Distance-coupling as an Approach to Position and Formation Control

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***Abstract*— Studies of coordinated motion in autonomous vehicle groups have become a significant topic of interest in recent years. In this letter, we will study the case when the vehicles operating in an environment only have access to distance measurements from predetermined reference points, or anchors. We propose a new method for approximating vehicle positions from distance measurements through a process referred to as distance-coupling. By taking the difference between squared distance functions, we can cancel out the quadratic term and obtain a linear function of the position for the point of interest. First, we prove this method is sufficient in the context of the homing problem via the Lyapunov stability criterion. We then extend our proof to the scenario when there is some degree of error in the position of the anchors: defining bounds on the region of attraction for the corresponding policy. To further increase complexity, we show how the policy can be implemented on the formation control problem for a set of autonomous vehicles. In this context we make deductions on the equilibria of the distance-coupled formation policy and its convergence under linear transformations of the initial conditions: validating our claims empirically.**

I. Introduction

To date, autonomously operated vehicles are commonly designed around high-cost measurement devices which are used to localize in real time. That said, there exists instances, especially in low-cost systems, where the measurements available to the vehicles are limited to basic geometric information. In this letter we study the case when the information available to the vehicle of interest is restricted to its Euclidean distance from p-anchors, or fixed points. Exploiting a linear, first-order vehicle model, we will first address the homing problem. In this example case, the only goal is to move a vehicle from an arbitrary initial position to some predetermined equilibrium position using the available measurement data. Despite being based on nonlinear measurements, our policy is a simple linear function, and its stability can be proven in a straight-forward manner. While all example cases studied here will be performed in $\mathbb{R}^2$, the distance-coupling approach will primarily be defined in $\mathbb{R}^n$ without sacrificing robustness.

We will also approach the scenario where $(m = p)$-independently driven vehicles exist in an open environment and must be autonomously driven to some predetermined formation. The only measurements available to a given vehicle is the distance it stands from each of the others. This is often referred to as the distance-based formation problem and has maintained extensive research over the last two decades (see next section for literature review). We will show that the distance-coupled controller defined for the homing problem can be implemented with minimal adjustments and will demonstrate its stability empirically.

A. Literature Review

Studies of distance-based position functions have primarily been investigated through barycentric coordinates [1]. These methods represent the positions of anchors as vertices of a polygon and utilize its geometry to derive the vehicle's position. While this method is robust in many applications, the point is usually limited to the bounds of a convex polyhedron and maintains ill-defined behaviors when this assumption is broken. To overcome this, researchers have implemented policy switching schemes to the formation control problem with success in 2-D [2] and 3-D [3] coordinate frames.

Another popular method utilizes a negative gradient control law on a graph-based representation of the communication network [4, 5]. The graph can take many forms (directed/undirected, fully connected, etc.), but the connections are predetermined and static. Papers by Lin, Dimarogonas and Choi are also among the first to demonstrate the stability of assorted gradient-based formation control policies through standard analysis methods [6, 7, 8].

Other papers address formation control through analytical means. A noteworthy example is that of [9] which is among the first written on the topic and discusses a catalog of methods which can be used to stabilize formations within small perturbations of their equilibrium.

Lastly, the papers [10, 11] discuss the formation control of second-order systems, and [12] defines a policy in terms of the complex Laplacian. The review paper [13] lays a good foundation for many of the methods available. In addition to formation control in a distance-based context, they also present methods for more idealistic position and displacement-based measurement cases.

B. Paper Contributions

We approach the aforementioned measurement problem through a new method for approximating position from distance functions which exploits the structure of the inner product. By taking the difference of available squared-distance measurements, we can construct a linear approximation of the vehicle's position in the environment. We refer to this method as *distance-coupling*.

The distance-coupled approach is novel in comparison to the literature as it requires much more relaxed constraints on the positions of the anchors (or, vehicles in the formation problem). Where preexisting methods primarily depend on the convexity of the anchor environment, our method makes no such claims and allows the user to select anchor positions which are non-convex. However, we impose restrictions on the anchor's collinearity and the number of anchors required (discussed in Section III-A). Even with these restrictions we show that the policies generated have regions of attraction which are global when the anchor positions are known, and large when the anchors are linearly offset from their intended positions.

Finally, unlike many of the preexisting practices, the distance-coupling approach is applicable to higher-order systems with little change in complexity. While we demonstrate our results in an $\mathbb{R}^2$ environment for simplicity, the methods discussed in Section III are generalized to $\mathbb{R}^n$.

II. System Definition

We will start by describing the dynamics and reference control policy of interest, along with the measurements available to the system.

A. System Dynamics

The dynamics of interest are described as a linear, time-invariant single integrator of the form

$$\dot{x} = u \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$ are the state and the control input, respectively. As common in the literature, we assume the environment is unbounded and without obstacles. A primary goal for the study will be to move a vehicle from some initial condition to an equilibrium position denoted by $x^{(\text{eq})}$.

In the case that there are m-vehicles operating in a single environment we can rewrite (1) as

$$\dot{X} = U \tag{2}$$

where $X \in \mathbb{R}^{n\times m}$ and $U \in \mathbb{R}^{n\times m}$ are the state and control for the set of vehicles defined by

$$X = \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix}, \text{ and } U = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix}. \tag{3}$$

B. Ideal Control Policy

Since the system is linear, a state feedback controller that brings the state to the goal position, $x^{(\text{eq})}$, can be defined as

$$u(x) = C(x - x^{(\text{eq})}). \tag{4}$$

Here, $x^{(\text{eq})} \in \mathbb{R}^n$ is the desired position and $C \in \mathbb{R}^{n\times n}$ describes the policy coefficients. By standard linear system theory, if $\text{Re}(\lambda) < 0 : C(x - x^{(\text{eq})}) = \lambda(x - x^{(\text{eq})})$ then the system is globally stable and the state will converge to the desired equilibrium from any initial condition [14].

Unfortunately, we are limited to the measurements described in the next section and cannot define a direct control policy from the state of the vehicle. That said, (4) will be referred to as the ideal controller.

