Macro PS4

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1 Question 1

The household solves the following problem:

$$\max_{\{c_t, l_t\}_{t=0}^{\infty}} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + \nu(l_t)\right]$$
$$\text{s.t. } (1+\tau_{ct})c_t + k_{t+1} + b_{t+1} = (1-\delta+r_t)k_t + R_t b_t + w_t(1-l_t)$$

Taking FOCs (Lagrange multiplier λ_t) yields the following:

$$\begin{aligned}
\beta^t c_t^{-\sigma} &= \lambda_t(1+\tau_{ct}) \\
\beta^t \nu'(l_t) &= \lambda_t w_t \\
\lambda_t &= (1-\delta+r_{t+1})\lambda_{t+1} \\
\lambda_t &= R_{t+1}\lambda_{t+1}.
\end{aligned}$$

Simplifying,

$$\begin{aligned}
w_t &= (1+\tau_{ct})\nu'(l_t)c_t^{\sigma} \\
1 &= \beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\frac{1+\tau_{ct+1}}{1+\tau_{ct}}R_{t+1} \\
R_t &= 1-\delta+r_t.
\end{aligned}$$

The above formulas represent the solution to the household's problem. Now we can set up the Ramsey problem. The resource constraint is the following:

$$c_t + k_{t+1} = (1-\delta)k_t + F(k_t, (1-l_t)).$$

Our implementability constraint takes the following form:

$$\sum_{t=0}^{\infty}\beta^t[c_t^{1-\sigma} - (1-l_t)\nu'(l_t)] = \frac{c_0^{-\sigma}}{1+\tau_{c0}}[(1-\delta+r_0)k_{-1} + R_0 b_{-1}],$$

where we make standard assumptions on τ_{c0} to rule out the effective lump-sum tax solution.

Defining $W(c_t, l_t, \lambda) = \frac{c_t^{1-\sigma}}{1-\sigma} + \nu(l_t) + \lambda[c_t^{1-\sigma} - (1-l_t)\nu'(l_t)]$, then the Ramsey problem consists of solving the following maximization problem:

$$\max \sum_t \beta^t [W(c_t, l_t, \lambda) - \lambda\frac{c_0^{-\sigma}}{1+\tau_{c0}}[(1-\delta+r_0)k_{-1} + R_0 b_{-1}]$$

The intertemporal first order condition is the following:

$$W_{ct} = \beta W_{ct+1} R_{t+1}$$

We are almost to our solution. We just need to look at W_{ct}:

$$c_t^{-\sigma} + \lambda c_t^{-\sigma}(1-\sigma) = c_t^{-\sigma}(1+\lambda(1-\sigma))$$
$$\Rightarrow \frac{W_{ct+1}}{W_{ct}} = \left(\frac{c_t}{c_{t+1}}\right)^{\sigma}$$

Coming back to our intertemporal FOC for the Ramsey problem,

$$W_{ct} = \beta W_{ct+1} R_{t+1}$$
$$1 = \beta \left(\frac{c_t}{c_{t+1}}\right)^{\sigma} R_{t+1}$$

Comparing to our first order conditions for the HH problem we can see immediately that $\tau_{ct} = \tau_{ct+1} \forall t$.

2 Question 2

2.1 Part A

A competitive equilibrium is a set of prices $\{p_t, w_t, R_t\}$ and allocations $\{c_{1t}, c_{2t}, n_t, M_t, B_t, T_t\}$ such that agents optimize, markets clear, and the government budget constraint holds.

Agents solve the following:

$$\max_{c_{1t}, c_{2t}, n_t} \sum_{t=0}^{\infty} \beta^t (log(c_{1t}) + \alpha log(c_{2t}) + \gamma log(1-n_t))$$

s.t. $p_t c_{1t} \leq M_t$

and $M_t + B_t \leq (M_{t-1} - p_{t-1}c_{1t-1}) - p_{t-1}c_{2t-1} + w_{t-1}n_{t-1} + R_{t-1}B_{t-1} - T_t$

Market clearing is the following:

$$c_{1t} + c_{2t} = n_t.$$

GBC:

$$M_t - M_{t-1} + B_t = R_{t-1}B_{t-1} - T_t$$

Finally, we are given that monetary policy acts to make $R_t = R \ \forall t$.

2.2 Part B

The consumer's FOCs are the following:

$$\beta^t c_{1t}^{-1} = \lambda_{t+1} p_t + \gamma_t p_t$$
$$\beta^t \alpha c_{2t}^{-1} = \lambda_{t+1} p_t$$
$$\beta^t \gamma (1-n_t)^{-1} = \lambda_{t+1} w_t$$
$$\lambda_t = \gamma_t + \lambda_{t+1}$$
$$\lambda_t = \lambda_{t+1} R_t.$$

Simplifying,

$$\beta^t c_{1t}^{-1} = p_t \lambda_t$$
$$\Rightarrow \frac{c_{2t}}{\alpha c_{1t}} = R_t = R,$$
$$\frac{\gamma c_{2t}}{\alpha(1-n_t)} = \frac{w_t}{p_t}$$

Note that the real wage $\frac{w_t}{p_t}$ must be one because that is the marginal productivity of labor from the firm side. We now have 3 equations in 3 unkowns we can solve for allocations:

$$\frac{c_{2t}}{\alpha c_{1t}} = R, \tag{1}$$

$$\frac{\gamma c_{2t}}{\alpha(1-n_t)} = 1, \tag{2}$$

$$c_{1t} + c_{2t} = n_t. \tag{3}$$

$$\begin{aligned}
n_t &= 1 - \frac{\gamma c_{2t}}{\alpha}, \\
c_{2t} &= \alpha R c_{1t}, \\
c_{1t}(1+\alpha R) &= 1 - \gamma R c_{1t} \\
\Rightarrow c_{1t} &= \frac{1}{1+(\alpha+\gamma)R} \\
\Rightarrow c_{2t} &= \frac{\alpha R}{1+(\alpha+\gamma)R} \\
\Rightarrow n_t &= 1 - \frac{\gamma R}{1+(\alpha+\gamma)R} \\
&= \frac{1+\alpha R}{1+(\alpha+\gamma)R}
\end{aligned}$$

From Wolfram Alpha,

$$\begin{aligned}
\frac{\partial n_t}{\partial R} &= -\frac{\gamma}{(1+(\alpha+\gamma)R)^2} \\
&< 0.
\end{aligned}$$

Therefore, n_t is decreasing in R.