Microeconomic Theory Notes

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1 Preferences, Choice and Utility Functions

1.1 Preferences

- Choice set X, with preferences $\succsim$ on X
- Complete: $\forall x, y \in X$, we have $x \succsim y$ or $y \succsim x$
- Transitive: $\forall x, y \in X$, $x \succsim y$ and $y \succsim z \Rightarrow x \succsim z$
- Rational: Complete and transitive
- Continuous: For any sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ if $x_n \to \bar{x}$, $y_n \to \bar{y}$, and $x_n \succsim y_n$ $\forall n$ then $\bar{x} \succsim \bar{y}$
- Monotone: $x \gg y \Rightarrow x \succ y$
- Strong monotone: $x > y \Rightarrow x \succ y$
- Locally nonsatiated: $\forall y \in X$ and $\forall \epsilon > 0$, $\exists x \in X$ such that $||x - y|| < \epsilon$ and $x \succsim y$
- Convex: $\forall t \in (0, 1)$, $x \succsim y$ and $x' \succsim y \Rightarrow tx + (1 - t)x' \succsim y$
- Strictly convex: $\forall t \in (0, 1)$, $x \succsim y$ and $x' \succsim y \Rightarrow tx + (1 - t)x' \succ y$
- Separable: Let $= Y \times Z$. Preferences over Y do not depend on z if $\forall y, y' \in Y$ and $z, z' \in Z$, $(y, z) \succsim (y', z) \iff (y, z') \succsim (y', z')$

1.2 Choice

- Let $\mathcal{B}$ be the power set of X
- Choice rule from primitive: $C : \mathcal{B} \to \mathcal{B}$ scuh that $\forall B \in \mathcal{B}$, $C(B) \subseteq B$
- Choice rule from preference: $C(B; \succsim) = \{x \in B : x \succsim y \forall y \in B\}$
- C is nonempty if $B \neq \emptyset \Rightarrow C(B) \neq \emptyset$

1.3 Weak Axiom of Revealed Preference

- Preference version: If $\succsim$ is complete and transitive, for all sets $A, B \subseteq X$, if $x, y \in A \cap B$, $x \in C(A; \succsim)$ and $(y \in C(B; \succsim)$, then $y \in C(A; \succsim)$ and $x \in C(B; \succsim)$.
- Choice version: If $x, y \in A \cap B$, then $x \in C(A)$ and $y \in C(B) \Rightarrow y \in C(A)$ and $x \in C(B)$.
- Proposition: Suppose $C(B)$ nonempty. Then there exists a complete and transitive preference relation $\succsim$ on X such that $C(\cdot) = C(\cdot; \succsim)$ if and only if C satisfies WARP.

1.4 Utility

- $\succsim$ on X is represented by a utility function $u : X \to \mathbb{R}$ if $\forall x, y \in X$, $x \succsim y \iff u(x) \geq u(y)$
- If X is finite, then any complete and transitive preference relation on X can be represented by a utility function.
- If $X \subseteq \mathbb{R}^k$, then any complete, transitive and continuous preference relation can be represented by a utility function.
- Separability: Suppose preferences on $X = Y \times Z$ are represented by a utility function $u(y, z)$. Then preferences over y do not depend on z if and only if there exist functions $v : Y \to \mathbb{R}$ and $U : \mathbb{R} \times Z \to \mathbb{R}$ such that U is increasing in its first argument and $u(y, z) = U(v(y), z)$.
- Quasilinearity: Suppose $X = \mathbb{R}_+ \times Y$. Suppose that: (1) preferences $\succsim$ are complete and transitive; (1) there is a worst element $\bar{y} \in Y$ such that $(0, y) \succsim (0, \bar{y})$ $\forall y \in Y$; (3) the first good is valuable, with $(a, \bar{y}) \succsim (a', \bar{y}) \iff a \succsim a'$; (4) compensation is possible, meaning $\forall y \in Y$, $\exists t$ such that $(0, y) \sim (t, \bar{y})$; and (5) there are no wealth effects, meaning $(a, y) \succsim (a', y') \iff (a + t, y) \succsim (a' + t, y')$. Then preferences over X can be represented by a utility function with the form $u(a, y) = a + v(y)$.

1.5 Expected Utility Representation

Utility representation for game theory and information economics

- Z a finite set of outcomes, and ΔZ the set of probability distributions (lotteries) over Z
- $\succsim$ on ΔZ admits an expected utility representation if there exists a (Bernoulli utility) function $u : Z \to \mathbb{R}$ such that
$$p \succsim q \iff \sum_{z \in Z} u(z)p(z) \geq \sum_{z \in Z} u(z)q(z)$$
- Axiom vNM1: $\succsim$ is complete and transitive
- Axiom vNM2: (Continuity) $\forall p, q, r$ with $p \succ q \succ r$, $\exists \delta, \epsilon \in (0, 1)$ such that $(1 - \delta)p + \delta r \succ (1 - \epsilon)r + \epsilon p$
- Axiom vNM3: (Independence) $\forall p, q, r$ and $\forall \alpha \in (0, 1)$, $p \succsim q \iff \alpha p + (1 - \alpha)r \succsim \alpha q + (1 - \alpha)r$
- Theorem (von Neumann-Morgenstern): Let Z be a finite set and $\succsim$ a preference relation on ΔZ. Then there exists a Bernoulli utility function $u : Z \to \mathbb{R}$ that provides an expected utility representation for $\succsim$ if and only if $\succsim$ satisfies axioms vNM1, vNM2 and vNM3. The function u is unique up to positive affine transformations.
- Bayesian rational (subjective expected utility preferences) if: (1) in settings with uncertainty, form beliefs about probabilities of all relevant events; (2) decision-making maximizes expected utility given belieffs; (3) given new information, update beliefs using conditional probabilities whenever possible.

2 Consumer Problem

2.1 Utility Maximization

- Budget set $B(p, w) = \{x \in \mathbb{R}^k_+ : p \cdot x \leq w$
- Budget set is homogeneous of degree zero (HD0): $\forall \lambda > 0$, $B(\lambda p, \lambda w) = B(p, w)$
- Consumer problem: $\max_{x \in B(p,w)} u(x) = \max_{x \in \mathbb{R}^k_+} u(x) \quad \text{s.t.} \quad p \cdot x \leq w$
- Marshallian demand: $x(p, w) = \arg\max_{x \in B(p,w)} u(x) = C(B(p, w); \succsim_u) = \{x \in B(p, w) : u(x) \geq u(y) \forall y \in B(p, w)\}$
- Marshallian demand is HD0: $\forall \lambda > 0$, $x(\lambda p, \lambda w) = x(p, w)$
- Walras' Law: If preferences are LNS, then $\forall (p, w)$ and any $x \in x(p, w)$, $p \cdot x = w$.

- If preferences are convex, then $x(p,w)$ is a convex set. If preferences are strictly convex, then $x(p,w)$ is a singleton.
- Indirect utility function: $v(p,w) = \max_{x \in B(p,w)} u(x)$
- If u is a conditnuous utility function representing LNS preferences on $\mathbb{R}^k_+$, then $v(p,w)$ is: (1) homogeneous of degree 0; (2) continuous on $\{(p,w) : p \gg 0, w \geq 0\}$; (3) nonincreasing in p and strictly increasing in w; (4) quasiconcave, meaning $\forall \bar{v}$, $\{(p,w) : v(p,w) \leq \bar{v}\}$ is a convex set.
- Roy's Identity: Suppose v is differentiable at $(p,w) \gg 0$, and $\frac{dv}{dw} > 0$. Then $x(p,w)$ is a singleton and for each i, $x_i(p,w) = -\frac{dv}{dp_i}(p,w) \Big/ \frac{dv}{dw}(p,w)$.

2.2 Solving Tools

$$\max u(x) \qquad \text{s.t.} \qquad p \cdot x \leq w, x \geq 0$$

$$\mathcal{L}(x, \lambda, \mu) = u(x) + \lambda(w - p \cdot x) + \mu \cdot x$$

Proposition: Fix $(p,w) \gg 0$. If $x^*, \lambda^*, \mu^*)$ is a saddle point of $\mathcal{L}$, then x^* solves the consumer problem. Moreover, if u is differentiable and concave and x^* solves the consumer problem, then $\exists \lambda^*, \mu^* \geq 0$ such that (x^*, λ^*, μ^*) is a saddle point of $\mathcal{L}$.
Kuhn-Tucker conditions:

1. (FOC w.r.t. x) $\frac{du}{dx_i}(x^*) - \lambda^* p_i + \mu_i^* = 0 \ \forall i$
2. (Original constraints) $p \cdot x^* \leq w$ and $x^* \geq 0$
3. (Nonnegative multipliers) $\lambda^* \geq 0$ and $\mu_i^* \geq 0 \ \forall i$
4. (Complementary slackness) $\lambda^*(w - p \cdot x^*) = 0$ and $\mu_i x_i = 0 \ \forall i$

Theorem (KT1): Assume u is continuously differentiable, $p \gg 0$ and $w > 0$. If x^* solves the consumer problem, then $\exists \lambda^* \geq 0$ and $\mu^* \geq 0$ such that (x^*, λ^*, μ^*) satisfy the Kuhn-Tucker conditions.
Theorem (KT2): Suppose u is continuously differentiable and quasiconcave, and that $u(x') > u(x) \Rightarrow \nabla u(x) \cdot (x' - x) > 0$. Then if $(x^* \lambda^*, \mu^*)$ satisfy the Kuhn-Tucker conditions, then x^* solves the consumer problem.

