of Galois groups fields in $\mathcal{K}_2$ which are Galois over $\mathbf{Q}$.

Fields in $\mathcal{K}_2$ and groups in $\mathcal{G}_2$ have been studied by several authors [Tat94,Har94,Bru01,Les06, Mar63]. In particular, fields in $\mathcal{K}_2$ of degree less than 9 are fully understood, and various non-solvable groups have been shown to not lie in $\mathcal{G}_2$. Here we extend these results for low degree fields.

The basic techniques used in the papers cited above are class field theory, exhaustive computer searches of number fields with particular discriminants, and discriminant bound arguments. In this paper, we will employ the third approach. We use well-known lower bounds for discriminants of number fields [Odl76]. Our upper bounds for discriminants come from a study of higher ramification groups. Preliminaries on discriminants of local fields are in Section 1 with the main results in Section 2.

In general, we will use $K$ to denote a number field and $F$ to denote a finite extension of $\mathbf{Q}_p$, for some prime $p$. Several notations apply to both situations. If $E$ is a finite degree $n$ extension of $\mathbf{Q}$ or of $\mathbf{Q}_p$, we let $(D_E)$ be the discriminant as an ideal over the base, choosing $D_E$ to be a positive integer.

Then the root discriminant for $E$ is $\mathrm{rd}(E) := D_E^{1/n}$. We will denote the Galois closure of $E$ over its base by $E^{gal}$. Then, the Galois root discriminant of $E$ is defined as $\mathrm{grd}(E) := \mathrm{rd}(E^{gal})$. We will use $\mathrm{ord}_p$ to denote the $p$-adic valuation on $\mathbf{Q}$ such that $\mathrm{ord}_p(p) = 1$.

When referring to Galois groups, we will use standard notations such as $C_n$ for a cyclic group of order $n$, and $D_n$ for dihedral groups of order $2n$. Otherwise, we will use the $T$-numbering introduced in [BM83], writing $nT_j$ for a degree $n$ field whose normal closure has Galois group $T_j$.

## 1. Local fields

Although we are particularly interested in extensions of $\mathbf{Q}_2$, throughout this section, we work over $\mathbf{Q}_p$ where $p$ is any prime.

### 1.1. Slope

If $F$ is a finite extension of $\mathbf{Q}_p$, then $D_F$ is a power of $p$. We define $c(F)$ to be the integer such that $D_F = p^{c(F)}$.

Now assume $F$ is Galois over $\mathbf{Q}_p$, with $G = \mathrm{Gal}(F/\mathbf{Q}_p)$. We let $G^\nu$ be the higher ramification groups of $G$ in upper numbering following the convention of [JR06], which is shifted by 1 from [Ser79]. In particular, $G = G^0$ and the inertia subgroup is $G^1$. We also define $G^{\nu+} := \bigcup_{\epsilon>0} G^{\nu+\epsilon}$. Slopes of $F/\mathbf{Q}_p$ are values $s$ where $G^{s+} \lneq G^s$, i.e. locations of jumps in the filtration.

Letting $F^{\mathrm{unr}}$ be the fixed field of $G^1 = G^{0+}$ and $F^{\mathrm{tame}}$ be the fixed field $G^{1+}$, we have that $F^{\mathrm{unr}}$ is the maximal unramified subextension of $F$ over $\mathbf{Q}_p$ with $\mathrm{Gal}(F^{\mathrm{unr}}/\mathbf{Q}_p) \cong G^0/G^{0+}$, and $F^{\mathrm{tame}}$ is the maximal tame extension of $F$ over $F^{\mathrm{unr}}$ with $\mathrm{Gal}(F^{\mathrm{tame}}/F^{\mathrm{unr}}) \cong G^1/G^{1+}$. Let $f = [F^{\mathrm{unr}} : \mathbf{Q}_p] = |G^0/G^{0+}|$ and $t = [F^{\mathrm{tame}} : F^{\mathrm{unr}}] = |G^1/G^{1+}|$ be the unramified and tame degrees respectively. These two integers completely describe the only slopes $\leqslant 1$.

Slopes greater than 1 correspond to wild ramification. The slope content of $F/\mathbf{Q}_p$ is then of the form $[s_1, \ldots, s_m]_t^f$ where $f$ and $t$ are the unramified and tame degrees defined above, and the $s_i$ are the wild slopes, sorted so that $s_i \leqslant s_{i+1}$. The ramification group $G^{1+}$ is a $p$-group, and so for slopes $s > 1$, the corresponding quotients $G^s/G^{s+}$ are finite $p$-groups. We repeat each $s_i$ with multiplicity $m_i$ where $p^{m_i} = |G^{s_i}/G^{s_i+}|$. In particular, if $F/\mathbf{Q}_p$ has slope content $[s_1, \ldots, s_m]_t^f$, then $|G| = p^m tf$.

