Macro PS4

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1 Question 1

The household maximizes their utility subject to their budget constraint. Equivalently, households minimize their costs subject to their utility constraint:

$$\min_{C_{ik}} \int \sum_i P_{ik} C_{ik} dk$$

$$\text{s.t.} \quad \left(\int C_k^{\frac{\rho-1}{\rho}} dk \right)^{\frac{\rho}{\rho-1}} = C,$$

$$\text{where} \quad \left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} = C_k.$$

We will then write the Lagrangian as follows:

$$\mathcal{L} = \int \sum_i P_{ik} C_{ik} dk - P\left(\left(\int C_k^{\frac{\rho-1}{\rho}} dk \right)^{\frac{\rho}{\rho-1}} - C \right) + \int P_k \left[C_k - \left(\sum_i C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} \right] dk$$

We solve this maximization problem by taking first order conditions with respect to our choice variables, in this case C_{ik}, C_k:

$$P_k = \frac{\rho}{\rho-1} \left(\int C_k^{\frac{\rho-1}{\rho}} dk \right)^{\frac{1}{\rho-1}} \frac{\rho-1}{\rho} C_k^{\frac{-1}{\rho}}$$

$$\Rightarrow C_k = \left(\frac{P_k}{P} \right)^{\rho} C.$$

$$P_{ik} = P_k \frac{\theta}{\theta-1} \left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{1}{\theta-1}} \frac{\theta-1}{\theta} C_{ik}^{-\frac{1}{\theta}}$$

$$\Rightarrow C_{ik} = \left(\frac{P_{ik}}{P_k} \right)^{\theta} C_k$$

We then can simplify our consumption first order condition to the following:

$$P_k \left(\frac{C_{ik}}{C_k} \right)^{\frac{1}{\theta}} = P C^{\frac{1}{\rho}} C_k^{\frac{-1}{\rho}} \left(\frac{C_{ik}}{C_k} \right)^{\frac{1}{\theta}}$$

$$\Rightarrow C_k = \left(\frac{P_k}{P} \right)^{-\rho} C,$$

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a familiar expression to what we found in lecture.
We can substitute in our expressions into the definitions of C, C_k:

$$\left(\int \left[\left(\frac{P_k}{P}\right)^{-\rho} C\right]^{\frac{\rho-1}{\rho}} dk\right)^{\frac{\rho}{\rho-1}} = C$$

$$\Rightarrow \left(\int \left(\frac{P_k}{P}\right)^{1-\rho} dk\right)^{\frac{\rho}{\rho-1}} = 1$$

$$\Rightarrow \left(\int P_k^{1-\rho} dk\right)^{\frac{1}{1-\rho}} = P,$$

$$\left(\sum_i \left[\left(\frac{P_{ik}}{P_k}\right)^{\theta} C_k\right]^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}} = C_k$$

$$\Rightarrow \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{1}{1-\theta}} = P_k$$

To summarize, we have the following:

$$P_k = \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{1}{1-\theta}} \tag{1}$$

$$P = \left(\int \left[\left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{1}{1-\theta}}\right]^{1-\rho} dk\right)^{\frac{1}{1-\rho}} \tag{2}$$

$$C_{ik} = P_{ik}^{-\theta} \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} P^{\rho} C \tag{3}$$

2 Question 2

The firms compete a la Cournot:

$$\max_{P_{ik}} P_{ik} C_{ik} - W L_{ik}$$
$$\text{s.t.} C_{ik} = P_{ik}^{-\theta} \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} P^{\rho} C$$
$$\text{and } C_{ik} = A_{ik} L_{ik}$$

Substituting, we form the following objective function:

$$\max_{P_{ik}} P_{ik}^{1-\theta} \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} P^{\rho} C - W A_{ik}^{-1} P_{ik}^{-\theta} \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} P^{\rho} C$$

$$\Rightarrow \max_{P_{ik}} P_{ik}^{1-\theta} \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} - W A_{ik}^{-1} P_{ik}^{-\theta} \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}}$$

We take first order conditions:

$$(1-\theta)P_{ik}^{-\theta}P_k^{\theta-\rho} + P_{ik}^{1-\theta}\frac{\theta-\rho}{1-\theta}P_k^{2\theta-\rho-1}(1-\theta)P_{ik}^{-\theta} = \frac{W}{A_{ik}}\left[(-\theta)P_{ik}^{-\theta-1}P_k^{\theta-\rho} + P_{ik}^{-\theta}\frac{\theta-\rho}{1-\theta}P_k^{2\theta-\rho-1}(1-\theta)P_{ik}^{-\theta}\right]$$
$$(1-\theta)P_k^{\theta} + P_{ik}^{1-\theta}(\theta-\rho)P_k^{2\theta-1} = \frac{W}{A_{ik}}\left[(-\theta)P_{ik}^{-1}P_k^{\theta} + P_{ik}^{-\theta}(\theta-\rho)P_k^{2\theta-1}\right]$$

3 Question 3

4 Question 4

5 Question 5

6 Question 6