IO Problem Set 1

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1 Question 1

The demand curve with constant elasticity can be written as $Q = aP^{-c}$. Rewriting, the corresponding inverse demand function is $P(Q) = a^{1/c}Q^{-1/c}$. Then, $P'(Q) = (-1/c)a^{1/c}Q^{-(1+c)/c}$, $P''(Q) = (-1/c)(-(1+c)/c)a^{1/c}Q^{-(1+2c)/c} = ((1+c)/c^2)a^{1/c}Q^{-(1+2c)/c}$ We then have the following:

$$\begin{aligned}
P'(Q) + QP''(Q) &= (-1/c)a^{1/c}Q^{-(1+c)/c} + Q(((1+c)/c^2)a^{1/c}Q^{-(1+2c)/c}) \\
&= (-1/c)a^{1/c}Q^{-(1+c)/c} + (((1+c)/c^2)a^{1/c}Q^{-(1+c)/c}) \\
&= (1/c^2)a^{1/c}Q^{-(1+c)/c} > 0.
\end{aligned}$$

Next, let N firms be competing a la Cournot.
Assumption (A1) is that $0 \geq P''(Y)y_i + P'(Y)\forall y_i < Y$.
Assumption (A2) states that $0 \geq P'(Y) - C_i''(y_i)\forall y_i < Q$. We are given that each firm has identical cost functions, so $C_i(y) = C(y)$. Note that (A2) therefore states that $C''(y_i) \geq P'(Y)$. With identical costs, in equilibrium $y_i = y = Y/N$. Using this, (A1) becomes
$0 \geq P''(Y)Y/N + P'(Y)$

Let us set up the maximization problem for each firm:

$$\begin{aligned}
&\max_{y_i} P(y_i + Y_{-i})y_i - C(y_i) \\
&\Rightarrow P'(Y)y_i + P(Y) - C'(y_i) = 0 \\
&\Rightarrow P(Ny) = C'(y) - P'(Ny)y
\end{aligned}$$

Differentiating both sides with respect to N,

$$\begin{aligned}
\frac{\partial P(Y)}{\partial N} &= -P''(Y)y^2 \\
&\geq P'(Y)y
\end{aligned}$$

2 Question 2

Each bidder chooses a bid $b_i \in \mathbb{R}$ to maximize their payoffs:

$$b_i = \arg\max_b \pi(b, b_{-1}),$$

where the payoff $\pi(b, b_{-i})$ is:

$$\pi(b, b_{-i}) = \begin{cases} V - b, b > b_{-i} \\ 0, b < b_{-i} \\ (1/2)(V - b), b = b_{-i} \end{cases}.$$

The equilibrium is $b_i = V \forall i$. Why is this the case? Suppose instead player i bid $b_i > V$. Then, their payoff would be $V - b_i < 0$. Moreover, suppose $b_i < V$. Then, person i still only gets 0 payoff. So, $b_i = V \forall i$ is an equilibrium. No other equilibrium can exist. To see why, first note that equilibria can only exist with $b_i = b_{-i}$ as otherwise the bidder with the largest bid is strictly better off reducing their bid by some ϵ. If $b_i = b_{-i} < V$ then bidder i is better off increasing their bid by an epsilon and winning positive payoff. If $b_i = b_{-i} > V$ then the best response for i is to reduce their bid by an ϵ such that they are sure to receive 0 payoff instead of negative expected payoff. So, the only equilibrium is $b_i = b_{-i} = V$.

Now consider the all-pay auction. The expected payoff $\pi(b, b_{-i})$ is given by the following:

$$\pi(b, b_{-i}) = \begin{cases} V - b, b > b_{-i}, \\ -b, b < b_{-i} \\ (1/2)V - b, b = b_{-i} \end{cases}$$

First we will show that a pure strategy Nash equilibrium does not exist. Suppose it does. Then, $b_i = b_{-i}$ because otherwise the highest bidder would be strictly better off by reducing their bid by an ϵ. Consider, then, $b_i = b_{-i} = b$. If $b < V$ then either bidder would be better off increasing their bid by an ϵ and winning V surely. If $b \geq V$ then either bidder would be better off not bidding (or bidding zero). Therefore no pure strategy Nash equilibrium can exist.

Consider an equilibrium where each bidder bids $b > 0$ with probability p and 0 with probability $1 - p$. Each player must be indifferent between bidding b and not bidding in order to mix. Taking as given that player $-i$ is playing this strategy, indifference of player i implies:

$$\begin{aligned} p((1/2)V - b) + (1 - p)(V - b) &= 0 \\ (p/2 + (1 - p))V &= b \end{aligned}$$

Moreover, the b must be such that one is weakly better off choosing 0 than $b - \epsilon$, with equality in the limit:

$$\begin{aligned} p(0) + (1 - p)V - b + \epsilon &\leq 0 \\ \Rightarrow b &\geq \epsilon + (1 - p)V \end{aligned}$$

Taking $\lim_{\epsilon \to 0}$, $b = (1 - p)V$. Combining with our previous expression, $(p/2 + (1 - p))V = b = (1 - p)V$. Simplifying, this expression yields a unique p: $p = 0$.