Econometrics HW6

Michael B. Nattinger*

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1 Question 1

1.1 Part A

$P(X=1) = p = p^1(1-p)^{1-1} = f(1)$. $P(X=0) = (1-p) = p^0(1-p)^{1-0} = f(0)$.

1.2 Part B

Our parameter is $\theta = p$. $l_n(\theta) = \sum_{i=1}^n log(f(X_i|\theta)) = \sum_{i=1}^n log(\theta^{X_i}(1-\theta)^{1-X_i}) = \sum_{i=1}^n X_i log(\theta) + (1-X_i)log(1-\theta)$.

1.3 Part C

$\frac{\partial l_n(\theta)}{\partial\theta} = 0 \Rightarrow \sum_{i=1}^n \frac{X_i}{\theta} - \frac{1-X_i}{1-\theta} = 0 \Rightarrow \sum_{i=1}^n X_i(1-\theta) = \sum_{i=1}^n \theta - X_i\theta \Rightarrow \hat{\theta}_n = \frac{1}{n}\sum_{i=1}^n X_i$.

2 Question 2

2.1 Part A

$l_n(\theta) = \sum_{i=1}^n log\left(\frac{\theta}{X_i^{1+\theta}}\right) = \sum_{i=1}^n log(\theta) - (1+\theta)log(X_i) = nlog(\theta) - (1+\theta)\sum_{i=1}^n log(X_i)$

2.2 Part B

$\frac{\partial l_n(\theta)}{\partial\theta} = 0 \Rightarrow \frac{n}{\theta} - \sum_{i=1}^n log(X_i) = 0 \Rightarrow \hat{\theta}_n = \frac{n}{\sum_{i=1}^n log(X_i)}$.

3 Question 3

3.1 Part A

$l_n(\theta) = \sum_{i=1}^n log\left(\frac{1}{\pi(1+(X_i-\theta)^2)}\right) = -nlog(\pi) - \sum_{i=1}^n log(1+(X_i-\theta)^2)$.

3.2 Part B

$\frac{\partial l_n(\theta)}{\partial\theta} = 0 \Rightarrow -\sum_{i=1}^n \frac{2(X_i-\hat{\theta}_n)}{1+(X_i-\hat{\theta}_n)^2} = 0$

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

4 Question 4

4.1 Part A

$l_n(\theta) = \sum_{i=1}^n log(\frac{1}{2}exp(-|X_i - \theta|)) = nlog(\frac{1}{2}) - \sum_{i=1}^n |X_i - \theta|$

4.2 Part B

The likelihood is maximized when the term $\sum_{i=1}^n |X_i - \theta|$ is minimized. This is minimized for $\theta = M$ where M is the median of the sample, which I will show below:

Let X_i be ordered from smallest to largest. If n is an odd number, define $m = \frac{n+1}{2}$. Then, by the triangle inequality,

$$\sum_{i=1}^n |X_i - \theta| \geq |X_n - \theta - (X_1 - \theta)| + |X_{n-1} - \theta - (X_2 - \theta)| + \cdots + |X_{m-1} - \theta - (X_{m+1} - \theta)| + |X_m - \theta|$$
$$= \sum_{i=1}^{m-1} |X_{n+1-i} - X_i| + |X_m - \theta|.$$

Clearly, this term is minimized when $\theta = X_m = M$, and the weak inequality holds with equality when θ is the median because $(X_{n+1-i} - M) \geq 0 \geq (X_i - M)$.

If n is even, we instead define $m = n/2$, and have:

$$\sum_{i=1}^n |X_i - \theta| \geq |X_n - \theta - (X_1 - \theta)| + \cdots + |X_{m-1} - \theta - (X_{m+1} - \theta)| + |X_m - \theta| + |X_{m+1} - \theta|$$
$$= \sum_{i=1}^{m-1} |X_{n+1-i} - X_i| + |X_m - \theta| + |X_{m+1} - \theta|,$$

where again our weak inequality holds with equality. In this case, the final expression is clearly minimized for any $\theta \in [X_m, X_{m+1}]$, and $M \in [X_m, X_{m+1}]$.

