Macro PS2

Michael B. Nattinger*

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1 Question 1

In this question we will model a 2-dimensional linear system, the Ramsey model of consumption and capital. We have the following:

$$k_{t+1} = zk_t^\alpha + (1-\delta)k_t - c_t \tag{1}$$

$$\frac{\beta}{c_{t+1}} = (c_t)^{-1}(1-\delta+\alpha z k_{t+1}^{\alpha-1})^{-1} \tag{2}$$

1.1 Solve for steady state $(\bar{k}, \bar{c})$

$$\begin{aligned}
\bar{k} &= z\bar{k}^\alpha + (1-\delta)\bar{k} - \bar{c} \\
\bar{c}/\beta &= \bar{c}(1-\delta+\alpha z\bar{k}^{\alpha-1}) \\
\Rightarrow \beta^{-1} &= 1-\delta+\alpha z\bar{k}^{\alpha-1} \\
\Rightarrow \bar{k} &= \left(\frac{\beta^{-1}-1+\delta}{\alpha z}\right)^{\frac{1}{\alpha-1}} \\
\Rightarrow \bar{c} &= z\left(\frac{\beta^{-1}-1+\delta}{\alpha z}\right)^{\frac{\alpha}{\alpha-1}} - \delta\left(\frac{\beta^{-1}-1+\delta}{\alpha z}\right)^{\frac{1}{\alpha-1}} \\
\Rightarrow \bar{k} &= 3.2690 \\
\Rightarrow \bar{c} &= 1.0998.
\end{aligned}$$

1.2 Linearize the system about its steady state

First we write $k_{t+1} = g(k_t, c_t), c_{t+1} = h(k_t, c_t)$. From (1) and (2) we have:

$$\begin{aligned}
k_{t+1} &= zk_t^\alpha + (1-\delta)k_t - c_t = g(k_t, c_t), \\
c_{t+1} &= \beta c_t(1-\delta+\alpha z(zk_t^\alpha + (1-\delta)k_t - c_t)^{\alpha-1}).
\end{aligned}$$

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

Now, we can write down our Jacobian J:

$$
\begin{aligned}
J &= \begin{pmatrix} dk_{t+1}/dk_t & dk_{t+1}/dc_t \\ dc_{t+1}/dk_t & dc_{t+1}/dc_t \end{pmatrix} \\
&= \begin{pmatrix} \alpha a k_t^{\alpha-1} + (1-\delta) & -1 \\ (\beta c_t)(\alpha z(\alpha-1)(zk_t^{\alpha} + (1-\delta)k_t - c_t)^{\alpha-2})(\alpha z k_t^{\alpha-1} + (1-\delta)) & dc_{t+1}/dc_t \end{pmatrix}
\end{aligned}
$$

where $dc_{t+1}/dc_t = \beta(1-\delta+\alpha z(zk_t^{\alpha}+(1-\delta)k_t - c_t)^{\alpha-1}) - \beta c_t \alpha z(\alpha-1)(zk_t^{\alpha}+(1-\delta)k_t - c_t)^{\alpha-2}$.

Then, for $\tilde{x}_t := x_t - \bar{x}$, we can write our first-order taylor approximation to the system:

$$
\begin{pmatrix} \tilde{k_{t+1}} \\ \tilde{c_{t+1}} \end{pmatrix} = J \begin{pmatrix} \tilde{k_t} \\ \tilde{c}_t \end{pmatrix}.
$$

1.3 Compute numerically the eigenvalues and eigenvectors of the Jacobian at the SS. Verify that the system has a saddle path. Find the slope of the saddle path at the SS.

We will write $J = E\Lambda E^{-1}$ where E is the matrix of eigenvectors and Λ is the diagonal matrix of corresponding eigenvalues. From Matlab,

$$
J = \begin{pmatrix} 0.9850 & 0.9848 \\ -0.1725 & 0.17634 \end{pmatrix} \begin{pmatrix} 1.2060 & 0 \\ 0 & 0.8548 \end{pmatrix} \begin{pmatrix} 0.9850 & 0.9848 \\ -0.1725 & 0.17634 \end{pmatrix}^{-1}
$$

Since the magnitude of the first eigenvalue is greater than one, and the magnitude of the second eigenvalue is less than one, the system has a saddle path. The slope of the saddle path at the SS is equal to the slope of the second eigenvector, $\frac{0.17634}{0.9848} = 0.1761$.

