Econometrics HW4

Michael B. Nattinger*

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1 Question 1

The probability of defying is the following:

$$\begin{aligned} P(D) &= Pr(X = 0|Z = 1 \cap X = 1|Z = 1) \\ &= Pr(-U_0 + U_1 \leq 0 \cap -U_0 > 0) \end{aligned}$$

By observing the above, it is clear that $P(D) > 0 \iff P(U_0 < 0 \cap U_1 \leq U_0) > 0$. Therefore, if $P(U_0 < 0 \cap U_1 \leq U_0) = 0$, $P(D) = 0$.

The probability of complying is the following:

$$\begin{aligned} P(C) &= Pr(X = 1|Z = 1 \cap X = 0|Z = 0) \\ &= Pr(-U_0 + U_1 > 0 \cap -U_0 \leq 0) \end{aligned}$$

Again by observing the above, it is clear that $P(C) = 0 \iff P(U_0 \geq 0 \cap U_1 > U_0) = 0$. Therefore, if $P(U_0 \geq 0 \cap U_1 > U_0) > 0$, $P(C) > 0$.

We are askeed to find conditions on U such that $P(C) > 0$ and $P(D) = 0$. This is satisfied in a variety of ways. If $U_0 \geq 0$, the conditions are satisfied. Moreover, if $U_0 < 0$, the conditions are still satisfied so long as $U_1 > U_0$.

2 Question 2

2.1 Part i

For notational convenience define $\theta_0 = 1$.

$$\begin{aligned} \gamma(k) &= Cov(Y_t, Y_{t-k}) \\ &= Cov(\mu + \epsilon_t + \theta_1\epsilon_{t-1} + \cdots + \theta_q\epsilon_{t-q}, \mu + \epsilon_{t-k} + \theta_1\epsilon_{t-1-k} + \cdots + \theta_q\epsilon_{t-q-k}) \\ &= \begin{cases} 0, |k| > q \\ \sigma^2 \sum_{i=0}^{q-|k|} \theta_i\theta_{k+i}, q \geq |k| \end{cases} . \end{aligned}$$

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2.2 Part ii

$$\begin{aligned}\rho(k) &= \frac{\gamma(k)}{\gamma(0)} \\ &= \frac{\gamma(k)}{\sigma^2(1+\theta_1^2)} \\ &= \begin{cases} 0, |k| > 1 \\ 1, k = 0 \\ \frac{\theta_1}{(1+\theta_1^2)}, |k| = 1 \end{cases}\end{aligned}$$

2.3 Part iii

Notice that θ only appears in the functional form of $\rho(k)$ if $|k| = 1$. Note that $\rho(1) = \rho(-1)$. WLOG consider $\rho(1)$. Note that $\theta_1 = 0 \iff \rho(1) = 0$. Now, assume $\theta_1 \neq 0$, Then, $\rho(1) = \frac{\theta_1}{1+\theta_1^2}$ is a quadratic function with two solutions, or one solution if $\theta_1 \in \{-1, 1\}$. Note, in particular, that θ_1 and θ_1^{-1} yield the same value of $\rho(1)$, and are the two solutions. Therefore, in general θ_1 is not identified from $\rho(1)$.

2.4 Part iv

If we are restricted to $\theta_1 \in [-1, 1]$, then we can rule out one of our two solutions for θ_1, namely the solution with $|\theta_1| > 1$ and can exactly identify θ_1.

3 Question 3

3.1 Part A

It is sufficient to find μ, τ such that $E[Y_1] = E[Y_0], Var(Y_1) = Var(Y_0)$.

$$\begin{aligned}E[Y_0] &= E[\mu + \epsilon_0 + \nu] \\ &= \mu \\ E[Y_1] &= E[\alpha_0 + Y_0\rho + U_1] \\ &= \alpha_0 + \mu\rho \\ \Rightarrow \mu &= \alpha_0 + \mu\rho \\ \Rightarrow \mu &= \frac{\alpha_0}{1-\rho}. \\ Var(Y_0) &= Var(\mu + \epsilon_0 + \nu) \\ &= \sigma^2 + \tau \\ Var(Y_1) &= Var(\alpha_0 + Y_0\rho + U_1) \\ &= Var(\epsilon_0\rho + \nu\rho + \epsilon_1 + \theta\epsilon_0) \\ &= Var(\epsilon_0(\rho+\theta) + \nu\rho + \epsilon_1) \\ &= (\rho+\theta)^2\sigma^2 + \tau\rho^2 + \sigma^2 \\ \Rightarrow \sigma^2 + \tau &= (\rho+\theta)^2\sigma^2 + \tau\rho^2 + \sigma^2 \\ \Rightarrow \tau &= \frac{(\rho+\theta)^2\sigma^2}{1-\rho^2}.\end{aligned}$$

3.2 Part B

To be a valid instrument, the instrument must satisfy $E[U_t|Y_{t-2}] = 0, Cov(Y_{t-1}, Y_{t-2}) \neq 0$.

$$\begin{aligned}
E[U_t|Y_{t-2}] &= E[\epsilon_t + \theta\epsilon_{t-1}|Y_{t-2}] \\
&= E[\epsilon_t + \theta\epsilon_{t-1}|\nu, \epsilon_0, \ldots, \epsilon_{t-2}] \\
&= 0, \\
Cov(Y_{t-1}, Y_{t-2}) &= Cov(\alpha_0 + Y_{t-2}\rho + \epsilon_{t-1} + \theta\epsilon_{t-2}, Y_{t-2}) \\
&= \rho Var(Y_{t-2}) + \theta\sigma^2.
\end{aligned}$$

By the covariance stationarity of Y, $Var(Y_{t-2}) = Var(Y_0) = \sigma^2 + \tau$. Therefore,

$$\begin{aligned}
Cov(Y_{t-1}, Y_{t-2}) &= \rho(\sigma^2 + \tau) + \theta\sigma^2 \\
&= \sigma^2(\rho + \theta) + \rho\frac{(\rho + \theta)^2\sigma^2}{1 - \rho^2} \\
&= \sigma^2(\rho + \theta)\left(1 + \frac{\rho}{1 - \rho^2}(\rho + \theta)\right) \\
&= \sigma^2(\rho + \theta)\left(\frac{1 - \rho\theta}{1 - \rho^2}\right)
\end{aligned}$$

We are given that $|\rho| < 1, |\theta| < 1$ so $Cov(Y_{t-1}, Y_{t-2}) \neq 0 \iff \rho + \theta \neq 0$.