Neoclassical Growth Model

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Primitives of the model:

1. preferences: $U = \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}-1}{1-\sigma}$ (CRRA) ,
2. technology: $Y_t = AK_t^\alpha = C_t + I_t = C_t + K_{t+1} - (1-\delta)K_t$,
3. endowment: K_0 is given.

SPP is to maximize welfare of a representative agent subject to the resource constraint:

$$\max_{\{C_t, K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}-1}{1-\sigma}$$

$$\text{s.t. } C_t = AK_t^\alpha + (1-\delta)K_t - K_{t+1}. \tag{1}$$

Denote the Lagrange multiplier with $\beta^t \lambda_t$ and write down the Lagrangian:

$$\sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}-1}{1-\sigma} + \lambda_t \big(AK_t^\alpha + (1-\delta)K_t - K_{t+1} - C_t \big) \right].$$

Taking the derivatives wrt C_t and K_{t+1}, we get the first-order conditions

$$\lambda_t = C_t^{-\sigma},$$

$$\lambda_t = \beta \lambda_{t+1} \big(\alpha A K_{t+1}^{\alpha-1} + (1-\delta) \big).$$

Substitute λ_t from the first equation into the second one to obtain the Euler equation:

$$C_t^{-\sigma} = \beta C_{t+1}^{-\sigma} \big(\alpha A K_{t+1}^{\alpha-1} + 1 - \delta \big). \tag{2}$$

Given K_0, the optimal trajectory $\{C_t, K_t\}$ is characterized by the (infinite) system of resource constraints (1) and intertemporal optimality conditions (2) combined with the transversality condition (TVC)

$$\lim_{t\to\infty} \beta^t C_t^{-\sigma} K_{t+1} = 0. \tag{3}$$

Competitive equilibrium coincides with the SPP because assumptions of the FWT hold. However, we still need to find prices that support the decentralized equilibrium. To this end, define the CE as a list of sequences $\{C_t, K_t, R_t, \Pi_t\}$ s.t.

1. Given $\{R_t, \Pi_t\}$, $\{C_t, K_t\}$ solve household problem:

$$\max \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma} - 1}{1-\sigma}$$

$$\text{s.t. } C_t + K_{t+1} - (1-\delta)K_t = R_t K_t + \Pi_t.$$

2. Given $\{R_t\}$, $\{\Pi_t, K_t\}$ solve firm's problem:

$$\max\ \Pi_t, \quad \Pi_t = AK_t^\alpha - R_t K_t.$$

3. Markets clear:

$$C_t + K_{t+1} - (1-\delta)K_t = AK_t^\alpha.$$

The firm's FOC implies $R_t = \alpha A K_t^{\alpha-1}$.

Phase diagrams help to illustrate the optimal trajectory. Strictly speaking, it applies only to continuous-time models, but we will assume that periods are short enough (months, weeks, days, minutes), so that the phase diagrams provide an accurate approximation to our discrete-time model. From capital law of motion (1) it follows

$$\Delta K_{t+1} = AK_t^\alpha - \delta K_t - C_t \quad \Rightarrow \quad \Delta K_{t+1} > 0 \text{ iff } C_t < AK_t^\alpha - \delta K_t,$$

while the Euler equation (2) implies

$$\left(\frac{C_{t+1}}{C_t}\right)^\sigma = \beta\big[\alpha A K_{t+1}^{\alpha-1} + 1 - \delta\big] \quad \Rightarrow \quad \Delta C_{t+1} > 0 \text{ iff } \beta\big[\alpha A K_{t+1}^{\alpha-1} + 1 - \delta\big] > 1.$$

Draw these two lines in the space $\{K_t, C_t\}$ and show with arrows the directions, in which capital and consumption evolve in each of the four quadrants. As usual, it is convenient to disentangle the long-run and the short-run dynamics of the system:

1. *Steady states* (SS) correspond to the long-run equilibria of the model and can be found from equations (1)-(2) by imposing the stationarity conditions $K_t = K_{t+1}$ and $C_t = C_{t+1}$. Using the upper bar to denote the SS values, we get

$$\beta\big[\alpha A \bar{K}^{\alpha-1} + 1 - \delta\big] = 1, \qquad \bar{C} = A\bar{K}^\alpha - \delta \bar{K}.$$

2. *Saddle path* describes transitional dynamics and is the only trajectory out of the ones satisfying (1)-(2), for which $K_t \geq 0$ and the TVC holds.

