IO Problem Set 1

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September 27, 2021

1 Question 1

1.1 Part (a)

The demand curve with constant elasticity can be written as $Q = aP^{-c}$. Rewriting, the corresponding inverse demand function is $P(Q) = a^{1/c}Q^{-1/c}$. Then, $P'(Q) = (-1/c)a^{1/c}Q^{-(1+c)/c}$, $P''(Q) = (-1/c)(-(1+c)/c)a^{1/c}Q^{-(1+2c)/c} = ((1+c)/c^2)a^{1/c}Q^{-(1+2c)/c}$ We then have the following:

$$\begin{aligned}
P'(Q) + QP''(Q) &= (-1/c)a^{1/c}Q^{-(1+c)/c} + Q(((1+c)/c^2)a^{1/c}Q^{-(1+2c)/c}) \\
&= (-1/c)a^{1/c}Q^{-(1+c)/c} + (((1+c)/c^2)a^{1/c}Q^{-(1+c)/c}) \\
&= (1/c^2)a^{1/c}Q^{-(1+c)/c} > 0.
\end{aligned}$$

1.2 Part (b)

Next, let N firms be competing a la Cournot.
Assumption (A1) is that $0 \geq P''(Y)y_i + P'(Y)\forall y_i < Y$.
Assumption (A2) states that $0 \geq P'(Y) - C_i''(y_i)\forall y_i < Q$. We are given that each firm has identical cost functions, so $C_i(y) = C(y)$. Note that (A2) therefore states that $C''(y_i) \geq P'(Y)$. With identical costs, in equilibrium $y_i = y = Y/N$. Using this, (A1) becomes
$0 \geq P''(Y)Y/N + P'(Y)$

Let us set up the maximization problem for each firm:

$$\begin{aligned}
&\max_{y_i} P(y_i + Y_{-i})y_i - C(y_i) \\
&\Rightarrow P'(Y)y_i + P(Y) - C'(y_i) = 0 \\
&\Rightarrow P(Y) = C'(Y/N) - P'(Y)Y/N \qquad (1)
\end{aligned}$$

Differentiating both sides with respect to N,

$$\begin{aligned}
\frac{\partial P(Y)}{\partial N} &= -C''(Y/N)(Y/N^2) + P'(Y)(Y/N^2) \\
&= (Y/N^2)(P'(Y) - C''(y)) \\
&\leq 0.
\end{aligned}$$

Now we can rewrite (1) for y and differentiate with respect to N:

$$\begin{aligned}
y &= \frac{C'(y) - P(Ny)}{P'(Ny)} \\
\frac{\partial y}{\partial N} &= -\frac{C'(y)}{(P'(Ny))^2} P''(Ny)y - y + \frac{P(Ny)}{(P'(Ny))^2} P''(Ny)y \\
&= -y - \frac{P''(Y)y}{P'(Y)^2}(P(Y) - C'(y)) \\
&= -y - \frac{P''(Y)y}{P'(Y)^2} P'(Y)y \\
&= y\left(-1 - \frac{P''(Y)y}{P'(Y)}\right) & \leq 0,
\end{aligned}$$

where the final inequality uses our rewritten form of (A1).

2 Question 2

Each bidder chooses a bid $b_i \in \mathbb{R}$ to maximize their payoffs:

$$b_i = \arg\max_b \pi(b, b_{-1}),$$

where the payoff $\pi(b, b_{-i})$ is:

$$\pi(b, b_{-i}) = \begin{cases} V - b, b > b_{-i} \\ 0, b < b_{-i} \\ (1/2)(V - b), b = b_{-i} \end{cases}.$$

The equilibrium is $b_i = V \forall i$. Why is this the case? Suppose instead player i bid $b_i > V$. Then, their payoff would be $V - b_i < 0$. Moreover, suppose $b_i < V$. Then, person i still only gets 0 payoff. So, $b_i = V \forall i$ is an equilibrium. No other equilibrium can exist. To see why, first note that equilibria can only exist with $b_i = b_{-i}$ as otherwise the bidder with the largest bid is strictly better off reducing their bid by some ϵ. If $b_i = b_{-i} < V$ then bidder i is better off increasing their bid by an epsilon and winning positive payoff. If $b_i = b_{-i} > V$ then the best response for i is to reduce their bid by an ϵ such that they are sure to receive 0 payoff instead of negative expected payoff. So, the only equilibrium is $b_i = b_{-i} = V$.

