Macro PS3

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1 Question 1

We study a similar environment to the OG model we studied in class. We now assume no commitment technology and we assume allocations w_1, w_2 in each generation's second period.

1.1 State and solve the planner's problem.

The planner maximizes utility subject to the resource constraints:

$$\begin{aligned} &\max_{c_t^{t-1}, c_t^t} \ln c_t^{t-1} + \ln c_t^t \\ &\text{s.t. } c_t^{t-1} + c_t^t \leq w_1 + w_2. \end{aligned} \tag{1}$$

Since utility is strictly increasing in consumption, the resource constraint will hold with equality so we can solve for c_t^t in terms of $c_t^{t+1} : c_t^t = w_1 + w_2 - c_t^{t+1}$. Then we can rewrite the maximization problem as the following:

$$\begin{aligned} &\max_{c_t^{t-1}} \ln c_t^{t-1} + \ln (w_1 + w_2 - c_t^{t-1}) \\ &\Rightarrow \frac{1}{c_{*,t}^{t-1}} - \frac{1}{w_1 + w_2 - c_t^{*,t-1}} = 0 \\ &\Rightarrow c_t^{*,t-1} = \frac{w_1 + w_2}{2} \\ &\Rightarrow c_t^{*,t} = \frac{w_1 + w_2}{2}. \end{aligned}$$

Thus, the social planner will set $c_t^{*,t-1} = c_t^{*,t} = \frac{w_1+w_2}{2}$.

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

1.2 State the representative household's problem in period $t \geq 0$.

The households maximize utility subject to their budget constraints:

$$\begin{aligned}&\max_{c_t^t, c_{t+1}^t, M_{t+1}^t} \ln c_t^t + \ln c_{t+1}^t\\&\text{s.t. } p_t c_t^t + M_{t+1}^t \leq p_t w_1\\&\text{and } p_{t+1} c_{t+1}^t \leq p_{t+1} w_2 + M_{t+1}^t\end{aligned}$$

Rewriting the budget constraints,

$$\begin{aligned}&\max_{c_t^t, c_{t+1}^t, M_{t+1}^t} \ln c_t^t + \ln c_{t+1}^t\\&\text{s.t. } c_t^t + \frac{M_{t+1}^t}{p_t} \leq w_1\\&\text{and } c_{t+1}^t \leq w_2 + \frac{M_{t+1}^t}{p_{t+1}}\end{aligned} \tag{2}$$

The initial old generation lives only one period and maximizes their utility in that period subject to their budget constraint in that period.

$$\begin{aligned}&\max_{c_1^0} \ln c_1^0\\&\text{and } p_1 c_1^0 \leq p_1 w_2 + M\end{aligned}$$

Again rewriting the budget constraint,

$$\begin{aligned}&\max_{c_1^0} \ln c_1^0\\&\text{s.t. } c_1^0 \leq w_2 + \frac{M}{p_1}.\end{aligned}$$

1.3 Define and solve for an autarkic equilibrium, assuming it exists.

In the Autarkic equilibrium, money is not traded across individuals - there is no belief that money will have value in the future. Thus, individuals will not trade their endowment in the first period for money, so $M_{t+1}^t = 0$ and M has no value for the initial old agent. Since utility is strictly increasing in consumption, the budget constraints will hold with equality so we have the following solution: $c_t^{**,t} = w_1, c_{t+1}^{**,t} = w_2 \ \forall i \in \mathbb{N} \cup \{0\}$.

1.4 Define and solve for a competitive equilibrium assuming valued money but with $w_2 = 0$.

Now we will solve for a competitive equilibrium assuming agents will expect money will carry value into the future, and assuming $w_2 = 0$. Our agents then solve the following:

$$\begin{aligned}
&\max_{c_t^t, c_{t+1}^t, M_{t+1}^t} \ \ln\, c_t^t + \ln\, c_{t+1}^t \\
&\text{s.t. } c_t^t + \frac{M_{t+1}^t}{p_t} \leq w_1 \\
&\text{and } c_{t+1}^t \leq \frac{M_{t+1}^t}{p_{t+1}}
\end{aligned}$$

Utility is strictly increasing with consumption so the budget constraints will hold with equality - that is, $c_{t+1}^t = \frac{M_{t+1}^t}{p_{t+1}}, c_t^t + \frac{M_{t+1}^t}{p_t} = w_1 \Rightarrow c_t^t = w_1 - \frac{M_{t+1}^t}{p_t}$. We then have:

$$\max_{M_{t+1}^t} \ln \left(w_1 - \frac{M_{t+1}^t}{p_t} \right) + \ln\, \frac{M_{t+1}^t}{p_{t+1}}$$

