

Econometrics HW1

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1 Question 13.1

We can use the moment conditions and rewrite as follows:

$$\begin{aligned}E[Xe] &= 0 \\E[X(Y - X'\beta)] &= 0 \\ \frac{1}{n} \sum_{i=1}^n X_i(Y_i - X_i'\hat{\beta}) &= 0 \\ \hat{\beta} &= \left(\sum_{i=1}^n X_i X_i' \right)^{-1} \left(\sum_{i=1}^n X_i Y_i \right) \\E[Z\eta] &= 0 \\E[Z(e^2 - Z'\gamma)] &= 0 \\E[Z((Y - X'\beta)^2 - Z'\gamma)] &= 0 \\ \frac{1}{n} \sum_{i=1}^n \left(Z_i((Y_i - X_i'\hat{\beta})^2 - Z_i'\hat{\gamma}) \right) &= 0 \\ \sum_{i=1}^n \left(Z_i(Y_i - X_i'\hat{\beta})^2 \right) &= \left(\sum_{i=1}^n Z_i Z_i' \right) \hat{\gamma} \\ \hat{\gamma} &= \left(\sum_{i=1}^n Z_i Z_i' \right)^{-1} \left(\sum_{i=1}^n \left(Z_i(Y_i - X_i'\hat{\beta})^2 \right) \right)\end{aligned}$$

2 Question 13.2

The GMM estimator is the following:

$$\begin{aligned}\hat{\beta}_{gmm} &= (X'Z(Z'Z)^{-1}Z'X)^{-1} X'Z(Z'Z)^{-1}Z'Y \\ \Rightarrow \sqrt{n}(\hat{\beta}_{gmm} - \beta) &= \left(\left(\frac{1}{n} X'Z \right) \left(\frac{1}{n} Z'Z \right)^{-1} \left(\frac{1}{n} Z'X \right) \right)^{-1} \left(\frac{1}{n} X'Z \right) \left(\frac{1}{n} Z'Z \right)^{-1} \left(\frac{1}{\sqrt{n}} Z'e \right)\end{aligned}$$

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, Katherine Kwok, and Danny Edgel.

By application of the CLT, LLN, and CMT we have the following:

$$\begin{aligned}\sqrt{n}(\hat{\beta}_{gmm} - \beta) &\rightarrow_d \left(E(XZ') E(ZZ')^{-1} E(ZX') \right)^{-1} E(XZ') E(ZZ')^{-1} N(0, E[ZZ'e^2]) \\ &=_d \left(E(XZ') E(ZZ')^{-1} E(ZX') \right)^{-1} E(XZ') E(ZZ')^{-1} N(0, \sigma^2 E[ZZ']) \\ &=_d N(0, V),\end{aligned}$$

where

$$\begin{aligned}V &= \left(E(XZ') E(ZZ')^{-1} E(ZX') \right)^{-1} E(XZ') E(ZZ')^{-1} \\ &\quad * \sigma^2 E[ZZ'] E(ZZ')^{-1} E(ZX') \left(E(XZ') E(ZZ')^{-1} E(ZX') \right)^{-1} \\ &= \sigma^2 \left(E(XZ') E(ZZ')^{-1} E(ZX') \right)^{-1} \\ &= \sigma^2 (Q'M^{-1}Q)^{-1}.\end{aligned}$$

3 Question 13.3

$$\begin{aligned}\hat{W} &= \left(\frac{1}{n} \sum_{i=1}^n Z_i Z_i' \tilde{e}_i^2 \right)^{-1} \\ &= \left(\frac{1}{n} \sum_{i=1}^n Z_i Z_i' (Y_i - X_i' \tilde{\beta})^2 \right)^{-1}.\end{aligned}$$

By the LLN, CMT, and consistency of $\tilde{\beta}$,

$$\begin{aligned}\hat{W} &\rightarrow_p E[ZZ'(Y - X'\beta)^2]^{-1} \\ &= E[ZZ'e^2]^{-1} \\ &= \Omega^{-1}.\end{aligned}$$

