

Econometrics HW4

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1 Question 1

1.1 Show that $\bar{X}_{n+1} = (n\bar{X}_n + X_{n+1})/(n+1)$.

$$\begin{aligned}\bar{X}_{n+1} &= \frac{1}{n+1} \sum_{i=1}^{n+1} X_i = \frac{1}{n+1} \left(\left(\sum_{i=1}^n X_i \right) + X_{n+1} \right) \\ &= \frac{1}{n+1} (n\bar{X}_n + X_{n+1}).\end{aligned}$$

1.2 Show that $s_{n+1}^2 = ((n-1)s_n^2 + (n/(n+1))(X_{n+1} - \bar{X}_n)^2)/n$.

$$\begin{aligned}s_{n+1}^2 &= \frac{1}{n} \sum_{i=1}^{n+1} (X_i - \bar{X}_{n+1})^2 = \frac{1}{n} \sum_{i=1}^{n+1} (X_i - \bar{X}_n + \bar{X}_n - \bar{X}_{n+1})^2 \\ &= \frac{1}{n} \left((n-1) \sum_{i=1}^n (X_i - \bar{X}_{n+1})^2 + n(\bar{X}_n - \bar{X}_{n+1})^2 + 2 \left(\sum_{i=1}^n (X_i - \bar{X}_n)(\bar{X}_n - \bar{X}_{n+1}) \right) + (X_{n+1} - \bar{X}_{n+1})^2 \right) \\ &= \frac{1}{n} \left((n-1) \sum_{i=1}^n (X_i - \bar{X}_{n+1})^2 + n(\bar{X}_n - \bar{X}_{n+1})^2 + (X_{n+1} - \bar{X}_{n+1})^2 \right) \\ &= \frac{1}{n} ((n-1)s_n^2 + n\bar{X}_n^2 - 2n\bar{X}_n\bar{X}_{n+1} + n\bar{X}_{n+1}^2 + X_{n+1}^2 - 2X_{n+1}\bar{X}_{n+1} + \bar{X}_{n+1}^2) \\ &= \frac{1}{n} \left((n-1)s_n^2 + \frac{n}{n+1}(X_{n+1} - \bar{X}_n)^2 \right).\end{aligned}$$

2 Question 2

Define $\hat{\mu}_k := \frac{1}{n} \sum_{i=1}^n X_i^k$. We will show that this is unbiased.

$$\begin{aligned}E[\hat{\mu}_k] &= E \left[\frac{1}{n} \sum_{i=1}^n X_i^k \right] = \frac{1}{n} \sum_{i=1}^n E[X_i^k] \\ &= \frac{1}{n} \sum_{i=1}^n \mu_k \\ &= \mu_k.\end{aligned}$$

Thus, $\hat{\mu}_k$ is an unbiased estimator for μ_k .

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

3 Question 3

Define $\hat{m}_k := \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^k$. This estimator is biased, which can be seen from the fact that $\hat{m}_2 = \hat{\sigma}^2 \neq s_n^2$, so \hat{m}_2 is not an unbiased estimator for $m_2 = \sigma^2$. There exists no general formula for an unbiased estimator of $m_k, k > 3$ to the best of my knowledge.

4 Question 4

$$\begin{aligned} \text{Var}(\hat{\mu}_k) &= \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i^k) = \frac{1}{n} (E[X_i^{2k}] - E[X_i^k]^2) \\ &= \frac{1}{n} (\mu_{2k} - \mu_k^2). \end{aligned}$$

5 Question 5

Note that $f(x) = x^2$ is convex. By Jensen's inequality,

$$E[s_n]^2 \leq E[s_n^2] = \sigma^2,$$

so $E[s_n] \leq \sigma$ as both are nonnegative.

