Econometrics HW5

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1 Question 1

1.1 $a_n = 1/n$

Let $\epsilon > 0$. Let N be the smallest integer such that $N > 1/\epsilon$. Then, $|a_n - 0| = 1/n < \epsilon \ \forall n > N$.

1.2 $a_n = \frac{1}{n} sin(n\pi/2)$

Let $\epsilon > 0$. Let N be the smallest integer such that $N > 1/\epsilon$. Then, $|a_n - 0| = |\frac{1}{n} sin(n\pi/2)| \leq \frac{1}{n} < \epsilon \ \forall n > N$.

2 Question 2

2.1 Does $X_n \rightarrow_p 0$ as $n \rightarrow 0$?

Let $\epsilon > 0$. Let N be the smallest integer such that $N > \epsilon$. Then, for $n > N$, $P(|X_n| \geq \epsilon) = 2/n$ so $\lim_{n\rightarrow\infty} P(|X_n| \geq \epsilon) = \lim_{n\rightarrow\infty} 2/n = 0$, so $X_n \rightarrow_p 0$.

2.2 Calculate $E(X_n)$.

$E(X_n) = -n(1/n) + 0(1 - 2/n) + n(1/n) = 0$.

2.3 Calculate $Var(X_n)$.

$Var(X_n) = E(X_n^2) - E(X_n)^2 = (n^2)(1/n) + (0^2)(1 - 2/n) + (n^2)(1/n) - 0^2 = 2n$.

2.4 Calculate X_n for the next distribution.

$E(X_n) = (0)(1 - 1/n) + (n)(1/n) = 1$.

2.5 Conclude that . . . .

Note that $\lim_{n\rightarrow\infty} E(X_n) = \lim_{n\rightarrow\infty} 1 = 1$. Now let $\epsilon > 0$. Note that $\lim_{n\rightarrow\infty} P(|X_n - 0| < \epsilon) = \lim_{n\rightarrow\infty} 1/n$ for $n > N$ where N is the smallest integer such that $N > 1/\epsilon$. Thus, $\lim_{n\rightarrow\infty} P(|X_n - 0| < \epsilon) = \lim_{n\rightarrow\infty} 1/n = 0$ so $X_n \rightarrow_p 0$, yet $E(X_n) \rightarrow 1$.

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

3 Question 3

3.1 Show that $\bar{Y}^*$

$E(\bar{Y}^*) = E(\frac{1}{n}\sum_{i=1}^n w_i Y_i) = \frac{1}{n}\sum_{i=1}^n w_i E(Y_i) = \frac{1}{n}\sum_{i=1}^n w_i \mu = \frac{\mu}{n}\sum_{i=1}^n w_i = \mu.$

3.2 Calculate $Var(\bar{Y}^*)$

$Var(\bar{Y}^*) = \frac{1}{n^2}Var(\sum_{i=1}^n w_i Y_i) = \frac{1}{n^2}\sum_{i=1}^n w_i^2 Var(Y_i) = \sigma_Y^2 \frac{1}{n^2}\sum_{i=1}^n w_i^2$ where σ_Y^2 is the variance of each draw of Y.

3.3 Show the first sufficient condition.

Let $\frac{1}{n^2}\sum_{i=1}^n w_i^2 \to 0$. Let $\epsilon > 0$. By Chebyshev's inequality, $P(|\bar{Y}^* - \mu| \geq \epsilon) \leq \frac{\sigma_Y^2 \sum_{i=1}^n w_i^2}{n^2 \epsilon^2} \to_{n\to\infty} 0$ so $\lim_{n\to\infty} P(|\bar{Y}^* - \mu| \geq \epsilon) = 0$.

3.4 Show the second sufficient condition.

Now, let $\max_{i \leq n} w_i/n \to 0$. Let $\epsilon > 0$. By Chebyshev's inequality, $P(|\bar{Y}^* - \mu| \geq \epsilon) \leq \frac{\sigma_Y^2 \sum_{i=1}^n w_i^2}{n^2}$ $\leq \frac{\sigma_Y^2 \sum_{i=1}^n w_i \max_{j\leq n} w_j}{n^2} = \frac{\sigma_Y^2 \max_{j\leq n} w_j \sum_{i=1}^n w_i}{n^2} = \frac{\sigma_Y^2 \max_{j \leq n} w_j}{n} \to_{n\to\infty} 0.$

4 Question 4

4.1 $\frac{1}{n}\sum_{i=1}^n X_i^2$

Assuming the moment exists, by the WLLN $\frac{1}{n}\sum_{i=1}^n X_i^2 \to_p E[X_i^2]$.

