

# IO Problem Set 2

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October 5, 2021

## 1 Question 1

Note: lots of omitted algebra done by hand on scratch paper.

### 1.1 Part (a)

We have  $Q = \frac{a_0 - P - \nu}{a_1}$ .  $\epsilon = -\frac{P \partial Q}{Q \partial P} = \frac{P}{Q a_1} = \frac{a_0 - a_1 Q + \nu}{a_1 Q} = \frac{a_0 + \nu}{a_1 Q} - 1$ . Thus,  $\frac{\partial \epsilon}{\partial Q} = -\frac{\nu + a_0}{a_1 Q^2} \leq 0$ . Similarly,  $\frac{\partial \epsilon}{\partial \nu} = \frac{1}{a_1 Q} \geq 0$ . Thus, as  $Q$  increases, elasticity of demand decreases; and as  $\nu$  increases, the elasticity of demand increases.

### 1.2 Part (b)

We have  $c = F + (b_0 + \nu)Q$ . The firm takes the quantities of the other firms as given and maximizes profit:

$$\begin{aligned} \max_q [a_0 - a_1(q + Q_{-i})]q - [F + (b_0 + \eta)q] \\ \Rightarrow a_0 - 2a_1q - a_1Q_{-i} - b_0 - \eta = 0 \end{aligned}$$

Applying symmetry,  $Q = Nq \Rightarrow q = \frac{a_0 + \nu - b_0 - \eta}{a_1(N+1)}$ . Quantity is  $Q = N \frac{a_0 + \nu - b_0 - \eta}{a_1(N+1)}$ ; prices are  $P = \frac{a_0 + \nu + N(b_0 + \eta)}{N+1}$ .

### 1.3 Part (c)

Individual firm profits are  $Pq - C = \frac{a_0 + \nu + N(b_0 + \eta)}{N+1} \frac{a_0 + \nu - b_0 - \eta}{a_1(N+1)} - F - \frac{(b_0 + \eta)(a_0 + \nu - b_0 - \eta)}{a_1(N+1)} = \frac{1}{a_1} \left[ \frac{a_0 + \nu - b_0 - \eta}{N+1} \right]^2 - F$ . Firms enter until profits are zero:  $0 = \frac{1}{a_1} \left[ \frac{a_0 + \nu - b_0 - \eta}{N+1} \right]^2 - F \Rightarrow N = \frac{1}{\sqrt{F a_1}} (a_0 + \nu - b_0 - \eta) - 1$ .

### 1.4 Part (d)

$$\begin{aligned} L_I &= \frac{P - mc}{P} = \frac{\frac{a_0 + \nu + N(b_0 + \eta)}{N+1} - (b_0 + \eta)}{\frac{a_0 + \nu + N(b_0 + \eta)}{N+1}} \\ &= \frac{a_0 + \nu - b_0 - \eta}{a_0 + \nu + N(b_0 + \eta)} \\ &= \frac{a_0 + \nu - b_0 - \eta}{a_0 + \nu + \left( \frac{1}{\sqrt{F a_1}} (a_0 + \nu - b_0 - \eta) - 1 \right) (b_0 + \eta)} \\ &= \frac{\sqrt{F a_1}}{\sqrt{F a_1} + b_0 + \eta}. \end{aligned}$$

Firms are identical so the Herfindahl index is  $H = \frac{1}{N} = \frac{\sqrt{F a_1}}{a_0 + \nu - b_0 - \eta - \sqrt{F a_1}}$ .

$$\begin{aligned}
\epsilon &= \frac{a_0 + \nu - N \left( \frac{a_0 + \nu - b_0 - \eta}{N+1} \right)}{N \left( \frac{a_0 + \nu - b_0 - \eta}{N+1} \right)} \\
&= \frac{b_0 + \eta + \sqrt{F a_1}}{a_0 + \nu - b_0 - \eta - \sqrt{F a_1}}.
\end{aligned}$$

### 1.5 Part (e)

$$\begin{aligned}
\frac{\partial \epsilon}{\partial F} &= \frac{\sqrt{a_1}(a_0 + \nu)}{2\sqrt{F}(a_0 + \nu - b_0 - \eta - \sqrt{F a_1})^2} \\
&\geq 0, \\
\frac{\partial \epsilon}{\partial \nu} &= -\frac{b_0 + \eta + \sqrt{F a_1}}{(a_0 + \nu - b_0 - \eta - \sqrt{F a_1})^2} \\
&\leq 0, \\
\frac{\partial \epsilon}{\partial \eta} &= \frac{a_0 + \nu}{(a_0 + \nu - b_0 - \eta - \sqrt{F a_1})^2} \\
&\geq 0.
\end{aligned}$$

