Micro HW1

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March 27, 2021

1 Question 1

1.1 Part A

The game consists of players $I = \{1, \ldots, N\}$. Each player is of type $\theta \sim F$. Each player plays a strategy $b : [0, 1] \to \mathbb{R}$ which is a function of their type. The a priori belief is that everyone's type is drawn from F. The interim belief is full knowledge of one's own type, and knowing that everyone else's type is drawn from F. The ex ante belief is full knowledge of all types. The person with the highest bid wins the auction and thus gains their valuation, and all players (including the winner) pays their bid regardless of winning or losing.

1.2 Part B

Each player faces the following utility maximization problem:

$$\max_{b_i} E[U(b_i)]$$

$$\max_{b_i} \theta_i Pr(b_i > \max_{j \neq i}\{b_j\}) - b$$

In our case, with *iid* observations, $Pr(b_i > \max_{j \neq i}\{b_j\}) = \Pi_{j \neq i} Pr(b_i > b_j)$. We can also now plug in the strategy of the other players as a function. Our maximization problem now takes the following form, where θ_{-i} is any draw from F:

$$\max_{b_i} \theta_i (Pr(b_i > b(\theta_{-i})))^{N-1} - b_i$$

$$\max_{b_i} \theta_i (Pr(b^{-1}(b_i) > \theta_{-i}))^{N-1} - b_i$$

$$\max_{b_i} \theta_i (b^{-1}(b_i))^{aN-a} - b_i$$

Taking first order conditions,

$$1 = \theta_i (aN - a)(b^{-1}(b_i))^{aN-a-1} \frac{1}{b'(b^{-1}(b_i))}$$

$$= \frac{(aN - a)(\theta_i)^{aN-a}}{b'(\theta_i))}$$

$$b'(\theta_i) = (aN - a)(\theta_i)^{aN-a}$$

$$\Rightarrow b(\theta_i) = \frac{aN - a}{aN - a + 1}(\theta_i)^{aN-a+1} + c_{\theta_i}$$

Players with a valuation of 0 will bid 0 so $c_{\theta_i} = 0$. Therefore, $b(\theta_i) = \frac{aN-a}{aN-a+1}(\theta_i)^{aN-a+1}$

1.3 Part C

We can verify by rewriting the optimization problem as we did originally with the known function b and show that the best reply is to follow said function.

$$\max_{b_i} \theta_i (Pr(b_i > b(\theta_{-i})))^{N-1} - b_i$$

$$\max_{b_i} \theta_i (Pr(b_i > \frac{aN - a}{aN - a + 1}(\theta_{-i})^{aN-a+1}))^{N-1} - b_i$$

$$\max_{b_i} \theta_i (Pr(\left(\frac{aN - a + 1}{aN - a}b_i\right)^{\frac{1}{aN-a+1}} > \theta_{-i})^{N-1} - b_i$$

$$\max_{b_i} \theta_i \left(\frac{aN - a + 1}{aN - a}b_i\right)^{\frac{aN-a}{aN-a+1}} - b_i$$

Taking FOCs:

$$1 = \theta_i \frac{aN - a}{aN - a + 1} \left(\frac{aN - a + 1}{aN - a}b_i\right)^{\frac{aN-a}{aN-a+1} - 1} \frac{aN - a + 1}{aN - a}$$

$$\left(\frac{aN - a + 1}{aN - a}b_i\right)^{\frac{1}{aN-a+1}} = \theta_i$$

$$b_i = \frac{aN - a}{aN - a + 1}(\theta_i)^{aN-a+1}$$

Therefore , $b(\theta) = \frac{aN-a}{aN-a+1}(\theta)^{aN-a+1}$ is a best response to the other players playing the same strategy. Thus, it constitutes an equilibrium.

1.4 Part D

As a increases, bids will shrink for given types less than 1, so the bidding will become less competitive for a given type. Intuitively, as a rises, more of the distribution mass is

near 1 so for a fixed θ value less than one the odds of winning decrease, and therefore the bid decreases. However, note that there is a competing effect here: the bids for a given type is decreasing, however as a increases the probability mass of types shifts to the right so the drawn types tend to be higher. In the limit as $a \to \infty$ this effect dominates and the bids tend to increase.

