Macro PS1

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1 Question 1

1.1 Part A

$$V(A_t, c_{t-1}) = \max_{c_t, A_{t+1}} u(c_t, c_{t-1}) + \beta V(A_{t+1}, c_t)$$
$$\text{s.t. } A_{t+1} = R(A_t - c_t)$$

We can rewrite this as the following:

$$V(A_t, c_{t-1}) = \max_{c_t} u(c_t, c_{t-1}) + \beta V(R(A_t - c_t), c_t) \tag{1}$$
$$V(A_t, c_{t-1}) = \max_{A_t} u\left(A_t - \frac{A_{t+1}}{R}, c_{t-1}\right) + \beta V\left(A_{t+1}, A_t - \frac{A_{t+1}}{R}\right) \tag{2}$$

To ensure the solution is unique, with a strictly increasing, strictly concave value function that is differentiable on the interior of the feasible set, we require the following assumptions:

- $u(.)$ is continuously differentiable, strictly concave, and strictly increasing in its arguments.
- $0 < \beta < 1$
- $\lim_{k\to 0} u'(k, u) = \lim_{k\to 0} u'(u, k) = \infty$
- $\lim_{k\to \infty} u'(k, u) = \lim_{k\to \infty} u'(u, k) = 0$
- The utility function is bounded?
- Do we need anything else?

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Assuming the above conditions hold, we can solve the value function as it is written in (1) by taking a first order condition with respect to c_t, and then applying the envelope theorem twice:

$$0 = \frac{\partial u}{\partial c_t}(c_t, c_{t-1}) - R\beta \frac{\partial V}{\partial R(A_t - c_t)}(R(A_t - c_t), c_t) + \beta \frac{\partial V}{\partial c_t}(R(A_t - c_t), c_t)$$

$$\frac{\partial V}{\partial R(A_t - c_t)}(R(A_t - c_t), c_t) = 999$$

$$\frac{\partial V}{\partial c_t}(R(A_t - c_t), c_t) = \frac{\partial u}{\partial c_t}(c_{t+1}, c_t)$$

$$\Rightarrow \frac{\partial u}{\partial c_t}(c_t, c_{t-1}) + \beta() = R\beta()$$

Instead we will use the value function as it is written in (2) by taking a first order condition with respect to A_{t+1}, and then applying the envelope theorem twice:

$$0 = \frac{-1}{R} \frac{\partial u}{\partial A_t - \frac{A_{t+1}}{R}} \left(A_t - \frac{A_{t+1}}{R}, c_{t-1} \right) + \beta \frac{\partial V}{\partial A_{t+1}} \left(A_{t+1}, A_t - \frac{A_{t+1}}{R} \right) - \frac{\beta}{R} \frac{\partial V}{\partial A_t - \frac{A_{t+1}}{R}} \left(A_{t+1}, A_t - \frac{A_{t+1}}{R} \right)$$

$$\frac{\partial V}{\partial R(A_t - c_t)}(R(A_t - c_t), c_t) = 999$$

$$\frac{\partial V}{\partial c_t}(R(A_t - c_t), c_t) = \frac{\partial u}{\partial c_t}(c_{t+1}, c_t)$$

$$\Rightarrow \frac{\partial u}{\partial c_t}(c_t, c_{t-1}) + \beta() = R\beta()$$

2 Question 2

2.1 Part A

The sequence problem is to maximize profits:

$$\max_{\{x_t\}_{i=1}^{\infty}} \left(\frac{1}{1+r} \right)^t \left(ax_t - \frac{b}{2}x_t^2 - \frac{c}{2}(x_{t+1} - x_t)^2 \right)$$

We can rewrite this as a bellman equation in the following way:

$$V(x) = \max_y ax - \frac{b}{2}x^2 - \frac{c}{2}(y - x)^2 + \delta V(y) \tag{3}$$

We can rewrite this as follows:

$$T(v)(x) = \max_y ax - \frac{b}{2}x^2 - \frac{c}{2}(y - x)^2 + \delta v(y) \tag{4}$$

where the fixed point of our T operator in (4) is the solution to the Bellman equation in (3).

2.2 Part B

Let $L < 0$ be arbitrary. If we set $y = 0, x < \frac{L}{a}$ then $F(x,y) = ax - \frac{b}{2}x^2 - \frac{c}{2}(y-x)^2 \leq ax < L$ so F is unbounded below.

