Macro PS1

Michael B. Nattinger*

November 7, 2020

1 Question 1

1.1 Part A

$$V(A_t, c_{t-1}) = \max_{c_t, A_{t+1}} u(c_t, c_{t-1}) + \beta V(A_{t+1}, c_t)$$
$$\text{s.t. } A_{t+1} = R(A_t - c_t)$$

We can rewrite this as the following:

$$V(A_t, c_{t-1}) = \max_{c_t} u(c_t, c_{t-1}) + \beta V(R(A_t - c_t), c_t) \tag{1}$$

To ensure the solution is unique, with a strictly increasing, strictly concave value function that is differentiable on the interior of the feasible set, we require the following assumptions:

- $u(.)$ is continuously differentiable, strictly concave, and strictly increasing in its arguments.
- $0 < \beta < 1$
- $\lim_{k \to 0} u'(k, u) = \lim_{k \to 0} u'(u, k) = \infty$
- $\lim_{k \to \infty} u'(k, u) = \lim_{k \to \infty} u'(u, k) = 0$
- The utility function is bounded?
- Do we need anything else?

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

Assuming the above conditions hold, we can solve the value function as it is written in (1) by taking a first order condition with respect to c_t, and then applying the envelope theorem twice:

$$0 = u_1(c', c) + \beta(-RV_1(RA - Rc', c') + V_2(RA - Rc', c')) \tag{2}$$
$$V_1(A, c) = R\beta V_1(RA - Rc', c') \tag{3}$$
$$V_2(A, c) = u_2(c', c) \tag{4}$$

Next, we can substitute in the envelope conditions (3), (4) into our first order condition (2) to find an expression for V_1, and substitute back into our initial first order condition (2):

$$\Rightarrow 0 = u_1(c', c) + \beta\left(-\frac{V_1(A, c)}{\beta} + u_2(c'', c')\right)$$

$$\Rightarrow V_1(A, c) = u_1(c', c) + \beta(c'', c') \tag{5}$$

We can now combine equations (2), (4), (5) to yield the following:

$$0 = u_1(c', c) + \beta(-R(u_1(c'', c') + \beta u_2(c''', c'')) + u_2(c'', c'))$$

$$\Rightarrow 0 = u_1(c', c) - \beta R u_1(c'', c') - \beta^2 R u_2(c''', c'') + \beta u_2(c'', c') \tag{6}$$

Equation (6) yields our optimality condition.

1.2 Part B

With the utility function as given, our value function becomes the following:

$$V(A, c) = \max_{c'} log(c') + \gamma log(c) + \beta V(RA - Rc', c')$$

The optimal choice c' takes the form of the arg max of the optimization problem.

$$\begin{aligned} c' &= \arg\max_{c'} log(c') + \gamma log(c) + \beta V(RA - Rc', c') \\ &= \arg\max_{c'} log(c') + \gamma log(c) + \beta \max_{c''} log(c'') + \gamma log(c') + \beta V(RA - Rc'', c'') \\ &= \arg\max_{c'} log(c') + \gamma log(c') + \gamma log(c) + \beta \max_{c''} log(c'') + \beta V(RA - Rc'', c'') \\ &= \arg\max_{c'} log(c') + \gamma log(c') + \beta \max_{c''} log(c'') + \beta V(RA - Rc'', c'') \end{aligned}$$

This arg max is independent of c.

We can rewrite this Bellman equation as the following, which will preserve the choice of c':

$$V(A) = \max_{A'} (1+\gamma) log\left(\frac{A'}{R} - A\right) + \beta V(A')$$

Taking FOC's,

$$\begin{aligned}
0 &= \frac{1+\gamma}{R\left(\frac{A'}{R} - A\right)} + \beta V'(A') \\
V'(A) &= -\frac{1+\gamma}{\left(\frac{A'}{R} - A\right)} \\
\Rightarrow \frac{1+\gamma}{R\left(\frac{A'}{R} - A\right)} &= \beta \frac{1+\gamma}{\left(\frac{A''}{R} - A'\right)} \\
\Rightarrow \left(\frac{A''}{R} - A'\right) &= \beta R \left(\frac{A'}{R} - A\right) \\
\Rightarrow A'' &= A'(1+\beta)R - \beta R^2 A
\end{aligned}$$

Now we will solve for our optimality conditions. Applying (6) to our new utility function yields the following:

$$\begin{aligned}
\beta R(c'')^{-1} + \gamma\beta^2 R(c'')^{-1} &= (c')^{-1} + \gamma\beta(c')^{-1} \\
\Rightarrow c'\beta R(1+\gamma\beta) &= c''(1+\gamma\beta)
\end{aligned}$$

$$\Rightarrow c'\beta R = c'' \tag{7}$$

Equation (7) yields our Euler conditions.