C. Anchor and Anchor-distance Sets

We assume the presence of p*-anchors* which are represented by the set

$$\mathcal{A} = \{a_i \in \mathbb{R}^n : \forall i \le p\} \tag{5}$$

where a_i is the position of the i-th anchor. From this, the vehicle with position denoted by x is given the distance from its position to each of the anchors such that the *anchor-distance* set is defined as

$$\begin{aligned} &\mathcal{D}(x,\mathcal{A}) = \{d(x,a_i) \in \mathbb{R} : a_i \in \mathcal{A}\} \\ &\text{where } d(x,a_i) = \sqrt{(x-a_i)^\intercal(x-a_i)}. \end{aligned} \tag{6}$$

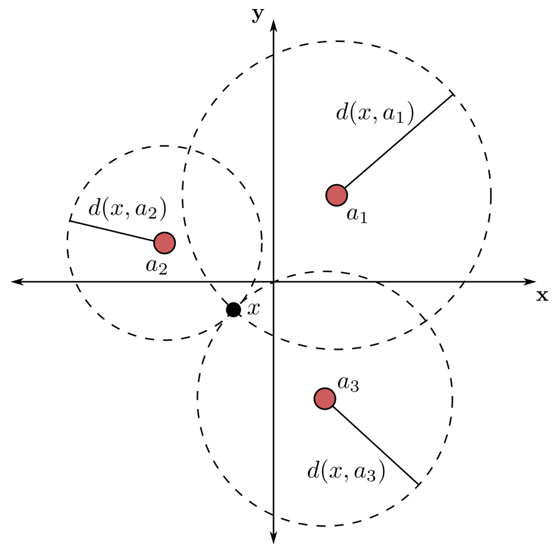

Fig. 1: Anchor set where $p = 3$ so that $\mathcal{A} = \{a_1, a_2, a_3\}$ and $\mathcal{D}(x, A) = \{d(x, a_1), d(x, a_2), d(x, a_3)\}$.

An example for when $p = 3$ is shown in Figure 1. In the interest of conciseness, the individual distance functions from the point x are referred to as $d_i(x) = d(x, a_i)$, and the anchor-distance set is referred to as $\mathcal{D}(x) = \mathcal{D}(x, \mathcal{A})$.

III. Anchor-based Position Control

In this section we show how (4) can be indirectly implemented as a function of the measurement set $\mathcal{D}(x)$ via what we refer to as *distance-coupled* functions. Then, using this new position function, we will prove stability, show the applicable region of attraction and study the behavior of the model in (1).

A. Distance-coupled Position and Control

Start by squaring and expanding an element of the set (6) for an individual anchor such that $a_i \in \mathcal{A}$ and

$$\begin{aligned} d_i^2(x) &= (x-a_i)^\intercal(x-a_i) \\ &= x^\intercal x - 2x^\intercal a_i + a_i^\intercal a_i. \end{aligned} \tag{7}$$

The key step to our approach is to note that by subtracting the squared-distance functions for two anchors, the quadratic terms $x^\intercal x$ cancel out such that

$$\begin{aligned} d_i^2(x) - d_j^2(x) &= (x^\intercal x - 2x^\intercal a_i + a_i^\intercal a_i) \\ &\quad - (x^\intercal x - 2x^\intercal a_j + a_j^\intercal a_j) \\ &= -2x^\intercal(a_i - a_j) + a_i^\intercal a_i - a_j^\intercal a_j. \end{aligned} \tag{8}$$

Moving the state-related terms to the left we obtain

$$-2(a_i - a_j)^\intercal x = d_i^2(x) - d_j^2(x) - a_i^\intercal a_i + a_j^\intercal a_j. \tag{9}$$

We reorganize the terms in (9) in matrix form with the intent of obtaining an expression for the full state x:

$$\begin{aligned} A &= \text{vstack}(\{-2(a_i - a_j)^\intercal : \forall(i,j) \in \mathcal{I}\}), \\ b &= \text{vstack}(\{a_i^\intercal a_i - a_j^\intercal a_j : \forall(i,j) \in \mathcal{I}\}), \\ h(x) &= \text{vstack}(\{d_i^2(x) - d_j^2(x) : \forall(i,j) \in \mathcal{I}\}), \end{aligned} \tag{10}$$

where $\mathcal{I}$ is the set of (i,j) indices which make up every combination of the anchors included in (5). Using $|\mathcal{I}|$ to denote the cardinality of $\mathcal{I}$, we then have $A \in \mathbb{R}^{|\mathcal{I}| \times n}$, and the vectors $b, h(x) \in \mathbb{R}^{|\mathcal{I}|}$.

Combining the equations of the form (9), the position of the vehicle, x, can be computed by solving a linear system such that

$$\begin{aligned} &Ax = h(x) - b \\ &\Rightarrow A^\intercal A x = A^\intercal (h(x) - b) \\ &\Rightarrow x = (A^\intercal A)^{-1} A^\intercal (h(x) - b) \\ &\quad\;\; = K(h(x) - b), \end{aligned} \tag{11}$$

where $K = (A^\intercal A)^{-1} A^\intercal$. Since the matrix $A^\intercal A$ is by definition positive definite, it is invertible if A has full column rank with a subset of $n+1$ noncollinear anchor positions. While we do not include the proof here for conciseness, it is trivial and follows from standard linear algebra theory.

Equation (11) is referred to as the distance-coupled position and can be substituted into (4) to obtain the corresponding distance-coupled control policy:

$$u(x) = C(K(h(x) - b) - x^{(\text{eq})}), \tag{12}$$

where $u(x)$ is solely dependent on the sets $\mathcal{A}$ and $\mathcal{D}(x)$.

B. Global Position Stability

We will use Lyapunov's method to prove stability of the policy defined in (12) for an arbitrary vehicle with model (1). Excluded from this letter for conciseness, a thorough definition of the Lyapunov stability criterion and the appropriate proofs can be found in [14].