2.3 Expenditure Minimization

- Expenditure minimization problem: $e(p,u) = \min_{x \geq 0} p \cdot x \qquad \text{s.t.} \qquad u(x) \geq u$
- Hicksian demand: $h(p,u) = \arg\min_{x \geq 0} p \cdot x \qquad \text{s.t.} \qquad u(x) \geq u$
- If u is a continuous utilit function on $\mathbb{R}^k_+$, then: (1) $h(p,u)$ is homogeneous of degree 0 in $p - h(\lambda p, u) = h(p,u)$; (2) there is no excess utility, meaning if $u \geq u(0)$ and $p \gg 0$ then $\forall x \in h(p,u)$, $u(x) = u$; (3) if preferences are convex, then $h(p,u)$ is convex; (4) if preferences are strictly convex and $p \gg 0$, then $h(p,u)$ is a singleton.
- If u is a continuous utility function representing LNS preferences on $\mathbb{R}^k_+$, then $e(p,u)$ is: (1) homogeneous of degree 1 in p, meaning $e(\lambda p, u) = \lambda e(p,u)$; (2) continuous in p and u; (3) nondecreasing in p, and strictly increasing in u if $p \gg 0$; (4) concave in p.
- Shepard's Lemma: Let u be a continuous utility function representing LNS preferences, and suppose $h(p,u)$ is single-valued. Then $e(p,u)$ is differentiable in p and $\frac{de}{dp_i}(p,u) = h_i(p,u)$.

Proposition: Suppose u represents continuous, LNS preferences on $\mathbb{R}^k_+$. Then $\forall p \gg 0, w \geq 0$, we have: (1) $x(p,w) = h(p, v(p,w))$; (2) $e(p, v(p,w)) = w$; (3) $h(p,u) = x(p, e(p,u))$; (4) $v(p, e(p,u)) = u$.

2.4 Slutsky Equation, Comparative Statics, and Types of Goods

- Law of Demand: Suppose $p, p' \geq 0$, $x \in h(p, u)$ and $x' \in h(p', u)$. Then $(p' - p) \cdot (x' - x) \leq 0$.
- Suppose u represents a preference relation and $h(p, u)$ is single-valued and differentiable at (p, u) with $p \gg 0$. Then $D_p h(p, u)$ is symmetric and negative semi-definite.
- Slutsky: Let u be a continuous utility function representing LNS preferences on $\mathbb{R}^k_+$. Pick $p \gg 0$ and let $w = e(p, u)$. If $h(p, u)$ and $x(p, w)$ are single-valued and differentiable at (p, u, w), then

$$\frac{dx_i}{dp_j}(p, w) = \frac{dh_i}{dp_j}(p, u) - \frac{dx_i}{dw}(p, w) x_j(p, w)$$

- Normal good: x_i increasing in w
- Inferior good: x_i decreasing in w
- Regular good: x_i decreasing in p_i
- Giffen good: x_i increasing in p_i
- Substitute: h_i increasing in p_j
- Complement: h_i decreasing in p_j
- Gross substitute: x_i increasing in p_j
- Gross complement: x_i decreasing in p_j

2.5 GARP, Rationalizability and Recoverability

- Suppose u represents LNS preferences, and $x(p, w)$ that solves the consumer problem is single-valued and differentiable. Then: (1) $x(p, w)$ is homogeneous of degree 0; (2) at every (p, w), $p \cdot x(p, w) = w$ (Walras' Law); (3) the matrix $\left[\frac{dx_i}{dp_j} + \frac{dx_i}{dw} x_j\right]$ is symmetric and negative semidefinite.
- Revealed preference definitions: x directly revealed preferred to x' ($x \succsim_D x'$) if $x \in x(p, w)$ and $p \cdot x' \leq w$; x revelaed preferred to x' ($x \succsim_R x'$) if there are observations such that $x \succsim_D x_1 \succsim_D ... \succsim_D x_n \succsim_D x'$; x directly strictly revealed preferred to x' ($x \succ_D x'$) if $x \in x(p, w)$ and $p \cdot x' < w$.
- Generalized Axiom of Revealed Preference: for any x, x', if $x \succsim_R x'$ then $x' \not\succ_D x$
- Afriat's Theorem: Let (p^t, x^t) for $t = 1, ..., T$ be a finite number of observations of price vectors and consumption bundles. Then the following are equivalent: (1) there exists a LNS utility function that rationalizes the data; (2) the data satisfy GARP; (2) there exists a LNS, continuous, concave and monotonic utility function that rationalizes the data.

Recoverability: Note that we don't do this formally, and we can only practice it.
Aggregation: Suppose there are n consumers and that consumer i has indirect utility function $v_i(p, w_i) = a_i(p) + b(p) w_i$ (Gorman form). Then aggregate demand $X = \sum_i x^i(p, w_i)$ is the same as the demand of a single consumer with indirect utility function $V(p, W) = \sum_i a_i(p) + b(p)W$ where $W = \sum_i w_i$.

2.6 Welfare Effects of Price Changes

Consider a shift that lowers prices.

- Compensating variation: how much more cheaply we can afford the old level of utility, or how much money we can take away ex post and leave you as well off as ex ante. Your willingness to "buy" a price change.

$$CV = e(p^0, u^0) - e(p^1, u^0) = e(p^1, u^1) - e(p^1, u^0)$$

- Equivalent variation: how much more cheaply we can affort the new level of utility relative to how muc h it would have cost before. How much to make you "give back" a price change after it's happened.

$$EV = e(p^0, u^1) - e(p^1, u^1)$$

- Suppose only the price of good i changes, with $p_i^1 < p_i^0$. Then

$$CV = \int_{p_i^1}^{p_i^0} h_i(p, u^0)dp_i \qquad EV = \int_{p_i^1}^{p_i^0} h_i(p, u^1)dp_i \qquad \Delta CS = \int_{p_i^1}^{p_i^0} x_i(p, w)dp_i$$

- If i is a normal good, then $EV \geq \Delta CS \geq CV$. If i is an inferior good, then $EV \leq \Delta CS \leq CV$. If preferences are quasilinear, then $EV = \Delta CS = CV$.

3 Firm Problem

Firms are price takers, and profit maximizers taking prices and technology as given.

3.1 Production Sets

- Production plan $y \in \mathbb{R}^k$, where positive numbers are outputs and negative numbers are inputs
- Production set $Y \subset \mathbb{R}^k$
- Standard properties: (1) Y is nonempty and closed; (2) free disposal, meaning $y \in Y \Rightarrow y' \in y \forall y' \leq y$; shutdown, with $0 \in Y$.
- Other possible assumptions: (1) constant returns to scale, with $y \in Y \Rightarrow \alpha y \in Y\ \forall \alpha > 0$; (2) increasing returns to scale, with $y \in Y \Rightarrow \alpha y \in Y\ \forall \alpha > 1$; (3) decreasing returns to scale, with $y \in Y \Rightarrow \alpha y \in Y$ $\forall \alpha \in (0, 1)$; (4) cannot create from nothing, meaning $Y \cap \mathbb{R}_+^k = \{0\}$; (5) irreversible production, with $Y \cap -Y = \{0\}$; (6) Y strictly convex, with $\forall t \in (0, 1)$, $y, y' \in Y$ with $y \neq y' \Rightarrow ty + (1-t)y' \in int(Y)$.
- Transformation function $T : \mathbb{R}^k \to \mathbb{R}$ such that $T(y) \leq 0$ if $y \in Y$ and $T(y) > 0$ if $y \notin Y$. The boundary $T(y) = 0$ is the transformation frontier.
- If T is differentiable, then the Marginal Rate of Transformation is $MRT_{i,j}(y) = \frac{dT/dy_i}{dT/dy_j}$.

