Micro HW3

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1 Question 1

1.1 Part A

$$u_i(w_i, w_j) = \gamma w_i - \beta(w_i - w_j)^2 - \rho w_i - \alpha w_i^2/2 - \alpha w_j w_i/2$$
$$\Rightarrow \frac{\partial u}{\partial w_i}(0,0) = \gamma - \rho > 0$$

The assumption guarantees that the marginal utility from $w_i > 0$ at $0, 0$, so the gangs will set $w_i > 0$.

1.2 Part B

The game is supermodular if the cross partials are (weakly) positive:

$$\frac{\partial^2 u}{\partial w_j \partial w_i} = 2\beta - \alpha/2 \geq 0$$
$$\Rightarrow \beta \geq \alpha/4.$$

1.3 Part C

From our parametric assumption we know $w_i > 0$ so we can take FOCs to optimize:

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, Ryan Mather, and Tyler Welch. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

$$
\begin{aligned}
0 &= \gamma - 2\beta(w_i - w_j) - \rho - \alpha w_i - \alpha w_j/2 \\
\Rightarrow w_i &= \frac{(\gamma-\rho) + (2\beta - \alpha/2)w_j}{2\beta+\alpha} \\
\Rightarrow w_i &= \frac{(\gamma-\rho) + (2\beta - \alpha/2)\frac{(\gamma-\rho)+(2\beta-\alpha/2)w_i}{2\beta+\alpha}}{2\beta+\alpha} \\
\Rightarrow (2\beta+\alpha)^2 w_i &= (2\beta+\alpha+2\beta-\alpha/2)(\gamma-\rho) + (2\beta-\alpha/2)^2 w_i \\
\Rightarrow (6\alpha\beta + (3/4)\alpha^2)w_i &= (4\beta+\alpha/2)(\gamma-\rho) \\
\Rightarrow w_i &= \left(\frac{4\beta+\alpha/2}{6\alpha\beta+(3/4)\alpha^2}\right)(\gamma-\rho) \\
\Rightarrow w_i &= \frac{2(\gamma-\rho)}{3\alpha}
\end{aligned}
$$

1.4 Part D

- ρ is the base value of the price of weapons, i.e. the y intercept of the weapon supply equation. Higher values of ρ result in directly higher prices of weapons, resulting in lower equilibrium quantities of weapons demanded.
- γ is the direct positive impact (for the gang) of buying a weapon. Perhaps an interpretation of it would be expected revenue per weapon owned. Higher values of γ increases the equilibrium quantity of weapons demanded.
- β is the disutility from having a different number of weapons than the other gang. In equilibrium this has no effect as, due to symmetry, the gangs will own the same number of guns.
- α is the slope of the weapon supply curve. Higher values of α result in price responding more to an increase in weapon demand. Therefore, an increase in α reduces the amount of weapons demanded.

1.5 Part E

With w_j as given, the best response of firm i satisfies the following condition:

$$0 = \gamma - 2\beta(w_i - w_j) - \rho - \alpha w_i - \alpha w_j/2$$

We will try to find a value of w_j which will result in the optimal w_i being 0:

$$
\begin{aligned}
0 &= \gamma + 2\beta w_j - \rho - \alpha w_j/2 \\
\Rightarrow w_j &= -\frac{\gamma-\rho}{2\beta-\alpha/2}.
\end{aligned}
$$

We know that $\gamma > \rho$ so this quantity is positive iff $2\beta < \alpha/2 \Rightarrow \beta < \alpha/4$. Note that this is the opposite condition from what we found to satisfy supermodularity. If the game is not supermodular then there exists equilibria where $w_i = 0$.

1.6 Part F

Can the game support a mixed strategy nash equilibrium?

1.7 Part G

Gang 1's weapon demand will increase as the weapons are more valueable now. This increase will be larger in magnitude than it would be if both gangs were able to respond because, if gang 2 could also buy weapons then they would buy more weapons which would increase the price of weapons. When the second gang cannot buy weapons, the weapons are relatively cheaper for gang 1 so gang 1 will buy more weapons.

2 Question 2

Each gang takes the average as given and sets w_i to satisfy their FOC:

$$\begin{aligned} 0 &= \gamma - 2\beta(w_i - \bar{w}) - \rho - \alpha\bar{w} \\ \Rightarrow 2\beta w_i &= (2\beta - \alpha)\bar{w} + (\gamma - \rho) \end{aligned}$$

In the symmetric equilibrium, $\bar{w} = w_i \forall i$, so:

$$w_i = \frac{(\gamma - \rho)}{\alpha}$$

We get a different equilibrium in this case because no individual gang has any ability to affect the average quantity. The $\gamma - \rho$ term is the same, and the part that changes is that α is less important when choosing prices because additional purchasing of weapons does not increase the average quantity of guns purchased, and thus does not affect the price of weapons through the α term in the same way that it does in part C.

