Econometrics Exam Notesheet

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1 First few lectures

1.1 Lecture 1

- $P(A) = 1 - P(A^c)$
- $P(A) \leq 1$
- If $A \subseteq B$ then $P(A) \leq P(B)$
- Boole's inequality: $P(A \cup B) \leq P(A) + P(B)$
- Bonferroni's inequality: $P(A \cap B) \geq P(A) + P(B) - 1$
- $P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A)} = \frac{P(A|B)P(B)}{P(A|B)P(B)+P(A|B^c)P(B^c)}$
- A, B are independent if $P(A \cap B) = P(A)P(B)$. If $P(A) > 0$ this implies $P(B) = P(B|A)$.
- A group of events are jointly independent if for any subset $J \subseteq \{1, \ldots, k\}$, $P(\cap_{j\in J} A_j) = \prod_{j\in J} A_j$.

1.2 Lecture 2

- $\lim_{x\to\infty} F(x) = 1, \lim_{x\to-\infty} F(x) = 0$; F is non-decreasing; F is right-cts.
- $F_Y(y) = P(g(X) \leq y) = P(X \leq g^{-1}(y) = F_X(g^{-1}(y))$ if g is strictly increasing. Differentiate to find the pdf. If decreasing then the inequality sign flips and to flip back you get $F_Y(y) = 1 - F_X(h(y))$

Let X have PDF $f_X(x), Y = g(X)$, where g is a monotone function. Suppose that $f_X(X)$ is continuous on X and that $g^{-1}(y)$ has a continuous derivative on Y. Then the PDF of Y is given by:

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)|\frac{d}{dy}g^{-1}(y)| \\ 0, \text{else} \end{cases}$$

- $E(X) = \sum_{x\in X} f_X x$ or $\int_{-\infty}^{\infty} x f_X(x) dx$
- Expectations are linear.
- $M_X(t) = E[exp(tX)]$
- $\frac{d^m}{dt^m} M(t)|_{t=0} = E(X^m)$

1.3 Lecture 3

- $f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$
- $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,v)dv$
- $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y}{f_X(x)}$
- A, B independent if $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ or $F_{X,Y}(x,y) = F_X(x)F_Y(y)$ or $f_{X,Y}(x,y) = f_X(x)f_Y(y)$
- X, Y independent then $E(g(X)h(Y)) = E(g(X))E(h(Y))$
- $E[Y|X=x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy$
- $E[Y|X=x]\frac{\int_{-\infty}^{\infty} y f_{X,Y}(x,y)dy}{\int_{-\infty}^{\infty} f_{X,Y}(x,y)dy}$
- $E(E[Y|X]) = E(Y)$
- Cauchy-Schwarz: $E|XY| \leq \sqrt{E(X^2)E(Y^2)}$
- $Var(Y) = E[Var(Y|X)] + Var(E(Y|X))$
- $Cov(X,Y) = E((X-EX)(Y-EY)) = E(X(Y-EY)) = E(XY) - EXEY$
- $Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$
- A variance-covariance matrix is symmetric and positive semi-definite.
- If g is one to one and $Y = g(X)$ then $f_Y(y) = f_X(g^{-1})(y))|J|$
- A matrix is psd if its eigenvalues are nonnegative and nsd if its eigenvalues are nonpositive.

2 New stuff

2.1 Lecture 4

- An estimator of θ is unbiased if $E(\hat{\theta}) = \theta$.
- Jensen's inequality: If X is a random variable and f is convex then $f(E[X]) \leq E[f(X)]$
- $Var(\bar{X}_n) = \sigma_X^2/n$
- $s^2 = \frac{n}{n-1}\sum_i (X_i - \bar{X}_n)^2$ is an unbiased estimator of the variance.
- t statistic: $t = \sqrt{n}(\bar{X}_n - \mu)/s$
- $I_n - n^{-1}1_n1_n'$ is idempotent.

2.2 Lecture 5

- A sequence of random variables converges in probability to Z as $n \to \infty$ if $\forall \epsilon > 0$ we have $\lim_{n\to\infty} P(|Z_n - Z| \geq \epsilon) = 0$. Notated $Z_n \to_p Z$ as $n \to \infty$
- WLLN: $\bar{X}_n \to_p \mu$ as $n \to \infty$.
- If an estimator $\hat{\theta}_n$ for θ converges in probability to θ then $\hat{\theta}_n$ is consistent for θ.

- Markov's inequality: $P(|X| \geq \lambda) \leq \frac{E(|X|)}{\lambda}$.
- Chebychev's inequality: $P(|X - \mu| \geq \lambda) \leq \frac{Var(X)}{\lambda^2}$
- CMT: If $Z_n \to_p z$ as $n \to \infty$ and g is continuous then $g(Z_n) \to_p g(z)$ as $n \to \infty$.
- A sequence of random variables converges in distribution to Z if $P(Z_n \leq x) \to P(Z \leq x)$
- CLT: If X_i iid with $E(X_i) = \mu, Var(X_i) \to_d N(0, \sigma^2)$ (multivariate version uses covariance matrix instead of σ^2).
- Delta Method: If $\sqrt{n}(\hat{\theta}_n - \theta) \to_d N(0, \sigma^2)$ and g is continuously differentiable in an open neighborhood of θ. Then $\sqrt{n}(g(\hat{\theta}_n) - g(\theta)) \to_d N(0, V)$ where $V = (g'(\theta))^2\sigma^2$.
- mulitvariate: $V = H(\theta)\Sigma H(\theta)'$ where $H(\theta) = \frac{\partial}{\partial \theta'} h(\theta) = \begin{pmatrix} \frac{\partial h_1(\theta)}{\partial \theta_1} & \dots & \frac{\partial h_1(\theta)}{\partial \theta_n} \\ \dots & \dots & \dots \\ \frac{\partial h_n(\theta)}{\partial \theta_1} & \dots & \frac{\partial h_n(\theta)}{\partial \theta_n} \end{pmatrix}$

2.3 Lecture 6

- Find MLE: write down log likelihood ($f(x|\theta)$) and take FOC, and check second order conditions to ensure negative!
- $S = \frac{\partial}{\partial \theta} log(f(X|\theta))$.
- $I_0 = E[SS'] = -E\left[\frac{\partial^2}{\partial \theta \partial \theta'} log f(X|\theta)|_{\theta=\theta_0}\right]$
- Note: for intuition note that we have log likelihood and $log''(x) = (\frac{1}{x})' = -\frac{1}{x^2}$ so $log''(x) = -(log'(x))^2$.
- CRLB: $Var(\hat{\theta}_n) \geq (nI_0)^{-1}$ and CR efficient is when this holds with equality.
- So long as taylor approx error does not matter in distribution (typically the case), $\sqrt{n}(\hat{\theta}_{n,MLE} - \theta) \to_d N(0, I_0^{-1})$