

Macro PS4

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April 13, 2021

1 Question 1

The household solves the following problem:

$$\begin{aligned} \max_{\{c_t, l_t\}_{t=0}^{\infty}} \quad & \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + \nu(l_t) \right] \\ \text{s.t.} \quad & (1 + \tau_{ct})c_t + k_{t+1} + b_{t+1} = (1 - \delta + r_t)k_t + R_t b_t + w_t(1 - l_t) \end{aligned}$$

Taking FOCs (Lagrange multiplier λ_t) yields the following:

$$\begin{aligned} \beta^t c_t^{-\sigma} &= \lambda_t(1 + \tau_{ct}) \\ \beta^t \nu'(l_t) &= \lambda_t w_t \\ \lambda_t &= (1 - \delta + r_{t+1})\lambda_{t+1} \\ \lambda_t &= R_{t+1}\lambda_{t+1}. \end{aligned}$$

Simplifying,

$$\begin{aligned} w_t &= (1 + \tau_{ct})\nu'(l_t)c_t^\sigma \\ 1 &= \beta \left(\frac{c_t}{c_{t+1}} \right)^\sigma \frac{1 + \tau_{ct+1}}{1 + \tau_{ct}} R_{t+1} \\ R_t &= 1 - \delta + r_t. \end{aligned}$$

The above formulas represent the solution to the household's problem. Now we can set up the Ramsey problem. The resource constraint is the following:

$$c_t + k_{t+1} = (1 - \delta)k_t + F(k_t, (1 - l_t)).$$

Our implementability constraint takes the following form:

$$\sum_{t=0}^{\infty} \beta^t [c_t^{1-\sigma} - (1 - l_t)\nu'(l_t)] = \frac{c_0^{-\sigma}}{1 + \tau_{c0}} [(1 - \delta + r_0)k_{-1} + R_0 b_{-1}],$$

where we make standard assumptions on τ_{c0} to rule out the effective lump-sum tax solution.

Defining $W(c_t, l_t, \lambda) = \frac{c_t^{1-\sigma}}{1-\sigma} + \nu(l_t) + \lambda[c_t^{1-\sigma} - (1 - l_t)\nu'(l_t)]$, then the Ramsey problem consists of solving the following maximization problem:

$$\max_t \sum_t \beta^t [W(c_t, l_t, \lambda) - \lambda \frac{c_0^{-\sigma}}{1 + \tau_{c0}} [(1 - \delta + r_0)k_{-1} + R_0 b_{-1}]]$$

The intertemporal first order condition is the following:

$$W_{ct} = \beta W_{ct+1} R_{t+1}$$

We are almost to our solution. We just need to look at W_{ct} :

$$\begin{aligned} c_t^{-\sigma} + \lambda c_t^{-\sigma}(1 - \sigma) &= c_t^{-\sigma}(1 + \lambda(1 - \sigma)) \\ \Rightarrow \frac{W_{ct+1}}{W_{ct}} &= \left(\frac{c_t}{c_{t+1}} \right)^\sigma \end{aligned}$$

Coming back to our intertemporal FOC for the Ramsey problem,

$$\begin{aligned} W_{ct} &= \beta W_{ct+1} R_{t+1} \\ 1 &= \beta \left(\frac{c_t}{c_{t+1}} \right)^\sigma R_{t+1} \end{aligned}$$

Comparing to our first order conditions for the HH problem we can see immediately that $\tau_{ct} = \tau_{ct+1} \forall t$.

2 Question 2

2.1 Part A

A competitive equilibrium is a set of prices $\{p_t, w_t, R_t\}$ allocations $\{c_{1t}, c_{2t}, n_t, M_t, B_t\}$ and policies $\{T_t\}$ such that agents optimize, markets clear, and the government budget constraint holds.

Agents solve the following:

$$\begin{aligned} \max_{c_{1t}, c_{2t}, n_t} \quad & \sum_{t=0}^{\infty} \beta^t (\log(c_{1t}) + \alpha \log(c_{2t}) + \gamma \log(1 - n_t)) \\ \text{s.t.} \quad & p_t c_{1t} \leq M_t \\ & \text{and } M_t + B_t \leq (M_{t-1} - p_{t-1} c_{1t-1}) - p_{t-1} c_{2t-1} + w_{t-1} n_{t-1} + R_{t-1} B_{t-1} - T_t \end{aligned}$$

Market clearing is the following:

$$c_{1t} + c_{2t} = n_t.$$

GBC:

$$M_t - M_{t-1} + B_t = R_{t-1} B_{t-1} - T_t$$

Finally, we are given that monetary policy acts to make $R_t = R \forall t$.

2.2 Part B

The consumer's FOCs are the following:

$$\begin{aligned} \beta^t c_{1t}^{-1} &= \lambda_{t+1} p_t + \gamma_t p_t \\ \beta^t \alpha c_{2t}^{-1} &= \lambda_{t+1} p_t \\ \beta^t \gamma (1 - n_t)^{-1} &= \lambda_{t+1} w_t \\ \lambda_t &= \gamma_t + \lambda_{t+1} \\ \lambda_t &= \lambda_{t+1} R_t. \end{aligned}$$

Simplifying,

$$\begin{aligned} \beta^t c_{1t}^{-1} &= p_t \lambda_t \\ \Rightarrow \frac{c_{2t}}{\alpha c_{1t}} &= R_t = R, \\ \frac{\gamma c_{2t}}{\alpha (1 - n_t)} &= \frac{w_t}{p_t} \end{aligned}$$

Note that the real wage $\frac{w_t}{p_t}$ must be one because that is the marginal productivity of labor from the firm side. We now have 3 equations in 3 unknowns we can solve for allocations:

$$\frac{c_{2t}}{\alpha c_{1t}} = R, \quad (1)$$

$$\frac{\gamma c_{2t}}{\alpha(1 - n_t)} = 1, \quad (2)$$

$$c_{1t} + c_{2t} = n_t. \quad (3)$$

$$\begin{aligned} n_t &= 1 - \frac{\gamma c_{2t}}{\alpha}, \\ c_{2t} &= \alpha R c_{1t}, \\ c_{1t}(1 + \alpha R) &= 1 - \gamma R c_{1t} \\ \Rightarrow c_{1t} &= \frac{1}{1 + (\alpha + \gamma)R} \\ \Rightarrow c_{2t} &= \frac{\alpha R}{1 + (\alpha + \gamma)R} \\ \Rightarrow n_t &= 1 - \frac{\gamma R}{1 + (\alpha + \gamma)R} \\ &= \frac{1 + \alpha R}{1 + (\alpha + \gamma)R} \end{aligned}$$

From Wolfram Alpha,

$$\begin{aligned} \frac{\partial n_t}{\partial R} &= -\frac{\gamma}{(1 + (\alpha + \gamma)R)^2} \\ &< 0. \end{aligned}$$

Therefore, n_t is decreasing in R .