Proposition 1: *For any n-order system, the distance-coupled policy,* (12)*, is globally asymptotically stable for* (1) *given that $C = \alpha \mathbf{I}$ with $\alpha < 0$ and the identity matrix $\mathbf{I}$.*

Proof: Define the Lyapunov candidate as the inner product of the position, x, such that

$$V(x) = (x - x^{(\text{eq})})^\intercal (x - x^{(\text{eq})}). \tag{13}$$

By definition, $V(x) = 0$ at $x = x^{(\text{eq})}$ and is positive definite for all $x \neq x^{(\text{eq})}$, thus satisfying the first component to Lyapunov stability. Taking its derivative and substituting (12) for the control gives

$$\begin{aligned} \dot{V}(x) &= 2(x - x^{(\text{eq})})^\intercal \dot{x} = 2(x - x^{(\text{eq})})^\intercal u(x) \\ &= 2(x - x^{(\text{eq})})^\intercal C(K(h(x) - b) - x^{(\text{eq})}) \end{aligned} \tag{14}$$

Finally, using (11):

$$\dot{V}(x) = 2(x - x^{(\text{eq})})^\intercal C(x - x^{(\text{eq})}). \tag{15}$$

Implying that $\dot{V} = 0$ when $x = x^{(\text{eq})}$ and is negative definite given that $C \prec 0$. Thus making the policy globally asymptotically stable via the Lyapunov stability criterion. ■

Note that the proof can be extended to the case where C is any stable matrix. However, we use the restriction $C = \alpha \mathbf{I}$ for the sake of simplicity.

C. Transformations of the Anchor Set

Let us now study the case when the set $\mathcal{A}$ is moved by some orthogonal rotation, $R \in \mathbb{R}^{n \times n}$, and offset $r \in \mathbb{R}^n$, with the distance-coupled policy components, (10), held constant. With the incorporation of error into the position of the anchors, we can differentiate between two frames of reference. We will define $\mathbb{W}$ as the world frame which corresponds to the true positions of each point in the system. Its approximation, denoted by a tilde, is defined in the anchor frame, $\mathbb{A}$.

We can thus denote the new anchor positions by

$$a_i = R\tilde{a}_i + r : \tilde{a}_i \in \mathcal{A} \subset \mathbb{A}, \tag{16}$$

where $\tilde{a}_i$ is the anchor frame coordinate which corresponds to, and is linearly offset from, $a_i \in \mathbb{W}$. We differentiate between the two frames as the rotation of the set $\mathcal{A}$ creates error in the vehicle position defined by (11) such that the distance-coupled function becomes

$$\tilde{x} = K(h(x) - b) = R^\intercal (x - r) \tag{17}$$

where $x \in \mathbb{W}$ represents the world frame position of the vehicle and $\tilde{x} \in \mathbb{A}$ is its anchor frame approximation. The distance-coupled control can similarly be redefined as

$$\begin{aligned} u(x) &= C(K(h(x) - b) - x^{(\text{eq})}) \\ &= C(\tilde{x} - x^{(\text{eq})}) = C(R^\intercal (x - r) - x^{(\text{eq})}). \end{aligned} \tag{18}$$

Notably, the distance-coupled function acts over the world frame positions, but gives coordinates in the anchor frame.

Similar to Proposition 1, we can prove stability in terms of a range of acceptable values for the rotation and offset of the anchors. In order to make concrete claims on the stability of a system in $\mathbb{R}^2$, we identify the range of acceptable θ which make R positive definite. This finding will be subsequently used in Proposition 2.

Lemma 1: *A 2-D rotation defined by $R(\theta) \in \mathbb{R}^{2\times 2} : \theta \in \mathbb{R}$ is positive definite for the range $\theta \in (2\pi k - \frac{\pi}{2}, 2\pi k + \frac{\pi}{2})$ $\forall k \in \mathbb{Z}$ where $\mathbb{Z}$ is the set of integers.*

Proof: Define the matrix R in terms of Rodrigues' formula for 2-D rotations:

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{19}$$

where θ is some scalar angle. For some arbitrary point, x, we can say $R \succ 0$ when $x^\intercal R x > 0$ for all $x \neq 0$. Expanding this condition we get

$$\begin{aligned} x^\intercal R x &= [x_1 \;\, x_2] \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > 0 \\ &= (x_1^2 + x_2^2)\cos(\theta) > 0. \end{aligned} \tag{20}$$

Letting us conclude that $R(\theta)$ is positive definite so long as $\cos(\theta) > 0$. This is true for the ranges

$$\theta \in (2\pi k - \tfrac{\pi}{2}, 2\pi k + \tfrac{\pi}{2}) \;\forall k \in \mathbb{Z}, \tag{21}$$

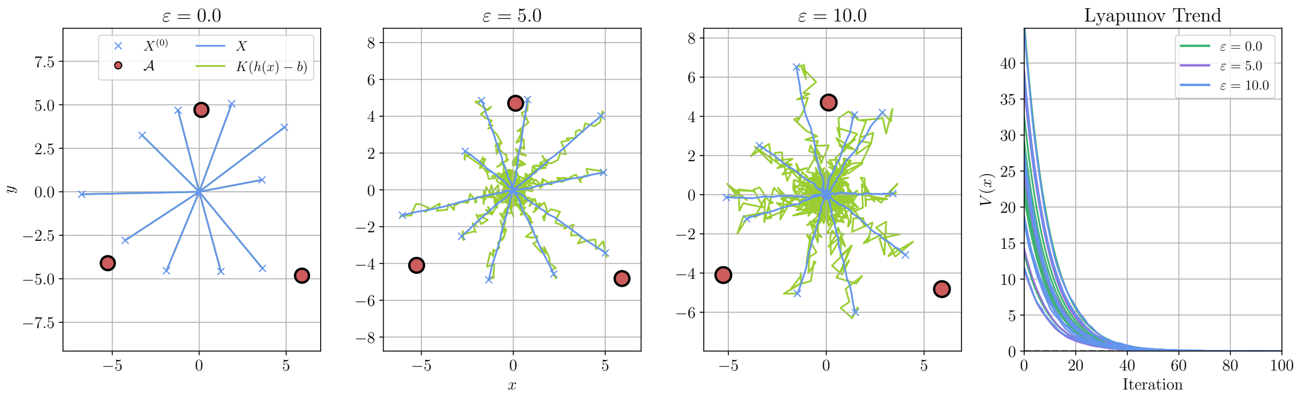

Fig. 2: Response of the distance-coupled controller, (12), with $x^{(\mathrm{eq})}$ at the origin and for varying magnitudes of randomly generated measurement noise, $\omega(\varepsilon)$. The vehicles are given arbitrary initial positions around the origin.

where $\mathbb{Z}$ is the set of integers. ■

Using this, the stability function developed in Proposition 1 can be reformulated for the rotated anchor case and the region of attraction for (18).