4 Question 13.4

4.1 Part A

$$\begin{aligned}V_0 &= (Q'WQ)^{-1} Q'W\Omega WQ(Q'WQ)^{-1} \\ &= (Q'\Omega^{-1}Q)^{-1} Q'\Omega^{-1}\Omega\Omega^{-1}Q(Q'\Omega^{-1}Q)^{-1} \\ &= (Q'\Omega^{-1}Q)^{-1}.\end{aligned}$$

4.2 Part B

If we let $A := WQ(Q'WQ)^{-1}$, $B := \Omega^{-1}Q(Q'\Omega^{-1}Q)^{-1}$ then $A'\Omega A = (Q'WQ)^{-1} Q'W\Omega WQ(Q'WQ)^{-1} = V$ and $B'\Omega B = (Q'\Omega^{-1}Q)^{-1} Q'\Omega^{-1}\Omega\Omega^{-1}Q(Q'\Omega^{-1}Q)^{-1} = (Q'\Omega^{-1}Q)^{-1} = V_0$ as desired.

4.3 Part C

$$\begin{aligned}B'\Omega A &= (Q'\Omega^{-1}Q)^{-1} Q'\Omega^{-1}\Omega WQ(Q'WQ)^{-1} \\ &= (Q'\Omega^{-1}Q)^{-1} Q'WQ(Q'WQ)^{-1} \\ &\quad (Q'\Omega^{-1}Q)^{-1} \\ &= B'\Omega B \\ \Rightarrow B'\Omega(A - B) &= 0.\end{aligned}$$

4.4 Part D

$$\begin{aligned}
V - V_0 &= A'\Omega A - B'\Omega B \\
&= (B + (A - B))'\Omega(B + (A - B)) - B'\Omega B \\
&= B'\Omega(A - B) + (A - B)'\Omega B + (A - B)'\Omega(A - B) \\
&= (B'\Omega(A - B))' + (A - B)'\Omega(A - B) \\
&= (A - B)'\Omega(A - B).
\end{aligned}$$

The above derivation shows that $V - V_0 = (A - B)'\Omega(A - B)$. Note that $(A - B)'\Omega(A - B)$ is a quadratic form matrix, i.e. it is positive semidefinite. Therefore, $V - V_0$ is positive semidefinite, so $V \geq V_0$ in the matrix sense.

5 Question 13.11

We can plug in our choice of Z into the formulas for GMM IV. The optimal weight matrix is the following:

$$\begin{aligned}
W &= E[ZZ'e^2] = \begin{pmatrix} E[X_i^2 e_i^2] & E[X_i^3 e_i^2] \\ E[X_i^3 e_i^2] & E[X_i^4 e_i^2] \end{pmatrix} \\
\Rightarrow \hat{W} &= \begin{pmatrix} \hat{E}[X_i^2 e_i^2] & \hat{E}[X_i^3 e_i^2] \\ \hat{E}[X_i^3 e_i^2] & \hat{E}[X_i^4 e_i^2] \end{pmatrix}
\end{aligned}$$

Our formula for efficient GMM then reads:

$$\begin{aligned}
\hat{\beta}_{gmm} &= (X'Z\hat{\Omega}^{-1}Z'X)^{-1}(X'Z\hat{\Omega}^{-1}Z'Y) \\
&= \frac{a_n \sum_i x_i y_i + b_n \sum_i x_i^2 y_i}{a_n \sum_i x_i^2 + b_n \sum_i x_i^3},
\end{aligned}$$

where

$$\begin{aligned}
a_n &= \frac{1}{n} \left(\sum_i x_i^2 \right) \hat{E}[X^4 e^2] - \frac{1}{n} \left(\sum_i x_i^3 \right) \hat{E}[X^3 e^2], \\
b_n &= -\frac{1}{n} \left(\sum_i x_i^2 \right) \hat{E}[X^3 e^2] + \frac{1}{n} \left(\sum_i x_i^3 \right) \hat{E}[X^2 e^2].
\end{aligned}$$

OLS and 2SLS yield the same estimated coefficient in this case, and given X is a scalar we have $\hat{\beta}_{2sls} = \hat{\beta}_{ols} = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}$.

In general, the estimators are different. However, there are some conditions under which the estimators are the same, such as if $b_n = 0$.

