Macro PS4

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February 20, 2021

1 Question 1

The household maximizes their utility subject to their budget constraint. Equivalently, households minimize their costs subject to their utility constraint:

$$\min_{C_{ik}} \int \sum_i P_{ik} C_{ik} dk$$
$$\text{s.t.} \quad \left(\int C_k^{\frac{\rho-1}{\rho}} dk \right)^{\frac{\rho}{\rho-1}} = C,$$
$$\text{where} \quad \left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} = C_k.$$

We will then write the Lagrangian as follows:

$$\mathcal{L} = \int \sum_i P_{ik} C_{ik} dk - P \left(\left(\int C_k^{\frac{\rho-1}{\rho}} dk \right)^{\frac{\rho}{\rho-1}} - C \right) + \int P_k \left[C_k - \left(\sum_i C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} \right] dk$$

We solve this maximization problem by taking first order conditions with respect to our choice variables, in this case C_{ik}, C_k:

$$P_k = \frac{\rho}{\rho-1} \left(\int C_k^{\frac{\rho-1}{\rho}} dk \right)^{\frac{1}{\rho-1}} \frac{\rho-1}{\rho} C_k^{\frac{-1}{\rho}}$$
$$\Rightarrow C_k = \left(\frac{P_k}{P} \right)^{-\rho} C.$$
$$P_{ik} = P_k \frac{\theta}{\theta-1} \left(\sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{1}{\theta-1}} \frac{\theta-1}{\theta} C_{ik}^{-\frac{1}{\theta}}$$
$$\Rightarrow C_{ik} = \left(\frac{P_{ik}}{P_k} \right)^{-\theta} C_k$$

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, Katherine Kwok, and Danny Edgel.

We can substitute in our expressions into the definitions of C, C_k:

$$\left(\int \left[\left(\frac{P_k}{P}\right)^{-\rho} C\right]^{\frac{\rho-1}{\rho}} dk\right)^{\frac{\rho}{\rho-1}} = C$$
$$\Rightarrow \left(\int \left(\frac{P_k}{P}\right)^{1-\rho} dk\right)^{\frac{\rho}{\rho-1}} = 1$$
$$\Rightarrow \left(\int P_k^{1-\rho} dk\right)^{\frac{1}{1-\rho}} = P,$$
$$\left(\sum_i \left[\left(\frac{P_{ik}}{P_k}\right)^{-\theta} C_k\right]^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}} = C_k$$
$$\Rightarrow \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{1}{1-\theta}} = P_k$$

To summarize, we have the following:

$$P_k = \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{1}{1-\theta}} \tag{1}$$
$$P = \left(\int \left[\left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{1}{1-\theta}}\right]^{1-\rho} dk\right)^{\frac{1}{1-\rho}} \tag{2}$$
$$C_{ik} = P_{ik}^{-\theta} \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} P^{\rho} C \tag{3}$$

2 Question 2

The firms compete a la Bertrand:

$$\max_{P_{ik}} P_{ik} C_{ik} - W L_{ik}$$
$$\text{s.t.} C_{ik} = P_{ik}^{-\theta} \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} P^{\rho} C$$
$$\text{and } C_{ik} = A_{ik} L_{ik}$$

Substituting, we form the following objective function:

$$\max_{P_{ik}} P_{ik}^{1-\theta} \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} P^{\rho} C - W A_{ik}^{-1} P_{ik}^{-\theta} \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} P^{\rho} C$$
$$\Rightarrow \max_{P_{ik}} P_{ik}^{1-\theta} \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}} - W A_{ik}^{-1} P_{ik}^{-\theta} \left(\sum_i P_{ik}^{1-\theta}\right)^{\frac{\theta-\rho}{1-\theta}}$$

