2-**group Belyi maps**

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- Mary, Jim, Matt, and Nicole

Outline

Motivation

Galois representations

Let X be an irreducible, smooth projective algebraic curve of genus $g \geq 1$ over a number field K. Let $G_K := \mathrm{Gal}(K^{\mathrm{al}} \mid K)$ be the absolute Galois group of K and let $\ell \in \mathbb{Z}$ be prime.

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The geometry of X and the arithmetic of ρ are inimately related.
For example, if X has good reduction at a prime $\mathfrak{p}$ above $p \neq \ell$, then $\mathfrak{p}$ will be unramified in the ℓ-**torsion field** $K(J[\ell])$.

Belyi's theorem

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Theorem (Belyi 1979)

An algebraic curve (smooth projective) X over $\mathbb{C}$ can be defined over a number field if and only if X admits a Belyi map.

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We can now state Beckmann's theorem to relate Belyi maps to our original motivation.

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Theorem (Beckmann 1989)

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Let p be a prime not dividing $\#G$.
Then there exists a number field M satisfying the following properties.

- *p is unramified in M*
- *ϕ is defined over M*
- *X is defined over M*
- *X has good reduction at all primes $\mathfrak{p}$ of M above p*

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The hope is that an explicit nonsolvable field ramified only at 2 can be obtained as $K(\mathrm{Jac}(X)[2])$ where X is the domain of a Galois Belyi map with monodromy group a 2-group.
We call these Belyi maps 2-**group Belyi maps**.

Main results

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Main results

Motivated by the applications of 2-group Belyi maps to arithmetic geometry, we now state the main results.

- implementation of an algorithm to enumerate isomorphism classes of 2-group Belyi maps
- implementation of an algorithm to compute equations for 2-group Belyi maps over finite fields
- implementation of a *method* to compute equations for 2-group Belyi maps over number fields
- computational and theoretical evidence supporting a conjecture that every 2-group Belyi map is defined over an abelian extension of the rationals

2-group Belyi maps as iterated quadratic extensions

$$
\begin{tikzcd}
X_r \arrow[d, "2"'] \arrow[ddddr, bend left] & \\
\vdots \arrow[d, "2"'] & \\
X_2 \arrow[d, "2"'] \arrow[ddr, bend left] & \\
X_1 \arrow[dr, "2"'] & \\
& X_0 = \mathbb{P}^1
\end{tikzcd}
\qquad
\begin{tikzcd}
K(X_r) \arrow[d, dash, "2"'] \arrow[ddddr, dash, bend left] & \\
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Background

Isomorphism of Belyi maps

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Let $\phi\colon X \to \mathbb{P}^1$ and $\phi'\colon X' \to \mathbb{P}^1$ be Belyi maps of degree d. ϕ and ϕ' are **isomorphic** (respectively **lax isomorphic**) if the diagrams

$$\begin{tikzcd} X \arrow[rr, "\sim"] \arrow[dr, "\phi"'] & & X' \arrow[dl, "\phi'"] \\ & \mathbb{P}^1 & \end{tikzcd}, \text{ respectively } \begin{tikzcd} X \arrow[r, "\sim"] \arrow[d, "\phi"'] & X' \arrow[d, "\phi'"] \\ \mathbb{P}^1 \arrow[r, "\sim", "\beta"'] & \mathbb{P}^1 \end{tikzcd}$$

commute where $\beta(\{0,1,\infty\}) = \{0,1,\infty\}$.

Permutation Triples

A **transitive permutation triple of degree** d is a triple

$$\sigma = (\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3$$

such that

- $\sigma_\infty \sigma_1 \sigma_0 = 1$
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The set of degree d Belyi maps up to isomorphism is in bijection with the set of degree d transitive permutation triples up to **simultaneous conjugation** and the group $\langle \sigma \rangle$ is the monodromy group of ϕ.

Passports

A **passport** $\mathcal{P}$ consists of the data (g, G, λ) where $g \geq 0$ is an integer, $G \leq S_d$ is a transitive subgroup, and $\lambda = (\lambda_0, \lambda_1, \lambda_\infty)$ is a triple of partitions of d.

