Econometrics HW3

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1 Question 1

Let a random point be distributed uniformly on the square with vertices $(1,1),(1,-1),(-1,1),(-1,-1)$.

1.1 Determine $P(X^2+Y^2<1)$.

$$\begin{aligned}P(X^2+Y^2<1) &= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\frac{1}{4}dxdy = \frac{1}{2}\int_{-1}^{1}\sqrt{1-y^2}dy = \frac{1}{2}((1/2)\text{arcsin } x + (1/2)x\sqrt{1-x^2}|_{-1}^{1}) \\ &= \frac{\pi}{4}.\end{aligned}$$

1.2 Determine $P(|X+Y|<2)$

$$P(|X+Y|<2) = \int_{-1}^{1}\int_{-1}^{1}\frac{1}{4}dxdy = \int_{-1}^{1}\frac{1}{2}dy = 1.$$

2 Question 2

2.1 What conditions should a, b satisfy in order for $f(x,y)$ to be a bivariate PDF?

For $f(x,y)$ to be a bivariate PDF, it must integrate to one:

$$\begin{aligned}1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x)h(y)dxdy = \int_{-\infty}^{\infty}g(x)dx\int_{-\infty}^{\infty}h(y)dy \\ &= ab.\end{aligned}$$

So, for $f(x,y)$ to be a bivariate PDF, $ab=1$.

2.2 Find the marginal PDF of X and Y.

$$\begin{aligned}f_X(x) &= \int_{-\infty}^{\infty}g(x)h(y)dy = g(x)\int_{-\infty}^{\infty}h(y)dy = bg(x), \\ f_Y(y) &= \int_{-\infty}^{\infty}g(x)h(y)dx = h(y)\int_{-\infty}^{\infty}g(x)dx = ah(y).\end{aligned}$$

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

2.3 Show that X, Y are independent.

$f_{X,Y}(x,y) = g(x)h(y) = (ab)g(x)h(y) = bg(x)ah(y) = f_X(x)f_Y(y)$ so X, Y are independent.

3 Question 3

3.1 Find c such that $f(x,y)$ is a joint PDF.

We will find the value of c such that the integral of $f(x,y)$ on its support is 1.

$$\begin{aligned}\int_0^1 \int_0^{1-y} cxydxdy = c\int_0^1 y(1-y)^2/2dy = (c/2)\int_0^1 y - 2y^2 + y^3 dy = (c/2)(y^2/2 - (2/3)y^3 + y^4/4|_0^1) \\ = (c/2)((1/2) - (2/3) + (1/4)) = c/24.\end{aligned}$$

Thus, for the integral of $f(x,y)$ on its support to be equal to 1, $c = 24$.

3.2 Find the marginal distributions of X and Y.

$$\begin{aligned}f_X(x) &= \int_0^{1-x} 24xydy = 24x(y^2/2|_0^{1-x}) = 12x(1-x)^2, \\ f_Y(y) &= \int_0^{1-y} 24xydx = 24y(x^2/2|_0^{1-y}) = 12y(1-y)^2.\end{aligned}$$

Note: This is for $x, y \in [0,1]$. For all other x, y, $f_X(x) = f_Y(y) = 0$.

3.3 Are X, Y independent?

X, Y are not independent. $f_{X,Y}(x,y) = 24xy \neq (12x(1-x)^2)(12y(1-y)^2) = f_X(x)f_Y(y)$. Note that our result from question 2 does not hold for this question because the region on which the joint PDF is nonzero is a function of X, Y. So, the joint distribution can not be separately factored into X, Y marginals as the support for the marginal for each random variable is a function of the realization of the other random variable.

4 Question 4

We will show that any random variable is uncorrelated with a constant.

$$P(k \leq y, X \leq x) = \begin{cases} 0, y < k \\ P(X \leq x), y \geq k \end{cases} \qquad = P(k \leq y)P(X \leq x)$$

so X, k independent and, therefore, X, k are uncorrelated.

5 Question 5

From the independence of X, Y, $E(XY) = EXEY = \mu_X\mu_Y$.

$$\begin{aligned}
\sigma_{XY}^2 &= E[(XY)^2] - E[XY]^2 = E[X^2Y^2] - \mu_X^2\mu_Y^2 = E[X^2]E[Y^2] - \mu_X^2\mu_Y^2 \\
&= (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_X^2\mu_Y^2 \\
Cov(XY, Y) &= E((XY)Y) - E(XY)EY = E(XY^2) - \mu_X\mu_Y^2 = EXE(Y^2) - \mu_X\mu_Y^2 \\
&= \mu_X(\sigma_Y^2 + \mu_Y^2) - \mu_X\mu_Y^2 = \mu_X\sigma_Y^2 \\
Corr(XY, Y) &= \frac{Cov(XY, Y)}{\sqrt{\sigma_{XY}^2\sigma_Y^2}} = \frac{\mu_X\sigma_Y^2}{\sqrt{((\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_X^2\mu_Y^2)\sigma_Y^2}} \\
&= \frac{\mu_X\sigma_Y}{\sqrt{\sigma_X^2\sigma_Y^2 + \mu_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2}}.
\end{aligned}$$

6 Question 6

Let $(X_1, \ldots, X_n)'$ be a random vector. We will prove via induction. First, let $n = 2$. Then, $Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)$.

