Macro PS5

Michael B. Nattinger[*]

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1 Question 1

We will begin with deriving the flexible price version of the model and add in the price stickiness after. Households maximize utility subject to their budget constraint. The first order conditions with respect to consumption, labor, and bond holdings yields the labor supply equation and Euler equation:

$$C_t = W_t/P_t \tag{1}$$

$$1 = E_t \left[\beta \frac{C_t}{C_{t+1}} \frac{P_t}{P_{t+1}} (1 + i_t) \right] \tag{2}$$

Note that, under flexible prices, disutility of labor drops out from the labor supply equation. Labor supplied will always equal $L_t = 1$. In other words, under flexible prices and the given utility formulation, the labor supply is perfectly inelastic and set to 1.

Under fully flexible prices, firms profit maximization yields the following optimal pricing equations a la Dixit-Stiglitz:

$$P_{it} = \frac{\theta}{\theta - 1} \frac{W_t}{A_t} \tag{3}$$

$$P_t = \left(\int P_{it}^{1-\theta} di \right)^{\frac{1}{1-\theta}} \tag{4}$$

$$Y_t = C_t = A_t L_t \tag{5}$$

Note that (**??**) and symmetry across firms implies that $P_{it} = P_t$. Note that we also have that

In the deterministic steady state of the economy, the above equations (**??**),(**??**),(**??**),(**??**) imply the following:

$$\bar{C} = \bar{W}/\bar{P} \tag{6}$$

$$1 = \beta(1 + \bar{i}) \tag{7}$$

$$\bar{P}_i = \frac{\theta}{\theta - 1} \frac{\bar{W}}{\bar{A}} \tag{8}$$

$$\bar{Y} = \bar{C} = \bar{A}\bar{L} \tag{9}$$

Note that (**??**),(**??**),(**??**) imply that $\bar{L} = \frac{\theta - 1}{\theta}$

Linearized dynamics of the optimality conditions (**??**),(**??**),(**??**),(**??**) about the steady state defined by (**??**),(**??**),(**??**),(**??**) are the following:

$$c_t + p_t = w_t$$
$$E_t[c_t - c_{t+1} + p_t - p_{t+1} + i_t] = 0$$
$$p_t = w_t - a_t$$
$$c_t = a_t + l_t$$

We can simplify the above equations to find the linearized consumption dynamics about the steady state:

$$c_t = a_t, \tag{10}$$
$$l_t = 0. \tag{11}$$

In other words, our consumption log deviation from steady state follows the productivity deviation exactly in the first order approximation, and labor does not deviate.

2 Question 2

We now consider the world of sticky prices. Our household dynamics (**??**),(**??**) still hold, however the firm dynamics change so we will explicitly solve the firm problem. From (**??**) we see that we can define the SDF as $\Theta_{t,t+j} := \beta^j \frac{C_t P_t}{C_{t+j} P_{t+j}}$.

The firm maximizes expected (discounted) profits, taking for granted the decision making of the individual:

$$\max_{P_{it}} E_t \sum_{j=0}^{\infty} \Theta_{t,t+j} \left(P_{it+j} C_{it+j} - W_{t+j} L_{it+j} - \frac{\varphi W_{t+j}}{2} \left(\frac{P_{it+j} - P_{it+j-1}}{P_{it+j-1}} \right)^2 \right)$$

$$\text{s.t.} C_{it} = \left(\frac{P_{it}}{P_t} \right)^{-\theta} C_t, C_{it} = A_t L_{it}$$

We substitute in the constraints to the maximization problem:

$$\max_{P_{it}} E_t \sum_{j=0}^{\infty} \beta^j \frac{C_t P_t}{C_{t+j} P_{t+j}} \left(P_{it+j}^{1-\theta} P_{t+j}^{\theta} C_{t+j} - \frac{W_{t+j}}{A_{t+j}} \left(\frac{P_{it+j}}{P_{t+j}} \right)^{-\theta} C_{t+j} - \frac{\varphi W_t}{2} \left(\frac{P_{it+j}}{P_{it+j-1}} - 1 \right)^2 \right)$$

We take first order conditions with respect to P_{it}:

$$(1-\theta)C_{it} + \theta \frac{W_t}{A_t} C_{it} P_{it}^{-1} - \frac{\varphi W_t}{P_{it-1}} \left(\frac{P_{it}}{P_{it-1}} - 1 \right) = -E_t \left[\Theta_{t,t+1} \frac{\varphi W_{t+1} P_{it+1}}{P_{it}^2} \left(\frac{P_{it+1}}{P_{it}} - 1 \right) \right]$$

$$\Rightarrow (1-\theta)C_{it}P_{it} + \theta \frac{W_t}{A_t} C_{it} - \varphi W_t \frac{P_{it}}{P_{it-1}} \left(\frac{P_{it}}{P_{it-1}} - 1 \right) = -E_t \left[\Theta_{t,t+1} \frac{\varphi W_{t+1} P_{it+1}}{P_{it}} \left(\frac{P_{it+1}}{P_{it}} - 1 \right) \right]$$

