Macro PS2

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1 Question 1

In this question we will model a 2-dimensional linear system, the Ramsey model of consumption and capital. We have the following:

$$k_{t+1} = zk_t^\alpha + (1-\delta)k_t - c_t \tag{1}$$

$$\frac{\beta}{c_{t+1}} = (c_t)^{-1}(1-\delta+\alpha z k_{t+1}^{\alpha-1})^{-1} \tag{2}$$

1.1 Solve for steady state $(\bar{k}, \bar{c})$

$$\begin{aligned}
\bar{k} &= z\bar{k}^\alpha + (1-\delta)\bar{k} - \bar{c} \\
\bar{c}/\beta &= \bar{c}(1-\delta+\alpha z\bar{k}^{\alpha-1}) \\
\Rightarrow \beta^{-1} &= 1-\delta+\alpha z\bar{k}^{\alpha-1} \\
\Rightarrow \bar{k} &= \left(\frac{\beta^{-1}-1+\delta}{\alpha z}\right)^{\frac{1}{\alpha-1}} \\
\Rightarrow \bar{c} &= z\left(\frac{\beta^{-1}-1+\delta}{\alpha z}\right)^{\frac{\alpha}{\alpha-1}} - \delta\left(\frac{\beta^{-1}-1+\delta}{\alpha z}\right)^{\frac{1}{\alpha-1}} \\
\Rightarrow \bar{k} &= 3.2690 \\
\Rightarrow \bar{c} &= 1.0998.
\end{aligned}$$

1.2 Linearize the system about its steady state

First we write $k_{t+1} = g(k_t, c_t), c_{t+1} = h(k_t, c_t)$. From (1) and (2) we have:

$$\begin{aligned}
k_{t+1} &= zk_t^\alpha + (1-\delta)k_t - c_t = g(k_t, c_t), \\
c_{t+1} &= \beta c_t(1-\delta+\alpha z(zk_t^\alpha + (1-\delta)k_t - c_t)^{\alpha-1}) = h(k_t, c_t).
\end{aligned}$$

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

Now, we can write down our Jacobian J:

$$
\begin{aligned}
J &= \begin{pmatrix} dk_{t+1}/dk_t & dk_{t+1}/dc_t \\ dc_{t+1}/dk_t & dc_{t+1}/dc_t \end{pmatrix} \\
&= \begin{pmatrix} \alpha z k_t^{\alpha-1} + (1-\delta) & -1 \\ (\beta c_t)(\alpha z(\alpha-1)(zk_t^{\alpha} + (1-\delta)k_t - c_t)^{\alpha-2})(\alpha z k_t^{\alpha-1} + (1-\delta)) & dc_{t+1}/dc_t \end{pmatrix}
\end{aligned}
$$

where $dc_{t+1}/dc_t = \beta(1-\delta+\alpha z(zk_t^{\alpha} + (1-\delta)k_t - c_t)^{\alpha-1}) - \beta c_t \alpha z(\alpha-1)(zk_t^{\alpha} + (1-\delta)k_t - c_t)^{\alpha-2}$.

Then, for $\tilde{x}_t := x_t - \bar{x}$, we can write our first-order taylor approximation to the system:

$$
\begin{pmatrix} \tilde{k_{t+1}} \\ \tilde{c_{t+1}} \end{pmatrix} = J \begin{pmatrix} \tilde{k}_t \\ \tilde{c}_t \end{pmatrix}.
$$

