Econometrics HW6

Michael B. Nattinger*

October 12, 2020

1 Question 1

1.1 Part A

$P(X=1) = p = p^1(1-p)^{1-1} = f(1)$. $P(X=0) = (1-p) = p^0(1-p)^{1-0} = f(0)$.

1.2 Part B

Our parameter is $\theta = p$. $l_n(\theta) = \sum_{i=1}^n log(f(X_i|\theta)) = \sum_{i=1}^n log(\theta^{X_i}(1-\theta)^{1-X_i}) = \sum_{i=1}^n X_i log(\theta) + (1-X_i)log(1-\theta)$.

1.3 Part C

$\frac{\partial l_n(\theta)}{\partial\theta} = 0 \Rightarrow \sum_{i=1}^n \frac{X_i}{\theta} - \frac{1-X_i}{1-\theta} = 0 \Rightarrow \sum_{i=1}^n X_i(1-\theta) = \sum_{i=1}^n \theta - X_i\theta \Rightarrow \hat{\theta}_n = \frac{1}{n}\sum_{i=1}^n X_i$.

2 Question 2

2.1 Part A

$l_n(\theta) = \sum_{i=1}^n log\left(\frac{\theta}{X_i^{1+\theta}}\right) = \sum_{i=1}^n log(\theta) - (1+\theta)log(X_i) = nlog(\theta) - (1+\theta)\sum_{i=1}^n log(X_i)$

2.2 Part B

$\frac{\partial l_n(\theta)}{\partial\theta} = 0 \Rightarrow \frac{n}{\theta} - \sum_{i=1}^n log(X_i) = 0 \Rightarrow \hat{\theta}_n = \frac{1}{n}\sum_{i=1}^n log(X_i)$.

3 Question 3

3.1 Part A

$l_n(\theta) = \sum_{i=1}^n log\left(\frac{1}{\pi(1+(X_i-\theta)^2)}\right) = -nlog(\pi) - \sum_{i=1}^n log(1+(X_i-\theta)^2)$.

3.2 Part B

$\frac{\partial l_n(\theta)}{\partial\theta} = 0 \Rightarrow -\sum_{i=1}^n \frac{2(X_i-\hat{\theta}_n)}{1+(X_i-\hat{\theta}_n)^2} = 0$

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

4 Question 4

4.1 Part A

$l_n(\theta) = \sum_{i=1}^{n} log(\frac{1}{2}exp(-|X_i - \theta|)) = nlog(\frac{1}{2}) - \sum_{i=1}^{n} |X_i - \theta|$

4.2 Part B

The likelihood is maximized when the term $\sum_{i=1}^{n} |X_i - \theta|$ is minimized. This is minimized for $\theta = E[X]$ and so $\hat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

5 Question 5

$I_0 = -E\left[\frac{\partial^2}{\partial\theta^2}log(f(X|\theta))|_{\theta=\theta_0}\right] = -E\left[\frac{\partial^2}{\partial\theta^2}log(\theta x^{-1-\theta})|_{\theta=\theta_0}\right] = -E\left[\frac{\partial^2}{\partial\theta^2}log(\theta) + (-1-\theta)log(x)|_{\theta=\theta_0}\right]$
$= -E\left[\frac{\partial}{\partial\theta}\frac{1}{\theta} - log(x)|_{\theta=\theta_0}\right] = -E\left[\frac{\partial}{\partial\theta}\frac{1}{\theta} - log(x)|_{\theta=\theta_0}\right] = \frac{1}{\theta_0^2}$

6 Question 6

6.1 Part A

$I_0 = -E\left[\frac{\partial^2}{\partial\theta^2}log(\theta exp(-\theta x))|_{\theta=\theta_0}\right] = -E\left[\frac{\partial^2}{\partial\theta^2}log(\theta) + log(exp(-\theta x))|_{\theta=\theta_0}\right]$
$= -E\left[\frac{\partial^2}{\partial\theta^2}log(\theta) - \theta x|_{\theta=\theta_0}\right] = \hat{\theta}_0^{-2} \Rightarrow Var(\bar{\theta}_n) \geq (n\hat{\theta}_0^{-2})^{-1} = \frac{\theta_0^2}{n}$

6.2 Part B

$l_n(\theta) = \sum_{i=1}^{n} log(f(X_i|\theta)) = \sum_{i=1}^{n} log(\theta exp(-\theta X_i)) = \sum_{i=1}^{n} log(\theta) + log(exp(-\theta X_i))$
$= nlog(\theta) - \theta\sum_{i=1}^{n} X_i \Rightarrow \frac{\partial l_n(\theta)}{\partial\theta} = 0 \Rightarrow \frac{n}{\theta} - \sum_{i=1}^{n} X_i = 0 \Rightarrow \hat{\theta}_n = \frac{n}{\sum_{i=1}^{n} X_i}$.

By the delta method, $\sqrt{n}(\hat{\theta}_n - \theta_0) \rightarrow_d N(0, V)$ where $V = (-1(\theta_0)^{-2})^2\sigma^2 = \theta_0^{-4}\sigma^2$ where $\sigma^2 = Var(X_i) = \frac{1}{\theta_0^2}$. Thus, $\sqrt{n}(\hat{\theta}_n - \theta_0) \rightarrow_d N(0, \theta_0^{-6})$

6.3 Part C

Our general formula is $\sqrt{n}(\hat{\theta}_n - \theta_0) \rightarrow_d N(0, I_0^{-1}) = N(0, \theta_0^2)$

7 Question 7

8 Question 8

9 Question 9

10 Question 10