Proposition 2: *For a 2-D system, let the set $\mathcal{A}$ be transformed by some rotation, R, and translation, r. Then (18) describes an asymptotically stable policy with equilibrium $x = Rx^{(eq)} + r$ for any r and any R given that $RC \prec 0$ where C is defined in Proposition 1 and R in Lemma 1.*

Proof: Start by assuming that the new equilibrium, $\tilde{x}^{(\mathrm{eq})}$, takes an equivalent transformation as that defined over the set $\mathcal{A}$, changing our candidate function from (13) to

$$\begin{aligned} V(x) &= (x - \tilde{x}^{(\mathrm{eq})})^\intercal (x - \tilde{x}^{(\mathrm{eq})}) \\ &= (x - (Rx^{(\mathrm{eq})} + r))^\intercal (x - (Rx^{(\mathrm{eq})} + r)). \end{aligned} \tag{22}$$

We thus have $V(x) = 0 : x = \tilde{x}^{(\mathrm{eq})} = Rx^{(\mathrm{eq})} + r$ and $V(x) > 0$ otherwise. Taking the derivative we get

$$\begin{aligned} \dot{V}(x) &= 2(x - (Rx^{(\mathrm{eq})} + r))^\intercal u(x) \\ &= 2(x - (Rx^{(\mathrm{eq})} + r))^\intercal C(K(h(x) - b) - x^{(\mathrm{eq})}), \end{aligned} \tag{23}$$

where $u(x)$ is the control policy defined in (18). Thus giving

$$\begin{aligned} \dot{V}(x) &= 2(x - (Rx^{(\mathrm{eq})} + r))^\intercal C(R^\intercal (x - r) - x^{(\mathrm{eq})}) \\ &= 2(x - (Rx^{(\mathrm{eq})} + r))^\intercal RC(x - (Rx^{(\mathrm{eq})} + r)). \end{aligned} \tag{24}$$

By inspection, we can conclude that $\dot{V}(x) = 0 : x = \tilde{x}^{(\mathrm{eq})} = Rx^{(\mathrm{eq})} + r$. Moreover, $\dot{V}(x)$ is negative definite so long as $RC \prec 0$. If we assume $C \prec 0$ by Proposition 1, then, if $x \in \mathbb{R}^2$, $RC \prec 0$ if R is defined as in Lemma 1. ■

D. Anchor-based Positioning Results

We will now validate the response of the distance-coupled policy, (12), with varying degrees of noise in the measurement terms. The claims made on the regions of attraction defined in (21) will also be demonstrated. Note that gray-dotted lines are used in figures to clearly indicate anchor orientations after linear transformations.

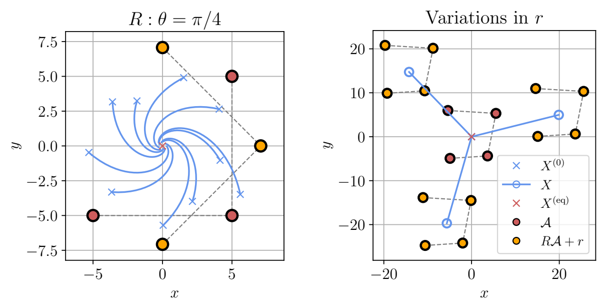

Fig. 3: Homing controller response with $x^{(\mathrm{eq})} = 0$ and with variations in R and r simulated separately for $x^{(0)}$ chosen outside $Rx^{(\mathrm{eq})} + r$.

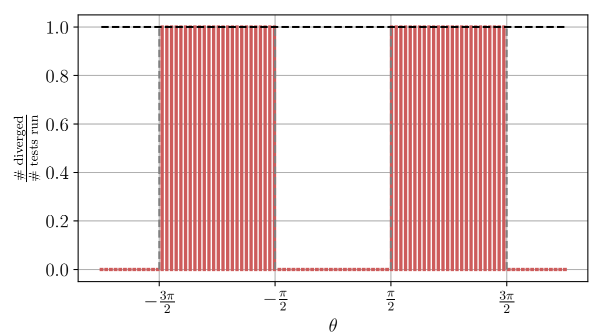

Fig. 4: Ratio of 100 tests diverged for 100 rotations defined by $R : \theta \in [-2\pi, 2\pi]$ of the anchor positions in $\mathcal{A}$.

Let us define the appropriately dimensioned noise function, $\omega(\varepsilon)$, as a uniformly distributed random error with elements in the range $[-\varepsilon, \varepsilon]$ on each distance measurement such that

$$d_i(x, \varepsilon) = d_i(x) + \omega(\varepsilon).$$

Figure 2 shows the response of the system with $(p = 3)$-anchors and for varying degrees of noise denoted by ε. The first case shows the ideal measurement scenario. That is, $\varepsilon = 0$ with no error in the positions of the anchors. We place

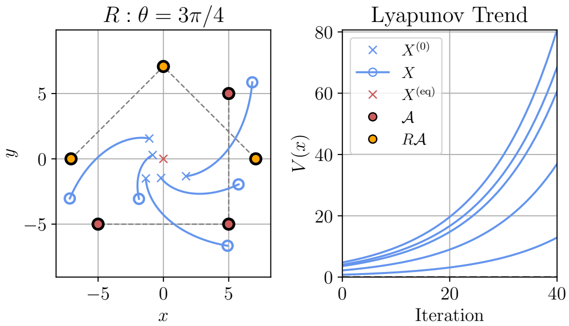

Fig. 5: Distance-coupled controller response with the rotation, $R : \theta = \frac{3\pi}{4}$, of the anchor set (policy diverges).

$x^{(\text{eq})}$ at the origin for simplicity. In this case, it is clear that there is no deviation between the true position (blue) and the anchor-based approximation (green). Likewise, the Lyapunov candidate converges, $V(x) \to 0$, as the simulation continues.

With an incorporation of a relatively large degree of measurement error ($\varepsilon = 5$ and $\varepsilon = 10$) we observe considerable differences between the approximation of the position from the distance-coupled function and the true position of the vehicle. That said, the vehicle is still capable of converging to a region around the equilibrium from any point.

