Macro PS5

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1 Question 1

The planner solves the following optimization problem:

$$\max_{x_t,\pi_t,i_t} \frac{1}{2} E \sum_{t=0}^{\infty} \beta^t (x_t^2 + \alpha \pi_t^2)$$
$$\text{s.t.} \sigma E_t \Delta x_{t+1} = i_t - E_t \pi_{t+1} - r_t^n,$$
$$\text{and } \pi_t = \kappa x_t + \beta E_t \pi_{t+1} + u_t$$

We consider the primal approach:

$$\max_{x_t,\pi_t} \frac{1}{2} E \sum_{t=0}^{\infty} \beta^t (x_t^2 + \alpha \pi_t^2)$$
$$\text{and } \pi_t = \kappa x_t + \beta E_t \pi_{t+1} + u_t$$

$$\mathcal{L} = E \sum_{t=0}^{\infty} \left[(1/2) \beta^t (x_t^2 + \alpha \pi_t^2) + \lambda_t (\pi_t - \kappa x_t - \beta \pi_{t+1} - u_t) \right]$$

$$\beta^t x_t = \lambda_t k$$
$$\beta^t \alpha \pi_t + \lambda_t - \beta \lambda_{t-1} = 0, t \geq 1$$
$$\beta^t \alpha \pi_t + \lambda_t = 0, t \geq 1$$

$$\alpha \kappa \pi_t + \Delta x_t = 0, t \geq 1$$
$$\alpha \kappa \pi_0 + x_0 = 0.$$

In class, at this point we defined $\hat{p}_t := p_t - p_{-1}$. However, in this question we are asked about commitment with the timeless perspective. Following Woodford (1999), we define $p_{-1} = 0 \Rightarrow \hat{p}_t = p_t$. Therefore, we can proceed just as we did in class without having to carry around the hats on p_t. Note that now, following class, we also $x_{-1} := 0$. Then, we can combine our above two equations into one that holds for all t:

$$\alpha \kappa \pi_t + \Delta x_t = 0.$$

Since the above holds for all t, it follows that $-\alpha \kappa p_t = x_t$ for all t, which can be easily shown via induction.

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, Katherine Kwok, and Danny Edgel.

We can plug this into our constraint, the NKPC curve:

$$
\begin{aligned}
p_t - p_{t-1} &= -\alpha\kappa^2 p_t + \beta E_t p_{t+1} - \beta p_t + u_t \\
-\beta E_t p_{t+1} &= (-1 - \alpha\kappa^2 - \beta) p_t + p_{t-1} + u_t \\
\begin{pmatrix} -\beta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E_t p_{t+1} \\ p_t \end{pmatrix} &= \begin{pmatrix} -1 - \alpha\kappa^2 - \beta & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_t \\
\begin{pmatrix} E_t p_{t+1} \\ p_t \end{pmatrix} &= \begin{pmatrix} -1/\beta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 - \alpha\kappa^2 - \beta & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \end{pmatrix} + \begin{pmatrix} -1/\beta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_t \\
\begin{pmatrix} E_t p_{t+1} \\ p_t \end{pmatrix} &= \begin{pmatrix} \frac{1}{\beta} + \frac{\alpha}{\beta}\kappa^2 + 1 & -\frac{1}{\beta} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \end{pmatrix} + \begin{pmatrix} -\frac{1}{\beta} \\ 0 \end{pmatrix} u_t
\end{aligned}
$$

We can find the eigenvalues of the matrix above, that satisfy the following:

$$
-\beta\lambda^2 + (1 + \alpha\kappa^2 + \beta)\lambda - 1 = 0
$$

This has two roots, one above and one below 1 in magnitude, corresponding to the fact that we have one state and one control variable. WLOG let $\lambda_1 > 1$. By using the quadratic formula and multiplying the roots you can easily show that $\lambda_1 \lambda_2 = \beta^{-1}$.

We can now write the equation in the following way:

$$
\begin{aligned}
-\beta(1 - \lambda_1 L)(1 - \lambda_2 L)L^{-1} p_t &= u_t \\
(\beta\lambda_1 - \beta L^{-1})(1 - \lambda_2 L) p_t &= u_t \\
(1 - \beta\lambda_2 L^{-1})(1 - \lambda_2 L) p_t &= \lambda_2 u_t \\
(1 - \lambda_2 L) p_t &= \lambda_2 (1 - \beta\lambda_2 L^{-1})^{-1} u_t
\end{aligned}
$$

We are given that $u_t \sim \text{iid}(\bar{u}, \sigma^2)$. We then rewrite the above equation as the following:

$$
\begin{aligned}
p_t &= \lambda_2 p_{t-1} + \lambda_2 E_t \sum_{j=0}^{\infty} (\beta\lambda_2)^j u_{t+j} \\
p_t &= \lambda_2 p_{t-1} + \lambda_2 \left(u_t + \bar{u} \frac{\beta\lambda_2}{1 - \beta\lambda_2} \right), \qquad (1) \\
x_t &= \lambda_2 x_{t-1} - \lambda_2 \alpha\kappa \left(u_t + \bar{u} \frac{\beta\lambda_2}{1 - \beta\lambda_2} \right). \qquad (2)
\end{aligned}
$$

The equations (1),(2) determine the dynamics of the price level and output gap.