6 Question 13.13

6.1 Part A

Consider the spectral decomposition of Ω : $\Omega = H\Lambda H'$ where H is orthonormal and Λ is diagonal and positive definite. Then, define $C := H\Lambda^{-1/2}$. Note that H is invertible and $\Lambda^{-1/2}$

is diagonal and positive definite so C is invertible. Then,

$$\begin{aligned} CC' &= H\Lambda^{-1/2}\Lambda^{-1/2}H' \\ &= H\Lambda^{-1}H' \\ &= \Omega^{-1} \\ \Rightarrow \Omega &= C^{-1'}C^{-1} \end{aligned}$$

6.2 Part B

$$\begin{aligned} J &= n\bar{g}_n(\hat{\beta})'\hat{\Omega}^{-1}\bar{g}_n(\hat{\beta}) \\ &= n\bar{g}_n(\hat{\beta})'(CC^{-1})\hat{\Omega}^{-1}(C^{-1}C)'\bar{g}_n(\hat{\beta}) \\ &= n(C'\bar{g}_n(\hat{\beta}))'(C'\hat{\Omega}C)^{-1}C'\bar{g}_n(\hat{\beta}) \end{aligned}$$

6.3 Part C

$$\begin{aligned} C'\bar{g}_n(\hat{\beta}) &= C'\frac{1}{n}Z'e \\ &= C'\frac{1}{n}Z'e - C'\frac{1}{n}Z'X \left(\left(\frac{1}{n}X'Z \right) \hat{\Omega}^{-1} \left(\frac{1}{n}Z'X \right) \right)^{-1} \left(\frac{1}{n}X'Z \right) \hat{\Omega}^{-1}C^{-1'}C' \left(\frac{1}{n}Z'e \right) \\ &= \left(I - C'\frac{1}{n}Z'X \left(\left(\frac{1}{n}X'Z \right) \hat{\Omega}^{-1} \left(\frac{1}{n}Z'X \right) \right)^{-1} \left(\frac{1}{n}X'Z \right) \hat{\Omega}^{-1}C^{-1'} \right) C'\bar{g}_n(\beta) \\ &= D_n C'\bar{g}_n(\beta). \end{aligned}$$

6.4 Part D

$$\begin{aligned} D_n &= I - C'\frac{1}{n}Z'X \left(\left(\frac{1}{n}X'Z \right) \hat{\Omega}^{-1} \left(\frac{1}{n}Z'X \right) \right)^{-1} \left(\frac{1}{n}X'Z \right) \hat{\Omega}^{-1}C^{-1'} \\ &\rightarrow_p I - C'E[ZX'] (E[ZX']'\Omega^{-1}E[ZX'])^{-1} E[ZX']'\Omega^{-1}C'^{-1} \\ &= I - C'E[ZX'] (E[ZX']'CC'E[ZX'])^{-1} E[ZX']'CC'C'^{-1} \\ &= I - R(R'R)^{-1}R', \end{aligned}$$

where $R = C'E[ZX']$.

6.5 Part E

By the CLT and CMT,

$$\begin{aligned} \sqrt{n}C'\bar{g}_n(\beta) &= C'\frac{1}{\sqrt{n}}Z'e \\ &\rightarrow_d C'N(0, \Omega) \\ &=_d C'C'^{-1}N(0, I) \\ &=_d u \sim N(0, I). \end{aligned}$$

6.6 Part F

By CLT, CMT:

$$\begin{aligned}
J &= n(C'\bar{g}_n(\hat{\beta}))'(C'\hat{\Omega}C)^{-1}C'\bar{g}_n(\hat{\beta}) \\
&= (\sqrt{n}C'\bar{g}_n(\beta))'D'_n(C'\hat{\Omega}C)^{-1}D_n\sqrt{n}C'\bar{g}_n(\beta) \\
&\rightarrow_d u'(I - R(R'R)^{-1}R')'(C'\Omega C)(I - R(R'R)^{-1}R')u \\
&=_d u'(I - R(R'R)^{-1}R')'(I - R(R'R)^{-1}R')u \\
&= u'(I - R(R'R)^{-1}R')u,
\end{aligned}$$

where the final step is due to the idempotency of $I - R(R'R)^{-1}R'$.

