Econometrics HW2

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November 19, 2020

1 3.2

$$
\begin{aligned}
\hat{\beta}_{ols} &= (X'X)^{-1}X'Y \\
\hat{\beta}_{mix} &= (Z'Z)^{-1}Z'Y \\
&= (C'X'XC)^{-1}C'X'Y \\
&= C^{-1}(X'X)^{-1}C'^{-1}C'X'Y \\
&= C^{-1}\hat{\beta}_{ols} \\
\hat{\epsilon}_{ols} &= (I - X(X'X)^{-1}X')Y \\
\hat{\epsilon}_{mix} &= (I - XCC^{-1}(X'X)^{-1}X')Y \\
&= (I - X(X'X)^{-1}X')Y \\
&= \hat{\epsilon}_{ols}
\end{aligned}
$$

2 3.5

$$
\begin{aligned}
\hat{\epsilon} &= Y - X'(X'X)^{-1}X'Y = (I - X(X'X)^{-1}X')Y \\
\hat{\beta}_e &= (X'X)^{-1}X'\hat{\epsilon} \\
&= (X'X)^{-1}X'(I - X(X'X)^{-1}X')Y \\
&= (X'X)^{-1}X'Y - (X'X)^{-1}X'X(X'X)^{-1}X'Y \\
&= (X'X)^{-1}X'Y - (X'X)^{-1}X'Y \\
&= 0.
\end{aligned}
$$

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

3 3.6

$$\begin{aligned}\hat{\beta}_{\hat{Y}} &= (X'X)^{-1}X'\hat{Y}\\ &= (X'X)^{-1}X'X(X'X)^{-1}X'Y\\ &= (X'X)^{-1}X'Y\\ &= \hat{\beta}_{ols}\end{aligned}$$

4 3.7

Note that $X_1 = X\Gamma$ where $\Gamma = \begin{pmatrix} I_{n_1} \\ \bar{0} \end{pmatrix}$ where $\bar{0}$ is an $n_2 \times n_1$ vector of zeros. Then,

$$\begin{aligned}PX_1 &= X(X'X)^{-1}X'X_1\\ &= X(X'X)^{-1}X'X\Gamma\\ &= X\Gamma\\ &= X_1.\end{aligned}$$

$$\begin{aligned}MX_1 &= (I - X(X'X)^{-1}X')X\Gamma\\ &= (X - X(X'X)^{-1}X'X)\Gamma\\ &= (X - X)\Gamma\\ &= 0.\end{aligned}$$

5 3.11

Let X contain a constant.

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\hat{Y}_i &= \frac{1}{n}\sum_{i=1}^{n} Y_i - \hat{\epsilon}_i\\ &= \frac{1}{n}\sum_{i=1}^{n} Y_i - \frac{1}{n}\sum_{i=1}^{n}\hat{\epsilon}_i\\ &= \frac{1}{n}\sum_{i=1}^{n} Y_i - X'\hat{\epsilon}\\ &= \frac{1}{n}\sum_{i=1}^{n} Y_i\end{aligned}$$

where $\frac{1}{n}\sum_{i=1}^{n}\hat{\epsilon}_i = X'\hat{\epsilon} = 0$ because X contains a column of ones.

6 3.12

Equation (3.53) cannot be estimated by OLS because $D_1 + D_2 = \vec{1}$ (a vector containing 1 in every element), and therefore $X'X$ is not invertible.

6.1 Part A

Equations (3.54) and (3.55) contain the same information, since $D_1 + D_2 = \vec{1}$, and so the $\hat{Y}$ from each regression would be identical. Ergo,

$$\begin{aligned} D_1\alpha_1 + D_2\alpha_2 + e &= (\vec{1} - D_2)\alpha_1 + D_2\alpha_2 \\ &= \vec{\alpha_1} + D_2(\alpha_2 - \alpha_1) \end{aligned}$$

Therefore, the regressions are the same with $\mu = \alpha_1$ and $\phi = \alpha_2 - \alpha_1$.