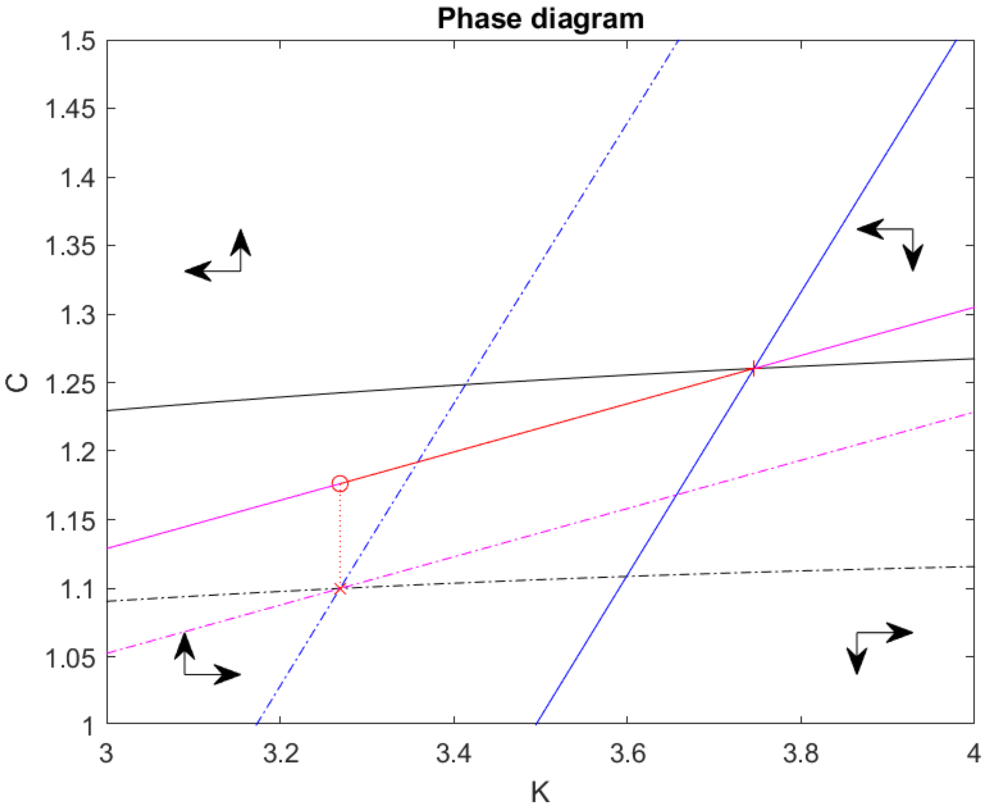
1.3 Compute numerically the eigenvalues and eigenvectors of the Jacobian at the SS. Verify that the system has a saddle path. Find the slope of the saddle path at the SS.

We will write $J = E\Lambda E^{-1}$ where E is the matrix of eigenvectors and Λ is the diagonal matrix of corresponding eigenvalues. From Matlab,

$$
J = \begin{pmatrix} 0.9850 & 0.9848 \\ -0.1725 & 0.1734 \end{pmatrix} \begin{pmatrix} 1.2060 & 0 \\ 0 & 0.8548 \end{pmatrix} \begin{pmatrix} 0.9850 & 0.9848 \\ -0.1725 & 0.1734 \end{pmatrix}^{-1}
$$

Since the magnitude of the first eigenvalue is greater than one, and the magnitude of the second eigenvalue is less than one, the system has a saddle path. The slope of the saddle path at the SS is equal to the slope of the second (i.e., nonexplosive) eigenvector, $\frac{0.1734}{0.9848} = 0.1761$.

1.4 Draw a phase diagram demonstrating how the system responds to an unexpected (permanent) productivity shock.



The above figure shows the phase diagram as computed via Matlab. The dot-dashed lines show the original, pre-shock, curves for $\Delta k = 0$ (black), $\Delta c = 0$ (blue), and the saddle path (magenta), while the solid lines show the same curves post shock. The system begins at its old steady state (the red x), and then after the shock the system must jump up in c to the new saddle path (jumps to the red o). Then the system will follow the saddle path, tracing out the red line, approaching the new steady state (red +) in the limit as $t \to \infty$. Note: the plotted saddle path is the saddle path implied by the taylor approximation. Vector field arrows are also drawn for completeness.

1.5 Compute numerically and plot trajectories of k_t, c_t if the productivity shock occurs at $t_0 = 5$ and $z^{'} = z + 0.1$.

We first will compute the new steady state values.
From Matlab, $\bar{k}^{'} = 3.7458, \bar{c}^{'} = 1.2602$; $J = \begin{pmatrix} 1.0309 & -1 \\ -0.0308 & 1.0299 \end{pmatrix}$.

