Macro PS1

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1 Question 1

1.1 Part A

$$V(A_t, c_{t-1}) = \max_{c_t, A_{t+1}} u(c_t, c_{t-1}) + \beta V(A_{t+1}, c_t)$$
$$\text{s.t. } A_{t+1} = R(A_t - c_t)$$

We can rewrite this as the following:

$$V(A_t, c_{t-1}) = \max_{c_t} u(c_t, c_{t-1}) + \beta V(R(A_t - c_t), c_t) \tag{1}$$

To ensure the solution is unique, with a strictly increasing, strictly concave value function that is differentiable on the interior of the feasible set, we require the following assumptions:

- $u(.)$ is continuously differentiable, strictly concave, and strictly increasing in its arguments.

- $0 < \beta < 1$

- $\lim_{k \to 0} u'(k, u) = \lim_{k \to 0} u'(u, k) = \infty$

- $\lim_{k \to \infty} u'(k, u) = \lim_{k \to \infty} u'(u, k) = 0$

- The utility function is bounded?

- Do we need anything else?

Assuming the above conditions hold, we can solve the value function as it is written in (1) by taking a first order condition with respect to c_t, and then applying the envelope theorem twice:

$$0 = u_1(c', c) + \beta(-RV_1(RA - Rc', c') + V_2(RA - Rc', c')) \tag{2}$$
$$V_1(A, c) = R\beta V_1(RA - Rc', c') \tag{3}$$
$$V_2(A, c) = u_2(c', c) \tag{4}$$

Next, we can substitute in the envelope conditions (3), (4) into our first order condition (2) to find an expression for V_1, and substitute back into our initial first order condition (2):

$$\Rightarrow 0 = u_1(c', c) + \beta\left(-\frac{V_1(A, c)}{\beta} + u_2(c'', c')\right)$$

$$\Rightarrow V_1(A, c) = u_1(c', c) + \beta(c'', c') \tag{5}$$

We can now combine equations (2), (4), (5) to yield the following:

$$0 = u_1(c', c) + \beta(-R(u_1(c'', c') + \beta u_2(c''', c'')) + u_2(c'', c'))$$

$$\Rightarrow 0 = u_1(c', c) - \beta R u_1(c'', c') - \beta^2 R u_2(c''', c'') + \beta u_2(c'', c') \tag{6}$$

Equation (6) yields our optimality condition.

1.2   Part B

With the utility function as given, our value function becomes the following:
$$V(A, c) = \max_{c'} log(c') + \gamma log(c) + \beta V(RA - Rc', c')$$

The optimal choice c' takes the form of the arg max of the optimization problem.

$$c' = \arg\max_{c'} log(c') + \gamma log(c) + \beta V(RA - Rc', c')$$
$$= \arg\max_{c'} log(c') + \beta V(RA - Rc', c')$$

This arg max is independent of c.

Now we will solve for our optimality conditions. Applying (6) to our new utility function yields the following:

$$\beta R(c'')^{-1} + \gamma \beta^2 R(c'')^{-1} = (c')^{-1} + \gamma\beta(c')^{-1}$$
$$\Rightarrow c'\beta R(1 + \gamma\beta) = c''(1 + \gamma\beta)$$

$$\Rightarrow c'\beta R = c'' \tag{7}$$

Equation (7) yields our Euler conditions. This implies that the optimal saving policy is to save a current fraction of current assets.

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1.3 Part C

No, in general this will not hold. The utility function given to us was a seperable utility function, and for a non-separable utility function the utility

2 Question 2

2.1 Part A

The sequence problem is to maximize profits:

$$\max_{\{x_t\}_{i=1}^{\infty}} \sum_{i=1}^{\infty} \left(\frac{1}{1+r}\right)^t \left(ax_t - \frac{b}{2}x_t^2 - \frac{c}{2}(x_{t+1} - x_t)^2\right)$$

We can rewrite this as a bellman equation in the following way:

$$V(x) = \max_{y} ax - \frac{b}{2}x^2 - \frac{c}{2}(y - x)^2 + \delta V(y) \tag{8}$$

We can rewrite this as follows:

$$T(v)(x) = \max_{y} ax - \frac{b}{2}x^2 - \frac{c}{2}(y - x)^2 + \delta v(y) \tag{9}$$

where the fixed point of our T operator in (9) is the solution to the Bellman equation in (8).

