Macro PS1

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1 Question 1

1.1 Part A

$$V(A_t, c_{t-1}) = \max_{c_t, A_{t+1}} u(c_t, c_{t-1}) + \beta V(A_{t+1}, c_t)$$
$$\text{s.t. } A_{t+1} = R(A_t - c_t)$$

We can rewrite this as the following:

$$V(A_t, c_{t-1}) = \max_{c_t} u(c_t, c_{t-1}) + \beta V(R(A_t - c_t), c_t) \tag{1}$$

To ensure the solution is unique, with a strictly increasing, strictly concave value function that is differentiable on the interior of the feasible set, we require the following assumptions:

- $u(.)$ is continuously differentiable, strictly concave, and strictly increasing in its arguments.
- $0 < \beta < 1$
- $\lim_{k\to 0} u'(k, u) = \lim_{k\to 0} u'(u, k) = \infty$
- $\lim_{k\to \infty} u'(k, u) = \lim_{k\to \infty} u'(u, k) = 0$
- The utility function is bounded?
- Do we need anything else?

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Assuming the above conditions hold, we can solve the value function as it is written in (1) by taking a first order condition with respect to c_t, and then applying the envelope theorem twice:

$$\begin{aligned} 0 &= u_1(c', c) + \beta(-RV_1(RA - Rc', c') + V_2(RA - Rc', c')) \quad (2) \\ V_1(A, c) &= R\beta V_1(RA - Rc', c') \quad (3) \\ V_2(A, c) &= u_2(c', c) \quad (4) \end{aligned}$$

Next, we can substitute in the envelope conditions (3), (4) into our first order condition (2) to find an expression for V_1, and substitute back into our initial first order condition (2):

$$\Rightarrow 0 = u_1(c', c) + \beta \left(-\frac{V_1(A, c)}{\beta} + u_2(c'', c') \right)$$

$$\Rightarrow V_1(A, c) = u_1(c', c) + \beta(c'', c') \quad (5)$$

We can now combine equations (2), (4), (5) to yield the following:

$$0 = u_1(c', c) + \beta(-R(u_1(c'', c') + \beta u_2(c''', c'')) + u_2(c'', c'))$$

$$\Rightarrow 0 = u_1(c', c) - \beta R u_1(c'', c') - \beta^2 R u_2(c''', c'') + \beta u_2(c'', c') \quad (6)$$

Equation (6) yields our optimality condition.

1.2 Part B

With the utility function as given, our value function becomes the following:

$$V(A, c) = \max_{c'} log(c') + \gamma log(c) + \beta V(RA - Rc', c')$$

The optimal choice c' takes the form of the arg max of the optimization problem.

$$\begin{aligned} c' &= \arg\max_{c'} log(c') + \gamma log(c) + \beta V(RA - Rc', c') \\ &= \arg\max_{c'} log(c') + \gamma log(c) + \beta \max_{c''} log(c'') + \gamma log(c') + \beta V(RA - Rc'', c'') \\ &= \arg\max_{c'} log(c') + \gamma log(c') + \gamma log(c) + \beta \max_{c''} log(c'') + \beta V(RA - Rc'', c'') \\ &= \arg\max_{c'} log(c') + \gamma log(c') + \beta \max_{c''} log(c'') + \beta V(RA - Rc'', c'') \end{aligned}$$

This arg max is independent of c.

We can rewrite this Bellman equation as the following, which will preserve the choice of c':

$$V(A) = \max_{A'} (1+\gamma) log \left(\frac{A'}{R} - A \right) + \beta V(A')$$

Taking FOC's,

$$\begin{aligned}
0 &= \frac{1+\gamma}{R\left(\frac{A'}{R} - A\right)} + \beta V'(A') \\
V'(A) &= -\frac{1+\gamma}{\left(\frac{A'}{R} - A\right)} \\
\Rightarrow \frac{1+\gamma}{R\left(\frac{A'}{R} - A\right)} &= \beta \frac{1+\gamma}{\left(\frac{A''}{R} - A'\right)} \\
\Rightarrow \left(\frac{A''}{R} - A' \right) &= \beta R \left(\frac{A'}{R} - A \right) \\
\Rightarrow A'' &= A'(1+\beta)R - \beta R^2 A
\end{aligned}$$

Now we will solve for our optimality conditions. Applying (6) to our new utility function yields the following:

$$\beta R(c'')^{-1} + \gamma \beta^2 R(c'')^{-1} = (c')^{-1} + \gamma \beta (c')^{-1}$$
$$\Rightarrow c' \beta R(1+\gamma\beta) = c''(1+\gamma\beta)$$

$$\Rightarrow c' \beta R = c'' \tag{7}$$

Equation (7) yields our Euler conditions.