3.2 Solving Firm Problems and Properties

$$\max_y p \cdot y \qquad \text{s.t.} \qquad y \in Y$$

- Profit function: $\pi(p) = \sup_{y \in Y} p \cdot y$
- Optimal supply correspondence: $y(p) = Y^*(p) = \{y \in Y : p \cdot y = \pi(p)\}$
- If Y has constant returns to scale, then $\pi(p) = 0$ or $\pi(p) = \infty$. If Y has increasing returns to scale and shutdown, then $\pi(p) = 0$ or $\pi(p) = \infty$.
- $\pi(p)$ is homogeneous of degree 1, with $\pi(\lambda p) = \lambda \pi(p)$
- $Y^*(p)$ is homogeneous of degree 0, with $Y^*(\lambda p) = Y^*(p)$
- π is convex.
- If Y is convex, then $Y^*(p)$ is convex.
- If Y is strictly convex, $p \neq 0$ and $Y^*(p)$ nonempty, then $Y^*(p)$ is single-valued.
- Law of Supply: For $y \in Y^*(p)$ and $y' \in Y^*(p')$, $(p' - p) \cdot (y' - y) \geq 0$
- Hotelling's Lemma: If Y closed with free disposal,, and $y(\cdot)$ is single-valued in a neighborhood of p, then π is differentiable at p and $\frac{d\pi}{dp_i}(p) = y_i(p)$.
- If Y closed with free disposal, and if $y(\cdot)$ single-valued and differentiable, then $D_p y(p)$ is symmetric and positive semidefinite, with $[D_p y(p)]p = 0$.

3.3 Rationalizability and Recoverability

3.3.1 Rationalizability

- Let $P \subseteq \mathbb{R}^k_+$ be the set of price vectors for which we have observations.
- Observe $\pi : P \to \mathbb{R}$ or $y : P \to \mathbb{R}^k$
- Inner bound: $Y^I = \bigcup_{p \in P} y(p)$
- Outer bound: $Y^O = \{y \in \mathbb{R}^k : p \cdot y \leq \pi(p)\ \forall p \in P\}$
- Weak Axiom of Profit Maximization: y and π are jointly rationalizable if and only if $p \cdot y = \pi(p) \forall y \in y(p)$, $\forall p \in P$, and $Y^I \subseteq Y^O$.

3.3.2 Recoverability

- Inner bound with free disposal: $Y^I_{FD} - \{y \in \mathbb{R}^k : y \leq x \text{ for some } x \in Y^I\}$
- Suppose Y is closed with free disposal, and $P = \mathbb{R}^k_+ - \{0\}$. If Y is convex, then $Y = Y^O$. If Y is convex and we observe all of a firm's optimal choices (meaning $y(p) = Y^*(p)\ \forall p \in P$), then $Y = Y^I_{FD}$.

3.3.3 Special Case

P is an open, convex subset of $\mathbb{R}^k_{++}$, $y(p)$ is single-valued, and $\pi(p)$ is differentiable. Then the following hold

- y and π are jointly rationalizable if and only if: (1) adding up holds, with $\pi(p) = p \cdot y(p)\ \forall p \in P$; (2) Hotelling's lemma holds, with $y_i(p) = \frac{d\pi}{dp_i}(p)\ \forall i, \forall p \in P$; (3) π is convex.
- A differentiable supply function y is rationalizable if and only if it is homogeneous of degree 0 and $D_p y(p)$ is symmetric and positive semidefinite.
- π is rationalizable if and only if it is homogeneous of degree 1 and convex.

3.4 Single Output and Cost Minimization

- Quantity aq of output, vector $z \in \mathbb{R}^m_+$ of inputs
- Price of output good $p \in \mathbb{R}_+$, and $w \in \mathbb{R}^m_+$ a vector of input prices
- Production function $f : \mathbb{R}^m_+ \to \mathbb{R}_+$
- With free disposal, $Y = \{(q, -z) : q \leq f(z)\}$
- Marginal rate of technical substitution: $MRTS_{k,\ell} = \frac{df/dz_\ell}{df/dz_k}$
- Firm problem, primitive: $\max_{q,z} pq - w \cdot z \qquad \text{s.t.} \qquad q \leq f(z)$
- Firm problem if $p > 0$ ($\Rightarrow q = f(z)$): $\max_z pf(z) - w \cdot z$
- Two-step: Solve $c(q) = \min_{z : f(z) \geq 0} w \cdot z$ and then $\max_q pq - c(q)$
- Cost function: $c(q, w) = \inf_z w \cdot z \qquad \text{s.t.} \qquad z \geq 0, f(z) \geq q$
- Conditional factor demand correspondence: $Z^*(q, w) = \{z : f(z) \geq q, w \cdot z = c(q, w)\}$
- $c(q, w)$ is homogeneous of degree 1 in w and increasing in q
- $c(q, w)$ is concave in w
- Shepard's Lemma: If $z(q, w)$ is single-valued, then c is differentiable with respect to w and $\frac{dc}{dw_i}(q, w) = z_i(q, w)$.
- If $z(q, w)$ is single-valued and differentiable, then $D_w z = D^2_w c$ is symmetric and negative semidefinite with $[D_w z]w = 0$.

- Law of Supply: If $z \in Z(q,w)$ and $z' \in Z(q,w')$ then $(z'-z)\cdot(w'-w) \leq 0$.
- If f is concave, then c is convex in q (increasing marginal costs.
- Average cost function: $\frac{c(q,w)}{q}$
- If f has increasing (decreasing) returns to scale, then the average cost c/q is decreasing (increasing) in q.

4 Monotone Comparative Statics

Choice variable $x \in X$, parameter $t \in T$, and function $g : X \times T \to \mathbb{R}$

$$x^*(t) = \arg\max_{x \in X} g(x,t)$$

4.1 Preliminaries and Defintions

- Strong set order: For $A, B \subset \mathbb{R}$, $A \geq_{SSO} B$ if $\forall a \in A, b \in B$, $\max\{a,b\} \in A$ and $\min\{a,b\} \in B$.
- Increasing differences: The following definitions are equivalent, for one-dimensional x: (1) $g(x',t) - g(x,t)$ is weakly increasing in t if $x' > x$; (2) $g(x',t') - g(x,t') \geq g(x',t) - g(x,t)$ whenever $x' > x$ and $t' > t$; (3) if g is differentiable, $\frac{dg}{dx}$ is increasing in t or $\frac{dg}{dt}$ is increasing in x; (4) if g is twice differentiable, $\frac{d^2g}{dxdt} \geq 0$.
- Componenet-wise maximum: For $a, b \in \mathbb{R}^m$, "a join b" $= a \vee b = \big(\max\{a_1,b_1\}, ..., \max\{a_m,b_m\}\big)$
- Componenet-wise minimum: For $a, b \in \mathbb{R}^m$, "a meet b" $= a \wedge b = \big(\min\{a_1,b_1\}, ..., \min\{a_m,b_m\}\big)$
- Supermodularity: For a product set X in $\mathbb{R}^m$, the following definitions are equivalent: (1) $g(x \vee y) + g(x \wedge y) \geq g(x) + g(y)$; (2) g has increasing differences in (x_i, x_j) for every pair (i,j), holding other variables fixed; (3) if g is twice differentiable, $\frac{d^2g}{dx_idx_j} \geq 0 \ \forall (i,j)$.
- Increasing differences (multidimensional): g has increasing differences in (X,T) if g has increasing differences in (x_i, t_j) $\forall i, j$

4.2 Theorems

- Theorem (Topkis 1-D): If g has increasing differences, then $x^*(t)$ is increasing in t via the strong set order.
- Corollary: If g has strictly increasing differences, then $\forall x \in x^*(t)$ and $x' \in x^*(t')$ with $t' > t$, we have $x' \geq x$.
- Theorem (Topkis M-D): Let X be a product set in $\mathbb{R}^m$ and $T \subseteq \mathbb{R}$. If g is supermodular in X and has increasing differences in X and T, then $x^*(t)$ is increasing in t, meaning that for any $x \in x^*(t)$ and $x' \in x^*(t')$ with $t' > t$, we have $x \vee x' \in x^*(t')$ and $x \wedge x' \in x^*(t)$.
- Corollary: If x^* is single-valued, then $x^*(t)$ is weakly increasing in every dimension.