Corresponding to the slope content $[s_1, \ldots, s_m]_t^f$ is a filtration on the Galois group which is just a slight modification of the filtration discussed above. For each wild slope $s > 1$ with multiplicity $k > 1$, we refine the step $G^{s+} \leqslant G^s$ with index $p^k$ into $k$ steps, each of degree $p$. Taking fixed fields, we get the tower

$$\mathbf{Q}_p \subseteq F^{\mathrm{unr}} \subseteq F^{\mathrm{tame}} = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F \tag{1}$$

where each extension $F_{i+1}/F_i$ has degree $p$. For any finite extension of local fields $F/E$, we define its average slope by

$$\mathit{Slope}_{avg}(F/E) := \frac{c(F) - c(E)}{[F : \mathbf{Q}_p] - [F : E]}. \tag{2}$$

For our tower (1), the average slopes give the wild slopes in the slope content for $F/\mathbf{Q}_p$, i.e., $s_i = \mathit{Slope}_{avg}(F_i/F_{i-1})$ for $i \geqslant 1$ (see [JR06]). So, the list of slopes can be discovered by working up through a particular chain of subfields of $F/\mathbf{Q}_p$.

From slope data $[s_1, \ldots, s_m]_t^f$, we can compute the discriminant, and hence the root discriminant of $F$. Specifically, $\mathrm{rd}(F) = p^{gms_p(F)}$ where

$$gms_p(F) = \frac{c(F)}{[F : \mathbf{Q}_p]} = \frac{1}{p^m}\left(\sum_{i=1}^{m} p^{i-1}(p-1)s_i + \frac{t-1}{t}\right). \tag{3}$$

Note, the unramified degree $f$ does not enter into formula (3). For the remainder of this paper, we will omit $f$ from the slope content of an extension and write simply $[s_1, \ldots, s_m]_t$. Since formula (3) is already a function of the slope content, we will also use it to define $gms_p([s_1, \ldots, s_m]_t)$. When comparing possible slope contents $\alpha$ and $\beta$, we say that $\alpha$ is an upper bound for $\beta$ if $gms_p(\alpha) \geqslant gms_p(\beta)$.

**Remark 1.1.** The notation *gms* stands for *Galois mean slope*, so named because it is a weighted average of slopes for a Galois extension. The terminology is similar to our use of *average slope*, denoted by *Slope$_{avg}$*, which is also a weighted sum of slopes from a Galois extension. However, we will not be making use of this latter fact here.

### 1.2. Composita

If we start with a global field $K$, we can compute $grd(K)$ locally. We decompose $K \otimes \mathbf{Q}_p \cong \prod_{i=1}^{g} K_{p,i}$ as a product of finite extensions of $\mathbf{Q}_p$. The algebra $K^{gal} \otimes \mathbf{Q}_p$ is a product of copies of $K_p^{gal} := (K_{p,1})^{gal} \cdots (K_{p,g})^{gal}$, the compositum of the Galois closures of the $K_{p,i}$. Picking a prime for $K^{gal}$ above each prime $p$, we let $gms_p(K) := gms_p(K_p^{gal})$, and then

$$grd(K) = rd(K^{gal}) = \prod_p p^{gms_p(K)}.$$

Naturally, in this product, the factor for each unramified prime is $p^0 = 1$.

An important, and somewhat subtle problem, then is to determine $gms_p$ for the compositum of fields $K_{p,i}^{gal}$. Proposition 1.2 below gives reasonable bounds on $gms_p$ for a compositum. Given a slope content $\alpha = [s_1, \ldots, s_m]_t$ and a rational number $s > 1$, we write $m_s(\alpha)$ for the multiplicity of $s$ in $\alpha$, i.e., the number of $s_i$ equal to $s$. Similarly, we write $m_{\geqslant s}(\alpha)$ for the number of slopes $s_i \geqslant s$. The following proposition is a straightforward consequence of Herbrand's theorem [Ser79].