Next, we will diagonalize the system using $J = E\Lambda E^{-1}, \hat{x} = E^{-1}x$:

$$\begin{aligned} \begin{pmatrix} \widetilde{k_{t+1}} \\ \widetilde{c_{t+1}} \end{pmatrix} &= J \begin{pmatrix} \tilde{k_t} \\ \tilde{c_t} \end{pmatrix} \\ \Rightarrow \begin{pmatrix} \widehat{k_{t+1}} \\ \widehat{c_{t+1}} \end{pmatrix} &= E^{-1}E\Lambda \begin{pmatrix} \hat{k_t} \\ \hat{c_t} \end{pmatrix} \\ \Rightarrow \begin{pmatrix} \widehat{k_{t+1}} \\ \widehat{c_{t+1}} \end{pmatrix} &= \Lambda \begin{pmatrix} \hat{k_t} \\ \hat{c_t} \end{pmatrix}. \end{aligned}$$

Next, we will write down non-explosive solution for $(\hat{k_t}, \hat{c_t})$, and then re-write in terms of the original variables (k_t, c_t).

$$\begin{aligned} \widehat{k_{t+1}} &= \lambda_1 \hat{k_t} = 1.2060\hat{k_t} \\ \widehat{c_{t+1}} &= \lambda_2 \hat{c_t} = 0.8548\hat{c_t} \\ \Rightarrow \hat{k_t} &= c_1\lambda_1^t, \hat{c_t} = c_2\lambda_2^t. \end{aligned}$$

Our non-explosive solution must have $c_1 = 0$. Re-writing in terms of our original variables,

$$\begin{aligned} k_t &= e_{1,2}c_2\lambda_2^t \\ c_t &= e_{2,2}c_2\lambda_2^t \\ \Rightarrow k_t^g &= e_{1,2}c_2\lambda_2^t + \bar{k}' \\ \Rightarrow c_t^g &= e_{2,2}c_2\lambda_2^t + \bar{c}'. \end{aligned}$$

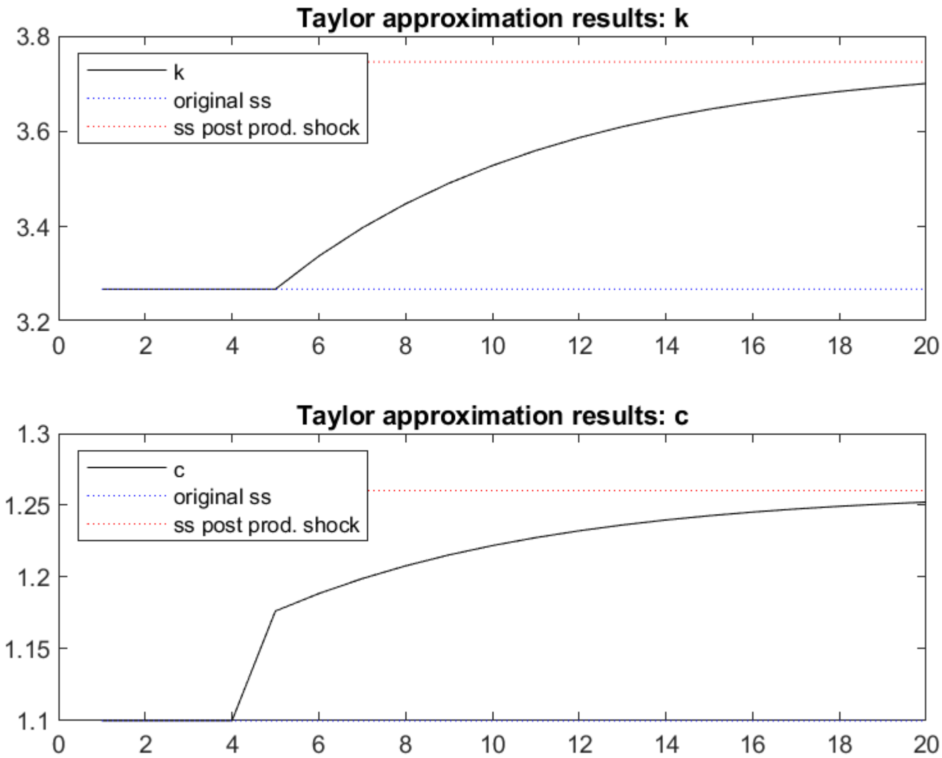
Note that we have 2 boundary conditions but only one constant to solve for. We will use our boundary condition for k and solve for an implied initial value of c. We have the following:

$$\begin{aligned} k_{t_0} &= 3.2690 = 0.9848c_2(0.8548)^5 + 3.7458 \Rightarrow c_2 = -1.0607 \\ c_{t_0} &= 0.1734(-1.0607)(0.8548)^5 + 1.2602 = 1.1762. \end{aligned}$$