6.2 Part B

$$\begin{aligned} \vec{1}'D_1 &= \sum_{i=1}^{n} 1\{\text{person } i \text{ is a man}\} \\ &= n_1, \\ \vec{1}'D_2 &= \sum_{i=1}^{n} 1\{\text{person } i \text{ is a woman}\} \\ &= n_2. \end{aligned}$$

7 3.13

7.1 Part A

Let $X = [D_1 D_2]$. Order our observations such that the first n_1 observations are men and the rest of the observations are women, then $X'X = \begin{pmatrix} \vec{1}'_{n_1}\vec{1}_{n_1} & \bar{0} \\ \bar{0} & \vec{1}'_{n_2}\vec{1}_{n_2} \end{pmatrix}$

$$\begin{aligned} \begin{pmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{pmatrix} &= \begin{pmatrix} \vec{1}'_{n_1}\vec{1}_{n_1} & \bar{0} \\ \bar{0} & \vec{1}'_{n_2}\vec{1}_{n_2} \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i=1}^{n_1} y_i \\ \sum_{i=n_1+1}^{n_1+n_2} y_i \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{n_1}\sum_{i=1}^{n_1} y_i \\ \frac{1}{n_2}\sum_{i=n_1+1}^{n_1+n_2} y_i \end{pmatrix} \\ &= \begin{pmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{pmatrix} \end{aligned}$$

7.2 Part B

The first transformation simplifies to $Y^* = \hat{u}$, in other words Y^* is the deviation from average for men and women.

The second transformation similarly transforms the X data, so X^* is the residual of the following regression: $X = D_1 b_1 + D_2 b_2$, which we know from Part A will yield $b_1 = \bar{X}_1, b_2 = \bar{X}_2$. X^* then is a matrix of regressors transformed to be in deviations from the average for whatever gender the individual identifies as.

7.3 Part C

$$\begin{aligned}
\tilde{\beta} &= (X'^* X^*)^{-1} X'^* Y^* \\
&= (X M_D X)^{-1} X' M_D Y \\
\hat{\beta} &= (X M_D X)^{-1} X' M_D Y \\
&= \tilde{\beta}
\end{aligned}$$

where we solved for $\hat{\beta}$ via theorem 3.4.

8 3.16

Let $X = [X_1 X_2], \hat{\beta} = [\hat{\beta}_1' \hat{\beta}_2']', \hat{\beta}^* = [\tilde{\beta}_1' \vec{0}_{n_2}']'$ where $\vec{0}_{n_2}$ is the n_2 sized matrix of zeros.

$$\begin{aligned}
R_2^2 &= 1 - \frac{\sum_{i=1}^n \hat{e}_i^2}{\sum_{i=1}^n (Y_i - \bar{Y})^2} \\
&= 1 - \frac{\hat{\epsilon}' \hat{\epsilon}}{\sum_{i=1}^n (Y_i - \bar{Y})^2} \\
&= 1 - \frac{(Y - X\hat{\beta})'(Y - X\hat{\beta})}{\sum_{i=1}^n (Y_i - \bar{Y})^2} \\
&\geq 1 - \frac{(Y - X\hat{\beta}^*)'(Y - X\hat{\beta}^*)}{\sum_{i=1}^n (Y_i - \bar{Y})^2} \\
&= R_1^2,
\end{aligned}$$

where the inequality comes from the fact that OLS minimizes the sum of squared residuals.

Yes, if X_2 is orthogonal to Y then $X_2' Y = 0 \Rightarrow \hat{\beta}_2 = 0 \Rightarrow \tilde{\beta} = \hat{\beta} \Rightarrow R_2^2 = R_1^2$.

9 3.21

If one or both of X_1, X_2 is orthogonal to Y, or if X_1, X_2 are orthogonal to each other, then $\tilde{\beta}_1 = \hat{\beta}_1, \tilde{\beta}_2 = \hat{\beta}_2$.

