Microeconomic Study Sheet for the Prelim Exam

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1 Microeconomic Theory (Quint, Quarter 1)

1.1 Producer Theory

Production Set $Y \subseteq \mathbb{R}^k$ is the set of possible production plans y.

- Typically assume that Y is (i) nonempty and closed, (ii) there is *free disposal* so that if $y \in Y$, $y' \leq y$, then $y' \in Y$, and (iii) *shutdown* is an option so that $0 \in Y$.
- Other possible assumptions include
  - *Constant Returns to Scale*: $y \in Y \implies \alpha y \in Y \ \forall \alpha > 0$
  - *Increasing Returns to Scale*: $y \in Y \implies \alpha y \in Y \ \forall \alpha > 1$
  - *Decreasing Returns to Scale*: $y \in Y \implies \alpha y \in Y \ \forall \alpha \in (0, 1)$
  - "Cannot create something from nothing": $Y \cap \mathbb{R}^k_+ = \{0\}$
  - *Irreversibility*: $Y \cap -Y = \{0\}$
  - *Strict Convexity*: $\forall t \in (0, 1)$, $y, y' \in Y$, $y \neq y' \implies ty + (1-t)y' \in Int(Y)$

Firm Problem: $\max_{y \in Y}\{p \cdot y\}$.

- $\pi(p) = \sup_{y \in Y} p \cdot y$ and $y(p) = Y^*(p) = \arg\max_{y \in Y} p \cdot y$
- Properties
  - $\pi(p)$ is (i) homogeneous of degree 1 and (ii) convex
  - $Y^*(p)$ is (i) homogenous of degree 0 and (ii) if Y is convex, then $Y^*(p)$ is convex. If Y is strictly convex, and $Y^*(p)$ nonempty for $p \neq 0$, then $Y^*(p)$ is a singleton.
  - Constant returns to scale $\implies \pi(p) = 0$ or ∞
- **Law of Supply**: For $y \in Y^*(p)$ and $y' \in Y^*(p')$, $(p' - p) \cdot (y' - y) \geq 0$
- **Hotelling's Lemma**: If Y is closed with free disposal and $y()$ is single-valued and differentiable in a neighborhood of p, then $\pi(p)$ is differentiable at p and $\frac{\partial \pi}{\partial p_i} = y_i(p)$.
- If Y is closed with free disposal and $y(\cdot)$ is single-valued and differentiable, then $D_p y(p)$ is symmetric and positive semidefinite, and $[D_p y(p)]_p = \vec{0}$.

Transformation Function: $T : \mathbb{R}^k \to \mathbb{R}$ is a function where $T(y) \leq 0$ if $y \in Y$ and $T(y) > 0$ if $y \notin Y$.

- The *marginal rate of transformation* between goods y_i and y_j is $MRT_{i,j} = \frac{\partial T}{\partial y_i} / \frac{\partial T}{\partial y_j}$.
- The *marginal rate of technical substitution* of a production function f is $MRTS_{k,l} = \frac{\partial f}{\partial z_l} / \frac{\partial f}{\partial z_k}$. (ARE THOSE INDEXES RIGHT FOR THESE LAST TWO BULLET POINTS?)

Rationalizability holds when observed data of p and y can be rationalized by some production set to be the behavior of a profit-maximizing firm.

- **Weak Axiom of Profit Maximization (WAPM)**: $p \cdot y(p') \leq p \cdot y(p)$. The data is rationalizable if it satisfies WAPM.
- Theorem: Suppose that Y is closed and satisfies free disposal, and that $P = \mathbb{R}^k_+ \setminus \{0\}$ (i.e., there is *perfect data*).
  1. Y is convex $\Leftrightarrow Y = Y^O$
  2. If Y is convex and $y(p) = Y^*(p) \ \forall p \in P$, then $Y = Y^I_{FD}$.
- Theorem: Given $y : P \to \mathbb{R}$ and $\pi : P \to \mathbb{R}$ satisfying adding up, Y rationalizes y and $\pi \Leftrightarrow Y^I \subseteq Y \subseteq Y^O$.
  - The *adding up* condition holds that $p \cdot y = \pi(p) \ \forall y \in y(p)$, $\forall p \in P$.
  - The *inner bound* $Y^I = \cup_{p \in P} y(p)$.
  - The *outer bound* $Y^O = \{y \in \mathbb{R}^k : p \cdot y \leq \pi(p) \quad \forall p \in P\}$
- Theorem: If P is an open, convex subset of $\mathbb{R}^k_{++}$, $y(p)$ is single-valued, and $\pi(p)$ is differentiable, then:
  1. y and π are jointly rationalizable if and only if (i) adding up holds so that $\pi(p) = p \cdot y \ \forall p \in P$, (ii) hotelling's lemma holds so that $y_i(p) = \partial\pi / \partial p_i$ for all i, and (iii) π is convex.
  2. A differentiable supply function is rationalizable if and only if it is homogenous of degree 0 and $D_p y(p)$ is symmetric and positive-semidefinite.
  3. π is rationalizable if and only if it is homogenous of degree 1 and convex.
- **Envelope Theorem**: Let $V(t) = \max_{x \in X} f(x, t)$, and $x^*(t) = \arg\max_{x \in X} f(x, t)$. If $V'(t)$ exists, then $V'(t) = \frac{\partial f}{\partial t)}(\hat{x}, t)$ for any $\hat{x} \in x^*(t)$.
- **Hotelling's Lemma**: Let Y be closed and satisfy free disposal. If $y()$ is single-valued in a neighborhood of p, then π is differentiable at p and $\frac{\pi}{\partial p_i} = y_i(p)$.

Cost Minimization

- $c(q, w) = \inf w \cdot z$ s.t. $f(z) \geq q$. $c(q, w)$ is (i) homogeneous of degree 1 in w, (ii) concave in input prices w, (iii) increasing in q, and (iv) if f is concave, c is convex in q.
- $z^*(q, w) = \arg\min_{f(z) \geq q} w \cdot z = \{z : f(z) \geq q \text{ and } w \cdot z = c(q, w)\}$
  - *Restated law of supply*: $z \in z^*(q, w)$ and $z' \in z^*(q, w) \implies (z' - z) \cdot (w' - w) \leq 0$.