2.2 Part B

Let $L < 0$ be arbitrary. If we set $y = 0, x < \frac{L}{a}$ then $F(x,y) = ax - \frac{b}{2}x^2 - \frac{c}{2}(y - x)^2 \leq ax < L$ so F is unbounded below.

This F function is of the form of a polynomial with limits of negative infinity in all directions, so we can find the global maximum by taking first order conditions:

$$0 = a - bx + c(y - x)$$
$$0 = -c(y - x) \Rightarrow y - x = 0 \Rightarrow y = x$$
$$\Rightarrow y = x = \frac{a}{b}$$
$$F\left(\frac{a}{b}, \frac{a}{b}\right) = a\left(\frac{a}{b}\right) - \frac{b}{2}\left(\frac{a}{b}\right)^2 - 0$$
$$= \frac{a^2}{2b}$$

Therefore, the maximum value F can take is $\frac{a^2}{2b}$

We can find bounds on $\hat{v}$ in the following way:

$$\hat{v} = \frac{a^2}{2b} + \delta\hat{v}$$

$$\Rightarrow \hat{v} = \frac{a^2}{2b(1-\delta)}$$

2.3 Part C

$$T\hat{v}(x) = \max_{y} ax - \frac{b}{2}x^2 - \frac{c}{2}(y-x)^2 + \delta\hat{v}$$

$$0 = -c(y-x) \Rightarrow y = x,$$

$$\Rightarrow T\hat{v}(x) = ax - \frac{b}{2}x^2 + \delta\hat{v}$$

$$\leq \frac{a^2}{2b} + \delta\frac{a^2}{2b(1-\delta)} = \frac{a^2}{2b(1-\delta)}$$

$$= \hat{v}.$$

2.4 Part D

We showed the base case of the induction in Part C. Now we will show the induction step.

Assume that $T^n\hat{v}(x)$ takes the form $T^n\hat{v}(x) = \alpha_n x - \frac{1}{2}\beta_n x^2 + \gamma_n$. Then,

$$T^{n+1}\hat{v}(x) = \max_{y} ax - \frac{b}{2}x^2 - \frac{c}{2}(y-x)^2 + \delta(\alpha_n x - \frac{1}{2}\beta_n x^2 + \gamma_n)$$

$$y = x \Rightarrow T^{n+1}\hat{v}(x) = ax - \frac{b}{2}x^2 + \delta\alpha_n x - \delta\frac{1}{2}\beta_n x^2 + \delta\gamma_n$$

$$= (a + \delta\alpha_n)x - \frac{b + \delta\beta_n}{2}x^2 + \delta\gamma_n$$

$$= \alpha_{n+1}x - \frac{1}{2}\beta_{n+1}x^2 + \gamma_{n+1}$$

where $\alpha_{n+1} = (a + \delta\alpha_n), \beta_{n+1} = b + \delta\beta_n, \gamma_{n+1} = \delta\gamma_n$.

2.5 Part E

Note that $\alpha_n = a + \delta a + \delta^2 a + \dots$, $\beta_n = b + \delta b + \delta^2 b + \dots$, $\gamma_n = \delta^n \hat{v}$. Thus, we can take the limit of α, β as geometric sums, and the limit of γ_n is 0. Therefore,

$$\tilde{V} = \lim_{n \to \infty} T^n \hat{v} = \frac{a}{1-\delta} x - \frac{1}{2} \frac{b}{1-\delta} x^2.$$

$$T\tilde{V} = \max_{y} ax - \frac{b}{2} x^2 - \frac{c}{2} (y-x)^2 + \delta \frac{a}{1-\delta} x - \frac{1}{2} \frac{b}{1-\delta} x^2,$$

$$y = x \Rightarrow T\tilde{V} = ax - \frac{b}{2} x^2 + \delta \frac{a}{1-\delta} x - \frac{1}{2} \frac{b}{1-\delta} x^2$$

$$= \frac{a}{1-\delta} x - \frac{1}{2} \frac{b}{1-\delta} x^2$$

$$= \tilde{V}.$$

Therefore, the limit function $\tilde{V}$ satisfies the Bellman equation.

