Econometrics HW3

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1 3.24

	beta
education	0.14431
experience	0.042633
experienceˆ2/100	-0.095056
constant	0.53089

	results
Rˆ2	0.38932
SSE	82.505

	reestimate
coefficient estimate	0.14431
Rˆ2	0.36874
SSE	82.505

From the above tables, we see that we have matched the ols coefficient from equation (3.50). The R^2 and SSE are listed as well in the second table. In the third table, we see our re-estimated coefficient is the same as in the original regression; however, the R^2 is lower in the re-estimated regression as part of the informational content was already regressed out of the response variable in the first stage of the two-stage regression. The SSE are identical, however, due to the residuals from the original regression being identical to the residuals from the second stage of the re-estimated regression.

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

2 3.25

	sums
a	4.4187e-14
b	-7.2209e-13
c	-2.0606e-13
d	133.1331
e	1.5575e-11
f	-8.249e-14
g	82.505

The above table yields the relevant sums. Note that a, b, c, e are 0 (to computational accuracy) reflecting the fact that these sums are the inner product of one of the columns of X and the residual estimates. These inner products are 0 by construction. f is also 0 by construction for similar reasons. d, g are not forced to be 0 by construction, and in this case they are clearly nonzero.

3 7.2

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} X_i X_i' &\to_p E[X_i X_i'] \\
\frac{1}{n}\lambda I_k &\to_p 0 \\
\hat{\beta} &= (X'X + \lambda I_k)^{-1} X'Y \\
&= (X'X + \lambda I_k)^{-1} X'(X\beta + \epsilon) \\
&= (X'X + \lambda I_k)^{-1} X'X\beta + (X'X + \lambda I_k)^{-1} X'\epsilon \\
&\to_p \left(E[X_i X_i'] + 0\right)^{-1} E[X_i X_i']\beta + (E[X_i X_i'] + 0)^{-1} E[X_i \epsilon] \\
&= \left(E[X_i X_i']\right)^{-1} E[X_i X_i']\beta + \left(E[X_i X_i']\right)^{-1} 0 \\
&= \beta
\end{aligned}$$

Thus, $\hat{\beta}$ is consistent for β.

4 7.3

$$\begin{aligned}
\frac{1}{n}\lambda I_k = \frac{1}{n} cn I_k &\to_p cI_k \\
\Rightarrow \hat{\beta} &\to_p \left(E[X_i X_i'] + cI_k\right)^{-1} E[X_i X_i']\beta + (E[X_i X_i'] + CI_k)^{-1} E[X_i \epsilon] \\
&= \left(E[X_i X_i'] + cI_k\right)^{-1} E[X_i X_i']\beta
\end{aligned}$$

So, in this case the estimator is not consistent as $\left(E[X_i X_i'] + cI_k\right)^{-1} E[X_i X_i'] \neq I_k$.

5 7.4

1. $E[X_1] = 1/2(1) + 1/2(-1) = 0$
2. $E[X_1]^2 = 1/2(1) + 1/2(1) = 1$
3. $E[X_1 X_2] = 3/4(1) + 1/4(-1) = 1/2$
4. $E[e^2] = (5/4)(3/4) + (1/4)(1/4) = 1$
5. $E[X_1^2 e^2] = (3/4)((1)(5/4)) + (1/4)((1)(1/4)) = 1$
6. $E[X_1 X_2 e^2] = (3/4)((1)(5/4)) + (1/4)((-1)(1/4)) = 7/8$

6 7.8

We know from (7.18) that $\hat{\sigma}^2 \to_p \sigma^2$. Moreover,

$$\begin{aligned}
\sqrt{n}(\hat{\sigma}^2 - \sigma^2) &= \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\hat{\epsilon}_i^2 - \sigma^2\right) \\
&= \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}(\epsilon_i - x_i'(\hat{\beta}-\beta))^2 - \sigma^2\right) \\
&= \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\epsilon_i^2 - \sigma^2\right) - 2\left(\frac{1}{n}\sum_{i=1}^{n}\epsilon_i x_i'\right)\sqrt{n}(\hat{\beta}-\beta) + \sqrt{n}(\hat{\beta}-\beta)'\left(\frac{1}{n}\sum_{i=1}^{n}x_i x_i'\right)(\hat{\beta}-\beta) \\
&= \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\epsilon_i^2 - \sigma^2\right) - 2o_p(1)O_p(1) + O_p(1)O_p(1)o_p(1) \\
&\to_d N(0, V),
\end{aligned}$$

where $V = Var(\epsilon_i^2) = E(\epsilon_i^4) - \sigma^4$. Note that we have implicitly assumed that the fourth moment of ϵ exists.