*Contributions made by Ryan Mather,... (add names here)

- If $z(q,w)$ s single-valued and differentiable, then $D_w z = D^2_w c$ is symmetric and negative semidefinite with $[D_w z]w = 0$.

- **Shepard's Lemma**: If $z(q,w)$ is single-valued, then c is differentiable with respect to w and $\frac{\partial c}{\partial w_i}(q,w) = z_i(q,w)$.

1.2 Monotone Comparative Statics

Definitions

- $A \geq_{SSO} B \Leftrightarrow \forall a \in A$ and $b \in B$, $\max\{a,b\} \in A$ and $\min\{a,b\} \in B$.

- **Component-wise maximum ("a join b")**: For $a, b \in \mathbb{R}^m$, $a \vee b = (max\{a_1,b_1\},\dots,max\{a_n,b_n\})$.

- **Component-wise minimum ("a meet b")**: For $a, b \in \mathbb{R}^m$, $a \wedge b = (min\{a_1,b_1\},\dots,min\{a_n,b_n\})$.

- **Increasing differences**: The following are equivalent:

  1. $g(x',t) - g(x,t)$ is increasing in t for any $x' > x$.
  2. $g(x',t') - g(x,t) \geq g(x',t) - g(x,t)$ $\forall x' \geq x$ and $t' \geq t$.
  3. If g is differentiable, $\frac{\partial g}{\partial x}$ is increasing in t or $\frac{\partial g}{\partial t}$ is increasing in x.
  4. If g is twice differentiable, $\frac{\partial^2 g}{\partial x \partial t} \geq 0$.

  Multi-dimensional increasing differences: g has increasing differences in (X,T) if g has increasing differences in (x_i,t_j) $\forall i,j$

- **Supermodularity**: For a product set $X \in \mathbb{R}^m$, the following are equivalent:

  1. $g(x \vee y) + g(x \wedge y) \geq g(x) + g(y)$ $\forall x, y \in X$
  2. g has increasing differences in (x_i,x_j) for every pair (i,j) holding other variables fixed.
  3. If g is twice differentiable, $\frac{\partial g}{\partial x_i \partial x_j} \geq 0$ $\forall i,j$

Baby Topkis Theorem: For $x^* = \arg\max_{x\in X} g(x,t)$, if $X \subseteq \mathbb{R}$, and g has weakly increasing differences, then x^* is increeasing in t via the strong set order. (i.e. $t' \geq t \implies x^*(t) \geq_{SSO} x^*(t)$.

- *(Corollary) Monotone Selection Theorem*: if x^* is single-valued, then $t' > t$,$x \in x^*(t)$ and $x' \in x^*(t') \implies x' \geq x$.

- If g has strictly increasing differences, then I believe the above can be strengthened so that $x' > x$

Topkis Theorem: Let X be a product set in $\mathbb{R}^m$, $T \subseteq \mathbb{R}^n$, $g : X \times T \to \mathbb{R}$, and $x^*(t) = \arg\max_{x\in X} g(x,t)$. If $g(x,t)$ is supermodular in X and has increasing differences in (X,T), then $x^*(t)$ is increasing in t. (i.e. For any $x \in x^*(t)$, $x' \in x^*(t')$, $t' > t$, we have that $x \vee x' \in x^*(t')$ and $x \wedge x' \in x^*(t)$)

- *(Corollary):* If x^* is single-valued, then $x^*(t)$ is weakly increasing in every dimension

- Note that the requirement for this theorem can be weakened to *single-crossing differences*, that is, that $g(x',t) - g(x,t) \geq 0 \implies g(x',t') - g(x,t') \geq 0$ for any $x' > x$ and $t' > t$

Le' Chatlier's Principle: The long-run adjustment of the firm is larger than the short-run adjustment of the firm.

1.3 Preferences and Utility

1.3.1 Properties of Preferences

- **Rational** preferences over a choice set X satisfy

  - **Complete**: $\forall x, y \in X$, $x \succsim y$ or $y \succsim x$ (or both). This implies *reflexivity*, that $x \sim x$
  - **Transitivity**: If $x \succsim y$ and $y \succsim z$, then $x \succsim z$

- **Continuous**: For any sequence $\{x_n, y_n\}_{n=1}^{\infty}$, if $x_n \to \bar{x}$, $y_n \to \bar{y}$, and $x_n \succsim y_n$ $\forall n$ then $\bar{x} \succsim \bar{y}$

- **Monotone**: $x >> y \implies x \succ y$

- **Strong monotone**: $x > y \implies x \succ y$

- **Locally non-satiated**: $\forall y \in X$ and $\forall \epsilon > 0$, $\exists x \in X$ s.t. $\|x - y\| < \epsilon$ and $x \succ y$.

- **Convex**: $\forall t \in (0,1)$, $x \succsim y$ and $x' \succsim y \implies tx + (1-t)x' \succsim y$

  - **Strictly Convex**: $\forall t \in (0,1)$,$x \succsim y$ and $x' \succsim y \implies tx + (1-t)x' \succ y$
  - If preferences are convex, then a utility function u representing those preferences is *quasi-concave* (that is, the upper contour set $\{x : u(x) \geq u(y)\}$ is a convex set). If preferences strictly convex, this can be extended to *strict quasi-concavity* (in which the upper contour set is strictly convex)

- **Separable**: Let $X = Y \times Z$. Preferences over Y do not depend on Z if for all $y, y' \in Y$ and $z, z' \in Z$, $(y,z) \succsim (y',z) \Leftrightarrow (y,z') \succsim (y',z')$

  - In the case of a utility function $u : Y \times Z \to \mathbb{R}$, preferences over Y do not depend on Z if and only if there exist functions $v : Y \to \mathbb{R}$ and $U : \mathbb{R} \times Z \to \mathbb{R}$ such that U is increasing in its first argument and $u(y,z) = U(v(y),z)$

- **Lexicographic**: $(x_1,x_2) \succ (y_1,y_2)$ if $x_1 > y_1$ or $x_1 = y_1$ and $x_2 > y_2$

Choice Rule: Given a power set $P(x) = 2^X\{A : A \subseteq X\}$, a choice rule is a function $C : P(X) \to P(X)$ such that $C(A) \subseteq A$ for all $A \in P(X)$