We can now write our general solution:

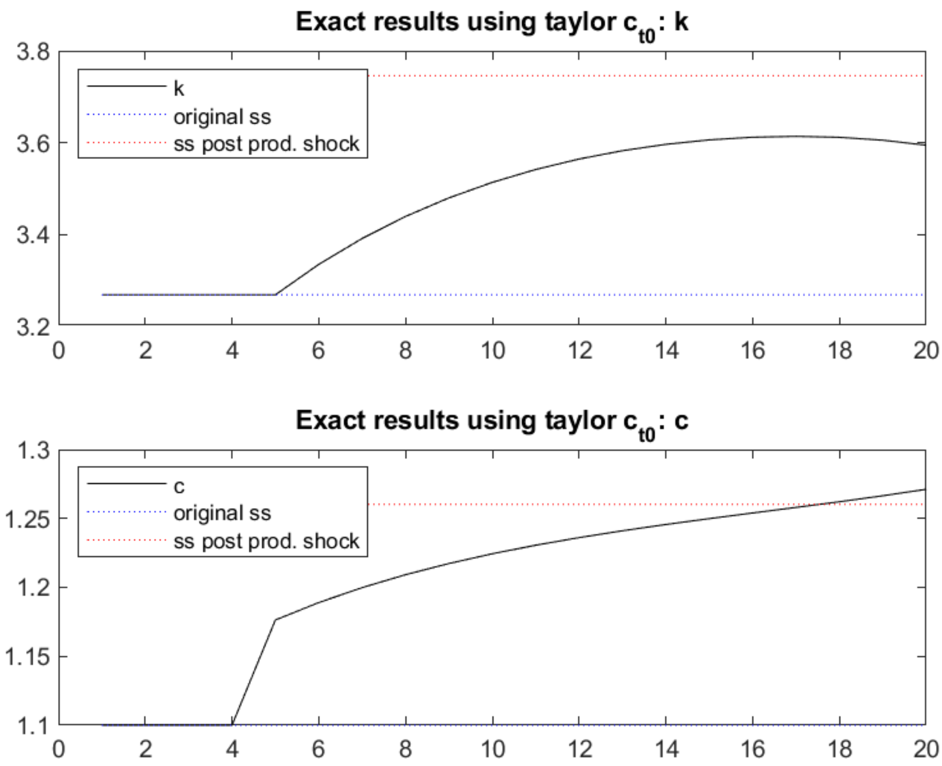
$$\begin{aligned} k_t^g &= e_{1,2}c_2\lambda_2^t + \bar{k} = -1.0447(0.8548)^t + \bar{k}' \\ c_t^g &= e_{2,2}c_2\lambda_2^t + \bar{c} = -0.1840(0.8548)^t + \bar{c}'. \end{aligned}$$

We will now use our particular solution to compute and plot k_t, c_t.

