

Micro HW2

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1 Question 1

1.1 Prove that if the production set $Y = \{(q, -z) : f(z) \geq q\} \subset \mathbb{R}^{m+1}$ is convex, the production function f is concave.

Let $q_1 = f(z_1), q_2 = f(z_2)$. $(q_1, -z_1), (q_2, -z_2) \in Y$ by definition and by convexity $t(q_1, -z_1) + (1-t)(q_2, -z_2) \in Y, t \in (0, 1)$. By definition, $f(t(z_1) + (1-t)(z_2)) \geq tq_1 + (1-t)q_2 = tf(z_1) + (1-t)f(z_2)$ so f is concave.

1.2 Prove that if f is concave, the cost function is convex in q .

We can fix $w \in \mathbb{R}_+^k$. Let $q_1, q_2 \in \mathbb{R}_+$. Let $z_1 \in Z_1^*, z_2 \in Z_2^*$ where $Z_1^* = \arg \min_{z: f(z) \geq q_1} w \cdot z, Z_2^* = \arg \min_{z: f(z) \geq q_2} w \cdot z$.

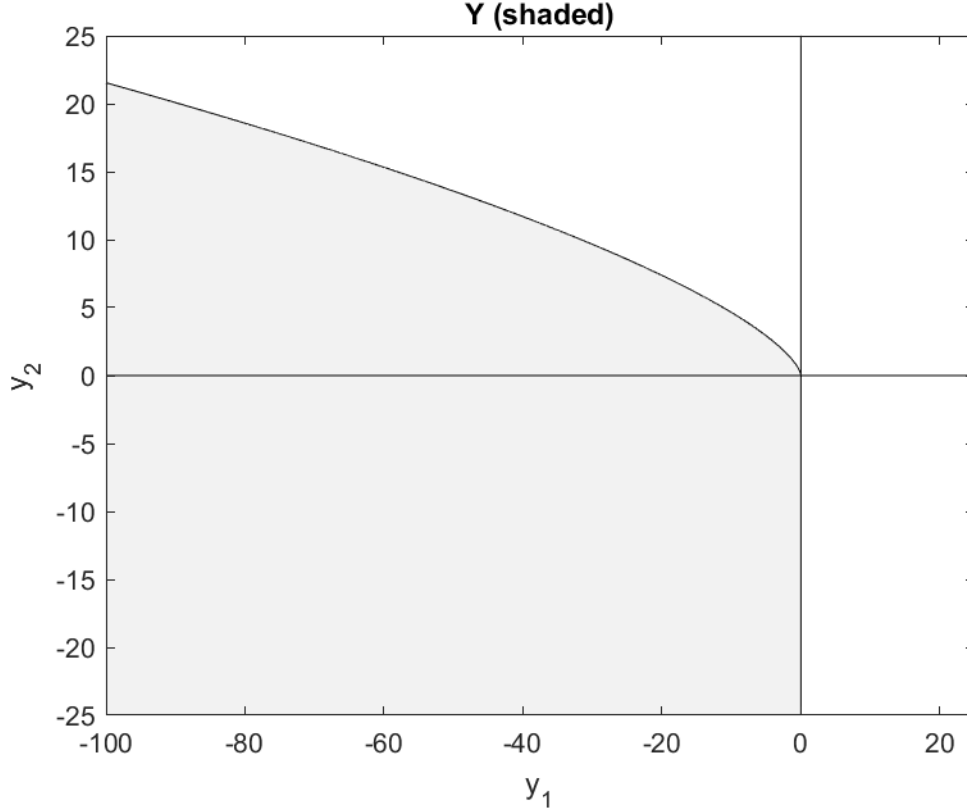
$\arg \min_{z: f(z) \geq q_2} w \cdot z$.

By the concavity of f , for $t \in (0, 1)$ we have $f(tz_1 + (1-t)z_2) \geq tf(z_1) + (1-t)f(z_2) \geq tq_1 + (1-t)q_2$. Thus, we can produce at least $tq_1 + (1-t)q_2$ by using $tz_1 + (1-t)z_2$ inputs, so the minimum cost of producing $tq_1 + (1-t)q_2$ goods cannot be higher than the cost of those inputs, $w \cdot (tz_1 + (1-t)z_2) = t(w \cdot z_1) + (1-t)(w \cdot z_2) = tc(q_1, w) + (1-t)c(q_2, w)$. Therefore, $tc(q_1, w) + (1-t)c(q_2, w) \geq c(tq_1 + (1-t)q_2, w)$.