1.4 Draw a phase diagram demonstrating how the system responds to an unexpected (permanent) productivity shock.

1.5 Compute numerically and plot trajectories of k_t, c_t if the productivity shock occurs at $t_0 = 5$ and $z^{'} = z + 0.1$.

We first will compute the new steady state values.
From Matlab, $\bar{k}^{'} = 3.7458, \bar{c}^{'} = 1.2602$; $J = \begin{pmatrix} 1.0309 & -1 \\ -0.0308 & 1.0299 \end{pmatrix}$.
Next, we will diagonalize the system using $J = E\Lambda E^{-1}, \hat{x} = E^{-1}x$:

$$
\begin{aligned}
\begin{pmatrix} \tilde{k_{t+1}} \\ \tilde{c_{t+1}} \end{pmatrix} &= J \begin{pmatrix} \tilde{k_t} \\ \tilde{c_t} \end{pmatrix} \\
\Rightarrow \begin{pmatrix} \hat{k_{t+1}} \\ \hat{c_{t+1}} \end{pmatrix} &= E^{-1}E\Lambda \begin{pmatrix} \hat{k_t} \\ \hat{c_t} \end{pmatrix} \\
\Rightarrow \begin{pmatrix} \hat{k_{t+1}} \\ \hat{c_{t+1}} \end{pmatrix} &= \Lambda \begin{pmatrix} \hat{k_t} \\ \hat{c_t} \end{pmatrix}.
\end{aligned}
$$

Next, we will write down non-explosive solution for $(\hat{k_t}, \hat{c_t})$, and then re-write in terms of the original variables (k_t, c_t).

$$\begin{aligned}
\hat{k_{t+1}} &= \lambda_1 \hat{k_t} = 1.2060\hat{k_t} \\
\hat{c_{t+1}} &= \lambda_2 \hat{c_t} = 0.8548\hat{c_t} \\
\Rightarrow \hat{k_t} &= c_1\lambda_1^t, \hat{c_t} = c_2\lambda_2^t.
\end{aligned}$$

Our non-explosive solution must have $c_1 = 0$. Re-writing in terms of our original variables,

$$\begin{aligned}
k_t &= e_{1,2}c_2\lambda_2^t \\
c_t &= e_{2,2}c_2\lambda_2^t \\
\Rightarrow k_t^g &= e_{1,2}c_2\lambda_2^t + \bar{k} \\
\Rightarrow c_t^g &= e_{2,2}c_2\lambda_2^t + \bar{c}.
\end{aligned}$$