$$\begin{aligned}
\log(L_I) &= (1/2)\log(F a_1) - \log(\sqrt{F a_1} + b_0 + \eta), \\
\log(H) &= (1/2)\log(F a_1) - \log(a_0 + \nu - b_0 - \eta - \sqrt{F a_1}).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \log(L_I)}{\partial F} &= \frac{1}{2F} - \frac{\sqrt{a_1}}{(\sqrt{F a_1} + b_0 + \eta)2\sqrt{F}}, \\
\frac{\partial \log(H)}{\partial F} &= \frac{1}{2F} + \frac{\sqrt{a_1}}{(-\sqrt{F a_1} + a_0 + \nu - b_0 - \eta)2\sqrt{F}}, \\
&\neq \frac{\partial \log(L_I)}{\partial F}, \\
\frac{\partial \log(L_I)}{\partial \nu} &= 0, \\
\frac{\partial \log(H)}{\partial \nu} &= -\frac{1}{a_0 + \nu - b_0 - \eta - \sqrt{F a_1}}, \\
&\neq \frac{\partial \log(L_I)}{\partial \nu}, \\
\frac{\partial \log(L_I)}{\partial \eta} &= -\frac{1}{\sqrt{F a_1} + b_0 + \eta}, \\
\frac{\partial \log(H)}{\partial \eta} &= \frac{1}{a_0 + \nu - b_0 - \eta - \sqrt{F a_1}}, \\
&\neq \frac{\partial \log(L_I)}{\partial \eta}.
\end{aligned}$$

Clearly,  $\log(L_I), \log(H)$  do not change at the same rate in responses to  $F, \nu, \eta$ .

## 1.6 Part (f)

Colluding firms have:

$$\begin{aligned}
& \max_Q (a_0 - a_1 Q + \nu)Q - (b_0 + \eta)Q - F \\
& \Rightarrow a_0 - 2a_1 Q + \nu - b_0 - \eta = 0 \\
& \Rightarrow Q = \frac{a_0 + \nu - b_0 - \eta}{2a_1}, \\
& \Rightarrow P = (1/2)(a_0 + \nu + b_0 + \eta), \\
& \Rightarrow \pi = \frac{(a_0 + \nu - b_0 - \eta)^2}{4Na_1} - F.
\end{aligned}$$

Firms enter until profits are zero  $\Rightarrow N = \frac{(a_0 + \nu - b_0 - \eta)^2}{4Fa_1}$ .  $L_I = \frac{P - mc}{P} = \frac{a_0 + \nu - b_0 - \eta}{a_0 + \nu + b_0 + \eta}$ .  $H = \frac{1}{N} = \frac{4Fa_1}{(a_0 + \nu - b_0 - \eta)^2}$ .  $\epsilon = -\frac{P \partial Q}{Q \partial P} = \frac{a_0 + \nu + b_0 + \eta}{a_0 + \nu - b_0 - \eta}$ .

## 1.7 Part (g)

We now have  $Q = \exp((1/c_1)(c_0 - \log(P) + \xi))$ .  $\epsilon = (P/Q)(\exp((1/c_1)(c_0 - \log(P) + \xi))) = (1/Q)\exp((1/c_1)(c_0 - \log(P) + \xi))$ .

$$\begin{aligned}
\frac{\partial \epsilon}{\partial Q} &= -\frac{1}{Q^2} \exp((1/c_1)(c_0 - \log(P) + \xi)) \\
&\leq 0 \\
\frac{\partial \epsilon}{\partial Q} &= (1/Q) \exp((1/c_1)(c_0 - \log(P) + \chi)) \\
&\geq 0.
\end{aligned}$$

Under the new inverse supply curve, the firm maximizes profit:

$$\begin{aligned}
& \max_q \exp(c_0 - c_1 \log(q + Q_{-i}) + \xi)q - (b_0 + \eta)q - F \\
& \Rightarrow \exp(c_0 - c_1 \log(q + Q_{-i}) + \xi) - \exp(c_0 - c_1 \log(q + Q_{-i}) + \xi) \frac{qc_1}{Q} - b_0 - \eta = 0
\end{aligned}$$

Applying symmetry across firms,

$$\begin{aligned}
\exp(c_0 - c_1 \log(Nq) + \xi) &= \frac{N}{N - c_1} (b_0 + \eta) \\
\Rightarrow q &= \frac{1}{N} \exp((1/c_1)(c_0 + \xi - \log((N/(N - c_1))(b_0 + \eta)))), \\
Q &= \exp((1/c_1)(c_0 + \xi - \log((N/(N - c_1))(b_0 + \eta)))), \\
P &= \exp(c_0 - c_1 \log(\exp((1/c_1)(c_0 + \xi - \log((N/(N - c_1))(b_0 + \eta))))) + \xi), \\
\Rightarrow P &= \frac{N}{N - c_1} (b_0 + \eta).
\end{aligned}$$

$$\begin{aligned}
L_I &= \frac{\frac{N}{N-c_1}(b_0 + \eta) - (b_0 + \eta)}{\frac{N}{N-c_1}(b_0 + \eta)} \\
&= \frac{c_1}{N}. \\
H &= \frac{1}{N}. \\
\epsilon &= (1/Q)\exp((1/c_1)(c_0 - \log(P) + \xi)) \\
&= \frac{1}{c_1}.
\end{aligned}$$

