RBC Model

Dmitry Mukhin

dmukhin@wisc.edu

Primitives of the model:

1. preferences: $U = \mathbb{E}\sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$,
2. technology: $Y_t = A_t F(K_t, L_t) = C_t + K_{t+1} - (1-\delta)K_t$,
3. endowment: K_0 is given.

SPP is a natural starting point given that the decentralized allocation is the same (FWT):

$$\max_{\{C_t, L_t, K_{t+1}\}} \mathbb{E}\sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$

$$\text{s.t. } C_t = A_t F(K_t, L_t) + (1-\delta)K_t - K_{t+1}. \tag{1}$$

Denote the Lagrange multiplier with λ_t , write down the Lagrangian and take the FOCs:

$$\beta^t U_{Ct} = \lambda_t,$$

$$-\beta^t U_{Lt} = \lambda_t A_t F_{Lt},$$

$$\lambda_t = \mathbb{E}_t \lambda_{t+1}\big(A_{t+1}F_{Kt+1} + 1 - \delta\big).$$

Substitute λ_t from the first equation into the second and third conditions to obtain the intra-temporal optimality condition for labor:

$$-\frac{U_{Lt}}{U_{Ct}} = A_t F_{Lt}. \tag{2}$$

and the inter-temporal optimality condition (the Euler equation):

$$U_{Ct} = \beta \mathbb{E}_t U_{Ct+1}\big(A_{t+1}F_{Kt+1} + 1 - \delta\big). \tag{3}$$

Competitive equilibrium is a list of sequences $\{C_t, L_t, K_t, R_t, W_t\}$ s.t.

1. Given $\{R_t, W_t\}$, $\{C_t, L_t, K_t\}$ solve household problem:

$$\max \sum_{t=0}^{\infty} \mathbb{E}\beta^t U(C_t, L_t)$$

$$\text{s.t. } C_t + K_{t+1} - (1-\delta)K_t = R_t K_t + W_t L_t.$$

2. Given $\{R_t, W_t\}$, $\{K_t, L_t\}$ solve firm's problem:

$$\max \ A_t F(K_t, L_t) - R_t K_t - W_t L_t.$$

3. Markets clear:

$$C_t + K_{t+1} - (1-\delta)K_t = A_t F(K_t, L_t).$$

The firm's FOCs imply $R_t = A_t F_{Kt}$ and $W_t = A_t F_{Lt}$, while the household optimality conditions are

$$-\frac{U_{Lt}}{U_{Ct}} = W_t, \qquad U_{Ct} = \beta \mathbb{E}_t U_{Ct+1}\big(R_{t+1} + 1 - \delta\big).$$

Dynamics of the economy can be characterized in closed form because of only one endogenous state variable K_t and one control variable C_t – in contrast, employment L_t is a static choice variables and can be substituted out from the dynamic system. To simplify the analysis, however, we first solve dynamics in the model without labor and then use the static conditions to discuss the properties of employment.

Assume that labor is fixed at exogenous value $L_t = 1$ and that the TFP follows AR(1) process $a_t = \rho a_{t-1} + \varepsilon_t$. The setup is then exactly the same as in the growth model except for the stochastic shocks A_t.

1. Log-linearize the equilibrium conditions (1) and (3):

$$k_{t+1} = \frac{1}{\beta} k_t - (\phi - \delta)\, c_t + \phi a_t,$$

$$\mathbb{E}_t c_{t+1} = c_t - \frac{\beta(1-\alpha)\alpha\phi}{\sigma} k_{t+1} + \frac{\beta\alpha\phi}{\sigma} \mathbb{E}_t a_{t+1},$$

where $\phi = \frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)$ and define $\eta = \frac{\beta(1-\alpha)\alpha\phi(\phi-\delta)}{\sigma}$. Use the fact that $\mathbb{E}_t a_{t+1} = \rho a_t$ and substitute k_{t+1} from the resource constraint into the EE, so that the system can be written in matrix form as $\mathbb{E}_t x_{t+1} = A x_t + B a_t$, where

$$x_t = \begin{pmatrix} k_t \\ c_t \end{pmatrix}, \quad A = \begin{pmatrix} \frac{1}{\beta} & -(\phi-\delta) \\ -\frac{(1-\alpha)\alpha\phi}{\sigma} & 1+\eta \end{pmatrix}, \quad B = \begin{pmatrix} \phi \\ \frac{\beta\alpha\phi}{\sigma}\big[\rho - (1-\alpha)\phi\big] \end{pmatrix}.$$