3 Question 3

3.1 Part A

There exist 3 pure strategy Nash equilibria: $x_i = 0, x_i = 1, x_i = \alpha$. If all other agents choose $x_i = 0$, then choosing any number in $(0, \alpha)$ will result in strictly lower utility as the $(x_i - \alpha)^2$ term will decrease and the $-(x_i - \bar{x})^2$ term will decrease. Moreover, choosing any number in $[\alpha, 1]$ will also result in strictly lower utility as the $(x_i - \alpha)^2$ term will increase by less than the $-(x_i - \bar{x})^2$ term will decrease. Therefore, $x_i = 0$ is a pure strategy nash equilibrium. Parallel logic shows that $x_i = 1$ is also a pure strategy nash equilibrium.

If everyone chooses α then any individual would be indifferent from following the nash equilibrium and deviating. Therefore a pure strategy nash equilibrium exists

3.2 Part B

Any distribution such that the average is at α is a nash equilibria as the disutility of being far away from the mean is equal to the utility gained by being close to the average, so the agents would be indifferent between staying with the equilibria or moving anywhere else.

We can also show that no value of $\bar{x} \in (0,1) \setminus \{\alpha\}$ would result in a nash equilibrium. For purpose of contradiction, say there exists a distributional nash equilibrium with $\bar{x} \in (0, \alpha)$. Then, there exists either some $x_i \geq \bar{x}$. If $x_i > \bar{x}$ then individual i would receive more utility by choosing $\bar{x}$. Similarly, if $x_i = \bar{x}$ then individual i would receive higher utility by setting $x_i = 0$. Thus, in either case individual i is better off by deviating, which is a contradiction. By parallel logic, there cannot be a nash equilibrium with $\bar{x} \in (\alpha, 1)$. Therefore, the nash equilibria are any distributions with $\bar{x} = \alpha$, and the symmetric nash equilibria at $\bar{x} = 1$, $\bar{x} = 0$.

3.3 Part C

If agents can choose any $x \in \mathbb{R}$ then nash equilibria would take the form of any distribution such that the average was at α. If that is the case, then the disutility of moving away from the mean value would exactly offset the utility of moving away from α for any individual agent, so the agent would be indifferent between staying with the equilibria or moving. If instead the mean is below α, then any agent could receive unboundedly more utility by moving to the left of the number line than they lose by moving away from the mean, and similarly if the mean is above α then they can receive unboundedly more utility by moving to the right of the number line than they lose utility from moving away from the mean.

4 Question 4

First we will check for symmetric nash equilibria. We start by taking first order conditions:

$$\begin{aligned} 1 + (q_j - 1)^{1/3} &= q_i \\ 1 + (1 + (q_i - 1)^{1/3} - 1)^{1/3} &= q_i \\ &\Rightarrow q_i = 1, q_i = 2. \end{aligned}$$

Thus, nash equilibria can only occur at the following choices: $(1, 1), (2, 2), (1, 2), (2, 1)$. We can check the utility at these points to see which form our nash equilibria:

$$u_i(1, 1) = 1 + 1(0) - 1/2 = 1/2$$
$$u_i(2, 1) = 2 + 2(0) - 1/2(4) = 0$$
$$u_i(1, 2) = 1 + 1(1) - 1/2 = 3/2$$
$$u_i(2, 2) = 2 + 2(1) - 1/2(4) = 2$$

Therefore, the nash equilibria are $(1, 1)$ and $(2, 2)$.

5 Question 5

There are no pure strategy nash equilibria. We can show this easily. Assume a pure strategy nash equilibria exists. Then, it is not the case that both players are receiving 10 utility. Therefore, at least one of the players would be better off by changing strategies to whatever would yield them 10 utility. This is a contradiction, so no pure strategy nash equilibria exist.

Assume a nash equilibrium exists where both player 1 and player 2 are mixing. Then, as the game is zero loss, with an extra cost of mixing for both players, at least one of the players must have negative utility. Then, that player would be better off by choosing a pure strategy where they purely play the strategy which wins against one of the strategies which recieves the largest weight from the other player's mixture. Therefore, a mixed strategy is not a best response to a mixed strategy.

Moreover, assume player 1 is playing a pure strategy. Then, if player 2 were to play a pure strategy that wins against player 1, they would receive 10 expected utility. If they instead were to mix, they would receive less than 9 expected utility. Therefore, the best response to a pure strategy is a pure strategy.

The above arguments show that a pure strategy is a best response to a mixed strategy, and that a pure strategy is a best response to a mixed strategy. Therefore, for any nash equilibria that were to exist, the equilibria could only contain pure strategies. However, we have also shown that no pure strategy nash equilibria can exist. Therefore, no nash equilibria can exist.