Given a set of assets in an initial period, A_1, $c_{t+1} = \beta R c_t \ \forall t \in \mathbb{N}$.

1.3 Part C

No, in general this will not hold. The utility function given to us was a seperable utility function, and for a non-separable utility function the utility

2 Question 2

2.1 Part A

The sequence problem is to maximize profits:

$$\max_{\{x_t\}_{i=1}^{\infty}} \left(\frac{1}{1+r} \right)^t \left(a x_t - \frac{b}{2} x_t^2 - \frac{c}{2} (x_{t+1} - x_t)^2 \right)$$

We can rewrite this as a bellman equation in the following way:

$$V(x) = \max_y ax - \frac{b}{2}x^2 - \frac{c}{2}(y-x)^2 + \delta V(y) \tag{8}$$

We can rewrite this as follows:

$$T(v)(x) = \max_y ax - \frac{b}{2}x^2 - \frac{c}{2}(y-x)^2 + \delta v(y) \tag{9}$$

where the fixed point of our T operator in (9) is the solution to the Bellman equation in (8).

2.2 Part B

Let $L < 0$ be arbitrary. If we set $y = 0, x < \frac{L}{a}$ then $F(x,y) = ax - \frac{b}{2}x^2 - \frac{c}{2}(y-x)^2 \leq ax < L$ so F is unbounded below.

This F function is of the form of a polynomial with limits of negative infinity in all directions, so we can find the global maximum by taking first order conditions:

$$\begin{aligned}
0 &= a - bx + c(y-x) \\
0 &= -c(y-x) \Rightarrow y - x = 0 \Rightarrow y = x \\
\Rightarrow y = x &= \frac{a}{b} \\
F\left(\frac{a}{b}, \frac{a}{b}\right) &= a\left(\frac{a}{b}\right) - \frac{b}{2}\left(\frac{a}{b}\right)^2 - 0 \\
&= \frac{a^2}{2b}
\end{aligned}$$

Therefore, the maximum value F can take is $\frac{a^2}{2b}$

We can find bounds on $\hat{v}$ in the following way:

$$\begin{aligned}
\hat{v} &= \frac{a^2}{2b} + \delta\hat{v} \\
\Rightarrow \hat{v} &= \frac{a^2}{2b(1-\delta)}
\end{aligned}$$

2.3 Part C

$$\begin{aligned}
T\hat{v}(x) &= \max_y ax - \frac{b}{2}x^2 - \frac{c}{2}(y-x)^2 + \delta\hat{v} \\
0 &= -c(y-x) \Rightarrow y = x, \\
\Rightarrow T\hat{v}(x) &= ax - \frac{b}{2}x^2 + \delta\hat{v} \\
&\leq \frac{a^2}{2b} + \delta\frac{a^2}{2b(1-\delta)} = \frac{a^2}{2b(1-\delta)} \\
&= \hat{v}.
\end{aligned}$$

2.4 Part D

We showed the base case of the induction in Part C. Now we will show the induction step.

Assume that $T^n\hat{v}(x)$ takes the form $T^n\hat{v}(x) = \alpha_n x - \frac{1}{2}\beta_n x^2 + \gamma_n$. Then,

$$\begin{aligned}
T^{n+1}\hat{v}(x) &= \max_y ax - \frac{b}{2}x^2 - \frac{c}{2}(y-x)^2 + \delta(\alpha_n x - \frac{1}{2}\beta_n x^2 + \gamma_n) \\
y = x \Rightarrow T^{n+1}\hat{v}(x) &= ax - \frac{b}{2}x^2 + \delta\alpha_n x - \delta\frac{1}{2}\beta_n x^2 + \delta\gamma_n \\
&= (a + \delta\alpha_n)x - \frac{b + \delta\beta_n}{2}x^2 + \delta\gamma_n \\
&= \alpha_{n+1}x - \frac{1}{2}\beta_{n+1}x^2 + \gamma_{n+1}
\end{aligned}$$

where $\alpha_{n+1} = (a + \delta\alpha_n), \beta_{n+1} = b + \delta\beta_n, \gamma_{n+1} = \delta\gamma_n$.

2.5 Part E

Note that $\alpha_n = a + \delta a + \delta^2 a + \ldots$, $\beta_n = b + \delta b + \delta^2 b + \ldots, \gamma_n = \delta^n\hat{v}$. Thus, we can take the limit of α, β as geometric sums, and the limit of γ_n is 0. Therefore,

$$\begin{aligned}
\tilde{V} &= \lim_{n\to\infty} T^n\hat{v} = \frac{a}{1-\delta}x - \frac{1}{2}\frac{b}{1-\delta}x^2. \\
T\tilde{V} &= \max_y ax - \frac{b}{2}x^2 - \frac{c}{2}(y-x)^2 + \delta\frac{a}{1-\delta}x - \frac{1}{2}\frac{b}{1-\delta}x^2, \\
y = x \Rightarrow T\tilde{V} &= ax - \frac{b}{2}x^2 + \delta\frac{a}{1-\delta}x - \frac{1}{2}\frac{b}{1-\delta}x^2 \\
&= \frac{a}{1-\delta}x - \frac{1}{2}\frac{b}{1-\delta}x^2 \\
&= \tilde{T}.
\end{aligned}$$

Therefore, the limit function $\tilde{V}$ satisfies the Bellman equation.