2. As before, the eigenvalues of matrix A are equal

$$\lambda_{1,2} = \frac{1}{2}\left(\frac{1}{\beta} + 1 + \eta \pm \sqrt{\left(\frac{1}{\beta} + 1 + \eta\right)^2 - \frac{4}{\beta}}\right)$$

and the dynamic system can be expressed as $\mathbb{E}_t y_{t+1} = \Lambda y_t + Q^{-1} B a_t$, where $y_t = Q^{-1} x_t$,

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad Q^{-1} = \frac{1}{\lambda_1 - \lambda_2}\begin{pmatrix} \frac{\frac{1}{\beta}-\lambda_2}{\phi-\delta} & -1 \\ \frac{\frac{1}{\beta}-\lambda_1}{\phi-\delta} & 1 \end{pmatrix}.$$

3. Consider the first equation with the eigenvalue greater than one. Iterating it forward we get

$$y_t^1 = -b\lambda_1^{-1} a_t + \lambda_1^{-1}\mathbb{E}_t y_{t+1}^1 = \cdots = -\frac{b}{\lambda_1}\sum_{j=0}^{\infty} \lambda_1^{-j}\mathbb{E}_t a_{t+j} + \lim_{j\to\infty} \lambda_1^{-j} y_{t+j}^1,$$

where b is the first element of $Q^{-1}Ba_t$. The last term is equal zero as the economy converges to the SS in the long-run in the absence of future shocks (TVC). Given $\mathbb{E}_t a_{t+j} = \rho^j a_t$, it follows $y_t^1 = -\frac{b}{\lambda_1 - \rho} a_t$ and we can express the control variable in terms of the state variables

$$c_t = \frac{\frac{1}{\beta} - \lambda_2}{\phi - \delta} k_t + \kappa a_t, \quad \kappa \equiv \frac{\phi}{\lambda_1 - \rho}\left[\frac{\frac{1}{\beta} - \lambda_2}{\phi - \delta} - \frac{\beta\alpha\big(\rho - (1-\alpha)\phi\big)}{\sigma}\right].$$

The law of motion for capital is then

$$k_{t+1} = \lambda_2 k_t + \chi a_t, \; \chi \equiv \frac{\phi}{\lambda_1 - \rho}\left[(\lambda_1 - \rho) - \left(\frac{1}{\beta} - \lambda_2\right) + \frac{\beta\alpha}{\sigma}(\phi - \delta)\big(\rho - (1-\alpha)\phi\big)\right].$$

In fact, we can go one step further and characterize the stochastic process for all endogenous variables. Define the lag operator as $\mathbb{L}x_t = x_{t-1}$ and rewrite the capital law of motion:

$$(1 - \lambda_2\mathbb{L})k_{t+1} = \chi a_t.$$

Multiplying both sides of the equation by $1 - \rho\mathbb{L}$, obtain

$$(1 - \rho\mathbb{L})(1 - \lambda_2\mathbb{L})k_{t+1} = \chi\varepsilon_t.$$

It follows that k_t follows AR(2) process and one can explicitly solve for the impulse response function $\frac{\mathrm{d}k_{t+j}}{\mathrm{d}\varepsilon_t}$. Similarly, from the cointegration relationship, c_t follows ARMA(2,1) process:

$$(1-\rho\mathbb{L})(1-\lambda_2\mathbb{L})c_t = \frac{\frac{1}{\beta}-\lambda_2}{\phi-\delta}\chi\varepsilon_{t-1} + \kappa(1-\lambda_2\mathbb{L})\varepsilon_t = \kappa\varepsilon_t + \left(\frac{\frac{1}{\beta}-\lambda_2}{\phi-\delta}\chi - \lambda_2\kappa\right)\varepsilon_{t-1}.$$

Employment is perhaps the most important business-cycle variable . To understand its dynamics in the RBC model, consider the standard functional forms:

$$U(C,L) = \frac{C^{1-\sigma}}{1-\sigma} - \frac{L^{1+\varphi}}{1+\varphi}, \qquad F(K,L) = K^\alpha L^{1-\alpha}.$$

