Micro HW7

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1 Question 1

1.1 Part A

Let u be linear, so for some $c, d \in \mathbb{R}$, $u(m) = cm + d$. Then, $U(a) = pu(w + 2a) + (1 - p)u(w-a) = p(c(w+2a)+d)+(1-p)(c(w-a)+d) = pcw+2pca+cw-ca+d-pcw+pca-pd = (3p-1)ca+cw+d-pd$. $a = \arg\max_{0\leq a\leq w}(3p-1)ca+cw+d-pd = \arg\max_{0\leq a\leq w}(3p-1)ca$. If $3p - 1 > 0 \Rightarrow p > \frac{1}{3}$ then the objective function is maximized when a is maximized, so $a = w$. If $3p - 1 < 0 \Rightarrow p < \frac{1}{3}$ then the objective function is maximized when a is minimized, so $a = 0$.

1.2 Part B

$\frac{\partial U}{\partial a}(0) = 2pu'(w) - (1 - p)u'(w) > 0$ because $u'(x)$ is positive as u is strictly increasing, and $2p > 1 - p$ as $p > \frac{1}{3}$.

1.3 Part C

Let $a, b \in (0, w)$ and let $t \in (0, 1)$. By the strict concavity of u, since $u'' < 0$, $U(ta + (1-t)b) = pu(w + 2(ta + (1-t)b)) + (1-p)u(w - (ta + (1-t)b)) = pu(t(w + 2a) + (1-t)(w + 2b)) + (1-p)u(t(w-a) + (1-t)(w-b)) > p(tu(w + 2a) + (1-t)u(w + 2b)) + (1-p)(tu(w-a) + (1-t)u(w + 2b)) = tU(a) + (1-t)U(b)$ so U is concave.

1.4 Part D

If $u'(0)$ is infinite, $\frac{\partial U}{\partial a}(w) = 2pu'(3w) - (1-p)u'(0) = -\infty$ so investing all of your wealth is not optimal. If $u'(0)$ is not infinite then $\frac{\partial U}{\partial a}(w) = 2pu'(3w) - (1-p)u'(0) \geq 0 \Rightarrow p(2u'(3w) + u'(0)) \geq u'(0) \Rightarrow p \geq \frac{u'(0)}{2u'(3w)+u'(0)}$ so if $p \geq \bar{p} = \frac{u'(0)}{2u'(3w)+u'(0)} = \bar{p}$ then it is optimal to invest all of your wealth.

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1.5 Part E

$\arg\max_{0\le a\le w} p(1-e^{-c(w+2a)})+(1-p)(1-e^{-c(w-a)}) = \arg\max_{0\le a\le w} p-pe^{-cw}e^{-2ca}+(1-p)-(1-p)e^{-cw}e^{ca} = \arg\max_{0\le a\le w} -pe^{-cw}e^{-2ca}-(1-p)e^{-cw}e^{ca} = \arg\max_{0\le a\le w} e^{-cw}(-pe^{-2ca}-(1-p)e^{ca}) = \arg\max_{0\le a\le w} -cw+log(-pe^{-2ca}-(1-p)e^{ca}) = \arg\max_{0\le a\le w} log(-pe^{-2ca}-(1-p)e^{ca})$ which does not depend on wealth.

1.6 Part F

Let $A(x)$ be decreasing. Then,

$$\frac{d}{dw}U'(a) = 2pu''(w+2a)-(1-p)u''(w-a) = -2pu'(w+2a)A(w+2a)+(1-p)u'(w-a)A(w-a).$$

At the optimum, $U'(x^*(w)) = 0 \Rightarrow 2pu'(w+2a) = (1-p)u'(w-a)$ so,

$$\frac{d}{dw}U'(a)|_{a=a^*(w)} = (1-p)u'(w+2a^*)(-A(w+2a^*)+A(w-a^*)).$$

Since $u'(x) > 0$ and A is decreasing $\Rightarrow (-A(w+2a^*)+A(w-a^*)) > 0$ so $\frac{d}{dw}U'(a)|_{a=a^*(w)} > 0$ so our marginal utility from a is strictly increasing in w, so a^* is strictly increasing in w.

1.7 Part G

$\arg\max_{0\le t\le 1} pu(w(1+2t))+(1-p)u(w(1-t)) = \arg\max_{0\le t\le 1} pu(w(1+2t))+(1-p)u(w(1-t)) = \arg\max_{0\le t\le 1} p\frac{1}{1-\rho}(w(1+2t))^{1-\rho}+(1-p)\frac{1}{1-\rho}(w(1-t))^{1-\rho} = \arg\max_{0\le t\le 1} w^{1-\rho}(p\frac{1}{1-\rho}(1+2t)^{1-\rho}+(1-p)\frac{1}{1-\rho}(1-t)^{1-\rho}) = \arg\max_{0\le t\le 1} log(w^{1-\rho})+log(p\frac{1}{1-\rho}(1+2t)^{1-\rho}+(1-p)\frac{1}{1-\rho}(1-t)^{1-\rho})$
$= \arg\max_{0\le t\le 1} log(p\frac{1}{1-\rho}(1+2t)^{1-\rho}+(1-p)\frac{1}{1-\rho}(1-t)^{1-\rho})$ which does not depend on w.

1.8 Part H

Let $R(x)$ be increasing.

$$\begin{aligned}
U'(t) &= pu'(w(1+2t))+(1-p)u'(w(1-t)) \\
&= pu'(w(1+2t))2w-(1-p)wu'(w(1-t)) \\
\frac{\partial U'(t)}{\partial w} &= 2wp(1+2t)u''(w(1+2t))-(1-p)w(1-t)u''(w(1-t)) \\
&= -2pu'(w(1+2t))R(w(1+2t))+(1-p)R(w(1-t))u'(w(1-t)) \\
\frac{\partial U'(t)}{\partial w}|_{t=t^*} &= -2pu'(w(1+2t^*))R(w(1+2t))+2pR(w(1-t^*))u'(w(1+2t^*)) \\
&= 2pu'(w(1+2t^*))(-R(w(1+2t^*))+R(w(1-t^*))).
\end{aligned}$$

This is negative as R is increasing.