Econometrics HW2

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1 Question 1

Suppose that $Y = X^3$ and $f_X(x) = 42x^5(1-x), x \in (0,1)$. We are asked to find the PDF of Y.

We will begin by finding the CDF of Y:

$$P(Y \leq y) = P(X^3 \leq y) = P(X \leq y^{1/3}) = \int_0^{y^{1/3}} f_X(x)dx = \int_0^{y^{1/3}} 42x^5 - 42x^6 dx$$
$$= (7x^6 - 6x^7)|_0^{y^{1/3}} = (7y^2 - 6y^{7/3}) - 0 = 7y^2 - 6y^{7/3}.$$

Thus, $F_Y(y) = 7y^2 - 6y^{7/3}$. $f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}7y^2 - 6y^{7/3} = 14y - \frac{42}{3}y^{4/3}$.

We will check that this integrates to 1: $\int_{-\infty}^{\infty} f_Y(y)dy = \int_0^1 f_Y(y)dy = F_Y(y)|_0^1 = (7-6) - (0) = 1$.

2 Question 2

Let $x \in [0,1]$ and define F_X, f_X, a as described in the problem. We then have 3 cases:

- $x < 0.5$: In this case, $\int_0^x f_X(t)dt = \int_0^x 1.2dx = 1.2x$.
- $x = 0.5$: In this case, $\int_0^x f_X(t)dt = \int_0^{0.5} 1.2dt + \int_{0.5}^{0.5} adt = 0.6 + 0 = 0.2 + 0.8(x)$.
- $x > 0.5$:In this case, $\int_0^x f_X(t)dt = \int_0^{0.5} 1.2dt + \int_{0.5}^{0.5} adt + \int_{0.5}^{x} 0.8dt = 0.6 + 0 + 0.8x - 0.4 = 0.2 + 0.8(x)$.

Thus, $F_X(x) = \int_0^x f_X(t)dt \ \forall x \in [0,1]$.

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*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

3 Question 3

We will begin by finding the CDF of Y. $P(Y \leq y) = P(X^2 \leq y) = P(|X| \leq \sqrt{y}) = P(-\sqrt{y} \leq X \leq \sqrt{y})$. Note: Y is weakly positive. Also, $Y \leq 4$ because $|X| \leq 2$.

For $y \in [0,1]$,

$$\begin{aligned} P(-\sqrt{y} \leq X \leq \sqrt{y}) &= \int_{-\sqrt{y}}^{\sqrt{y}} (2/9)(x+1)dx = ((1/9)x^2 + (2/9)x)|_{-\sqrt{y}}^{\sqrt{y}} \\ &= ((1/9)y + (2/9)\sqrt{y}) - ((1/9)y - (2/9)\sqrt{y}) = (4/9)\sqrt{y} \\ \Rightarrow f_Y(y) &= \frac{d}{dy}(4/9)\sqrt{y} = (2/9)y^{-1/2}. \end{aligned}$$

For $y \in (1,4]$,

$$\begin{aligned} P(-\sqrt{y} \leq X \leq \sqrt{y}) &= \int_{-1}^{\sqrt{y}} (2/9)(x+1)dx = ((1/9)x^2 + (2/9)x)|_{-1}^{\sqrt{y}} \\ &= ((1/9)y + (2/9)\sqrt{y}) - ((1/9) - (2/9)) = (1/9)y + (2/9)\sqrt{y} + (1/9) \\ \Rightarrow f_Y(y) &= \frac{d}{dy}((1/9)y + (2/9)\sqrt{y} + (1/9)) = (1/9) + (1/9)y^{-1/2} \end{aligned}$$

4 Question 4

We will find the median of the given distribution.

$$\begin{aligned} P(X \leq m) &= \int_{-\infty}^{m} \frac{1}{\pi(1+x^2)}dx = \frac{1}{\pi}(\tan^{-1}x|_{\infty}^{m}) = \frac{1}{\pi}\left(\tan^{-1}(m) - \tan^{-1}(\infty)\right) = \frac{1}{\pi}\left(\tan^{-1}(m) + \frac{\pi}{2}\right) \\ \Rightarrow P(X \leq m) &= 0.5 \to \frac{1}{\pi}\left(\tan^{-1}(m) + \frac{\pi}{2}\right) = 0.5 \Rightarrow m = \tan\left(\frac{\pi}{2} - \frac{\pi}{2}\right) \\ &\Rightarrow m = 0. \end{aligned}$$

5 Question 5

We will begin by finding $E|X-a|, a \in \mathbb{R}$. We have that $E|X-a| = \int_{-\infty}^{\infty} |t-a| f_X(t)dt = \int_{-\infty}^{a}(a-t)f_X(t)dt + \int_{a}^{\infty}(t-a)f_X(t)dt$. Taking the derivative with respect to a, $\frac{d}{da}E|X-a| = ((a-t)f_X(t)|_a) + \int_{-\infty}^{a} f_X(t)dt + ((t-a)f_X(t)|_a) - \int_{a}^{\infty} f_X(t)dt = \int_{-\infty}^{a} f_X(t)dt - \int_{a}^{\infty} f_X(t)dt$. At the minimum, the derivitave with respect to a is 0 so $\int_{-\infty}^{a} f_X(t)dt = \int_{a}^{\infty} f_X(t)dt \Rightarrow P(X \leq a) = P(X \geq a) = 0.5$ so $\min_a E|X-a| = E|X-m|$ where m is the median of X..