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, Ryan Mather, and Tyler Welch. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

2 Question 2

2.1 Draw Y



The shaded area in the above figure is Y graphed in Matlab, for a sample value of $B = 1$.

2.2 Solve the firm's profit maximization problem to find $\pi(p)$ and $Y^*(p)$.

The firm chooses production to maximize profit: $\max_{-y_1, y_2 \in \mathbb{R}_+} p \cdot (y_1, y_2)'$ s.t. $y_2 \leq B(-y_1)^{2/3}$.

Since profits are strictly increasing in y_2 the profit maximizing firm will set $y_2 = B(-y_1)^{2/3}$. We will also write $-y_1 = z$. Our optimization problem thus becomes: $\max_{q \in \mathbb{R}_+} p \cdot (-q, Bq^{2/3})'$. Taking the firm's first order conditions, we find that $0 = \frac{d\pi(q)}{dq} =$

$0 \Rightarrow \frac{d}{dq} (-p_1 q + p_2 B q^{2/3}) = 0 \Rightarrow -p_1 + (2/3)p_2 B q^{-1/3} = 0 \Rightarrow q = \left(\frac{B p_2}{(3/2)p_1} \right)^3$. This production yields the maximum profits given p , which we can compute as:

$$\pi(p) = -p_1 \left(\frac{B p_2}{(3/2)p_1} \right)^3 + p_2 B \left(\frac{B p_2}{(3/2)p_1} \right)^2 = \frac{4B^3 p_2^3}{27 p_1^2} \text{ and } Y^*(p) = \left(- \left(\frac{B p_2}{(3/2)p_1} \right)^3, B \left(\frac{B p_2}{(3/2)p_1} \right)^2 \right)'.$$

2.3 Verify that $\pi(p)$ is homogeneous of degree 1, and $y(p)$ is homogeneous of degree 0.

$\pi(\lambda p) = \lambda p_1 \left(\frac{B\lambda p_2}{(3/2)\lambda p_1} \right)^3 + \lambda p_2 B \left(\frac{B\lambda p_2}{(3/2)\lambda p_1} \right)^2 = \lambda \left(p_1 \left(\frac{Bp_2}{(3/2)p_1} \right)^3 + p_2 B \left(\frac{Bp_2}{(3/2)p_1} \right)^2 \right) = \lambda \pi(p)$
so $\pi(p)$ is homogeneous of degree 1.

$y(\lambda p) = \left(- \left(\frac{B\lambda p_2}{(3/2)\lambda p_1} \right)^3, B \left(\frac{B\lambda p_2}{(3/2)\lambda p_1} \right)^2 \right)' = \left(- \left(\frac{Bp_2}{(3/2)p_1} \right)^3, B \left(\frac{Bp_2}{(3/2)p_1} \right)^2 \right)' = y(p)$ so $y(p)$ is homogeneous of degree 0.

2.4 Verify that $y_1(p) = \frac{\partial \pi}{\partial p_1}$ and $y_2(p) = \frac{\partial \pi}{\partial p_2}$.

$$\begin{aligned} \frac{\partial \pi}{\partial p_1} &= \frac{\partial}{\partial p_1} \left(\frac{4B^3 p_2^3}{27p_1^2} \right) = (-2) \left(\frac{4B^3 p_2^3}{27p_1^3} \right) = - \left(\frac{Bp_2}{(3/2)p_1} \right)^3 = y_1(p) \\ \frac{\partial \pi}{\partial p_2} &= \frac{\partial}{\partial p_2} \left(\frac{4B^3 p_2^3}{27p_1^2} \right) = 3 \frac{4B^3 p_2^2}{27p_1^2} = B \left(\frac{Bp_2}{(3/2)p_1} \right)^2 = y_2(p) \end{aligned}$$