6.7 Part G

Again since $I - R(R'R)^{-1}R'$ is idempotent, and because $u \sim N(0, I_l)$, $u'(I - R(R'R)^{-1}R')u$ is χ^2 with degree of freedom equal to $\text{tr}(I_l - R(R'R)^{-1}R') = l - \text{tr}(R(R'R)^{-1}R') = l - \text{tr}(R'R(R'R)^{-1}) = l - k$.

7 Question 13.18

Let us use $Z = (X, Q)$ as an instrument. Then,

$$\Omega = E[Z_i Z_i' e_i^2] = \begin{pmatrix} E[X_i X_i' e_i^2] & E[X_i Q_i' e_i^2] \\ E[Q_i X_i' e_i^2] & E[Q_i Q_i' e_i^2] \end{pmatrix}$$

If $\hat{\Omega}$ is a consistent estimator of Ω then $\hat{\Omega}$ is the efficient weight matrix and $\hat{\beta} = (X'Z\hat{\Omega}^{-1}Z'X)^{-1}X'Z\hat{\Omega}^{-1}Z'Y$ is the efficient GMM estimator for β .

8 Question 13.19

We can use the two given moments and apply GMM:

$$\begin{aligned}
g_i(\mu) &= \begin{pmatrix} y_i - \mu \\ x_i \end{pmatrix} \\
\Omega &= E[g_i(\mu)g_i(\mu)'] \\
&= \begin{pmatrix} \text{Var}(y_i) & \text{Cov}(y_i, x_i) \\ \text{Cov}(y_i, x_i) & \text{Var}(x_i) \end{pmatrix} \\
&= \begin{pmatrix} \sigma_y^2 & \sigma_{x,y} \\ \sigma_{x,y} & \sigma_x^2 \end{pmatrix}.
\end{aligned}$$

Efficient GMM uses as the weight matrix Ω^{-1}

$$\begin{aligned}
J(\mu) &= \bar{g}_n(\mu)\Omega^{-1}\bar{g}_n(\mu) \\
&= \frac{\sigma_x^2(\bar{y} - \mu)^2 - 2\sigma_{x,y}\bar{x}(\bar{y} - \mu) + \sigma_y^2\bar{x}^2}{\sigma_x^2\sigma_y^2 - \sigma_{x,y}^2}.
\end{aligned}$$

We can take first order conditions with respect to μ , and this is 0 at our estimate $\hat{\mu}$:

$$\begin{aligned}
J'(\hat{\mu}) &= -2\sigma_x^2(\bar{y} - \hat{\mu}) + 2\bar{x}\sigma_{x,y} = 0 \\
\Rightarrow \hat{\mu} &= \bar{y} - \frac{\sigma_{x,y}}{\sigma_x^2}\bar{x}.
\end{aligned}$$

Note however that the variance and covariance in the above expression are unknown. We can replace with a consistent estimator of these values and our estimator $\hat{\mu}$ will still have the same asymptotic variance, i.e. it will still be efficient. Therefore, our optimal estimator becomes:

$$\hat{\mu} = \bar{y} - \frac{\hat{\sigma}_{x,y}}{\hat{\sigma}_x^2} \bar{x}.$$

9 Question 13.28

9.1 Part A

Output from the 2SLS and GMM outputs are below. Code for all parts will follow at the end of the question.

```
. ivregress 2sls lwage exp exp2per south black urban (edu = public private), r
```

Instrumental variables (2SLS) regression	Number of obs	=	3,010
	Wald chi2(6)	=	717.93
	Prob > chi2	=	0.0000
	R-squared	=	0.1447
	Root MSE	=	.41037

lwage	Coef.	Robust Std. Err.	z	P> z	[95% Conf. Interval]	
edu	.1610916	.0404709	3.98	0.000	.0817702	.2404131
exp	.1193108	.0181653	6.57	0.000	.0837075	.1549141
exp2per	-.2305416	.0367518	-6.27	0.000	-.3025738	-.1585094
south	-.0950355	.0217387	-4.37	0.000	-.1376427	-.0524283
black	-.1017274	.0439722	-2.31	0.021	-.1879113	-.0155435
urban	.1164481	.02627	4.43	0.000	.0649599	.1679363
_cons	3.268014	.6821174	4.79	0.000	1.931088	4.604939

