Econometrics HW2

Michael B. Nattinger*

February 16, 2021

1 Question 1

1.1 Part i

$$
\begin{aligned}
E[ZX'] &= E\left[\begin{pmatrix} Z_1 \\ X_2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}'\right] \\
&= \begin{pmatrix} E[Z_1X_1] & E[Z_1X_2'] \\ E[X_2X_1] & E[X_2X_2'] \end{pmatrix} \\
E[ZZ'] &= E\left[\begin{pmatrix} Z_1 \\ X_2 \end{pmatrix}\begin{pmatrix} Z_1 \\ X_2 \end{pmatrix}'\right] \\
&= \begin{pmatrix} E[Z_1^2] & E[Z_1X_2'] \\ E[X_2Z_1] & E[X_2X_2'] \end{pmatrix}
\end{aligned}
$$

Note that $E[X_2X_2']$ must be invertible for either $E[ZX']$ or $E[ZZ']$ to be invertible.[1]

Block inversion implies that $E[ZX']$ is invertible iff $E[Z_1X_1]-E[Z_1X_2']E[X_2X_2']^{-1}E[X_2X_1] \neq 0$, and similarly $E[ZZ']$ is invertible iff $E[Z_1^2]-E[Z_1X_2']E[X_2X_2']^{-1}E[X_2Z_1] \neq 0$. We can rewrite these expressions as follows: $E[\tilde{Z}_1X_1] \neq 0, E[\tilde{Z}_1^2] \neq 0$ for $\tilde{Z}_1 := Z_1 - X_2'E[X_2X_2']^{-1}E[X_2Z_1]$. From FWL, for $\pi_1 = E[\tilde{Z}_1X_1]E[\tilde{Z}_1^2]$. Together, $E[\tilde{Z}_1^2] \neq 0$ and $\pi_1 \neq 0$ imply that $E[\tilde{Z}_1X_1] \neq 0$, and the reverse direction comes from Cauchy-Schwarz:

$$0 < E[Z_1X_1]^2 \leq E[\tilde{Z}_1^2]E[X_1^2].$$

1.2 Part ii

Under homoskedasticity, $\Omega = \sigma_U^2 E[ZX']^{-1}E[ZZ']E[XZ']^{-1}$. We again go back to block inversion and find that:

$$E[ZX']^{-1} = E[\tilde{Z}_1]^{-1}\begin{pmatrix} 1 & -E[Z_1X_2']E[X_2X_2']^{-1} \\ \cdots & \cdots \end{pmatrix},$$

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, Katherine Kwok, and Danny Edgel.

[1] $E[X_2X_2']$ not being invertible implies the existence of some t such that $E[X_2X_2']t = 0 \Rightarrow E[(X_2't)^2] = 0 \Rightarrow E[ZX'](0,t')' = E[ZX'](0,t')' = 0$ so $E[ZX']$ and $E[ZZ']$ are not invertible.

where the second block row does not enter into the upper left entry of Ω. We then have the following:

$$\begin{aligned}\Omega_{1,1} &= \frac{\sigma_U^2}{E[Z_1X_1']^2}(E[Z_1]^2 - E[Z_1X_2']^{-1}E[X_2Z_1]) \\ &= \frac{\sigma_U^2 E[\tilde{Z}_1^2]}{E[Z_1X_1']^2} \\ &= \frac{\sigma_U^2]}{E[\tilde{Z}_1^2]\pi_1^2}\end{aligned}$$

1.3 Part iii

π is the population projection (regression) coefficients mapping Z into X_1, and $\tilde{Z}$ is Z residualized at the population level with respect to X_2.