Shocks can be classified in a number of ways: (i) temporary vs. permanent, (ii) unexpected (MIT) vs. anticipated, (iii) productivity vs. preference shocks. Although this does not make the model truely stochastic, this is the first step towards a model with uncertainty. The basic principles are:

1. state variables always evolve continuously (unless there is some exogenous shock that changes them),

2. the Euler equation fails in the period one the new (unexpected) information arrives, but has to hold for all future periods (optimal consumption smoothing).

Shooting algorithm is one of the three main methods to solve the growth model. It is intuitive, easy to implement, and provides the full non-linear dynamics. The main limitation of the method is that it cannot be generalized to a stochastic environment.

1. set maximum and minimum values for C_0, e.g. if $K_0 < \bar{K}$, then $C_0 \in [0, \bar{C}]$,

2. make initial guess $C_0 \in [C_{min}, C_{max}]$, e.g. $C_0 = (C_{min} + C_{max})/2$,

3. compute the optimal trajectory iterating the system (1)-(2) for T periods,

4. stop if K_t, C_t do not converge to the SS – in the growth model, this is equivalent to either variable exhibiting a non-monotonic dynamics,

5. update the minimum value or the maximum values of C_0 depending on whether K_t or C_t is non-monotonic and go back to step 2,

6. iterate until convergence.

Blanchard-Kahn method relies on the first-order approximations to the equilibrium conditions to solve the model. The accuracy of the results depends on the type of model. It is generalizable to stochastic environments, allows for a large number of state variables, and works even when CE$\neq$SPP. Importantly, the BK method allows to solve simple models (including most of the models in our course) in closed form!

1. Pick the point of the approximation – usually the SS of the model.

2. Log-linearize the equilibrium conditions around the point of approximation. The following approach works in most cases:

$$X_t = \bar{X}\frac{X_t}{\bar{X}} = \bar{X}e^{\log \frac{X_t}{\bar{X}}} \equiv \bar{X}e^{x_t} \approx \bar{X}(1 + x_t),$$

where x_t is the log-deviation of X_t from the SS value. A few useful properties:

$$\begin{aligned} Z_t = X_t^a &\quad \Rightarrow \quad z_t = a x_t, \\ Z_t = X_t Y_t &\quad \Rightarrow \quad z_t = x_t + y_t, \\ Z_t = X_t + Y_t &\quad \Rightarrow \quad z_t = a x_t + (1-a) y_t, \; a \equiv \frac{\bar{X}}{\bar{Z}} = 1 - \frac{\bar{Y}}{\bar{Z}}. \end{aligned}$$

Apply this technique to log-linearize resource constraint (1):

$$\bar{K}(1 + k_{t+1}) = A\bar{K}^{\alpha}(1 + \alpha k_t) + (1-\delta)\bar{K}(1 + k_t) - \bar{C}(1 + c_t).$$

Note that constants (zero-order terms) always go away – this follows from the SS values of the variables. Simplifying, we get a linear law of motion for capital:

$$k_{t+1} = \left[\alpha A \bar{K}^{\alpha-1} + 1 - \delta\right] k_t - \frac{\bar{C}}{\bar{K}} c_t.$$

Finally, denote $\phi \equiv A\bar{K}^{\alpha-1}$ and use the SS values to obtain

$$k_{t+1} = \frac{1}{\beta} k_t - (\phi - \delta)\, c_t, \quad \phi = \frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right).$$

Following the same steps, one can log-linearize the Euler equation (2):

$$1 + \sigma(c_{t+1} - c_t) = \beta\left[\alpha A \bar{K}^{\alpha-1}(1 - (1-\alpha)k_{t+1}) + 1 - \delta\right],$$

$$c_{t+1} = c_t - \frac{1}{\sigma}\beta\alpha(1-\alpha)A\bar{K}^{\alpha-1}k_{t+1},$$

$$c_{t+1} = c_t - \frac{\beta(1-\alpha)\alpha\phi}{\sigma} k_{t+1}.$$

3. If there are any static variables in the system, substitute them out. For example, this would be the case in a growth model with endogenous supply of labor.

4. Write down the dynamic system

$$\begin{pmatrix} 1 & 0 \\ \frac{\beta(1-\alpha)\alpha\phi}{\sigma} & 1 \end{pmatrix} \begin{pmatrix} k_{t+1} \\ c_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} & \delta - \phi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k_t \\ c_t \end{pmatrix}.$$