6 Question 6

$$\begin{aligned} \hat{\sigma}^2 &= \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2 = \frac{1}{n} \sum_{i=1}^n (X_i^2 - 2X_i\bar{X}_n + \bar{X}_n^2) \\ &= \frac{1}{n} \left(\sum_{i=1}^n X_i^2 - 2\bar{X}_n \sum_{i=1}^n X_i + \bar{X}_n^2 \sum_{i=1}^n 1 \right) \\ &= \frac{1}{n} \left(\sum_{i=1}^n X_i^2 - 2n\bar{X}_n^2 + n\bar{X}_n^2 \right) = \frac{1}{n} \left(\sum_{i=1}^n X_i^2 - n\bar{X}_n^2 \right). \\ \frac{1}{n} \sum_{i=1}^n (X_i - \mu)^2 - (\bar{X}_n - \mu)^2 &= \frac{1}{n} \sum_{i=1}^n X_i^2 - 2X_i\mu + \mu^2 - \bar{X}_n^2 + 2\mu\bar{X}_n - \mu^2 \\ &= \frac{1}{n} \sum_{i=1}^n X_i^2 - \bar{X}_n^2 + 2\mu(\bar{X}_n - X_i) \\ &= \frac{1}{n} \left(\sum_{i=1}^n X_i^2 - \bar{X}_n^2 \sum_{i=1}^n (1) + \sum_{i=1}^n 2\mu(\bar{X}_n - X_i) \right) \\ &= \frac{1}{n} \left(\sum_{i=1}^n X_i^2 - n\bar{X}_n^2 \right) = \hat{\sigma}^2. \end{aligned}$$

7 Question 7

$$\begin{aligned}
Cov(\hat{\sigma}^2, \bar{X}_n) &= E[\hat{\sigma}^2(\bar{X}_n - E\bar{X}_n)] \\
&= E\left[\left(\frac{1}{n} \sum_{i=1}^n (X_i - \mu)^2 - (\bar{X}_n - \mu)^2\right)(\bar{X}_n - \mu)\right] \\
&= \frac{1}{n^2} E\left[\left(\sum_{i=1}^n (X_i - \mu)^2\right)\left(\sum_{i=1}^n X_i - n\mu\right)\right] - E[(\bar{X}_n - \mu)^3] \\
&= \frac{1}{n^2} \left(\sum_{i=1}^n E[(X_i - \mu)^3]\right) + \frac{1}{n^2} \left(\sum_{i \neq j} E[(X_i - \mu)^2(X_j - \mu)]\right) - E[(\bar{X}_n - \mu)^3] \\
&= \frac{1}{n} E[(X_i - \mu)^3] - E[(\bar{X}_n - \mu)^3] \\
&= \frac{1}{n} E[(X_i - \mu)^3] \\
&\quad - \frac{1}{n^3} \left(\sum_{i=1}^n E[(X_i - \mu)^3] + 3 \sum_{i \neq j} E[(X_i - \mu)^2(X_j - \mu)] + 3 \sum_{i \neq j \neq k} E[(X_i - \mu)(X_j - \mu)(X_k - \mu)]\right) \\
&= \frac{1}{n} E[(X_i - \mu)^3] - \frac{1}{n^2} E[(X_i - \mu)^3] \\
&= \left(\frac{1}{n} - \frac{1}{n^2}\right) E[(X_i - \mu)^3]
\end{aligned}$$

This quantity will be 0 when X_i has no skewness.

8 Question 8

8.1 Find $E[\bar{X}_n]$

$$\begin{aligned}
E[\bar{X}_n] &= E\left[\frac{1}{n} \sum_{i=1}^n X_i\right] = \frac{1}{n} \sum_{i=1}^n E[X_i] \\
&= \frac{1}{n} \sum_{i=1}^n \mu_i.
\end{aligned}$$

8.2 Find $Var(\bar{X}_n)$.

$$\begin{aligned}
Var(\bar{X}_n) &= \frac{1}{n^2} Var\left(\sum_{i=1}^n X_i\right) = \frac{1}{n^2} \sum_{i=1}^n Var(X_i) \\
&= \frac{1}{n^2} \sum_{i=1}^n \sigma_i^2.
\end{aligned}$$

