Econometrics HW6

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1 Question 1

1.1 Part i

$\hat{\mu}_{ols}$ can be computed as the average of all observations in the sample:

$$\begin{aligned}\hat{\mu}_{ols} &= \frac{1}{\sum_{i=1}^n T_i}\sum_{i=1}^n\sum_{t=1}^{T_i} Y_{it} \\ &= \frac{\sum_{i=1}^n 1_i' Y_i}{\sum_{i=1}^n 1_i' 1_i}.\end{aligned}$$

1.2 Part ii

We write the estimator as signal plus noise:

$$\begin{aligned}\hat{\mu}_{iv} &= \frac{\sum_{i=1}^n Z_i' Y_i}{\sum_{i=1}^n Z_i' 1_i} \\ &= \frac{\sum_{i=1}^n Z_i'(\mu_0 1_i + \alpha_i 1_i + \epsilon_i)}{\sum_{i=1}^n Z_i' 1_i} \\ &= \mu_0 + \frac{\sum_{i=1}^n Z_i'(\alpha_i 1_i + \epsilon_i)}{\sum_{i=1}^n Z_i' 1_i}.\end{aligned}$$

Now, we can find the variance:

$$\begin{aligned}Var(\hat{\mu}_{iv}) &= Var\left(\frac{\sum_{i=1}^n Z_i'(\alpha_i 1_i + \epsilon_i)}{\sum_{i=1}^n Z_i' 1_i}\right) \\ &= \frac{\sum_{i=1}^n Z_i' Var(\alpha_i 1_i + \epsilon_i) Z_i}{(\sum_{i=1}^n Z_i' 1_i)^2} \\ &= \frac{\sum_{i=1}^n Z_i' \Omega_i Z_i}{(\sum_{i=1}^n Z_i' 1_i)^2},\end{aligned}$$

where

$$\begin{aligned}\Omega_i &= Var(\alpha_i 1_i + \epsilon_i) \\ &= \sigma_\alpha^2 1_i 1_i' + \sigma^2 I_{T_i} \\ &= \sigma_\alpha^2 1_i 1_i' + \sigma^2 I_{T_i}.\end{aligned}$$

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, Katherine Kwok, and Danny Edgel.

1.3 Part iii

We will apply Cauchy-Schwarz.

$$Z_i'1_i = Z_i'\Omega_i^{1/2}\Omega_i^{-1/2} \leq ||Z_i\Omega_i^{1/2}||||\Omega_i^{-1/2}1_i||,$$

$$\left(\sum_{i=1}^{n} Z_i'1_i\right)^2 \leq \left(\sum_{i=1}^{n} ||\Omega_i^{1/2}Z_i||||\Omega_i^{-1/2}1_i||\right)^2 \leq \sum_{i=1}^{n} Z_i'\Omega_i Z_i \cdot \sum_{i=1}^{n} 1_i'\Omega_i^{-1}1_i.$$

We then apply the latter to the formula for variance and get:

$$Var(\hat{\mu}_{iv}) \geq \frac{\sum_{i=1}^{n} Z_i'\Omega_i Z_i}{\sum_{i=1}^{n} Z_i'\Omega_i Z_i \sum_{i=1}^{n} Z_i'\Omega_i^{-1} Z_i}$$

If we use $\tilde{Z}_i = \Omega_i^{-1}1_i$ as an instrument then that has variance:

$$\frac{\sum_{i=1}^{n} \tilde{Z}_i\Omega_i\tilde{Z}_i}{(\sum_{i=1}^{n} \tilde{Z}_i'1_i)^2} = \frac{1}{\sum_{i=1}^{n} 1_i'\Omega_i'1_i}$$

1.4 Part iv

GLS and OLS are the same estimators when the panel is balanced. Therefore, GLS is not more efficient.

$$\Omega_i^{-1} = \frac{1}{\sigma^2} I_{T_i} - \frac{1}{\sigma 4} \frac{\sigma_a^2 I_{T_i} 1_i 1_i' I_{T_i}}{1 + \sigma_a^2 1_i' I_{T_i} 1_i/\sigma^2} = \frac{1}{\sigma^2}\left(I_{T_i} - \frac{T_i\sigma_a^2}{T_i\sigma_a^2 + \sigma^2}\frac{1_i 1_i'}{T_i}\right)$$

$$\tilde{Z}_i = \Omega_i^{-1}1_i = \frac{1_i}{T_i\sigma_a^2 + \sigma^2}.$$

Now we impose $T_i = T$:

$$\hat{\mu}_{gls} = \frac{\sum \tilde{Z}_i'Y_i}{\tilde{Z}_i'1_i} = \frac{\sum \frac{1}{T\sigma_a^2+\sigma^2}1_i'Y_i}{\sum \frac{1}{T\sigma_a^2+\sigma^2}1_i'1_i} = \frac{\sum 1_i'Y_i}{\sum 1_i'1_i} = \hat{\mu}_{ols}.$$

1.5 Part v

Let $\bar{\epsilon}_i = \frac{1}{T_i}\sum_{t=1}^{T_i} \epsilon_{it}$.