Given a set of assets in an initial period, A_1, $c_{t+1} = \beta R c_t \ \forall t \in \mathbb{N}$.

1.3 Part C

No, in general this will not hold. The utility function given to us was a seperable utility function, and for a non-separable utility function the utility

2 Question 2

2.1 Part A

The sequence problem is to maximize profits:

$$\max_{\{x_t\}_{i=1}^{\infty}} \left(\frac{1}{1+r}\right)^t \left(ax_t - \frac{b}{2}x_t^2 - \frac{c}{2}(x_{t+1} - x_t)^2\right)$$

We can rewrite this as a bellman equation in the following way:

$$V(x) = \max_y ax - \frac{b}{2}x^2 - \frac{c}{2}(y-x)^2 + \delta V(y) \tag{8}$$

We can rewrite this as follows:

$$T(v)(x) = \max_y ax - \frac{b}{2}x^2 - \frac{c}{2}(y-x)^2 + \delta v(y) \tag{9}$$

where the fixed point of our T operator in (9) is the solution to the Bellman equation in (8).

2.2 Part B

Let $L < 0$ be arbitrary. If we set $y = 0, x < \frac{L}{a}$ then $F(x,y) = ax - \frac{b}{2}x^2 - \frac{c}{2}(y-x)^2 \leq ax < L$ so F is unbounded below.

This F function is of the form of a polynomial with limits of negative infinity in all directions, so we can find the global maximum by taking first order conditions:

$$\begin{aligned}
0 &= a - bx + c(y-x) \\
0 &= -c(y-x) \Rightarrow y - x = 0 \Rightarrow y = x \\
\Rightarrow y = x &= \frac{a}{b} \\
F\left(\frac{a}{b}, \frac{a}{b}\right) &= a\left(\frac{a}{b}\right) - \frac{b}{2}\left(\frac{a}{b}\right)^2 - 0 \\
&= \frac{a^2}{2b}
\end{aligned}$$

Therefore, the maximum value F can take is $\frac{a^2}{2b}$
We can find bounds on $\hat{v}$ in the following way:

$$\begin{aligned}
\hat{v} &= \frac{a^2}{2b} + \delta \hat{v} \\
\Rightarrow \hat{v} &= \frac{a^2}{2b(1-\delta)}
\end{aligned}$$

2.3 Part C

$$\begin{aligned}
T\hat{v}(x) &= \max_y ax - \frac{b}{2}x^2 - \frac{c}{2}(y-x)^2 + \delta\hat{v} \\
0 &= -c(y-x) \Rightarrow y = x, \\
\Rightarrow T\hat{v}(x) &= ax - \frac{b}{2}x^2 + \delta\hat{v} \\
&\leq \frac{a^2}{2b} + \delta\frac{a^2}{2b(1-\delta)} = \frac{a^2}{2b(1-\delta)} \\
&= \hat{v}.
\end{aligned}$$

2.4 Part D

We showed the base case of the induction in Part C. Now we will show the induction step.

Assume that $T^n\hat{v}(x)$ takes the form $T^n\hat{v}(x) = \alpha_n x - \frac{1}{2}\beta_n x^2 + \gamma_n$. Then,

$$\begin{aligned}
T^{n+1}\hat{v}(x) &= \max_y ax - \frac{b}{2}x^2 - \frac{c}{2}(y-x)^2 + \delta(\alpha_n x - \frac{1}{2}\beta_n x^2 + \gamma_n) \\
y = x \Rightarrow T^{n+1}\hat{v}(x) &= ax - \frac{b}{2}x^2 + \delta\alpha_n x - \delta\frac{1}{2}\beta_n x^2 + \delta\gamma_n \\
&= (a + \delta\alpha_n)x - \frac{b + \delta\beta_n}{2}x^2 + \delta\gamma_n \\
&= \alpha_{n+1}x - \frac{1}{2}\beta_{n+1}x^2 + \gamma_{n+1}
\end{aligned}$$

where $\alpha_{n+1} = (a + \delta\alpha_n), \beta_{n+1} = b + \delta\beta_n, \gamma_{n+1} = \delta\gamma_n$.