7 7.9a

The first estimator, $\hat{\beta}$ is the univariate version of OLS. We know that this is therefore a consistent estimator. It is less immediate that $\tilde{\beta}$ is consistent, but we will show below that this is the case.

$$
\begin{aligned}
\tilde{\beta} &= \frac{1}{n}\sum_{i=1}^{n}\frac{Y_i}{X_i} = \frac{1}{n}\sum_{i=1}^{n}\frac{X_i\beta + e_i}{X_i} \\
&= \frac{1}{n}\sum_{i=1}^{n}\beta + \frac{e_i}{X_i} = \beta + \frac{1}{n}\sum_{i=1}^{n}\frac{e_i}{X_i} \\
&\to_p \beta + E\left[\frac{e_i}{X_i}\right] = \beta + E\left[\frac{E[e_i|X_i]}{X_i}\right] \\
&= \beta
\end{aligned}
$$

Therefore, $\tilde{\beta}$ is also a consistent estimator of β.

8 7.10

8.1 Point forecast

Let $\hat{Y}_{n+1} = x'\hat{\beta}$. We will show that this estimator of Y_{n+1} yields, in expectation conditional on X, x, the expectation of Y_{n+1} conditional on x.

$$
\begin{aligned}
\hat{Y}_{n+1} &= x'\hat{\beta} = x'((X'X)^{-1}X'Y) \\
&= x'(X'X)^{-1}X'(X\beta + e) \\
&= x'\beta + x'(X'X)^{-1}X'e. \\
E[\hat{Y}_{n+1}|X, x] &= E[x'\beta + x'(X'X)^{-1}X'e|X, x] \\
&= x'\beta + E[x'(X'X)^{-1}X'E[e|X]|X, x] \\
&= x'\beta \\
&= E[Y_{n+1}|x]
\end{aligned}
$$

8.2 Variance estimator

$$
\begin{aligned}
Var(\hat{Y}_{n+1}) &= E[\hat{e}_{n+1}^2] \\
&= E[(e_{n+1} - x'(\hat{\beta} - \beta))^2] \\
&= E[e_{n+1}^2] - 2E[e_{n+1}x'(\hat{\beta} - \beta)] + E[x'(\hat{\beta} - \beta)(\hat{\beta} - \beta)'x] \\
&= \sigma^2 + x'V_{\hat{\beta}}x
\end{aligned}
$$

These are not known, however. Yet, we do have estimates of these quantities. Therefore,

$$
\hat{Var}(\hat{Y}_{n+1}) = \hat{\sigma}^2 + x'\hat{V}_{\beta}x
$$

is an estimator of the variance of our forecast.

9 7.13

We propose $\hat{\gamma} = \frac{1}{n}\sum_{i=1}^{n} X_i/Y_i$. Naturally, this leads to an estimator for $\theta : \hat{\theta} = 1/\hat{\gamma}$. $Var(\hat{\gamma}) = \frac{1}{n^2}\sum_{i=1}^{n} Var\left(\frac{X_i}{Y_i}\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var\left(\gamma + \frac{u_i}{Y_i}\right) = \frac{1}{n}\left(\frac{Var(u_i)}{Var(Y_i)}\right) := \frac{1}{n}V$. Therefore, $\sqrt{n}(\hat{\gamma} - \gamma) \to_d N(0, V)$. Thus, we can apply the delta method and find that $\sqrt{n}(\hat{\theta} - \theta) \to_d N(0, W)$ where $W = \frac{V}{\gamma^2} = \theta^2 V$.

The asymptotic standard error for $\hat{\theta}$ is $\sqrt{W} = \theta\sqrt{V} = \theta\sqrt{\frac{Var(u_i)}{Var(Y_i)}}$.

10 7.14

We can retrieve OLS estimates of β_1, β_2 $(\hat{\beta}_1, \hat{\beta}_2)$ and then define $\hat{\theta} = \hat{\beta}_1\hat{\beta}_2$. Next, we know the asymptotic distribution for OLS: $\sqrt{n}(\hat{\beta} - \beta) \to_d N(0, V_\beta)$ where $V_\beta = E[x_i x_i']^{-1}E[\epsilon_i^2 x_i x_i']E[x_i x_i']^{-1}$Then, we can apply the delta method and find:

$$\sqrt{n}(\hat{\theta} - \theta) \to_d N(0, V),$$

where $V = [\beta_2\beta_1]V_\beta[\beta_2\beta_1]'$.