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The **passport of a Belyi map** $\phi : X \to \mathbb{P}^1$ is $(g(X), \mathrm{Mon}(\phi), (\lambda_0, \lambda_1, \lambda_\infty))$ with $g(X)$ the genus of X, $\mathrm{Mon}(\phi)$ the monodromy group of ϕ, and the partitions from ramification.

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The **passport of a permutation triple** σ is $(g(\sigma), \langle\sigma\rangle, \lambda(\sigma))$ where

$$g(\sigma) = 1 - d + (e(\sigma_0) - e(\sigma_1) - e(\sigma_\infty))/2$$

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$$e(\tau) = d - \#\text{cycles of } \tau,$$

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We now discuss the importance of organizing triples by passport.

Fields of moduli, fields of definition, and passports

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The situation improves, however, in the Galois setting...

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Then

- ϕ and X are defined over $M(\phi)$,
- $\#G = d$,
- all cycles of σ_s have the same length for $s \in \{0, 1, \infty\}$,
- and if we let a, b, c be the orders of $\sigma_0, \sigma_1, \sigma_\infty$ respectively, we have

$$g(X) = 1 + \frac{\#G}{2}\left(1 - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}\right).$$

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For K perfect, $K(X)$ has a primitive element and can be written as

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The monodromy group in this setting corresponds to field automorphisms of the Galois closure of $K(X)$ fixing $K(x)$.

Computing permutation triples

Setup

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- $\sigma_\infty \sigma_1 \sigma_0 = \mathrm{id}$;
- $G := \langle \sigma_0, \sigma_1 \rangle$ is a transitive subgroup of S_d; and
- G is a 2-group of order d embedded in S_d via its left regular representation.

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We say two degree d 2-group permutation triples σ, σ' are **simultaneously conjugate** if there exists $\tau \in S_d$ such that

$$\sigma^\tau := (\tau^{-1}\sigma_0\tau, \tau^{-1}\sigma_1\tau, \tau^{-1}\sigma_\infty\tau,) = \sigma'$$

Lifting permutation triples

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$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} \widetilde{G} \xrightarrow{\pi} \langle \sigma \rangle \longrightarrow 1$$

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For a 2-group permutation triple σ, we denote the set of lifts of σ by $\mathrm{Lifts}(\sigma)$ and $\mathrm{Lifts}(\sigma)/\sim$ denotes the set of lifts up to simultaneous conjugation.

Algorithm to compute Lifts$(\sigma)/\sim$

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1. Let $G = \langle \sigma \rangle$ and compute representatives of $H^2(G, A)$ where $A := \mathbb{Z}/2\mathbb{Z}$ with the trivial G-module structure
2. For each $f \in H^2(G, A)$ compute the corresponding extension

$$1 \longrightarrow A \xrightarrow{\iota_f} \widetilde{G}_f \xrightarrow{\pi_f} G \longrightarrow 1$$

Algorithm to compute Lifts$(\sigma)/\sim$

Input: σ a 2-group permutation triple of degree d
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1. Let $G = \langle\sigma\rangle$ and compute representatives of $H^2(G, A)$ where $A := \mathbb{Z}/2\mathbb{Z}$ with the trivial G-module structure
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Choosing

$$\alpha := (1\ d+1)(2\ d+2)\dots(d-1\ 2d-1)(d\ 2d)$$

allows us to label blocks by reducing modulo d.

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Lastly, we quotient by simultaneous conjugation to obtain
$\mathrm{Lifts}(\sigma)/\sim = \Big\{((1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)),$
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Bipartite graphs of permutation triples

Now that we can lift permutation triples, we now describe some notation for the bipartite graphs that organize these triples.

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For $i \in \mathbb{Z}_{\geq 1}$ we define the bipartite graph denoted $\mathscr{G}_{2^i}$ with the following node sets.

