Macro PS4

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February 19, 2021

1 Question 1

The household maximizes their utility subject to their budget constraint. Equivalently, households minimize their costs subject to their utility constraint:

$$\min_{C_{ik}} \int \sum_i P_{ik} C_{ik} dk$$
$$\text{s.t. } \left(\int C_k^{\frac{\rho-1}{\rho}} dk \right)^{\frac{\rho}{\rho-1}} = C,$$
$$\text{where } \left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} = C_k.$$

We will then write the Lagrangian as follows:

$$\mathcal{L} = \int \sum_i P_{ik} C_{ik} dk - P\left(\left(\int C_k^{\frac{\rho-1}{\rho}} dk \right)^{\frac{\rho}{\rho-1}} - C \right) + \int P_k \left[C_k - \left(\sum_i C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} \right] dk$$

We solve this maximization problem by taking first order conditions with respect to our choice variables, in this case C_{ik}, C_k:

$$P_k = \frac{\rho}{\rho-1} \left(\int C_k^{\frac{\rho-1}{\rho}} dk \right)^{\frac{1}{\rho-1}} \frac{\rho-1}{\rho} C_k^{\frac{-1}{\rho}}$$
$$\Rightarrow C_k = \left(\frac{P_k}{P} \right)^{\rho} C.$$
$$P_{ik} = P_k \frac{\theta}{\theta-1} \left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{1}{\theta-1}} \frac{\theta-1}{\theta} C_{ik}^{-\frac{1}{\theta}}$$
$$\Rightarrow C_{ik} = \left(\frac{P_{ik}}{P_k} \right)^{\theta} C_k$$

We then can simplify our consumption first order condition to the following:

$$P_k \left(\frac{C_{ik}}{C_k} \right)^{\frac{1}{\theta}} = P C^{\frac{1}{\rho}} C_k^{\frac{-1}{\rho}} \left(\frac{C_{ik}}{C_k} \right)^{\frac{1}{\theta}}$$
$$\Rightarrow C_k = \left(\frac{P_k}{P} \right)^{-\rho} C,$$

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, Katherine Kwok, and Danny Edgel.

a familiar expression to what we found in lecture.
We can substitute in our expressions into the definitions of C, C_k:

$$\begin{aligned}
\left(\int \left[\left(\frac{P_k}{P}\right)^{-\rho} C\right]^{\frac{\rho-1}{\rho}} dk\right)^{\frac{\rho}{\rho-1}} &= C \\
\Rightarrow \left(\int \left(\frac{P_k}{P}\right)^{1-\rho} dk\right)^{\frac{\rho}{\rho-1}} &= 1 \\
\Rightarrow \left(\int P_k^{1-\rho} dk\right)^{\frac{1}{1-\rho}} &= P, \\
\left(\sum_i \left[\left(\frac{P_{ik}}{P_k}\right)^{\theta} C_k\right]^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}} &= C_k \\
\Rightarrow \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{1}{1-\theta}} &= P_k
\end{aligned}$$

To summarize, we have the following:

$$P_k = \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{1}{1-\theta}} \tag{1}$$

$$P = \left(\int \left[\left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{1}{1-\theta}}\right]^{1-\rho} dk\right)^{\frac{1}{1-\rho}} \tag{2}$$

$$C_{ik} = P_{ik}^{-\theta} \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} P^{\rho} C \tag{3}$$

2 Question 2

The firms compete a la Cournot:

$$\begin{aligned}
&\max_{P_{ik}} P_{ik} C_{ik} - W L_{ik} \\
&\text{s.t.} C_{ik} = P_{ik}^{-\theta} \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} P^{\rho} C \\
&\text{and } C_{ik} = A_{ik} L_{ik}
\end{aligned}$$

Substituting, we form the following objective function:

$$\begin{aligned}
&\max_{P_{ik}} P_{ik}^{1-\theta} \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} P^{\rho} C - W A_{ik}^{-1} P_{ik}^{-\theta} \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} P^{\rho} C \\
\Rightarrow &\max_{P_{ik}} P_{ik}^{1-\theta} \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} - W A_{ik}^{-1} P_{ik}^{-\theta} \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}}
\end{aligned}$$

We take first order conditions:

$$
\begin{aligned}
(1-\theta)P_{ik}^{-\theta}P_k^{\theta-\rho} + P_{ik}^{1-\theta}\frac{\theta-\rho}{1-\theta}P_k^{2\theta-\rho-1}(1-\theta)P_{ik}^{-\theta} &= \frac{W}{A_{ik}}\left[(-\theta)P_{ik}^{-\theta-1}P_k^{\theta-\rho} + P_{ik}^{-\theta}\frac{\theta-\rho}{1-\theta}P_k^{2\theta-\rho-1}(1-\theta)P_{ik}^{-\theta}\right] \\
(1-\theta) + P_{ik}^{1-\theta}(\theta-\rho)P_k^{\theta-1} &= \frac{W}{A_{ik}}\left[(-\theta)P_{ik}^{-1} + P_{ik}^{-\theta}(\theta-\rho)P_k^{\theta-1}\right]
\end{aligned}
$$

Denote the weighted price ratio $s_{ik} := \left(\frac{P_{ik}}{P_k}\right)^{1-\theta}$:

$$
\begin{aligned}
P_{ik}[(1-\theta) + s_{ik}(\theta-\rho)] &= \frac{W}{A_{ik}}\left[(-\theta) + s_{ik}(\theta-\rho)\right] \\
\Rightarrow P_{ik} &= \frac{W}{A_{ik}}\left[1 - \frac{1}{(1-\theta) + s_{ik}(\theta-\rho)}\right]
\end{aligned}
$$

We can derive demand elasticities $\frac{P_{ik}\partial C_{ik}}{C_{ik}\partial P_{ik}}$:

$$
\begin{aligned}
\frac{P_{ik}\partial C_{ik}}{C_{ik}\partial P_{ik}} &= \frac{P_{ik}}{C_{ik}}\left((-\theta)P_{ik}^{-1-\theta}P_k^{\theta-\rho} + P_{ik}^{-\theta}\frac{\theta-\rho}{1-\theta}P_k^{2\theta-\rho-1}(1-\theta)P_{ik}^{-\theta}\right)P^{\rho}C \\
&= \left(P_{ik}^{1+\theta}P_k^{\rho-\theta}P^{-\rho}C^{-1}\right)\left((-\theta)P_{ik}^{-1-\theta}P_k^{\theta-\rho} + P_{ik}^{-2\theta}(\theta-\rho)P_k^{2\theta-\rho-1}\right)P^{\rho}C \\
&= \left((-\theta) + P_{ik}^{1-\theta}(\theta-\rho)P_k^{\theta-1}\right) \\
&= (\theta-\rho)s_{ik} - \theta.
\end{aligned}
$$

3 Question 3

4 Question 4

5 Question 5

6 Question 6