1.5 Part E

Before drawing a type, the expected bid is the following:

$$
\begin{aligned}
E[b_i(\theta)] &= \int_0^1 \frac{aN - a}{aN - a + 1}(\theta)^{aN - a + 1} a\theta^{a-1} d\theta \\
&= \frac{a(aN - a)}{aN - a + 1} \int_0^1 \theta^{aN} d\theta \\
&= \frac{a(aN - a)}{aN - a + 1} \frac{1}{(aN + 1)}
\end{aligned}
$$

I originally interpreted the question as being the expected return of the bidder before and after drawing the type. This is not what the question asks but I still have it below.

After drawing a type, person i with type $\theta_i \sim F$ bids $b(\theta_i) = \frac{aN-a}{aN-a+1}(\theta_i)^{aN-a+1}$ and wins payoff θ_i with some probability:

$$
\begin{aligned}
E[\pi|\theta] &= \theta_i Pr(\theta_i > \theta_{-i})^{N-1} - b(\theta_i) \\
&= \theta_i^{aN-a+1} - \frac{aN - a}{aN - a + 1}(\theta_i)^{aN - a + 1} \\
&= \frac{1}{aN - a + 1}\theta_i^{aN-a+1}
\end{aligned}
$$

We can find the expected winnings prior to drawing a type by taking the expectation of winnings by type across the distribution of types. Note that $F(\theta) = \theta^a \Rightarrow f(\theta) = a\theta^{a-1}$. Then, we have the following:

$$
\begin{aligned}
E[\pi] &= \int_0^1 E[\pi|\theta] f(\theta) d\theta \\
&= \int_0^1 \frac{1}{aN - a + 1}\theta^{aN-a+1} a\theta^{a-1} d\theta \\
&= \frac{a}{aN - a + 1} \int_0^1 \theta^{aN} d\theta \\
&= \frac{a}{aN - a + 1} \frac{1}{aN + 1}.
\end{aligned}
$$

2 Question 2

2.1 Part A

Suppose there exists bidding strategies $b_1(x), b_2(x)$ for individuals $F_1(x) = x, F_2(x) = x^2$ which are known to all individuals and form an equilibrium (non-symmetric due to non-

iid types). For each bidder with valuation x_i we can think of the bidder as choosing the X_i they pretend to be by submitting bid $b_i(X_i)$. The bidder will win iff $X_i > x_j$. The payoff for person 1 is then the following:

$$(x_1 - b_1(X_1))Pr(X_1 > x_2) = (x_1 - b_1(X_1))X_1^2$$

Taking FOCs wrt X_1:

$$2X_1 x_1 - 2X_1 b_1(X_1) - X_1^2 b_1'(X_1) = 0$$

If b_1 is an equilibrium then $X_1 = x_1$ so our FOC reads $2x_1^2 - 2x_1 b_1(x_1) - x_1^2 b_1'(x_1) = 0$. This is a first order differential equation. A bidder with value r cannot earn a positive surplus and thus will bid r, this forms our boundary condition by which we can solve:

$$2x_1^2 = 2x_1 b_1(x_1) + x_1^2 b_1'(x_1)$$
$$2x_1^2 = (x_1^2 b_1(x_1))'$$
$$\frac{2}{3}x_1^3 + c = x_1^2 b_1(x_1)$$
$$b_1(x_1) = \frac{2}{3}x_1 + \frac{c}{x_1^2}$$
$$b_1(r) = r = \frac{2}{3}r + c/(r^2)$$
$$\frac{r^3}{3} = c$$
$$b_1(x_1) = \frac{2}{3}x_1 + \frac{r^3}{3x_1^2}.$$

The payoff for person 2 is then the following:

$$(x_2 - b_1(X_2))Pr(X_2 > x_1) = (x_2 - b_2(X_2))X_2$$

Taking FOCs wrt X_2:

$$x_2 - b_2(X_2) - X_2 b_2'(X_2) = 0$$

If b_2 is an equilibrium then $X_2 = x_2$ so our FOC reads $x_2 - b_2(x_2) - x_2 b_2'(x_2) = 0$. This is a first order differential equation. A bidder with value r cannot earn a positive surplus

and thus will bid r, this forms our boundary condition by which we can solve:

$$x_2 = b_2(x_2) + x_2 b_2'(x_2)$$
$$= (x_2 b_2(x_2))'$$
$$\frac{x_2^2}{2} + c = x_2 b_2(x_2)$$
$$b_2(x_2) = \frac{x_2}{2} + \frac{c}{x_2}$$
$$b_2(r) = r = \frac{r}{2} + \frac{c}{r}$$
$$c = r^2/2$$
$$b_2(x_2) = \frac{x_2^2 + r^2}{2x_2}$$

b_1, b_2 form the equilibrium bidding strategies for the two people. The highest type agent will win, so long as their type is above r.

