

# Econometrics HW6

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## 1 Question 1

### 1.1 Part A

$$P(X = 1) = p = p^1(1 - p)^{1-1} = f(1). \quad P(X = 0) = (1 - p) = p^0(1 - p)^{1-0} = f(0).$$

### 1.2 Part B

Our parameter is  $\theta = p$ .  $l_n(\theta) = \sum_{i=1}^n \log(f(X_i|\theta)) = \sum_{i=1}^n \log(\theta^{X_i}(1-\theta)^{1-X_i}) = \sum_{i=1}^n X_i \log(\theta) + (1 - X_i) \log(1 - \theta)$ .

### 1.3 Part C

$$\frac{\partial l_n(\theta)}{\partial \theta} = 0 \Rightarrow \sum_{i=1}^n \frac{X_i}{\theta} - \frac{1-X_i}{1-\theta} = 0 \Rightarrow \sum_{i=1}^n X_i(1-\theta) = \sum_{i=1}^n \theta - X_i\theta \Rightarrow \hat{\theta}_n = \frac{1}{n} \sum_{i=1}^n X_i.$$

## 2 Question 2

### 2.1 Part A

$$l_n(\theta) = \sum_{i=1}^n \log\left(\frac{\theta}{X_i^{1+\theta}}\right) = \sum_{i=1}^n \log(\theta) - (1+\theta) \log(X_i) = n \log(\theta) - (1+\theta) \sum_{i=1}^n \log(X_i)$$

### 2.2 Part B

$$\frac{\partial l_n(\theta)}{\partial \theta} = 0 \Rightarrow \frac{n}{\theta} - \sum_{i=1}^n \log(X_i) = 0 \Rightarrow \hat{\theta}_n = \frac{n}{\sum_{i=1}^n \log(X_i)}.$$

## 3 Question 3

### 3.1 Part A

$$l_n(\theta) = \sum_{i=1}^n \log\left(\frac{1}{\pi(1+(X_i-\theta)^2)}\right) = -n \log(\pi) - \sum_{i=1}^n \log(1 + (X_i - \theta)^2).$$

### 3.2 Part B

$$\frac{\partial l_n(\theta)}{\partial \theta} = 0 \Rightarrow - \sum_{i=1}^n \frac{2(X_i - \hat{\theta}_n)}{1 + (X_i - \hat{\theta}_n)^2} = 0$$

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\*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

## 4 Question 4

### 4.1 Part A

$$l_n(\theta) = \sum_{i=1}^n \log\left(\frac{1}{2} \exp(-|X_i - \theta|)\right) = n \log\left(\frac{1}{2}\right) - \sum_{i=1}^n |X_i - \theta|$$

### 4.2 Part B

The likelihood is maximized when the term  $\sum_{i=1}^n |X_i - \theta|$  is minimized. This is minimized for  $\theta = M$  where  $M$  is the median of the sample, which I will show below:

Let  $X_i$  be ordered from smallest to largest. If  $n$  is an odd number, define  $m = \frac{n+1}{2}$ . Then, by the triangle inequality,

$$\begin{aligned} \sum_{i=1}^n |X_i - \theta| &\geq |X_n - \theta - (X_1 - \theta)| + |X_{n-1} - \theta - (X_2 - \theta)| + \cdots + |X_{m-1} - \theta - (X_{m+1} - \theta)| + |X_m - \theta| \\ &= \sum_{i=1}^{m-1} |X_{n+1-i} - X_i| + |X_m - \theta|. \end{aligned}$$

Clearly, this term is minimized when  $\theta = X_m = M$ , and the weak inequality holds with equality when  $\theta$  is the median because  $(X_{n+1-i} - \theta) \geq 0 \geq (X_i - \theta) \forall \theta \in \mathbb{R}$ .