Instrumented: edu

Instruments: exp exp2per south black urban public private

```
. ivregress gmm lwage exp exp2per south black urban (edu = public private), r
```

```
Instrumental variables (GMM) regression      Number of obs   =      3,010
                                             Wald chi2(6)    =      715.88
                                             Prob > chi2     =      0.0000
                                             R-squared       =      0.1433
GMM weight matrix: Robust                 Root MSE       =      .41071
```

lwage	Coef.	Robust Std. Err.	z	P> z	[95% Conf. Interval]	
edu	.1615162	.0405052	3.99	0.000	.0821275	.2409049
exp	.1195553	.018182	6.58	0.000	.0839192	.1551913
exp2per	-.2315108	.036812	-6.29	0.000	-.3036609	-.1593607
south	-.0953557	.0217546	-4.38	0.000	-.1379939	-.0527175
black	-.1011997	.0440045	-2.30	0.021	-.187447	-.0149524
urban	.1150211	.0262525	4.38	0.000	.0635671	.1664751
_cons	3.261881	.6827035	4.78	0.000	1.923806	4.599955

Instrumented: edu

Instruments: exp exp2per south black urban public private

The above outputs show almost no change from estimating the IV models via 2SLS vs GMM.

9.2 Part B

Output from the 2SLS and GMM outputs are below. Code for all parts will follow at the end of the question.

```
. ivregress 2sls lwage exp exp2per south black urban (edu = public private pubage pubage2), r
```

```
Instrumental variables (2SLS) regression      Number of obs   =      3,010
                                             Wald chi2(6)    =     1018.86
                                             Prob > chi2     =      0.0000
                                             R-squared       =      0.2891
                                             Root MSE       =      .37412
```

lwage	Coef.	Robust Std. Err.	z	P> z	[95% Conf. Interval]	
edu	.0825386	.0062178	13.27	0.000	.070352	.0947252
exp	.087094	.0070498	12.35	0.000	.0732766	.1009115
exp2per	-.2247205	.0319734	-7.03	0.000	-.2873872	-.1620538
south	-.1219401	.0154109	-7.91	0.000	-.152145	-.0917352
black	-.1810215	.0180273	-10.04	0.000	-.2163544	-.1456887
urban	.1570178	.0152781	10.28	0.000	.1270732	.1869623
_cons	4.590107	.1106351	41.49	0.000	4.373266	4.806948

Instrumented: edu

Instruments: exp exp2per south black urban public private pubage pubage2

```
. ivregress gmm lwage exp exp2per south black urban (edu = public private pubage pubage2), r
```

```
Instrumental variables (GMM) regression      Number of obs   =    3,010
                                           Wald chi2(6)    =   1020.21
                                           Prob > chi2     =    0.0000
                                           R-squared       =    0.2886
GMM weight matrix: Robust                 Root MSE       =    .37425
```

lwage	Coef.	Robust Std. Err.	z	P> z	[95% Conf. Interval]	
edu	.0838525	.0062069	13.51	0.000	.0716872	.0960177
exp	.0876355	.0070531	12.43	0.000	.0738116	.1014594
exp2per	-.2249305	.0320052	-7.03	0.000	-.2876595	-.1622015
south	-.1244904	.0153914	-8.09	0.000	-.1546571	-.0943237
black	-.1774914	.017986	-9.87	0.000	-.2127433	-.1422396
urban	.152938	.0152107	10.05	0.000	.1231255	.1827505
_cons	4.569933	.1105307	41.35	0.000	4.353296	4.786569

```
Instrumented:  edu
```

```
Instruments:  exp exp2per south black urban public private pubage pubage2
```

As before, the above outputs show almost no change from estimating the IV models via 2SLS vs GMM.

