Macro PS2

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1 Question 1

The planner solves the following maximization problem subject to the capital law of motion and the resource constraint:

$$\max_{\{C_t, I_t, K_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t logC_t$$
$$\text{s.t. } K_{t+1} = K_t^{1-\delta} I_t^{\delta}$$
$$\text{and } AK_t^{\alpha} = C_t + I_t$$

We can solve the resource constraint for I_t and plug it into the capital law of motion. Using this simplification, we can write down our Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t logC_t + \lambda_t \left(-K_{t+1} + K_t^{1-\delta} \left(AK_t^{\alpha} - C_t \right)^{\delta} \right)$$

Taking first order conditions with respect to C_t, K_{t+1} we find the following:

$$\begin{aligned}
\frac{\beta^t}{C_t} &= \lambda_t \delta K_t^{1-\delta} (AK_t^{\alpha} - C_t)^{\delta-1} \\
\lambda_t &= \lambda_{t+1} (K_{t+1}^{1-\delta} \delta (AK_{t+1}^{\alpha} - C_{t+1})^{\delta-1} A\alpha K_{t+1}^{\alpha-1} + (1-\delta) K_{t+1}^{-\delta} \left(AK_{t+1}^{\alpha} - C_{t+1} \right)^{\delta}) \\
\Rightarrow \lambda_t &= \frac{\beta^t}{\delta C_t K_t^{1-\delta} I_t^{\delta-1}} \\
\Rightarrow \frac{1}{C_t K_t^{1-\delta} I_t^{\delta-1}} &= \frac{\beta}{C_{t+1} K_{t+1}^{1-\delta} I_{t+1}^{\delta-1}} (A\alpha\delta K_{t+1}^{\alpha-\delta} I_{t+1}^{\delta-1} + (1-\delta) K_{t+1}^{-\delta} I_{t+1}^{\delta})
\end{aligned}$$

$$\frac{1}{C_t K_t^{1-\delta} I_t^{\delta-1}} = \frac{\beta}{C_{t+1}} (A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta) K_{t+1}^{-1} I_{t+1}) \tag{1}$$

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, Katherine Kwok, and Danny Edgel.

The above equation forms our Euler equation.

Assume we are on the optimal trajectory at time t, and consider a one-period deviation in consumption by an amount D. Our resource constraint implies that this results in a decrease in I_t by an equal amount, D. Then, our K_{t+1} is reduced (to first order approximation) by $-\delta D K_t^{1-\delta} I_t^{\delta-1}$. Then, our consumption in the second equation is reduced by two effects: reduced K_{t+1} leads to less production at time $t+1$, and a larger gap to make up via I_{t+1} to get back onto the optimal trajectory at time $t+2$. The net effect of the first of these terms, to first order expansion, is $-(\delta D K_t^{1-\delta} I_t^{\delta-1})(A\alpha K_{t+1}^{\alpha-1})$, in other words, the reduction in C_{t+1} from the (first order approximation of the) decrease in production in period (t+1). Now we must address the second of these turns. $K_{t+2} = K_{t+1}^{1-\delta} I_{t+1}^{\delta}$ is fixed and we know the value of K_{t+1} so we can determine the value of I_{t+1}. To first order approximation, small deviations of capital and investment $(\Delta K_{t+1}), (\Delta I_{t+1})$ satisfy $(1-\delta)((\Delta K_{t+1}))(K_{t+1}^{-\delta} I_{t+1}^{\delta}) = -\delta(\Delta I_{t+1})(K_{t+1}^{1-\delta} I_{t+1}^{\delta-1}) \Rightarrow (\Delta I_{t+1}) = -\frac{1-\delta}{\delta}(I_{t+1} K_{t+1}^{-1})(\Delta K_{t+1})$. This is taken away from C_{t+1}. Therefore, our second effect of the reduction in K_{t+1} on C_{t+1} is $-(\delta \Delta K_t^{1-\delta} I_t^{\delta-1}) \frac{1-\delta}{\delta} \frac{I_{t+1}}{K_{t+1}}$.

Our marginal utility by making this move is thus

$$dU = \beta^t C_t^{-1} D - \beta^{t+1} C_{t+1}^{-1} \left((\delta K_t^{1-\delta} I_t^{\delta-1}) \left(A\alpha K_{t+1}^{\alpha-1} + \frac{1-\delta}{\delta} \frac{I_{t+1}}{K_{t+1}} \right) \right) D$$

$$dU = 0 \Rightarrow C_t^{-1} = \beta C_{t+1}^{-1} (K_t^{1-\delta} I_t^{\delta-1}) \left(A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)\frac{I_{t+1}}{K_{t+1}} \right)$$

This yields (1), our euler condition. Therefore, the euler condition represents a no-profitable-deviation condition.

