Macro PS4

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1 Question 1

1.1 State and solve the SPP

The social planner maximizes total utility subject to the resource constraint. This problem is represented as:

$$\begin{aligned}&\max_{c_i^{i-1}, c_i^i, i\in\mathbb{N}} U(c_1^0) + \sum_{t=1}^{\infty} U(c_t^t, c_{t+1}^t)\\ &\text{s.t. } (1+n)c_t^t + c_t^{t-1} \leq (1+n)w_1 + w_2 \; \forall t \in \mathbb{N}\end{aligned}$$

The resource constraint inequality will bind as utility is strictly increasing in consumption. Then, we can substitute $c_t^t = w_1 + (w_2 - c_t^{t-1})/(1+n)$ into our optimization problem, as well as our functional form for utility, and we are left with the following:

$$\max_{c_i^{i-1}, i\in\mathbb{N}} \ln\, c_1^0 + \sum_{t=1}^{\infty} \ln\, (w_1 + (w_2 - c_t^{t-1})/(1+n) + \ln\, c_t^{t-1}$$

Taking first order conditions with respect to c_t^{t-1}, $\frac{-1}{(1+n)w_1+w_2-c_t^{t-1}} + \frac{1}{c_t^{t-1}} = 0$
$\Rightarrow c_t^{t-1} = \frac{w_1(1+n)+w_2}{2}, c_t^t = \frac{(1+n)w_1+w_2}{2} \; \forall t \in \mathbb{N}$.

1.2 Set up the competitive equilibrium

The competitive equilibrium is a set of allocations and prices such that agents optimize and markets clear. Note that I will write agents' budget constraints as binding as utility is strictly increasing in consumption. The first old agent solves the following:

$$\begin{aligned}&\max_{c_1^0} \ln\, c_1^0\\ &\text{s.t. } p_1 c_1^0 = p_1 w_2 + \bar{M}_1.\end{aligned}$$

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

All other agents live for 2 periods, so they instead solve:

$$\max_{c_1^0} \ln c_t^t + \ln c_{t+1}^t$$
$$\text{s.t. } p_t c_t^t + M_{t+1}^t = p_t w_1$$
$$\text{and } p_{t+1} c_{t+1}^t = (1+z) M_{t+1}^t + p_{t+1} w_2.$$

In the competitive equilibrium, the goods market and money market clear:

$$(n+1)c_t^t + c_t^{t-1} = (n+1)w_1 + w_2,$$
$$M_{t+1}^t = (1+z)^{t-1}\bar{M}_1,$$

where the money supply is the amount given to the initial old, growing exogenously at a rate of $(1+z)$ in each period thereafter.

1.3 Solve for an autarkic equilibrium

Money has no value in this equilibrium. Thus, our agents' problems change to be the following:

$$\max_{c_1^0} \ln c_1^0$$
$$\text{s.t. } c_1^0 = w_2.$$

All other agents live for 2 periods, so they instead solve:

$$\max_{c_1^0} \ln c_t^t + \ln c_{t+1}^t$$
$$\text{s.t. } c_t^t = w_1$$
$$\text{and } c_{t+1}^t = w_2.$$

In the competitive equilibrium, the goods market and money market clear:

$$(n+1)c_t^t + c_t^{t-1} = (n+1)w_1 + w_2,$$
$$(n+1)^t M_{t+1}^t = (1+z)^{t-1}\bar{M}_1,$$

Thus, $c_t^t = w_1, c_t^{t-1} = w_2, M_{t+1}^t = (1+z)^{t-1}\bar{M}_1 \ \forall t \in \mathbb{N}$.

1.4 Solve for a steady state (non-autarkic) monetary equilibrium

The initial old's consumption is determined by their budget constraint: $c_1^0 = w_2 + \frac{\bar{M}_1}{p_1}$.

For all other agents, we can solve for c_t^t, c_{t+1}^t as a function of M_{t+1}^t and prices using their budget constraints:

$$c_t^t = w_1 - \frac{M_{t+1}^t}{p_t}$$
$$c_{t+1}^t = (1+z)\frac{M_{t+1}^t}{p_{t+1}} + w_2.$$

Plugging this into the maximization problem we have the following optimization problem and first order condition with respect to M_{t+1}^t :

$$\begin{aligned}
&\max_{M_{t+1}^t} \ln\left(w_1 - \frac{M_{t+1}^t}{p_t}\right) + \ln\left((1+z)\frac{M_{t+1}^t}{p_{t+1}} + w_2\right)\\
&\Rightarrow 0 = -\left(\frac{1}{p_t}\right)\frac{1}{w_1 - M_{t+1}^t/p_t} + \left(\frac{1}{p_{t+1}}\right)\frac{1}{M_{t+1}^t/p_{t+1} + w_2/(1+z)}\\
&\Rightarrow M_{t+1}^t = \left(p_t w_1 - \frac{p_{t+1}}{1+z}w_2\right)/2\\
&\Rightarrow c_t^t = w_1 - \frac{1}{2p_t}\left(p_t w_1 - \frac{p_{t+1}}{1+z}w_2\right) = \frac{w_1}{2} + \frac{p_{t+1}}{2p_t(1+z)}w_2\\
&\quad = \frac{w_1}{2} + \frac{1}{2q_t(1+z)}w_2\\
&\Rightarrow c_{t+1}^t = w_2 + \frac{1+z}{2p_{t+1}}\left(p_t w_1 - \frac{p_{t+1}}{1+z}w_2\right) = \frac{w_2}{2} + \frac{p_t(1+z)}{2p_{t+1}}w_1\\
&\quad = \frac{w_2}{2} + \frac{q_t(1+z)}{2}w_1,
\end{aligned}$$

where $q_t := \frac{p_t}{p_{t+1}}$.

To clear goods, we find:

$$(1+n)w_1 + w_2 = (1+n)\left(\frac{w_1}{2} + \frac{1}{2q_t(1+z)}w_2\right) + \frac{w_2}{2} + \frac{q_{t-1}(1+z)}{2}w_1$$