9.3 Part C

Below, we report the J statistic for overidentification for both of the GMM models.

```
. estat overid
```

```
Test of overidentifying restriction:
```

```
Hansen's J chi2(1) = .869261 (p = 0.3512)
```

```
. estat overid
```

```
Test of overidentifying restriction:
```

```
Hansen's J chi2(3) = 10.4389 (p = 0.0152)
```

Our J statistics show a major difference. The P value from the first model is quite large, while the P value from the second model is quite small, and significant at a 5 percent level. This may be caused by small sample distortions, but also may indicate that the model can be improved.

Below we display our code that generates the results presented above.


```

use "C:\Users\micha\OneDrive\Documents\HOMEWORK\Y1S1\Metrics\Q4\PS1\Card1995.dta", clear
gen lwage = lwage76
gen edu = ed76
gen exp = age76 - edu - 6
gen exp2per = exp^2/100
gen south = reg76r
gen urban = smsa76r
gen public = nearc4a
gen private = nearc4b
gen pubage = nearc4a*age76
gen pubage2 = nearc4a*age76^2/100

ivregress 2sls lwage exp exp2per south black urban (edu = public private), r
ivregress gmm lwage exp exp2per south black urban (edu = public private), r

estat overid

ivregress 2sls lwage exp exp2per south black urban (edu = public private pubage pubage2), r
ivregress gmm lwage exp exp2per south black urban (edu = public private pubage pubage2), r

estat overid

```

10 Question 17.15

10.1 Part A

Output from the Arellano-Bond estimator is below. As in the previous question, code for all parts will follow at the end of the question.

```

. xtabond k, lags(1) vce(robust)

```

Arellano-Bond dynamic panel-data estimation

Group variable: **id**

Time variable: **year**

Number of obs = **751**

Number of groups = **140**

Obs per group:

min = **5**

avg = **5.364286**

max = **7**

Number of instruments = **29**

Wald chi2(1) = **77.63**

Prob > chi2 = **0.0000**

One-step results

(Std. Err. adjusted for clustering on id)

	Coef.	Robust Std. Err.	z	P> z	[95% Conf. Interval]	
k						
L1.	.9357448	.1062022	8.81	0.000	.7275923	1.143897
_cons	-.062468	.0439518	-1.42	0.155	-.1486119	.023676

Instruments for differenced equation

GMM-type: **L(2/.)k**

Instruments for level equation

Standard: **_cons**

10.2 Part B

Output from the Blundell-Bond estimator is below.

```
. xtdpdsys k, lags(1) vce(robust)
```

System dynamic panel-data estimation

Group variable: **id**

Time variable: **year**

Number of obs = **891**

Number of groups = **140**

Obs per group:

min = **6**

avg = **6.364286**

max = **8**

Number of instruments = **36**

Wald chi2(1) = **2213.64**

Prob > chi2 = **0.0000**

One-step results

	Coef.	Robust Std. Err.	z	P> z	[95% Conf. Interval]	
k						
L1.	1.100816	.0233971	47.05	0.000	1.054958	1.146673
_cons	.0057373	.0173146	0.33	0.740	-.0281986	.0396732

Instruments for differenced equation

GMM-type: **L(2/.)k**

Instruments for level equation

GMM-type: **LD.k**

Standard: **_cons**

10.3 Part C

Arellano-Bond suffers from a weak instrument problem if the true coefficient is near 1, i.e. if the true law of motion is close to a random walk. Our estimated coefficient from this model is in fact near 1, which means that this weak instrument issue seems to be a potential issue. As we saw in lecture, the Blundell-Bond estimator adds an extra assumption of stationarity, but avoids the weak instrument problem. In the results from the Blundell-Bond model, we see that the coefficient is again near 1. There are some differences between the results from the Blundell-Bond model and Arellano-Bond model. The differences are either due to (1) the weak instrument issue caused by the near-one lag coefficient, (2) the additional assumption of stationarity made by the Blundell-Bond model, which may or may not be true, or some combination of the two.

```
use "C:\Users\micha\OneDrive\Documents\HOMEWORK\Y1S1\Metrics\Q4\PS1\AB1991.dta", clear

xtabond k, lags(1) vce(robust)

xtdpdsys k, lags(1) vce(robust)
```