**Proposition 1.2.** *Suppose $F_1$ and $F_2$ are finite Galois extensions of $\mathbf{Q}_p$, with slope contents $\alpha_1$ and $\alpha_2$. Let $\beta$ be the slope content of the compositum $F_1 F_2$. Then,*

(1) *for all $s > 1$, $m_s(\beta) \geqslant \max(m_s(\alpha_1), m_s(\alpha_2))$;*
(2) *for all $s > 1$, $m_{\geqslant s}(\beta) \leqslant m_{\geqslant s}(\alpha_1) + m_{\geqslant s}(\alpha_2)$.*

*Moreover, the tame degree for $F_1 F_2$ is the least common multiple of the tame degrees of $F_1$ and $F_2$.*

Given two finite Galois extensions $F_1$ and $F_2$ of $\mathbf{Q}_p$, Proposition 1.2 gives upper bounds for the slope content of $F_1 F_2$, and hence for $gms_p(F_1 F_2)$, which are easy to compute. Namely, one combines the tame degrees as described in the proposition, and just concatenates (and sorts) the lists of wild slopes. To bound the slope content of the compositum subject to an upper bound on the number of wild slopes, one removes slopes from the combination which occur in the slope contents of both fields, starting with the smallest such slopes.

For example, given slope contents $[2, 3, 7/2]_9$ and $[2, 3, 4]_{15}$, an upper bound for the slope content of the compositum is $[2, 2, 3, 3, 7/2, 4]_{45}$. The maximal combinations with 5 and 4 wild slopes are $[2, 3, 3, 7/2, 4]_{45}$ and $[2, 3, 7/2, 4]_{45}$ respectively. One cannot have a combination with less than 4 wild slopes in this case by Proposition 1.2. We will refer to this process as computing the *crude upper bound* for slope content. In some cases, one can certainly obtain better bounds by using more knowledge of the fields involved, e.g., Proposition 2.5 below.

### 1.3. Individual slope bounds

Our first lemma follows from basic facts about ramification [Ser79, Chap. 3], and some simple algebra to translate statements from discriminant exponents to slopes.

**Lemma 1.3.** *If $F \supseteq E \supseteq \mathbf{Q}_p$ are finite extensions where $F/E$ is totally ramified of degree $p^n$, and $[E : \mathbf{Q}_p] = ef$ where $f$ is the residue field degree for $E/\mathbf{Q}_p$, then*

$$c(F) = p^n \cdot c(E) + f\nu$$

*where $\nu$ is an integer, $ep^n \leqslant \nu \leqslant p^n - 1 + nep^n$. Moreover, the average slope for $F/E$ equals*

$$Slope_{avg}(F/E) = \frac{c(E)}{[E : \mathbf{Q}_p]} + \frac{\nu}{(p^n - 1)e}. \tag{4}$$

**Remark 1.4.** In Lemma 1.3, given a field $E$, there will exist extensions $F$ of degree $p^n$ satisfying both extremes of the inequalities for $\nu$. If $\pi$ is a uniformizer for $E$, one can use $x^{p^n} + \pi x + \pi$ and $x^{p^n} + \pi$ to define extensions achieving the lower and upper bounds respectively.

We now apply Lemma 1.3 to bound the average slopes in a tower.

**Lemma 1.5.** *Given a tower of finite extensions*

$$\mathbf{Q}_p \subseteq F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_m = F$$

*where $[F_0 : \mathbf{Q}_p] = ef$, with $f$ being the residue field degree, $p \nmid e$, and each $F_i/F_{i-1}$ totally ramified of degree $p$, then for all $i$, $1 \leqslant i \leqslant m$,*

$$Slope_{avg}(F_i/F_{i-1}) \leqslant i + \frac{p}{p - 1}.$$

**Proof.** We will abbreviate $Slope_{avg}(F_i/F_{i-1})$ by $S_i$. Applying Lemma 1.3, we get the recursions

$$c(F_i) = pc(F_{i-1}) + f\nu_i \tag{5}$$

and

$$S_i = \frac{c(F_{i-1})}{p^{i-1}ef} + \frac{\nu_i}{(p - 1)p^{i-1}e}. \tag{6}$$

Here $\nu_i$ is the value of $\nu$ in Lemma 1.3 for the extension $F_i/F_{i-1}$. Note that in tower, we have

$$S_{i+1} - S_i = \frac{c(F_{i+1}) - c(F_i)}{(p - 1)p^i ef} - \frac{c(F_i) - c(F_{i-1})}{(p - 1)p^{i-1}ef} = \frac{\nu_{i+1} - \nu_i}{(p - 1)p^i e}. \tag{7}$$