Next, assume that $Var\left(\sum_{i=1}^n X_i\right) = \left(\sum_{i=1}^n Var(X_i)\right) + 2\left(\sum_{1\le i<j\le n} Cov(X_i, X_j)\right)$. Then,

$$\begin{aligned}
Var\left(\sum_{i=1}^{n+1} X_i\right) &= Var\left(\sum_{i=1}^{n} X_i\right) + Var(X_{n+1}) + 2Cov\left(\left(\sum_{i=1}^{n} X_i\right), X_{n+1}\right) \\
&= \left(\sum_{i=1}^{n} Var(X_i)\right) + 2\left(\sum_{1\le i<j\le n} Cov(X_i, X_j)\right) + Var(X_{n+1}) + 2Cov\left(\left(\sum_{i=1}^{n} X_i\right), X_{n+1}\right) \\
&= \left(\sum_{i=1}^{n+1} Var(X_i)\right) + 2\left(\sum_{1\le i<j\le n} Cov(X_i, X_j)\right) + 2Cov\left(\left(\sum_{i=1}^{n} X_i\right), X_{n+1}\right).
\end{aligned}$$

Next, note that, for random variables X, Y, Z,

$$\begin{aligned}
Cov(X + Y, Z) &= E[(X + Y)Z] - E(X + Y)E(Z) = E[XZ + YZ] - (EX + EY)EZ \\
&= E(XZ) - EXEZ + E(YZ) - EYEZ = Cov(X, Z) + Cov(Y, Z).
\end{aligned}$$

We then have,

$$\begin{aligned}
Var\left(\sum_{i=1}^{n+1} X_i\right) &= \left(\sum_{i=1}^{n+1} Var(X_i)\right) + 2\left(\sum_{1\le i<j\le n} Cov(X_i, X_j)\right) + 2\sum_{i=1}^{n} Cov\left(X_i, X_{n+1}\right) \\
&= \left(\sum_{i=1}^{n+1} Var(X_i)\right) + 2\left(\sum_{1\le i<j\le n+1} Cov(X_i, X_j)\right).
\end{aligned}$$

Therefore, by induction, $Var\left(\sum_{i=1}^n X_i\right) = \left(\sum_{i=1}^n Var(X_i)\right) + 2\left(\sum_{1\le i<j\le n} Cov(X_i, X_j)\right)$.

7 Question 7

Let X, Y be jointly normal.

7.1 Derive the marginal distribution of X, Y and observe that both are normal distributions.

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-(2(1-\rho^2))^{-1}((x^2/\sigma_X^2) - 2\rho xy/(\sigma_X\sigma_Y) + (y^2/\sigma_Y^2))\right) dy \\
&= \frac{1}{\sqrt{2\pi}\sigma_X} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}\sigma_Y} \exp\left(-(2(1-\rho^2))^{-1}((x^2/\sigma_X^2) + (y/\sigma_Y - \rho x/\sigma_X)^2 - (\rho^2 x^2/\sigma_X^2))\right) dy \\
&= \frac{\exp\left(-(2(1-\rho^2))^{-1}((x^2/\sigma_X^2) - (\rho^2 x^2/\sigma_X^2))\right)}{\sqrt{2\pi}\sigma_X} \\
&* \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \exp\left(-(2(1-\rho^2))^{-1}((y/\sigma_Y - \rho x/\sigma_X)^2)\right) dy \\
&= \frac{\exp\left(-(2(1-\rho^2))^{-1}((x^2/\sigma_X^2) - (\rho^2 x^2/\sigma_X^2))\right)}{\sqrt{2\pi}\sigma_X} = \frac{\exp\left(-(x^2/(2\sigma_X^2))\right)}{\sqrt{2\pi}\sigma_X}.
\end{aligned}
$$

Note that this is the form of a normal distribution with mean 0 and variance σ_X^2. By symmetry, $f_Y(y) = \frac{\exp\left(-(y^2/(2\sigma_Y^2))\right)}{\sqrt{2\pi}\sigma_Y}$ is also a normal distribution with mean 0 and variance σ_Y^2.