- $\mathscr{G}_{2^i}^{\text{above}}$: the set of isomorphism classes of 2-group Belyi maps of degree 2^i indexed by 2-group permutation triples $\widetilde{\sigma}$ up to simultaneous conjugation in S_{2^i}
- $\mathscr{G}_{2^i}^{\text{below}}$: the set of isomorphism classes of 2-group Belyi maps of degree 2^{i-1} indexed by 2-group permutation triples σ up to simultaneous conjugation in $S_{2^{i-1}}$

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For every pair of nodes $(\widetilde{\sigma}, \sigma) \in \mathscr{G}_{2^i}^{\text{above}} \times \mathscr{G}_{2^i}^{\text{below}}$ there is an edge between σ and $\widetilde{\sigma}$ if and only if $\widetilde{\sigma}$ is simultaneously conjugate to a lift of σ.

Algorithm to compute $\mathscr{G}_{2^i}$

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4. For every pair $(\widetilde{\sigma}, \sigma) \in \mathscr{G}_{2^i}^{\text{above}} \times \mathscr{G}_{2^i}^{\text{below}}$ place an edge between $\widetilde{\sigma}$ and σ if and only if there is a triple in the equivalence class $[\widetilde{\sigma}] \in \mathrm{Lifts}(\mathscr{G}_{2^{i-1}})/\sim$ that is a lift of σ

Results : number of triples and passports

Theorem (M.)

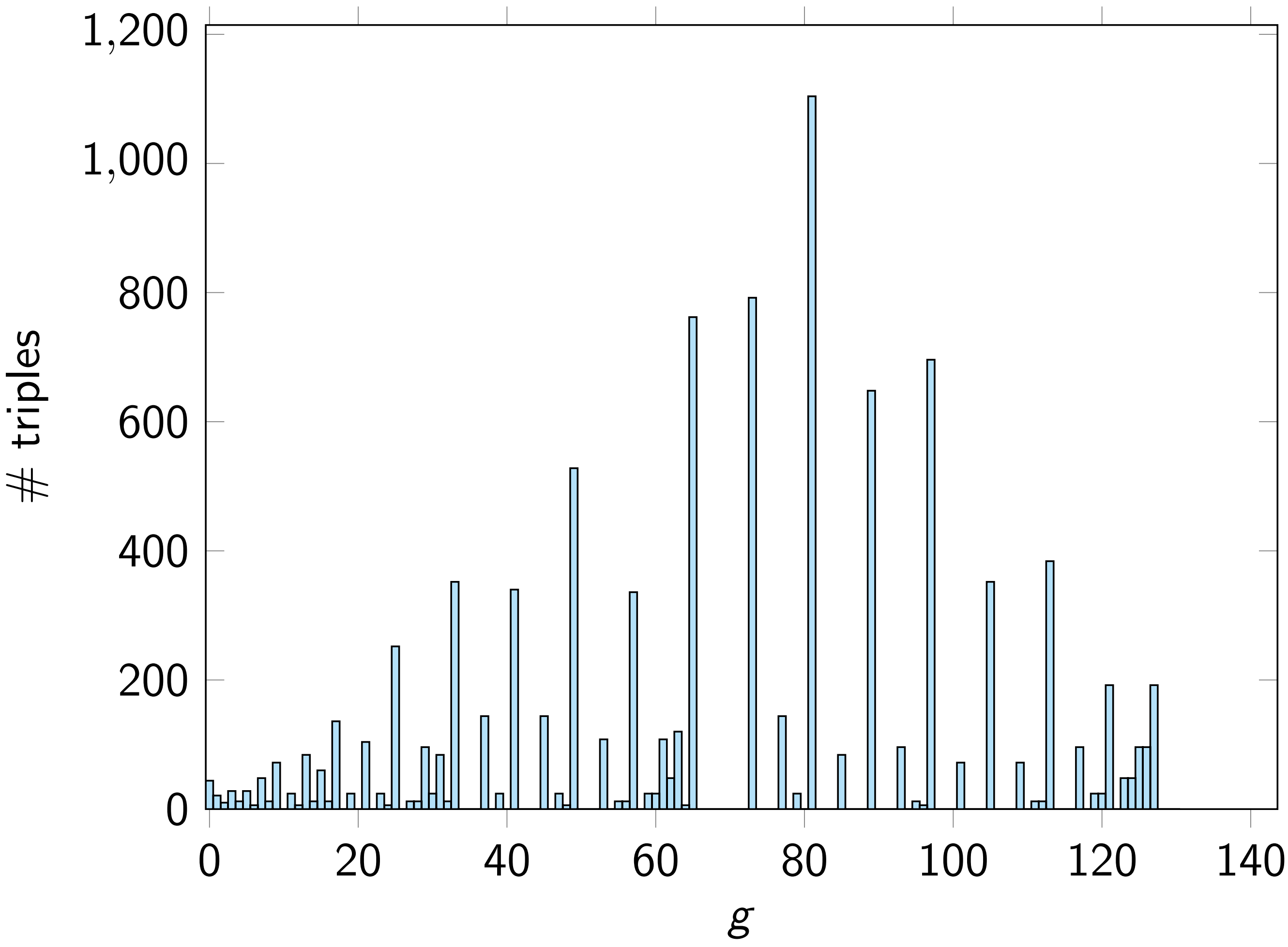
The following tables list the number of isomorphism classes of 2-group Belyi maps, the number of passports, and number of lax passports respectively up to degree 256.