5 Question 5

$I_0 = -E\left[\frac{\partial^2}{\partial\theta^2} log(f(X|\theta))|_{\theta=\theta_0}\right] = -E\left[\frac{\partial^2}{\partial\theta^2} log(\theta x^{-1-\theta})|_{\theta=\theta_0}\right] = -E\left[\frac{\partial^2}{\partial\theta^2} log(\theta) + (-1-\theta)log(x)|_{\theta=\theta_0}\right]$
$= -E\left[\frac{\partial}{\partial\theta}\frac{1}{\theta} - log(x)|_{\theta=\theta_0}\right] = -E\left[\frac{\partial}{\partial\theta}\frac{1}{\theta} - log(x)|_{\theta=\theta_0}\right] = \frac{1}{\theta_0^2}$

6 Question 6

6.1 Part A

$I_0 = -E\left[\frac{\partial^2}{\partial\theta^2} log(\theta exp(-\theta x))|_{\theta=\theta_0}\right] = -E\left[\frac{\partial^2}{\partial\theta^2} log(\theta) + log(exp(-\theta x))|_{\theta=\theta_0}\right]$
$= -E\left[\frac{\partial^2}{\partial\theta^2} log(\theta) - \theta x|_{\theta=\theta_0}\right] = \hat{\theta}_0^{-2} \Rightarrow Var(\bar{\theta}_n) \geq (n\hat{\theta}_0^{-2})^{-1} = \frac{\theta_0^2}{n}$

6.2 Part B

$l_n(\theta) = \sum_{i=1}^n log(f(X_i|\theta)) = \sum_{i=1}^n log(\theta exp(-\theta X_i)) = \sum_{i=1}^n log(\theta) + log(exp(-\theta X_i))$
$= nlog(\theta) - \theta\sum_{i=1}^n X_i \Rightarrow \frac{\partial l_n(\theta)}{\partial\theta} = 0 \Rightarrow \frac{n}{\theta} - \sum_{i=1}^n X_i = 0 \Rightarrow \hat{\theta}_n = \frac{n}{\sum_{i=1}^n X_i}$.

By the delta method, $\sqrt{n}(\hat{\theta}_n - \theta_0) \rightarrow_d N(0, V)$ where $V = (-1(\theta_0^{-1})^{-2})^2\sigma^2 = \theta_0^4\sigma^2$ where $\sigma^2 = Var(X_i) = \frac{1}{\theta_0^2}$. Thus, $\sqrt{n}(\hat{\theta}_n - \theta_0) \rightarrow_d N(0, \theta_0^2)$

6.3 Part C

Our general formula is $\sqrt{n}(\hat{\theta}_n - \theta_0) \to_d N(0, I_0^{-1}) = N(0, \theta_0^2)$.

7 Question 7

7.1 Part A

Via the delta method, $\sqrt{n}(\hat{\theta}_n - \theta_0) \to_d N(0, V)$ where $V = Var(X_i) = p(1-p)$. Thus, a consistent estimator for V will be a consistent estimator of $Var(X_i)$. A consistent estimator of the variance is $\hat{V} := (\frac{1}{n}\sum_{i=1}^n (X_i))(1 - \frac{1}{n}\sum_{i=1}^n (X_i))$.

7.2 Part B

The WLLN and CMT imply that $\hat{V} \to_p p(1-p) = Var(X_i) = V$. Thus, $\hat{V}$ is a consistent estimator of V.

7.3 Part C

We have that the asymptotic variance of $\sqrt{n}(\hat{\theta}_n - \theta_0)$ is consistently estimated by $(\frac{1}{n}\sum_{i=1}^n (X_i))(1 - \frac{1}{n}\sum_{i=1}^n (X_i))$. Therefore, an approximation of $Var(\hat{\theta}_n) = \frac{1}{n}Var(\sqrt{n}\hat{\theta}_n) = \frac{1}{n}Var(\sqrt{n}(\hat{\theta}_n - \theta_0))$ so an estimator of $Var(\hat{\theta}_n)$ is $\frac{1}{n}((\frac{1}{n}\sum_{i=1}^n (X_i))(1 - \frac{1}{n}\sum_{i=1}^n (X_i)))$.

