Q2 Macro Study Guide

2020 Entering Cohort*

June 20, 2021

Lectures 2 and 3 - Intro to Bellmans & Consumption Savings

Sequence problem:

$$V \sup_{x_{t+1}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \text{ F is usually utility}$$
$$\text{s.t. } x_{t+1} \text{ is feasible (budget constraint)}$$

Bellman:

$$V(x) = \sup_{x'} F(x, x') + \beta V(x')$$

Contraction mapping theorems in notes. Feasibility conditions in notes.

Blackwell's sufficient conditions: $B(X)$ is a set of bounded functions and $T : B(X) \to B(X)$. T is a contraction mod β if:

- T is monotone, $f(x) < g(x) \to Tf(x) < Tg(x)$
- Discounting, exists some β s.t. $T(f + a)(x) \leq Tf(x) + \beta a$
- See PS2-Q2 for example of contractions.

Solving Bellmans:

- Take first order conditions over maximizing variable
- Take envelope condition
- Combine for Euler equation
- Consumption savings (Stokey Lucas Prescott) example at the end of the notes

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*Contributions made by Sarah Bass, (add names here)

Lectures 4 and 5 - Optimal Growth Model

There are not typed notes for this lecture!
Sequence problem:

$$\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$
$$\text{s.t. } c_t + k_{t+1} - (1-\delta)k_t = F(K_t, 1) = f(k_t) \text{ (constant labor supply)}$$

Note, investment $i = k_{t+1} - (1-\delta)k_t$. The above constraint says that demand for goods = supply of goods when markets clear.

Bellman:

$$V(k) = \max_{x'} U(f(k) + (1-\delta)k - k') + \beta V(k')$$

Solving the Bellman gives the following laws of motion, which can be used to find the steady state:

$$U'(c) = \beta U'(c')(F'(k') + 1 - \delta)$$
$$F(k) = c + k' - (1-\delta)k$$

Cobb-Douglas example in Lecture 4 slides

Phase Diagrams:

- Lines: see slides, set $\Delta c = 0 \rightarrow F'(k^*) = \delta + \theta$, $\Delta k = 0 \rightarrow c = F(k) - \delta k$ (or what choice vars are in question)
- Arrows: see lecture 5 slides

Balanced Growth Path

We are given that technology (A) is growing at an exogenous fixed rate γ. The social planner's problem is:

$$\max \sum_{t=0}^{\infty} \beta^t u(C_t, N_t)$$
$$\text{s.t. } C_t + K_{t+1} = (1-\delta)K_t + F(K_t, A_t N_t)$$

Next, define $x_t = X_t / A_t$. Then our maximization problem becomes:

$$\max \sum_{t=0}^{\infty} \beta^t u(A_t c_t, N_t)$$

And our RC becomes:

$$\begin{aligned}
C_t/A_t + K_{t+1}/A_t &= (1-\delta)K_t/A_t + F(K_t, A_tN_t)/A_t \\
\Rightarrow C_t/A_t + (K_{t+1}/A_t)(A_{t+1}/A_{t+1}) &= (1-\delta)K_t/A_t + F(K_t, A_tN_t)/A_t \\
\Rightarrow c_t + k_{t+1}(A_{t+1}/A_t) &= (1-\delta)k_t + F(K_t, A_tN_t)/A_t \\
\Rightarrow c_t + k_{t+1}(1+\gamma) &= (1-\delta)k_t + F(K_t, A_tN_t)/A_t
\end{aligned}$$

Note, there are constant returns to scale, so $F(K_t, A_tN_t)/A_t = F(k_t, N_t)$. So we have:

$$\Rightarrow c_t + k_{t+1}(1+\gamma) = (1-\delta)k_t + F(k_t, N_t)$$

Next take FOCs w.r.t c_t, N_t, k_{t+1} to get Euler and labor supply. The household/firm problems can be solved using the same normalization process.

Now consider an economy where technology is fixed but the population is growing at an exogenous fixed rate, n. As before, the social planner's problem is:

$$\begin{aligned}
&\max \sum_{t=0}^{\infty} \beta^t u(c_t, N_t) \\
\text{s.t. } &C_t + K_{t+1} = (1-\delta)K_t + F(K_t, N_t)
\end{aligned}$$

Consumption in the utility function is usually per capita consumption, so it is already c_t (instead of C_t). Next, define $x_t = X_t/N_t$. Then our maximization problem becomes:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, (1+n)^t N_0)$$

And our RC becomes:

$$\begin{aligned}
C_t/N_t + K_{t+1}/N_t &= (1-\delta)K_t/N_t + F(K_t, N_t)/N_t \\
\Rightarrow C_t/N_t + (K_{t+1}/N_t)(N_{t+1}/N_{t+1}) &= (1-\delta)K_t/N_t + F(K_t, N_t)/N_t \\
\Rightarrow c_t + k_{t+1}(N_{t+1}/N_t) &= (1-\delta)k_t + F(K_t, N_t)/N_t \\
\Rightarrow c_t + k_{t+1}(1+n) &= (1-\delta)k_t + F(K_t, N_t)/N_t
\end{aligned}$$

Note, there are constant returns to scale, so $F(K_t, N_t)/N_t = F(k_t, N_0) = f(k_t)$. So we have:

$$\Rightarrow c_t + k_{t+1}(1+\gamma) = (1-\delta)k_t + f(k_t)$$

Next take FOCs w.r.t c_t, k_{t+1} to get Euler. The household/firm problems can be solved using the same normalization process.