1.4 Part iv

$$E[\tilde{Z}_1X_2] = E[X_2Z_1 - X_2X_2'E[X_2X_2']^{-1}E[X_2Z_1]] = 0.$$

The above expression implies the following:

$$\begin{aligned}E[\tilde{Z}_1X_2E[X_2X_2']^{-1}E[X_2E[X_1|Z]]] &= 0 \\ E[\tilde{Z}_1X_1] = E[\tilde{Z}_1E[X_1|Z]] - E[\tilde{Z}_1X_2E[X_2X_2']^{-1}E[X_2E[X_1|Z]]] &= E[\tilde{Z}Z_*]\end{aligned}$$

We apply Cauchy-Schwarz to achieve the desired inequality:

$$\begin{aligned}\Omega_{1,1} &= \frac{\sigma_U^2 E[\tilde{Z}_1^2]}{E[\tilde{Z}Z_*]^2} \\ &\geq \frac{\sigma_U^2 E[\tilde{Z}_1^2]}{E[\tilde{Z}^2]E[Z_*^2]} \\ &= \frac{\sigma_U^2}{E[Z_*^2]}\end{aligned}$$

We will achieve the lower bound when $Z_* = \tilde{Z}_1\pi_1$. This occurs when $E[X_1|Z] = Z_1\pi_1 + X_2'\pi_2$.

1.5 Part v

If X_2 is just a constant, $\tilde{Z} = Z - E[Z], E[\tilde{Z}_1^2] = Var(Z_1), E[\tilde{Z}_1X_1] = Cov(Z_1, X_1)$. Thus,

$$\Omega_{1,1} = \frac{\sigma_U^2 Var(Z_1)}{Cov(Z_1, X_1)^2}.$$

2 Question 2

2.1 Part i

$$\begin{aligned}E[h(Z)(Y - X\beta)] &= E[h(Z)(X(\beta_1 - \beta)) + U)] \\ &= E[h(Z)X](\beta_1 - \beta) + E[h(Z)U].\end{aligned}$$

If exogeneity holds, $E[h(Z)U] = E[h(Z)E[U|Z]] = 0$ so $E[h(Z)(Y - X\beta)] = 0 \iff \beta_1 = \beta$.

2.2 Part ii

Define $\hat{\beta}_1^h$ as the solution to the following:

$$\frac{1}{n}\sum_i (h(Z_i)(Y_i - X_i\hat{\beta}_1^h)) = 0 \tag{1}$$

$$\Rightarrow \frac{1}{n}\sum_i (h(Z_i)(Y_i)) - \frac{1}{n}\sum_i (h(Z_i)X_i)\hat{\beta}_1^h = 0 \tag{2}$$

$$\Rightarrow \hat{\beta}_1^h = \left[\frac{1}{n}\sum_i (h(Z_i)X_i)\right]^{-1} \frac{1}{n}\sum_i (h(Z_i)(Y_i)) \tag{3}$$

2.3 Part iii

By applying the law of large numbers twice and continuous mapping theorem, we reach the following probability limit:

$$\begin{aligned}
\hat{\beta}_1^h &\to_p [E[h(Z)X]]^{-1}E[h(Z)Y] \\
&= [E[h(Z)X]]^{-1}E[h(Z)(X\beta_1 + U)] \\
&= [E[h(Z)X]]^{-1}E[h(Z)X]\beta_1 + [E[h(Z)X]]^{-1}E[h(Z)U] \\
&= \beta_1 + [E[h(Z)X]]^{-1}E[h(Z)E[U|Z]] \\
&= \beta_1.
\end{aligned}$$

Now, we have the following:

$$\sqrt{n}(\hat{\beta}_1^h - \beta_1) = \left[\frac{1}{n}\sum_i (h(Z_i)X_i)\right]^{-1} \frac{1}{\sqrt{n}}\sum_i (h(Z_i)(U_i))$$