From Eqs. (5) and (6), it is clear that the sequences of discriminant exponents, $c(F_i)$, and average slopes, $S_i$, are each bounded by the corresponding sequences where we use the upper bound for each $\nu_i \leqslant p - 1 + p^i e$. For the sequence of $S_i$ where $\nu_i$ is maximal for all $i$,

$$S_{i+1} - S_i \leqslant \frac{(p - 1 + p^{i+1}e) - (p - 1 + p^i e)}{(p - 1)p^i e} = 1. \tag{8}$$

So $S_i \leqslant S_1 + i - 1$, and it is easy to check from Eq. (6) and the bound for $\nu_1$ that $S_1 \leqslant 1 + p/(p - 1)$, giving the result. $\square$

**Remark 1.6.** It is not always the case that in a sequence of slopes, $s_i \leqslant s_{i-1} + 1$ for all $i$. For example, there are extensions of $\mathbf{Q}_2$ with Galois group $8T_{23}$ with slope content $[4/3, 4/3, 3]_3$ (see [JR08]).

We have two applications of Lemma 1.5. First, we can apply it directly to the tower in display (1) to get bounds on the wild slopes of a Galois extension $F/\mathbf{Q}_p$.

**Proposition 1.7.** *Let $F/\mathbf{Q}_p$ be a Galois extension with slope content $[s_1, \ldots, s_m]_t$. Then for $1 \leqslant i \leqslant m$,*

$$s_i \leqslant i + \frac{p}{p-1}.$$

Specializing Proposition 1.7 to the case $p = 2$, we have the following.

**Corollary 1.8.** *If a Galois extension $F/\mathbf{Q}_2$ has slope content $[s_1, \ldots, s_m]_t$, then $s_i \leqslant i + 2$ for $1 \leqslant i \leqslant m$.*

**Remark 1.9.** The bound in Corollary 1.8 is achieved by a cyclic extension of degree $2^k$ over $\mathbf{Q}_2$ given by $\mathbf{Q}_2(\zeta_{2^k} + \zeta_{2^k}^{-1})$.

We now apply Lemma 1.5 to a non-Galois extension, and slopes of its Galois closure.

**Proposition 1.10.** *If $F$ is a finite extension of $\mathbf{Q}_p$, then all slopes $s$ for $F^{gal}/\mathbf{Q}_p$ satisfy*

$$s \leqslant \frac{p}{p-1} + \mathrm{ord}_p\big([F : \mathbf{Q}_p]\big).$$

**Proof.** Let $\bar{s}$ be the largest slope for $F^{gal}/\mathbf{Q}_p$, so we need to show that $\bar{s} \leqslant \frac{p}{p-1} + \mathrm{ord}_p([F : \mathbf{Q}_p])$. This is clear if $\bar{s} \leqslant 1$, so we can assume that $\bar{s} > 1$, i.e. there is wild ramification.

Let $G = \mathrm{Gal}(F^{gal}/\mathbf{Q}_p)$ and let $H$ be the subgroup fixing $F$. From [JR06, §3.6], the extension $F/\mathbf{Q}_p$ has a distinguished chain of subfields corresponding to subgroups $HG^s$; we will denote the fixed field of $HG^s$ by $F_s$, and define $F_{s+}$ analogously. For values of $s$ where $HG^s \neq HG^{s+}$, $s = Slope_{avg}(F_{s+}/F_s)$. Since $G^{\bar{s}+}$ is trivial and $H$ cannot contain a non-trivial normal subgroup of $G$, $HG^{\bar{s}} \neq H = HG^{\bar{s}+}$. Hence, $\bar{s} = Slope_{avg}(F/F_{\bar{s}})$.

Among extensions $F/\mathbf{Q}_p$ of a given degree, it is clear geometrically from [JR06, §3.6], or algebraically from Lemma 1.3, that the value of $\bar{s}$ for any extension is bounded by its value for an extension having intermediate fields of index $p^j$ for all $0 \leqslant j \leqslant \mathrm{ord}_p([F : \mathbf{Q}_p])$. So, we can apply Lemma 1.5 to obtain $\bar{s} \leqslant \frac{p}{p-1} + \mathrm{ord}_p([F : \mathbf{Q}_p])$.  □

**Remark 1.11.** Proposition 1.10 will be applied below to extensions of $\mathbf{Q}_2$ of degrees 12 and 14, showing that the Galois closure in each case has wild slopes bounded by 4 and 3 respectively.