2.2 Part B

In the second-price auction, both people will bid their valuations - we saw in class that the strategy was weakly dominant in the second price auction without reserve price, and the same logic holds here. If both bids are above r, then the seller with the higher bid will pay the seller the second-highest price, and will receive the object. If one valuation is above r then that individual will pay r and receive the object. If both valuations are below r then no trade will occur.

2.3 Part C

The revenue equivalence theorem does not hold here as the agents are drawn from different distributions, and the type 2 agent is more likely to win as their type distribution first order stochastically dominates the type 1 agent. We can find the optimal r to pick for both types of auction, and then calculate the revenue at the optimums and compare the two types of auctions.

First we will maximize profit of the first-price auction. The highest type will win
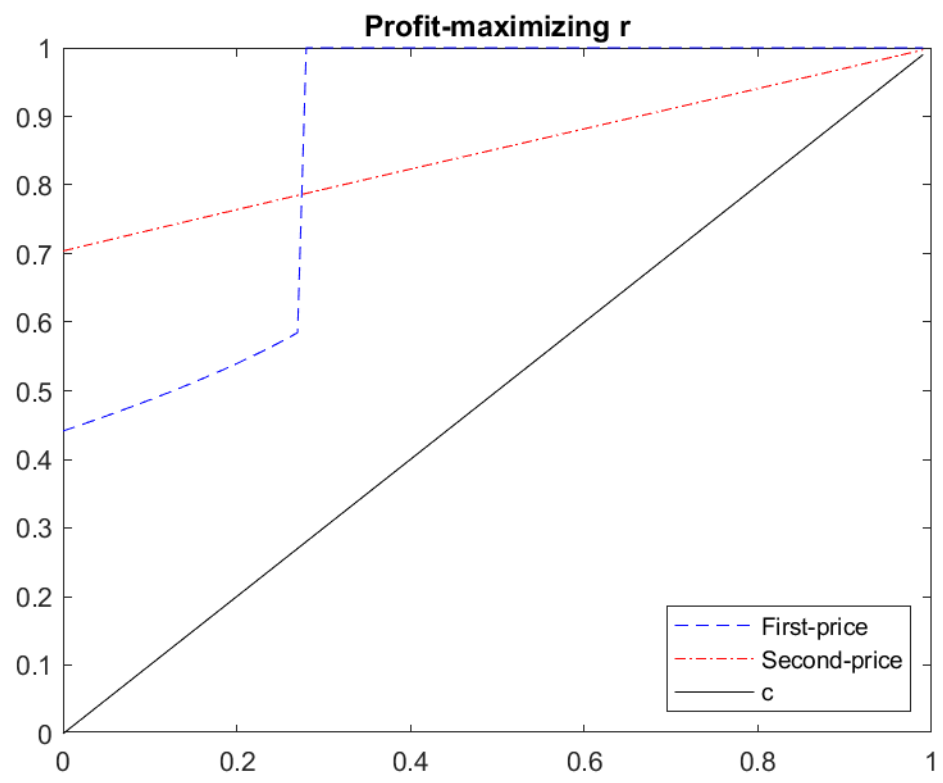
conditional on having a type above r:

$$\pi^1 = E[b_1(x_1) - c|x_1 > r > x_2]P(x_1 > r > x_2) + E[b_1(x_1) - c|x_1 > x_2 > r]P(x_1 > x_2 > r)$$
$$+ E[b_2(x_2) - c|x_2 > r > x_1]P(x_2 > r > x_1) + E[b_2(x_2) - c|x_2 > x_1 > r]P(x_2 > x_1 > r)$$

$$= E\left[\frac{2}{3}x_1 + \frac{r^3}{3x_1^2} - c|x_1 > r\right]P(x_1 > r > x_2) + E\left[\frac{2}{3}x_1 + \frac{r^3}{3x_1^2} - c|x_1 > x_2 > r\right]P(x_1 > x_2 > r)$$

$$+ E\left[\frac{x_2^2 + r^2}{2x_2} - c|x_2 > r\right]P(x_2 > r > x_1) + E\left[\frac{x_2^2 + r^2}{2x_2} - c|x_2 > x_1 > r\right]P(x_2 > x_1 > r)$$

$$= \left(\int_r^1 \frac{2}{3}x + \frac{r^3}{3x^2}dx - c\right)\int_r^1\int_0^r 2x_2 dx_2 dx_1$$