If  $n$  is odd, we instead define  $m = n/2$ , and have:

$$\begin{aligned} \sum_{i=1}^n |X_i - \theta| &\geq |X_n - \theta - (X_1 - \theta)| + \cdots + |X_{m-1} - \theta - (X_{m+1} - \theta)| + |X_m - \theta| + |X_{m+1} - \theta| \\ &= \sum_{i=1}^{m-1} |X_{n+1-i} - X_i| + |X_m - \theta| + |X_{m+1} - \theta|, \end{aligned}$$

where again our weak inequality holds with equality. In this case, the final expression is clearly minimized for any  $\theta \in [X_m, X_{m+1}]$ , and  $M \in [X_m, X_{m+1}]$ .

## 5 Question 5

$$\begin{aligned} I_0 &= -E \left[ \frac{\partial^2}{\partial \theta^2} \log(f(X|\theta)) \Big|_{\theta=\theta_0} \right] = -E \left[ \frac{\partial^2}{\partial \theta^2} \log(\theta x^{-1-\theta}) \Big|_{\theta=\theta_0} \right] = -E \left[ \frac{\partial^2}{\partial \theta^2} \log(\theta) + (-1-\theta) \log(x) \Big|_{\theta=\theta_0} \right] \\ &= -E \left[ \frac{\partial}{\partial \theta} \frac{1}{\theta} - \log(x) \Big|_{\theta=\theta_0} \right] = -E \left[ \frac{\partial}{\partial \theta} \frac{1}{\theta} \Big|_{\theta=\theta_0} \right] = \frac{1}{\theta_0^2} \end{aligned}$$

## 6 Question 6

### 6.1 Part A

$$\begin{aligned} I_0 &= -E \left[ \frac{\partial^2}{\partial \theta^2} \log(\theta \exp(-\theta x)) \Big|_{\theta=\theta_0} \right] = -E \left[ \frac{\partial^2}{\partial \theta^2} \log(\theta) + \log(\exp(-\theta x)) \Big|_{\theta=\theta_0} \right] \\ &= -E \left[ \frac{\partial^2}{\partial \theta^2} \log(\theta) - \theta x \Big|_{\theta=\theta_0} \right] = \hat{\theta}_0^{-2} \Rightarrow \text{Var}(\hat{\theta}_n) \geq (n \hat{\theta}_0^{-2})^{-1} = \frac{\theta_0^2}{n} \end{aligned}$$

### 6.2 Part B

$$\begin{aligned} l_n(\theta) &= \sum_{i=1}^n \log(f(X_i|\theta)) = \sum_{i=1}^n \log(\theta \exp(-\theta X_i)) = \sum_{i=1}^n \log(\theta) + \log(\exp(-\theta X_i)) \\ &= n \log(\theta) - \theta \sum_{i=1}^n X_i \Rightarrow \frac{\partial l_n(\theta)}{\partial \theta} = 0 \Rightarrow \frac{n}{\theta} - \sum_{i=1}^n X_i = 0 \Rightarrow \hat{\theta}_n = \frac{n}{\sum_{i=1}^n X_i}. \end{aligned}$$

By the delta method,  $\sqrt{n}(\hat{\theta}_n - \theta_0) \rightarrow_d N(0, V)$  where  $V = (-1(\theta_0^{-1})^{-2})^2 \sigma^2 = \theta_0^4 \sigma^2$  where  $\sigma^2 = \text{Var}(X_i) = \frac{1}{\theta_0^2}$ . Thus,  $\sqrt{n}(\hat{\theta}_n - \theta_0) \rightarrow_d N(0, \theta_0^2)$

### 6.3 Part C

Our general formula is  $\sqrt{n}(\hat{\theta}_n - \theta_0) \rightarrow_d N(0, I_0^{-1}) = N(0, \theta_0^2)$ .

## 7 Question 7

### 7.1 Part A

Via the delta method,  $\sqrt{n}(\hat{\theta}_n - \theta_0) \rightarrow_d N(0, V)$  where  $V = Var(X_i)$ . Thus, a consistent estimator for  $V$  will be a consistent estimator of  $Var(X_i)$ . A consistent estimator of the variance is  $\hat{V} := \frac{1}{n} \sum_{i=1}^n (X_i - \hat{\theta}_n)^2$ .