Weak Axiom of Revealed Preference (WARP): For any $A, B \subseteq X$ and $x, y \in A \cap B$, if $x \in C(A)$ and $y \in C(B)$, then $x \in C(B)$ and $y \in C(A)$

- *Proposition*: If $\succsim$ is complete and transitive, then $C(A, \succsim) = \{x \in A : x \succsim y \forall y \in A\}$ satisfies WARP

- *Proposition*: Let $C : P(X) \to P(X)$ be a nonempty choice rule. There exists a complete and transitive preference relation $\succsim$ such that $C(\cdot, \succsim) = C(\cdot) \Leftrightarrow C$ satisfies WARP

Utility Representation of Preferences: $u : X \to \mathbb{R}$ represents $\succsim$ if for all $x, y \in X$, $x \succsim y \Leftrightarrow u(x) \geq u(y)$

- *Proposition*: If X is finite, then any composite and transitive preference relation $\succsim$ on X can be represented by a utility function.

- *Proposition*: If $X \subseteq \mathbb{R}$, then and $\succsim$ which is complete, transitive, and continuous can be represented by a utility function.

- **Quasi-linearity**: Let $X = \mathbb{R}_+ Y$. If (i) preferences are complete and transitive, (ii) there is a worst element $\bar{y} \in Y$ such that $(0, y) \succsim (0, \bar{y}) \forall y \in Y$, (iii) the first good is valuable so that $(a, \bar{y}) \succsim (a', \bar{y}) \Leftrightarrow a \geq a'$, (iv) compensation is possible so that $\forall y \in Y \exists t$ such that $(0, y) \sim (t, \bar{y})$, (v) and there are no wealth effects meaning $(a, y) \succsim (a', y') \Leftrightarrow (a+t, y) \succsim (a'+t, y')$, then preferences over X can be represented by a utility function of the form $u(a, y) = a + v(y)$.

1.3.2 Rationalizability of Preferences

Direct preference $\succ_D$ and revealed preference $\succ_R$

- $x \succsim_D x'$ if x is chosen and $p \cdot x' \leq p \cdot x$
- $x \succ_D x'$ if x is chosen and $p \cdot x' < p \cdot x$.
- $x \succsim_R x'$ if there exists a chain of bundles such that $x \succsim_D x_1 \succsim_D x_2 \succsim_D \cdots \succsim_D x'$

Generalized Axiom of Revealed Preference (GARP): If $x \succsim_R x'$, then it must not be that $x' \succ_D x$

Afriat Theorem: For observations $(p_i, x_i)_{i \in \{1,\ldots,n\}}$, the following are equivalent:

1. There exists a locally nonsatiated utility function which rationalizes the data.
2. The data satisfies GARP
3. There exist a series of numbers $\{\mu_i, \lambda_i\}_{i=1}^n$ with $\lambda_i > 0$ such that for every (i, j), $\mu_i \leq \mu_j + \lambda_j p_j (x_i - x_j)$
4. There exists a locally nonsatiated, continuous, concave, and monotone utility function that rationalizes the data.

Rationalizing aggregate preferences: If consumers have indirect utility $v_i(p, w) = u_i(p) + b(p) w_i$, then aggregate demand is equivalent to the demand of a single consumer with preferences $V(p, w) = \sum_i a_i(p) + b(p) W$ where $W = \sum_i w_i$

Theorem: A differentiable demand function $x(p, w)$ is consistent with utility maximization over locally nonstatiated pereferences if and only if

1. $x(p, w)$ is homogenous of degree 0
2. $p \cdot x(p, w) = w \; \forall (p, w)$
3. At every (p, w), the matrix $D_p h(p, u)$ calculated based on the marshallian demand as $\frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w} x_j$ is symmetric and positive semidefinite

1.4 Consumer Problem

1.4.1 Utility Maximization

$\max_{x \in B(p,w)} u(x)$ s.t. $x \in B(p, w) = \{X \in \mathbb{R}_+^k : p \cdot x \leq w\}$

- $B(p, w)$ is homogenous of degree 0 so that $\forall \lambda > 0$, $B(\lambda p, \lambda w) = B(p, w)$.
- *Proposition*: If $p >> 0$ and u is continuous, then the consumer problem has a solution.

Marshallian Demand: $x(p, w) = \arg\max_{x \in B(p,w)} u(x) = C(B(p, w), \succsim_u)$

- Homogenous of degree 0: $\forall \lambda > 0$, $x(\lambda p, \lambda w) = x(p, w)$
- If preferences are convex, then $x(p, w)$ is a convex set. If preferences are strictly convex, then $x(p, w)$ is a singleton.
- **Walras' Law**: If preferences are locally non-satiated, for any $x \in x(p, w)$, $p \cdot x = w$.

Indirect utility function: $v(p, w) = \max_{x \in B(p,w)} u(x) = u(x^*)$. Note that this is always single-valued. If u is a continuous utility function representing locally non-satiated preferences $\succsim$ on $\mathbb{R}^k$, then

- $v(p, w)$ is homogeneous of degree 0 so that $v(\lambda p, \lambda w) = v(p, w)$
- $v(p, w)$ is continuous on $\{(p, w) : p >> 0, w \geq 0\}$
- $v(p, w)$ is nondecreasing in p and strinctly increasing in w
- $v(p, w)$ is quasi-concave, meaning $\{(p, w) : v(p, w) < \bar{v}\}$ is a convex set

Roy's Identity: Suppose v is differentiable at $(p, w) >> 0$ and $\frac{\partial v}{\partial w} > 0$. Then $x(p, w)$ is a singleton for each good i, and $x_i(p, w) = \frac{-\partial v(p,w)}{\partial p_i} / \frac{\partial v(p,w)}{\partial w}$

1.4.2 Expenditure Minimization

$e(p, u) = \min_{x \geq 0} p \cdot x$ such that $u(x) \geq u$. *Properties*: If u is a continuous utility function representing locally nonsatiated $\succsim$, then $e(p, u)$ is

1. Homogeneous of degree 1 in p so that $e(\lambda p, u) = \lambda e(p, u)$.
2. Continuous in p and u
3. Weakly increasing in p, and strictly increasing in u if $p >> 0$
4. $e(p, u)$ is concave in p