To run a test, we would estimate $V : \hat{V} = [\hat{\beta}_2\hat{\beta}_1]\hat{V}_\beta[\hat{\beta}_2\hat{\beta}_1]'$ and calculate the 95 percent CI as $\left[\hat{\theta} - 1.96\sqrt{\hat{V}/n}, \hat{\theta} + 1.96\sqrt{\hat{V}/n}\right]$.

11 7.15

$$\begin{aligned}
\hat{\beta} &= \frac{\sum_{i=1}^{n} X_i^3 Y_i}{\sum_{i=1}^{n} X_i^4} \\
&= \frac{\sum_{i=1}^{n} X_i^3(X_i\beta + e_i)}{\sum_{i=1}^{n} X_i^4} \\
&= \frac{\sum_{i=1}^{n} X_i^4\beta + \sum_{i=1}^{n} X_i^3 e_i}{\sum_{i=1}^{n} X_i^4} \\
&= \beta + \frac{\sum_{i=1}^{n} X_i^3 e_i}{\sum_{i=1}^{n} X_i^4} \\
\Rightarrow \sqrt{n}(\hat{\beta} - \beta) &\to_d \frac{1}{E[X_i^4]} N(0, E[X_i^6 e_i^2]) \\
&= N\left(0, , \frac{E[X_i^6 e_i^2]}{E[X_i^4]}\right)
\end{aligned}$$

12 7.17

12.1 Part A

Under the null hypothesis that $\theta = 0$, $\sqrt{n}(\hat{\theta}-\theta) \rightarrow_d N(0, Var(\hat{\theta})) = N(0, Var(\hat{\beta}_1-\hat{\beta}_2)) = N(0, Var(\hat{\beta}_1) + Var(\hat{\beta}_2) - 2Cov(\hat{\beta}_1, \hat{\beta}_2)) \sim N(0, s(\hat{\beta}_1)^2 + s(\hat{\beta}_2)^2 - 2\hat{\rho}s(\hat{\beta}_1)s(\hat{\beta}_2))$. Therefore, the 95% CI for $\hat{\theta}$
$= \left[\hat{\theta} - 1.96\sqrt{s(\hat{\beta}_1)^2 + s(\hat{\beta}_2)^2 - 2\hat{\rho}s(\hat{\beta}_1)s(\hat{\beta}_2)}, \hat{\theta} + 1.96\sqrt{s(\hat{\beta}_1)^2 + s(\hat{\beta}_2)^2 - 2\hat{\rho}s(\hat{\beta}_1)s(\hat{\beta}_2)}\right]$
$= \left[0.2 - 1.96\sqrt{2(0.07)^2(1-\hat{\rho})}, 0.2 + 1.96\sqrt{2(0.07)^2(1-\hat{\rho})}\right]$.

12.2 Part B

We are not given the estimated covariance of $\hat{\beta}_1, \hat{\beta}_2$ so we cannot calculate the estimated correlation.

12.3 Part C

Correlation is in $[-1, 1]$ so an upper bound for the width of the confidence interval is when the estimated correlation is $-1 : [0.2 - 1.96 * 2 * (0.07), 0.2 + 1.96 * 2 * (0.07)] = [-0.0744, 0.4744]$. This bound contains 0 so we cannot reject the null hypothesis given the reported information.

13 7.19

$$\sqrt{n}(\hat{\beta}_1 - \hat{\beta}_2) = \sqrt{n}(\hat{\beta}_1 - \beta) - \sqrt{n}(\hat{\beta}_2 - \beta).$$

Let us add an indicator d_i : 1{is in the first split}. Then, the regression equation is of the form:

$$y_i = d_i x_i' \beta + (1 - d_i)x_i\beta + \epsilon_i$$

$$\begin{aligned}
\sqrt{n}\left[\begin{pmatrix}\hat{\beta}_1\\ \hat{\beta}_2\end{pmatrix} - \begin{pmatrix}\beta\\ \beta\end{pmatrix}\right] &= \left[\frac{1}{2n}\sum_{i=1}^{n}\begin{pmatrix}d_i x_i\\ (1-d_i)x_i\end{pmatrix}\begin{pmatrix}d_i x_i\\ (1-d_i)x_i\end{pmatrix}'\right]\frac{1}{2\sqrt{n}}\sum_{i=1}^{n}\begin{pmatrix}d_i x_i \epsilon_i\\ (1-d_i)x_i\epsilon_i\end{pmatrix}\\
&= \left[\frac{1}{2n}\begin{pmatrix}\sum_{i=1}^{\infty} d_i x_i x_i' & \sum_{i=1}^{\infty} d_i(1-d_i)x_i x_i'\\ \sum_{i=1}^{\infty} d_i(1-d_i)x_i x_i' & \sum_{i=1}^{\infty}(1-d_i)x_i x_i'\end{pmatrix}\right]^{-1}\frac{1}{2\sqrt{n}}\sum_{i=1}^{n}\begin{pmatrix}d_i x_i\epsilon_i\\ (1-d_i)x_i\epsilon_i\end{pmatrix}
\end{aligned}$$

