# Micro HW4

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## 1 Question 1

Let $A, B \subset X$, and let $x, y \in A \cap B$. Let $x \in C(A), y \in C(B)$. Then, $x \succeq y, y \succeq x$ because $y \in A, x \in B$. Then, $\forall z \in A$, $y \succeq x \succeq z \Rightarrow y \in C(A)$ by transitivity. Similarly, $\forall z \in B, x \succeq y \succeq z \Rightarrow x \in C(B)$ by transitivity. Therefore, $C(\cdot)$ satisfies WARP.

## 2 Question 2

Let $C(\cdot)$ satisfy WARP.

Let $x, y \in A$. Then either $x \in C(\{x, y\})$ or $y \in C(\{x, y\})$ because $C$ is nonempty so $x \succeq y$ or $y \succeq x$. Thus, $\succeq$ is complete.

Let $x, y, z \in X$ such that $x \succeq y, y \succeq z$. Then, $\exists A, B \subset X s.t. x, y \in A, x \in C(A); y, z \in B, y \in C(B)$. Assume for the purpose of contradiction that $z \succ x$. Then, $z \in C(A \cup B), y \in C(B) \Rightarrow y \in C(A \cup B)$ by WARP, and similarly by WARP $x \in C(A \cup B)$. Then, $x \sim z$ which is a contradiction.

Let $A \subset X$ and define the choice rule implied by $\succeq$ to be $C^I(A) := \{x \in A : x \succeq y \forall y \in A\}$. Let $x \in C^I(A)$. Then, $\forall y \in A, x \succeq Y \Rightarrow x \in C(A) \Rightarrow C(A) \subseteq C^I(A)$. Now let $x \in C(A)$. Since $x \in C(A)$ then $x \in A$. Then, $A$ is nonempty. For any $y \in A, x, y \in A$ and $x \in C(A)$ imply $x \succeq y \Rightarrow x \in C^I(A) \Rightarrow C^I(A) \subseteq C(A) \Rightarrow C^I(A) = C(A)$.

## 3 Question 3

### 3.1 Show that the induced choice rule is nonempty.

We fix finite $X$. Let $A \subseteq X$ have one element, $a$. Then, $C(A) = a$ because $a \succeq a$.

Now assume $A_{n+1} \subseteq X$ has $n+1$ elements, and that $C(A_n)$ is nonempty for any $A_n \subset X$ where $A_n$ contains $n$ elements. Then, for any $a \in A_{n+1}, C(A_{n+1} \setminus \{a\})$ is nonempty so let $c \in C(A_{n+1} \setminus \{a\})$. Then, either $a \succeq c \Rightarrow a \in C(A_{n+1})$ or $c \succeq a \Rightarrow c \in C(A_{n+1})$ by completeness. In either case, $C(A_{n+1})$ is nonempty.

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### 3.2 Show that a utility representation exists.

Assume $X$ has one element, $x$. Then we can set $u(x) = 1$ so immediately $\forall x, y \in X, u(x) = u(y) = 1$ and $x = y$ so $\succeq$ can be represented by a utility representation with range $\{1\}$.

Assume that for all $X_n$ with $n$ elements, $\succeq$ can be represented by a utility representation with range $\{1, \ldots, n\}$. Now, let $X_{n+1}$ have $n+1$ elements. Then, let $x \in C(X_{n+1})$. $X_{n+1} \setminus \{x\}$ can be represented by a utility representation with range $\{1, \ldots, n\}$. Denote this representation as $u$. For each element $y$ of $X_{n+1} \setminus \{x\}$, $x \succeq y$. Now define $v$ as the following:

$$v(z) = \begin{cases} u(z), z \in X_{n+1} \setminus \{x\} \\ n+1, z = x \end{cases}$$

Now, let $a, b \in X_{n+1}$. If $a, b \in X_{n+1} \setminus \{x\}$, $v(a) = u(a), v(b) = u(b)$ so $a \succeq b$ iff $u(a) \geq u(b)$. Otherwise, one or both of $a, b$ is $x$. WLOG, say $a = x$. Then $a \in C(X_{n+1})$ so $a \succeq b$, $v(a) \geq v(b)$ so the iff required for preferences to be represented by a utility function is trivially satisfied. Therefore, $v$ is a utility representation with range $\{1, \ldots, n+1\}$. Therefore, a utility representation exists.