This F function is of the form of a polynomial with limits of negative infinity in all directions, so we can find the global maximum by taking first order conditions:

$$\begin{aligned}
0 &= a - bx + c(y-x) \\
0 &= -c(y-x) \Rightarrow y - x = 0 \Rightarrow y = x \\
\Rightarrow y = x &= \frac{a}{b} \\
F\left(\frac{a}{b}, \frac{a}{b}\right) &= a\left(\frac{a}{b}\right) - \frac{b}{2}\left(\frac{a}{b}\right)^2 - 0 \\
&= \frac{a^2}{2b}
\end{aligned}$$

Therefore, the maximum value F can take is $\frac{a^2}{2b}$

We can find bounds on $\hat{v}$ in the following way:

$$\begin{aligned}
\hat{v} &= \frac{a^2}{2b} + \delta\hat{v} \\
\Rightarrow \hat{v} &= \frac{a^2}{2b(1-\delta)}
\end{aligned}$$

2.3 Part C

$$\begin{aligned}
T\hat{v}(x) &= \max_y ax - \frac{b}{2}x^2 - \frac{c}{2}(y-x)^2 + \delta\hat{v} \\
0 &= -c(y-x) \Rightarrow y = x, \\
\Rightarrow T\hat{v}(x) &= ax - \frac{b}{2}x^2 + \delta\hat{v} \\
&\leq \frac{a^2}{2b} + \delta\frac{a^2}{2b(1-\delta)} = \frac{a^2}{2b(1-\delta)} \\
&= \hat{v}.
\end{aligned}$$

2.4 Part D

We showed the base case of the induction in Part C. Now we will show the induction step.

Assume that $T^n\hat{v}(x)$ takes the form $T^n\hat{v}(x) = \alpha_n x - \frac{1}{2}\beta_n x^2 + \gamma_n$. Then,

$$\begin{aligned}
T^{n+1}\hat{v}(x) &= \max_y ax - \frac{b}{2}x^2 - \frac{c}{2}(y-x)^2 + \delta(\alpha_n x - \frac{1}{2}\beta_n x^2 + \gamma_n) \\
y = x \Rightarrow T^{n+1}\hat{v}(x) &= ax - \frac{b}{2}x^2 + \delta\alpha_n x - \delta\frac{1}{2}\beta_n x^2 + \delta\gamma_n \\
&= (a + \delta\alpha_n)x - \frac{b+\delta\beta_n}{2}x^2 + \delta\gamma_n \\
&= \alpha_{n+1}x - \frac{1}{2}\beta_{n+1}x^2 + \gamma_{n+1}
\end{aligned}$$

where $\alpha_{n+1} = (a + \delta\alpha_n), \beta_{n+1} = b + \delta\beta_n, \gamma_{n+1} = \delta\gamma_n$.

2.5 Part E

Note that $\alpha_n = a + \delta a + \delta^2 a + \ldots$, $\beta_n = b + \delta b + \delta^2 b + \ldots$, $\gamma_n = \delta^n \hat{v}$. Thus, we can take the limit of α, β as geometric sums, and the limit of γ_n is 0. Therefore,

$$\begin{aligned}
\tilde{V} &= \lim_{n\to\infty} T^n\hat{v} = \frac{a}{1-\delta}x - \frac{1}{2}\frac{b}{1-\delta}x^2. \\
T\tilde{V} &= \max_y ax - \frac{b}{2}x^2 - \frac{c}{2}(y-x)^2 + \delta\frac{a}{1-\delta}x - \frac{1}{2}\frac{b}{1-\delta}x^2, \\
y = x \Rightarrow T\tilde{V} &= ax - \frac{b}{2}x^2 + \delta\frac{a}{1-\delta}x - \frac{1}{2}\frac{b}{1-\delta}x^2 \\
&= \frac{a}{1-\delta}x - \frac{1}{2}\frac{b}{1-\delta}x^2 \\
&= \tilde{T}.
\end{aligned}$$

Therefore, the limit function $\tilde{V}$ satisfies the Bellman equation.

3 Question 3

$$V(k) = \max_{k'} \pi(k) - \gamma(k' - (1-\delta)k) + \frac{1}{R}V(k')$$

$$\begin{aligned}
0 &= -\gamma'(k' - (1-\delta)k) + \frac{1}{R}\left(\pi'(k') - (1-\delta)\gamma'(k'' - (1-\delta)k')\right) \\
\Rightarrow \pi'(k') &= R\gamma'(k' - (1-\delta)k) + (1-\delta)\gamma'(k'' - (1-\delta)k'))
\end{aligned}$$

4 Question 4