5 Normal Form Games

5.1 Notation

- Players $\mathcal{P} = \{1, ..., n\}$
- Strategy set S_i
- Strategy profile $S = \prod_{i \in \mathcal{P}} S_i$
- Game $G = \big\{\mathcal{P}, \{S_i\}_{i \in \mathcal{P}}, \{u_i\}_{i \in \mathcal{P}}\big\}$
- Mixed strategy $\sigma_i \in \Delta S_i$
- Correlated strategy $\rho \in \Delta\Big(\prod_{i \in \mathcal{P}} S_i\Big) = \Delta S$

- Opponents' strategy profile $s_{-i} \in S_{-i} = \prod_{j \neq i} S_j$
- Conjecture $\mu_i \in \Delta S_{-i}$

5.2 Dominance and Iterated Dominance

- σ_i strictly dominant if $u_i(\sigma_i, \mu_i) > u_i(\sigma_i', \mu_i)\ \forall \sigma_i' \neq \sigma_i, \forall \mu_i \in \Delta S_{-i}$
- σ_i' strictly dominated by σ_i if $u_i(\sigma_i, \mu_i) > u_i(\sigma_i', \mu_i)\ \forall \mu_i \in \Delta S_{-i}$
- Only a pure strategy can be strictly dominant. If a pure strategy s_i is strictly dominated, so is any mixed strategy with s_i in its support.
- Iterated strict dominance: iteratively remove all dominated pure strategies, then check all remaining mixed strategies.
- σ_i' weakly dominated by σ_i if $u_i(\sigma_i, \mu_i) \geq u_i(\sigma_i', \mu_i)\ \forall \mu_i \in \Delta S_{-i}$ and $u_i(\sigma_i, \mu_i') > u_i(\sigma_i', \mu_i')$ for some $\mu_i' \in \Delta S_{-i}$.

5.3 Rationalizability

- Best response: $\sigma_i \in B_i(\mu_i)$ if $u_i(\sigma_i, \mu_i) \geq u_i(\sigma_i', \mu_i)\ \forall \sigma_i' \in \Delta S_i$
- $\sigma_i \in B_i(\mu_i) \Rightarrow$ every pure strategy in the support of σ_i is a best response
- Rationalizable strategies: those remaining after iteratively removing all strategies that are not best responses to any allowable conjectures
- In a 2-player game, σ_i is strictly dominated if and only if σ_i is not a best response to any conjecture.
- In a 2-player game, a strategy survives iterated strict dominance if and only if it is rationalizable.
- In a finite-player game, σ_i is strictly dominated if and only if σ_i is not a best response to any correlated conjectures.
- In a finite-player game, a strategy survives iterated strict dominance if and only if it is correlated rationalizable.

5.4 Nash Equilibrium

- A strategy profile σ is a Nash equilibrium supported by full conjectures $\{v_i\}_{i \in \mathcal{P}}$ if $\forall i \in \mathcal{P}$, $v_i(\sigma_{-i}) = 1$ (correct beliefs) and $\sigma_i \in B_i(v_i)$.
- Assuming correct beliefs, this can be characterized as $\sigma_i \in B_i(\sigma_{-i})\ \forall i \in \mathcal{P}$.
- Any pure strategy with a positive probability in a Nash equilibrium is rationalizable. If each player has a unique rationalizable strategy, the profile of these strategies is a Nash equilibrium.
- Theorem: Any finite normal form game has at least one Nash equilibrium.
- Theorem: If each S_i is a compact, convex subset of $\mathbb{R}^k$, each u_i is continuous in s, and each u_i is quasiconcave in S_i, then G has at least one pure strategy Nash equilibrium.

5.5 Correlated Equilibrium

- Let $\rho \in \Delta\left(\prod_{i \in \mathcal{P}} S_i\right)$ be a correlated strategy. If $\rho(s_i) \equiv \sum_{s_{-i} \in S_{-i}} \rho(s_i, s_{-i}) > 0$, let $\rho(s_{-i}|s_i) \equiv \frac{\rho(s_i, s_{-i})}{\rho(s_i)}$.
- Correlated equilibrium, definition 1:

$$\sum_{s_{-i} \in S_{-i}} \rho(s_{-i}|s_i) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \rho(s_{-i}|s_i) u_i(s_i', s_{-i}) \qquad \forall s_i : \rho(s_i) > 0, \forall s_i' \in S_i, \forall i \in \mathcal{P}$$

- Correlated equilibrium, definition 2:

$$\sum_{s_{-i}\in S_{-i}} \rho(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i}\in S_{-i}} \rho(s_i, s_{-i}) u_i(s_i, s_{-i}) \qquad \forall s_i, s_i' \in S_i, \forall i \in \mathcal{P}$$

- A correlated strategy ρ is a Nash equilibrium if and only if ρ is a correlated equilibrium and a product measure (players' signals are independent).

5.6 Minmax Theorem and Zero-Sum Games

- We can write $u_2(\sigma) = -u_1(\sigma)$.
- Given σ_1, player 2 chooses a punishment strategy: $\alpha_2(\sigma_1) = \arg\min_{\sigma_2\in\Delta S_2} u_1(\sigma_1, \sigma_2)$
- Then player 1 chooses a maxmin strategy: $\bar{\sigma}_1 = \arg\max_{\sigma_1\in\Delta S_1} u_1(\sigma_1, \alpha_2(\sigma_1))$
- Alternatively, given σ_2, player 1 can choose a best response: $\beta_1(\sigma_2) = \arg\max_{\sigma_1\in\Delta S_1} u_1(\sigma_1, \sigma_2)$
- Then player 2 chooses a minmax strategy: $\underline{\sigma}_2 = \arg\min_{\sigma_2\in\Delta S_2} u_1(\beta_1(\sigma_2), \sigma_2)$
- These give the results $v_1^{maxmin} = u_1(\bar{\sigma}_1, \alpha_2(\bar{\sigma}_1))$ and $v_1^{minmax} = u_1(\beta_1(\underline{\sigma}_2), \underline{\sigma}_2)$
- Minmax Theorem: If G is a 2-player zero-sum game, then: (1) $v_1^{maxmin} = v_1^{minmax}$; (2) (σ_2^*, σ_2^*) is a Nash equilibrium of G if and only if σ_1^* is a maxmin strategy for player 1 and σ_2^* is a minmax strategy for player 2; (3) for any Nash equilibrium of G, player 1 obtains the payoff $v_1^{maxmin} = v_1^{minmax}$.

6 Extensive Form Games

6.1 Notation

- Players $\mathcal{P}$
- Set of actions A, with action set A_x available at node x
- Set of decision nodes D, with player i's decision nodes $D_i \subseteq D$, and nodes for Nature D_0
- Bernoulli utility functions $u_i : Z \to \mathbb{R}$
- Probability distribution $p_x \in \Delta A_x$ over Nature's actions at $x \in D_0$
- Information sets $I \subseteq D_i$
- Strategy σ_i specifies one action at each information set
- Strategy profile $\sigma = (\sigma_1, ..., \sigma_n)$
- Pure strategies: $s_i \in S_i \equiv \prod_{I\in\mathbb{I}_i} A_I$
- Mixed strategies (randomize over pure strategies): $\sigma_i \in \Delta S_i = \Delta\big(\prod_{I\in\mathbb{I}_i} A_I\big)$
- Behavior strategies (randomize separately at each information set): $\beta_i \in \prod_{I\in\mathbb{I}_i} \Delta A_i$
- With perfect recall, every mixed strategy σ_i is outcome-equivalent to some behavior strategy β_i.
- Beliefs $\mu_i : D_i \to [0,1]$ such that $\mu_i(x) \geq 0 \forall x$ and $\sum_{x\in I} \mu_i(x) = 1 \; \forall I \in \mathbb{I}_i$
- Profile of beliefs $\mu = (\mu_1, ..., \mu_n)$
- Assessment (σ, μ)

6.2 Perfect Information

- Every information set is a singleton, and Γ has no moves by Nature.
- In a finite perfect information game, the following are equivalent: (1) σ is a subgame perfect equilibrium (all players' strategies are optimal); (2) σ is sequentially rational (strategy of the owner of each node x is optimal); (3) σ admits no profitable one-shot deviations (owner's choice at each node x is optimal); (4) σ survives backward induction.
- Continuity at infinity: $\forall \epsilon > 0$, $\exists K$ such that choices made after the K^{th} node cannot alter any player's payoffs by more than ϵ. This is satisfied if payoffs are discounted and undiscounted payoffs per period are bounded.
- Let Γ be an arbitrary perfect information game with payoffs continuous at infinity. Then the following are equivalent: (1) σ is a subgame perfect equilibrium (all players' strategies are optimal); (2) σ is sequentially rational (strategy of the owner of each node x is optimal); (3) σ admits no profitable one-shot deviations (owner's choice at each node x is optimal).

6.3 Imperfect Information

6.3.1 Beliefs and Rationality

Bayesian beliefs:

- Let $P_\sigma(x)$ be the probability x reached under strategy profile σ, and let $P_\sigma(I) = \sum_{x \in I} P_\sigma(x)$.
- μ is Bayesian given σ if $\mu_i(x) = \frac{P_\sigma(x)}{P_\sigma(I)}$ whenever $P_\sigma(I) > 0$

Sequential rationality:

- σ_i is *rational starting from $I \in \mathbb{I}_i$ given σ_{-i} and μ_i* if
$$\sum_{x \in I} \mu_i(x) u_i(\sigma_i, \sigma_{-i}|x) \geq \sum_{x \in I} \mu_i(x) u_i(\hat{\sigma}_i, \sigma_{-i}|x) \qquad \forall \hat{\sigma}_i$$
- If this holds for every information set, then σ_i is *sequentially rational given σ_{-i} and μ_i*.
- If this is true for all players, then σ is *sequentially rational given μ*.