2.5 Calculate $D_p y(p)$ and verify it is symmetric, positive semidefinite, and $[D_p y]p = 0$

$$D_p y(p) = \begin{pmatrix} \frac{\partial y_1(p)}{\partial p_1} & \frac{\partial y_2(p)}{\partial p_1} \\ \frac{\partial y_1(p)}{\partial p_2} & \frac{\partial y_2(p)}{\partial p_2} \end{pmatrix} = \begin{pmatrix} \frac{8B^3 p_2^3}{9p_1^4} & -\frac{8B^3 p_2^2}{9p_1^3} \\ -\frac{8B^3 p_2^2}{9p_1^3} & \frac{8B^3 p_2}{9p_1^2} \end{pmatrix}$$

Upon observation, it is clear that $D_p y(p)$ is symmetric. Furthermore, the first element of $D_p y(p)$, $\frac{8B^3 p_2^3}{9p_1^4}$, is positive because B, p_1, p_2 are all positive. Next we will check the matrix's determinant:

$$\det D_p y(p) = \frac{8B^3 p_2^3}{9p_1^4} \frac{8B^3 p_2}{9p_1^2} - \left(-\frac{8B^3 p_2^2}{9p_1^3} \right) \left(-\frac{8B^3 p_2^2}{9p_1^3} \right) = \frac{128B^6 p_2^4}{81p_1^6}.$$

The determinant is positive, so $D_p y(p)$ is positive semidefinite.

$$[D_p y(p)]p = \begin{pmatrix} \frac{8B^3 p_2^3}{9p_1^4} & -\frac{8B^3 p_2^2}{9p_1^3} \\ -\frac{8B^3 p_2^2}{9p_1^3} & \frac{8B^3 p_2}{9p_1^2} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \frac{8B^3 p_2^3}{9p_1^3} - \frac{8B^3 p_2^3}{9p_1^3} \\ -\frac{8B^3 p_2^2}{9p_1^2} + \frac{8B^3 p_2^2}{9p_1^2} \end{pmatrix} = \vec{0}.$$

3 Question 3

3.1 What conditions must hold for this profit function to be rationalizable?

From lecture, the profit function is rationalizable if and only if it is homogenous of degree 1 and convex.

3.2 Show that $y \in Y^O$ must have $y_1 \leq 0$.

Let $(y_1, y_2) \in Y^O$. Then $p_1 y_1 + p_2 y_2 \leq A p_1^{-2} p_2^3 \forall p_1, p_2 \in \mathbb{R}$. Assume for the purpose of contradiction that $y_1 > 0$. Now fix $p_2 > 0$, we can make $p_1 y_1 + p_2 y_2$ unboundedly large by increasing p_1 while $A p_1^{-2} p_2^3$ goes to 0, so we can force a contradiction. Thus $y_1 \leq 0$.

3.3 Solve the minimization problem and describe the production set Y^O .

We want to solve the following problem: $\min_{r>0} A r^2 - \frac{y_1}{r}$. Taking first order conditions, $2Ar + \frac{y_1}{r^2} = 0 \Rightarrow r = \left(\frac{-y_1}{2A}\right)^{1/3}$. So, $\min_{r>0} A r^2 - \frac{y_1}{r} = A \left(\frac{-y_1}{2A}\right)^{2/3} - \frac{y_1}{\left(\frac{-y_1}{2A}\right)^{1/3}} =$

$$2^{-2/3} A^{1/3} (-y_1)^{2/3} + 2^{1/3} A^{1/3} (-y_1)^{2/3} = (2^{-2/3} + 2^{1/3}) A^{1/3} (-y_1)^{2/3} .$$

Therefore, $Y^O = \{(y_1, y_2) : y_2 \leq (2^{-2/3} + 2^{1/3}) A^{1/3} (-y_1)^{2/3}\}$.

3.4 Verify that $Y = Y^O$ would generate the data we started with.

Let $p_1, p_2 \in \mathbb{R}_{++}$. We want to find $\max_{(y_1, y_2) \in Y} p_1 y_1 + p_2 y_2$. We know that profits are increasing in y_2 so $y_2 = (2^{-2/3} + 2^{1/3}) A^{1/3} (-y_1)^{2/3}$. From the previous question we know that $\pi(p) = \frac{4B^3 p_2^3}{27p_1^2}$ where $B = (2^{-2/3} + 2^{1/3}) A^{1/3}$. Simplifying,

$$\pi(p) = \frac{4(27/4) A p_2^3}{27 p_1^2} = A p_1^{-2} p_2^3$$

Thus, Y generates $\pi(p) = A p_1^{-2} p_2^3$.