Hicksian Demand: $h(p, u) = \arg\min_{x \geq 0} p \cdot x$ such that $u(x) \geq u$. *Properties*: If u is a continuous utility function on $\mathbb{R}_+^k$, then

1. $h(p, u)$ is homogeneous of degree 0 in p so that for $\lambda > 0$, $h(\lambda p, u) = h(p, u)$.
2. There is no excess utility, so that if $u \geq u(0)$ and $p >> 0$, then $u(x) = u$ for $x \in h(p, u)$
3. If $\succsim$ is convex, $h(p, u)$ is convex. If $\succsim$ is strictly convex and $p >> 0$, then $h(p, u)$ is a singleton

Theorem: If $p >> 0$, u is continuous, and there exists x such that $u(x) \geq u$, a solution exists to expenditure minimization

Law of Compensating Demand: If $p, p' \geq 0$, $x \in h(p, u')$, and $x' \in h(p, u)$, then $(p' - p) \cdot (x' - x) \leq 0$

Shepard's Lemma: If u is a continous representation of locally nonsatiated preferences and $h(p, u)$ is single-valued at (p, u), the e is differentiable at (p, u) and $\frac{\partial e(p,u)}{\partial p_i} = h_i(p, u)$

1.4.3 Relating Hicksian and Marshallian Demand

Slutsky: If u is a continuous representation of locally nonsatiated preferences on $\mathbb{R}_+^k$, then pick $p >> 0$ and let $w = e(p, u)$. If $h(p, u)$ and $x(p, w)$ are single-valued and differentiable at (p, u, w), then

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i(p, u)}{\partial p_j} - \frac{\partial x_i(p, w)}{\partial w} x_j(p, w)$$

Theorem: If u is a continuous representation of locally nonsatiated preferences, $p >> 0$ and $w \geq 0$, then

1. $e(p, v(p, w)) = w$ and $v(p, e(p, u)) = u$
2. $h(p, u) = x(p, e(p, u))$ and $x(p, w) = h(p, v(p, w))$

Theorem: If u represents $\succsim$ and $h(p, u)$ is single-valued and differentiable with $p >> 0$, then $D_p h(p, u)$ is symmetric and negative semidefinite

1.4.4 Classifying Goods

Good i is a

- *normal good* if $x_i(p, w)$ is increasing in wealth
- *inferior good* if $x_i(p, w)$ is decreasing in wealth
- *regular good* if $x_i(p, w)$ is decreasing in p_i
- *giffen good* if $x_i(p, w)$ is increasing in p_i
- *substitute* for good j if $h_i(p, u)$ is increasing in p_j
- *complement* for good j if $h_i(p, u)$ is decreasing in p_j
- *gross substitute* for good j if $x_i(p, w)$ is increasing in p_j
- *gross substitute* for good j if $x_i(p, w)$ is decreasing in p_j

1.5 Objective Probability Theory

A *lottery* $L = (p_1, ..., p_n)$ is a vector of probabilities where $\sum_i p_i = 1$ and $p_i \geq 0 \quad \forall i$

Properties of Preferences:

- Linearity: $U(pL \oplus (1-p)L') = pU(L) + (1-p)U(L')$
- Continuity: For any $\{L, L', L''\}$, the set $\{p : pL \oplus (1+p)L' \succsim L''\}$ is closed
- Independence: For $\{L, L', L''\}$, if $l \succsim L'$, then $pL \oplus (1-p)L'' \succsim pL' \oplus (1-p)L''$. It therefore follows that $pL \oplus (1-p)L' \succsim L'$

Von Noyman and Morgenstern Proposition: Suppose preferences over lotteries are complete, transitive, and satisfy continuity and independence. Then they are represented by a utility function $U(L) = \sum_i p_i u(x_i)$. (Note that this is an if and only if condition, but only one direction is typically used)

Proposition: Suppose U is an expected utility representation of preferences over lotteries. V is too if and only if $V(\cdot) = a + bU(\cdot)$ for $a \in \mathbb{R}$ and $b \in \mathbb{R}_{++}$

1.5.1 Risk

- A decision maker is *risk averse* if for any lottery F with expected utility E_F, $\partial_{E_F} \succsim F$ where ∂_x is a *degenerate lottery* in which x is received for certain. This decision maker is *strictly risk averse* if $\partial_{E_F} \succ F$ whenever F is non-degenerate
- A decision maker is *risk-loving* if u is convex
- $c(F, u)$ is set so that $\partial_{c(F,u)} \sim F$
- The *risk premium* $E_F - c(F, u)$ is ≥ 0 if u is risk averse.
- *Arrow-Pratt coefficient of absolute risk aversion* $A = \frac{-u''(x)}{u'(x)}$.
  - A utility function has *decreasing absolute risk aversion* $\frac{-u''(x)}{u'(x)}$ is decreasing in x
- *Coefficient of Relative Risk Aversion* $R(x) = \frac{-xu''(x)}{u'(x)}$.
  - The *constant relative risk aversion (CRRA)* utility is $u(x) = (\frac{1}{1-\rho}x^{1-\rho}$ for $\rho \geq 0$, $\rho \neq 1$.
- Lottery cdf $F \geq_{FOSD} G$ *(first-order stochastically dominates)* if $F(x) \leq G(x)\ \forall x$
  - *Proposition*: $F \geq_{FOSD} G \Leftrightarrow \int u(x)dF(x) \geq \int u(x)dG(x)$ for every non-decreasing u
- Lottery cdf $F \geq_{SOSD} G$ *(second-order stochastically dominates)* if the expected utility of lottery F $E_F = E_G$ and $\int_{-\infty}^{x} F(s)ds \leq \int_{-\infty}^{x} G(s)ds$ for all x

Theorem: Let u and v be two bernoulli utility functions. THe following definitions of "u is more risk averse than v" are equivalent:

1. For any lottery F and ∂_x, if $F \succsim_u \partial_x$ then $F \succsim_v \partial_x$
2. For any lottery F, $c(F, u) \leq C(F, v)$
3. $u = g \circ v$ for some increasing and concave function g.
4. $\forall x$, $\frac{-u''(x)}{u'(x)} \geq \frac{-v''(x)}{v'(x)}$