$$+ \left(\int_r^1\int_r^{x_1} 2x_2 dx_2 \left(\frac{2}{3}x_1 + \frac{r^3}{3x_1^2}\right)dx_1 - c\right)\int_r^1\int_r^{x_1} 2x_2 dx_2 dx_1$$

$$+ \left(\int_r^1 x^2 + r^2 dx - c\right)\int_r^1\int_0^r dx_1 2x_2 dx_2 + \left(\int_r^1\int_r^{x_2} dx_1 \left(x_2^2 + r^2\right)dx_2 - c\right)\int_r^1\int_r^{x_2} dx_1 2x_2 dx_2$$

$$= \left(\frac{1}{3}(1 - r^3) - c\right)(r^2 - r^3) + \left(\frac{1}{6}(1 - r)^2(2r^3 + r^2 + 2r + 1) - c\right)\left(\frac{1}{3}(1 - r)^2(2r + 1)\right)$$

$$+ \left(r^2 - \frac{4}{3}r^3 + \frac{1}{3} - c\right)(r - r^3) + \left(\frac{1}{12}(1 - r)^2(7r^2 + 2r + 3) - c\right)\frac{1}{3}\left(r^3 - 3r + 2\right)$$
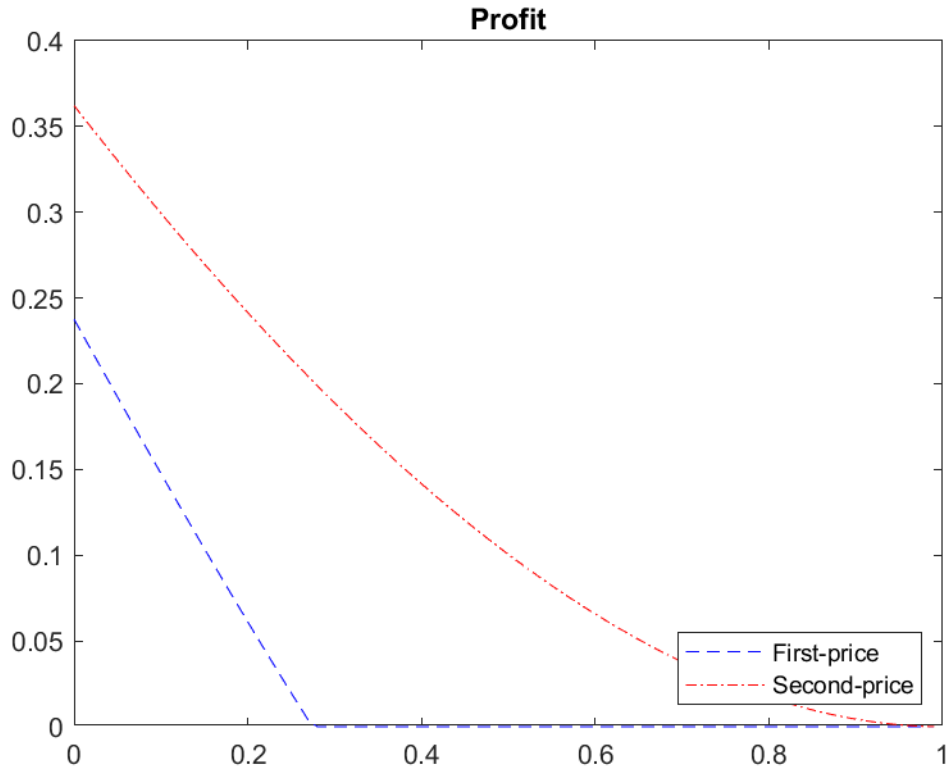
r^{1*} satisfies the first order condition of the above expression with respect to r. I will solve computationally. Then, maximum profits can be computed as a function of c.

Next we will maximize profit of the second-price auction.

$$\pi^2 = E[r - c|x_1 > r > x_2]P(x_1 > r > x_2) + E[x_2 - c|x_1 > x_2 > r]P(x_1 > x_2 > r)$$
$$+ E[r - c|x_2 > r > x_1]P(x_2 > r > x_1) + E[x_1 - c|x_2 > x_1 > r]P(x_2 > x_1 > r)$$

$$= (r - c)\int_r^1\int_0^r 2x_2 dx_2 dx_1 + \left(\int_r^1\int_r^{x_1} 2x_2^2 dx_2 dx_1 - c\right)\int_r^1\int_r^{x_1} 2x_2 dx_2 dx_1$$