4.2 $\frac{1}{n}\sum_{i=1}^n X_i^3$

Assuming the moment exists, by the WLLN $\frac{1}{n}\sum_{i=1}^n X_i^3 \to_p E[X_i^3]$.

4.3 $\max_{i \leq n} X_i$

We cannot say anything using WLLN or CMT.

4.4 $\frac{1}{n}\sum_{i=1}^n X_i^2 - (\frac{1}{n}\sum_{i=1}^n X_i)^2$

Assuming the necessary moments exist, by WLLN, $\frac{1}{n}\sum_{i=1}^n X_i^2 \to_p E[X_i^2], \frac{1}{n}\sum_{i=1}^n X_i \to_p E[X_i]$ so by continuity and the CMT, $\frac{1}{n}\sum_{i=1}^n X_i^2 - (\frac{1}{n}\sum_{i=1}^n X_i)^2 \to_p Var(X_i)$.

4.5 $\frac{\sum_{i=1}^n X_i^2}{\sum_{i=1}^n X_i}$

Assuming the moments exist, by WLLN, $\frac{1}{n}\sum_{i=1}^n X_i^2 \to_p E[X_i^2], \frac{1}{n}\sum_{i=1}^n X_i \to_p E[X_i]$ so by continuity and the CMT, $\frac{\sum_{i=1}^n X_i^2}{\sum_{i=1}^n X_i} \to_p E[X_i^2]/E[X_i]$.

4.6 $\mathbb{1}(\sum_{i=1}^n X_i)$

By WLLN, $\frac{1}{n}\sum_{i=1}^n X_i \to_p E[X_i]$ so by CMT $\mathbb{1}(\sum_{i=1}^n X_i) \to_p \mathbb{1}(E[X_i] > 0)$ unless $E[x_i] = 0$ in which case the indicator function is not continuous at that point, and CMT cannot be applied.

5 Question 5

Note that $\hat{\mu} = exp(log(\hat{\mu})) = exp(log((\pi_{i=1}^{n} X_i)^{1/n})) = exp((1/n)\sum_{i=1}^{n} log(X_i))$. Due to continuity of log, exp on $(0, \infty)$, the WLLN and CMT gives us $\hat{\mu} = exp((1/n)\sum_{i=1}^{n} log(X_i)) \rightarrow_p exp(E(log(X_i))) = \mu$.

6 Question 6

6.1 Find the natural moment estimator for μ_k

Define $\hat{\mu}_k := \frac{1}{n}\sum_{i=1}^{n} X_i^k$. By WLLN this is a consistent estimator for μ_k.

6.2 Find the asymptotic distribution of $\sqrt{n}(\hat{\mu}_k - \mu)$ as $n \rightarrow \infty$.

Assuming the necessary moments exist, by CLT $Var(X_i^k) = E(X_i^{2k}) - (E(X_i^k))^2$ so $\sqrt{n}(\hat{\mu}_k - \mu_k) \rightarrow_d N(0, \mu_{2k} - \mu_k^2)$.

7 Question 7

7.1 Find a consistent estimator

By continuity, assuming the moment exists, $\hat{m}_k = (\hat{\mu}_k)^{1/k}$ is a consistent estimator for m_k.

7.2 Find the distribution

Using the delta method, $\sqrt{n}(g(\hat{m}_k - m_k)) \rightarrow_d N(0, V)$ where $V = ((1/k)(\mu_k)^{\frac{1-k}{k}})^2(\mu_{2k} - \mu_k^2) = \frac{1}{k^2}\mu_k^{2\frac{1-k}{k}}(\mu_{2k} - \mu_k^2)$.

8 Question 8

8.1 Use the Delta Method

Using the Delta Method, $\sqrt{n}(\hat{\beta} - \beta) \rightarrow_d N(0, V)$ where $V = 4\mu^2 v^2$.

8.2 What happens when μ is 0?

If $\mu = 0$ we get a degenerate normal with no variance; in the limit the distribution collapses into a unit point mass at 0.

8.3 Improve your answer.

$\sqrt{n}\hat{\mu} \rightarrow_d N(0, v^2) \Rightarrow \sqrt{n}\hat{\mu}/v \rightarrow_d N(0, 1) \Rightarrow n\hat{\mu}^2/v^2 \rightarrow_d \chi_1^2 \Rightarrow n\hat{\beta} \rightarrow_d v^2\chi_1^2$

8.4 Why do we get this difference?

It seems to me that, when $\beta = 0$, the estimator $\hat{\beta}$ converges to 0 at a rate of n rather than a rate of $\sqrt{n}$. So, by checking the convergence at the rate of $\sqrt{n}$ we find that $\hat{\beta}$ has already converged to 0, though if we check the convergence at the rate of n we find a distribution at that rate, which takes the form of a scaled χ_1^2.