Theorem: Let F and G be two lotteries with the same mean. These definitions of "G is riskier than F" are equivalent:

1. $F \geq_{SOSD} G$
2. $\int u(x)dF(x) \geq \int u(x)dG(x)$ for every nondecreasing concave u
3. "$G = F$+noise." If $X \sim F$, $Y \sim G$, then $Y = X + Z$ where $\mathbb{E}[z|X = x] = 0$

1.6 Welfare Effects of Price Changes

For $p' < p^o$,

- **Compensating Variation (CV)**$= e(p^o, u^o) - e(p', u^o) = e(p', u') - e(p', u^o)$. In the case that only one price i changes, this can be represented as $\int_{p_i'}^{p_i^O} h_i(p, u^o)dp_i$
- **Equivalent Variation (EV)**$= e(p^o, u') - e(p', u')$. In the case that only one price i changes, this can be represented as $\int_{p_i'}^{p_i^o} h_i(p, u')dp_i$

Proposition: If i is a normal good, $EV \geq \Delta CS \geq CV$. If i is inferior, $EV \leq \Delta CS \leq CV$. If preferences are quasilinear, $EV = \Delta CS = CV$

1.7 Lagrangians

$\mathscr{L}(x, \lambda, \mu) = u(x) + \lambda * (w - p \cdot x) + \mu \cdot x$.

Saddle Point: (y^*, z^*) such that $f(y, z^*) \leq f(y^*, z^*) \leq f(y^*, z)$ for all $y \in Y$ and $z \in Z$. For $(p, w) >> 0$,

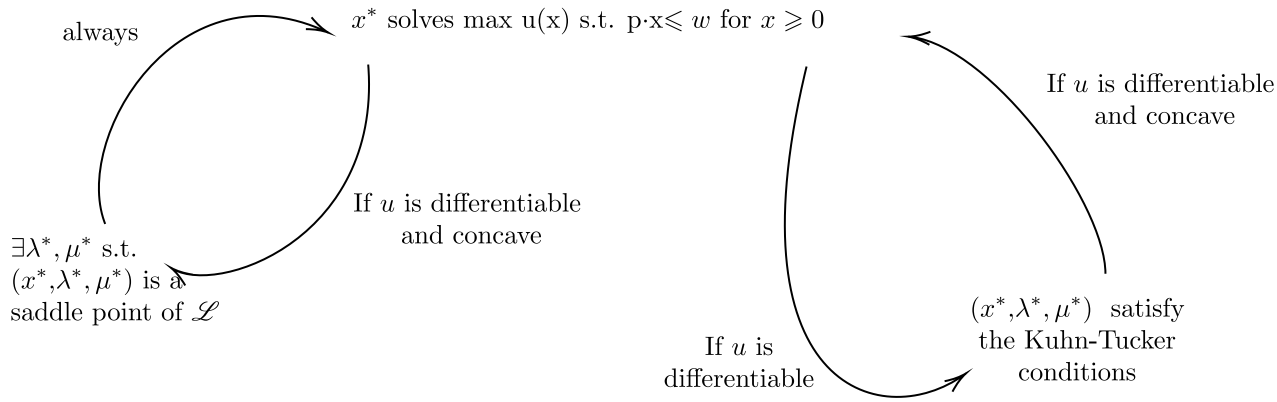
1. If $(x^*,^*, \mu^*)$ is a saddlepoint of $\mathscr{L}$, x^* solves the consumer problem.
2. If u is continuous and differentiable and concave, and if x^* solves the consumer problem, then there exist $\lambda^*, \mu^* \geq 0$ such that $(x^*,^*, \mu^*)$ is a saddlepoint of $\mathscr{L}$

Kuhn-Tucker Conditions:

1. First order condition with respect to x: $\frac{\partial u(x^*}{\partial x_i} - \lambda^* = 0\ \forall i$
2. Original Constraints: $p \cdot x^* \leq w$ and $x^* \geq 0$
3. Non-negativity multipliers: $\lambda^* \geq 0$ and $\mu_i^* \geq 0\ \forall i$
4. Complementary Slackness conditions: $\lambda^*(w - p \cdot x^*) = 0$ and $\mu_i x_i = 0\ \forall i$

Results relating Kuhn-Tucker conditions and the consumer problem:

- If u is continuously differentiable, $p >> 0$ and $w > 0$, and x^* solves the consumer problem, then $\exists \lambda^* \geq 0$ and $\mu^* \geq 0$ such that $(x^*,^*, \mu^*)$ satisfy the Kuhn-Tucker conditions.
- Suppose that u is continuously differentiable and quasi-concave, and that $u(x') > u(x)\nabla u(x) \cdot (x' - x) > 0$. Then if $(x^*,^*, \mu^*)$ satisfy the Kuhn-Tucker conditions, x^* satisfies the consumer problem.



2 Game Theory (Lones, Quarter 2)

2.1 Game Setups

Normal Form: Defined by a set of players N along with their strategies S and payoffs u.

Bayesian Game: Defined by

1. A set of players $N = \{1, \dots, i, \dots, N\}$
2. Action sets A_i for all $i \in \{1, ..., N\}$
3. A type set Θ
4. A probability of any type profiles $p : \Theta \to \mathbb{R}$
5. A payoff function $u_i : A \times \Theta \to \mathbb{R}$

A strategy specifies actions at each possible type $\theta \in \Theta$.

Extensive Form Game: Characterized by a game tree. A pure strategy specifies actions at all nodes.

2.2 Solution Concepts

Nash Equilibrium: Each person's strategy is their best reply to what everyone else does. Nearly all finite games have an odd number of Nash Equilibria.

Bayesian Nash Equilibrium: A mixed strategy profile $\sigma = (\sigma_i)_{i \in N}$ such that for all players $i \in N$ and and every type $\theta_i \in \Theta_i$ we have $\sigma_i(\cdot|\theta_i) \in$

$$\arg\max_{\tilde{\sigma} \in \Delta(A_i)} \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) \sum_{a \in A} \left(\prod_{j \in N \setminus \{i\}} \sigma_j(a_j|\theta_j)) \right) \tilde{\sigma}(a_i) u_i(a, \theta)$$

In other words, given a type, every agent optimizes their expected payoff, where the expectation is taken over the possible types of their opponents.

