

Econometrics HW2

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1 3.2

$$\begin{aligned}\hat{\beta}_{ols} &= (X'X)^{-1}X'Y \\ \hat{\beta}_{mix} &= (Z'Z)^{-1}Z'Y \\ &= (C'X'XC)^{-1}C'X'Y \\ &= C^{-1}(X'X)^{-1}C'^{-1}C'X'Y \\ &= C^{-1}\hat{\beta}_{ols} \\ \hat{\epsilon}_{ols} &= (I - X(X'X)^{-1}X')Y \\ \hat{\epsilon}_{mix} &= (I - XCC^{-1}(X'X)^{-1}X')Y \\ &= (I - X(X'X)^{-1}X')Y \\ &= \hat{\epsilon}_{ols}\end{aligned}$$

2 3.5

$$\begin{aligned}\hat{\epsilon} &= Y - X'(X'X)^{-1}X'Y = (I - X(X'X)^{-1}X')Y \\ \hat{\beta}_e &= (X'X)^{-1}X'\hat{\epsilon} \\ &= (X'X)^{-1}X'(I - X(X'X)^{-1}X')Y \\ &= (X'X)^{-1}X'Y - (X'X)^{-1}X'X(X'X)^{-1}X'Y \\ &= (X'X)^{-1}X'Y - (X'X)^{-1}X'Y \\ &= 0.\end{aligned}$$

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3 3.6

$$\begin{aligned}
\hat{\beta}_{\hat{Y}} &= (X'X)^{-1}X'\hat{Y} \\
&= (X'X)^{-1}X'X(X'X)^{-1}X'Y \\
&= (X'X)^{-1}X'Y \\
&= \hat{\beta}_{ols}
\end{aligned}$$

4 3.7

Note that $X_1 = X\Gamma$ where $\Gamma = \begin{pmatrix} I_{n_1} \\ \bar{0} \end{pmatrix}$ where $\bar{0}$ is an $n_2 \times n_1$ vector of zeros. Then,

$$\begin{aligned}
PX_1 &= X(X'X)^{-1}X'X_1 \\
&= X(X'X)^{-1}X'X\Gamma \\
&= X\Gamma \\
&= X_1.
\end{aligned}$$

$$\begin{aligned}
MX_1 &= (I - X(X'X)^{-1}X')X\Gamma \\
&= (X - X(X'X)^{-1}X'X)\Gamma \\
&= (X - X)\Gamma \\
&= 0.
\end{aligned}$$

5 3.11

Let X contain a constant.

$$\begin{aligned}
\frac{1}{n} \sum_{i=1}^n \hat{Y}_i &= \frac{1}{n} \sum_{i=1}^n Y_i - \hat{\epsilon}_i \\
&= \frac{1}{n} \sum_{i=1}^n Y_i - \frac{1}{n} \sum_{i=1}^n \hat{\epsilon}_i \\
&= \frac{1}{n} \sum_{i=1}^n Y_i - X'\hat{\epsilon} \\
&= \frac{1}{n} \sum_{i=1}^n Y_i
\end{aligned}$$

where $\frac{1}{n} \sum_{i=1}^n \hat{\epsilon}_i = X'\hat{\epsilon} = 0$ because X contains a column of ones.

6 3.12

Equation (3.53) cannot be estimated by OLS because $D_1 + D_2 = \vec{1}$ (a vector containing 1 in every element), and therefore $X'X$ is not invertible.

6.1 Part A

Equations (3.54) and (3.55) contain the same information, since $D_1 + D_2 = \vec{1}$, and so the \hat{Y} from each regression would be identical. Ergo,

$$\begin{aligned} D_1\alpha_1 + D_2\alpha_2 + e &= (\vec{1} - D_2)\alpha_1 + D_2\alpha_2 \\ &= \vec{\alpha}_1 + D_2(\alpha_2 - \alpha_1) \end{aligned}$$

Therefore, the regressions are the same with $\mu = \vec{\alpha}_1$ (the vector with α_1 as every element) and $\phi = \alpha_2 - \alpha_1$.