2 Question 2

Under discretion, we have that $\alpha\kappa\pi_t + x_t = 0$ in each period. We also know that NKPC holds:

$$\begin{aligned}
\pi_t &= \kappa x_t + \beta E_t \pi_{t+1} + u_t \\
&= -\alpha\kappa^2 \pi_t + \beta E_t \pi_{t+1} + u_t \\
&= \frac{\beta}{1+\alpha\kappa^2} E_t \pi_{t+1} + \frac{1}{1+\alpha\kappa^2} u_t \\
&= \frac{1}{1+\alpha\kappa^2} E_t \sum_{j=0}^{\infty} \left(\frac{\beta}{1+\alpha\kappa^2} \right)^j u_{t+j} \\
&= \frac{u_t}{1+\alpha\kappa^2} + \frac{\beta}{1+\alpha\kappa^2} \frac{\bar{u}}{1 - \frac{\beta}{1+\alpha\kappa^2}} \\
\pi_t &= \frac{u_t}{1+\alpha\kappa^2} + \frac{\beta\bar{u}}{1+\alpha\kappa^2-\beta}, \\
x_t &= -\alpha\kappa \frac{u_t}{1+\alpha\kappa^2} - \alpha\kappa \frac{\beta\bar{u}}{1+\alpha\kappa^2-\beta}.
\end{aligned}$$

3 Question 3

Under the inflation targeting rule, $\pi_t = 0$ and our NKPC curve states the following:

$$x_t = -\frac{u_t}{\kappa}.$$

4 Question 4

Under output targeting rule, $x_t = 0$ and our NKPC curve states the following:

$$\begin{aligned}
\pi_t &= \beta E_t \pi_{t+1} + u_t \\
&= E_t \sum_{j=0}^{\infty} \beta^j u_{t+j} \\
&= u_t + \frac{\beta\bar{u}}{1-\beta}.
\end{aligned}$$

5 Question 5

We can consider the welfare implications of the two regimes to determine the optimal policy. Under inflation targeting, our welfare losses are the following:

$$\begin{aligned}
\mathcal{W}^{\pi} &= \frac{1}{2} E \sum_{t=0}^{\infty} \beta^t \frac{u_t^2}{\kappa^2} \\
&= \frac{1}{2\kappa^2} \sum_{t=0}^{\infty} \beta^t E[u_t^2] \\
&= \frac{\bar{u}^2 + \sigma^2}{2\kappa^2(1-\beta)}.
\end{aligned}$$

Under discretion, our welfare losses are the following:

$$
\begin{aligned}
\mathcal{W}^D &= \frac{\alpha(1+\alpha\kappa^2)}{2} E \sum_{t=0}^{\infty} \beta^t \left(\frac{u_t}{1+\alpha\kappa^2} + \frac{\beta\bar{u}}{1+\alpha\kappa^2-\beta} \right)^2 \\
&= \frac{\alpha(1+\alpha\kappa^2)}{2} E \sum_{t=0}^{\infty} \beta^t \left[\frac{u_t^2}{(1+\alpha\kappa^2)^2} + \frac{2\beta\bar{u}u_t}{(1+\alpha\kappa^2)(1+\alpha\kappa^2-\beta)} + \frac{\beta^2\bar{u}^2}{(1+\alpha\kappa^2-\beta)^2} \right] \\
&= \frac{\alpha(1+\alpha\kappa^2)}{2} \sum_{t=0}^{\infty} \beta^t \left[\frac{\bar{u}^2+\sigma^2}{(1+\alpha\kappa^2)^2} + \frac{2\beta\bar{u}^2}{(1+\alpha\kappa^2)(1+\alpha\kappa^2-\beta)} + \frac{\beta^2\bar{u}^2}{(1+\alpha\kappa^2-\beta)^2} \right] \\
&= \frac{\alpha(1+\alpha\kappa^2)}{2(1-\beta)} \left[\left(\frac{1}{(1+\alpha\kappa^2)^2} + \frac{2\beta}{(1+\alpha\kappa^2)(1+\alpha\kappa^2-\beta)} + \frac{\beta^2}{(1+\alpha\kappa^2-\beta)^2} \right) \bar{u}^2 + \frac{\sigma^2}{(1+\alpha\kappa^2)^2} \right] \\
&= \frac{\alpha}{2(1-\beta)} \left[\left(\frac{1}{1+\alpha\kappa^2} + \frac{2\beta}{1+\alpha\kappa^2-\beta} + \frac{(1+\alpha\kappa^2)\beta^2}{(1+\alpha\kappa^2-\beta)^2} \right) \bar{u}^2 + \frac{\sigma^2}{(1+\alpha\kappa^2)} \right]
\end{aligned}
$$

6 Question 6

7 Question 7