**Remark 1.12.** The proof of Proposition 1.10 shows that the extension $F/\mathbf{Q}_p$ must have certain intermediate fields, including a subfield corresponding the largest slope for $F^{gal}/\mathbf{Q}_p$. A nice illustration of this comes from sextic extensions of $\mathbf{Q}_2$. If the extension is wildly ramified, then $F/\mathbf{Q}_2$ must have a cubic subfield. Checking the appropriate table at [JR04], we see that there is exactly one sextic extension of $\mathbf{Q}_2$ which does not have a cubic subfield, but it is not wildly ramified. For fields which are wildly ramified, the slope of $F/F_3$ where $F_3$ is the cubic subfield is the largest slope for $F^{gal}$. For the field $F$ with no cubic subfield, there is tame ramification and $F$ has a quadratic subfield $F'$ so that $F/F'$ corresponds to the maximum slope of 1 for $F^{gal}$.

## 2. Number fields of degree less than 16

In Sections 2.1 and 2.2, we prove the following theorem.

**Table 1**

Unconditional root discriminant bounds. A field $K$ with $[K : \mathbf{Q}] \geqslant n$ has $\mathrm{rd}(K)$ greater than or equal to the given value. If $K \in \mathcal{K}_2$, then $gms_2(K)$ is greater than or equal to the given bounds.

| $gms_2$ for $\mathcal{K}_2$ | $\mathrm{rd}(K)$ | $n$ | $gms_2$ for $\mathcal{K}_2$ | $\mathrm{rd}(K)$ | $n$ |
|---|---|---|---|---|---|
| 4.002 | 16.032 | 88 | 4.303 | 19.742 | 400 |
| 4.066 | 16.756 | 110 | 4.428 | 21.535 | 2400 |
| 4.216 | 18.597 | 220 | 4.449 | 21.843 | 4800 |
| 4.231 | 18.788 | 240 | 4.460 | 22.021 | 8862 |

**Theorem 2.1.** *There do not exist any degree n extensions of* $\mathbf{Q}$ *which are unramified away from* 2 *where* $9 \leqslant n \leqslant 15$.

We consider each degree $n$, and within each degree we consider the possible Galois groups among the transitive subgroups of $S_n$. To minimize the number of cases we need to consider in detail, we note that if $G$ is the Galois group of $K^{gal} \in \mathcal{K}_2$ where $[K : \mathbf{Q}] > 8$, then $G$ must satisfy the following two properties:

(1) $|G|$ is a multiple of $2^4$;
(2) all proper quotients of $G$ are in $\mathcal{G}_2$.

The first property is a consequence of Theorem 2.23 of [Har94]; the second is clear. Progressing successively through degrees, there will only be a small number of groups which satisfy both conditions. For reference, we state here previously known results of groups which are not in $\mathcal{G}_2$ based on [Tat94, Har94, Mar63, Bru01, Les06]. They provide the starting point for applying property (2) above. Suppose $K \in \mathcal{K}_2$ and $G \in \mathcal{G}_2$. Then,

(1) $[K : \mathbf{Q}] \neq 3, 5, 6, 7$;
(2) if $[K : \mathbf{Q}] \leqslant 8$, then $\mathrm{Gal}(K^{gal}/\mathbf{Q})$ is a 2-group;
(3) if $|G| < 272$, then $G$ is a 2-group;
(4) $G \neq PSL_2(2^j)$ for $j \geqslant 1$;
(5) if $G$ is a 2-group, then $G$ can be generated by two elements, one of which is 2-torsion.

Markšaǐtis's result [Mar63] carries even more information. If $G_{\mathbf{Q},2}$ is the Galois group of the maximal extension of $\mathbf{Q}$ unramified away from 2, he shows that the maximal pro-2 quotient of $G_{\mathbf{Q},2}$ is the pro-2 completion of the free product $\mathbf{Z} * C_2$.

For lower bounds on root discriminants, we will refer to Table 1. These values are simply an extract from [Odl76] and are provided for easy reference. We have added the values in the column "$gms_2$ for $\mathcal{K}_2$" which are simply log base 2 of the values in the $\mathrm{rd}(K)$ column, and then rounded down.

### 2.1. Degrees 9–11

The main goal of this section is to prove Proposition 2.4 below. First, we establish some preliminaries.