Subgame Perfect Nash Equilibrium: Of an extensive form game induces a Nash Equilibrium in every subgame of the game.

- A subgame is a part of a game tree such that
  1. It starts with an information set ι that is a single decision node
  2. It contains every successor to this node
  3. If it contains a node in an information set, then it includes all nodes in that information set (no "broken" information sets).

Sequential Equilibrium: A set of strategies β and beliefs μ that obey sequential rationality and consistency. *Sequential Rationality* players do what they ought to rationally do at each decision node given their beliefs. (β, μ) satisfies *consistency* if there exists a sequence $(\beta_n, \mu_n) \to (\beta, \mu)$ where $\beta_n >> 0$ and each $\beta_n, \mu_n)$ is weakly consistent.[1]

Types of Equilibria

- **Pooling** equilibria are where agents of different types take different actions.
- **Separating** equilibria occur when agents of different types take different actions.

2.3 Solution Methods

Iterated Strict Dominance: ISD_k is the set of strategies that survive k rounds of deletion of all possible strictly dominated strategies. Not that *strict* dominance is required for removal—removing weakly dominated strategies does not yield consistent results.

Zermelo's Theorem (backward induction): At each preterminal decision node, choose an optimal action. Then, repeat at the unique precursor nodes with the "trimmed tree." The theorem states that this method identifies a Nash Equilibrium.

Simplex Method: Useful for finding mixed strategies in one-time games.

Games of Pre-emption/Attrition: COME BACK!

Repeated Games:

- $1 + \delta + \delta^2 + \dots = \frac{1}{1-\delta}$ for $\delta \in [0, 1)$
- Remember to check that the punishment phase is sustainable, and also that the equilibrium continues to be sustainable in the future.

2.4 Theorems

Minmax Theorem: Suppose that S_1 and S_2 are finite. Then

$$\max_{\sigma_1 \in \Delta(S_1)} \min_{\sigma_2 \in \Delta(S_2)} u_1(\sigma_1, \sigma_2) = \min_{\sigma_2 \in \Delta(S_2)} \max_{\sigma_1 \in \Delta(S_1)} u_1(\sigma_1, \sigma_2)$$

in zero-sum games.

- **Minmax Strategy**: Player i's minmax strategy m^i satisfies $u_i(m^i) = \min_{a_{-i}} \max_{a_i} u_i(a_i, a_{-i})$.

[1] Weakly consistent essentially means "Bayes' rule whenever possible."

Glicksberg Fixed Point Theorem: Suppose that all player action spaces are compact, convex sets of $\mathbb{R}^k$ and that each payoff function is continuous. Then the game has at least one (possibly mixed) equilibrium.

Two-Player Folk Theorem: For any two-player normal form game, any payoff that strictly exceeds the minmax payoff can support a subgame perfect equilibrium if the players are patient enough.

Krepps and Wilson Theorem: A finite extensive form game of perfect recall has a sequential equilibrium. A sequential equilibrium is a subgame perfect equilibrium.

2.5 Misc.

Supermodular and Submodular Games

- **Supermodular games** occur when all payoffs $u(s_i, s_{-i})$ have increasing differences ($\frac{\partial^2 u}{\partial s_i \partial s_{-i}} \geq 0$) for all i. These games have "strategic complements."
- **Submodular games** feature decreasing differences and are characterized by "strategic substitutes." By the *attenuation effect*, the effect of a parameter change in a submodular game is less than the private effect.

Relevant Statistics

- **Jensen's Inequality**: u is convex on $[a, b]$ iff $u(\mathbb{E}[x]) \geq \mathbb{E}[u(x)]$ for all random X on $[a, b]$. The opposite inequality holds for concave u, and this holds with equality for linear u.
- **Bayes' Rule** is given by

$$P(A|B) = \frac{P(A)P(B|A)}{P(B|A)P(A) + P(B|A^C)P(A^C)}$$

Key Reminders

- If continuous payoffs are concave, mixed strategies will never be optimal.
- Remember to make sure that the best response function maps to a feasible set.
- Remember to use Bayes' rule when checking beliefs in sequential equilibria testing.

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3 General Equilibrium (Lones, Quarter 3)

3.1 Matching Models

Differed acceptance algorithm (DAA): Say that the men propose. Then the algorithm is (i) All men start unengaged and women start with no suitors, (ii) each man proposes to his most preferred woman among those he has not already proposed to (if he prefers matching with this woman to being single), (iii) each woman gets engaged to the suitor she most prefers, including any prior engagements, assuming that she prefers the match to remaining single, (iv) repeat until no more proposals are possible.

Properties:

- The DAA stops in finite time. In particular, for n "men" and n "women," the maximum number of proposals is $n^2 - 2n + 2$.
- Given an equal number of "men" and "women," if matching with someone is always better than remaining single then everyone matches.
- The DAA matching is stable. A corollary is that a stable matching always exists.
- Given strict preferences, the DAA yields a unique matching.
- The DAA outcome is male-optimal if the males propose and female-optimal if the females propose first.
  - In a *male-optimal* matching, every man pairs with his best valid partner. A *valid partner* of x is some y such that there exists a stable matching in which x and y are paired.
  - Corollary: The DAA produces the same outcome regardless of whether males or females propose if and only if there is a unique stable matching.

3.1.1 Matching Welfare Theorems

Efficient matching: An efficient matching $m \in \mathscr{M}$ maximizes the sum of all match benefits $\sum_x \sum_y m(x, y)h(x, y)$ over $\mathscr{M}$.

Competitive Equilibrium: A competitive equilibrium (in which wages/returns are paid to x and w for matching) satisfies feasibility and

1. Free Entry: $v(x) + w(y) \geq h(x, y)$ for any (x, y)
2. Free Exit: $v(x) + w(y) \leq h(x, y)$ if $m(x, y) > 0$

Note that this implies that $v(x) + w(y) = h(x, y)$ if $m(x, y) > 0$.

1st welfare theorem of matching: A Competitive equilibrium yields an efficient matching.

2nd welfare theorem of matching: An efficient matching is a competitive equilibrium for a suitable set of wages.

