Econometrics HW4

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September 28, 2020

1 Question 1

1.1 Show that $\bar{X}_{n+1} = (n\bar{X}_n + X_{n+1})/(n+1)$.

$$\begin{aligned}
\bar{X}_{n+1} &= \frac{1}{n+1}\sum_{i=1}^{n+1} X_i = \frac{1}{n+1}\left(\left(\sum_{i=1}^{n} X_i\right) + X_{n+1}\right) \\
&= \frac{1}{n+1}\left(n\bar{X}_n + X_{n+1}\right).
\end{aligned}$$

1.2 Show that $s_{n+1}^2 = ((n-1)s_n^2 + (n/(n+1))(X_{n+1} - \bar{X}_n)^2)/n$.

$$\begin{aligned}
s_{n+1}^2 &= \frac{1}{n}\sum_{i=1}^{n+1}(X_i - \bar{X}_{n+1})^2 = \frac{1}{n}\sum_{i=1}^{n+1}(X_i - \bar{X}_n + \bar{X}_n + \bar{X}_{n+1})^2 \\
&= \frac{1}{n}\left((n-1)\sum_{i=1}^{n+1}(X_i - \bar{X}_{n+1})^2 + n(\bar{X}_n - \bar{X}_{n+1}) + 2(\sum_{i=1}^{n}(X_j - \bar{X}_n)(\bar{X}_n - \bar{X}_{n+1})) + (X_{n+1} - \bar{X}_{n+1})\right) \\
&= \frac{1}{n}\left((n-1)\sum_{i=1}^{n+1}(X_i - \bar{X}_{n+1})^2 + n(\bar{X}_n - \bar{X}_{n+1})^2 + (X_{n+1} - \bar{X}_{n+1})^2\right) \\
&= \frac{1}{n}\left((n-1)s_n^2 + n\bar{X}_n^2 - 2n\bar{X}_n\bar{X}_{n+1} + n\bar{X}_{n+1}^2 + X_{n+1}^2 - 2X_{n+1}\bar{X}_{n+1} + \bar{X}_{n+1}^2\right) \\
&= \frac{1}{n}\left((n-1)s_n^2 + \frac{n}{n+1}(X_{n+1} - \bar{X}_n)^2\right).
\end{aligned}$$

2 Question 2

Define $\hat{\mu}_k := \frac{1}{n}\sum_{i=1}^{n} X_i^k$. We will show that this is unbiased.

$$\begin{aligned}
E[\hat{\mu}_k] &= E\left[\frac{1}{n}\sum_{i=1}^{n} X_i^k\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i^k] \\
&= \frac{1}{n}\sum_{i=1}^{n}\mu_k \\
&= \mu_k.
\end{aligned}$$

Thus, $\hat{\mu}_k$ is an unbiased estimator for μ_k.

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

3 Question 3

Define $\hat{M}_k := \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^k$. This estimator is biased, which can be seen from the fact that $\hat{M}_2 = \hat{\sigma}^2 \neq s_n^2$, so $\hat{M}_2$ is not an unbiased estimator for $M_2 = \sigma^2$. There exists no general formula for an unbiased estimator of M_k.

4 Question 4

$$\begin{aligned} E[(\hat{\mu}_k - E[\hat{\mu}_k])^2] &= E\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i^k\right)^2\right] - E\left[\frac{1}{n}\sum_{i=1}^{n} X_i^k\right]^2 \\ &= \frac{1}{n^2}E\left[\left(\sum_{i=1}^{n} X_i^k\right)^2\right] - \mu_k^2 \end{aligned}$$