d	1	2	4	8	16	32	64	128	256
$\#$ *triples*	1	3	7	19	55	151	503	1799	7175

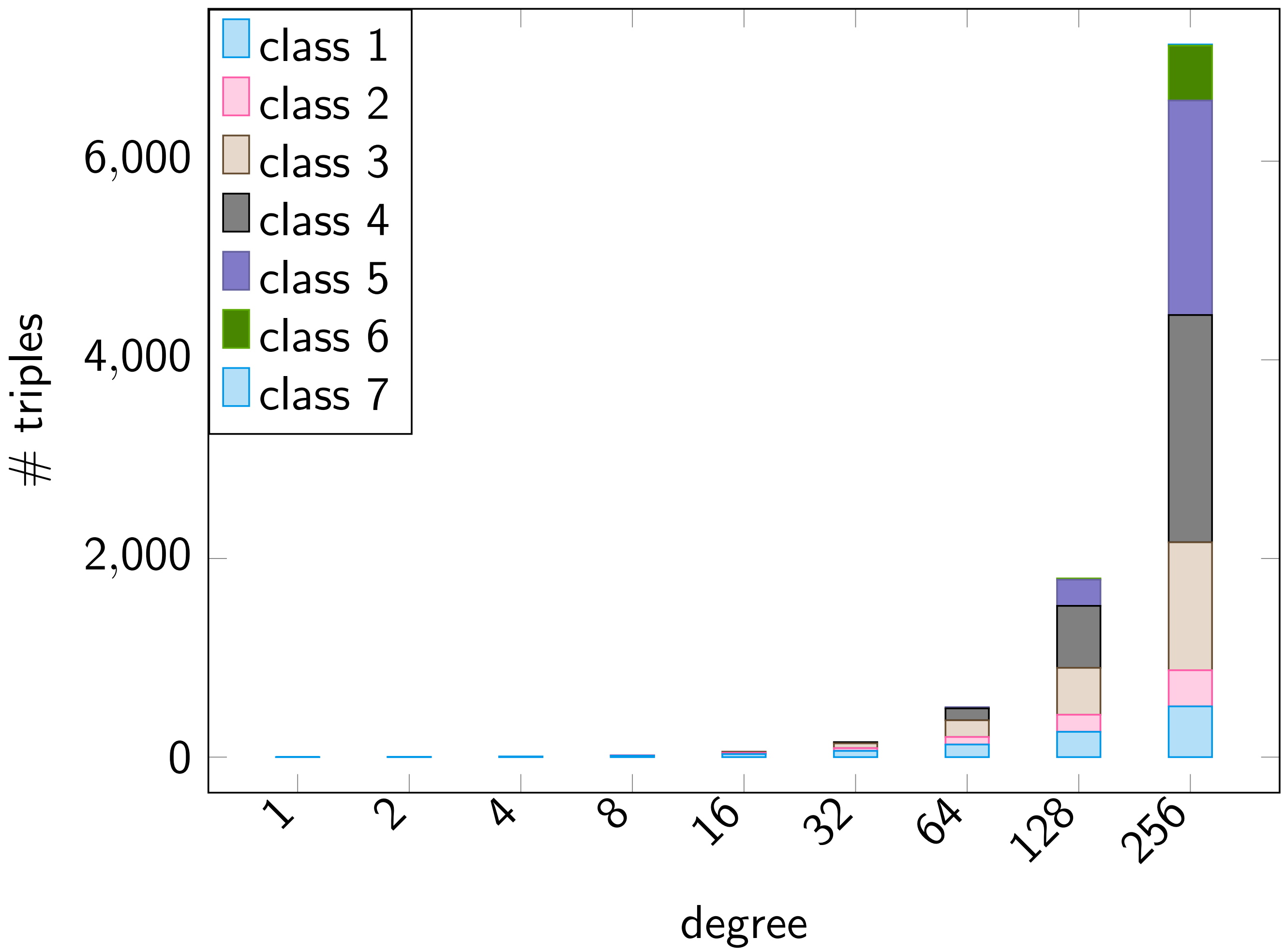
d	1	2	4	8	16	32	64	128	256
$\#$ *passports*	1	3	7	16	41	96	267	834	2893