In this case, since we are told to keep  $N$  as exogenously determined, all of the partial derivatives are zero:

$$\frac{\partial \epsilon}{\partial F} = \frac{\partial \epsilon}{\partial \nu} = \frac{\partial \epsilon}{\partial \eta} = \frac{\partial \log(L_I)}{\partial F} = \frac{\partial \log(L_I)}{\partial \nu} = \frac{\partial \log(L_I)}{\partial \eta} = \frac{\partial \log(H)}{\partial F} = \frac{\partial \log(H)}{\partial \nu} = \frac{\partial \log(H)}{\partial \eta} = 0.$$

Therefore,  $\log(L_I), \log(H)$  do change at the same rate in response to changes in  $F, \nu, \eta : 0$ .

## 2 Question 2

I computed in Matlab. Results follow.

Figure 1: Regression tables and F test results for price given by equation (3).

	Pooled	Collusion	No Collusion
cons	-0.469*** (0.0536)	-0.857*** (0.0828)	-0.106*** (0.003)
log(H)	0.543*** (0.0322)	0.0714 (0.0498)	1*** (0.0018)
F	201	348	0.000646
p(f>F)	0	0	0.98

In our initial regressions, we see that results differ greatly across groups. For cities with no collusion, the coefficient is tightly identified to be near 1. In fact, we cannot reject the F test with the null hypothesis being that this coefficient is exactly 1. With collusion, however, the coefficient is not statistically different from zero, and we can thoroughly reject the null hypothesis that the coefficient is 1. The pooled case, unsurprisingly, is somewhere in the middle, but we do still reject that the coefficient is equal to one.

Note that, without collusion, the  $H$  index and true  $L_I$  index are identical. This is why the regression coefficients are 1 in this case.

Figure 2: Regression tables and F test results for price given by equation (1).

	Pooled	Collusion	No Collusion
cons	-0.777*** (0.0255)	-0.973*** (0.0373)	-0.594*** (0.00668)
log(H)	0.294*** (0.0153)	0.0568** (0.0224)	0.524*** (0.00402)
F	2.11e+03	1.77e+03	1.4e+04
p(f>F)	0	0	0

With the different price equation, note that we no longer have identical  $H$  index and true  $L_I$  index. Thus, none of the coefficients are close to 1, and the F test rejects that the true coefficient is 1 for all cases. However, we do still have a positive relationship between  $H$  and true  $L_I$ . Our formula for  $L_I = \frac{a_0 + \nu - b_0 - \eta}{a_0 + \nu + N(b_0 + \eta)}$  without collusion compared to  $H = \frac{1}{N}$ . It is clear that  $Cov(\log(H), \log(L_I)) > 0$  from the functional form.

What can we do that is positive from what we have found? In both cases, the OLS coefficients are higher without collusion than with. If we want to show that a subset of markets features collusion then we can run these regressions on those markets and the magnitude of the coefficient can be indicative of likely presence of collusion. I would hesitate to extrapolate this fact to more general model.

### 3 Question 3

	$\nu \sim U[-1, 1]$	$\eta \sim U[-1, 1]$
cons	-0.652*** (1.5e-08)	-2.23*** (0.0121)
log(H)	-8.57e-17 (1.37e-08)	-1.51*** (0.011)

Above are the results from the simulated regressions. The differences are quite stark, and in line with theory. Our theory predicts that  $H = \frac{\sqrt{Fa_1}}{a_0 + \nu - b_0 - \eta - \sqrt{Fa_1}}$ ,  $L_I = \frac{\sqrt{Fa_1}}{\sqrt{Fa_1} + b_0 + \eta}$ . The only differences across markets are in  $\nu$  in the first case, and in  $\eta$  in the second case. In the first case, as  $\nu$  varies,  $H$  is affected but  $L_I$  is unaffected as it is independent of  $\nu$ . Therefore, we expect to see no statistically significant relationship between  $H, L_I$ . In the second hand, in contrast,  $H$  is increasing in  $\eta$  and  $L_I$  is decreasing in  $\eta$ , so we expect a negative sign on the regression coefficient - and our results do show a negative relationship.

The elasticities are increasing in  $\eta$  and decreasing in  $\nu$ , so we would expect  $\epsilon$  would positively correlate with  $\log(H)$  in both experiments, and negatively correlate with  $L_I$  only in the experiment where  $\eta$  varies and  $\nu$  is fixed.