**Proposition 2.2.** *If $K$ is an extension of* $\mathbf{Q}$ *of degree $n < 12$, and $m$ is the number of wild slopes for $p = 2$ for* $K^{gal} \otimes \mathbf{Q}_2$, *then*

$$gms_2(K) \leqslant \begin{cases} 97/24 < 4.042 & \text{if } m \leqslant 4, \\ 101/24 < 4.209 & \text{if } m \leqslant 5, \\ 53/12 < 4.417 & \text{if } m \leqslant 6, \\ 71/16 < 4.438 & \text{for any } m. \end{cases}$$

**Table 2**
Maximum slope combinations for octic extensions of $\mathbf{Q}_2$.

| # slopes | Slope content | $gms_2$ |
|---|---|---|
| 3 | $[3, 4, 5]_1$ | $31/8$ |
| 4 | $[2, 3, 4, 5]_1$ | $4$ |
| 5 | $[2, 3, 7/2, 4, 5]_1$ | $67/16$ |
| 6 | $[2, 3, 7/2, 4, 17/4, 5]_1$ | $141/32$ |

**Proof.** We consider the possible decompositions of $K \otimes \mathbf{Q}_2 \cong \prod_i K_{p,i}$. If no $K_{p,i}$ has degree 8 over $\mathbf{Q}_2$, then all slopes $s$ for $K^{gal} \otimes \mathbf{Q}_2$ satisfy $s \leqslant 4$ by [JR06]. Hence, $gms_2(K) \leqslant 4$ which implies the asserted bounds.

Now suppose some $K_{p,i}/\mathbf{Q}_2$ is an octic extension. There can be at most one other non-trivial extension among the $K_{p,i}/\mathbf{Q}_2$, and its degree over $\mathbf{Q}_2$ is at most 3. A complete summary of all candidates of the slope content of an octic over $\mathbf{Q}_2$ is given in [JR08]. Table 2 gives maximal slope content for $m$ wild slopes, for $m \geqslant 3$. Using this, we compute the crude bound for an octic and a quadratic (maximal slope content being $[3]_1$) and for an octic with a cubic (maximal slope content being $[\ ]_3$). The statement of the theorem lists the resulting values of the Galois mean slope.

For example, the first entry arises from the maximum contribution by an octic with 3 slopes, plus a single slope of 3 from a quadratic to give slope content $[3, 3, 4, 5]_1$. On the other hand, the maximum for 5 slopes arises from an octic with slope content $[2, 3, 7/2, 4, 5]_1$ and a tame cubic to give content $[2, 3, 7/2, 4, 5]_3$.  □

If $K/\mathbf{Q}$ is unramified away from 2, then we can compare $gms_2(K)$ with values in Table 1 to get the following.

**Corollary 2.3.** *If $K/\mathbf{Q}$ is unramified away from 2, $[K : \mathbf{Q}] < 12$, and $G = \mathrm{Gal}(K^{gal}/\mathbf{Q})$, let $m = \mathrm{ord}_2(|G|)$, then*

$$|G| < \begin{cases} 110 & \text{if } m \leqslant 4, \\ 220 & \text{if } m \leqslant 5, \\ 2400 & \text{if } m \leqslant 6, \\ 4800 & \text{in all cases.} \end{cases}$$

**Proposition 2.4.** *If $K \in \mathcal{K}_2$, then $[K : \mathbf{Q}]$ is not equal to 9, 10, or 11.*

**Proof.** We consider each of the possible Galois groups $G$ of polynomials of degree 9, 10, and 11, of which there are 34, 45, and 8 groups respectively. By Theorem 2.23 of [Har94], we can eliminate $G$ if $|G|$ is not a multiple of 16. By Corollary 2.3, we eliminate groups where $|G| \geqslant 4800$. Next, we eliminate groups which have a quotient which has already been eliminated. Note, this already eliminates all groups in degree 11. Each of the remaining groups is then eliminated by Corollary 2.3 by comparing $|G|$ with $\mathrm{ord}_2(|G|)$:

$$|9T_{19}| = 144 = 2^4 \cdot 9, \qquad |10T_{28}| = 400 = 2^4 \cdot 25, \qquad |10T_{30}| = 720 = 2^4 \cdot 45,$$

$$|10T_{31}| = 720 = 2^4 \cdot 45, \qquad |10T_{33}| = 800 = 2^5 \cdot 25, \qquad |10T_{35}| = 1440 = 2^5 \cdot 45. \qquad □$$

*2.2. Degrees 12–15*

The structure of this section is similar to that of Section 2.1, although bounding $gms_2$ is more complicated. We start with a bound on the slope content of composita of certain quartic extensions.

