Computational Problem Set 7

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1 Asymptotics

The true data generating process is given by the following:

$$x_t = \rho_0 x_{t-1} + \epsilon_t, \tag{1}$$

where $\epsilon_t \sim N(0, \sigma^2)$. Note first that, given $x_0 = 0$, we can substitute backwards and write $x_t = \sum_{i=0}^{t} \rho_0^i \epsilon_{t-i}$. This immediately implies covariance stationarity of x_t. It is then trivial that $E[x_t] = \bar{x} = 0$. We now can apply covariance stationity and LIE to solve for the other expectations: $E[(x_t - \bar{x})^2] = Var(x_t) = Var(\rho_0 x_{t-1} + \epsilon_t) = \rho_0^2 Var(x_{t-1}) + \sigma_0^2 \Rightarrow Var(x_t) = \frac{\sigma_0^2}{1-\rho_0^2}$. $E[(x_t - \bar{x})(x_{t-1} - \bar{x})] = E[(\rho_0 x_{t-1} + \epsilon_t)x_{t-1}] = E[\rho_0 x_{t-1}^2 + \epsilon_t x_{t-1}] = \rho_0 E[x_{t-1}^2] = \frac{\rho_0 \sigma_0^2}{1-\rho_0^2}$.

We can now ask ourselves, do these moments identify the parameters? For the mean, clearly this is unhelpful as no parameters show up in the expression. Note that is because we are imposing that the constant on the autoregressive process is zero and that therefore its unconditional mean is zero. If we relax this assumption this moment would allow us to estimate the mean of the autoregression process - it would no longer be uninformative. What about the other moments? Together they do identify the parameters, note that $\rho_0 = \frac{E[(x_t - \bar{x})(x_{t-1} - \bar{x})]}{E[(x_t - \bar{x})^2]}$, $\sigma_0^2 = E[(x_t - \bar{x})^2](1 - \rho_0^2)$.

The population moments are therefore:

$$\mu(x) = \begin{pmatrix} 0 \\ \frac{\sigma_0^2}{1-\rho_0^2} \\ \frac{\rho_0 \sigma_0^2}{1-\rho_0^2} \end{pmatrix},$$

and population model moments:

$$\mu(y(b)) = \begin{pmatrix} 0 \\ \frac{\sigma^2}{1-\rho^2} \\ \frac{\rho \sigma^2}{1-\rho^2} \end{pmatrix}.$$

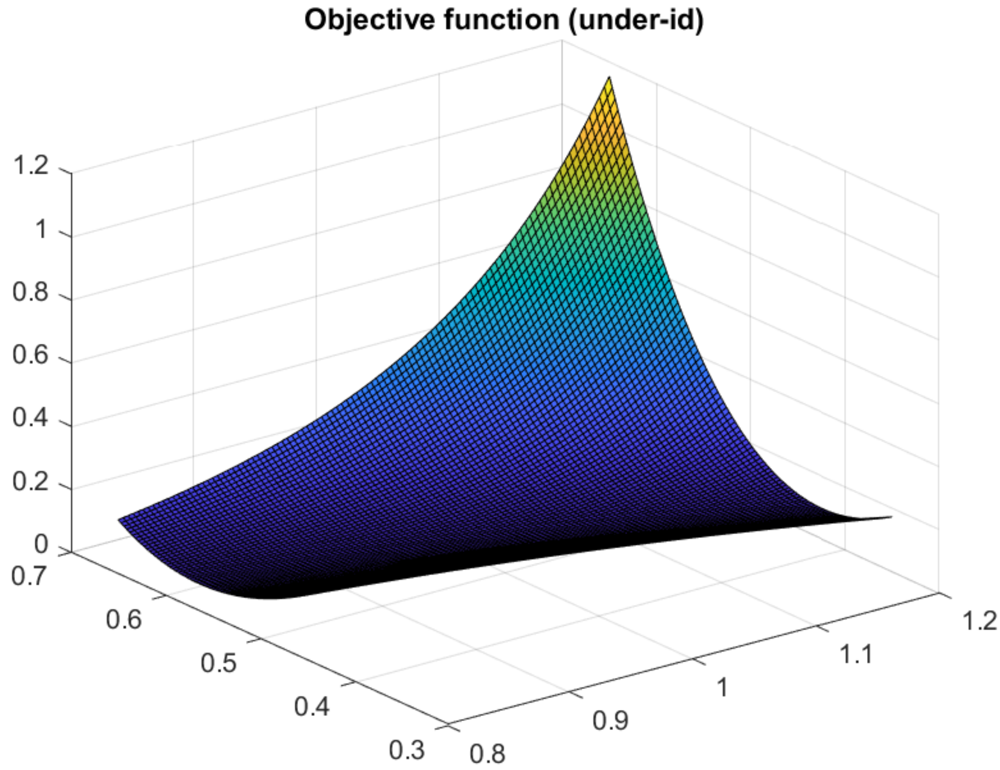
We now can solve for the Jacobian of g:

$$\begin{aligned}
\Delta_b g(b_0) &= \Delta_b \mu(y(b_0)) \\
&= \begin{pmatrix} \frac{\partial}{\partial \rho}(0) & \frac{\partial}{\partial \sigma}(0) \\ \frac{\partial}{\partial \rho}\left(\frac{\sigma^2}{1-\rho^2}\right) & \frac{\partial}{\partial \sigma}\left(\frac{\sigma^2}{1-\rho^2}\right) \\ \frac{\partial}{\partial \rho}\left(\frac{\rho\sigma^2}{1-\rho^2}\right) & \frac{\partial}{\partial \sigma}\left(\frac{\rho\sigma^2}{1-\rho^2}\right) \end{pmatrix}_{|b=b_0} \\
&= \begin{pmatrix} 0 & 0 \\ \frac{2\rho_0\sigma_0^2}{(1-\rho_0^2)^2} & \frac{2\sigma_0}{1-\rho_0^2} \\ \frac{\sigma_0^2}{1-\rho_0^2} + \frac{2\rho_0^2\sigma_0^2}{(1-\rho_0^2)^2} & \frac{2\rho_0\sigma_0}{1-\rho_0^2} \end{pmatrix}.
\end{aligned}$$

The informative moments for identifying b are the variance and first autocovariance moments.

1.1 Under-identified

In this section we present results related to the underidentified model (technically $l = n$ but one of the moments is uninformative so effectively $l > n$). First the objective function:



Note that this objective function doesn't achieve zero, and also the shape is strange - it kind of has a trench moving diagonally across the $\rho \times \sigma$ space. This means that we cannot achieve identification, which is obvious. Here are the other objects that we are asked to report:

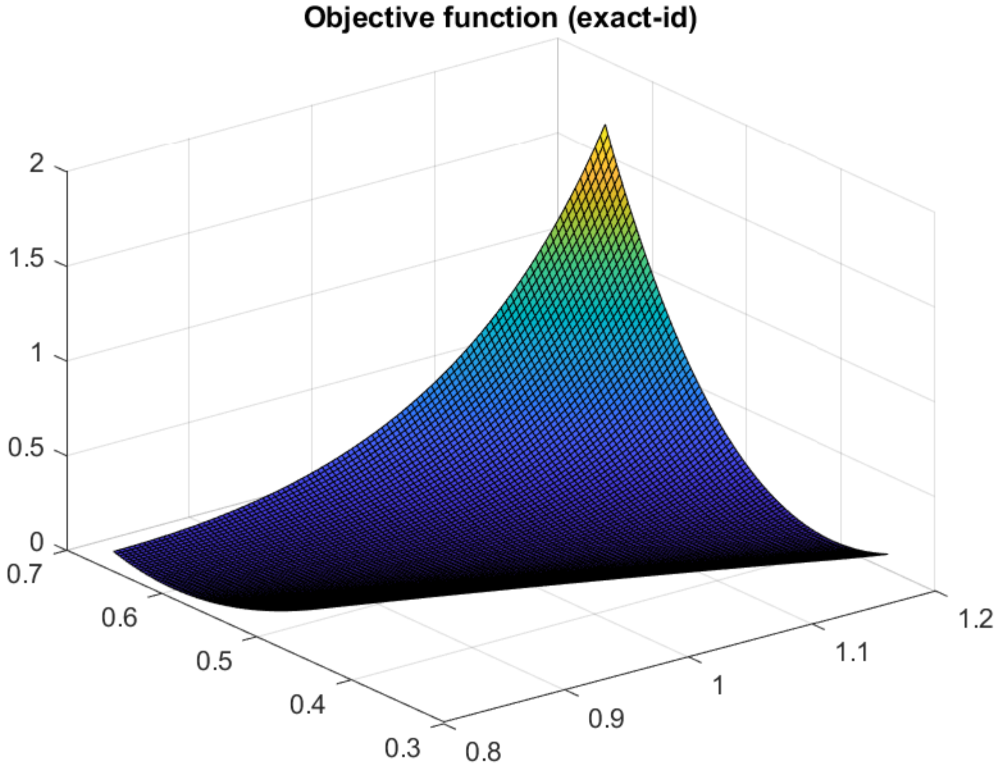
	b1	b2	SE
rho	-0.989	-1	326
sigma	0.184	-0.106	8.57e+03

Jacobian	
-0.341	-0.013
-225	-8.55

Note that the "standard errors" here are massive because we don't have 'real' identification. The Jacobian is nearly collinear in its columns, so the only reason we can invert $\Delta g' \hat{S}^{-1} \Delta g$ is because of numerical precision issues making the Jacobian slightly not exactly collinear. Our J-test (which should be exactly 0) is 47.5! Needless to say, the underidentified estimation doesn't work.

1.2 Exactly-identified

We now present the exactly-identified model. Results are substantially improved.



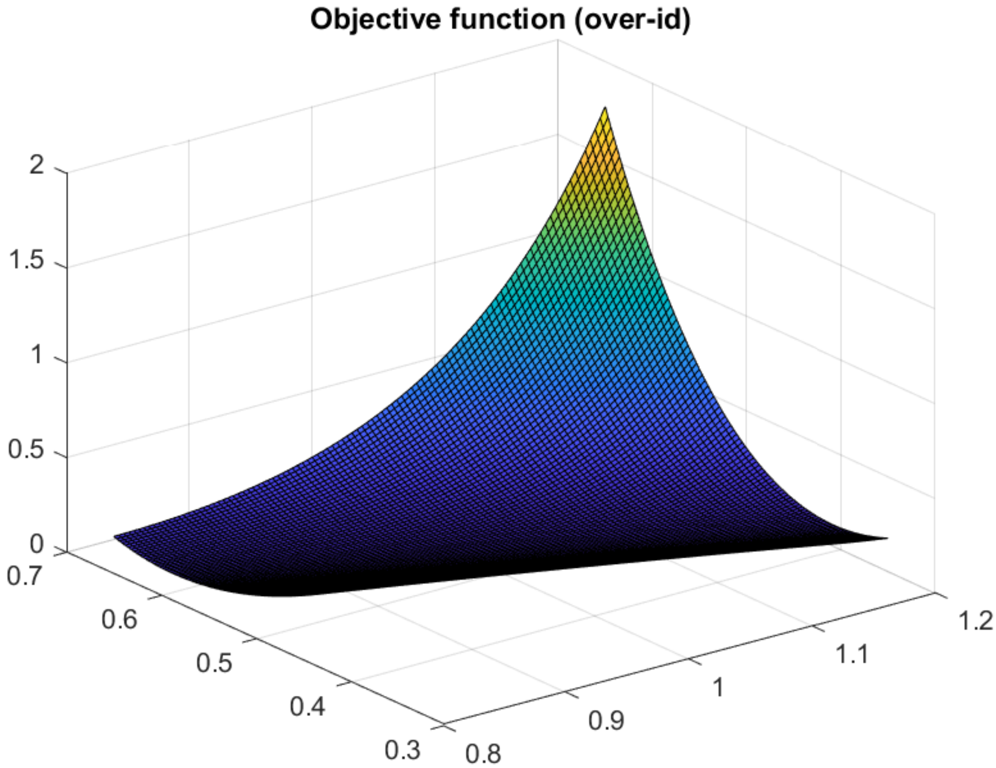
	b1	b2	SE
rho	0.543	0.543	0.0612
sigma	0.984	0.984	0.0552