Due to symmetry, all producers act the same $P_{it} = P_t, C_{it} = C_t \forall i, t.$[1] Moreover, define $\pi_t = \frac{P_t}{P_{t-1}} - 1$. We can rewrite the first order condition as follows:

$$(1-\theta)C_{it}P_{it} + \theta \frac{W_t}{A_t} C_{it} = \varphi E_t \left[W_t(\pi_t^2 + \pi_t) - W_{t+1}\Theta_{t,t+1} \frac{P_{t+1}}{P_t} \pi_{t+1} \right]$$

$$(1-\theta)P_{it} + \theta \frac{W_t}{A_t} = \varphi E_t \left[\frac{W_t}{C_t}(\pi_t^2 + \pi_t) - \frac{\beta W_{t+1}}{C_{t+1}} \pi_{t+1} \right]$$

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[1]Let $P_{it} = K_t \forall i.$ $P_t = \left(\int_0^1 P_{kt}^{1-\theta} dk \right)^{\frac{1}{1-\theta}} = K_t \left(\int_0^1 dk \right)^{\frac{1}{1-\theta}} = K_t = P_{it}.$ A similar argument shows $C_{it} = C_t.$

Note that when $\varphi = 0$ (flexible prices), the right hand side drops out and we are left with the Dixit-Stiglit price equation (??). We will now log-linearize the above expression as follows:

$$(1 - \theta)P_t + \theta \frac{W_t}{A_t} = \varphi E_t \left[\frac{W_t}{C_t}(\pi_t^2 + \pi_t) - \frac{\beta W_{t+1}}{C_{t+1}} \pi_{t+1} \right]$$

$$X_t = \varphi E_t Z_t$$

$$X_t = (1 - \theta)P_t + \theta \frac{W_t}{A_t}$$

Note that $\bar{\pi} = 0 \Rightarrow \bar{X} = 0, \bar{Y} = 0$.

$$x_t = (1 - \theta)\bar{P}p_t + \theta \bar{W}\bar{A}^{-1}(w_t - a_t),$$
$$Z_t = \frac{W_t}{C_t}(\pi_t^2 + \pi_t) - \frac{\beta W_{t+1}}{C_{t+1}}\pi_{t+1}$$
$$= H_t - Q_t$$
$$H_t = W_t \frac{\pi_t^2 + \pi_t}{C_t}$$
$$= W_t I_t / C_t$$
$$I_t = \pi_t^2 + \pi_t$$
$$i_t = \pi_t$$
$$h_t = i_t \frac{\bar{W}}{\bar{C}}(1 + w_t - c_t)$$
$$= \bar{W}\bar{C}i_t$$
$$Q_t = \frac{\beta W_{t+1}}{C_{t+1}}\pi_{t+1}$$
$$q_t = \beta \bar{W}\bar{C}^{-1}\pi_{t+1}(1 + w_{t+1} - c_{t+1})$$
$$= \beta \bar{W}\bar{C}^{-1}\pi_{t+1}$$
$$(1 - \theta)\bar{P}p_t + \theta\bar{W}\bar{A}^{-1}(w_t - a_t) = \varphi E_t[\bar{W}\bar{C}^{-1}\pi_t - \beta\bar{W}\bar{C}^{-1}\pi_{t+1}]$$

From our log linearized labor supply equation and market clearing for the consumption good, $c_t + p_t = w_t, c_t = y_t$:

$$(1 - \theta)\bar{P}p_t + \theta\bar{W}\bar{A}^{-1}(y_t + p_t - a_t) = \varphi E_t[\bar{W}\bar{C}^{-1}\pi_t - \beta\bar{W}\bar{C}^{-1}\pi_{t+1}]$$
$$((1 - \theta)\bar{P} + \theta\bar{W}\bar{A}^{-1})p_t + \theta\bar{W}\bar{A}^{-1}(y_t - a_t) = \varphi E_t[\bar{W}\bar{C}^{-1}\pi_t - \beta\bar{W}\bar{C}^{-1}\pi_{t+1}]$$

Noting that $\bar{X} = 0 \Rightarrow (1 - \theta)\bar{P} + \theta\bar{W}\bar{A}^{-1} = 0$, $\bar{C} = \bar{L}\bar{A}$, and $\bar{L} = \frac{\theta - 1}{\theta}$, we have our NKPC: inflation written as a function of the output gap and expected future inflation.