Consistency: μ is consistent given σ if there exists a sequence of completely mixed strategy profiles $\{\sigma^k\}_{k=1}^{\infty}$ such that: (1) $\lim_{k \to \infty} \sigma^k = \sigma$; (2) if μ^k are the unique Bayesian beliefs for σ^k, then $\lim_{k \to \infty} \mu^k = \mu$.
Implications of consistency:

- μ is Bayesian given σ
- Preconsistency: If by changing his strategy a player can force one of his own information sets to be reached, his beliefs there should be as if he did so.
- Parsimony: Let D_x be the set of deviations from σ required to reach x. If $x, y \in I \in \mathbb{I}_i$ and $D_y \subset D_x$, then $\mu_i(x) = 0$.
- Stagewise consistency: Let Γ_x be a subgame of Γ that begins with "simultaneous moves" among some subset of players. Then at information sets in this simultaneous move game, each player's beliefs are determined by the others' strategies in the simultaneous move game.
- Cross-player consistency: Players with the same information must have the same beliefs about opponents' deviations.

6.3.2 Equilibrium Concepts

- Weak sequential equilibrium: (σ, μ) is a weak SE if μ is Bayesian given σ and σ is sequentially rational given μ.
- Sequential equilibrium: (σ, μ) is a SE if μ is consistent given σ and σ is sequentially rational given μ.
- In a finite extensive form game with perfect recall, a strategy profile σ and beliefs profile μ that is preconsistent given σ, σ is sequentially rational given μ if and only if no player i has a profitable one-shot deviation from σ_i given σ_{-i} and μ_i.

7 Signalling Games

7.1 Notation

- Players $\mathcal{P} = \{1, 2\}$, with player 1 the sender and player 2 the receiver
- T finite set of player 1's types, with prior distribution $\pi(t) > 0 \; \forall t \in T$
- $A_1 = M$ player 1's finite action set (messages). $A_2 = R$ player 2's finite action set (responses), with responses $R^m \subseteq R$ available after message m.
- Pure strategy sets $S_1 = \{s_1 : T \to M\}$ and $S_2 = \{s_2 : M \to R\}$
- Bernoulli utility functions $u_{1a}(m, r)$ and $u_2(t_a, m, r)$ for sender of type t_a, message m and response r
- $\sigma_{1a}(m)$ probability type t_a sender chooses message m. $\sigma_2^m(r)$ probability receiver observing m chooses response r.
- $\mu_2^m \in \Delta T$ receiver's beleifs after observing m

7.2 Equilibrium and Cho-Kreps

(σ, μ) is a weak Sequential Equilibrium of Γ if

1. $\forall t_a \in T$, $\sigma_{1a}(m) > 0 \Rightarrow m$ optimal for sender of type t_a given σ_2
2. $\forall m \in M$, $\sigma_2^m(r) > 0 \Rightarrow r$ optimal for receiver after m given μ_2^m
3. μ_2 is Bayesian given σ

In a signalling game, any Bayesian beliefs are consistent, so every weak sequential equilibrium is a sequential equilibrium.
Cho-Kreps: Let u_{1a}^* be the payoff to sender of type t_a in a sequential equilibrium.

1. For each unused message m, let

$$D^m = \{t_a \in T : u_{1a} > \max_{r \in BR_2^m(T)} u_{1a}(m, r)\}$$

   This is the set of types for which message m is dominated by the equilibrium payoff, given that the receiver is playing a best response to some beliefs (any beliefs). In other words, this is the set of types for which sending the unused message m certainly leads to a lower payoff than what they're getting in the equilibrium.

2. If m is unused with $D^m \neq T$ and there is some type t_b such that $u_{1b} < \min_{r \in BR_2^m(T - D^m)} u_{1b}(m, r)$, then this component of equilibrium fails the Cho-Kreps criterion.

   In other words, after eliminating some types for whom the unused message m is equilibrium dominated (part 1), we can restrict the receiver's beliefs to exclude those types given m. Then if all remaining rational responses given the remaining beliefs would certainly increase the payoffs for a different type of sender, this other type would thus deviate, which fails Cho-Kreps.

8 Repeated Games

8.1 Notation

Stage game $G = \{\mathcal{P}, \{A_i\}_{i \in \mathcal{P}}, \{u_i\}_{i \in \mathcal{P}}\}$ has

- Pure actions $a_i \in A_i$
- Mixed actions $\alpha_i \in \Delta A_i$
- Mixed action profile $\alpha \in \prod_{i \in \mathcal{P}} \Delta A_i$

Repeated game $G^\infty(\delta) = \{\mathcal{P}, \{S_i\}_{i \in \mathcal{P}}, \{\pi_i\}_{i \in \mathcal{P}}, \delta\}$ has

- Null history $H^0 = \{h^0\}$
- Histories as of $t \geq 1$: $H^t = \{(a^0, a^1, ..., a^{t-1}) : a^s \in A\}$
- Finite (countable) histories: $H = \bigcup_{t=0}^{\infty} H^t$
- Infinite (uncountable) histories: $H^\infty = \{(a^0, a^1, ...) : a^s \in A\}$
- Pure strategy sets $S_i = \{s_i : H \to A_i\}$
- Behavior strategy sets $\Sigma_i = \{\sigma_i : H \to \Delta A_i\}$
- Strategy profile σ, with continuation strategy profile $\sigma_{|h^t}$ generated by σ after history h^t
- Discount rate $\delta \in (0, 1)$
- Payoff functions $\pi_i : H^\infty \to \mathbb{R}$, with $\pi_i(h^\infty) = (1-\delta)\sum_{t=0}^{\infty} \delta^t u_i(a^t)$

8.2 Results

- Nash equilibrium: σ is a NE of $G^\infty(\delta)$ if no player has a profitable unilateral deviation from σ.
- Subgame perfect equilibrium: σ is a SPE of $G^\infty(\delta)$ if $\forall h^t \in H$, $\sigma_{|h^t}$ is a Nash equilibrium of $G^\infty(\delta)$
- Theorem: Strategy profile σ is a SPE of $G^\infty(\delta)$ if and only if σ admits no profitable one-shot deviation.
- For finitely repeated Prisoner's Dilemma $G^T(\delta)$, "Always defect" is the unique SPE.
- For infinitely repeated Prisoner's Dilemma $G^\infty(\delta)$: (1) "Always defect is a SPE of $G^\infty(\delta)$ for all δ; (2) if $\delta \geq 1/2$, the grim trigger strategy (cooperate if and only if no one has ever defected) is a SPE of $G^\infty(\delta)$.

Folk Theorem

- Constraint sustainable payoffs using minmax: $\underline{v}_i \equiv \min_{\alpha_{-i}} \max_{\alpha_i} u_i(\alpha_i, \alpha_{-i})$
- Then individually rational payoff vectors are $F^* = \{v \in F : v_i > \underline{v}_i \; \forall i \in \mathcal{P}\}$
- Theorem: Let $v \in F^*$ and suppose there are exactly 2 players or that no 2 players have identical preferences (up to affine transformations). Then for δ close enough to 1, there exists a subgame perfect equilibrium of $G^\infty(\delta)$ with payoffs v.

9 Centipede Game

Something here too?

10 Matching

- Let $\mathcal{M}$ be the set of all possible matchings, with sides $x \in X$ and $y \in Y$.
- $m \in \mathcal{M}$ is feasible if if obeys: (1) symmetry, with $m(x, y) = m(y, x)$; (2) no overmatching, meaning $\forall x \in X$, $m(x, y) = 1$ for at most one $y \in Y$.
- Let $f(y|x)$ be the payoff of x when matched with y, and $g(x|y)$ be the payoff of y when matched with x. Let unmatched payoffs be $f(0|x)$ and $g(0|y)$.
- m is x-optimal if for all $x \in X$, given any other stable matching m', each x prefers his match in m to his match in m'. m is x-pessimal if for all $x \in X$, he prefers his match in m' to his match in m.

10.1 Nontransferable Utility

Stability

- Blocking pair: (x, y) not matched but x prefers y to his current partner and y prefers x to his current partner.
- m is unstable if there exists a blocking pair.
- m is stable if it is not unstable.

Gale-Shapley Deferred Acceptance Algorithm

1. All start unmatched.
2. Each unengaged man proposes to his most-preferred woman among those he has not yet proposed to, if matching with her is better than being single.
3. Each woman gets engaged to her most-preferred suitor (including any current engagement), if matching with him is better than being single.
4. Repeat steps 2 and 3 until no more proposals can be made. Engagements then become matches.