2.5 Part E

Note that $\alpha_n = a + \delta a + \delta^2 a + \dots$, $\beta_n = b + \delta b + \delta^2 b + \dots, \gamma_n = \delta^n\hat{v}$. Thus, we can take the limit of α, β as geometric sums, and the limit of γ_n is 0. Therefore,

$$\begin{aligned}
\tilde{V} &= \lim_{n\to\infty} T^n\hat{v} = \frac{a}{1-\delta}x - \frac{1}{2}\frac{b}{1-\delta}x^2. \\
T\tilde{V} &= \max_y ax - \frac{b}{2}x^2 - \frac{c}{2}(y-x)^2 + \delta\frac{a}{1-\delta}x - \frac{1}{2}\frac{b}{1-\delta}x^2, \\
y = x \Rightarrow T\tilde{V} &= ax - \frac{b}{2}x^2 + \delta\frac{a}{1-\delta}x - \frac{1}{2}\frac{b}{1-\delta}x^2 \\
&= \frac{a}{1-\delta}x - \frac{1}{2}\frac{b}{1-\delta}x^2 \\
&= \tilde{T}.
\end{aligned}$$

Therefore, the limit function $\tilde{V}$ satisfies the Bellman equation.

3 Question 3

3.1 Part A

$$V(k) = \max_{k'} \pi(k) - \gamma(k' - (1-\delta)k) + \frac{1}{R}V(k')$$

$$0 = -\gamma'(k' - (1-\delta)k) + \frac{1}{R}\left(\pi'(k') + (1-\delta)\gamma'(k'' - (1-\delta)k')\right)$$
$$\Rightarrow \pi'(k') + (1-\delta)\gamma'(k'' - (1-\delta)k') = R\gamma'(k' - (1-\delta)k)$$

3.2 Part B

Letting $k = k' = k'' = \bar{k}$, we know that $\bar{I} = \delta\bar{k}$ and, moreover, we can rewrite our conditions for optimization in the following way:

$$\pi'(\bar{k}) + (1-\delta)\gamma'(\bar{I}) = R\gamma'(\bar{I}))$$
$$\Rightarrow \pi'(\bar{k}) = (R - 1 + \delta)\gamma'(\bar{I})$$

By the strict convexity of γ and strict concavity of π, in addition to our Inada conditions, the solution exists and is unique.

If R were to increase, the steady state level of $\pi'(\bar{k})$ would increase, resulting in a reduction in $\bar{k}$, and since $\bar{I} = \delta\bar{k}$, $\bar{I}$ will fall as well.

3.3 Part C

Our optimality conditions become:

$$-(k' - k^*) = R(k' - (1-\delta)k) + (1-\delta)(I')$$
$$\Rightarrow -(k' - k^*) = RI - (1-\delta)I'$$

4 Question 4

4.1 Part A

The Bellman equation takes the following form:

$$V(k) = \max_{c} \frac{1}{1-\gamma}(cG^{\eta})^{1-\gamma} + \beta V((1-\delta)k + f(k) - c)$$

We will find the conditions for maximization by taking first order conditions and applying the envelope conditions:

$$
\begin{aligned}
c^{-\gamma}(G^{\eta})^{1-\gamma} - \beta V'((1-\delta)k + f(k) - c) &= 0 \\
V'(k) &= \beta V'((1-\delta)k + f(k) - c)(1 - \delta + f'(k)) \\
\Rightarrow c^{-\gamma}(G^{\eta})^{1-\gamma} &= \frac{V'(k)}{1-\delta+f'(k)} \\
\Rightarrow V'(k) &= c^{-\gamma}(G^{\eta})^{1-\gamma}(1-\delta+f'(k)) \\
\Rightarrow c^{-\gamma}(G^{\eta})^{1-\gamma} &= \beta c^{-\gamma}(G^{\eta})^{1-\gamma}(1-\delta+f'((1-\delta)k+f(k)-c)) \\
\Rightarrow \frac{1}{\beta} &= 1-\delta+f'((1-\delta)k+f(k)-c) \\
\Rightarrow f'((1-\delta)k+f(k)-c) &= \frac{1}{\beta}+\delta-1.
\end{aligned}
$$

4.2 Part B

Yes, we can still have a unique steady state regardless of g. We can solve for $\bar{c}, \bar{k}$ as follows:

$$
\begin{aligned}
\bar{c} &= (1-\delta)\bar{k} + f(\bar{k}) - f'^{-1}(\frac{1}{\beta}+\delta-1) \\
\delta\bar{k} - f(\bar{k}) &= -\bar{c}
\end{aligned}
$$

The above equations form 2 equations in 2 unknowns which pin down our steady state values.

4.3 Part C