6.2 Part B

$$\begin{aligned} \vec{1}'D_1 &= \sum_{i=1}^n 1\{\text{person } i \text{ is a man}\} \\ &= n_1, \\ \vec{1}'D_2 &= \sum_{i=1}^n 1\{\text{person } i \text{ is a woman}\} \\ &= n_2. \end{aligned}$$

7 3.13

7.1 Part A

7.2 Part B

8 3.16

Let $X = [X_1 X_2]$, $\hat{\beta} = [\hat{\beta}_1' \hat{\beta}_2']'$, $\hat{\beta}^* = [\tilde{\beta}_1' \vec{0}_{n_2}']'$ where $\vec{0}_{n_2}$ is the n_2 sized matrix of zeros.

$$\begin{aligned}
 R_2^2 &= 1 - \frac{\sum_{i=1}^n \hat{e}_i^2}{\sum_{i=1}^n (Y_i - \bar{Y})^2} \\
 &= 1 - \frac{\hat{e}'\hat{e}}{\sum_{i=1}^n (Y_i - \bar{Y})^2} \\
 &= 1 - \frac{(Y - X\hat{\beta})'(Y - X\hat{\beta})}{\sum_{i=1}^n (Y_i - \bar{Y})^2} \\
 &\geq 1 - \frac{(Y - X\hat{\beta}^*)'(Y - X\hat{\beta}^*)}{\sum_{i=1}^n (Y_i - \bar{Y})^2} \\
 &= R_1^2,
 \end{aligned}$$

where the inequality comes from the fact that OLS minimizes the sum of squared residuals.

Yes, if X_2 is orthogonal to Y then $X_2'Y = 0 \Rightarrow \hat{\beta}_2 = 0 \Rightarrow \tilde{\beta} = \hat{\beta} \Rightarrow R_2^2 = R_1^2$.

9 3.21

If one or both of X_1, X_2 is orthogonal to Y , or if X_1, X_2 are orthogonal to each other, then $\tilde{\beta}_1 = \hat{\beta}_1, \tilde{\beta}_2 = \hat{\beta}_2$.

The first condition is nearly immediate, as whichever regressor is orthogonal will have estimated coefficients of 0 in both equations, and the equation with both regressors included reduces to the "one regressor at a time" estimator equation, so the coefficients in both have the same estimated value. Moreover, if both regressors are orthogonal to Y then all of the coefficient estimates will be 0.

Now we concern ourselves with the final case, where X_1, X_2 are orthogonal. Then, by theorem 3.4 we have that:

$$\begin{aligned}
 \hat{\beta}_1 &= (X_1' M_2 X_1)^{-1} (X_1' M_2 Y) \\
 &= ((M_2 X_1)' (M_2 X_1))^{-1} ((M_2 X_1)' Y) \\
 &= (X_1' X_1)^{-1} (X_1' Y) \\
 &= \tilde{\beta}_1
 \end{aligned}$$

By symmetry, the same condition ensures $\hat{\beta}_2 = \tilde{\beta}_2$

10 3.22

$$\begin{aligned}
\tilde{\beta} &= (X_1'X_1)^{-1}X_1'Y \\
\tilde{u} &= Y - X_1\tilde{\beta} \\
\tilde{\beta}_2 &= (X_2'X_2)^{-1}X_2'\tilde{u} \\
&= (X_2'X_2)^{-1}X_2'(Y - X_1\tilde{\beta}_1) \\
\hat{\beta}_2 &= (X_2'X_2)^{-1}X_2'(Y - X_1\hat{\beta}_1)
\end{aligned}$$

Therefore, this is only the case when $\tilde{\beta}_1 = \hat{\beta}_1$. As we showed in the previous problem, this occurs when X_1, X_2 are orthogonal (or when one (or both) of the regressors is orthogonal to Y).

11 3.23

The residuals are the same from both equations, which I will show below, and therefore the residual variance estimates, a function of the estimated residuals, are the same from both regressions. Therefore, $\hat{\sigma}^2 = \tilde{\sigma}^2$.

Now we will show that the residuals are the same.

$$\begin{aligned}
\tilde{\beta}_2 &= ((X_2 - X_1)'M_1(X_2 - X_1))^{-1}((X_2 - X_1)'M_1Y) \\
&= (X_2'X_2)^{-1}X_2'Y \\
&= \hat{\beta}_2. \\
\tilde{\beta}_1 &= (X_1'X_1)^{-1}X_1'(Y - (X_2 - X_1)\tilde{\beta}_2) \\
&= (X_1'X_1)^{-1}X_1'Y - (X_1'X_1)^{-1}X_1'(X_2 - X_1)\tilde{\beta}_2 \\
&= (X_1'X_1)^{-1}X_1'(Y - X_2\hat{\beta}_2) + (X_1'X_1)^{-1}X_1'X_1\hat{\beta}_2 \\
&= \hat{\beta}_1 + \hat{\beta}_2. \\
\Rightarrow \tilde{\epsilon} &= X_1\tilde{\beta}_1 + (X_2 - X_1)\tilde{\beta}_2 \\
&= X_1(\hat{\beta}_1 + \hat{\beta}_2) + (X_2 - X_1)\hat{\beta}_2 \\
&= X_1\hat{\beta}_1 + X_2\hat{\beta}_2 \\
&= \hat{\epsilon}.
\end{aligned}$$

12 Question 7