Taking first order conditions:

$$\begin{aligned}
0 &= -(p_t^{***})^{-1} \frac{1}{w_1 - M_{t+1}^{***,t}/p_t^{***}} + (p_{t+1}^{***})^{-1} \frac{1}{M_{t+1}^{***,t}/p_{t+1}^{***}} \\
\Rightarrow M_{t+1}^{***,t} &= \frac{p_t^{***} w_1}{2} \\
\Rightarrow c_t^{***,t} &= \frac{w_1}{2}, c_{t+1}^{***,t} = \frac{p_t^{***}}{p_{t+1}^{***}} \frac{w_1}{2}
\end{aligned}$$

In the first period, the initial old solves the following:

$$\begin{aligned}
&\max_{c_1^0} \ln\, c_1^0 \\
&\text{s.t. } c_1^0 \leq \frac{M}{p_1}.
\end{aligned}$$

Utility is increasing in consumption so the budget constraint holds with equality: $c_1^{***,0} = \frac{M}{p_1^{***}}$.

In the competitive equilibrium, our money market must clear so $M_{t+1}^{***,t} = M \Rightarrow \frac{p_t^{***} w_1}{2} = M \Rightarrow p_t^{***} = \frac{2M}{w_1}$ $\forall t \in \mathbb{N}$. This value is constant so $p_i^{***} = p_j^{***}$ $\forall i, j \in \mathbb{N}$. By Walras' law, this also clears the goods market, which we can verify: $c_t^{***,t} + c_t^{***,t-1} = w_1 \Rightarrow \frac{w_1}{2} + \frac{p_{t-1}^{***}}{p_t^{***}} \frac{w_1}{2} = w_1 \Rightarrow w_1 = w_1$.

Therefore, $c_t^{***,t} = c_{t+1}^{***,t} = \frac{w_1}{2}$, $p_t^{***} = \frac{2M}{w_1}$, $M_{t+1}^{***,t} = M$.

1.5 Compare the solutions to the planner's problem, the autarky equilibrium, and the stationary monetary competitive equilibrium with valued money, all with $w_2 = 0$.

The solution to the planner's problem was $c_t^{*,t-1} = c_t^{*,t} = \frac{w_1+w_2}{2}$, so for $w_2 = 0$ the solution is $c_t^{*,t-1} = c_t^{*,t} = \frac{w_1}{2} = c_t^{***,t-1} = c_t^{***,t}$ so the solutions for the planner's problem and stationary monetary competitive equilibrium with valued money are equivalent. Note however that, for $w_2 = 0$, $c_t^{**,t} = w_1, c_{t+1}^{**,t} = 0$ which is a strictly worse solution for the agents as this yields $-\infty$ utility in the second period for the agents. Thus, the autarky equilibrium does not yield the same result as the planner's problem.

1.6 What happens to consumption, money demand, and prices in a competitive equilibrium with valued money if the initial money supply is halved?

Denote with primes the new values of the variables in the equilibrium with halved money supply.

$$c_t^{',***,t} = c_{t+1}^{',***,t} = \frac{w_1}{2}, M_{t+1}^{',***,t} = \frac{M}{2}, p_t^{',***} = \frac{M}{w_1}.$$

If the initial money supply is halved, the consumption allocations are unchanged, and the prices and money demand adjust accordingly.

2 Question 2

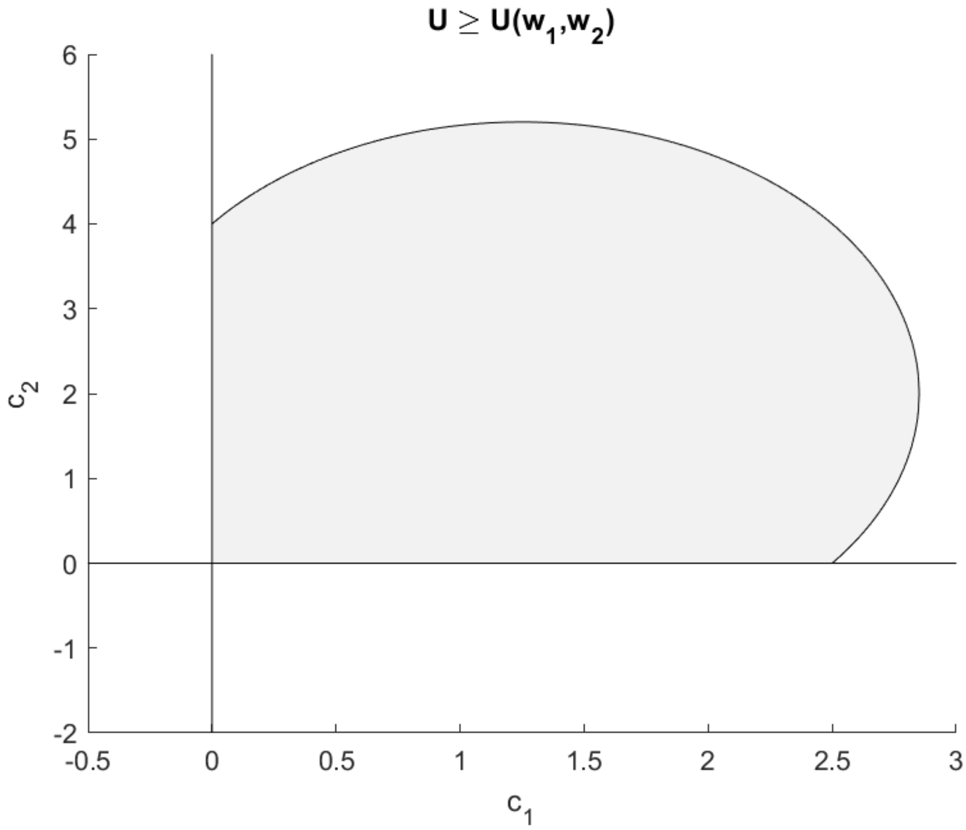
Plot the trade offer curves for the following utility functions where the endowment is (w_1, w_2) for goods 1 and 2, respectively:

2.1 $U = 10c_1 - 4c_1^2 + 4c_2 - c_2^2, (w_1, w_2) = (0, 2)$

The agent will trade to (weakly) increase their utility: $U(c_1, c_2) \geq U(w_1, w_2) = 0$. We will solve for a convenient expression for c_1, c_2:

$$\begin{aligned}
&\Rightarrow 10c_1 - 4c_1^2 + 4c_2 - c_2^2 \geq 0 \Rightarrow -10c_1 + 4c_1^2 + (c_2 - 2)^2 \leq 4 \\
&\Rightarrow 4(c_1 - \frac{5}{4})^2 + (c_2 - 2)^2 \leq 4 + \frac{25}{4} = \frac{41}{4} \\
&\Rightarrow \frac{16}{41}(c_1 - \frac{5}{4})^2 + \frac{4}{41}(c_2 - 2)^2 \leq 1.
\end{aligned}$$

It is clear that our expression for c_1, c_2 is of the form of a filled ellipse centered at $(\frac{5}{4}, 2)$ with semi-major axis in the c_2 direction with length $\frac{\sqrt{41}}{2}$ and semi-minor axis in the c_1 direction with length $\frac{\sqrt{41}}{4}$. This ellipse is plotted below. Note: $c_1, c_2 \geq 0$.