### 7.2 Part B

The WLLN and CMT imply that  $\hat{V} \rightarrow_p E(X - EX)^2 = Var(X_i) = V$ . Thus,  $\hat{V}$  is a consistent estimator of  $V$ .

### 7.3 Part C

We have that the asymptotic variance of  $\sqrt{n}(\hat{\theta}_n - \theta_0)$  is consistently estimated by  $\frac{1}{n} \sum_{i=1}^n (X_i - \hat{\theta}_n)^2$ . Therefore, an approximation of  $Var(\hat{\theta}_n) = \frac{1}{n} Var(\sqrt{n}\hat{\theta}_n) = \frac{1}{n} Var(\sqrt{n}(\hat{\theta}_n - \theta_0))$  so an estimator of  $Var(\hat{\theta}_n)$  is  $\frac{1}{n^2} \sum_{i=1}^n (X_i - \hat{\theta}_n)^2$ .

## 8 Question 8

### 8.1 Part A

$$F_X(c) = \int_{-\infty}^c f_X(x)dx = \begin{cases} 0, c < 0 \\ G(c), 0 \leq c \leq 1 \\ 1, c > 1 \end{cases} \quad \text{where } G(c) = \int_0^c \frac{1}{\theta} dx = \frac{c}{\theta}.$$

### 8.2 Part B

$$F_{n(\hat{\theta}_n - \theta)}(x) = Pr(\max_{i=1, \dots, n}(n(X_i - \theta)) \leq x) = Pr(n(X_1 - \theta) \leq x, \dots, n(X_n - \theta) \leq x) = \prod_{i=1}^n Pr(n(X_i - \theta) \leq x) = \prod_{i=1}^n Pr(X_i \leq \theta + \frac{x}{n}) = Pr(X_i \leq \theta + \frac{x}{n})^n = (F_X(\theta + \frac{x}{n}))^n.$$

### 8.3 Part C

Fix  $x$ . For  $x < 0$ ,  $F_{n(\hat{\theta}_n - \theta)}(x) = (F_X(\theta + \frac{x}{n}))^n = (F_X(\theta(1 + \frac{x/\theta}{n})))^n \rightarrow_{n \rightarrow \infty} \lim_{n \rightarrow \infty} ((\theta(1 + \frac{x/\theta}{n}))/\theta)^n = e^{x/\theta}$ .

For  $x > 0$ ,  $F_{n(\hat{\theta}_n - \theta)}(x) = (F_X(\theta + \frac{x}{n}))^n = 1^n = 1$  so  $\lim_{n \rightarrow \infty} F_{n(\hat{\theta}_n - \theta)} = 1$ .

### 8.4 Part D

$\lim_{n \rightarrow \infty} f_{n(\hat{\theta}_n - \theta)}(x) = \lim_{n \rightarrow \infty} \frac{\partial}{\partial x} F_{n(\hat{\theta}_n - \theta)}(x) = \frac{1}{\theta} e^{x/\theta}$  for  $x \leq 0 \Rightarrow \lim_{n \rightarrow \infty} f_{n(\hat{\theta}_n - \theta)}(-x) = \frac{1}{\theta} e^{-x/\theta}$  so  $n(\hat{\theta}_n - \theta) \rightarrow_d -A$  where distribution A is an exponential with parameter  $\theta$ .

## 9 Question 9

We should use a two-sided test. We will calculate  $t = \frac{\bar{X}_n - 1}{se}$ ,  $se = \sqrt{s^2/n}$ . For a chosen significance level  $\alpha$  we can reject the null hypothesis if  $P(|T| > t) < \alpha$ , where  $t \sim t_{n-1}$ .

## 10 Question 10

$$Pr(T > c | \mu = 0) = Pr()$$