9 Question 9

$$\begin{aligned}
 E[Q] &= E \left[\sum_{i=1}^r X_i^2 \right] = \sum_{i=1}^r E[X_i^2] \\
 &= \sum_{i=1}^r \mu_X^2 + \sigma_X^2 = \sum_{i=1}^r 0^2 + 1^2 \\
 &= r.
 \end{aligned}$$

$$\begin{aligned}
 Var(Q) &= E[Q^2] - E[Q]^2 = E \left[\left(\sum_{i=1}^r X_i^2 \right)^2 \right] - r^2 \\
 &= \sum_{i=1}^r E[X_i^4] + 2 \sum_{1 \leq i < j \leq r} E[X_i^2 X_j^2] - r^2 \\
 &= \sum_{i=1}^r 3 + 2 \sum_{1 \leq i < j \leq r} (1)(1) - r^2 \\
 &= 3r + 2 \left(\frac{r!}{2(r-2)!} \right) - r^2 = 3r + r(r-1) - r^2 \\
 &= 2r.
 \end{aligned}$$

Note: we calculated $E[X^4] = 3$ for $X \sim N(0, 1)$ in the previous problem set.

10 Question 10

We will first show that a sum of independent normal random variables is normal. Let $Z_i \sim N(\mu_i, \sigma_i^2) \forall i \in [1, \dots, n]$ for some $n \in \mathbb{N}$. Then the MGF of Z_i is $M_{Z_i}(t) = \exp(\mu_i t + \frac{\sigma_i^2 t^2}{2})$.¹ Then,²

$$M_{\sum_{i=1}^n Z_i}(t) = \prod_{i=1}^n M_{Z_i}(t) = \exp \left(\sum_{i=1}^n \left(\mu_i t + \frac{\sigma_i^2 t^2}{2} \right) \right) = \exp \left(\left(\sum_{i=1}^n \mu_i \right) t + \left(\sum_{i=1}^n \sigma_i^2 \right) \frac{t^2}{2} \right).$$

So $\sum_{i=1}^n Z_i$ is of the form of a normal random variable with mean $\sum_{i=1}^n \mu_i$ and variance $\sum_{i=1}^n \sigma_i^2$.

10.1 Find $E[\bar{X}_n - \bar{Y}_n]$

From the above, \bar{X}_n, \bar{Y}_n are of the form of normal variables with means μ_X, μ_Y and variances $\frac{1}{n_1} \sigma_X^2, \frac{1}{n_2} \sigma_Y^2$. Then, $\bar{X}_n - \bar{Y}_n$ is also normal with mean $\mu_X - \mu_Y$.

10.2 Find $Var[\bar{X}_n - \bar{Y}_n]$

From the above, $\bar{X}_n - \bar{Y}_n$ is normal with variance $\frac{1}{n_1} \sigma_X^2 + \frac{1}{n_2} \sigma_Y^2$.

10.3 Find the distribution of $\bar{X}_n - \bar{Y}_n$

From the above, $\bar{X}_n - \bar{Y}_n$ is of the form of a normal random variable with mean $\mu_X - \mu_Y$ and variance $\frac{1}{n_1} \sigma_X^2 + \frac{1}{n_2} \sigma_Y^2$.

¹ $M_{Z_i}(t) = \int_{-\infty}^{\infty} \exp(xt) \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp(-(x - \mu_i)^2/(2\sigma_i^2)) dx$. Define $z = (x - \mu_i)/\sigma_i$. Then $M_{Z_i}(t) = \exp(\mu_i t) \int_{-\infty}^{\infty} \exp(z\sigma_i t) \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp(-z^2/2) (dx/dz) dz = \exp(\mu_i t) \int_{-\infty}^{\infty} \exp(z\sigma_i t) \frac{1}{\sqrt{2\pi}} \exp(-z^2/2) dz = \exp(\mu_i t) \exp(\sigma_i^2 t^2/2) = \exp(\mu_i t + \frac{\sigma_i^2 t^2}{2})$.

² $M_{\sum_{i=1}^n Z_i}(t) = E[\exp(\sum_{i=1}^n t Z_i)] = E[\prod_{i=1}^n \exp(t Z_i)] = \prod_{i=1}^n E[\exp(t Z_i)] = \prod_{i=1}^n M_{Z_i}(t)$