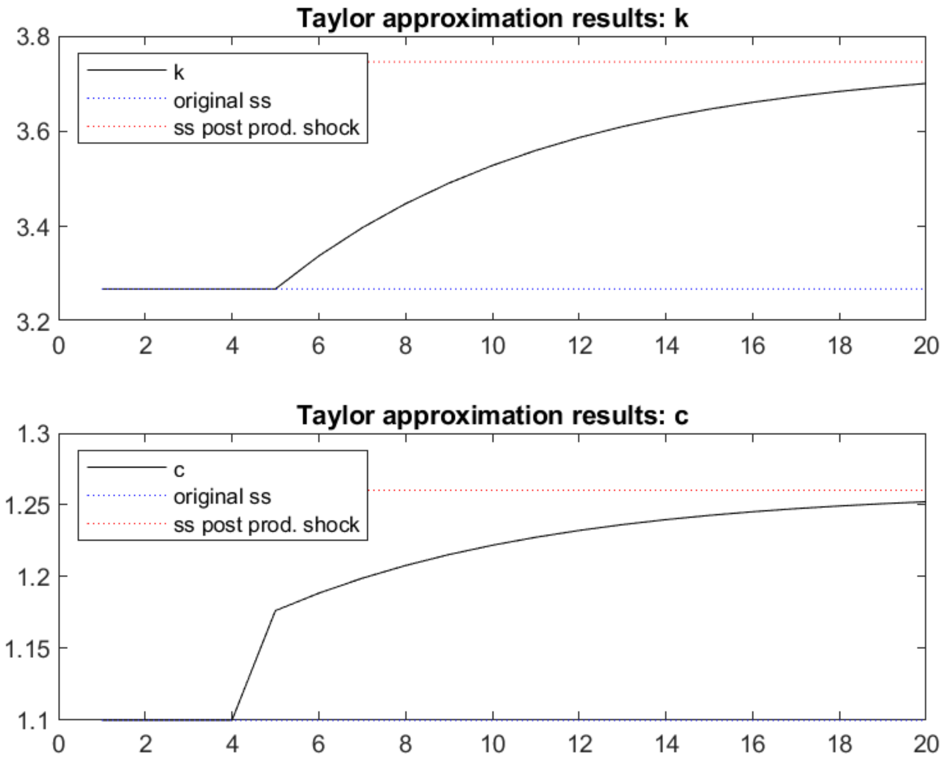
Note that we have 2 boundary conditions but only one constant to solve for. We will use our boundary condition for k and solve for an implied initial value of c. We have the following:

$$\begin{aligned}
k_{t_0} &= 3.2690 = 0.9848c_2(0.8548)^5 + 3.7458 \Rightarrow c_2 = -1.0607 \\
c_{t_0} &= 0.1734(-1.0607)(0.8548)^5 + 1.2602 = 1.1762.
\end{aligned}$$

We can now write our general solution:

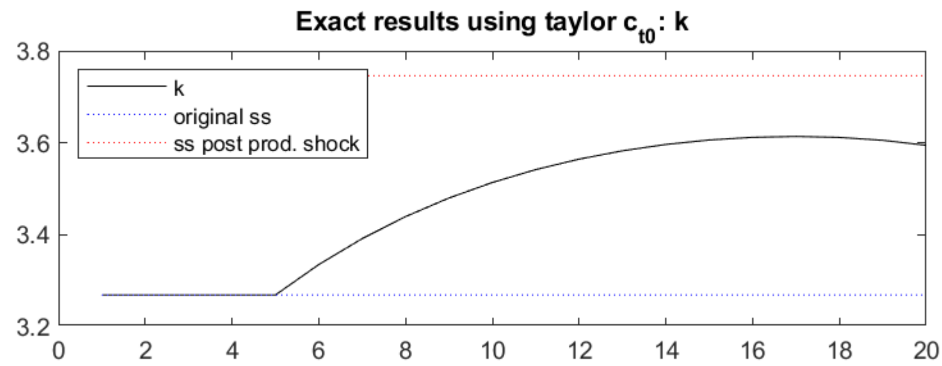
$$\begin{aligned}
k_t^g &= e_{1,2}c_2\lambda_2^t + \bar{k} = -1.0447(0.8548)^t + \bar{k} \\
c_t^g &= e_{2,2}c_2\lambda_2^t + \bar{c} = -0.1840(0.8548)^t + \bar{c}.
\end{aligned}$$

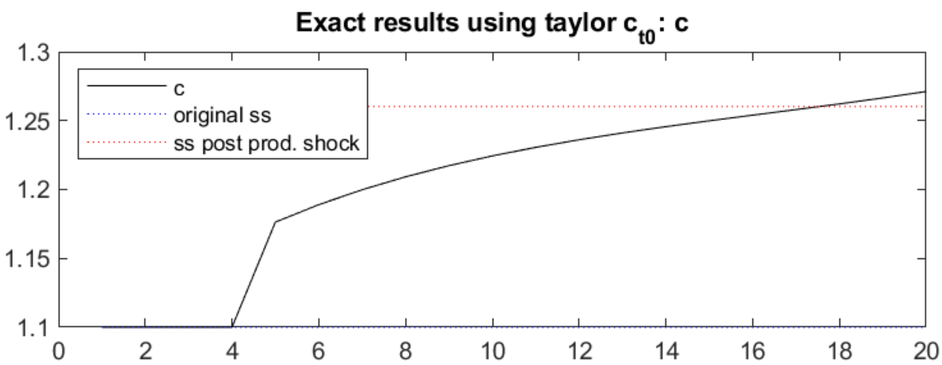
We will now use our particular solution to compute and plot k_t, c_t.