We take first order conditions:

$$(1-\theta)P_{ik}^{-\theta}P_k^{\theta-\rho}+P_{ik}^{1-\theta}\frac{\theta-\rho}{1-\theta}P_k^{2\theta-\rho-1}(1-\theta)P_{ik}^{-\theta}=\frac{W}{A_{ik}}\left[(-\theta)P_{ik}^{-\theta-1}P_k^{\theta-\rho}+P_{ik}^{-\theta}\frac{\theta-\rho}{1-\theta}P_k^{2\theta-\rho-1}(1-\theta)P_{ik}^{-\theta}\right]$$
$$(1-\theta)+P_{ik}^{1-\theta}(\theta-\rho)P_k^{\theta-1}=\frac{W}{A_{ik}}\left[(-\theta)P_{ik}^{-1}+P_{ik}^{-\theta}(\theta-\rho)P_k^{\theta-1}\right]$$

Denote the weighted price ratio $s_{ik} := \left(\frac{P_{ik}}{P_k}\right)^{1-\theta}$:

$$P_{ik}[(1-\theta)+s_{ik}(\theta-\rho)]=\frac{W}{A_{ik}}\left[(-\theta)+s_{ik}(\theta-\rho)\right]$$
$$\Rightarrow P_{ik}=\frac{W}{A_{ik}}\left[1-\frac{1}{(1-\theta)+s_{ik}(\theta-\rho)}\right]$$

This yields a recursive expression for P_{ik}. In other words, we have a set of $i\times k$ nonlinear equations in $i\times k$ unknowns.

We can derive demand elasticities $\frac{P_{ik}\partial C_{ik}}{C_{ik}\partial P_{ik}}$:

$$\begin{aligned}\frac{P_{ik}\partial C_{ik}}{C_{ik}\partial P_{ik}}&=\frac{P_{ik}}{C_{ik}}\left((-\theta)P_{ik}^{-1-\theta}P_k^{\theta-\rho}+P_{ik}^{-\theta}\frac{\theta-\rho}{1-\theta}P_k^{2\theta-\rho-1}(1-\theta)P_{ik}^{-\theta}\right)P^{\rho}C\\&=\left(P_{ik}^{1+\theta}P_k^{\rho-\theta}P^{-\rho}C^{-1}\right)\left((-\theta)P_{ik}^{-1-\theta}P_k^{\theta-\rho}+P_{ik}^{-2\theta}(\theta-\rho)P_k^{2\theta-\rho-1}\right)P^{\rho}C\\&=\left((-\theta)+P_{ik}^{1-\theta}(\theta-\rho)P_k^{\theta-1}\right)\\&=(\theta-\rho)s_{ik}-\theta.\end{aligned}$$

3 Question 3

The markup of firm i in industry k, μ_{ik}, with marginal cost M_{ik} is the following:

$$\begin{aligned}\mu_{ik}&=P_{ik}/M_{ik}\\&=\frac{W}{A_{ik}}\left[1-\frac{1}{(1-\theta)+s_{ik}(\theta-\rho)}\right]\Big/\frac{W}{A_{ik}}\\&=\left[1-\frac{1}{(1-\theta)+s_{ik}(\theta-\rho)}\right]\end{aligned}$$

Taking the derivative with respect to A_{ik}:

$$\begin{aligned}\frac{\partial\mu_{ik}}{\partial A_{ik}}&=-\frac{\partial}{\partial A_{ik}}\left(\frac{1}{(1-\theta)+(\theta-\rho)s_{ik}}\right)\\&=\left(\frac{1}{(1-\theta)+(\theta-\rho)s_{ik}}\right)^2(\theta-\rho)\frac{\partial s_{ik}}{\partial A_{ik}}.\end{aligned}$$

Note that:

$$\begin{aligned}\frac{\partial s_{ik}}{\partial A_{ik}}&=(1-\theta)P_k^{1-\theta}P_{ik}^{-\theta}\frac{\partial P_{ik}}{\partial A_{ik}}\\\Rightarrow\frac{\partial\mu_{ik}}{\partial A_{ik}}&=\left(\frac{1}{(1-\theta)+(\theta-\rho)s_{ik}}\right)^2(\theta-\rho)(1-\theta)P_k^{1-\theta}P_{ik}^{-\theta}\frac{\partial P_{ik}}{\partial A_{ik}}\\&>0,\end{aligned}$$

where we have concluded that this term is positive by noting that the squared fraction, $(\theta-\rho)$, and price terms are positive and the $(1-\theta)$ and $\frac{\partial P_{ik}}{\partial A_{ik}}$ terms are negative.