d	1	2	4	8	16	32	64	128	256
$\#$ *lax passports*	1	1	3	6	14	31	85	257	882

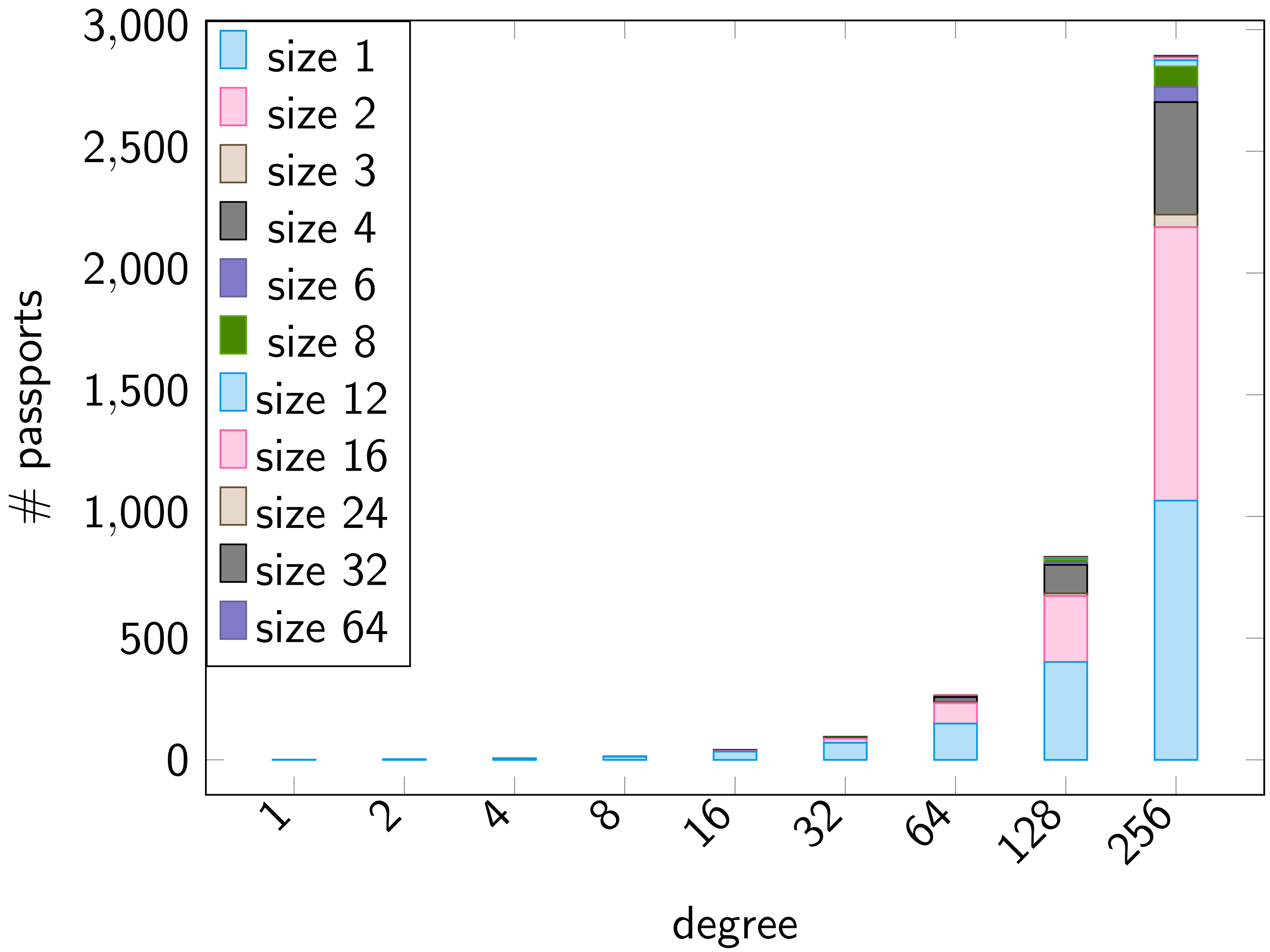
Results : distribution of genera



Results : groups by nilpotency class



Results : passport sizes



A refined conjecture

Passports

Recall that a passport $\mathcal{P}$ consists of the data (g, G, λ) where $g \in \mathbb{Z}_{\geq 0}$, G is a transitive subgroup of S_d and $\lambda = (\lambda_0, \lambda_1, \lambda_\infty)$ is a triple of partitions of d corresponding to conjugacy classes (C_0, C_1, C_∞) of S_d.

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The size of $\mathcal{P}$ is the cardinality of the set $\Sigma_{\mathcal{P}}$ defined by

$$\Big\{(\sigma_0, \sigma_1, \sigma_\infty) \in C_0 \times C_1 \times C_\infty : \sigma_\infty \sigma_1 \sigma_0 = 1 \text{ and } \langle \sigma_0, \sigma_1 \rangle = G\Big\}/\!\sim$$

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As a result of the action of $G_{\mathbb{Q}}$ on $\mathcal{P}$, the size of $\mathcal{P}$ bounds the degree of the field of moduli of any Belyi map with passport $\mathcal{P}$.

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To instead analyze $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}} \,|\, \mathbb{Q}^{\mathrm{ab}})$ we *refine* the notion of a passport.

Refined passports

A **refined passport** $\mathscr{P}$ consists of the data (g, G, c) where $g \in \mathbb{Z}_{\geq 0}$, G is a transitive subgroup of S_d and $c = (c_0, c_1, c_\infty)$ is a triple of conjugacy classes of G.

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where $(\sigma_0, \sigma_1, \sigma_\infty) \sim (\sigma_0', \sigma_1', \sigma_\infty')$ if there exists $\alpha \in \mathrm{Aut}(G)$ with $\alpha(\sigma_s) = \sigma_s'$ for every $s \in \{0, 1, \infty\}$.

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As was the case with passport, every permutation triple σ determines a refined passport $\mathscr{P}(\sigma)$.

A refined conjecture

Theorem (M.)

The size of $\mathscr{P}(\sigma)$ is equal to 1 for every 2-group permutation triple σ with degree ≤ 256.

A refined conjecture

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The size of $\mathscr{P}(\sigma)$ *is equal to* 1 *for every* 2*-group permutation triple* σ *with degree* ≤ 256.

Conjecture (ARC)

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Theorem (M.)

ARC is true for 2*-group permutation triples* σ *with* $\langle\sigma\rangle$ *dihedral.*

Computing equations