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My student # ~~starts~~ ends w/a 7 $\Rightarrow$ Exam B

1) (a) $1 = \int_{-1}^{2} f_x(x)\,dx = \int_{-1}^{2} \frac{x^2}{c}\,dx = \frac{1}{c}\left(\frac{x^3}{3}\Big|_{-1}^{2}\right) = \frac{1}{c}\left(\frac{8}{3} - \left(-\frac{1}{3}\right)\right) = \frac{3}{c}$
$\Rightarrow c = 3$

(b) $P(Z \le z) = P(X^2 \le z) = \begin{cases} 0, & z \le 0 \\ P(-\sqrt{z} \le X \le \sqrt{z}), & z \in [0,1] \\ P(-1 \le X < \sqrt{z}), & z \in (1,4] \end{cases}$

$z \in (-\infty, 0]$: $F_Z(z) = 0 \Rightarrow f_Z(z) = 0$

$z \in [0,1]$: $F_Z(z) = \int_{-\sqrt{z}}^{\sqrt{z}} \frac{x^2}{3}\,dx = \left(\frac{x^3}{9}\Big|_{-\sqrt{z}}^{\sqrt{z}}\right) = \frac{2z^{3/2}}{9} \Rightarrow f_Z(z) = \frac{z^{1/2}}{3}$

$z \in (1,4]$: $F_Z(z) = \int_{-1}^{\sqrt{z}} \frac{x^2}{3}\,dx = \left(\frac{x^3}{9}\Big|_{-1}^{\sqrt{z}}\right) = \frac{z^{3/2}}{9} + \frac{1}{9} \Rightarrow f_Z(z) = z^{1/2}/6$

$z \in (4, \infty)$: $F_Z(z) = 1 \Rightarrow f_Z(z) = 0$

So $f_Z(z) = \begin{cases} z^{1/2}/3 & z \in (0,1] \\ z^{1/2}/6 & z \in (1,4] \\ 0 & \text{otherwise} \end{cases}$

Note: 0 is a single point so, given the pdf is continuous, we can choose to either group it with $z^{1/3}/3$ or 0, and the answer is unchanged.

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2) (a) Propose : $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} \log Y_i$

Prove unbiased: $E\hat{\theta} = E\frac{1}{n}\sum_{i=1}^{n}\log Y_i = \frac{1}{n}\sum_{i=1}^{n} E\log Y_i = \frac{1}{n}(n\theta) = \theta$

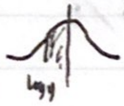
$E\hat{\theta} = \theta$ so $\hat{\theta}$ is an unbiased estimator of θ.

(b) Find $P(Y > y \mid Y \leq 1)$ for $y < 1$

Let $\log Y_i = X_i \Rightarrow Y_i = \exp(X_i)$, $X_i \sim N(\theta, 1)$.

$P(\log Y > y \mid Y \leq 1) = P(\exp(X_i) > y \mid \exp(X_i) \leq 1)$

$= P(X_i > \log(y) \mid X_i \leq 0)$



Bayes: $= \dfrac{P(X_i > \log(y) \cap X_i \leq 0)}{P(X_i \leq 0)} = \dfrac{\Phi(0) - \Phi(\log(y))}{\Phi(0)}$

By symmetry $\Phi(0) = \frac{1}{2} \Rightarrow P(Y > y \mid Y \leq 1) = 1 - 2\Phi(\log(y))$.

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3) (a) The sum of independent random normals X_1+X_2 is normal with $\mu = \mu_1 + \mu_2$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2$ where $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$

$\Rightarrow \sum_{i=1}^{n_x} x_i \sim N(n_x\mu_x, n_x\sigma_x^2)$, $\sum_{i=1}^{n_y} y_i \sim N(n_y\mu_y, n_y\sigma_y^2)$

Therefore, $\bar{x}_{n_x} \sim N\left(\mu_x, \frac{\sigma_x^2}{n_x}\right)$, $\bar{y}_{n_y} \sim N\left(\mu_y, \frac{\sigma_y^2}{n_y}\right)$.

Thus, $\hat{\theta} \sim N\left(\mu_x + \mu_y, \frac{\sigma_x^2}{n_x} + \frac{\sigma_y^2}{n_y}\right)$.

(b) It is known that $\hat{\theta} \sim N\left(\mu_x + \mu_y, \frac{1}{n_x} + \frac{1}{n_y}\right)$ for some $\mu_x + \mu_y$. The null hypothesis is that $\mu_x + \mu_y = 0$. The alternative is that $\mu_x + \mu_y \neq 0$.

Propose $T = (\hat{\theta} - 0) / \sqrt{\frac{1}{n_x} + \frac{1}{n_y}}$. We will reject if $|T| > t$ where t is the $1-\frac{\alpha}{2}$ quantile of $N(0,1)$.

$$\begin{aligned} P(|T| > t \mid H_0) &= P(T > t \mid H_0) + P(-T < t \mid H_0) \\ &= \frac{\alpha}{2} + \frac{\alpha}{2} \\ &= \alpha. \end{aligned}$$

(c) $T = -0.9\sqrt{50} = 6.36$

This is a very high value of T and we will reject under any reasonable value for α.