4 Question 4

It is relatively straightforward to code the problem numerically as a fixed point problem. Given a tolerance, draws of A_{ik}, tuning parameter $\gamma \in [0, 1)$ and starting guess $s_{ik}^{0,1}$, one can proceed using the following algorithm:

1. For all ik, calculate $P_{ik}^n = (W/A_{ik}) \left[1 - \frac{1}{(1-\theta)+s_{ik}^{n-1,1}(\theta-\rho)}\right]$.
2. For all k, calculate $P_k^n = \left(\sum_i (P_{ik}^n)^{1-\theta}\right)^{\frac{1}{1-\theta}}$.
3. For all ik, calculate $s_{ik}^{n,0} = \left(\frac{P_{ik}^n}{P_k^n}\right)^{1-\theta}$.
4. Check for convergence: if $\sum_k \sum_i |s_{ik}^{n,0} - s_{ik}^{n-1,1}|$ is less than the tolerance, stop. Otherwise, set $s_{ik}^{n,1} = (1-\gamma)s_{ik}^{n,0} + \gamma s_{ik}^{n-1,1}$ and return to step (1).

I note a couple of details here. First, note that the higher the tuning parameter γ, the slower s_{ik} will move in-between iterations. This is particularly important when ρ is near 1. For $\gamma = 2$, the algorithm converges rapidly with a tuning parameter near 0, but when $\gamma = 1.1$ then the algorithm will converge many orders of magnitude slower with a tuning parameter near 0 than with an appropriately chosen tuning parameter, such as $\gamma = 0.5$. For this exercise, we were told to use $\rho = 1$. This causes numerical issues as we end up dividing by 0 in our algorithm above. To get around this issue, we instead use $\rho = 1 + \epsilon$, with $\epsilon << 1$. Specifically, I use $\epsilon = 10^{-9}$, which converges in under fifteen seconds using a tuning parameter of $\gamma = 0.6$. The numerical results change very little as ϵ decreases further towards 0, with no change to any reported digits in moving from $\epsilon = 10^{-6}$ to $\epsilon = 10^{-9}$.

5 Question 5

With our solution, computed in Question 4, for all P_{ik}, we can calculate P and use the fact that $W = PC$ to solve for $C = WP^{-1}$. Given my set of productivity draws, I calculated that $C = 4.6062$ (note: here, C is equal to the real wage).

In the first-best outcome, firms charge their marginal cost: $P_{ik} = W/A_{ik}$. Given my set of draws, I calculated that $C_{SPP} = 7.2417$. Unlike the original Dixit-Stiglitz model where the result of the competitive equilibrium is equivalent to the first-best outcome, here there is a significant wedge owing to the market power firms have within their sectors.

6 Question 6

In the limit as $\theta \to \infty$, goods within a sector become infinitely substitutable. Therefore, in the limit, consumers only buy goods from the cheapest (highest productivity) firm in the sector. The intra-sectoral dynamic then drops out, and the only competition is inter-sectoral. In other words, when $\theta \to \infty$, the problem collapses to the one under Bertrand competition with homogeneous goods.

Note that there is a subtle difference between these models, in that the models are equivalent assuming the draws of the highest-productivity firms within each sector of this sectoral model are equal to the productivity draws of the firms in the original Dixit-Stiglitz case. In other words, if each firm in the sectoral model have productivities drawn from a distribution $\mathcal{F}$, then the distribution of productivities of the highest-productivity firms within each sector, $\mathcal{G}$, is defined as follows:

$$\mathcal{G} \sim \max\{A_{1k}, \ldots, A_{N_k k} : A_{1k}, \ldots, A_{N_k k} \sim \mathcal{F}\}$$

Then, the sectoral model with firm productivities drawn from the $\mathcal{F}$ in the limit as $\theta \to \infty$ converges to the original Dixit-Stiglitz model with firm productivities drawn from the $\mathcal{G}$ distribution.