2 Question 2

The system of equations that pins down the law of motion for the system are the following:

$$\begin{aligned} \frac{1}{C_t K_t^{1-\delta} I_t^{\delta-1}} &= \frac{\beta}{C_{t+1}} (A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta) K_{t+1}^{-1} I_{t+1}) \\ A K_t^{\alpha} &= C_t + I_t \\ K_{t+1} &= K_t^{1-\delta} I_t^{\delta} \end{aligned}$$

We can use the resource constraint to rewrite the system of equations without I_t:

$$C_{t+1} = \beta C_t K_t^{1-\delta} (A K_t^{\alpha} - C_t)^{\delta-1} (A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta) K_{t+1}^{-1} (A K_{t+1}^{\alpha} - C_{t+1})) \quad (2)$$

$$K_{t+1} = K_t^{1-\delta} (A K_t^{\alpha} - C_t)^{\delta} \quad (3)$$

Equations (2) and (3) determine the law of motion of the system. We can use these equations and impose stationarity ($K_t = K_{t+1} = \bar{K}, C_t = C_{t+1} = \bar{C}$) to determine the steady state:

$$
\begin{aligned}
1 &= \beta \bar{K}^{1-\delta}(A\bar{K}^{\alpha}-\bar{C})^{\delta-1}(A\alpha\delta\bar{K}^{\alpha-1}+(1-\delta)\bar{K}^{-1}(A\bar{K}^{\alpha}-\bar{C}))\\
1 &= \bar{K}^{-\delta}(A\bar{K}^{\alpha}-\bar{C})^{\delta}
\end{aligned}
$$

The above equations pin down the steady state of the model.

3 Question 3

We will log linearize about the steady state defined in Question 2. We first will define $I = AK^{\alpha} - C$. Log linearizing I we get:

$$
\begin{aligned}
\bar{I}(1+i_t) &= A\bar{K}^{\alpha}(1+\alpha k_t) - \bar{C}(1+c_t)\\
\Rightarrow i_t &= A\frac{\bar{K}^{\alpha}}{\bar{I}}k_t - \frac{\bar{C}}{\bar{I}}c_t\\
\Rightarrow i_t &= A\alpha\bar{K}^{\alpha-1}k_t - \frac{\bar{C}}{\bar{I}}c_t,
\end{aligned}
$$

where we have used the fact that equation (3) implies that $\bar{K} = \bar{I}$.

Using this we can log linearize equation (3):

$$
\begin{aligned}
\bar{K}(1+k_{t+1}) &= \bar{K}^{1-\delta}(1+(1-\delta)k_t)\bar{I}^{\delta}(1+\delta i_t)\\
\Rightarrow k_{t+1} &= (1-\delta)k_t + \delta i_t\\
&= (1-\delta)k_t + \delta\left(A\alpha\bar{K}^{\alpha-1}k_t - \frac{\bar{C}}{\bar{I}}c_t\right)\\
&= (1-\delta + A\alpha\bar{K}^{\alpha-1}\delta)k_t - \delta\frac{\bar{C}}{\bar{I}}c_t
\end{aligned}
$$

Now we can log linearize equation (2):

$$
\begin{aligned}
(1+c_{t+1}) &= \beta(1+c_t)(1+(1-\delta)k_t)(1+(\delta-1)i_t)\\
&\quad * (A\alpha\delta\bar{K}^{\alpha-1}(1+(\alpha-1)k_{t+1}) + (1-\delta)(1-k_{t+1})(1+i_{t+1}))\\
c_{t+1} &= \beta(A\alpha\delta\bar{K}^{\alpha-1}(\alpha-1)k_{t+1} + (1-\delta)(i_{t+1}-k_{t+1})\\
&\quad + (A\alpha\delta\bar{K}^{\alpha-1} + (1-\delta))(c_t + (1-\delta)k_t + (\delta-1)i_t))\\
c_{t+1} &= \frac{A\alpha\delta\bar{K}^{\alpha-1}(\alpha-1)}{(A\alpha\delta\bar{K}^{\alpha-1}+(1-\delta))}k_{t+1} + \frac{(1-\delta)}{(A\alpha\delta\bar{K}^{\alpha-1}+(1-\delta))}i_{t+1}\\
&\quad - \frac{(1-\delta)}{(A\alpha\delta\bar{K}^{\alpha-1}+(1-\delta))}k_{t+1} + c_t + (1-\delta)k_t + (\delta-1)i_t
\end{aligned}
$$