7.2 Derive the conditional distribution of Y given $X = x$. Observe that it is also a normal distribution.

$$
\begin{aligned}
f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-(2(1-\rho^2))^{-1}((x^2/\sigma_X^2) - 2\rho xy/(\sigma_X\sigma_Y) + (y^2/\sigma_Y^2))\right) \\
&* \left(\frac{\exp\left(-(x^2/(2\sigma_X^2))\right)}{\sqrt{2\pi}\sigma_X}\right)^{-1} \\
&= \frac{\exp(-(2(1-\rho^2))^{-1}((x^2/\sigma_X^2) - 2\rho xy/(\sigma_X\sigma_Y) + (y^2/\sigma_Y^2)) + (x^2/(\sigma_X^2)))}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \\
&= \frac{\exp(-(2(1-\rho^2))^{-1}((x^2/\sigma_X^2) - 2\rho xy/(\sigma_X\sigma_Y) + (y^2/\sigma_Y^2) - ((1-\rho^2)(x^2/\sigma_X^2))))}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \\
&= \frac{\exp(-(2(1-\rho^2))^{-1}((x^2/\sigma_X^2)(\rho^2) - 2\rho xy/(\sigma_X\sigma_Y) + (y^2/\sigma_Y^2)))}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \\
&= \frac{\exp(-(2(1-\rho^2))^{-1}((x^2/\sigma_X^2)(\rho^2) + (y/\sigma_Y - \rho x/\sigma_X)^2 - \rho^2 x^2/\sigma_X^2))}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \\
&= \frac{\exp(-(2(1-\rho^2))^{-1}(y/\sigma_Y - \rho x/\sigma_X)^2}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \\
&= \frac{\exp(-(2)^{-1}(y/(\sqrt{1-\rho^2}\sigma_Y) - \rho x/(\sqrt{1-\rho^2}\sigma_X))^2}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \\
&= \frac{\exp(-(2)^{-1}((y - (\rho x(\sigma_Y/\sigma_X)))/(\sqrt{1-\rho^2}\sigma_Y))^2}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}}
\end{aligned}
$$

This is also a normal distribution, with mean $\rho x(\sigma_Y/\sigma_X)$ and variance $\sigma_Y^2(1-\rho^2)$.

7.3 Derive the joint distribution of (X, Z) where $Z = (Y/\sigma_Y) - (\rho X/\sigma_X)$, and then show that X, Z are independent.

Note that our mapping from $\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} X \\ Y/\sigma_Y - \rho X/\sigma_X \end{pmatrix}$ has an inverse mapping $\begin{pmatrix} X \\ Z \end{pmatrix} \to \begin{pmatrix} X \\ \sigma_Y Z + \sigma_Y(\rho X/\sigma_X) \end{pmatrix}$ with Jacobian determinant $\left| \begin{pmatrix} 1 & 0 \\ \rho\sigma_Y/\sigma_X & \sigma_Y \end{pmatrix} \right| = \sigma_Y$. Thus, the joint density of X, Z is $f_{X,Z}(x, z) = f_{X,Y}(x, \sigma_Y Z + \rho X\sigma_Y/\sigma_X)\sigma_Y$:

$$\begin{aligned}
f_{X,Z}(x,z) &= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - 2\frac{x(z+\rho x/\sigma_X)\rho}{\sigma_X} + (z + (\rho x/\sigma_X))^2\right)\right) \\
&= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - 2\frac{xz\rho}{\sigma_X} - 2\frac{x^2\rho^2}{\sigma_X^2} + z^2 + 2z(\rho x/\sigma_X) + (\rho x/\sigma_X)^2\right)\right) \\
&= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}}\exp\left(-\frac{x^2}{2\sigma_X^2} - \frac{z^2}{2(1-\rho^2)}\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma_X}\exp\left(-\frac{x^2}{2\sigma_X^2}\right)\frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\left(-\frac{z^2}{2(1-\rho^2)}\right).
\end{aligned}$$

This clearly separates into separate distributions for X, Z so X, Z are independent.

8 Question 8

If X, Y are independent random variables, then $P(X \le x \cap Y \le y) = P(X \le x)P(Y \le y)$. Then, we have the following:

$$\begin{aligned}
P(Z \le z \cap W \le w) &= P(g_1(X) \le z \cap g_2(Y) \le w) \\
&= P(X \le g_1^{-1}(z) \cap Y \le g_2^{-1}(w)) \\
&= P(X \le g_1^{-1}(z))P(Y \le g_2^{-1}(w)) \\
&= P(g_1(X) \le g_1(g_1^{-1}(z)))P(g_2(Y) \le g_2(g_2^{-1}(w))) \\
&= P(Z \le z)P(W \le w).
\end{aligned}$$

Thus, Z, W are independent.