Invert the former matrix

$$\begin{pmatrix} 1 & 0 \\ \frac{\beta(1-\alpha)\alpha\phi}{\sigma} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{\beta(1-\alpha)\alpha\phi}{\sigma} & 1 \end{pmatrix}$$

and rewrite the system as $x_{t+1} = Ax_t$, where

$$x_t = \begin{pmatrix} k_t \\ c_t \end{pmatrix}, \quad A = \begin{pmatrix} \frac{1}{\beta} & -(\phi - \delta) \\ -\frac{(1-\alpha)\alpha\phi}{\sigma} & 1 + \eta \end{pmatrix}, \quad \eta \equiv \frac{\beta(1-\alpha)\alpha\phi(\phi - \delta)}{\sigma}.$$

5. Factorize matrix A:

$$\det \begin{pmatrix} \frac{1}{\beta} - \lambda & -(\phi - \delta) \\ -\frac{(1-\alpha)\alpha\phi}{\sigma} & 1 + \eta - \lambda \end{pmatrix} = \lambda^2 - \left(\frac{1}{\beta} + 1 + \eta\right)\lambda + \frac{1+\eta}{\beta} - \frac{(1-\alpha)\alpha\phi(\phi-\delta)}{\sigma} = 0$$

$$\lambda_{1,2} = \frac{1}{2}\left(\frac{1}{\beta} + 1 + \eta \pm \sqrt{\left(\frac{1}{\beta} + 1 + \eta\right)^2 - \frac{4}{\beta}}\right).$$

It is easy to check that $\lambda_1 > 1$ and $\lambda_2 < 1$. Write down a matrix with columns corresponding to the eigenvectors of matrix A and find the inverse matrix:

$$Q = \begin{pmatrix} \phi - \delta & \phi - \delta \\ \frac{1}{\beta} - \lambda_1 & \frac{1}{\beta} - \lambda_2 \end{pmatrix}, \qquad Q^{-1} = \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} \frac{\frac{1}{\beta} - \lambda_2}{\phi - \delta} & -1 \\ \frac{\frac{1}{\beta} - \lambda_1}{\phi - \delta} & 1 \end{pmatrix}.$$

6. The dynamic system can now be rewritten as follows:

$$x_{t+1} = Q\Lambda Q^{-1}x_t, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Since Λ is diagonal, it follows that dynamics is characterized a system of two independent equations for y_t^1 and y_t^2:

$$y_{t+1} = \Lambda y_t, \quad y_t = Q^{-1}x_t.$$

Blachard-Kahn principle: the model has unique (local) solution if the number of eigenvalues with absolute values greater than one is equal to the number of control variables.

Consider the first equation with the eigenvalue greater than one. It follows that $y_t^1 = \lambda_1^t y_0^1$ and if $y_0^1 \neq 0$, y_t^1 goes to infinity as $t \to \infty$. In other words, the system goes away from the steady state, which corresponds to $y = x = 0$. This is clearly in contradiction with the TVC. Hence, the only possible solution is $y_0^1 = 0$. This provides a cointegration relationship between c_t and k_t and allows to express control variable as a function of the state variable:

$$c_t = \frac{\frac{1}{\beta} - \lambda_2}{\phi - \delta}k_t.$$

Combined with the linearized resource constraint and k_0, this equation is sufficient to pin down the whole transition path. In particular, substituting c_t into the capital law of motion, we get that λ_2 governs the speed of convergence to the SS:

$$k_{t+1} = \lambda_2 k_t.$$

Calibration of parameters is required to use the model for quantitative predictions. E.g. suppose we want to do the following counterfactual: given K_0, evaluate how long it would take the U.S. to converge to the long-run SS after a positive productivity shock A. To evaluate λ_2 we need to know the values of structural parameters $\beta, \alpha, \delta, \sigma$. Although not directly observable, we can calibrate these parameters to reproduce some moments from the data:

1. The average annual gross real interest rate is about 1.04, which from the SS Euler equation implies $\beta = 0.99$ at quarterly horizon.
2. The share of capital income in GDP is about $1/3$, which implies $\frac{\alpha A K^{\alpha-1} K}{A K^{\alpha}} = \alpha = 1/3$.
3. The share of investment in GDP is about 20% and the ratio of capital to (quarterly) GDP is around 10. This implies $\frac{I}{K} = \frac{K-(1-\delta)K}{K} = \delta = 0.02$.
4. Much more uncertainty about IES: cannot be inferred from the SS ratios and often requires micro evidence. The "standard values" are $\sigma \approx 1$.

Combining these values for quarterly calibration, we get $\phi = 0.09$, $\eta = 0.0154$ and $\lambda \approx 0.89$. One way to evaluate the speed of convergence is to focus on the time it takes the economy to cut the distance to the SS by half. Since $k_t = \lambda_2^t k_0$, it follows the "half-life" is equal $T = -\frac{\log 2}{\log \lambda_2} \approx 5.8$ quarters, i.e. about 1.5 years.