3 Question 3

3.1 Part A

$$V(k) = \max_{k'} \pi(k) - \gamma(k' - (1-\delta)k) + \frac{1}{R}V(k')$$

$$0 = -\gamma'(k' - (1-\delta)k) + \frac{1}{R}\left(\pi'(k') + (1-\delta)\gamma'(k'' - (1-\delta)k')\right)$$
$$\Rightarrow \pi'(k') + (1-\delta)\gamma'(k'' - (1-\delta)k') = R\gamma'(k' - (1-\delta)k)$$

3.2 Part B

Letting $k = k' = k'' = \bar{k}$, we know that $\bar{I} = \delta\bar{k}$ and, moreover, we can rewrite our conditions for optimization in the following way:

$$\pi'(\bar{k}) + (1-\delta)\gamma'(\bar{I}) = R\gamma'(\bar{I}))$$
$$\Rightarrow \pi'(\bar{k}) = (R - 1 + \delta)\gamma'(\bar{I})$$

By the strict convexity of γ and strict concavity of π, in addition to our Inada conditions, the solution exists and is unique.

If R were to increase, the steady state level of $\pi'(\bar{k})$ would increase, resulting in a reduction in $\bar{k}$, and since $\bar{I} = \delta\bar{k}$, $\bar{I}$ will fall as well.

3.3 Part C

Our optimality conditions become:

$$-(k' - k^*) = R(k' - (1-\delta)k) + (1-\delta)(I')$$
$$\Rightarrow -(k' - k^*) = RI - (1-\delta)I'$$

4 Question 4

4.1 Part A

We will write our Bellman equation in the following form:

$$V(k) = \max_{k'}(((1-\delta)k + f(k) - k')G^{\eta})^{1-\gamma}/(1-\gamma) + \beta V(k') \tag{10}$$

By taking first order conditions and applying the envelope theorem, we get the following:

$$\begin{aligned}\beta V'(k') &= ((1-\delta)k + f(k) - k')^{-\gamma} G^{\eta(1-\gamma)} \\ V'(k') &= ((1-\delta)k' + f(k') - k'')^{-\gamma} (G')^{\eta(1-\gamma)} ((1-\delta) + f'(k')) \\ \Rightarrow ((1-\delta)k + f(k) - k')^{-\gamma} G^{\eta(1-\gamma)} &= \beta((1-\delta)k' + f(k') - k'')^{-\gamma} (G')^{\eta(1-\gamma)} ((1-\delta) + f'(k'))\end{aligned}$$

$$\Rightarrow \left(\frac{c'}{c}\right)^{\gamma} = \left(\frac{G'}{G}\right)^{\eta(1-\gamma)} \beta((1-\delta) + f'(k')) \tag{11}$$

Equation (11) along with the identity $k' = (1-\delta)k + f(k) - c$ form our difference equations for k', c' (2 equations, 2 variables).

4.2 Part B

If government spending grows at a constant rate, g, then we can find steady state values $\bar{k}, \bar{c}$ from our difference equations:

$$\begin{aligned}1 &= g^{\eta(1-\gamma)} \beta((1-\delta) + f'(\bar{k})) \\ \Rightarrow \bar{k} &= f'^{-1}(g^{-\eta(1-\gamma)} \beta^{-1} - 1 + \delta) \\ \Rightarrow \bar{c} &= f(\bar{k}) - \delta \bar{k}.\end{aligned}$$

4.3 Part C

Our identity $k' = (1-\delta)k + f(k) - c$ shows that in the period where g increases, the level of capital was already predetermined in the period before so it will remain at its initial level. Our value for c then makes the initial adjustment, and as g increases then c'/c increases, so (assuming consumption is positive), consumption increases. The increase in consumption results in lower capital in the next period. This lower capital increases $f'(k)$, resulting in a further increase in consumption. Eventually, the system will approach its new steady state. The new steady state value for capital will be lower as an increase in g will result in a decrease in f'^{-1}, and therefore a decrease in $\bar{k}$. The lower value for the steady state of capital has an indeterminate effect on the steady state of consumption as it will increase the $f(\bar{k})$ term but decrease the $-\delta\bar{k}$ term.