

# Macro PS1

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## 1 Question 1: Exercise 8.1

The first order conditions to the Pareto problem is the following:

$$\begin{aligned}\theta u'(c^1) &= \lambda \\ (1 - \theta)w'(c^2) &= \lambda \\ \Rightarrow \theta u'(c^1) &= (1 - \theta)w'(c^2).\end{aligned}$$

From envelope conditions, we get the following:

$$\begin{aligned}v'_\theta(c) &= \theta u'(c^1) \frac{\partial c^1}{\partial c} + (1 - \theta)w'(c^2) \frac{\partial c^2}{\partial c} \\ &= \theta u'(c^1) \frac{\partial(c^1 + c^2)}{\partial c} \\ &= \theta u'(c^1) = (1 - \theta)w'(c^2).\end{aligned}$$

Since  $u, w$  are twice differentiable, from the envelope theorem  $v_\theta$  is twice differentiable and  $v''_\theta = \theta u''(c^1) + (1 - \theta)w''(c^2) < 0$  since  $u, w$  are concave. Thus,  $v_\theta$  is concave.

## 2 Question 2: Exercise 8.3

### 2.1 Part A

A competitive equilibrium is a set of prices  $\{Q_t\}_{t=0}^\infty$  and endowments  $\{c_t^1, c_t^2\}_{t=0}^\infty$  such that both consumers optimize (maximize the sum of discounted utility) and markets clear ( $c_t^1 + c_t^2 = y_t^1 + y_t^2 = 1 \forall t$ ).

### 2.2 Part B

Agent  $i$  solves the following optimization problem:

$$\begin{aligned}\max_{\{c_t^i\}_{t=0}^\infty} & \sum_{t=0}^\infty \beta^t u(c_t^i) \\ \text{s.t.} & \sum_{t=0}^\infty Q_t c_t^i \leq \sum_{t=0}^\infty Q_t y_t^i\end{aligned}$$

Denoting the Lagrange multiplier of agent  $i$  as  $\mu_i$ , first order conditions take the following form:

$$\begin{aligned}\beta u'(c_t^i) &= \mu_i Q_t \\ \Rightarrow \frac{u'(c_t^1)}{u'(c_t^1)} &= \mu_i / \mu_j\end{aligned}$$

Note that the right hand side is independent of  $t$ , and since the total endowment of the economy is also constant (1), the consumption of each agent must also be constant for all time, i.e.  $c_t^1 = c^1, c_t^2 = c^2$ . Market clearing also implies  $c^1 + c^2 = 1$ .

Moreover, the first order conditions also yield the following:

$$\beta \frac{u'(c_{t+1}^1)}{u'(c_t^1)} = \frac{Q_{t+1}}{Q_t}$$

Constant consumption implies that  $Q_{t+1} = \beta Q_t$ . We can normalize  $Q_0 = 1$  and then we have that  $Q_t = \beta^t$ . Now we have the following:

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t c^1 &= \sum_{t=0}^{\infty} \beta y_t^i \\ \frac{c^1}{1-\beta} &= \frac{1}{1-\beta^3} \\ \Rightarrow c^1 &= \frac{1-\beta}{1-\beta^3}, \\ c^2 &= \frac{\beta-\beta^3}{1-\beta^3}. \end{aligned}$$

## 2.3 Part C

We can price the asset  $p^A$  using  $Q_t = \beta^t$ :

$$\begin{aligned} p^A &= \sum_{i=0}^{\infty} \frac{\beta^t}{20} \\ &= \frac{1}{20(1-\beta)}. \end{aligned}$$

## 3 Question 3: Exercise 8.4

### 3.1 Part I

#### 3.1.1 Part A

A competitive equilibrium is a set of prices  $\{Q_t(s^t)\}_{t=0}^{\infty}$  and allocations  $\{c_t(s^t)\}_{t=0}^{\infty}$  such that agents optimize and markets clear ( $c_t(s^t) = d_t(s^t)$ ).

I will quickly derive first order conditions that will help us later.

The agent maximizes:

$$\begin{aligned} &\max E_0 \sum_{t=0}^{\infty} \frac{c_t^{1-\gamma}}{1-\gamma} \\ \text{s.t. } &\sum_{t=0}^{\infty} \sum_{s^t} Q_t(s^t) c_t(s^t) \leq \sum_{t=0}^{\infty} Q_t(s^t) d_t(s^t) \end{aligned}$$

FOC (lagrange multiplier  $\mu$ ):

$$\begin{aligned} \beta^t \pi_t(s^t) u'(c_t(s^t)) &= \mu Q_t(s^t) \\ \Rightarrow \frac{\beta^t \pi_t(s^t) u'(c_t(s^t))}{u'(c_0(s_0))} &= Q_t(s^t) \end{aligned}$$

$$\beta^t \pi_t(s^t) u'(c_t(s^t)) = Q_t(s^t) \tag{1}$$

We can use the above expression to price claims in the sections that follow. Note that we are interested in prices in terms of the period 0 good, i.e. we have assumed  $Q_0(s_0) = 0$ . Note also that  $u'(c_0) = u'(d_0) = u'(1) = 1$ .