4 Question 4

Define, for sake of convenience, $\psi = A\alpha\delta\bar{K}^{\alpha-1}, \gamma = (1-\delta), \phi = A\bar{K}^{\alpha-1}$. Note that the euler equation steady state yields $\gamma + \psi = 1/\beta$, and the investment steady state yields

$\bar{C}/\bar{K} = \phi - 1$. Then, we can reduce the above equation to the following:

$$\begin{aligned}
c_{t+1} &= \beta(\psi(\alpha-1)-\gamma)k_{t+1} + \beta\gamma i_{t+1} + c_t + \gamma k_t - \gamma i_t, \\
&= \beta(\psi(\alpha-1)-\gamma)k_{t+1} + \beta\gamma\left(\frac{\psi}{\delta}k_{t+1} - (\phi-1)c_{t+1}\right) + c_t + \gamma k_t - \gamma\left(\frac{\psi}{\delta}k_t - (\phi-1)c_t\right)
\end{aligned}$$

$$\Rightarrow c_{t+1}\left(1+\beta\gamma(\phi-1)\right) = \beta(\psi(\alpha-1)+\gamma(\psi/\delta-1))k_{t+1} + (2-\phi)c_t + \gamma\left(1-\frac{\psi}{\delta}\right)k_t$$

$$\begin{aligned}
c_{t+1}\left(1+\beta\gamma(\phi-1)\right) &= \beta(\psi(\alpha-1)+\gamma(\psi/\delta-1))\left(\beta^{-1}k_t - \delta(\phi-1)c_t\right) \\
&\quad + (2-\phi)c_t + \gamma\left(1-\frac{\psi}{\delta}\right)k_t \\
&= ((\psi(\alpha-1)+\gamma(\psi/\delta-1)) + \gamma - \gamma\phi\alpha)k_t \\
&\quad + (2-\phi-\delta(\phi-1)\beta(\psi(\alpha-1)+\gamma(\psi/\delta-1)))c_t \\
&= \phi(\alpha-1)k_t \\
&\quad + (2-\phi-\delta(\phi-1)\beta(\psi(\alpha-1)+\gamma(\psi/\delta-1)))c_t
\end{aligned}$$

$$\begin{aligned}
c_{t+1} &= \frac{\phi(\alpha-1)}{1+\beta\gamma(\phi-1)}k_t \\
&\quad + \frac{2-\phi-\delta(\phi-1)\beta(\psi(\alpha-1)+\gamma(\psi/\delta-1))}{1+\beta\gamma(\phi-1)}c_t
\end{aligned}$$

5 Question 5

Outline: From (4) find η such that $c_t = \eta k_t \Rightarrow C_t = ZK_t$ for some Z. Plug into euler and LOM and solve to show that euler is satisfied.

6 Question 6

Let us first write the planner's problem. We will jump immediately to the lagrangian formulation:

$$\mathcal{L} = E_t \sum_{t=0}^{\infty} \beta^t logC_t + \lambda_t\left(-K_{t+1} + K_t^{1-\delta}\left(A_t K_t^{\alpha} - C_t\right)^{\delta}\right)$$

This yields the following first order conditions:

$$\begin{aligned}
\frac{\beta^t}{C_t} &= \lambda_t \delta K_t^{1-\delta}(A_t K_t^\alpha - C_t)^{\delta-1} \\
\lambda_t &= E_t \lambda_{t+1}(K_{t+1}^{1-\delta}\delta(A_{t+1}K_{t+1}^\alpha - C_{t+1})^{\delta-1} A_{t+1}\alpha K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-\delta}\left(A_{t+1}K_{t+1}^\alpha - C_{t+1}\right)^\delta) \\
\Rightarrow \lambda_t &= \frac{\beta^t}{\delta C_t K_t^{1-\delta} I_t^{\delta-1}} \\
\Rightarrow \frac{1}{C_t K_t^{1-\delta} I_t^{\delta-1}} &= E_t\left[\frac{\beta}{C_{t+1}}(A_{t+1}\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-1} I_{t+1})\right]
\end{aligned}$$

The above expression forms our euler condition for the stochastic case.

Outline: Guess small change for solution to (6) adjusting for stochastic case and show that this guess satisfies the euler condition above.

7 Question 7

Our solutions for Question 5 and Question 6 show that consumption and capital levels are perfectly correlated. This comes from the formulation for capital law of motion, which ensures that investment is perfectly correlated with capital levels, and therefore consumption will also be perfectly correlated with capital levels via the resource constraint.