Taxes and Subsidies

- The largest non-distortative tax is the difference between the highest price for which the equilibrium quantity is demanded and the lowest price for which that quantity is supplied.
- The largest non-distortative subsidy is the difference between the highest price for which the equilibrium quantity is supplied and the lowest price for which that quantity is demanded.

3.1.2 Matching in the continuum setting

Non-transferable utility (NTU): In the case of NTU, employ the comonotone theorem.

- f and g are comonotone if
  - $\forall y_2 > y_1$ and $x_2 > x_1$, $[f(y_2|x) - f(y_1|x)] \cdot [g(x_2|y) - g(x_1|y)] > 0 \; \forall x, y$. The opposite inequality is called *reverse comonotone*.
  - If f and g are differentiable, then both $\frac{f(y|x)}{\partial y}$ and $\frac{g(x|y)}{\partial x}$ have the same sign if comonotone.
- Theorem: The NTU unique stable matching is positive assortative matching (PAM) if f and g are comonotone, and negative assortative matching (NAM) if f and g are reverse comonotone.

Transferable Utility (TU): In the case of transferable utility, employ the Beckmer Marriage model.

- A match function $h(x, y)$ is *(strictly) supermodular* if $h(x', y') + h(x, y) \geq (>) h(x', y) + h(x, y')$ for any pair $x' \geq x$ and $y' \geq y$.

- The reverse inequality is called *(strict) submodularity*, and the case of equality is called *modular*.
- $h(x, y)$ is differentiable, then $h(x, y)$ is supermodular if $\frac{^2h(x,y)}{\partial x \partial y} > 0$ and submodular if $\frac{^2h(x,y)}{\partial x \partial y} < 0$.

3.2 Supply and Demand Markets

Excess demand: $x^i - \bar{x}_i$, where x^i is an individual's demand for good i and $\bar{x}_i$ is that individual's endowment of good i. Net demand is the sum of excess demands across individuals.

Walrasian Price Stability: Implement this rule: If net demand is positive at some price, then raise the price. If net demand is negative at some price, then decrease the price. If this rule pushes toward the equilibrium point in some neighborhood of the equilibrium price, then that equilibrium is Walrasian price stable.

Marshallian Quantity Stability: Implement this rule: Given a quantity, if the seller's price exceeds the demand price, then decrease the quantity. If the demand price exceeds the supply price, increase the quantity. If this rule pushes toward the equilibrium point in some neighborhood of the equilibrium quantity, then that equilibrium is Marshallian quantity stable.

3.2.1 Elasticity

The **elasticity of demand** $\epsilon_d = \frac{\partial Q}{\partial P}\frac{P}{Q} = \frac{\partial log(Q)}{\partial log(P)} \approx \frac{\text{\% change in quantity}}{\text{\% change in price}}$

Le Chatlier's Principle: The absolute change in any choice variable is weakly higher in the long run that in the short rune. (The |long run elasticity| > |short run elasticity|).

Tax Incidence: The more inelastic side of the market pays more of a tax and benefits more from a subsidy.

Lerner Index: A monopolist prices where $L = \frac{1}{|\epsilon|} < 1$ because it is never profit-maximizing to set a price along the inelastic portion of the demand curve where $|\epsilon| < 1$. Did I copy that down right?

3.2.2 Market Power

Cournot competition: (Simultaneous actions anticipating a Nash equilibrium). All firms optimize simultaneously given the quantities produced by others.

Stackelberg competition: (Sequential actions anticipating a sequential partial Nash equilibrium). Solve using backwards induction from what the last-acting firm will do, maximizing iteratively over the "trimmed tree."

Herfindahl Index: $H = \sum_i s_i^2 = \sum_i \left(\frac{q_i}{\sum_i q_i}\right)^2$

Price discrimination

1. First-degree price discrimination gives every consumer an individual price.

2. Second-degree price discrimination occurs when sellers charge a different per-unit price for different quantities purchased.

3. Third-degree price discrimination occurs when a seller chargers different prices to different consumers.

3.3 Market Failures

3.3.1 Externalities

In general for these problems, find the socially optimal outcome and enforce it through one of the following:

- **Pigouvian tax**: Forces the offender to internalize the social effects of their actions through a tax.

- **Coase solution**: Some group has a kind of property right/ability/voucher for some amount of production, and then there are transfers between groups to reach the efficient outcome.

Theorem: Assume well-defined property rights, negotiations that freely realize all gains from trade, and transfers that do not affect marginal values. Then, the efficient outcome arises irrespective of property rights.

3.3.2 Public Goods

Samuelson Condition The optimal consumption of the public good occurs where $\sum_{i=1}^{n} MRS_{G,m} = \sum_{i=1}^{n} \frac{\partial u_i/\partial G}{\partial u_i \partial m} = MRT_{G,m} = \frac{\partial/\partial G(G=f(m))}{\partial/\partial m(G=f(m))} = \frac{1}{f'(m)}$

Lindahl Equilibrium: an allocation of public and private goods $(G^*, x_1^*, \ldots, x_n^*)$ and individual goods prices $(p_1, \ldots, p_n)$ with sum $p = \sum_{i=1}^{n} p_i$ such that every consumer i chooses (G^*, x_i^*) given price p_i for G. That is, $(G^*, x_i^*) \in \arg\max_{x_i,G} u_i(G, x_i)$ such that $x_i + p_i G = w$

3.4 General Equilibrium

3.4.1 Endowment economies

Edgeworth boxes

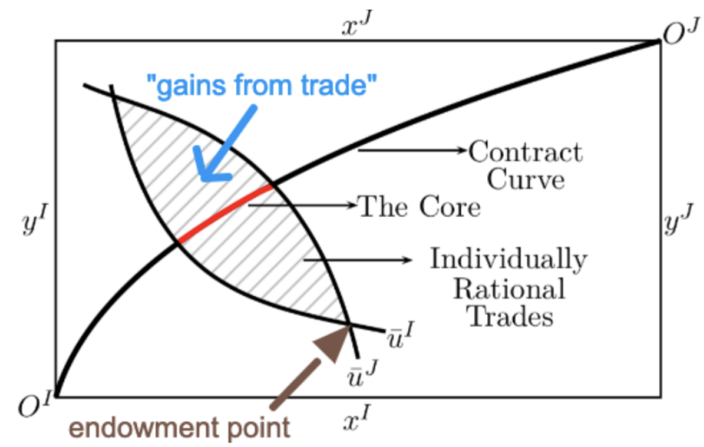

Figure 1: Edgeworth box

Trade Offer Curves plot optimal consumption allocations as prices vary, fixing endowments.