Properties of DAA

- DAA stops in finite time.
- With an equal number of men and women, if matching beats remaining single, then everybody matches.
- DAA produces stable matches.
- With strict preferences, DAA yields a unique matching.
- DAA chooses a male-optimal and female-pessimal matching.
- DAA produces the same matching regardless of who proposes if and only if there is a unique stable matching.

10.2 Transferable Utility

- Let $h(x, y) = f(y|x) + g(x|y)$
- Matching is pairwise efficient if for all matched pairs (x, y) and (x', y'), $h(x, y) + h(x', y') \geq h(x', y) + h(x, y')$
- Efficient matching maximizes the sum of all outputs. An efficient matching is pairwise efficient.
- Competitive Equilibrium: A matching and wage profile (m, w, v) satisfying: (1) Feasibility, meaning $m(x, y) \geq 0$, $\sum_x m(x, y) \leq 1 \ \forall y \in Y$ and $\sum_y m(x, y) \leq 1 \ \forall x \in X$; (2) Free entry, meaning $v(x) + w(y) \leq h(x, y)$ $\forall (x, y) \in X \times Y$; (3) $v(x) + w(y) \leq h(x, y)$ for any matched $(x, y) \in X \times Y$ (i.e. with $m(x, y) > 0$).
- First Welfare Theorem of Matching: Any CE (m, v, w) yields an efficient matching.
- An efficient matching exists.
- Second Welfare Theorem of Matching: An efficient matching m is a CE (m, v, w) for some prices (v, w).
- A CE is immune to bribes.

10.3 Assortative Matching

Definitions:

- Positive Assortative Matching (PAM):
  - The relationship between matched types is monotone increasing.
  - Let M and N be the masses and X and Y, and $M(x)$ and $N(y)$ denote the indexes. A matching has PAM if $M - M(x) = N - N(y(x))$ for all matched x.
- Negative assortative matching
  - The relationship between matched types is monotone decreasing.
  - A matching has NAM if $M - M(x) = N(y(x))$ for all matched x.
- Strictly comonotone: $\big(f(y_2|x) - f(y_1|x)\big)\big(g(x_2|y) - g(x_1|y)\big) > 0 \quad \forall x, y$ and $y_2 > y_1, x_2 > x_1$
- Reverse comonotone: $\big(f(y_2|x) - f(y_1|x)\big)\big(g(x_2|y) - g(x_1|y)\big) < 0 \quad \forall x, y$ and $y_2 > y_1, x_2 > x_1$

Nontransferable Utility:

- PAM if the unique stable matching if f and g are strictly comonotone.
- NAM is the unique stable matching if f and g are reverse comonotone.

Transferable Uility:

- If $h(x, y)$ is supermodular, then PAM is efficient.
- If $h(x, y)$ is strictly supermodular, then PAM is uniquely efficient.
- If $h(x, y)$ is submodular, then NAM is efficient.
- If $h(x, y)$ is strictly submodular, then NAM is uniquely efficient.
- If $h(x, y)$ is modular, then any matching is efficient.
- If $h(x, y)$ is modular for a set of agents that match in an efficient matching, then any rematching among them is also efficient.

10.4 Double Auctions

- Let ϵ_j denote buyer j's value from any seller i's good, and let c_i be the cost to seller i.
- Let $h(\epsilon, c) \equiv \max\{0, \epsilon - c\}$ be the gains from trade for buyer with valuation ϵ and seller with cost c. h is submodular.
- The efficient allocation maximizes $\sum_i \sum_j x_{ij} h(\epsilon_j, c_i)$, where $x_{ij} = 1$ if i trades with j and 0 otherwise.
- We can order buyers and seller, with buyers in $\epsilon_1 > ...\epsilon_k > \epsilon_{k+1} > ... > \epsilon_N$ and sellers in $c_1 < ... < c_k < c_{k+1} < ... < c_M$
- The Law of One Price holds.
- An efficient allocation exists, with trades by the k highest valuation buyers and the k lowest cost sellers, with positive trade if $v_1 > c_1$. This allocation is a competitive equilibrium for any price

$$p \in \big[\max\{c_k, \epsilon_{k+1}\}, \min\{c_{k+1}, \epsilon_k\}\big]$$

  Any CE is efficient and maximizes the sum of gains from trade. It is immune to side bribes.

11 Market Exchange

11.1 Equilibrium Stability

- Walrasian price stability: If at some price, net demand is positive ($Q_D - Q_S > 0$), then the price increases; if net demand is negative ($Q_D - Q_S < 0$), then the price decreases. The change in the price sharess the sign of $Q_D - Q_S$.
- Marshallian price stability: If at some quantity, demand price exceeds supply price ($P_D > P_S$), then quantity supplied increases. The change in quantity shares the sign of $P_D - P_S$.

11.2 Tax Irrelevance and Incidence

- Tax Irrelevance Theorem: Regardless of whether demand or supply pays the tax, demand and supply prices, market quantity and efficiency loss are the same.
- Deadweight loss: $\frac{d}{d\tau}\big(Q(P+\tau)\tau\big) = Q(P+\tau)(1+\epsilon)$
- Ramsey problem: Minimize the social cost of raising revenue R using tax tools
- Ramsey inverse elasticity rule: Taxes should be proportional to the sum of the reciprocals of its supply and demand elasticities.
- Incidence Theorem: The share of a small tax paid by demand is $\frac{\eta}{\eta-\epsilon}$.

11.3 Firm Entry and Exit

- A cost is escapable if it can be avoided, otherwise it is sunk.
- Fixed cost is invariant to the quantity. Variable cost is not.
- Short run: fixed costs inescapable. Cost function is just variable costs.
- Long run: All costs escapable, so included in the cost function. Firms enter if there are positive profits to be made and exit otherwise.
- Industry supply curve: Price-quantity locus (P, Q) such that allowable firms (existing firms in short run, all potential firms in long run) profitably produce Q taking P as given. Long-run industry supply curve is a sawtooth curve.

12 Market Power

Barriers to Entry:

- Technical: minimum efficient scale (aircraft makers); ownership of unique resources (De Beers); special knowledge of a low cost technique (Coca Cola); network externalities (Facebook).
- Legal: monopoly via franchise (public utilities); patents and copyrights.
- Noncompete agreements
- Illegal barriers to entry (criminal enterprises using violence)

12.1 Monopoly

$$\begin{aligned} \Pi(Q) &= R(Q) - C(Q) = P(Q)Q - C(Q) \\ \text{FOC} \qquad R'(Q) &= P(Q) + QP'(Q) = C'(Q) \\ \Rightarrow P(Q)\left(1 - \frac{1}{|\epsilon|}\right) &= C'(Q) \end{aligned}$$

Inverse elasticity rule:

$$\text{Lerner index} = L = \frac{P(Q) - C'(Q)}{P(Q)} = \frac{1}{|\epsilon|}$$

Monopolist never sells for any price along the inelastic portion of the demand curve, where $|\epsilon| < 1$.

12.2 Monopsony

- Market power on the buying sude
- Rising labor supply wage $w(L)$, with $w'(L) > 0$
- Production function $f(L)$ with fixed price p

$$\text{FOC} \qquad w(L) + Lw'(L) = pf'(L)$$

- Value marginal product of labor $VMP(L)$ is the marginal revenue from hiring labor
- Inverse elasticity rule:

$$VMP(L) = w(L)\left(1 + \frac{1}{\eta}\right) \Rightarrow \frac{VMP(L) - w(L)}{w(L)} = \frac{1}{\eta}$$

12.3 Price Discimination

- First degree: personalized prices. This is efficient, as seller gets all the surplus.
- Second degree: different prices for different quantities consumed. May be implemented using a two-part tariff, with a fixed fee for the right to trade at a linear price, or using quantity discounts.
- Third degree: different prices for different consumer groups.

12.4 Cartels and Oligopolies

- $n < \infty$ firms facing demand $P(Q)$, with $Q \equiv \sum_{i=1}^{n} q_i$. Each firm has cost function $C_i(q_i)$.
- Perfect competition: $C'(q_i) = P$
- If firms act as a monopoly (illegal cartel), they solve

$$\max_{\{q_i\}_{i=1}^n} \Big(P(Q)Q - \sum_{i=1}^{n} C_i(q_i) = \max_{\{q_i\}_{i=1}^n} \Big(R(Q) - \sum_{i=1}^{n} C_i(q_i)$$

$$\text{FOC} \qquad R'(Q) = P(Q) + QP'(Q) = P(Q) + Q\frac{dP(Q)}{dq_i} = C_i'(q_i)$$

- However, each firm's marginal revenue exceeds its marginal cost:

$$R_i'(Q) = P(Q) + q_i\frac{dP(Q)}{dq_i} > P(Q) + QP'(Q) = R'(Q) = C_i'(q_i)$$

- Chiseling produces a Cournot-Nash equilibrium

12.5 Oligopoly

Cournot competition:

- Firms optimize taking as given others' actions.

$$\text{FOC} R_i'(q_i) = P + q_i P_I'(Q) = C_i'(q_i)$$

- As $n \to \infty$, this converges to perfect competition.