$$\frac{1}{2n}\begin{pmatrix}\sum_{i=1}^{\infty} d_i x_i x_i' & \sum_{i=1}^{\infty} d_i(1-d_i)x_i x_i'\\ \sum_{i=1}^{\infty} d_i(1-d_i)x_i x_i' & \sum_{i=1}^{\infty}(1-d_i)x_i x_i'\end{pmatrix} \rightarrow_p \begin{pmatrix}\frac{1}{2}E[x_i x_i'] & 0\\ 0 & \frac{1}{2}E[x_i x_i']\end{pmatrix}$$

$$\frac{1}{2\sqrt{n}}\sum_{i=1}^{n}\begin{pmatrix}d_i x_i\epsilon_i\\ (1-d_i)x_i\epsilon_i\end{pmatrix} \rightarrow_d N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\frac{1}{2}E[e_i^2 x_i x_i'] & 0\\ 0 & \frac{1}{2}E[e_i^2 x_i x_i']\end{pmatrix}\right)$$

$$\Rightarrow \sqrt{n}\left[\begin{pmatrix}\hat{\beta}_1\\ \hat{\beta}_2\end{pmatrix} - \begin{pmatrix}\beta\\ \beta\end{pmatrix}\right] = N(0, V \otimes I_2)$$

where I_2 is the 2×2 identity matrix, $\otimes$ is the kronecker product, and

$$V = E[x_i x_i']^{-1} E[\epsilon_i^2 x_i x_i'] E[x_i x_i']^{-1}.$$

Then, $\sqrt{n}(\hat{\beta}_1 - \hat{\beta}_2) = \sqrt{n}(\hat{\beta}_1 - \beta) - \sqrt{n}(\hat{\beta}_2 - \beta) \to_d N(0, 2V)$.

14 Q 9

14.1 Part A

$$\begin{aligned}
\hat{\beta} &= \left[\frac{1}{n}\sum_{i=1}^{n} w_i w_i' 1\{x_i \in \{1,2\}\}\right]^{-1} \frac{1}{n}\sum_{i=1}^{n} w_i y_i 1\{x_i \in \{1,2\}\} \\
&= \left[\frac{1}{n}\sum_{i=1}^{n} w_i w_i' 1\{x_i \in \{1,2\}\}\right]^{-1} \frac{1}{n}\sum_{i=1}^{n} w_i (w_i'\beta + \epsilon_i) 1\{x_i \in \{1,2\}\} \\
&= \beta + \left[\frac{1}{n}\sum_{i=1}^{n} w_i w_i' 1\{x_i \in \{1,2\}\}\right]^{-1} \frac{1}{n}\sum_{i=1}^{n} w_i \epsilon_i 1\{x_i \in \{1,2\}\} \\
&\to_p \beta + E[w_i w_i' 1\{x \in \{1,2\}\}]^{-1} E[w_i \epsilon_i 1\{x \in \{1,2\}] \\
&= \beta + E[w_i w_i' 1\{x \in \{1,2\}\}]^{-1} E[w_i E[\epsilon_i | w_i] 1\{x \in \{1,2\}] \\
&= \beta.
\end{aligned}$$

Therefore, $\hat{\beta} \to_p \beta$.

14.2 Part B

$$\hat{\beta} = \beta + E[w_i w_i' 1\{x \in \{1,2\}\}]^{-1} E[w_i E[\epsilon_i | w_i] 1\{x \in \{1,2\}]$$

(A1') does not give us enought to deal with the indicator function inside the second expectation. So, in general, no.