First Welfare Theorem If (p, x) is a competitive equilibrium of endowment economy $\mathscr{E}$, and preferences are locally nonsatiated, then x is socially efficient.

Second Welfare Theorem Assume that consumers have continuous, monotone, and quasiconcave utility functions. If $x \in \mathbb{R}_+^{ln}$ is a socially efficient allocation, then there exists a price $p \in \mathbb{R}_+^{l}$ such that (x, p) is a competitive equilibrium of exchange economy $\mathscr{E} = (\{u_i\}, \bar{x})$

Walras' Law: If trader's consumer their entire income at an allocation $x(p)$, then the market value of excess demand vanishes (i.e., $\sum_{l=1}^{L} p_l ED_l(p)$).

- *Corollary* It suffices to check that $L - 1$ of L markets clear when solving for a competitive equilibrium.

- **Uniqueness of Walrasian Equilibrium**: Deman has the *gross substitutes property* if an increase in some price p_k raises the demand for every other good x_l with $l \neq k$. If the aggregate excess demand function has this property, then the economy has at most one Walrasian Equilibrium.

3.4.2 Production Economies

Arrow and Debreu Theorem: Assume every consumer $i \in \{1, \dots, n\}$ has a continuous, locally nonsatiated, and strictly quasiconcave u_i, some endowment $\bar{x}^i \in \mathbb{R}^l_+$, and dividend shares θ_{ij}. Assume firms $j \in \{1, \dots, m\}$ have closed and convex production technologies. A competitive equilibrium exists.

- *Corollary (Existence in General Exchange Markets*: Assume every consumer $i \in \{1, \dots, n\}$ has a continuous, locally nonsatiated, and strictly quasiconcave u_i, as well as some endowment $\bar{x}^i \in \mathbb{R}^l_+$. Then a competitive equilibrium exists.

Theorem: Efficiency $\Leftrightarrow$ Equilibrium: (The first and second welfare theorems in the context of a production economy)

- If (x, y, p) is a competitive equilibrium and preferences are locally nonsatiated, then (x, y) is a socially efficient allocation.

- Conversely, assume monotonic and convex preferences, as well as closed and convex technologies. If (x, y) is socially efficient, then (x, y, p) is a competitive equilibrium for some prices p, endowments $\bar{x}$, and ownership shares θ.

3.5 Risk

Complete Markets: Markets in which there exists at least one Arrow security for every state-contingent claim (or if the securities that exist span the states).

Fair prices for Arrow securities are those for which the price is proportional to the probability of the state in which the security pays (i.e., $p_i = \pi_i$)

Fundamental Theorem of Risk Bearing: $\frac{\pi_1 u'(x_1)}{p_1} = \cdots = \frac{\pi_s u'(x_s)}{p_s}$

3.6 Spatial Competition and Models of Firm Entry

3.6.1 Hotelling Model

Stages: Firms choose a location $a_i \in [0, 1]$ and then choose a price.

Solve by backwards induction:

- Find the location of the indifferent consumer t as a function of the locations and prices of each firm.

- Maximize profits for each firm through their choice of prices.

- Maximize profits for each firm through their choice of location, given the result for their subsequent choices of prices.

3.6.2 Salop Circle Market

Stages:

1. Firms decide on entry given some fixed cost F.

2. Firms decide on location.

3. Firms choose prices.

Solve by backwards induction:

- For an arbitrary firm j, find the indifferent consumer t on either side and use this to construct a profit function $\pi_j(p_j | p_{j+1}, p_{j-1})$. Maximize this with respect to p_j and, in the symmetric case, find prices such that $p_j^* = p_i^* \; \forall i, j$

- If cost functions are convex, it can be shown that firms always choose to be arranged evenly around the circle.

- Then, find the number of firms "n" using the given fixed cost F so that $\pi_n(p^*) \geq F$ (ensures that n firms enter) and $\pi_{n+1}(p^*) < F$ (so that the $n + 1$ firm will not enter)

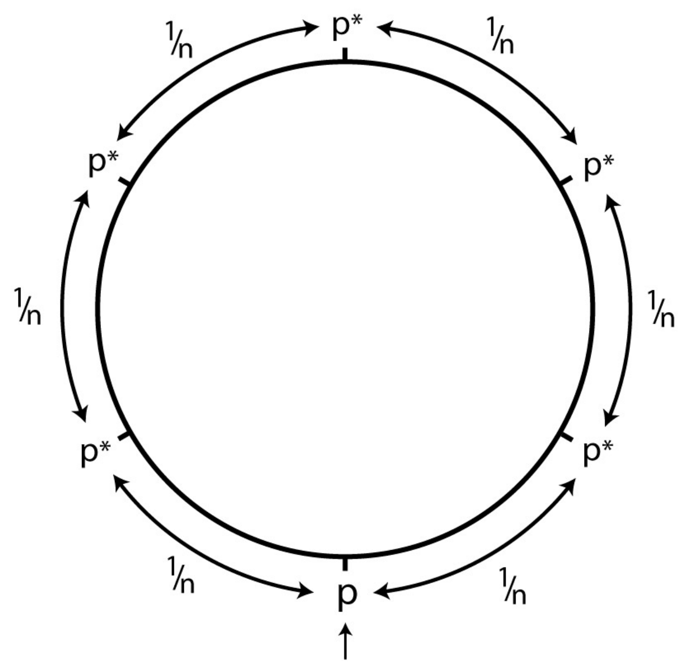

Figure 2: Salop Circle

3.7 Broad reminders

- For the Cobb-Douglas case of $\max x^\alpha y^{1-\alpha}$ s.t. $p_x x + p_y y \leq w$, $x^* = \frac{\alpha w}{p_x}$ and $y^* = \frac{(1-\alpha)w}{p_y}$