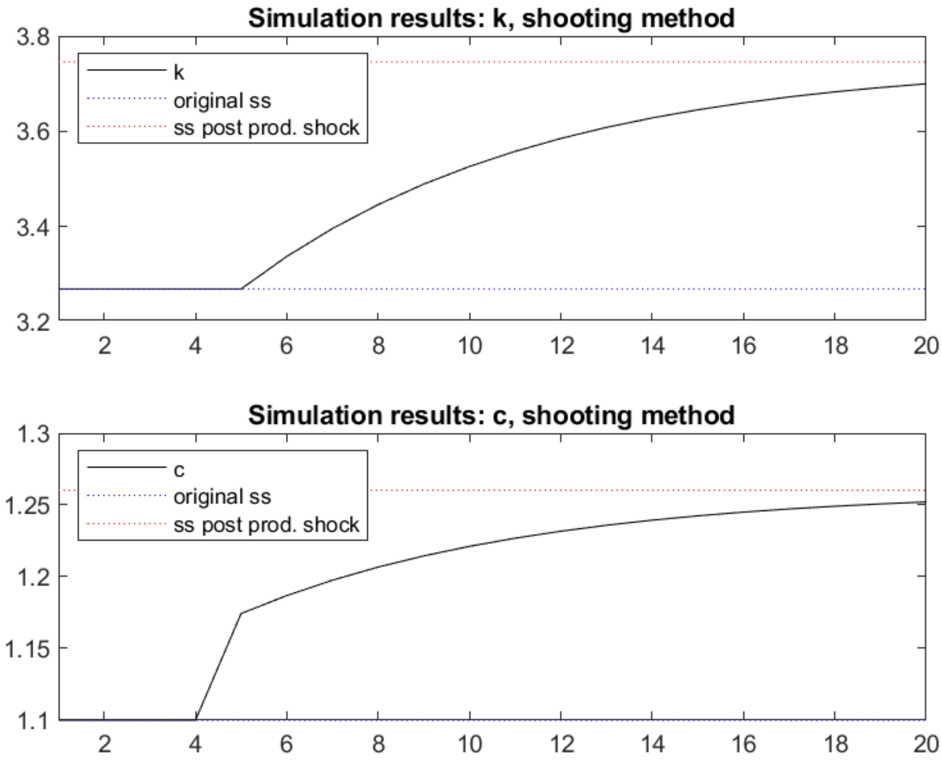
1.6 Numerically solve the actual transition path using the "shooting method".

If we put (k_{t_0}, c_{t_0}) into the nonlinear system (1) and (2), we will show that the system does not converge to a steady state:





Now we will instead use the shooting method to find the actual c_{t_0} needed to converge to the steady state.



2 Question 2

We are given three scenarios. For each scenario, we will state the SPP, CP, and CE.

2.1 Modified 2-period OG model

In this question we consider a model quite similar to the 2-period OG model we studied in class, but with a few differences as described in the problem set.

SPP: The social planner maximizes the utilities of the agents given the resource constraint:

$$\max_{c_t^t, c_{t+1}^t} N \ln c_t^t + N \ln c_{t+1}^t$$
$$\text{s.t. } Nc_t^t + Nc_{t+1}^t \leq \frac{N}{2} w_1 + \frac{N}{2} w_2$$

CP: Each consumer maximizes their own utility over the two periods, subject to their income constraints.

$$\max_{c_t^t, c_{t+1}^t} N \ln c_t^t + N \ln c_t^t$$
$$\text{s.t. } Nc_t^t + Nc_t^t \leq \frac{N}{2} w_1 + \frac{N}{2} w_2$$

2.2 3-period OG model

2.3 Cake eating problem