Macro Notesheet

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1 Blackwell, Contractions, Fixed Points, etc.

$$
\begin{aligned}
v(x) &= \max_{y \in \Gamma(A)} u(x, y) + \beta v(y); \\
T(z)(x) &= \max_{y \in \Gamma(A)} u(x, y) + \beta T(z)(y) \text{ (contraction-form)}
\end{aligned}
$$

For existence, uniqueness, strict monotonicity, and concavity of v^* (as fixed point of contraction form of Bellman), we need X (set of possible x choices) convex; $\Gamma(X)$ non-empty, compact-valued, continuous, and convex; $u(\cdot)$ continuous, bounded, strictly increasing and strictly concave; $\beta < 1$. Concavity of V is given by the theorem of the maximum. We may be expected to explicitly show Blackwell holds, i.e. prove discounting and monotonicity. For differentiablility of V, we need that for $x_0 \in INT(X), D(x_0)$ neighborhood, $\exists W(x) : D(x_0) \to \mathbb{R}$, concave, diff'l function s.t. $W(x_0) = V(x_0)$ and $W(x) \leq V(x) \forall x \in D$ (then V is differentiable by B-S).

T maps continuous bounded functions into continuous bounded functions. $T : C(\mathbb{C}) \to C(\mathbb{C})$. By the above assumptions, T is a contraction of modulus β by Blackwell sufficient conditions. $C(X)$ is a complete metric space, so by the contraction mapping theorem, there exists a unique v^* (fixed point); and by CMT corollary v^* is strictly increasing and strictly concave.

If we are not given that $u(\cdot)$ is bounded (as we typically are not), one possibility (unlikely on the exam) is if the utility is unbounded below (for example, see pset 1 question 2). The method is to find the upper bound of V, apply T n times (probably have to guess and verify the form this takes), then take the limit as $n \to \infty$. This will yield the fixed point solution.

Alternatively (and more likely to be on the exam, although still unlikely), if the utility function is homogenous of degree $0 < 1 - \gamma < 1$, if the constraint set is a convex cone, then V is homogenous of degree $1 - \gamma$. That is, under a specific norm and set of functions on X, $H(X)$, $T : H \to H$ is a contraction mapping, v is continuous via theorem of the maximum.

If you need to explicity use Blackwell to show a contraction, or something else specific, follow closely the steps in handout 9 or 10.