Macro PS2

Michael B. Nattinger*

February 2, 2021

1 Question 1

The planner solves the following maximization problem subject to the capital law of motion and the resource constraint:

$$\max_{\{C_t,I_t,K_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t logC_t$$
$$\text{s.t. } K_{t+1} = K_t^{1-\delta} I_t^{\delta}$$
$$\text{and } AK_t^{\alpha} = C_t + I_t$$

We can solve the resource constraint for I_t and plug it into the capital law of motion. Using this simplification, we can write down our Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t logC_t + \lambda_t \left(-K_{t+1} + K_t^{1-\delta} \left(AK_t^{\alpha} - C_t\right)^{\delta}\right)$$

Taking first order conditions with respect to C_t, K_{t+1} we find the following:

$$\frac{\beta^t}{C_t} = \lambda_t K_t^{1-\delta}(AK_t^{\alpha} - C_t)^{\delta-1}$$
$$\lambda_t = \lambda_{t+1}(K_{t+1}^{1-\delta}\delta(AK_{t+1}^{\alpha} - C_{t+1})^{\delta-1} A\alpha K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-\delta}\left(AK_{t+1}^{\alpha} - C_{t+1}\right)^{\delta})$$
$$\Rightarrow \lambda_t = \frac{\beta^t}{C_t K_t^{1-\delta} I_t^{\delta-1}}$$
$$\Rightarrow \frac{1}{C_t K_t^{1-\delta} I_t^{\delta-1}} = \frac{\beta}{C_{t+1} K_{t+1}^{1-\delta} I_{t+1}^{\delta-1}}(A\alpha\delta K_{t+1}^{\alpha-\delta} I_{t+1}^{\delta-1} + (1-\delta)K_{t+1}^{-\delta} I_{t+1}^{\delta})$$

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, Katherine Kwok, and Danny Edgel.

$$\frac{1}{C_t K_t^{1-\delta} I_t^{\delta-1}} = \frac{\beta}{C_{t+1}} (A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta) K_{t+1}^{-1} I_{t+1}) \tag{1}$$

The above equation forms our Euler equation.

Assume we are on the optimal trajectory at time t, and consider a one-period deviation in consumption by an amount Δ. Our resource constraint implies that this results in a decrease in I_t by an equal amount, Δ. Then, our K_{t+1} is reduced (to first order approximation) to $K_t^{1-\delta} I_t^{\delta} - \delta\Delta K_t^{1-\delta} I_t^{\delta-1}$. Then, our consumption in the second equation is reduced by two effects: reduced K_{t+1} leads to less production at time $t+1$, and a larger gap to make up via I_{t+1} to get back onto the optimal trajectory at time $t+2$. The net effect of the first of these terms, to first order expansion, is $-(\delta\Delta K_t^{1-\delta} I_t^{\delta-1})(A\alpha K_{t+1}^{\alpha-1})$, in other words, the reduction in C_{t+1} from the (first order approximation of the) decrease in production in period (t+1). Now we must address the second of these turns. $K_{t+2} = K_{t+1}^{1-\delta} I_{t+1}^{\delta}$ is fixed and we know the value of K_{t+1} so we can determine the value of I_{t+1}.

Our marginal utility by making this move is thus

$$\begin{aligned} dU &= \beta^t C_t^{-1} \Delta - \beta^{t+1} C_{t+1}^{-1} \left(\right) \Delta \\ dU/\Delta = 0 \Rightarrow C_t^{-1} &= \beta C_{t+1}^{-1} (\delta\Delta K_t^{1-\delta} I_t^{\delta-1}) \end{aligned}$$