Macro PS1

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1 Question 1: Exercise 8.1

The first order conditions to the Pareto problem is the following:

$$\theta u'(c^1) = \lambda$$
$$(1-\theta)w'(c^2) = \lambda$$
$$\Rightarrow \theta u'(c^1) = (1-\theta)w'(c^2).$$

From envelope conditions, we get the following:

$$\begin{aligned} v'_\theta(c) &= \theta u'(c^1)\frac{\partial c^1}{\partial c} + (1-\theta)w'(c^2)\frac{\partial c^2}{\partial c} \\ &= \theta u'(c^1)\frac{\partial(c^1+c^2)}{\partial c} \\ &= \theta u'(c^1) = (1-\theta)w'(c^2). \end{aligned}$$

Now we will think about concavity. This is slightly more involved as the envelope theorem holds only for the first derivative.

Define the compact set $B(c) = \{x = (c_1, c_2) \in \mathbb{R}^2 : c_1 + c_2 \leq c, c_1 \geq 0, c_2 \geq 0\}$. Define on this set the function $V(x) = \theta u(c^1) + (1-\theta)c^2$. Then, observe that $v(c) = \max_{x \in B(c)} V(x)$. Since u, w are continuous, V is continuous so it achieves its maximum on the compact set $B(c)$. Define $X(c)$ as the corresponding argmax - since V is strictly concave, it achieves its max at a unique point. Now, let $c, C \geq 0, \lambda \in [0,1]$. Then,

$$\begin{aligned} \lambda v(c) + (1-\lambda)v(C) &= \lambda V(X(c)) + (1-\lambda)V(X(C)) \\ &\leq V(\lambda X(c) + (1-\lambda)X(C)) \\ &\leq v(\lambda c + (1-\lambda)C). \end{aligned}$$

Therefore, $v(c)$ is concave.

2 Question 2: Exercise 8.3

2.1 Part A

A competitive equilibrium is a set of prices $\{Q_t\}_{t=0}^\infty$ and allocations $\{c_t^1, c_t^2\}_{t=0}^\infty$ such that both consumers optimize (maximize the sum of discounted utility) and markets clear ($c_t^1 + c_t^2 = y_t^1 + y_t^2 = 1 \forall t$).

2.2 Part B

Agent i solves the following optimization problem:

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{i=1}^{\infty} \beta^t u(c_t^i)$$
$$\text{s.t.} \sum_{t=0}^{\infty} Q_t c_t^i \leq \sum_{t=0}^{\infty} Q_t y_t^i$$

Denoting the Lagrange multiplier of agent i as μ_i, first order conditions take the following form:

$$\beta u'(c_t^i) = \mu_i Q_t$$
$$\Rightarrow \frac{u'(c_t^1)}{u'(c_t^2)} = \mu_1/\mu_2$$

Note that the right hand side is independent of t, and since the total endowment of the economy is also constant (1), the consumption of each agent must also be constant for all time, i.e. $c_t^1 = c^1, c_t^2 = c^2$. Market clearing also implies $c^1 + c^2 = 1$.

Moreover, the first order conditions also yield the following:

$$\beta \frac{u'(c_{t+1}^1)}{u'(c_t^1)} = \frac{Q_{t+1}}{Q_t}$$

Constant consumption implies that $Q_{t+1} = \beta Q_t$. We can normalize $Q_0 = 1$ and then we have that $Q_t = \beta^t$. Now we have the following:

$$\sum_{t=0}^{\infty} \beta^t c^1 = \sum_{t=0}^{\infty} \beta^t y_t^i$$
$$\frac{c^1}{1-\beta} = \frac{1}{1-\beta^3}$$
$$\Rightarrow c^1 = \frac{1-\beta}{1-\beta^3},$$
$$c^2 = \frac{\beta-\beta^3}{1-\beta^3}.$$

2.3 Part C

We can price the asset p^A using $Q_t = \beta^t$:

$$p^A = \sum_{i=0}^{\infty} \frac{\beta^t}{20}$$
$$= \frac{1}{20(1-\beta)}.$$

3 Question 3: Exercise 8.4

3.1 Part I

3.1.1 Part A

A competitive equilibrium is a set of prices $\{Q_t(s^t)\}_{t=0}^{\infty}$ and allocations $\{c_t(s^t)\}_{t=0}^{\infty}$ such that agents optimize and markets clear ($c_t(s^t) = d_t(s^t)$).

I will quickly derive first order conditions that will help us later.
The agent maximizes:

$$\max E_0 \sum_{t=0}^{\infty} \frac{c_t^{1-\gamma}}{1-\gamma}$$

$$\text{s.t.} \sum_{t=0}^{\infty} \sum_{s^t} Q_t(s^t) c_t(s^t) \leq \sum_{t=0}^{\infty} Q_t(s^t) d_t(s^t)$$

FOC (lagrange multiplier μ):

$$\beta^t \pi_t(s^t) u'(c_t(s^t)) = \mu Q_t(s^t)$$

$$\Rightarrow \frac{\beta^t \pi_t(s^t) u'(c_t(s^t))}{u'(c_0(s_0))} = Q_t(s^t)$$

$$(0.95)^t \pi_t(s^t)(d_t(s^t))^{-2} = Q_t(s^t) \tag{1}$$

We can use the above expression to price claims in the sections that follow. First, note that $c_t \leq d_t \Rightarrow c_t = d_t$ will maximize utility. Also, note that we are interested in prices in terms of the period 0 good, i.e. we have assumed $Q_0(s_0) = 0$. Note finally that $u'(c_0) = u'(d_0) = u'(1) = 1$.