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4) (a) $\pi(y) = \begin{cases} 0.2, & y=1 \\ 0.6, & y=2 \\ 0.2, & y=0 \end{cases}$

$$E(Y) = 0.2(1) + 0.6(2) + 0.2(0)$$
$$= 0.2 + 1.2 = 1.4.$$

(b) $\pi(y|x) = \begin{cases} 0.7, & x=2, y=1 \\ 0.3, & x=2, y=2 \\ 0, & x=2, y=0 \\ 0.4, & x=4, y=1 \\ 0.45 & x=4, y=2 \\ 0.15 & x=4, y=0 \\ 5/70 & x=0, y=1 \\ 48/70 & x=0, y=2 \\ 17/70 & x=0, y=0 \end{cases}$

$$E[Y|X] = \begin{cases} 0.7(1) + 0.3(2) + 0 = 1.3, & x=2 \\ 0.4(1) + 0.45(2) + 0 = 1.3, & x=4 \\ 5/70(1) + \frac{48}{70}(2) + 0 = \frac{101}{70}, & x=0 \end{cases}$$

$$E[E[Y|X]] = 1.3(0.1) + 1.3(0.2) + \frac{101}{70}(0.7) = \frac{140}{100} = 1.4$$

$\Rightarrow E[E[Y|X]] = E[Y]$ so we have shown LIE holds.

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5) (a) $l_n(X|\theta) = \sum_{i=1}^{n} \log (1-p_1-p_2)^{1(x=0)} p_1^{1(x=1)} p_2^{1(x=2)}$
$= \sum_{i=1}^{n} 1(x=0)\log(1-p_1-p_2) + 1(x=1)\log(p_1) + 1(x=2)\log(p_2)$

(b) $0 = \frac{\partial l_n}{\partial p_1} \Rightarrow \sum_{i=1}^{n} -\frac{1(x=0)}{1-p_1-p_2} + \frac{1(x=1)}{p_1} = 0 \Rightarrow \left(\sum_{i=1}^{n} 1(x=0)\right)/(1-p_1-p_2) = \left(\sum_{i=1}^{n} 1(x=1)\right)/p_1$
$0 = \frac{\partial l_n}{\partial p_2} \Rightarrow \sum_{i=1}^{n} -\frac{1(x=0)}{1-p_1-p_2} + \frac{1(x=2)}{p_2} = 0 \Rightarrow \left(\sum_{i=1}^{n} 1(x=0)\right)/(1-p_1-p_2) = \left(\sum_{i=1}^{n} 1(x=2)\right)/p_2$

Clearly these conditions hold if $\hat{p}_1 = \left(\sum_{i=1}^{n} 1(x=1)\right)/n$, $\hat{p}_2 = \left(\sum_{i=1}^{n} 1(x=2)\right)/n$.

(c) $s = \frac{\partial}{\partial\theta} \overset{\log(f(x))}{\cancel{\sum l_n}} = \begin{pmatrix} \frac{1(x=1)}{p_1} - \frac{1(x=0)}{1-p_1-p_2} \\ \frac{1(x=2)}{p_2} - \frac{1(x=0)}{1-p_1-p_2} \end{pmatrix}$

(cross terms = 0 because $1(x=1) = 1 \Rightarrow x \neq 0, x \neq 2$, etc.

$I_0 = E[s s'] = E\begin{bmatrix} \left(\frac{1(x=1)}{p_1}\right)^2 + \left(\frac{1(x=0)}{1-p_1-p_2}\right)^2 & \left(\frac{1(x=0)}{1-p_1-p_2}\right)^2 \\ \left(\frac{1(x=0)}{1-p_1-p_2}\right)^2 & \left(\frac{1(x=2)}{p_2}\right)^2 + \left(\frac{1(x=0)}{1-p_1-p_2}\right)^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{p_1} + \frac{1}{1-p_1-p_2} & \frac{1}{1-p_1-p_2} \\ \frac{1}{1-p_1-p_2} & \frac{1}{p_2} + \frac{1}{1-p_1-p_2} \end{bmatrix}$

$I_0 = -E\begin{bmatrix} -\frac{1(x=1)}{p_1^2} - \frac{1(x=0)}{(1-p_1-p_2)^2} & -\frac{1(x=0)}{(1-p_1-p_2)^2} \\ -\frac{1(x=0)}{(1-p_1-p_2)^2} & -\frac{1(x=2)}{p_2^2} - \frac{1(x=0)}{(1-p_1-p_2)^2} \end{bmatrix}$
$= \begin{bmatrix} \frac{1}{p_1} + \frac{1}{1-p_1-p_2} & \frac{1}{1-p_1-p_2} \\ \frac{1}{1-p_1-p_2} & \frac{1}{p_2} + \frac{1}{1-p_1-p_2} \end{bmatrix}$

so we have verified the information identity holds.