$$+ (r - c)\int_r^1\int_0^r dx_1 2x_2 dx_2 + \left(\int_r^1\int_r^{x_2} 2x_2 x_1 dx_1 dx_2 - c\right)\int_r^1\int_r^{x_2} dx_1 2x_2 dx_2$$

$$= (r - c)(r^2 - r^3) + \left(\frac{1}{6}(3r^4 - 4r^3 + 1) - c\right)\left(\frac{1}{3}(r - 1)^2(2r + 1)\right)$$

$$+ (r - c)(r - r^3) + \left(\frac{1}{4}(r^2 - 1)^2 - c\right)\frac{1}{3}(r^3 - 3r + 2)$$

r^{2*} satisfies the first order condition of the above expression with respect to r. I will solve computationally. Then, maximum profits can be computed as a function of c. We can then compare to the first price auction.

Profit-maximizing r

Legend:
- First-price (blue dashed)
- Second-price (red dash-dot)
- c (black solid)

Profit

We can see from the above figures that the profit-maximizing seller will choose the second price auction as they receive strictly higher profit for $c \in [0, 1]$.

2.4   Part D

The discount should be offered to the type that tends to have the lowest valuation $F(x) = x$. The idea is to try to get them to bid over their valuation, while staying the second-highest bidder, raising the sale price of the good. We calculate the optimal discount.

The type that receives the discount will bid such that they would receive no surplus buying the item at that price: $\alpha b - x = 0 \Rightarrow b = \frac{x}{\alpha}$. The other type bids their valuation. The seller chooses α to maximize expected revenue:

$$\max_{\alpha} E\left[\frac{x_1}{\alpha}|\frac{x_1}{\alpha} < x_2\right] P\left(\frac{x_1}{\alpha} < x_2\right) + E\left[x_2|\frac{x_1}{\alpha} > x_2\right] P\left(\frac{x_1}{\alpha} > x_2\right)$$

$$\max_{\alpha} \frac{1}{\alpha}\int_0^1 \int_0^{\alpha x_2} x_1 dx_1 2x_2 dx_2 \int_0^1 \int_0^{\alpha x_2} dx_1 2x_2 dx_2 + \int_0^1 \int_0^{x_1/\alpha} 2x_2^2 dx_2 dx_1 \int_0^1 \int_0^{x_1/\alpha} 2x_2 dx_2 dx_1$$

$$\max_{\alpha} 2\alpha^2 \int_0^1 x_2^3 dx_2 \int_0^1 x_2^2 dx_2 + \frac{2}{3\alpha^5}\int_0^1 x_1^3 dx_1 \int_0^1 x_1^2 dx_1$$

$$\max_{\alpha} \frac{\alpha^2}{6} + \frac{1}{18\alpha^5}$$

Taking FOCs:

$$\frac{\alpha}{3} = \frac{5}{18\alpha^6}$$
$$\alpha^7 = \frac{5}{6}$$
$$\alpha = \left(\frac{5}{6}\right)^{1/7}.$$

3 Question 3

3.1 Part A

The game consists of players $I = \{1, 2, 3\}$. Each player is of type $v \sim [0, 1]$. Each player plays a strategy $b : [0, 1] \to \mathbb{R}$ which is a function of their type. The a priori belief is that everyone's type is drawn from $[0, 1]$. The interim belief is full knowledge of one's own type, and knowing that everyone else's type is drawn from $[0, 1]$. The ex ante belief is full knowledge of all types. The person with the highest bid wins the auction and thus gains their valuation, and that player pays the lowest bid.

3.2 Part B

Player $i \in I$ maximizes their expected value. We will assume that players $-i$ play the strategy $b(v) = \frac{n-1}{n-2}v$, then prove that the best response is the same strategy.

The optimization problem is the following:

$$\max_{b_i} E[v_i - \frac{n-1}{n-2}\text{second}_{j \neq i}v_j | b_i > \frac{n-1}{n-2}\max_{j \neq i}v_j]Pr(b_i > \frac{n-1}{n-2}\max_{j \neq i}v_j)$$

$$Pr(b_i > \frac{n-1}{n-2} \max_{j \neq i} v_j) = Pr\left(b_i > \frac{n-1}{n-2} v_{-i}\right)^{n-1}$$

$$= \left(b_i \frac{n-2}{n-1}\right)^{n-1}$$

$$E[v_i - \frac{n-1}{n-2}\text{second}_{j \neq i} v_j | b_i > \frac{n-1}{n-2} \max_{j \neq i} v_j] = E[v_i - \frac{n-1}{n-2}\text{second}_{j \neq i} v_j | \frac{n-2}{n-1} b_i > v_j, j \neq i]$$

$$= v_i - \frac{n-1}{n-2} E[\text{second}_{j \neq i} v_j | \frac{n-2}{n-1} b_i > v_j, j \neq i]$$