3.1.2 Part B

Using equation (1), we can price the claim. $c_5 = d_5 = 0.97*0.97*1.03*0.97*1.03 = 0.968$. $\beta^5 = 0.774$. $\pi_t(s^t) = 0.8*0.8*0.2*0.1*0.2 = 0.00256$. Therefore, $Q_5 = (0.774)(0.00256)(0.968)^{-2} = 0.00211$.

3.1.3 Part C

Using equation (1), we can price the claim. $c_5 = d_5 = 1.03*1.03*1.03*1.03*0.97 = 1.092$. $\beta^5 = 0.774$. $\pi_t(s^t) = 0.2*0.9*0.9*0.9*0.1 = 0.01458$. Therefore, $Q_5 = (0.774)(0.01458)(1.092)^{-2} = 0.00946$.

3.1.4 Part D

The price is the sum of the prices and endowments across states and time:

$$\begin{aligned} P^e &= \sum_{t=0}^{\infty} \sum_{s^t} d_t(s^t) Q_t(s^t) \\ &= \sum_{t=0}^{\infty} \sum_{s^t} (0.95)^t \pi_t(s^t)(d_t(s^t))^{-1} \end{aligned}$$

3.1.5 Part E

The price is the sum of the prices and endowments across state histories at time 5, conditional on the state at time $t = 5$ being $\lambda_5 = 0.97$:

$$P^5 = \sum_{s^5 | s_5 = 0.97} (0.95)^5 \pi_5(s^5)(d_t(s^5))^{-1}$$

3.2 Part II

3.2.1 Part F

A recursive competitive equilibrium is a pricing kernel $\{q_t(s^t|s_{t+1})\}_{t=0}^{\infty}$ and decision rules $c(s_t, a_t), a_{t+1}(s_t, a_t)$ such that agents optimize $(v(s_t, a_t) = \max_{c,a_{t+1}} u(c) + \beta E[v(s_{t+1}, a_{t+1})])$ and markets clear $c_t = d_t, a_t = 0 \forall t$.

3.2.2 Part G

The natural debt limit for a state in the future $A_{t+1}(s^t, s_{t+1})$ is the maximum amount one can repay eventually, i.e. present discounted value of future income. It takes a recursive form:

$$A(s_t) = d_t + \beta \sum_{s_{t+1}} Q(s_{t+1}|s_t) A(s_{t+1})$$

3.2.3 Part H

In each period, the agent solves the following maximization problem:

$$\max_{c_t(s^t), \{a_{t+1}(s^t, s_{t+1})\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t(s^t))$$
$$\text{s.t.} c_t(s^t) + \sum_{s_t} a_{t+1}(s^t, s_{t+1}) q_t(s^t, s_{t+1}) \leq d_t(s^t) + a_t(s^t)$$

Taking first order conditions, we have the following:

$$q_t(s^t, s_{t+1}) = \beta \frac{u'(c_{t+1}(s^t, s_{t+1}))}{u'(c_t(s^t))} \pi(s_{t+1}|s^t)$$

Since the endowments are governed by a Markov process, and since we know that the feasible allocations satisfy $c_t \leq d_t \Rightarrow c_t = d_t$ optimizes utility, we can rewrite the first order conditions as follows:

$$\begin{aligned} q_t(s^t, s_{t+1}) &= \beta \frac{u'(d_{t+1}(s^t, s_{t+1}))}{u'(d_t(s^t))} \pi(s_{t+1}|s_t) \\ &= \beta \left(\frac{d_t(s^t)}{d_{t+1}(s^t, s_{t+1})} \right)^2 \pi(s_{t+1}|s_t) \\ &= \beta (\lambda_{t+1}(s_{t+1}))^{-2} \pi(s_{t+1}|s_t) \\ &= q_t(s_t, s_{t+1}). \end{aligned}$$

The above expression is our pricing kernel.

Finally, since we know $c_t(s^t) = d_t(s^t)$, it must immediately hold by induction that $a_t(s^t) = 0$.

3.2.4 Part I

We can use our pricing kernel to price this bond.

$$p^b(s_t) = \sum_{s^{t+1}} \sum_{s^{t+2}} \beta^2 (\lambda_{t+1}(s_{t+1}))^{-2} \pi(s_{t+1}|s_t) (\lambda_{t+2}(s_{t+2}))^{-2} \pi(s_{t+2}|s_{t+1})$$
$$\Rightarrow p^b(\lambda_t) = \begin{cases} (0.95)^2((0.97)^{-2}(0.8)(0.97^{-2}(0.8) + (1.03)^{-2}(0.2)) + (1.03)^{-2}(0.2)((0.9)(1.03)^{-2} + (0.1)(0.97)^{-2})) \\ (0.95)^2((0.97)^{-2}(0.1)(0.97^{-2}(0.8) + (1.03)^{-2}(0.2)) + (1.03)^{-2}(0.9)((0.9)(1.03)^{-2} + (0.1)(0.97)^{-2})) \end{cases}$$
$$p^b(\lambda_t) = \begin{cases} 0.96, \lambda_t = 0.97 \\ 0.83, \lambda_t = 1.03 \end{cases}.$$