Next, we validate the claims made in Section III-C for transformations of the anchors without updating (10). For the translation case, we define various r as uniform translations of the anchor set and observe the policy's performance over time. In the rotation case, we likewise define $R : \theta = \frac{\pi}{4}$ and show that the vehicles converge with a policy defined by RC. Both sets of results are shown in Figure 3.

To show the region of attraction with respect to the rotation, R, of the anchors we can look at the divergence of the system as the anchors are rotated by some angle θ about the origin. Figure 4 shows the divergence rate for a vehicle with initial conditions

$$x^{(0)} = x^{(\text{eq})} + \omega(\varepsilon),$$

where $\varepsilon = 0.001$ is selected to be very small. We then initialized 100 vehicles for 100 uniformly spaced $\theta \in [-2\pi, 2\pi]$ and simulated until the candidate function diverged ($V(x) \to \infty$) or converged ($V(x) \to 0$). The system clearly stabilizes for all θ defined by the bounds in (21), and diverges otherwise. For a more explicit example, Figure 5 shows a series of vehicles initialized with $\omega(\varepsilon = 2.5)$ and $R : \theta = \frac{3\pi}{4}$ diverging from the origin as expected from (21).

IV. Distance-Based Formation Control

Here, the anchor-based control policy in (12) will be expanded to a system of m-vehicles which are attempting to reach a predetermined formation. The only measurement each vehicle can make on its surroundings is the Euclidean distance it stands from each of the other vehicles in the environment. The following sections will discuss the new system's structure and the corresponding distance-coupled formation control policy, before demonstrating its properties in simulation.

A. Distance-coupled Formation Control

We define ($m = p$)-vehicles, with positions denoted by (3), and an intended formation described by the set, $\mathcal{A}$. By defining our formation in terms of $\mathcal{A}$ we can exploit the syntax of (11) to approximate the position of the vehicle group in the anchor frame, $\mathbb{A}$.

Each vehicle will act as an anchor for the others and vice versa. Thus, for a vehicle, x_k, we have the *vehicle-distance* set, $\mathcal{D}(x_k, X)$, with equilibrium positions at

$$X^{(\text{eq})} = \begin{bmatrix} x_1^{(\text{eq})} & \cdots & x_m^{(\text{eq})} \end{bmatrix}. \tag{25}$$

It makes sense to set $X^{(\text{eq})} = \text{hstack}(\mathcal{A})$ as we want the vehicles to maintain their position in the group. The full matrix of distance measurements shared between each vehicle in the formation can be denoted by

$$H(X) = \begin{bmatrix} h(x_1) & \cdots & h(x_m) \end{bmatrix} \tag{26}$$

where $h(x_k)$ is redefined from (10) as the distance-coupled vehicle measurements such that

$$h(x_k) = \text{hstack}(\{d^2(x_k, x_i) - d^2(x_k, x_j) : \forall (i,j) \in \mathcal{I}\}). \tag{27}$$

We can similarly arrange the squared anchor difference vector from (10) as a matrix by taking

$$B = b\mathbf{1}_m^\intercal, \tag{28}$$

where $\mathbf{1}_m \in \mathbb{R}^m$ is a vector of all ones. Using the matrix notations defined above we can reformulate (11) and (12):

$$X = K(H(X) - B), \text{ and} \tag{29}$$

$$U(X) = C(K(H(X) - B) - X^{(\text{eq})}), \tag{30}$$

where C and K are unchanged from (12).

It is important to note that between (11) and (29) the matrices, C, A, and the vector, b, do not undergo any change. The only parameter which differs between the vehicles is their intended equilibrium position, identified by $X^{(\text{eq})}$.

B. Formation Equivalence in the Anchor Frame

Next, we will define formation equivalence between the two frames of reference, $\mathbb{A}$ and $\mathbb{W}$, using selected tools from Procustes analysis [15]. Similar to the offsets defined in Section III-C, we have the vehicle positions in the world frame, $X \in \mathbb{W}$, with corresponding anchor frame coordinates $\tilde{X} \in \mathbb{A}$.

Definition 1: *Procrustes analysis defines the two sets, X and $\tilde{X}$, as formation equivalent when*

$$\exists \Psi \textit{ s.t. } X - \langle x \rangle = \Psi(\tilde{X} - \langle \tilde{x} \rangle), \tag{31}$$

given the orthogonal rotation $\Psi(X, \tilde{X}) = R(\theta)$ and the centroid of the formation $\langle x \rangle = \frac{1}{m}\Sigma_{k=1}^m x_k$.

In practice, we can calculate an approximation of Ψ using the Kabsch algorithm [16]. The resulting matrix is the most optimal rotation which superposes the set $\tilde{X}$ onto X.

Another method of denoting the transformation (31) is to combine the constant terms on the left and right sides into a single linear translation moving one formation to the other. This turns (31) into

$$\begin{aligned} &X - \langle x \rangle = \Psi(\tilde{X} - \langle \tilde{x} \rangle) \\ &\Rightarrow X = \Psi \tilde{X} + \psi \text{ where } \psi = \langle x \rangle - \Psi \langle \tilde{x} \rangle. \end{aligned} \tag{32}$$

The later sections will primarily use the shorthand definition of formation error, (32). It can be further noted that the uniform scaling component to Procrustes analysis is omitted since it is required for the systems in future sections to be scaled equally.

In order to measure policy convergence in Section IV-E, we will define the formation error between X and $X^{(\text{eq})}$ as the inner product of the function (32) on individual vehicle positions. That is,

$$W(X) = \Sigma_{k=1}^{p} (\Psi x_k + \psi - x_k^{(\text{eq})})^{\intercal} (\Psi x_k + \psi - x_k^{(\text{eq})}). \tag{33}$$

C. Transformation Invariance

In order to discuss the equilibria of (30) we define transformation invariance and prove its application to the distance function. This conclusion will be used to similarly prove the invariance of (29) and (30) in the next section.

Definition 2: *A function $f(x, y) \in \mathbb{R}^q : x, y \in \mathbb{R}^n$ is transformation invariant if $f(x, y)$ is both rotationally and translationally invariant. That is,*

$$f(Rx + r, Ry + r) = f(x, y), \tag{34}$$

given that R is an orthogonal rotation in $\mathbb{R}^n$ and $r \in \mathbb{R}^n$.