14.3 Part C

$$\begin{aligned}
\sqrt{n}(\hat{\beta}-\beta) &= \left[\frac{1}{n}\sum_{i=1}^{n} w_i w_i' 1\{x_i \in \{1,2\}\}\right]^{-1} \frac{1}{\sqrt{n}}\sum_{i=1}^{n} w_i \epsilon_i 1\{x_i \in \{1,2\}\} \\
&\rightarrow_d E[w_i w_i' 1\{x_i \in \{1,2\}\}]^{-1} N(0, Var(w_i \epsilon_i 1\{x_i \in \{1,2\}\})) \\
Var(w_i \epsilon_i 1\{x_i \in \{1,2\}) &= E[\epsilon_i^2 w_i w_i' 1\{x_i \in \{1,2\}\}] \\
&= E[E\epsilon_i^2 | w_i] w_i w_i' 1\{x_i \in \{1,2\}\}] \\
&= \sigma^2 E[w_i w_i' 1\{x_i \in \{1,2\}\}] \\
&= \sigma^2 \begin{pmatrix} 1/2 & 3/4 \\ 3/4 & 5/4 \end{pmatrix} \\
\Rightarrow \sqrt{n}(\hat{\beta}-\beta) &\rightarrow_d E[w_i w_i' 1\{x_i \in \{1,2\}]^{-1} N(0, Var(w_i \epsilon_i 1\{x_i \in \{1,2\}\})) \\
&\sim N\left(0, \sigma^2 \begin{pmatrix} 1/2 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}^{-1}\right) \\
&\sim N\left(0, \sigma^2 \begin{pmatrix} 20 & -12 \\ -12 & 8 \end{pmatrix}\right)
\end{aligned}$$

14.4 Part D

From identical logic to that which we used in Part A, we know that $\hat{\hat{\beta}}_2$ is a consistent estimator for γ. As we have shown in Part A that $\hat{\beta}_2$ is also consistent, we can choose estimators by comparing asymptotic variances. We showed in Part D that this variance is $8\sigma^2$ for $\hat{\beta}_2$, while by replicating the same steps we followed in Part C with the inequality in the indicator function flipped, we find that the asymptotic variance of $\hat{\hat{\beta}}_2$ is $72\sigma^2 > 8\sigma^2$. Thus, we should use $\hat{\beta}^2$ as it yields more precise estimates of the slope coefficient.

14.5 Part E

$$\begin{aligned}
\hat{\alpha} &= \left[\frac{1}{n}\sum_{i=1}^{n} x_i x_i' 1\{x_i \in \{1,2\}\}\right]^{-1} \frac{1}{n}\sum_{i=1}^{n} x_i y_i 1\{x_i \in \{1,2\}\} \\
&\rightarrow_p E[x_i x_i' 1\{x_i \in \{1,2\}\}]^{-1} E[x_i y_i 1\{x_i \in \{1,2\}\}] \\
&= E[x_i x_i' 1\{x_i \in \{1,2\}\}]^{-1} (E[x_i 1\{x_i \in \{1,2\}\}] + \gamma E[x_i x_i' 1\{x_i \in \{1,2\}\}] + E[x_i \epsilon_i 1\{x_i \in \{1,2\}\}]) \\
&= (5/4)^{-1}((3/4) + \gamma(5/4) + 0) \\
&= \gamma + 3/5
\end{aligned}$$

14.6 Part F

$$
\begin{aligned}
\sqrt{n}(\hat{\alpha}-\alpha) &= \left[\frac{1}{n}\sum_{i=1}^{n} x_i x_i' 1\{x_i \in \{1,2\}\}\right]^{-1} \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(x_i + x_i^2(\gamma-\alpha) + x_i\epsilon_i)1\{x_i \in \{1,2\}\} \\
&\to_d N(0,(4/5)^2 Var[(x_i + x_i^2(\gamma-\alpha) + x_i\epsilon_i)1\{x_i \in \{1,2\}\}])
\end{aligned}
$$

$$
Var[(x_i + x_i^2(\gamma-\alpha) + x_i\epsilon_i)1\{x_i \in \{1,2\}\}] = E[(x_i + x_i^2(\gamma-\alpha) + x_i\epsilon_i)^2 1\{x_i \in \{1,2\}\}]
$$

$$
\begin{aligned}
&= E[x_i^2 1\{x_i \in \{1,2\}\}] + (9/25)E[x_i^4 1\{x_i \in \{1,2\}\}] + \sigma^2 E[x_i^2 1\{x_i \in \{1,2\}\}] \\
&\quad - (6/5)E[x_i^3 1\{x_i \in \{1,2\}\}] + 2E[x_i^2\epsilon_i 1\{x_i \in \{1,2\}\}] - (6/5)E[x_i^3\epsilon_i 1\{x_i \in \{1,2\}\}] \\
&= (5/4) - (6/5)(9/4) + (9/25)(17/4) + \sigma^2(5/4) \\
\Rightarrow \sqrt{n}(\hat{\alpha}-\alpha) &\to_d N(0,(4/5)^2((5/4) - (6/5)(9/4) + (9/25)(17/4) + \sigma^2(5/4))) \\
&\sim N(0,(16/25)(2/25 + (5/4)\sigma^2))
\end{aligned}
$$