**Proposition 2.5.** *Let $F$ be the compositum of all quartic extensions of $\mathbf{Q}_2$ whose Galois closures have Galois groups which are 2-groups. Then $[F : \mathbf{Q}_2] = 2^8$, $F$ has residue field degree 4 and slope content $[2, 2, 3, 3, 7/2, 4]_1$.*

**Proof.** Clearly the tame degree is 1 since the compositum has Galois group a 2-group. Let $\mathbf{G}_2$ be the Galois group of the composita of all 2-group extensions of $\mathbf{Q}_2$. The group $\mathbf{G}_2$ is the pro-2 completion of the group with presentation $\langle x, y, z \mid x^2 y^3 z^{-1} yz = 1 \rangle$ [NSW00]. Using this description, one can compute with gap [GAP06] the intersection of the kernels of all homomorphisms to the groups $V_4$, $C_4$, and $D_4$, the three Galois groups of quartics which are 2-groups. The quotient of $\mathbf{G}_2$ by this kernel has order $2^8$, hence $[F : \mathbf{Q}_2] = 2^8$.

Naturally, this compositum contains the unramified extension of $\mathbf{Q}_2$ of degree 4, and from the tables in [JR06], we see that the wild slopes include $[2, 2, 3, 7/2, 4]$ since there are $D_4$ quartic fields with at least each slope once, and one with two slopes of 2.

To find the final slope, we consider the group $C_2 : C_4 : (C_4 \times C_2) = G(64, 61)$, meaning group number 61 among groups of order 64 in the numbering of gap. From the presentation above, one can check that $G(64, 61)$ appears as a Galois group over $\mathbf{Q}_2$. From the group itself, one can verify that a field with Galois group $G(64, 61)$ is the compositum of its $D_4$ subfields. Hence, there is an extension of $\mathbf{Q}_2$ with Galois group $G(64, 61)$ which is a subfield of $F$. But, the group $G(64, 61)$ has $8T_{11}$ as a quotient. Consulting [JR08, Table 5.1], we see that there are $8T_{11}$ fields with slope content $[2, 3, 3]_1$ in the notation used here (it is listed there as $[0, 2, 3, 3]$). In particular, there are two slopes of 3 for $F$. □

**Remark 2.6.** One can see the two slopes of 3 explicitly as follows. Consider the polynomials $x^4 + 2x^2 - 2$, $x^4 + 6x^2 + 3$, and $x^4 + 6x^2 + 18$, which each have Galois group $D_4$ both over $\mathbf{Q}_2$ and over $\mathbf{Q}$. One can compute using gp [PAR00] their compositum over $\mathbf{Q}$, $K_{64}$, which is a degree 64 extension with discriminant $2^{196}$. The extension $K_{64}$ also has a single prime above 2, so its global Galois group equals its decomposition group for the prime above 2. As a result, all subfields of the 2-adic field are seen globally. Computing subfields of $K_{64}$ and the 2-parts of their discriminants shows that $K_{64}$ contains a quadratic unramified extension and has slope content $[2, 2, 3, 3, 7/2]_1$.

**Proposition 2.7.** *If $K$ is an extension of $\mathbf{Q}$ of degree $n < 16$, and $m$ is the number of wild slopes for $p = 2$ for $K^{gal} \otimes \mathbf{Q}_2$, then*

$$gms_2(K) \leqslant \begin{cases} 203/48 < 4.230 & \text{if } m \leqslant 4, \\ 413/96 < 4.303 & \text{if } m \leqslant 5, \\ 495/112 < 4.420 & \text{if } m \leqslant 6, \\ 107/24 < 4.459 & \text{for any } m. \end{cases}$$

**Proof.** As in Proposition 2.2, the local algebra $K \otimes \mathbf{Q}_2 = \prod_i K_{p,i}$ must have an octic field $K_{p,i}$ or all slopes would be $\leqslant 4$, here using Proposition 1.10 to rule out local fields $K_{p,i}$ with $9 \leqslant [K_{p,i} : \mathbf{Q}_2] \leqslant 15$.

Note that a degree 6 field can always be replaced with its twin algebra. From [JR04], all 2-adic sextic fields have twin algebras which split as a product of fields of degrees less than or equal to 4. Hence, we do not need to consider sextic factors.

The cases with $m \leqslant 6$ work just like in Proposition 2.2, where we use the crude bound for the slope content of the composita. For example, our bound for 5 slopes comes from $[3, 4, 5]_1$ for the octic, $[3, 4]_1$ for a quartic, and a tame cubic combining to yield $gms_2([3, 3, 4, 4, 5]_3) = 413/96$.