Stackelberg competition:

- Two firms, with a leader (first mover) and a follower. The leader chooses their quantity before the second firm does.
- Backward induction: first solve the follower firm's best response to any initial quantity, $BR_2(Q_1)$, then solve for firm 1's FOC:

$$R_1'(Q_1) = P\big(Q_1 + BR_2(Q_1)\big) + Q_1 P\big(Q_1 + BR_2(Q_1)\big) = C_1'(Q_1)$$

13 Externalities

13.1 Pigouvian Taxation

- Benefits $B(q)$, costs $C(q)$ and external damage $\Delta(q)$
- Private optimum: $\hat{q} \in \arg\max_q B(q) - C(q) \qquad \Rightarrow B'(\hat{q}) = C'(\hat{q})$
- Social optimum: $q^* \in \arg\max_q B(q) - C(q) - \Delta(q) \qquad \Rightarrow B'(q^*) - C'(q^*) = \Delta'(q^*)$
- Pigouvian tax: $\tau = \Delta'(q^*)$

13.2 Coasean Bargaining

Assume well-defined property rights, negotiation that allows participants to freely realize all gains from trade, and transfers that do not affect anyone's marginal values. Then the efficient outcome arises irrespective of who has property rights. If a Pigouvian tax is imposed, efficiency is lost.

13.3 Arrow Markets

Endow one or both market participants with the right to the activity and create a market for trading these permits. This ends up trading at a price $t^* = \Delta'(q^*)$, which gives the efficient allocation.

14 Political Economy

14.1 Price Floors

- Price floor $\underline{P} > P^*$
- Short side of the market (demand) determines quantity $\underline{Q}$ satisfying $\underline{P} = P_D(\underline{Q})$
- If the long side of the market (supply) competes for right to sell (or they use ration coupons), it pays a market-clearing transfer $t = \underline{P} - P_s(\underline{Q})$
- With linear supply and demand, $DWL = \frac{1}{2}(\underline{P} - P_s(\underline{Q}))(Q^* - \underline{Q})$
- If right to trade is obtained by rent-seeking or randomness, then DWL much larger than standard triangle.

14.2 Price Ceilings

- Price ceiling $\bar{P} < P^*$
- Short side (supply) determines quantity $\bar{Q}$ satisfying $\bar{P} = P_s(\bar{Q})$
- A transfer to clear the market solves $t = P_D(\bar{Q}) - \bar{P}$
- Often cleared by queuing or searching for hard-to-find goods.

14.3 Quantity Restrictions

- Quantity ceilings may be equivalent to either quantity ceilings or price floors.
- Quantity floors are minimum supply requirements.

15 Public Goods

- Rival goods: Use by one consumer reduces another's benefit from it.
- Excludable goods: One can prevent others from jointly consuming a goods once it is produced.
- Private goods: Rival and Excludable.
- Congestion public good: Rival and Nonexcludable. Leads to tragedy of the commons. Includes city roads, internet in evening hours, etc.
- Club goods: Nonrival and Excludable. Includes golf courses, toll roads, etc.
- Pure public goods: Nonrival and Nonexcludable.

15.1 Optimal Provision

- Let G be the amount of the public good.
- Let agents have utility over the good and money, $U^i(G, m_i)$.
- Let G be produced from transfers t with $G = f(t)$.
- **Pareto Efficiency Rule:** If G is discrete (1 or 0), choose $G = 1$ if $\exists$ transfers $t_1, ..., t_n$ from consumers sufficient to provide the good ($\sum_i t_i \geq c$) such that: (1) everyone is weakly better off, with $U^i(m_i - t_i, 1) \geq U^i(m_i, 0)$ $\forall i$; (2) some j is strictly better off, with $U^j(m_j - t_j, 1) > U^j(m_j, 0)$.
- **Samuelson Condition:** Optimal provision of the public good (for a continuous good) obeys

$$\sum_{i=1}^{n} MRS^i_{G,m} = MRT_{G,m}$$

$$\Rightarrow \sum_{i=1}^{n} \frac{dU^i/dG}{dU^i/dm} = \frac{1}{f'(t)}$$

  With quasilinear preferences or additive utility $U^i(G, m) = \phi_i(G) + w$, this reduces to $\sum_{i=1}^{n} MB^i = MC$, where $MB^i = \phi_i'(G)$.

15.2 Lindahl Equilibrium

- Single private good x and public good G, with an initial endowment of private goods $(w_1, ..., w_n)$ across consumers.
- Assume public good sold at linear price p.
- Lindahl Equilibrium: An allocation of public and private goods $(G^*, x_1^*, ..., x_n^*)$ and individual public goods prices $(p_1, ..., p_n)$, with $p = p_1 + ... + p_n$, such that every consumer i chooses (x_i^*, G^*) given a price p_i for G:

$$(x_i^*, G^*) \in \arg\max_{x_i, G} U^i(x_i G) \qquad \text{s.t.} \qquad x_i + p_i G = w_i$$

- Knowing he must pay a share p_i of the price, consumer i agrees on the public good G^*.
- A Lindahl Equilibrium exists and is efficient.

16 GE in Exchange Economies

16.1 Market Fundamentals

- $L \geq 2$ goods, indexed $\ell \in \{1, 2, ..., L\}$
- $n \geq 2$ traders/consumers, indexed $i \in \{1, 2, ..., n\}$
- i has endowment $\bar{x}^i = (\bar{x}^i_1, ..., \bar{x}^i_L) \in \mathbb{R}^L_+$
- Consumption x^i, and allocation $x = (x^1, ..., x^n) \in \mathbb{R}^{nL}_+$
- Utility functions $u^i : \mathbb{R}^L_+ \to \mathbb{L}$
- Budget set $\mathcal{B}^i(p, \bar{x}^i) = \{x^i \in \mathbb{R}^L_+ : p \cdot x^i \leq p \cdot \bar{x}^i\}$
- Prices $p = (p_1, ..., p_L) \in \mathbb{R}^L_+$
- An exchange economy is $\mathcal{E} = \left(\{u^i\}^n_{i=1}, \bar{x}\right)$
- Consumers solve $\quad \max_{x^i} u^i(x^i) \quad$ s.t. $\quad x^i \in \mathcal{B}^i(p, \bar{x}^i)$
- An allocation is feasible if markets clear: $\sum_{i=1}^n x^i_\ell \leq \sum_{i=1}^n \bar{x}^i_\ell \qquad \forall \ell$
- A feasible allocation is efficient if $\nexists$ a feasible allocation $\hat{x}$ such that: (1) everybody if weakly better off, with $u^i(\hat{x}^i) \geq u^i(x^i)$ $\forall i$; (2) somebody is strictly better off, with $u^j(\hat{x}^j) > u^j(x^j)$.
- Competitive Equilibrium: A price and allocation pair (x, p) such that x is feasible and all consumers maximize their utility given p.

16.2 Edgeworth Box, Contract Curves, Individual Rationality, Trade Offer Curves

- 2 goods, 2 consumers
- Contract curve: Set of all efficient allocations
- Individually rational allocations obey $u^i(x^i) \geq u^i(\bar{x}^i)$
- Core: Set of efficient and individually rational allocations
- Trade offer curve: Optimal consumption bundle given prices. With $L = 2$, it is tangent to the indifference curve through at endowment and more curved.

16.3 First Welfare Theorem

Theorem: If (x, p) is a competitive equilibrium of $\mathcal{E}$ and preferences are locally nonsatiated, then x is efficient.
Proof: Suppose this is false. Then there exists a feasible allocation $\hat{x}$ with utility $u^i(\hat{x}^i) \geq u^i(x^i)$ $\forall i$ and $u^j(\hat{x}^j) > u^j(x^j)$ for some j.
If $p \cdot \hat{x}^i < p \cdot x^i$, then by LNS there exists another bundle $\tilde{x}^i$ that is feasible but strictly preferred to x^i, which contradicts that x^i is individually optimal given p. Therefore, $p \cdot \hat{x}^i \geq p \cdot x^i$ $\forall i$.
Also, because $u^j(\hat{x}^j) > u^j(x^j)$ but j optimizing given p chose x^i, so it must be that $p \cdot \hat{x}^j > p \cdot x^j$.
By LNS, the optimal consumption bundles x^i must be on the boundary of the budget set for each i. Adding up across all consumers, and using that $p \geq 0$,

$$\sum_{i=1}^n p \cdot \hat{x}^i > \sum_{i=1}^n p \cdot x^i \qquad \Rightarrow \qquad p \cdot \sum_{i=1}^n \hat{x}^i > p \cdot \sum_{i=1}^n x^i \qquad \Rightarrow \qquad \sum_{i=1}^n \hat{x}^i > \sum_{i=1}^n x^i = \sum_{i=1}^n \bar{x}^i$$

Therefore, $\hat{x}$ is not feasible, which is a contradiction. $\square$

16.4 Other Equilibrium Theorems

- **Second Welfare Theorem:** Assume consumers have continuous, monotone and quasiconcave utility functions. If $x \in \mathbb{R}^{nL}_+$ is an efficient allocation, then $\exists p \in \mathbb{R}^L_+$ such that (x, p) is a competitive equilibrium of $\mathcal{E}$.
- Excess demand function for i: $ED^i_\ell(p) = x^i_\ell(p) - \bar{x}^i_\ell$
- Excess demand for x_ℓ: $ED_\ell(p) - \sum_{i=1}^n ED^i_\ell(p)$
- **Walras Law:** If traders consume their entire income at an allocation $x(p)$, then the market value of net excess demand vanishes: $\sum_{\ell=1}^L p_\ell ED_\ell(p) = 0$
- **Existence Theorem** (2 goods): Assume every consumer owns a positive endowment $(\bar{x}^i, \bar{y}^i)$. Given strictly monotone and strictly convex preferences over x and y, there exists a competitive equilibrium (x, y, p). It is Walrasian stable.