Now consider the all-pay auction. The expected payoff $\pi(b, b_{-i})$ is given by the following:

$$\pi(b, b_{-i}) = \begin{cases} V - b, b > b_{-i}, \\ -b, b < b_{-i} \\ (1/2)V - b, b = b_{-i} \end{cases}$$

First we will show that a pure strategy Nash equilibrium does not exist. Suppose it does. Then, $b_i = b_{-i}$ because otherwise the highest bidder would be strictly better off by reducing their bid by an ϵ. Consider, then, $b_i = b_{-i} = b$. If $b < V$ then either bidder would be better off increasing their bid by an ϵ and winning V surely. If $b \geq V$ then either bidder would be better off not bidding (or bidding zero). Therefore no pure strategy Nash equilibrium can exist.

We will now search for a mixed-strategy Nash equilibrium. The equilibrium solves for a pair of distributions from which bids are drawn, $(F_1(b), F_2(b)$ are the corresponding CDFs. We know that the distributions satisfy boundary conditions $F_i(0) = 0, F_i(V) = 1$ by nonnegativity of our bids and from the value of the object being V. Given these distributions, and assuming differentiability, the maximization problem and first order conditions are the following:

$$\max_b F_{-i}(b)V - b$$
$$Vf_{-i}(b) = 1$$

From this first order condition we see that the pdf $f_j(b)$ is constant across the support of the distribution, implying a uniform distribution across our support. Our CDFs then take the functional form $F_i(b) = b/V$. The seller's expected revenue is $E[R] = 2E[b_i] = 2V/2 = V$.

It is important to note that the bidders are just indifferent between mixing in this way and not bidding (or bidding zero surely). That is, each individual's $E[\pi] = P(\text{winning}_i)V - E[b_i] = (1/2)V - V/2 = 0$. If the seller sets a reserve price then (applying symmetry) the probability of winning for each bidder will not change, so their expected bid will not increase (or they would be strictly better off by not bidding at all). Since each bidder's expected bid is will not change, the expected revenue for the seller will not increase.

3 Question 3

Consumers derive $v = 3$ from coffee. Starbucks coffee locations are at locations $0, 1$ and have prices p_0, p_1 respectively. Esquire is at location 0.5 and has price q. Marginal cost is 0. Travel cost of the consumer is d^2 where d is the distance (in miles) from the consumer to the coffee shop. We first solve for the indifferent consumers, under the assumption of existence. The indifferent consumer between Starbucks 0 and Esquire has the following expressions:

$$\begin{aligned} v - p_0 - x^2 &= v - q - (0.5 - x)^2 \\ q - p_0 &= x^2 - x^2 + x - 0.25 \\ x &= (q - p_0) + (1/4) \end{aligned}$$

The indifferent consumer between Starbucks 1 and Esquire has the following similar expression:

$$\begin{aligned} v - q - (0.5 - y)^2 &= v - p_1 - (1 - y)^2 \\ p_1 - q - y^2 + y - 0.25 &= -y^2 + 2y - 1 \\ (3/4) + p_1 - q &= y. \end{aligned}$$

Given p_0, p_1, Esquire sets q to maximize profits:

$$\begin{aligned} &\max_q q(y - x) \\ &\max_q q((1/2) + p_1 - 2q + p_0) \\ &\Rightarrow (1/2) + p_1 + p_0 = 4q \\ &\Rightarrow q = 1/8 + (1/4)(p_1 + p_0) \end{aligned}$$

Given q, Starbucks sets p_1, p_0 to maximize profits:

$$\begin{aligned} &\max_{p_0, p_1} p_0 x + p_1(1 - y) \\ &\max_{p_0, p_1} p_0((q - p_0) + (1/4)) + p_1(1 - ((3/4) + p_1 - q)) \\ &\Rightarrow p_0 = q/2 + 1/8 \\ &\Rightarrow p_1 = q/2 + 1/8 = p_0 \end{aligned}$$

Combining our expressions, we reach the following:

$$\begin{aligned} q &= 1/8 + (1/2)(q/2 + 1/8) \\ q &= 1/4 \\ p_0 = p_1 &= 1/4 \end{aligned}$$

Given these prices, the indifferent consumers are at $x = 1/4, y = 3/4$. From this we can see that the market shares are split evenly between Esquire and Starbucks, each with a share of 1/2 of the market.