Jacobian	
1.84	2.69
2.27	1.39

Note that our estimated parameters look identical but they are different in the first non-reported digit. Note also that we now no longer have collinear columns of the Jacobian. Our standard errors are much better now, unsurprisingly. Our J-test value is 2.77×10^{-7} which is on an order smaller than the convergence parameters of the numerical minimizer in Matlab. So, numerically insignificant from zero.

1.3 Over-identified

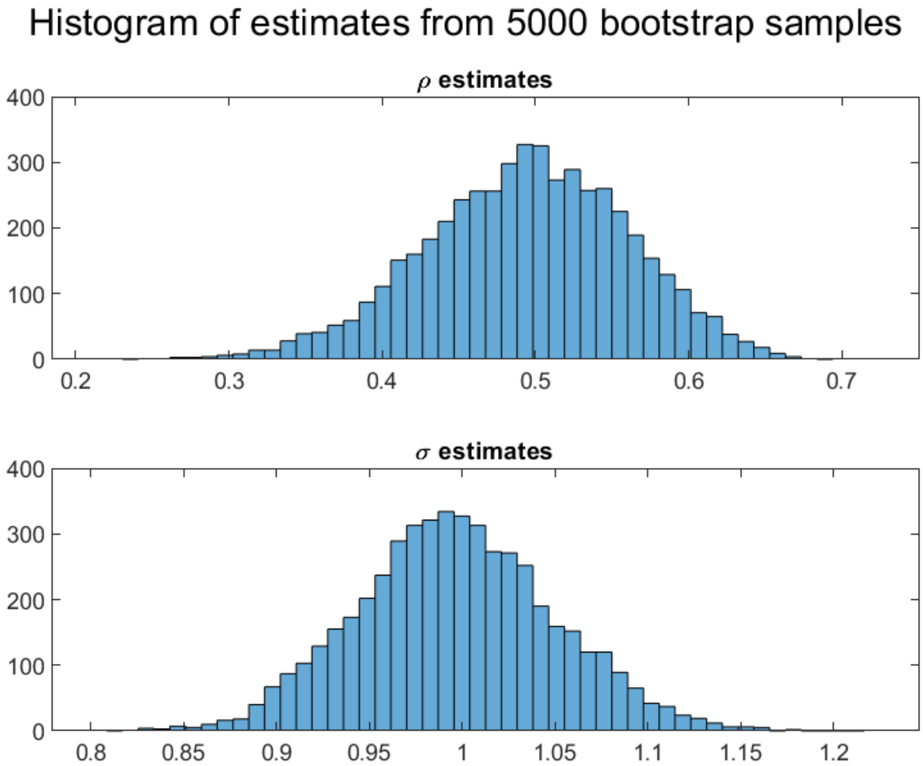
We now present the overidentified model. Results are different than in the exactly-identified case, but not substantially different and not substantially better or worse.



	b1	b2	SE
rho	0.484	0.477	0.063
sigma	0.97	0.965	0.0586

Jacobian	
0.105	0.0614
1.6	2.62
2.04	1.28

In this version, we bootstrap the procedure. Histogram is below. Our histograms show that the distribution of bootstrapped estimates are approximately centered at the truth. Displayed are 50-bin histograms.


Histogram of estimates from 5000 bootstrap samples
ρ estimates
400
300
200
100
0
0.2
0.3
0.4
0.5
0.6
0.7
σ estimates
400
300
200
100
0
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2