$$\frac{\theta - 1}{\varphi}(y_t - a_t) + \beta E_t[\pi_{t+1}] = \pi_t. \tag{12}$$

In class, we derived a different NKPC using a Calvo pricing model:

$$\frac{(1 - \lambda)(1 - \beta\lambda)(\sigma + \phi)}{\lambda}\left(y_t - \frac{1 + \phi}{\sigma + \phi}a_t\right) + \beta E_t[\pi_{t+1}] = \pi_t$$

$$\frac{(1 - \lambda)(1 - \beta\lambda)}{\lambda}(y_t - a_t) + \beta E_t[\pi_{t+1}] = \pi_t,$$

where we have substituted our values of σ, ϕ. The differences between the curves are a different formulation for the coefficient on the output gap, owing to the different mechanisms for price stickiness. In our case we do not have the 'Calvo fairy' linking inflation to output, but rather have the price adjustment cost, with labor needing to be hired to adjust prices.

3 Question 3

The mechanism is quite different. In the Calvo model, the source of inflation costs is the labor misallocation across firms - some of the firms are not being allowed by the 'Calvo fairy' to change their prices, which causes labor to be misallocated. In this model, the market is completely symmetric so there is no misallocation in this dimension. Instead, some of the workers are paid to change prices rather than produce goods, and this reduces the effective labor level when prices change.

4 Question 4

If we define $x_t = y_t - a_t$ as our output gap, and further define $\kappa = \frac{\theta - 1}{\varphi}$, then our NKPC becomes the following:

$$\kappa x_t + \beta E_t[\pi_{t+1}] = \pi_t.$$

Now, our EE is the following:

$$E_t(c_{t+1} - c_t) = i_t - E_t[\pi_{t+1}]$$

If we define the natural rate r_t^n to be the real rate that prevails under flexible prices:

$$r_t^n = E_t(a_{t+1} - a_t)$$
$$\Rightarrow E_t(x_{t+1} - x_t) = i_t - E_t[\pi_{t+1}] - r_t^n$$

Assume monetary policy follows the Taylor rule: $i_t = \phi x_t + u_t$. Then, our Euler becomes the following:

$$E_t(x_{t+1} - x_t) = \phi x_t - E_t[\pi_{t+1}] + u_t - r_t^n$$

We can write down our law of motion and apply the Blanchard-Kahn method:

$$E_t \begin{pmatrix} \pi_{t+1} \\ x_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} \\ -\frac{1}{\beta} & \phi + 1 + \frac{\kappa}{\beta} \end{pmatrix} \begin{pmatrix} \pi_t \\ x_t \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (u_t - r_t^n)$$

We know that the eigenvalues of the matrix describing the law of motion must both have magnitude greater than 1. Denote $F(\lambda)$ as follows:

$$F(\lambda) = \left| \begin{pmatrix} \frac{1}{\beta} - \lambda & -\frac{\kappa}{\beta} \\ -\frac{1}{\beta} & \phi + 1 + \frac{\kappa}{\beta} - \lambda \end{pmatrix} \right|$$
$$= \left(\frac{1}{\beta} - \lambda \right) \left(\phi + 1 + \frac{\kappa}{\beta} - \lambda \right) - \frac{\kappa}{\beta^2}$$
$$= \lambda^2 - \left(\phi + 1 + \frac{\kappa}{\beta} + \frac{1}{\beta} \right) \lambda + \frac{\phi + 1}{\beta}$$

Similarly to class, for the model to have a unique solution, the following two conditions must hold: $F'(1) < 0, F(1) > 0$. Note that $F'(1) = 1 - \phi - \frac{\kappa}{\beta} - \frac{1}{\beta}$ so our first inequality holds so long as $\phi > 1 - \frac{\kappa+1}{\beta}$. Moreover,

$$F(1) = 1 - \left(\phi + 1 + \frac{\kappa}{\beta} + \frac{1}{\beta} \right) + \frac{\phi + 1}{\beta}$$
$$= \phi \left(\frac{1}{\beta} - 1 \right) - \frac{\kappa}{\beta}$$
$$> 0 \iff \phi > \frac{\kappa}{1 - \beta}$$

Therefore, we have a unique solution so long as $\phi > \max\left\{1 - \frac{\kappa+1}{\beta}, \frac{\kappa}{1-\beta}\right\}$. Note that $\frac{\kappa}{1-\beta} > 0 > 1 - \frac{\kappa+1}{\beta}$, so $\phi > \frac{\kappa}{1-\beta}$ is necessary and sufficient for the model to have a unique solution.