Lemma 2: *The distance function,*

$$d(x, y) = \sqrt{(x - y)^{\intercal}(x - y)}, \tag{35}$$

is a transformation invariant function by Definition 2.

Proof: Define an arbitrary orthogonal rotation, R, and translation, r, of the original states x and y such that

$$x \to Rx + r, \text{ and } y \to Ry + r. \tag{36}$$

Taking the distance between the transformed points:

$$\begin{aligned} &d(Rx + r, Ry + r) = \\ &\quad \sqrt{(Rx + r - (Ry + r))^{\intercal}(Rx + r - (Ry + r))} \\ &\qquad = \sqrt{(x - y)^{\intercal} R^{\intercal} R (x - y)} \\ &\qquad = \sqrt{(x - y)^{\intercal}(x - y)}, \end{aligned}$$

which is equivalent to (35). Thus concluding that the distance function is transformation invariant by Definition 2. ■

D. Equilibria of the Distance-coupled Formation Controller

Using Lemma 2 we can establish the invariance of the distance-coupled functions in $H(X)$. Subsequently, we will use this claim to derive the equilibria of (30).

Proposition 3: *Using Lemma 2, the distance-coupled position,* (29)*, is transformation invariant for any formation equivalent set of positions $X = \Psi \tilde{X} + \psi$.*

Proof: Let us assume there exists an orthogonal rotation $\Psi(X, \tilde{X})$ with translation ψ such that X and $\tilde{X}$ are formation equivalent by Definition 1. Identifying that the dependent variables in (29) are solely contained in $H(X)$, we can isolate a single row of the measurement terms for the k-th vehicle, denoted $h(x_k)$. That is,

$$h_{i,j}(x_k) = d^2(x_k, x_i) - d^2(x_k, x_j), \tag{37}$$

where the indices (i, j) are used to denote the distance-coupling of the vehicles x_i and x_j with respect to x_k. Applying the transformation of the vehicle group we get

$$\begin{aligned} h_{i,j}(\Psi \tilde{x}_k + \psi) = d^2(&\Psi \tilde{x}_k + \psi, \Psi \tilde{x}_i + \psi) \\ &- d^2(\Psi \tilde{x}_k + \psi, \Psi \tilde{x}_j + \psi). \end{aligned} \tag{38}$$

Extrapolating the transformation invariance of the distance function from Lemma 2:

$$\begin{aligned} h_{i,j}(\Psi \tilde{x}_k + \psi) &= d^2(\tilde{x}_k, \tilde{x}_i) - d^2(\tilde{x}_k, \tilde{x}_j) \\ &= h_{i,j}(\tilde{x}_k). \end{aligned} \tag{39}$$

We can therefore conclude that, for a transformation on the entire vehicle group, (39) will be true for all $(i, j) \in \mathcal{I}$ and for any vehicle, x_k. Implying that every element in the measurement matrix, $H(X)$, is also transformation invariant. Combining this with (29) we get that

$$\begin{aligned} &H(X) = H(\Psi \tilde{X} + \psi) = H(\tilde{X}) \\ &\Rightarrow K(H(X) - B) = K(H(\Psi \tilde{X} + \psi) - B) \\ &\qquad\qquad\qquad\quad = K(H(\tilde{X}) - B) = \tilde{X}. \end{aligned} \tag{40}$$

Proving that (29) is transformation invariant. ■

Proposition 4: *Following from Proposition 3,* (30) *is transformation invariant for any formation equivalent X and $\tilde{X}$.*

Proof: Let us assume that Proposition 3 is true for the rotation, $\Psi(X, \tilde{X})$, and translation, ψ. Then we have that

$$\begin{aligned} U(X) &= C(K(H(\Psi \tilde{X} + \psi) - B) - X^{(\text{eq})}) \\ &= C(K(H(\tilde{X}) - B) - X^{(\text{eq})}) \\ &= U(\tilde{X}). \end{aligned} \tag{41}$$

Implying that (30) is transformation invariant for any transformation equivalent X and $\tilde{X}$. ■

Corollary 1: *Resulting from Proposition 4, the formation control policy is at an equilibrium, $U(X) = 0$ for all formation equivalent transformations given that $U(\tilde{X}) = 0$.*

Proof: Assume that $\tilde{X} = X^{(\text{eq})}$ such that $U(\tilde{X}) = 0$. Then, for the world frame positions, $X = \Psi \tilde{X} + \psi$, and by Proposition 4 we have that

$$U(X) = U(\Psi \tilde{X} + \psi) = U(\tilde{X}) = U(X^{(\text{eq})}) = 0. \tag{42}$$

Proving that the positions X are at an equilibrium given they are formation equivalent to $\tilde{X}$. ■

We can further show that for any initial conditions which are not formation equivalent to X_{eq}, both the equilibrium and the convergence rate are dependent on the initial positions $X^{(0)}$. We provide some additional insight through the following proposition.