### 3.1.2 Part B

Using equation (1), we can price the claim.  $c_5 = d_5 = 0.97 * 0.97 * 1.03 * 0.97 * 1.03 = 0.968$ .  $\beta^5 = 0.774$ .  $\pi_t(s^t) = 0.8 * 0.8 * 0.2 * 0.1 * 0.2 = 0.00256$ . Therefore,  $Q_5 = (0.774)(0.00256)(0.968)^{-2} = 0.00211$ .

### 3.1.3 Part C

Using equation (1), we can price the claim.  $c_5 = d_5 = 1.03 * 1.03 * 1.03 * 1.03 * 0.97 = 1.092$ .  $\beta^5 = 0.774$ .  $\pi_t(s^t) = 0.2 * 0.9 * 0.9 * 0.9 * 0.1 = 0.01458$ . Therefore,  $Q_5 = (0.774)(0.01458)(1.092)^{-2} = 0.00946$ .

### 3.1.4 Part D

The price is the sum of the prices and endowments across states and time:

$$\begin{aligned} P^e &= \sum_{t=0}^{\infty} \sum_{s^t} d_t(s^t) Q_t(s^t) \\ &= \sum_{t=0}^{\infty} \sum_{s^t} d_t(s^t) \beta^t \pi_t(s^t) u'(d_t(s^t)) \end{aligned}$$

### 3.1.5 Part E

The price is the sum of the prices and endowments across state histories at time 5, conditional on the state at time  $t = 5$  being  $\lambda_5 = 0.97$ :

$$P^5 = \sum_{s^5 | s_5=0.97} d(s^5) \beta^5 \pi_t(s^5) u'(d(s^5))$$

## 3.2 Part II

### 3.2.1 Part F

A competitive equilibrium is a set of prices  $\{q_t(s^t | s_{t+1})\}_{t=0}^{\infty}$  and allocations  $\{c_t(s^t), a_{t+1}(s^t, s_{t+1})\}_{t=0}^{\infty}$  such that agents optimize and markets clear  $c_t(s^t) = d_t(s^t) \forall t$ .

### 3.2.2 Part G

The natural debt limit for a state in the future  $A_{t+1}(s^t, s_{t+1})$  is the maximum amount one can repay in that state, i.e. the amount they could repay if they chose no consumption and no buying of future assets:

$$A_{t+1}(s^t, s_{t+1}) = d_{t+1}(s^{t+1})$$

Is this right? Check with others.

### 3.2.3 Part H

In each period, the agent solves the following maximization problem:

$$\begin{aligned} \max_{c_t(s^t), \{a_{t+1}(s^t, s_{t+1})\}} & E_0 \sum_{t=0}^{\infty} \beta^t u(c_t(s^t)) \\ \text{s.t. } & c_t(s^t) + \sum_{s_t} a_{t+1}(s^t, s_{t+1}) q_t(s^t, s_{t+1}) \leq d_t(s^t) + a_t(s^t) \end{aligned}$$

Taking first order conditions, we have the following:

$$q_t(s^t, s_{t+1}) = \beta \frac{u'(c_{t+1}(s^t, s_{t+1}))}{u'(c_t(s^t))} \pi(s_{t+1}|s^t)$$

Since the endowments are governed by a markov process, and since we know that the feasible allocations satisfy  $c_t \leq d_t \Rightarrow c_t = d_t$  optimizes utility, we can rewrite the first order conditions as follows:

$$\begin{aligned} q_t(s^t, s_{t+1}) &= \beta \frac{u'(d_{t+1}(s^t, s_{t+1}))}{u'(d_t(s^t))} \pi(s_{t+1}|s_t) \\ &= \beta \left( \frac{d_t(s^t)}{d_{t+1}(s^t, s_{t+1})} \right)^2 \pi(s_{t+1}|s_t) \\ &= \beta (\lambda_{t+1}(s_{t+1}))^{-2} \pi(s_{t+1}|s_t) \\ &= q_t(s_t, s_{t+1}). \end{aligned}$$

The above expression is our pricing kernel.

Finally, since we know  $c_t(s^t) = d_t(s^t)$ , it must immediately hold by induction that  $a_t(s^t) = 0$ .

### 3.2.4 Part I

We can use our pricing kernel to price this bond.

$$\begin{aligned} p^b(s_t) &= \sum_{s_{t+1}} \beta (\lambda_{t+1}(s_{t+1}))^{-2} \pi(s_{t+1}|s_t) \\ \Rightarrow p^b(\lambda_t) &= \begin{cases} (0.95)((0.97)^{-2}(0.8) + (1.03)^{-2}(0.2)), & \lambda_t = 0.97 \\ (0.95)((0.97)^{-2}(0.1) + (1.03)^{-2}(0.9)), & \lambda_t = 1.03 \end{cases} , \\ p^b(\lambda_t) &= \begin{cases} 0.987, & \lambda_t = 0.97 \\ 0.907, & \lambda_t = 1.03 \end{cases} . \end{aligned}$$