3 Question 3

3.1 Part A

$$V(k) = \max_{k'} \pi(k) - \gamma(k' - (1-\delta)k) + \frac{1}{R} V(k')$$

$$\gamma'(k' - (1-\delta)k) = \frac{1}{R} \beta V'(k')$$

$$V'(k) = \pi'(k) + (1-\delta)\gamma'(k' - (1-\delta)k)$$

$$\Rightarrow \pi'(k') + (1-\delta)\gamma'(k'' - (1-\delta)k') = R\gamma'(k' - (1-\delta)k)$$

3.2 Part B

Letting $k = k' = k'' = \bar{k}$, we know that $\bar{I} = \delta\bar{k}$ and, moreover, we can rewrite our conditions for optimization in the following way:

$$\pi'(\bar{k}) + (1-\delta)\gamma'(\bar{I}) = R\gamma'(\bar{I}))$$

$$\Rightarrow \pi'(\bar{k}) = (R - 1 + \delta)\gamma'(\delta\bar{k})$$

$$\Rightarrow \frac{\pi'(\bar{k})}{\gamma'(\delta\bar{k})} = (R - 1 + \delta)$$

By the strict convexity of γ and strict concavity of π, in addition to our Inada conditions, the solution exists and is unique.

If R were to increase, the steady state level of $\pi'(\bar{k})$ would increase, resulting in a reduction in $\bar{k}$, and since $\bar{I} = \delta\bar{k}$, $\bar{I}$ will fall as well.

3.3 Part C

Our optimality conditions become:

$$-(k' - k^*) = R(k' - (1 - \delta)k) + (1 - \delta)(I')$$
$$\Rightarrow -(k' - k^*) = RI - (1 - \delta)I'$$

This, in addition to our law of motion for capital, $k' = I + (1 - \delta)k$, forms the law of motion for capital and investment.

If the interest rate were to rise, there would not be an initial change in the capital level as k' is already exactly determined by I, k. Investment spending would instead adjust and would have to increase.

4 Question 4

4.1 Part A

We will write our Bellman equation in the following form:

$$V(k) = \max_{k'}(((1 - \delta)k + f(k) - k')G^\eta)^{1-\gamma}/(1 - \gamma) + \beta V(k') \tag{10}$$

By taking first order conditions and applying the envelope theorem, we get the following:

$$\beta V'(k') = ((1 - \delta)k + f(k) - k')^{-\gamma}G^{\eta(1-\gamma)}$$
$$V'(k') = ((1 - \delta)k' + f(k') - k'')^{-\gamma}(G')^{\eta(1-\gamma)}((1 - \delta) + f'(k'))$$
$$\Rightarrow ((1 - \delta)k + f(k) - k')^{-\gamma}G^{\eta(1-\gamma)} = \beta((1 - \delta)k' + f(k') - k'')^{-\gamma}(G')^{\eta(1-\gamma)}((1 - \delta) + f'(k'))$$

$$\Rightarrow \left(\frac{c'}{c}\right)^\gamma = \left(\frac{G'}{G}\right)^{\eta(1-\gamma)} \beta((1 - \delta) + f'(k')) \tag{11}$$

Equation (11) along with the identity $k' = (1 - \delta)k + f(k) - c$ form our difference equations for k', c' (2 equations, 2 variables).

4.2 Part B

If government spending grows at a constant rate, g, then we can find steady state values $\bar{k}, \bar{c}$ from our difference equations:

$$1 = g^{\eta(1-\gamma)}\beta((1 - \delta) + f'(\bar{k}))$$
$$\Rightarrow \bar{k} = f'^{-1}(g^{-\eta(1-\gamma)}\beta^{-1} - 1 + \delta)$$
$$\Rightarrow \bar{c} = f(\bar{k}) - \delta\bar{k}.$$

4.3   Part C

Our identity $k' = (1-\delta)k + f(k) - c$ shows that in the period where g increases, the level of capital was already predetermined in the period before so it will remain at its initial level. Our value for c then makes the initial adjustment, and as g increases then c'/c increases, so (assuming consumption is positive), consumption increases. The increase in consumption results in lower capital in the next period. This lower capital increases $f'(k)$, resulting in a further increase in consumption. Eventually, the system will approach its new steady state. The new steady state value for capital will be lower as an increase in g will result in a decrease in f'^{-1}, and therefore a decrease in $\bar{k}$. The lower value for the steady state of capital has an indeterminate effect on the steady state of consumption as it will increase the $f(\bar{k})$ term but decrease the $-\delta\bar{k}$ term.