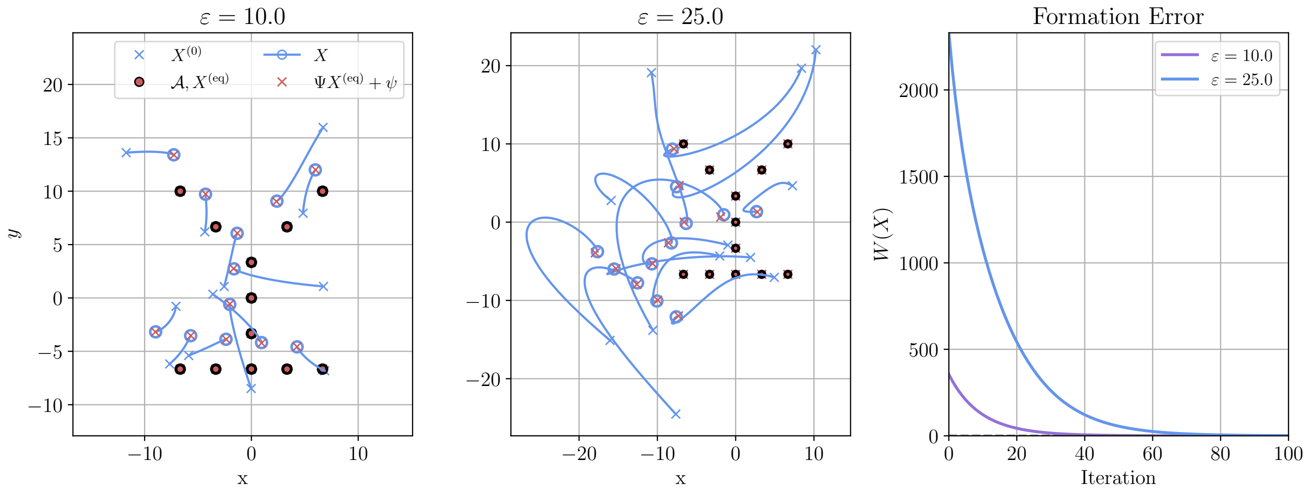

Fig. 6: Distance-coupled controller response for an arbitrary formation pattern and with varying levels of randomly generated offsets from the equilibrium positions: $X^{(0)} = X^{(\mathrm{eq})} + \omega(\varepsilon)$.

Corollary 2: *By Proposition 4, for a formation equivalent set, $X = \Psi\tilde{X} + \psi$, the vehicles in the world frame converge to an equilibrium which is linearly offset such that $\tilde{X}^{(eq)} = \Psi X^{(eq)} + \psi$ with a convergence rate ΨC.*

Proof: Taking X and $\tilde{X}$ as formation equivalent, and following from Proposition 4 we get that

$$\begin{aligned} U(X) &= U(\tilde{X}) \\ &= C(K(H(\tilde{X}) - B) - X^{(\mathrm{eq})}) \\ &= C(\tilde{X} - X^{(\mathrm{eq})}) = C(\Psi^{\mathsf{T}}(X - \psi) - X^{(\mathrm{eq})}) \\ &= \Psi C(X - (\Psi X^{(\mathrm{eq})} + \psi)). \end{aligned} \tag{43}$$

Implying that the vehicle list X converges towards an offset equilibrium described by Ψ and ψ with a convergence rate that is correlated with ΨC. ■

From this it can be reasonably deduced that, for vehicle positions X which are not formation equivalent to $\tilde{X}$, we get similar bounds on the region of attraction as those shown in (21). This is demonstrated in the results.

E. Distance-based Formation Results

While we do not present a proof for the stability of the distance-coupled formation controller, we can demonstrate empirically its behavior under various initial conditions. We first show the response of the policy when the initial conditions are randomly offset around $X^{(\mathrm{eq})}$. This will also help to validate our claims on the equilibrium positions defined in Corollary 1. Then, the region of attraction with respect to the orientation will be plotted similarly to Figure 4, and the results will be compared.

To validate the response of (30) we initialize the vehicle group such that

$$X^{(0)} = X^{(\mathrm{eq})} + \omega(\varepsilon),$$

with $\varepsilon = 10$ and $\varepsilon = 25$. The simulation is then iterated and the behavior of the policy is shown. The performance of the

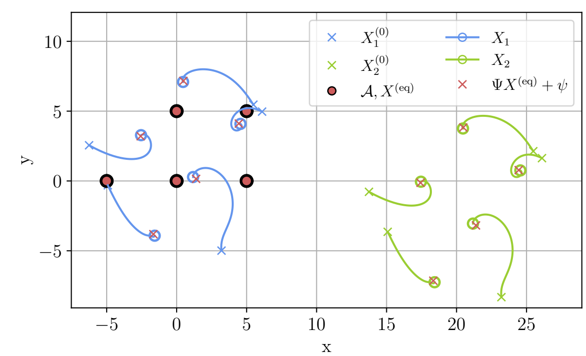

Fig. 7: Response of formation controller with offset $\omega(\varepsilon = 10)$ and for the two vehicle sets with positions denoted by $X_2 = X_1 + r$.

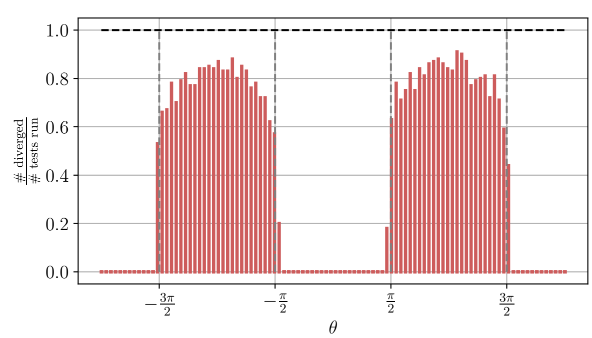

Fig. 8: Stability bounds results with $p = 4$ and 100 repetitions at each $R : \theta \in [-2\pi, 2\pi]$ for a randomly generated offset with $\omega(\varepsilon = 0.1)$.

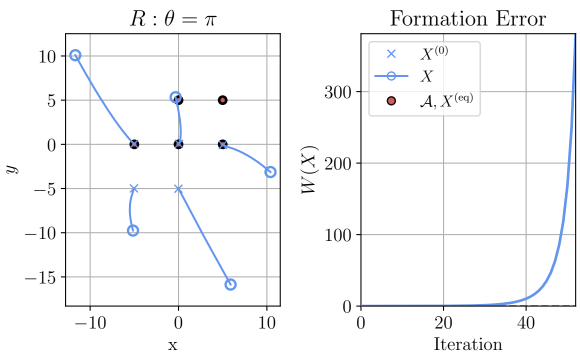

Fig. 9: Divergence of (30) with initial conditions $x^{(0)} = Rx^{(\mathrm{eq})}$ given $R : \theta = \pi$ and $\omega(\varepsilon = 0.1)$.

formation error function $W(X)$ is plotted over time for each test, showing that it converges in these two cases.

Figure 7 uses two sets of vehicle positions to validate Corollary 2. The first set, X_1, is the reference set and has initial conditions which are generated similar to the Figure 6 example with $\varepsilon = 10$. The second set is linearly offset from X_1 by a constant r. As expected, the two sets follow the same trajectory and end in equilibrium positions which differ by the initial translation, r.