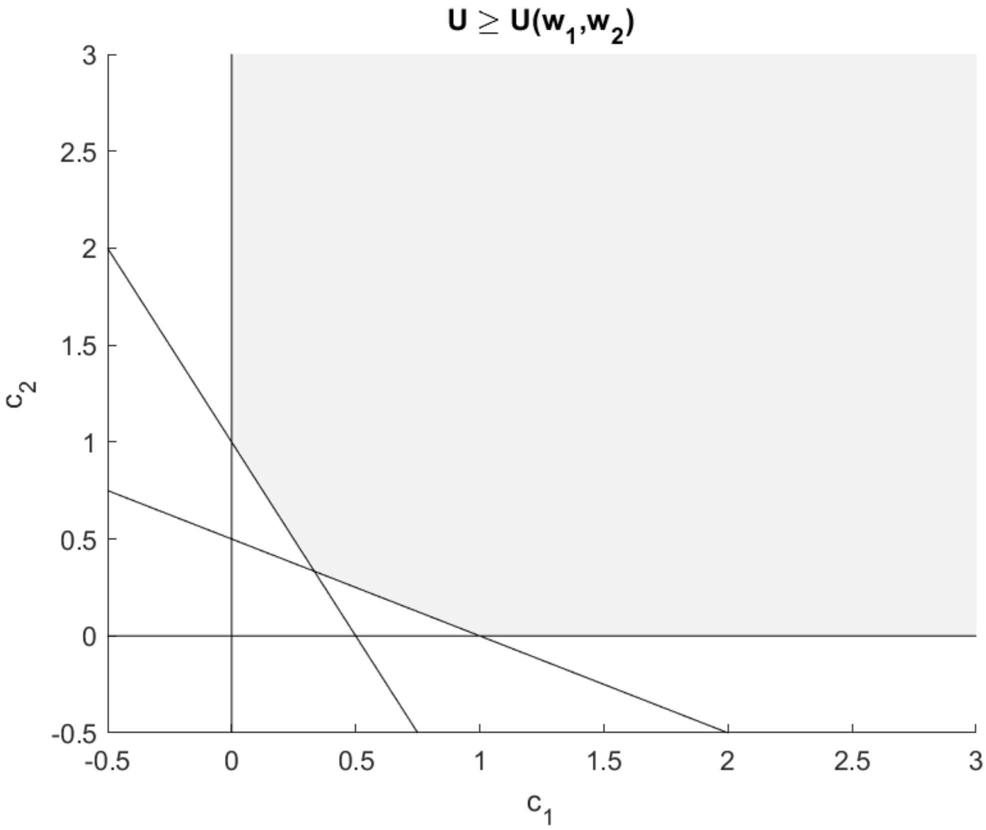


2.2 $U = \min\{2c_1 + c_2, c_1 + 2c_2\}, (w_1, w_2) = (1, 0)$

Our agent will trade to (weakly) increase their utility: $U(c_1, c_2) \geq U(w_1, w_2) = 1$. We will solve for a convenient expression for c_1, c_2:

$$\begin{aligned}
&2c_1 + c_2 \geq 1 \\
&c_1 + 2c_2 \geq 1 \\
&\quad \Rightarrow c_2 \geq 1 - 2c_1 \\
&\quad \Rightarrow c_2 \geq \frac{1 - c_1}{2}
\end{aligned}$$

Thus, our agent will trade to any point which is on or above both curves. This is plotted below. Note: $c_1, c_2 \geq 0$.

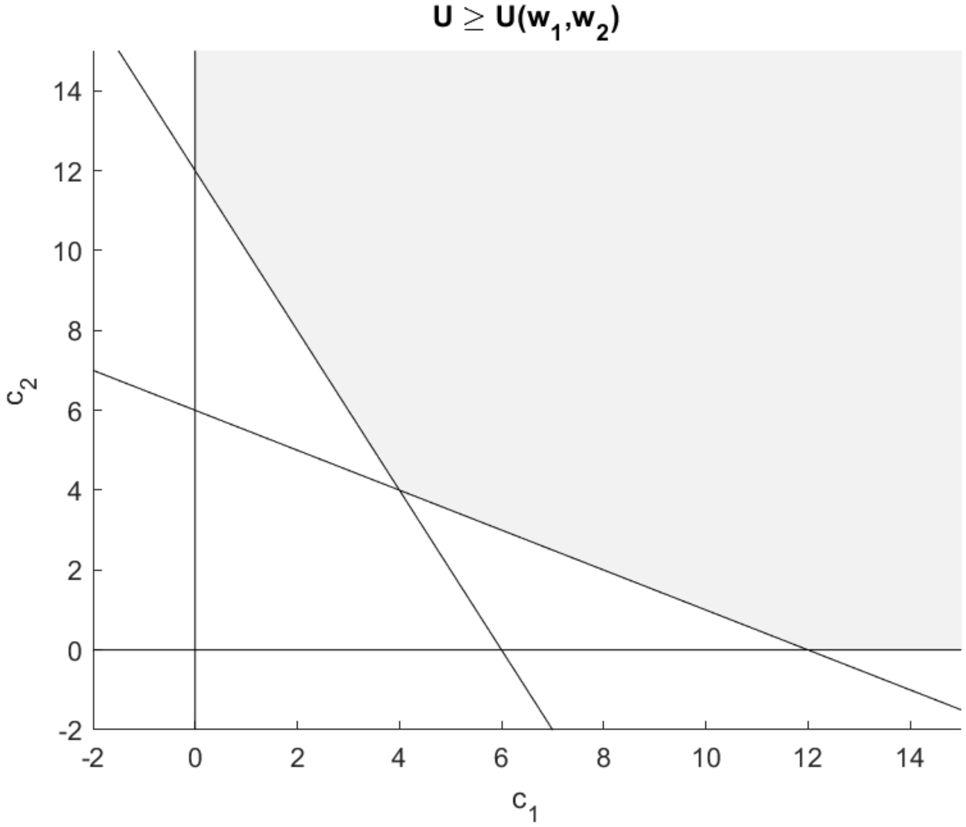


2.3 $U = \min\{2c_1 + c_2, c_1 + 2c_2\}, (w_1, w_2) = (1, 10)$

Our agent will trade to (weakly) increase their utility: $U(c_1, c_2) \geq U(w_1, w_2) = 12$. We will solve for a convenient expression for c_1, c_2:

$$\begin{aligned} 2c_1 + c_2 &\geq 12 \\ c_1 + 2c_2 &\geq 12 \\ \Rightarrow c_2 &\geq 12 - 2c_1 \\ \Rightarrow c_2 &\geq \frac{12 - c_1}{2} \end{aligned}$$

As in the previous subsection, we find that our agent will trade to any point which is on or above both curves. This is plotted below. Note: $c_1, c_2 \geq 0$.


U ≥ U(w1,w2)
14
12
10
8
6
4
2
0
-2
c2
-2
0
2
4
6
8
10
12
14
c1