Now we can consider what would happen if Starbucks and Esquire were to swap houses. WLOG let Starbucks control the coffee houses at points $(1/2), 1$ and let Esquire control the coffee house at 0. Given p_0, p_1 Esquire sets q to maximize profits:

$$\begin{aligned} &\max_q qx \\ &\max_q q((p_0 - q) + 1/4) \\ &\Rightarrow q = p_0/2 + 1/8. \end{aligned}$$

Given q, Starbucks sets p_0, p_1 to solve the following:

$$\begin{aligned} &\max_{p_0,p_1} p_0(y - x) + p_1(1 - y) \\ &\max_{p_0,p_1} p_0(p_1 - p_0 + (3/4) - (1/4) - (p_0 - q)) + p_1(1 - (3/4) - p_1 + p_0) \\ &\Rightarrow p_1 - 4p_0 + (1/2) + q = 0 \\ &\Rightarrow p_0 + (1/4) - 4p_1 + p_0 = 0 \\ &\Rightarrow p_0 = (5/28) + (2/7)q \\ &\Rightarrow p_1 = (3/14) + (1/7)q \end{aligned}$$

Combining our expressions,

$$\begin{aligned} q &= (1/8) + (1/7)q + (5/56) \\ q &= 1/4 \\ p_0 &= (5/28) + (2/7)(1/2) \\ p_0 &= 9/28 \\ p_1 &= (3/14) + 1/28 \\ p_1 &= 1/4 \end{aligned}$$

In terms of market shares, Esquire now controls $x = 1/4 + p_0 - q = 1/4 + 1/14 = 9/28$. Starbucks controls $1 - x = 19/28$. Starbucks sets higher prices to earn more profit, and as a result loses on market share.

Considering sales, the result is the same as the result just derived. Prices will be set in the same way, and therefore market share will be equivalent; the sale will just result in a transfer between Starbucks and Esquire.

4 Question 4

Agents Jack and Jim select locations to maximize profits. With prices p fixed, denote $D_i(x, x_{-i})$ as demand for agent i given they decide to locate at x. Taking as given the location of the other agent, agent i sets location to maximize demand:

$$\max_x D_i(x, x_{-i}).$$

We must deduce the form of $d_i(x, x_{-i})$. First, note that with equal prices the consumers will attend the bar which is closest to them:

$$D_i(x, x_{-i}) = \begin{cases} \int_0^{(1/2)x_{-i}+(1/2)x} 1dx, x \leq x_{-i} \\ \int_{(1/2)x_{-i}+(1/2)x}^1 dx, x > x_{-i} \end{cases}.$$

In equilibrium, both establishments will be at (1/2). First note that this is indeed an equilibrium - deviation away will only reduce profits by the deviating party. Next note that any other set of locations cannot be an equilibrium. Suppose not. Then, for any other set of locations there will either be one bar with more market share, in which case the losing bar will be best off moving such that they instead control the majority of market power, or the market share will be split. If the market share is split then 2 cases are possible: the bars are symmetrically located about 1/2, in which case either bar is better off deviating towards (1/2) to increase market power, or the bars are located on the same location (not at 1/2), again at which point either bar has incentive to deviate by moving an ϵ towards 1/2 which will strictly increase market share. So, the only equilibrium is for both establishments to be located at 1/2.

No, I do not think this is optimal, the social planner would like to distribute the bars in different places to reduce total costs. The planner solves:

$$\min_{a,b} \int_0^{1/2} (i-a)^2 di + \int_{1/2}^1 (i-b)^2 di$$
$$\min_{a,b} (1/24) - a/4 + a^2/2 + (7/24) - (3/4)b + b^2/2$$

Taking FOCs,

$$(1/4) = a$$
$$(3/4) = b.$$

So, the social planner would locate one bar at 1/4 and the other at 3/4.

5 Question 5