You need to normalize such that everything is stationary. Normalize using the non-stationary variable.

Lectures 6 and 7 - Consumption Savings under Uncertainty

Sequence Problem:

$$\max_{c_t, a_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t. } c_t + a_{t+1} = Ra_t + y_t$$

In this case, y is stochastic and follows a Markov process with a transition matrix Q.

$$E[f(y')|y] = \int f(y')Q(y, dy')$$

Bellman equation:

$$v(a, y) = \max_{a'} u(Ra + y - a') + \beta E[v(a', y')|y]$$
$$v(a, y) = \max_{a'} u(Ra + y - a') + \beta \int v(a', y')Q(y, dy')$$

Optimal policy correspondence: The values of consumption and assets (and other choice vars) such that the bellman holds.

Euler equation:

$$u'(c_t) = \beta R E_t u'(c_{t+1})$$

See lecture 6 notes for dynamics of consumption.

Lecture 8 - Arrow Debreu

Looking at states of the world at an individual level. Markov state s_t with a finite transition function $P(s'|s)$. A given sequence of states can be defined as $s^t = \{s_0, s_1, ..., s_t\}$

$$P(s^t|s_0) = P(s_t|s_{t-1})P(s_{t-1}|s_{t-2})...P(s_1|s_0)$$

Preferences (for individual i):

$$U^i(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} (\beta_i)^t u^i(c_t^i(s^t))P(s^t|s_0)$$

Budget constraint:

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t)y^i(s^t)$$

Feasible allocation:

$$\sum_{i=1}^{I} c_t^i(s^t) \leq \sum_{i=1}^{I} y^i(s^t)$$

See lecture 8 notes for more detail on competitive equilibrium. See PS3-Q1 for example.

Lectures 9, 10, 11, 12 & 13 - Asset Pricing

Large number of identical agents, single nonstorable consumption good, given off by productive units. Owners of productive units receive stochastic dividends s_t with transition function $Q(s, ds')$.

Representative agent problem:

$$\max_{c_t, a_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{s.t. } c_t + p_t a_{t+1} = (p_t + s_t) a_t$$

We can conjecture that prices are a function of dividends (and other stochastic shocks): $p_t = p(s_t)$.

Bellman:

$$v(a, s) = \max_{a'} u((p(s) + s)a - p(s)a') + \beta \int v(a', s') Q(s, ds')$$

At the competitive equilibrium, for all s, $v(1, s)$ is attained by $c = s, a = a' = 1$.

Euler equation:

$$u'(c(s)) = \beta \int u'c(s')) \frac{p(s') + s'}{p(s)} Q(s, ds')$$

For a lucas tree, if $p_t = p(s_t) \rightarrow R_{t+1} = p_{t+1} + s_{t+1}$, return on investment (payoff tomorrow / price today). So,

$$u'(c(s)) = \beta E_t[u'(c_{t+1}) R_{t+1}]$$

Then the equilibrium pricing function of the tree is:

$$p(s) = \beta \int \frac{u'(s')(p(s') + s')}{u'(s)} Q(s, ds')$$

In general, remember $p = E[mx]$ for individual assets, where $p = p(s)$ is the price today, $m = \beta \frac{u'(s')}{u'(s)}$ is the stochastic discount factor, and

$x = g(s')$ is the payoff tomorrow.

In a Lucas tree, the payoff is $x = p(s') + s'$.

A risk free asset is $p = E[mx]$ where $x = 1$. In other words $p = E[m]$.
See PS3-Q2 and 2020 Final Q2 for example.

Multi period claims pricing kernel *(lecture 11)*:

$$q^j(s, s^j) = \int q(s, s')q^{j-1}(s', s^j)ds'$$

Multi period price expression:

$$p_t = E_t\left[\sum_{j=1}^{\infty} \beta^j \frac{u'(s_{t+j})}{u'(s_t)} s_{t+j}\right]$$

Risk neutrality: with linear utility, $u'(c)$ is constant, so the risk free rate is $1 = E_t(\beta R) \to R = 1/\beta$. Then:

$$\begin{aligned} p_t &= E_t\left[\sum_{j=1}^{\infty} \beta^j \frac{u'(s_{t+j})}{u'(s_t)} s_{t+j}\right] \\ &= E_t\left[\sum_{j=1}^{\infty} \beta^j s_{t+j}\right] \\ &= E_t\left[\sum_{j=1}^{\infty} \frac{s_{t+j}}{R^j}\right] \end{aligned}$$

Risk corrections:

$$\begin{aligned} 1 &= E\left[\beta \frac{u'(c')}{u'(c)} R\right] \\ \Rightarrow R &= \frac{1}{E\left[\beta \frac{u'(c')}{u'(c)}\right]} \\ &= \frac{1}{E_t m_{t+1}} \end{aligned}$$

SPP and CE example in lecture 12 notes

A recursive competitive equilibrium is a pricing function $p(s)$, policy functions $a'(a, s), c(a, s)$, and a value function $V(a, s)$ such that

1. **Given the pricing function, V solves their Bellman equation and a', c maximize V**
2. **Markets clear: $a' = 1, c = s$**