By LLN, $\frac{1}{n}\sum_i (h(Z_i)X_i) \to_p E[h(Z)X]$. Also, by the CLT,

$$\frac{1}{\sqrt{n}}\sum_i (h(Z_i)(U_i)) \to_d N(0, V)$$

where $V = Var(h(Z)U) = E[(h(Z))^2U^2] - E[h(Z)U]^2 = E[(h(Z))^2U^2] - E[h(Z)E[U|Z]]^2 = E[(h(Z))^2U^2]$. By the continuous mapping theorem,

$$\sqrt{n}(\hat{\beta}_1^h - \beta_1) \to_d N(0, E[h(Z)X]^{-2}E[(h(Z))^2U^2])$$

2.4 Part iv

We have from part (iii) that $\Omega^h = E[h(Z)X]^{-2}E[(h(Z))^2U^2]$. By applying the law of iterated expectations and then Cauchy-Schwarz,

$$\begin{aligned}
\Omega^h &= \frac{E[(h(Z))^2U^2]}{E[h(Z)X]^2} \\
&= \frac{E[(h(Z))^2E[U^2|Z]]}{E\left[[h(Z)\sqrt{E[U^2|Z]}\frac{E[X|Z]}{\sqrt{E[U^2|Z]}}\right]^2} \\
&\geq \frac{E[(h(Z))^2E[U^2|Z]]}{E\left[(h(Z))^2E[U^2|Z]\right]\left(\frac{E[X|Z]^2}{E[U^2|Z]}\right)} \\
&= \left(\frac{E[X|Z]^2}{E[U^2|Z]}\right)^{-1}
\end{aligned}$$

We can find h_* which achieves this lower bound. Let $h_* = E[X|Z](E[U^2|Z])^{-1}$. Then,

$$\begin{aligned}
\Omega^{h_*} &= \frac{E[(h_*(Z))^2 U^2]}{E[h_*(Z)X]^2} \\
&= \frac{E[(E[X|Z](E[U^2|Z])^{-1})^2 E[U^2|Z]]}{E[E[X|Z](E[U^2|Z])^{-1}E[X|Z]]^2} \\
&= \frac{E[E[X|Z]^2(E[U^2|Z])^{-1}]}{E[E[X|Z]^2(E[U^2|Z])^{-1}]^2} \\
&= \left(\frac{E[X|Z]^2}{E[U^2|Z]}\right)^{-1}.
\end{aligned}$$

3 Question 3

	Results
Beta hat	0.10842
SE	0.01948

```
clear; close all; clc
reload = 0;
if reload
xl=readtable('AK91.csv');
y = xl.lwage; ed = xl.educ; n = length(y);
yob = zeros(n,9); sob = zeros(n,50); qob = zeros(n,3);
for t=1:n
    if xl.yob(t)>30&&xl.yob(t)<40
        yob(t,xl.yob(t) - 30) = 1;
    end
    if xl.sob(t)>0 && xl.sob(t)< 51
        sob(t,xl.sob(t)) = 1;
    end
    if xl.qob(t)>1 && xl.qob(t)<5
        qob(t,xl.qob(t)-1) = 1;
    end
end
save 'cleanAK91'
else
load 'cleanAK91'
end
ssob = sum(sob); ssobi = ssob>0;
sob = sob(:,ssobi); % remove columns with no data
x = [ed ones(n,1) yob sob]; z = [qob ones(n,1) yob sob];
% formulas
bhat2sls = (x'*z*inv(z'*z)*z'*x)\(x'*z*inv(z'*z)*z'*y);
e = y-x*bhat2sls; Qzz = z'*z/n; Qxz = x'*z/n;
Om = 0*Qzz;
for i=1:n
    Om = Om + z(i,:)'*z(i,:)*(e(i)^2);
end
Om = Om/n;
minv = inv(Qxz*inv(Qzz)*Qxz');
VhB = minv*(Qxz*inv(Qzz)*Om*inv(Qzz)*Qxz')*minv/n;
tab = table([bhat2sls(1,1); sqrt(VhB(1,1))],'VariableNames', ...
    {'Results'},'RowNames',{'Beta hat' 'SE'});
table2latex(tab,'ps3.tex')
```