8 Question 8

8.1 Part A

$$F_X(c) = \int_{-\infty}^c f_X(x)dx = \begin{cases} 0, c < 0 \\ G(c), 0 \leq c \leq \theta \\ 1, c > \theta \end{cases} \quad \text{where } G(c) = \int_0^c \frac{1}{\theta}dx = \frac{c}{\theta}.$$

8.2 Part B

$F_{n(\hat{\theta}_n - \theta)}(x) = Pr(\max_{i=1,\ldots,n}(n(X_i - \theta)) \leq x) = Pr(n(X_1 - \theta) \leq x, \ldots, n(X_n - \theta) \leq x) = \prod_{i=1}^n Pr(n(X_i - \theta) \leq x) = \prod_{i=1}^n Pr(X_i \leq \theta + \frac{x}{n}) = Pr(X_i \leq \theta + \frac{x}{n})^n = (F_X(\theta + \frac{x}{n}))^n$.

8.3 Part C

Fix x. For $x < 0$, $F_{n(\hat{\theta}_n - \theta)}(x) = (F_X(\theta + \frac{x}{n}))^n = (F_X(\theta(1 + \frac{x/\theta}{n})))^n \to_{n\to\infty} \lim_{n\to\infty}((\theta(1 + \frac{x/\theta}{n}))/\theta)^n = e^{x/\theta}$.

For $x > 0, F_{n(\hat{\theta}_n - \theta_0)} = (F_X(\theta + \frac{x}{n}))^n = 1^n = 1$ so $\lim_{n\to\infty} F_{n(\hat{\theta}_n - \theta)} = 1$.

8.4 Part D

$\lim_{n\to\infty} f_{n(\hat{\theta}_n - \theta)}(x) = \lim_{n\to\infty} \frac{\partial}{\partial x} F_{n(\hat{\theta}_n - \theta)}(x) = \frac{1}{\theta}e^{x/\theta}$ for $x \leq 0 \Rightarrow \lim_{n\to\infty} f_{n(\hat{\theta}_n - \theta)}(-x) = \frac{1}{\theta}e^{-x/\theta}$ so $n(\hat{\theta}_n - \theta) \to_d -A$ where distribution A is an exponential with parameter θ.

9 Question 9

We should use a two-sided test. We will calculate $t = \frac{\bar{X}_n - 1}{se}, se = \sqrt{s^2/n}$. For a chosen significance level α we can reject the null hypothesis if $P(|T| > t) < \alpha/2$, where $t \sim t_{n-1}$.

10 Question 10

Assume $\mu = 1$. Then, $X_i \sim N(1,1) \Rightarrow \sqrt{n}(\bar{X}_n - 1) \sim N(0,1)$ by WLLN, CLT $\Rightarrow |\sqrt{n}(\bar{X}_n - 1)| \sim |N(0,1)|$. Also, $\sqrt{n}(\bar{X}_n - 1) \sim N(0,1) \Rightarrow \sqrt{n}\bar{X}_n \sim N(\sqrt{n}, 1) \Rightarrow |\sqrt{n}\bar{X}_n| \sim |N(\sqrt{n}, 1)| = |N(0,1) + \sqrt{n}|$.

Therefore, $P(T > c|\mu = 1) = P(\min\{|\sqrt{n}\bar{X}_n|, |\sqrt{n}(\bar{X}_n - 1)|\} > c|\mu = 1) = P(\min\{|Z|, |Z - \sqrt{n}|\}) = \alpha$.

Assume $\mu = 0$. Then, $X_i \sim N(0,1) \Rightarrow \sqrt{n}(\bar{X}_n) \sim N(0,1)$ by WLLN, CLT $\Rightarrow |\sqrt{n}(\bar{X}_n)| \sim |N(0,1)|$. Also, $\sqrt{n}(\bar{X}_n) \sim N(0,1) \Rightarrow \sqrt{n}\bar{X}_n - \sqrt{n} \sim N(-\sqrt{n}, 1) \Rightarrow |\sqrt{n}\bar{X}_n - \sqrt{n}| \sim |N(-\sqrt{n}, 1)| = |N(\sqrt{n}, 1)| = |N(0,1) + \sqrt{n}|$.

Therefore, $P(T > c|\mu = 0) = P(\min\{|\sqrt{n}\bar{X}_n|, |\sqrt{n}(\bar{X}_n - 1)|\} > c|\mu = 0) = P(\min\{|Z|, |Z - \sqrt{n}|\}) = \alpha$.

Thus, the size of the test is α.