Because of their close relationship, we tested if the policy would behave similarly to rotations about the origin as it does in the homing problem. We thus modified the initial condition formula for the vehicle set such that

$$X^{(0)} = R(X^{(\mathrm{eq})} + \omega(\varepsilon)),$$

with $\varepsilon = 0.1$ and for the same list of θ described in Section III-D. The simulation was then iterated until the policy converged ($W(X) \to 0$) or diverged ($W(X) \to \infty$), and the proportion of tests which diverged (out of 100) are shown in Figure 8. As expected, there is a strong correlation to the bounds defined in (21), with more complicated interplay in the formation policy case. Figure 9 shows an example of the policy diverging with $R : \theta = \pi$ and $\omega(\varepsilon = 0.1)$.

V. Conclusion

In this letter we define the distance-coupling method as a suitable approach to the position and formation control problems. In the homing case we prove stability and show that it is held for linear transformations of the anchor positions, even when the policy components, (10), are not updated. The formation policy is also studied, and the equilibrium conditions are established under the definition of formation equivalence. Claims are made on the general performance of the system under arbitrary transformations, and the resulting convergence properties are demonstrated empirically.

Generally speaking, the approach described here is framed using a single-integrator model, (1), for simplicity. That said, the distance-coupled position functions (11) and (17) can be applied to any localization problem in which the available measurements are a function of the distance from known reference points. In the context of systems with more complex dynamics, the distance-coupled policies defined here can be used to calculate suitable waypoints for the vehicles to follow by a more appropriate controller.

References

[1] Michael S. Floater. "Generalized barycentric coordinates and applications". en. In: *Acta Numerica* 24 (May 2015). Publisher: Cambridge University Press, pp. 161–214. ISSN: 0962-4929, 1474-0508. DOI: 10.1017/S0962492914000129.

[2] Kaveh Fathian et al. "Distributed Formation Control via Mixed Barycentric Coordinate and Distance-Based Approach". In: *2019 American Control Conference (ACC)*. ISSN: 2378-5861. July 2019, pp. 51–58. DOI: 10.23919/ACC.2019.8814890.

[3] Tingrui Han et al. "A Barycentric Coordinate-Based Approach to Formation Control Under Directed and Switching Sensing Graphs". In: *IEEE Transactions on Cybernetics* 48.4 (Apr. 2018). Conference Name: IEEE Transactions on Cybernetics, pp. 1202–1215. ISSN: 2168-2275. DOI: 10.1109/TCYB.2017.2684461.

[4] Dimos V. Dimarogonas and Kostas J. Kyriakopoulos. "A connection between formation infeasibility and velocity alignment in kinematic multi-agent systems". In: *Automatica* 44.10 (Oct. 2008), pp. 2648–2654. ISSN: 0005-1098. DOI: 10.1016/j.automatica.2008.03.013.

[5] Dimos V. Dimarogonas, Panagiotis Tsiotras, and Kostas J. Kyriakopoulos. "Leader–follower cooperative attitude control of multiple rigid bodies". In: *Systems & Control Letters* 58.6 (June 2009), pp. 429–435. ISSN: 0167-6911. DOI: 10.1016/j.sysconle.2009.02.002.

[6] Zhiyun Lin, B. Francis, and M. Maggiore. "Necessary and sufficient graphical conditions for formation control of unicycles". In: *IEEE Transactions on Automatic Control* 50.1 (Jan. 2005). Conference Name: IEEE Transactions on Automatic Control, pp. 121–127. ISSN: 1558-2523. DOI: 10.1109/TAC.2004.841121.

[7] Dimos V. Dimarogonas and Karl H. Johansson. "On the stability of distance-based formation control". In: *2008 47th IEEE Conference on Decision and Control*. ISSN: 0191-2216. Dec. 2008, pp. 1200–1205. DOI: 10.1109/CDC.2008.4739215.

[8] Yun Ho Choi and Doik Kim. "Distance-Based Formation Control With Goal Assignment for Global Asymptotic Stability of Multi-Robot Systems". In: *IEEE Robotics and Automation Letters* 6.2 (Apr. 2021). Conference Name: IEEE Robotics and Automation Letters, pp. 2020–2027. ISSN: 2377-3766. DOI: 10.1109/LRA.2021.3061071.

[9] J. Baillieul and A. Suri. "Information patterns and Hedging Brockett's theorem in controlling vehicle formations". en-US. In: *Proceedings of the 42nd IEEE*. Dec. 2003.

[10] Wei Ren and Ella Atkins. "Distributed multi-vehicle coordinated control via local information exchange". en. In: *International Journal of Robust and Nonlinear Control* 17.10-11 (2007). _eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/rnc.1147, pp. 1002–1033. ISSN: 1099-1239. DOI: 10.1002/rnc.1147.

[11] Ting Chen et al. "Formation control for second-order nonlinear multi-agent systems with external disturbances via adaptive method". In: *2022 China Automation Congress (CAC)*. ISSN: 2688-0938. Nov. 2022, pp. 5616–5620. DOI: 10.1109/CAC57257.2022.10055393.

[12] Zhiyun Lin et al. "Distributed Formation Control of Multi-Agent Systems Using Complex Laplacian". en. In: *IEEE Transactions on Automatic Control* 59.7 (July 2014), pp. 1765–1777. ISSN: 0018-9286, 1558-2523. DOI: 10.1109/TAC.2014.2309031.

[13] Kwang-Kyo Oh, Myoung-Chul Park, and Hyo-Sung Ahn. "A survey of multi-agent formation control". In: *Automatica* 53 (Mar. 2015), pp. 424–440. ISSN: 0005-1098. DOI: 10.1016/j.automatica.2014.10.022.

[14] Hassan Khalil. *Nonlinear Systems*. English. 3rd. Pearson, Dec. 2001. ISBN: 978-0-13-067389-3.

[15] Mikkel B Stegmann and David Delgado Gomez. "A Brief Introduction to Statistical Shape Analysis". en. In: ().

[16] W. Kabsch. "A discussion of the solution for the best rotation to relate two sets of vectors". en. In: *Acta Crystallographica Section A: Crystal Physics, Diffraction, Theoretical and General Crystallography* 34.5 (Sept. 1978). Number: 5 Publisher: International Union of Crystallography, pp. 827–828. ISSN: 0567-7394. DOI: 10.1107/S0567739478001680.