Econometrics HW2

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1 Question 1

1.1 Part i

$$
\begin{aligned}
E[ZX'] &= E\left[\begin{pmatrix} Z_1 \\ X_2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}'\right] \\
&= \begin{pmatrix} E[Z_1X_1] & E[Z_1X_2'] \\ E[X_2X_1] & E[X_2X_2'] \end{pmatrix} \\
E[ZZ'] &= E\left[\begin{pmatrix} Z_1 \\ X_2 \end{pmatrix}\begin{pmatrix} Z_1 \\ X_2 \end{pmatrix}'\right] \\
&= \begin{pmatrix} E[Z_1^2] & E[Z_1X_2'] \\ E[X_2Z_1] & E[X_2X_2'] \end{pmatrix}
\end{aligned}
$$

Note that $E[X_2X_2']$ must be invertible for either $E[ZX']$ or $E[ZZ']$ to be invertible.[1]

Block inversion implies that $E[ZX']$ is invertible iff $E[Z_1X_1]-E[Z_1X_2']E[X_2X_2']^{-1}E[X_2X_1] \neq 0$, and similarly $E[ZZ']$ is invertible iff $E[Z_1^2]-E[Z_1X_2']E[X_2X_2']^{-1}E[X_2Z_1] \neq 0$. We can rewrite these expressions as follows: $E[\hat{Z}_1X_1] \neq 0, E[\tilde{Z}_1^2] \neq 0$ for $\hat{Z}_1 := Z_1 - X_2'E[X_2X_2']^{-1}E[X_2Z_1]$. From FWL, for $\pi_1 = E[\tilde{Z}_1X_1]$. Together, $E[\tilde{Z}_1^2] \neq 0$ and $\pi_1 \neq 0$ imply that $E[\tilde{Z}_1X_1] \neq 0$, and the reverse direction comes from Cauchy-Schwarz:

$$0 < E[Z_1X_1]^2 \leq E[\hat{Z}_1^2]E[X_1^2].$$

1.2 Part ii

Under homoskedasticity, $\Omega = \sigma_U^2 E[ZX']^{-1}E[ZZ']E[XZ']^{-1}$. We again go back to block inversion and find that:

$$E[ZX']^{-1} = E[\tilde{Z}_1]^{-1}\begin{pmatrix} 1 & -E[Z_1X_2']E[X_2X_2']^{-1} \\ \dots & \dots \end{pmatrix},$$

[*] I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, Katherine Kwok, and Danny Edgel.

[1] $E[X_2X_2']$ not being invertible implies the existence of some t such that $E[X_2X_2']t = 0 \Rightarrow E[(X_2't)^2] = 0 \Rightarrow E[ZX'](0,t')' = E[ZX'](0,t')' = 0$ so $E[ZX']$ and $E[ZZ']$ are not invertible.

where the second block row does not enter into the upper left entry of Ω. We then have the following:

$$\begin{aligned}\Omega_{1,1} &= \frac{\sigma_U^2}{E[Z_1X_1']^2}(E[Z_1]^2 - E[Z_1X_2']^{-1}E[X_2Z_1]) \\ &= \frac{\sigma_U^2 E[\tilde{Z}_1^2]}{E[Z_1X_1']^2} \\ &= \frac{\sigma_U^2]}{E[\tilde{Z}_1^2]\pi_1^2}\end{aligned}$$

1.3 Part iii

π is the population projection (regression) coefficients mapping Z into X_1, and $\tilde{Z}$ is Z residualized at the population level with respect to X_2.

1.4 Part iv

$$E[\tilde{Z}_1X_2] = E[X_2Z_1 - X_2X_2'E[X_2X_2']^{-1}E[X_2Z_1]] = 0.$$

The above expression implies the following:

$$\begin{aligned}E[\tilde{Z}_1X_2E[X_2X_2']^{-1}E[X_2E[X_1|Z]]] = 0 \\ E[\tilde{Z}_1X_1] = E[\tilde{Z}_1E[X_1|Z]] - E[\tilde{Z}_1X_2E[X_2X_2']^{-1}E[X_2E[X_1|Z]]] = E[\tilde{Z}Z_*]\end{aligned}$$

We apply Cauchy-Schwarz to achieve the desired inequality:

$$\begin{aligned}\Omega_{1,1} &= \frac{\sigma_U^2 E[\tilde{Z}_1^2]}{E[\tilde{Z}Z_*]^2} \\ &\geq \frac{\sigma_U^2 E[\tilde{Z}_1^2]}{E[\tilde{Z}^2]E[Z_*^2]} \\ &= \frac{\sigma_U^2}{E[Z_*^2]}\end{aligned}$$

We will achieve the lower bound when $Z_* = \tilde{Z}_1\pi_1$. This occurs when $E[X_1|Z] = Z_1\pi_1 + X_2'\pi_2$.

1.5 Part v

If X_2 is just a constant, $\tilde{Z} = Z - E[Z], E[\tilde{Z}_1^2] = Var(Z_1), E[\tilde{Z}_1X_1] = Cov(Z_1, X_1)$. Thus,

$$\Omega_{1,1} = \frac{\sigma_U^2 Var(Z_1)}{Cov(Z_1, X_1)^2}.$$

2 Question 2

2.1 Part i

$$\begin{aligned}E[h(Z)(Y - X\beta)] &= E[h(Z)(X(\beta_1 - \beta)) + U)] \\ &= E[h(Z)X](\beta_1 - \beta) + E[h(Z)U].\end{aligned}$$

If exogeneity holds, $E[h(Z)U] = E[h(Z)E[U|Z]] = 0$ so $E[h(Z)(Y - X\beta)] = 0 \iff \beta_1 = \beta$.

2.2 Part ii

2.3 Part iii

2.4 Part iv

2.5 Part v

3 Question 3

3.1 Part i

3.2 Part ii

3.3 Part iii

3.4 Part iv

3.5 Part v