16.5 Core Theory

- Valuation function $V : 2^n \to \mathbb{R}$, where $V(S)$ measures the best that the coalition $S \subseteq N = \{1, 2, ..., n\}$ secures.
- Core: $C(V)$ is the set of payoff vectors $x \in \mathbb{R}^n$ blocked by no coalition $S \subset N$.
- V is supermodular if $V(S \cup T) + V(S \cap T) \geq V(S) + V(T)\ \forall S, T \subseteq N$.
  Equivalently, V is supermodular $\iff V(S \cup \{i\}) - V(S) \leq V(T \cup \{i\}) - V(T)\ \forall S \subseteq T \subseteq N - \{i\}, \forall i \in N$.
- If V is supermodular and $V(\emptyset) = 0$, then V is superadditive: $S \cap T = \emptyset \Rightarrow V(S \cup T) \geq V(S) + V(T)$
- A covex game has a supermodular characteristic function V.
- Bondareva-Shapley Theorem: A convex game has a nonempty core.
- Welfare Theorem: If (x, p) is a competitive equilibrium, then x is in the core.
- For M replications, if $x^* \in C_M\ \forall M$, then x^* is a competitive equilibrium. Hence the limit of the M replica cores $\bigcap_{M=1}^\infty C_M$ is a competitive equilibrium.

17 GE with Production

17.1 New fundamentals

- Now have m firms
- Production sets $Y^j \in \mathbb{R}^L$ with: (1) free exit, $0 \in Y^j$; (2) no free lunch, $Y \cap \mathbb{R}^L_+ = \{0\}$; (3) free disposal, $Y \supset \mathbb{R}^L_-$; (4) closed convex technology.
- Consumers own firms: i owns a share θ_{ij} if profits from firm j
- Economy: $\mathcal{E} = \left(\{Y^j\}_{j=1}^m, \{X^i, u^i, \bar{x}^i, \theta_{i1}, ..., \theta_{im}\}\right)$

17.2 Competitive Equilibrium and Theorems

- Competitive Equilibrium: Allocation $(x, y) \in \mathbb{R}^{nL}_+ \times \mathbb{R}^{mL}$ and price $p \in \mathbb{R}^L$ such that
  1. $\forall j$, $y^j \in Y^j$ maximizes profits of firm j given p, meaning $p \cdot \hat{y}^j \leq p \cdot y^j\ \forall \hat{y}^j \in Y^j$
  2. $\forall i$, x^i maximizes u^i in the budget set $\mathcal{B}^i(p) = \left\{x^i \in X^i : p \cdot x^i \leq p \cdot \bar{x}^i + \sum_{j=1}^m \theta_{ij} p \cdot y^j\right\}$
  3. Markets clear, with the excess demand vector nonpositive: $z \equiv \sum_{i=1}^n x^i - \sum_{i=1}^n \bar{x}^i - \sum_{j=1}^m y^j \leq 0$
- Theorem (Arrow-Debreu): A competitive equilibrium (x, y, p) exists.
- FWT: If (x, y, p) is competitive equilibrium, then (x, y) is an efficient allocation.
- SWT: If (x, y) is an efficient allocation, then (x, y, p) is a competitive equilibrium for some prices p, endowments $\bar{x}$ and firm shares θ.

18 GE under Uncertainty

- Exchange economy with n traders and L goods
- At $t = 1$, a state of the world $s \in \mathcal{S} = \{1, ..., S\}$ is realized.
- At $t = 0$, the probabilities π_s for each $s \in \mathcal{S}$ are known.
- State-contingent claim $x_{\ell s} \in \mathbb{R}^{LS}$ is a title to a unit of consumption of good ℓ in state s.
- Consumption for i is $x^i \in \mathbb{R}^{LS}_+$
- At $t = 0$, there are LS forward markets.
- Complete markets: a security for every possible state
- Fundamental Theorem of Risk-Bearing: $\frac{\pi_1 u'(x_1)}{p_s} = ... = \frac{\pi_s u'(x_s)}{p_s} = \lambda$
- In rational expectations equilibrium, agents fully extract information from prices.
- Note that a rational expectations equilibrium may not exist.

19 Spatial Competition

Maybe put something here?

20 Auctions

20.1 Open Bid Auctions

20.2 First Price Sealed Bid Auctions

20.3 Second Price Sealed Bid Auctions

I

21 Double Auctions

First Lones, then Marzena

21.1 Solving for Equilibrium

21.2 Equilibrium Stability

21.3 Tax Irrelevance and Incidence Theorems

21.4 Double Auctions with Asymmetric Information

21.5 Adverse Selection

J

22 Signaling

- Individuals have types θ, and choose costly signal e, incurring a cost $c(e)$.
- Utility functions $U(w, e) = u(w) - c(e, \theta)$
- Firms form beliefs $p^*(\theta|e)$ and pay expected wages given belefs: $w^*(e) = \sum_\theta \theta p^*(\theta|e)$
- On-path beliefs must be Bayesian. Off-path beliefs are arbitrary, and usually pessimistic.

- Given w^*, effort levels are chosen to satisfy the incentive constraints for each type.
- Incentive constraints must apply for any e chosen, along with the wages resulting from that e. In general, ICs bind most for $e = 0$ and are slack elsewhere.

$$u\big(w^*(e^*(\theta))\big) - c\big(e^*(\theta), \theta\big) \geq u\big(w^*(e)\big) - c\big(e, \theta\big) \qquad \forall e, \forall \theta \tag{IC}$$

$$u\big(w^*(e^*(\theta))\big) - c\big(e^*(\theta), \theta\big) \geq \bar{u}_\theta \qquad \forall \theta \tag{IR}$$

22.1 Separating Equilibria

- For types $\theta \neq \theta'$, $e^*(\theta) \neq e^*(\theta')$
- Separating equilibria exist if the single crossing property holds.
-
-

22.2 Pooling Equilibria

- $e^*(\theta) = e^* \ \forall \theta$
- Pooling equilibrium always exists
-
-

22.3 Equilibrium Dominance and Cho-Kreps

Equilibrium Dominance:

- Action e is equilibrium dominated for type θ if

$$u\big(w^*(e^*(\theta))\big) - c\big(e^*(\theta), \theta\big) > \hat{w}(e) - c(e, \theta) \qquad \forall p(\theta|e) \in [0, 1]$$

  where $\hat{w}$ is the optimal wage (expected productivity) given any arbitrary beliefs.
- e is equilibrium dominated if there is no beliefs $p(\theta|e)$ off-path such that e will be chosen by θ.

Cho-Kreps (Intuitive) Criterion: A Perfect Bayesian Equilibrium satisfies the criterion if $\forall e$, if e is equilibrium dominated for some type θ but not for some other type θ', beliefs must be $p^*(\theta|e) = 0$.

- No pooling equilibrium survives the intuitive criterion.
- Only the least costly separating equilibria survive the intuitive criterion.

23 Screening

23.1 General Setup

- Informed player is the agent, uninformed player is the principal
- Prior to the moment of contracting, information is asymmetric. After the contract is signed, information is symmetric.
- For separating equilibria, we generally need the single crossing property to hold.

23.2 Monopolistic Screening

A monopolist has to design menus (q, t) for different types of consumers.

23.3 Labor Market Screening

A firm has to design contracts (w, e) to hire workers with different productivities.

23.4 Insurance Screening?

A firm has to design insurance policies (b, p) to offer to the market.

24 Moral Hazard

L

25 Cheap Talk

M

26 Revenue Equivalence and the Revelation Principle

N