

# Macro PS4

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## 1 Question 1

The household maximizes their utility subject to their budget constraint. Equivalently, households minimize their costs subject to their utility constraint:

$$\begin{aligned} \min_{C_{ik}} \quad & \int \sum_i P_{ik} C_{ik} dk \\ \text{s.t.} \quad & \left( \int C_k^{\frac{\rho-1}{\rho}} dk \right)^{\frac{\rho}{\rho-1}} = C, \\ \text{where} \quad & \left( \sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} = C_k. \end{aligned}$$

We will then write the Lagrangian as follows:

$$\mathcal{L} = \int \sum_i P_{ik} C_{ik} dk - P \left( \left( \int C_k^{\frac{\rho-1}{\rho}} dk \right)^{\frac{\rho}{\rho-1}} - C \right) + \int P_k \left[ C_k - \left( \sum_i C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} \right] dk$$

We solve this maximization problem by taking first order conditions with respect to our choice variables, in this case  $C_{ik}, C_k$ :

$$\begin{aligned} P_k &= \frac{\rho}{\rho-1} \left( \int C_k^{\frac{\rho-1}{\rho}} dk \right)^{\frac{1}{\rho-1}} \frac{\rho-1}{\rho} C_k^{-\frac{1}{\rho}} \\ \Rightarrow C_k &= \left( \frac{P_k}{P} \right)^{\rho} C. \\ P_{ik} &= P_k \frac{\theta}{\theta-1} \left( \sum_{i=1}^{N_k} C_{ik}^{\frac{\theta-1}{\theta}} \right)^{\frac{1}{\theta-1}} \frac{\theta-1}{\theta} C_{ik}^{-\frac{1}{\theta}} \\ \Rightarrow C_{ik} &= \left( \frac{P_{ik}}{P_k} \right)^{\theta} C_k \end{aligned}$$

We then can simplify our consumption first order condition to the following:

$$\begin{aligned} P_k \left( \frac{C_{ik}}{C_k} \right)^{\frac{1}{\theta}} &= P C^{\frac{1}{\rho}} C_k^{-\frac{1}{\rho}} \left( \frac{C_{ik}}{C_k} \right)^{\frac{1}{\theta}} \\ \Rightarrow C_k &= \left( \frac{P_k}{P} \right)^{-\rho} C, \end{aligned}$$

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\*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, Katherine Kwok, and Danny Edgel.

a familiar expression to what we found in lecture.

We can substitute in our expressions into the definitions of  $C, C_k$ :

$$\begin{aligned}
\left( \int \left[ \left( \frac{P_k}{P} \right)^{-\rho} C \right]^{\frac{\rho-1}{\rho}} dk \right)^{\frac{\rho}{\rho-1}} &= C \\
\Rightarrow \left( \int \left( \frac{P_k}{P} \right)^{1-\rho} dk \right)^{\frac{\rho}{\rho-1}} &= 1 \\
\Rightarrow \left( \int P_k^{1-\rho} dk \right)^{\frac{1}{\rho-1}} &= P, \\
\left( \sum_i \left[ \left( \frac{P_{ik}}{P_k} \right)^{\theta} C_k \right]^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} &= C_k \\
\Rightarrow \left( \sum_i P_{ik}^{\theta-1} \right)^{\frac{1}{\theta-1}} &= P_k
\end{aligned}$$

To summarize, we have the following:

$$P_k = \left( \sum_i P_{ik}^{\theta-1} \right)^{\frac{1}{\theta-1}} \quad (1)$$

$$P = \left( \int \left[ \left( \sum_i P_{ik}^{\theta-1} \right)^{\frac{1}{\theta-1}} \right]^{1-\rho} dk \right)^{\frac{1}{\rho-1}} \quad (2)$$

$$C_{ik} = P_{ik}^{-\theta} \left( \sum_i P_{ik}^{\theta-1} \right)^{\frac{\theta-\rho}{\theta-1}} P^{\rho} C \quad (3)$$

## 2 Question 2

The firms compete a la Cournot:

$$\begin{aligned}
&\max_{P_{ik}} P_{ik} C_{ik} - W L_{ik} \\
&\text{s.t. } C_{ik} = P_{ik}^{-\theta} \left( \sum_i P_{ik}^{\theta-1} \right)^{\frac{\theta-\rho}{\theta-1}} P^{\rho} C \\
&\text{and } C_{ik} = A_{ik} L_{ik}
\end{aligned}$$

Substituting, we form the following objective function:

$$\begin{aligned}
&\max_{P_{ik}} P_{ik}^{1-\theta} \left( \sum_i P_{ik}^{\theta-1} \right)^{\frac{\theta-\rho}{\theta-1}} P^{\rho} C - W A_{ik}^{-\rho} P_{ik}^{-\theta} \left( \sum_i P_{ik}^{\theta-1} \right)^{\frac{\theta-\rho}{\theta-1}} P^{\rho} C \\
&\Rightarrow \max_{P_{ik}} P_{ik}^{1-\theta} \left( \sum_i P_{ik}^{\theta-1} \right)^{\frac{\theta-\rho}{\theta-1}} - W A_{ik}^{-\rho} P_{ik}^{-\theta} \left( \sum_i P_{ik}^{\theta-1} \right)^{\frac{\theta-\rho}{\theta-1}}
\end{aligned}$$

We take first order conditions:

$$\begin{aligned}
(1 - \theta)P_{ik}^{-\theta}P_k^{\theta-\rho} + \frac{\theta - \rho}{\theta - 1}P_{ik}^{1-\theta}P_k^{1-\rho} &= wA_{ik}^{-1} \left[ (-\theta)P_{ik}^{-\theta-1}P_k^{\theta-\rho} + \frac{\theta - \rho}{\theta - 1}P_{ik}^{-\theta}P_k^{1-\rho} \right] \\
(1 - \theta)P_k^\theta + \frac{\theta - \rho}{\theta - 1}P_{ik}P_k &= wA_{ik}^{-1} \left[ (-\theta)P_{ik}^{-1}P_k^\theta + \frac{\theta - \rho}{\theta - 1}P_k \right] \\
(1 - \theta)P_{ik} + \frac{\theta - \rho}{\theta - 1}P_{ik}^2P_k^{1-\theta} &= wA_{ik}^{-1} \left[ (-\theta) + \frac{\theta - \rho}{\theta - 1}P_k^{1-\theta}P_{ik} \right] \\
0 &= \frac{\theta - \rho}{\theta - 1}P_{ik}^2P_k^{1-\theta} + \left( 1 - \theta - wA_{ik}^{-1}\frac{\theta - \rho}{\theta - 1}P_k^{1-\theta} \right) P_{ik} + \theta wA_{ik}^{-1}
\end{aligned}$$

$$\Rightarrow P_{ik} = \frac{wA_{ik}^{-1}\frac{\theta - \rho}{\theta - 1}P_k^{1-\theta} + (\theta - 1) \pm \sqrt{\left( 1 - \theta - wA_{ik}^{-1}\frac{\theta - \rho}{\theta - 1}P_k^{1-\theta} \right)^2 - 4\frac{\theta - \rho}{\theta - 1}P_k^{1-\theta}\theta wA_{ik}^{-1}}}{2\frac{\theta - \rho}{\theta - 1}P_k^{1-\theta}}$$

3 Question 3

4 Question 4

5 Question 5

6 Question 6