This conditional expectation is equivalent to the expectation of the second largest of $n-1$ draws from the uniform distribution $U\left(0, \frac{n-2}{n-1} b_i\right)$. The expectation of this is the expectation of the second largest of $n-1$ draws from a standard uniform distribution, scaled by $\frac{n-2}{n-1} b_i$. This expectation is, therefore, $\frac{n-2}{n-1} b_i \frac{n-2}{n}$.[1] We then have the following maximization problem:

$$\max_{b_i} v_i \left(b_i \frac{n-2}{n-1}\right)^{n-1} - \frac{n-1}{n-2} \frac{(n-2)^2}{n-1} \frac{b_i}{n} \left(b_i \frac{n-2}{n-1}\right)^{n-1}$$

$$\max_{b_i} v_i \left(b_i \frac{n-2}{n-1}\right)^{n-1} - \frac{n-2}{n} b_i^n \left(\frac{n-2}{n-1}\right)^{n-1}$$

Taking FOCs:

$$(n-1) v_i b_i^{n-2} \left(\frac{n-2}{n-1}\right)^{n-1} = (n-2) b_i^{n-1} \left(\frac{n-2}{n-1}\right)^{n-1}$$

$$\Rightarrow b_i = \frac{n-1}{n-2} v_i$$

Thus, the best response to the other players playing $b(v_i) = \frac{n-1}{n-2} v_i$ is also to play $(v_i) = \frac{n-1}{n-2} v_i$, so the bid function b is a symmetric Bayes-Nash equilibrium of the third-price auction.

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[1]See these lecture slides from Duke which contain formulas from which the expectation of the kth order statistic of a standard uniform can be constructed. Specifically, slides 8 and 11.

3.3 Part C

The expected revenue of the seller is the expectation of the third largest bid:

$$\begin{aligned}
R^3 &= E[\text{third}_{i \in I} b_i] \\
&= E[\text{third}_{i \in I} \frac{n-1}{n-2} v_i] \\
&= \frac{n-1}{n-2} E[\text{third}_{i \in I} v_i] \\
&= \frac{n-1}{n-2} \frac{n-2}{n+1} \\
&= \frac{n-1}{n+1}.
\end{aligned}$$

3.4 Part D

We know from class that for $k = 1$ the equilibrium strategy is to underbid and for $k = 2$ the equilibrium strategy is to bid the valuation. For $k = 3$ we showed above that the equilibrium strategy is to overbid. As k continues to increase above 3, the equilibrium strategy is to overbid by an increasing amount. One can show that $b_i^k(v_i) = \frac{n-1}{n-k+1}$ is the equilibrium strategy for the k'th price auction. I do this below:

$$\max_{b_i} E[v_i - \frac{n-1}{n-k+1} \text{k-1'th}_{j \neq i} v_j | b_i > \frac{n-1}{n-k+1} \max_{j \neq i} v_j] Pr(b_i > \frac{n-1}{n-k+1} \max_{j \neq i} v_j)$$

$$\max_{b_i} (v_i - \frac{n-1}{n-k+1} E[\text{k-1'th}_{j \neq i} v_j | b_i > \frac{n-1}{n-k+1} \max_{j \neq i} v_j]) \left(b_i \frac{n-k+1}{n-1} \right)^{n-1}$$

$$\max_{b_i} (v_i - \frac{n-1}{n-k+1} \frac{n-k+1}{n-1} b_i \frac{n-k+1}{n}) \left(b_i \frac{n-k+1}{n-1} \right)^{n-1}$$

$$\max_{b_i} v_i \left(b_i \frac{n-k+1}{n-1} \right)^{n-1} - b_i \frac{n-k+1}{n} \left(b_i \frac{n-k+1}{n-1} \right)^{n-1}$$

Taking FOCs:

$$(n-1) v_i b_i^{n-2} \left(\frac{n-k+1}{n-1} \right)^{n-1} = b_i^n (n-k+1) \left(\frac{n-k+1}{n-1} \right)^{n-1}$$

$$\Rightarrow b_i = \frac{n-1}{n-k+1} v_i.$$

Therefore, $b_i^k(v_i) = \frac{n-1}{n-k+1}$ is the equilibrium strategy for the k'th price auction.