Taking the derivative with respect to a, $\frac{d}{da}E|X-a| = 2F(a) + 2af(a) - 1 + \frac{d}{da}(-E[X|X < a] + E[X|X \geq a]) = 2F(a) + 2af(a) - 1 + (-xf(x)|_a + xf(x)|_a) = 2F(a) + 2af(a) - 1$.

6 Question 6

6.1 Show that if a density function is symmetric about a point a, then $\alpha_3 = 0$.

Let X be a random variable, with a density function symmetric about point a. Define $Y = X - a$, a random variable. Notice that $E[Y^3] = E[(-Y)^3]$ by the symmetry of the distribution of X. This implies that $E[Y^3] = 0$. Also, by symmetry, $E[X] = a$ so $\mu_3 = E(X - E[X])^3 = 0$. Thus $\alpha_3 = 0$.

6.2 Calculate α_3 for $f(x) = e^{-x}, x \geq 0$

By the chain rule, $E[X] = \int_0^\infty te^{-t}dt = -te^{-t} - e^{-t}|_0^\infty = 1$.

$E(X^2) = \int_0^\infty t^2e^{-x}dt = (-t^2e^t)|_0^\infty + 2\int_0^\infty te^{-t}dt = 2$. Thus, $\mu_2 = E(X^2) - E(X)^2 = 2 - 1 = 1$.

$E(X - E(X))^3 = \int_0^\infty (t-1)^3e^{-x}dt = \int_0^\infty (t^3 - 3t^2 + 3t - 1)e^{-x}dt$
$= \int_0^\infty t^3e^{-x}dt - 3\int_0^\infty t^2e^{-x}dt + 3\int_0^\infty te^{-x}dt - \int_0^\infty e^{-x}dt = \int_0^\infty t^3e^{-x}dt - 3(2) + 3(1) - (1)$
$= (-t^3e^{-t})|_0^\infty + 3\int_0^\infty t^2e^{-t}dt - 4 = 0 + 3(2) - 4 = 2 = \mu_3$.

Thus, $\alpha_3 = \frac{2}{1^{3/2}} = 2$.

6.3 Calculate α_4 for the listed densities and comment on the peakedness of the distributions.

- From lecture we have the moment generating function of $f(x)$: $M(t) = e^{t^2/2}$ and several derivatives including $M''(t) = e^{t^2/2} + t^2e^{t^2/2}, M'''(t) = 3te^{t^2/2} + t^3e^{t^2/2}$. We take a fourth derivative of M: $M''''(t) = 3e^{t^2/2} + 3t^2e^{t^2/2} + 3t^2e^{t^2/2} + t^4e^{t^2/2} = 3e^{t^2/2} + 6t^2e^{t^2/2} + t^4e^{t^2/2}$. Thus, $M''''(0) = 3, M''(0) = 1 \Rightarrow \alpha_4 = 3$.

- By symmetry, $E[X] = 0$. $E(X^2) = \int_{-1}^1 t^2/2dt = (t^3/6)|_{-1}^1 = (1/6) - (-1/6) = 1/3$ so the second central moment of the distribution is $1/3$. Because the mean of the distribution is 0, $E(X - E[X])^4 = E[X^4] = \int_{-1}^1 t^4/2dt = t^5/10|_{-1}^1 = 1/5 \Rightarrow \alpha_4 = \frac{1/5}{1/3} = \frac{3}{5}$. This distribution is, therefore, less peaked than the standard normal distribution, which is intuitive.

- By symmetry $E[X] = 0 \Rightarrow E(X - E[X])^k = E(X^k)$. $E(X^2) = \int_{-\infty}^\infty t^2(1/2)e^{-|t|}dt = 2\int_0^\infty t^2(1/2)e^{-t}dt = \int_0^\infty t^2e^{-t}dt$. We calculated this quantity in the previous subsection to be 2. $E(X^4) = \int_{-\infty}^\infty t^4(1/2)e^{-|t|}dt = \int_0^\infty t^4e^{-t}dt = ((-t^3e^{-t})|_0^\infty) + (1/3)\int_0^\infty t^3e^{-t}dt = 0 + (1/3)(6) = 2$. Thus, $\alpha_4 = \frac{2}{2} = 1$. Thus, this distribution is less peaked than the standard normal distribution, but more peaked than the uniform distribution.