The first condition is nearly immediate, as whichever regressor is orthogonal will have estimated coefficients of 0 in both equations, and the equation with both regressors included reduces to the "one regressor at a time" estimator equation, so the coefficients in both have the same estimated value. Moreover, if both regressors are orthogonal to Y then all of the coefficient estimates will be 0.

Now we concern ourselves with the final case, where X_1, X_2 are orthogonal. Then, by theorem 3.4 we have that:

$$\begin{aligned}
\hat{\beta}_1 &= (X_1' M_2 X_1)^{-1}(X_1' M_2 Y) \\
&= ((M_2 X_1)'(M_2 X_1))^{-1}((M_2 X_1)'Y) \\
&= (X_1' X_1)^{-1}(X_1' Y) \\
&= \tilde{\beta}_1
\end{aligned}$$

By symmetry, the same condition ensures $\hat{\beta}_2 = \tilde{\beta}_2$

10 3.22

$$\begin{aligned}
\tilde{\beta} &= (X_1' X_1)^{-1} X_1' Y \\
\tilde{u} &= Y - X_1 \tilde{\beta} \\
\tilde{\beta}_2 &= (X_2' X_2)^{-1} X_2' \tilde{u} \\
&= (X_2' X_2)^{-1} X_2' (Y - X_1 \tilde{\beta}_1) \\
\hat{\beta}_2 &= (X_2' X_2)^{-1} X_2' (Y - X_1 \hat{\beta}_1)
\end{aligned}$$

Therefore, this is only the case when $\tilde{\beta}_1 = \hat{\beta}_1$. As we showed in the previous problem, this occurs when X_1, X_2 are orthogonal (or when one (or both) of the regressors is orthogonal to Y).

11 3.23

The residuals are the same from both equations, which I will show below, and therefore the residual variance estimates, a function of the estimated residuals, are the same from both regressions. Therefore, $\hat{\sigma}^2 = \tilde{\sigma}^2$.

Now we will show that the residuals are the same.

$$\begin{aligned}
\tilde{\beta}_2 &= ((X_2 - X_1)'M_1(X_2 - X_1))^{-1}((X_2 - X_1)'M_1 Y) \\
&= (X_2'X_2)^{-1}X_2'Y \\
&= \hat{\beta}_2. \\
\tilde{\beta}_1 &= (X_1'X_1)^{-1}X_1'(Y - (X_2 - X_1)\tilde{\beta}_2) \\
&= (X_1'X_1)^{-1}X_1'Y - (X_1'X_1)^{-1}X_1'(X_2 - X_1)\tilde{\beta}_2 \\
&= (X_1'X_1)^{-1}X_1'(Y - X_2\hat{\beta}_2) + (X_1'X_1)^{-1}X_1'X_1\hat{\beta}_2 \\
&= \hat{\beta}_1 + \hat{\beta}_2. \\
\Rightarrow \tilde{\epsilon} &= X_1\tilde{\beta}_1 + (X_2 - X_1)\tilde{\beta}_2 \\
&= X_1(\hat{\beta}_1 + \hat{\beta}_2) + (X_2 - X_1)\hat{\beta}_2 \\
&= X_1\hat{\beta}_1 + X_2\hat{\beta}_2 \\
&= \hat{\epsilon}.
\end{aligned}$$

12 Question 7

12.1 Part A

$$\begin{aligned}
E[\hat{\beta}|X] &= E[(X'X)^{-1}X'Y|X] \\
&= (X'X)^{-1}X'E[Y|X] \\
&= (X'X)^{-1}X'X\beta \\
&= \beta \\
\Rightarrow E[\hat{\beta}_1|X] &= \beta_1
\end{aligned}$$