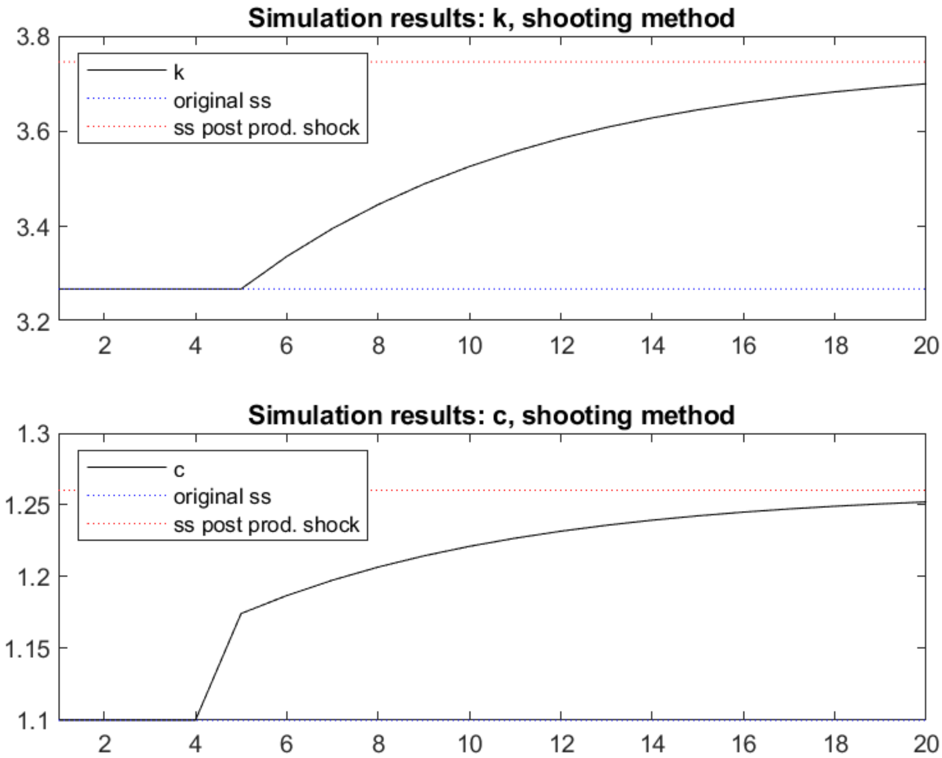


1.6 Numerically solve the actual transition path using the "shooting method".

If we put (k_{t_0}, c_{t_0}) into the nonlinear system (1) and (2), we will show that the system does not converge to a steady state:



Now we will instead use the shooting method to find the actual c_{t_0} needed to converge to the steady state.



2 Question 2

We are given three scenarios. For each scenario, we will state the SPP, CP, and CE.

2.1 Modified 2-period OG model

In this question we consider a model quite similar to the 2-period OG model we studied in class, but with a few differences as described in the problem set. Since N is constant, WLOG we can set $N = 1$.[1] Note that in this question we have two types of consumers. The first are given an allocation in their first period, and the second are given an allocation in the second period. We will number these consumers with the identifiers $1, 2$, respectively.

SPP: The social planner maximizes the utilities of the agents given the resource

[1] This is without loss of generality because we can divide our equations through by N whenever N would show up, and N will disappear from the equations, yielding a result equivalent to setting $N = 1$.

constraint:

$$\max_{c_t^{1,t}, c_t^{1,t-1}, c_t^{2,t}, c_t^{2,t-1}} \ln c_t^{1,t} + \ln c_t^{1,t-1} + \ln c_t^{2,t} + \ln c_t^{2,t-1}$$
$$\text{s.t. } c_t^{1,t} + c_t^{1,t-1} + c_t^{2,t} + c_t^{2,t-1} \leq \frac{1}{2}w_1 + \frac{1}{2}w_2.$$

CP: Each consumer maximizes their own utility over the two periods, subject to their income constraints. Assuming some full commitment technology, we will define risk-free bonds $B_{t+1}^{2,t}$ which can be borrowed by the second type of agent in period t and repaid in period $t+1$. We similarly define $B_{t+1}^{1,t}$ as the bonds leant by the first agent, which are repaid to that agent in the second period. Each bond costs q_t to buy or sell, and the bonds are repaid as one unit of the consumption good. The first type of consumer solves the following maximization problem:

$$\max_{c_t^{1,t}, c_{t+1}^{1,t}, M_{t+1}^{1,t}, D_{1,t+1}^t} \ln c_t^{1,t} + \ln c_{t+1}^{1,t}$$
$$\text{s.t. } p_t c_t^{1,t} + M_{t+1}^{1,t} + q_t D_{1,t+1}^t \leq p_t w_1$$
$$\text{and } p_{t+1} c_{t+1}^{1,t} \leq M_{t+1}^{1,t} + p_{t+1} D_{1,t+1}^t$$

The next type of consumer solves the following:

$$\max_{c_t^{2,t}, c_{t+1}^{2,t}, D_{t+1}^{2,t}} \ln c_t^{2,t} + \ln c_{t+1}^{2,t}$$
$$\text{s.t. } p_t c_t^{2,t} \leq q_t D_{2,t+1}^t$$
$$\text{and } p_{t+1} c_{t+1}^{2,t} + p_{t+1} D_{2,t+1}^t \leq p_{t+1} w_2.$$