$$\begin{aligned}
\hat{\sigma}_i^2 &= \frac{1}{T_i - 1}\sum_{t=1}^{T_i}(Y_{it} - \bar{Y}_i)^2 \\
&= \frac{1}{T_i - 1}\sum_{t=1}^{T_i}(\epsilon_{it} - \bar{\epsilon}_i)^2 \\
&= \frac{1}{T_i - 1}\sum_{t=1}^{T_i}\epsilon_{it}(\epsilon_{it} - \bar{\epsilon}_i) \\
&= \frac{1}{T_i - 1}\sum_{t=1}^{T_i}\epsilon_{it}^2 - \frac{1}{T_i(T_i - 1)}\sum_{t=1}^{T_i}\sum_{s=1}^{T_i}\epsilon_{is}\epsilon_{it} \\
&= \frac{1}{T_i}\sum_{t=1}^{T_i}\epsilon_{it}^2 - \frac{1}{T_i(T_i - 1)}\sum_{t=1}^{T_i}\sum_{s=1,s\neq t}^{T_i}\epsilon_{is}\epsilon_{it}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
E[\hat{\sigma}_i^2] &= \frac{1}{T_i}\sum_{t=1}^{T_i} E[\epsilon_{it}^2] \\
&= \sigma^2.
\end{aligned}
$$

The estimator $\hat{\sigma}^2$ is an average of independent random variables, so under very mild assumptions (existence of fourth moment), consistency holds.

1.6 Part vi

$$
\begin{aligned}
E[\hat{\sigma}_{\alpha,i}^2(\mu)] &= E\left[\frac{1}{T_i}\sum_{t=1}^{T_i}(Y_{it}-\mu)^2 - \hat{\sigma}_i^2\right] \\
&= E\left[\frac{1}{T_i}\sum_{t=1}^{T_i}(\alpha_i+\epsilon_{it})^2 - \hat{\sigma}_i^2\right] \\
&= E\left[\frac{1}{T_i}\sum_{t=1}^{T_i}(\alpha_i^2+2\alpha_i\epsilon_{it}+\epsilon_{it}^2)\right] - \sigma^2 \\
&= \sigma_\alpha^2 + \sigma^2 - \sigma^2 \\
&= \sigma_\alpha^2
\end{aligned}
$$

By the same logic as in part (v), we have the average of uncorrelated random variables as our estimator, so the estimator is consistent under very mild assumptions.

1.7 Part vii

Extending (iv),

$$
\left(\sum_{i=1}^{n} 1_i'\Omega_i^{-1}1_i\right)^{-1} = \left(\sum_{i=1}^{n}\frac{T_i}{T_i\sigma_\alpha^2+\sigma^2}\right)^{-1}
$$

We can use a plug-in estimator,

$$
\hat{V} = \left(\sum_{i=1}^{n}\frac{T_i}{T_i\hat{\sigma}_\alpha^2+\hat{\sigma}^2}\right)^{-1}.
$$

This will be consistent by the continuous mapping theorem.

2 Question 2

2.1 Part i

$$\begin{aligned}
\hat{\beta}_{FE} &\rightarrow_p \beta_0 + \frac{E[\sum_{t=1}^{T}(X_{it} - \bar{X}_i)\epsilon_{it}]}{E[\sum_{t=1}^{T}(X_{it} - \bar{X}_i)^2]} \\
&= \beta_0 + \frac{E[\sum_{t=1}^{T} X_{it}\epsilon_{it}] - E[\sum_{t=1}^{T} \bar{X}_i\epsilon_{it}]}{(T-1)\sigma_x^2} \\
&= \beta_0 + \frac{-E[\sum_{t=1}^{T} \bar{X}_i\epsilon_{it}]}{(T-1)\sigma_x^2} \\
&= \beta_0 + \frac{-E[\sum_{t=1}^{T}\sum_{s=1}^{T} X_{is}\epsilon_{it}]}{T(T-1)\sigma_x^2} \\
&= \beta_0 + \frac{-(T-1)\delta\sigma_x^2}{T(T-1)\sigma_x^2} \\
&= \beta_0 - \frac{\delta}{T}
\end{aligned}$$

Therefore, the asymptotic bias of $\hat{\beta}_{FE}$ is $-\frac{\delta}{T}$.

$$\begin{aligned}
\hat{\beta}_{FD} &\rightarrow_p \beta_0 + \frac{E[\sum_{t=2}^{T}(X_{it} - X_{i,t-1})(\epsilon_{it} - \epsilon_{i,t-1})]}{E[\sum_{t=1}^{T}(X_{it} - X_{i,t-1})^2]} \\
&= \beta_0 + \frac{E[\sum_{t=2}^{T} X_{it}\epsilon_{it} - X_{i,t-1}\epsilon_{it} - X_{it}\epsilon_{i,t-1} + X_{i,t-1}\epsilon_{i,t-1}]}{E[\sum_{t=2}^{T} X_{it}^2 - 2X_{it}X_{i,t-1} + X_{i,t-1}^2)^2]} \\
&= \beta_0 + \frac{-(T-1)\delta\sigma_x^2}{2(T-1)\sigma_x^2} \\
&= \beta_0 + \frac{-\delta}{2}
\end{aligned}$$

Therefore, the asymptotic bias of $\hat{\beta}_{FD}$ is $-\frac{\delta}{2}$.

2.2 Part ii

For $T = 2$, the asymptotic biases are the same.

3 Question 3