12.2 Part B

$$\begin{aligned}
E[\hat{\hat{\beta}}_1|X] &= E[(X_1'X_1)^{-1}X_1'\hat{Y}|X] \\
&= E[(X_1'X_1)^{-1}X_1'X\hat{\beta}|X] \\
&= E[(X_1'X_1)^{-1}X_1'X(X'X)^{-1}X'Y|X] \\
&= (X_1'X_1)^{-1}X_1'X(X'X)^{-1}X'E[Y|X] \\
&= (X_1'X_1)^{-1}X_1'X(X'X)^{-1}X'X\beta \\
&= (X_1'X_1)^{-1}X_1'X\beta \\
&= (X_1'X_1)^{-1}X_1'(X_1\beta_1 + X_2\beta_2) \\
&= \beta_1 + (X_1'X_1)^{-1}X_1'X_2\beta_2
\end{aligned}$$

This is equal to β_1 if either $\beta_2 = 0$ or X_1, X_2 are guaranteed to be orthogonal (so $X_1'X_2 = 0$).

12.3 Part C

$$
\begin{aligned}
\tilde{\tilde{\beta}} &= (X'X)^{-1}X'\tilde{Y} \\
&= (X'X)^{-1}X'X_1\tilde{\beta}_1 \\
&= \begin{pmatrix} I_{k_1} \\ \bar{0} \end{pmatrix} \tilde{\beta}_1
\end{aligned}
$$

12.4 Part D

Let $\tilde{\tilde{Y}} = X\tilde{\tilde{\beta}}, \tilde{\epsilon} = \tilde{Y} - \tilde{\tilde{Y}}$.

$$
\begin{aligned}
\tilde{\tilde{Y}} &= X\tilde{\tilde{\beta}} \\
&= X \begin{pmatrix} I_{k_1} \\ \bar{0} \end{pmatrix} \tilde{\beta}_1 \\
&= X_1\tilde{\beta}_1 \\
&= \tilde{Y} \\
\Rightarrow \tilde{\epsilon} &= 0 \\
\Rightarrow R^2 &= 1 - \frac{\tilde{\epsilon}'\tilde{\epsilon}}{\sum_{i=1}^{n}(\tilde{Y}_i - \bar{\tilde{Y}})^2} \\
&= 1 - \frac{0}{\sum_{i=1}^{n}(\tilde{Y}_i - \bar{\tilde{Y}})^2} \\
&= 1.
\end{aligned}
$$

12.5 Part E

$$
\begin{aligned}
Var(\tilde{\tilde{\beta}}|X) &= Var(\begin{pmatrix} I_{k_1} \\ \bar{0} \end{pmatrix} (X_1'X_1)^{-1}X_1'Y|X) \\
&= \begin{pmatrix} I_{k_1} \\ \bar{0} \end{pmatrix} (X_1'X_1)^{-1}X_1'Var[Y|X](\begin{pmatrix} I_{k_1} \\ \bar{0} \end{pmatrix} (X_1'X_1)^{-1}X_1')' \\
&= \begin{pmatrix} I_{k_1} \\ \bar{0} \end{pmatrix} (X_1'X_1)^{-1}X_1'\sigma^2 I X_1(X_1'X_1)^{-1} \begin{pmatrix} I_{k_1} \\ \bar{0} \end{pmatrix}' \\
&= \sigma^2 \begin{pmatrix} I_{k_1} \\ \bar{0} \end{pmatrix} (X_1'X_1)^{-1}X_1'X_1(X_1'X_1)^{-1} \begin{pmatrix} I_{k_1} \\ \bar{0} \end{pmatrix}' \\
&= \sigma^2 \begin{pmatrix} I_{k_1} \\ \bar{0} \end{pmatrix} (X_1'X_1)^{-1} \begin{pmatrix} I_{k_1} \\ \bar{0} \end{pmatrix}' \\
&= \begin{pmatrix} \sigma^2(X_1'X_1)^{-1} & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}
\end{aligned}
$$