We also have the special case of the initial period. The old in this first period only live for a single period. Those who are given an allocation in this period solve the following maximization problem:

$$\max_{c_1^{2,0}, M_1^{2,0}} \ln c_1^{2,0}$$
$$\text{s.t. } p_1 c_1^{2,0} \leq p_1 w_2 + M.$$

Meanwhile, the initial old with no resources solve the following maximization problem:

$$\max_{c_1^0, M_1^{1,0}} \ln c_1^{1,0}$$
$$\text{s.t. } p_1 c_1^{1,0} \leq M.$$

CE: For the competitive equilibrium, prices will adjust so that markets clear, in the goods market, money market, and bond market:

$$c_t^{1,t} + c_t^{1,t-1} + c_t^{2,t} + c_t^{2,t-1} = \frac{1}{2}w_1 + \frac{1}{2}w_2$$
$$M_{t+1}^{1,t} = M$$
$$D_{t+1}^{1,t} = D_{t+1}^{2,t}.$$

2.2 3-period OG model

We consider an OG model consisting of agents that live 3 periods. This time, generation sizes are no longer necessarily fixed.

SPP: The social planner maximizes the utilities of the agents given the resource constraint:

$$\begin{aligned}
&\max_{c_t^t, c_t^{t-1}, c_t^{t-1}} N_t \ln c_t^t + N_{t-1} \ln c_t^{t-1} + N_{t-2} \ln c_t^{t-2} \\
&\text{s.t. } N_t c_t^t + N_{t-1} c_t^{t-1} + N_{t-2} c_t^{t-2} \leq N_t w_1 + N_{t-1} w_2 + N_{t-2} w_3.
\end{aligned}$$

CP: Outside of the initialization, agents solve the following maximization problem:

$$\begin{aligned}
&\max_{c_t^t, c_{t+1}^t, c_{t+2}^t, M_{t+1}^t, M_{t+2}^t} \ln c_t^t + \ln c_{t+1}^t + \ln c_{t+2}^t \\
&\text{s.t. } p_t c_t^t + M_{t+1}^t \leq p_t w_1 \\
&\text{and } p_{t+1} c_{t+1}^t + M_{t+2}^t \leq p_{t+1} w_2 + M_{t+1}^t \\
&\text{and } p_{t+2} c_{t+2}^t \leq p_{t+2} w_3 + M_{t+2}^t.
\end{aligned}$$

We next must consider the initial case. There are 2 'special' generations (i.e. generations which live less than 3 periods). First we will consider the 2-period generation. They solve the following maximization problem:

$$\begin{aligned}
&\max_{c_1^0, c_2^0, M_2^0} \ln c_1^0 + \ln c_2^0 \\
&\text{s.t. } p_1 c_1^0 + M_2^0 \leq w_2 \\
&\text{and } p_2 c_2^0 \leq p_2 w_3 + M_2^0.
\end{aligned}$$

Finally, we also consider the 1-period generation. They solve the following:

$$\begin{aligned}
&\max_{c_1^{-1}} \ln c_1^{-1} \\
&\text{s.t. } p_1 c_1^{-1} \leq p_1 w_3 + 1.
\end{aligned}$$

CE: For the competitive equilibtium, prices will adjust so that markets clear, both in the goods market and money market:

$$\begin{aligned}
N_t c_t^t + N_{t-1} c_t^{t-1} + N_{t-2} c_t^{t-2} &= N_t w_1 + N_{t-1} w_2 + N_{t-2} w_3. \\
N_{t-1} M_t^{t-1} + N_{t-2} M_t^{t-2} &= N_{-1}.
\end{aligned}$$

2.3 Cake eating problem

We now consider a single agent, given an allocation in period 1, with perfect storage. In this case, the SPP, CP, and CE all collapse to a single optimization problem.

The single agent solves the following maximization problem:

$$\begin{aligned}
&\max_{c_i, i\in\mathbb{N}} \sum_{t=1}^{\infty} \beta^t u(c_t)\\
&\text{s.t. } \sum_{t=1}^{\infty} c_t = k_1\\
&\text{and } c_i \geq 0 \forall i \in \mathbb{N}.
\end{aligned}$$