For $m \geqslant 7$, we divide into several cases. If 5 is not a slope of the octic factor, we can apply the crude bound for the maximum slope content for the compositum of an octic (if 5 is not a slope, $[3, 7/2, 4, 17/4, 19/4]_1$ has the largest $gms_2$), a quartic with slope content $[2, 3, 4]_1$, and a quadratic with slope content $[3]_1$. The result is $gms_2([2, 3, 3, 3, 7/2, 4, 4, 17/4, 19/4]_1) = 421/96 < 107/24$.

Now assume that 5 is a slope for the octic. If $17/4$ is *not* a slope of the octic factor, then the maximum slope content of the octic is $[2, 3, 7/2, 4, 5]_1$. Again, the crude bound for this with a quartic and a quadratic is $gms_2([2, 2, 3, 3, 3, 7/2, 4, 4, 5]) = 1125/256 < 107/24$.

Finally, we have the case where 5 and $17/4$ are both slopes of the octic. From [JR08], the slope content of such an octic is $[2, 3, 7/2, 4, 17/4, 5]_1$ and only possibilities for the Galois group are $8T_{27}$, $8T_{28}$, are $8T_{35}$, each of which is a 2-group. In each case, the bottom 4 slopes $[2, 3, 7/2, 4]$ are visible in the compositum of quartic subfields.

If we combine with quartics whose Galois groups are 2-groups, then the maximal slope content of the quartic part is $[2, 2, 3, 3, 7/2, 4]$ by Proposition 2.5, so maximum combination in this case is $gms_2([2, 2, 3, 3, 7/2, 4, 17/4, 5]_3) = 427/96 < 107/24$. Finally, if we use the crude bound for the composita of an octic with slope content $[2, 3, 7/2, 4, 17/4, 5]_1$, a quartic whose Galois group is not a 2-group, so maximal slope content of $[8/3, 8/3]_3$, and a quadratic (slope content $[3]_1$), we get $gms_2([2, 8/3, 8/3, 3, 3, 7/2, 4, 17/4, 5]_3) = 107/24$.   $\square$

Now, we can combine Proposition 2.7 with bounds from Table 1 to get the following.

**Corollary 2.8.** *If $K/\mathbf{Q}$ is unramified away from 2, $[K : \mathbf{Q}] < 16$, and $G = \mathrm{Gal}(K^{gal}/\mathbf{Q})$, let $m = \mathrm{ord}_2(|G|)$, then*

$$|G| < \begin{cases} 240 & \textit{if } m \leqslant 4, \\ 400 & \textit{if } m \leqslant 5, \\ 2400 & \textit{if } m \leqslant 6, \\ 8862 & \textit{in all cases.} \end{cases}$$

**Proposition 2.9.** *If $K \in \mathcal{K}_2$, then $[K : \mathbf{Q}]$ is not equal to 12, 13, 14, or 15.*

**Proof.** As before, we consider each of the possible Galois groups $G$ of polynomials of the stated degrees. For $n = 12$, 13, 14, and 15, there are 301, 9, 63, and 104 conjugacy classes of subgroups in $S_n$ respectively. By Theorem 2.23 of [Har94], we can eliminate $G$ if $|G|$ is not a multiple of 16. By Corollary 2.8, we eliminate groups where $|G| \geqslant 8862$, and then eliminate groups which have a quotient which has already been eliminated. For the remaining groups, we give their orders with partial factorization to show that they too are eliminated by 2.8:

$$|12T_j| = 1296 = 2^4 \cdot 81 \quad \text{for } j = 215, 216,$$

$$|12T_j| = 2592 = 2^5 \cdot 81 \quad \text{for } 244 \leqslant j \leqslant 249,$$

$$|12T_j| = 5184 = 2^6 \cdot 81 \quad \text{for } 262 \leqslant j \leqslant 264,$$

$$|13T_7| = 5616 = 2^4 \cdot 351,$$

$$|14T_{16}| = 336 = 2^4 \cdot 21. \quad \square$$

Note, no transitive subgroups of $S_{15}$ passed through the various filters discussed in the proof of Theorem 2.9, and only one group needed to be considered in each of degrees 13 and 14.

Since there is particular interest in whether or not simple groups are in $\mathcal{G}_2$, we extract the new cases covered by Theorem 2.1. Additional results on simple groups excluded from $\mathcal{G}_2$ by a combination of root discriminant bounds and group theoretic techniques, see [Jon].

**Corollary 2.10.** *The following simple groups are not elements of $\mathcal{G}_2$: alternating groups $A_j$ for $9 \leqslant j \leqslant 15$, Mathieu groups $M_{11}$ and $M_{12}$, and $PSL_3(3)$.*

## Acknowledgment

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