The optimal labor supply is given by $C_t^\sigma L_t^\varphi = W_t$ – the elasticity of labor wrt to real wage $\frac{1}{\varphi}$ is called the Frisch elasticity. Take logs and use firm's optimality condition $w_t = y_t - l_t$ to obtain

$$l_t = \frac{1}{1+\varphi}\Big(y_t - \sigma c_t\Big).$$

1. GHH preferences (Greenwood, Hercowitz and Huffman 1988) eliminate the income effect on labor supply:

$$U = \frac{1}{1-\sigma}\Big[C - \frac{L^{1+\varphi}}{1+\varphi}\Big]^{1-\sigma} \quad \Rightarrow \quad -\frac{U_{Lt}}{U_{Ct}} = L^\varphi \quad \Rightarrow \quad l_t = \frac{1}{1+\varphi}y_t.$$

2. Discrete labor supply with lotteries (Rogerson 1988) can explain the large differences between Frisch elasticities at the micro level ≈ 0 and at the macro level > 1. Assume that each individual $i \in [0,1]$ can choose to work either full day $L_i = 1$ or stay unemployed $L_i = 0$. The social planner chooses the fraction of individuals that works η:

$$\max \mathbb{E}\sum_{t=0}^{\infty}\beta^t\Big[\frac{C_t^{1-\sigma}}{1-\sigma} - \eta_t\frac{1}{1+\varphi} - (1-\eta_t)\cdot 0\Big].$$

Thus, the micro elasticity φ can take arbitrary small values, while the macro elasticity is equal ∞! Can be decentralized using lotteries.

Calibration and simulation allow to quantitatively access the fit of the model. Preferences and technology are calibrated the same as in the growth model: at quarterly frequency, we have $\beta = 0.99$, $\alpha = 1/3$, $\delta = 0.02$, $\sigma = 1$, $\varphi = 1$ (see Chetty et al. AER'2011). In addition, one needs to calibrate stochastic process for TFP shocks. Assume AR(1) process $a_t = \rho a_{t-1} + \varepsilon_t$ with $\varepsilon_t \sim i.i.d.(0,\sigma^2)$ and calibrate either using the Solow residual $\hat{a}_t = y_t - \alpha k_t - (1-\alpha)l_t$ or targeting the volatility and persistence of GDP.

An additional challenge here is the trend in the data, which is not present in the model. The Hodrick-Prescott filter is a bit outdated and has been heavily critisized (see Hamilton 2018), but is still widely used in practice to filter the cyclical component of time series:

$$\min_{\{\bar{x}_t\}} \sum_{t=2}^{T} \Big[(x_t - \bar{x}_t)^2 + \lambda(\Delta\bar{x}_t - \Delta\bar{x}_{t-1})^2 \Big],$$

where λ is the smoothing parameter with the standard value of 1600 for quarterly data.

Table 1
Business cycle statistics for the US Economy

	Standard deviation	Relative standard deviation	First-order autocorrelation	Contemporaneous correlation with output
Y	1.81	1.00	0.84	1.00
C	1.35	0.74	0.80	0.88
I	5.30	2.93	0.87	0.80
N	1.79	0.99	0.88	0.88
Y/N	1.02	0.56	0.74	0.55
w	0.68	0.38	0.66	0.12
r	0.30	0.16	0.60	−0.35
A	0.98	0.54	0.74	0.78

Table 3
Business cycle statistics for basic RBC model[a,b]

	Standard deviation	Relative standard deviation	First-order autocorrelation	Contemporaneous correlation with output
Y	1.39	1.00	0.72	1.00
C	0.61	0.44	0.79	0.94
I	4.09	2.95	0.71	0.99
N	0.67	0.48	0.71	0.97
Y/N	0.75	0.54	0.76	0.98
w	0.75	0.54	0.76	0.98
r	0.05	0.04	0.71	0.95
A	0.94	0.68	0.72	1.00

Wedge accounting at the aggregate level was first introduced by Chari, Kehoe and McGrattan (ECM'2007) and has since then expanded to micro-level misallocation analysis (Restuccia and Rogerson JED'2008, Hsieh and Klenow QJE'2009). The dynamics of $\{Y_t, C_t, L_t, K_t\}$ is determined in the RBC model by four equations: production function, the resource constraint, the labor supply, and the EE. Introduce one wedge in each of the equilibrium conditions:

$$Y_t = A_t F(K_t, L_t),$$

$$A_t F(K_t, L_t) = C_t + I_t + G_t,$$

$$-\frac{U_{Lt}}{U_{Ct}} = (1 - \tau_{Lt}) A_t F_{Lt},$$

$$U_{Ct}(1 + \tau_{It}) = \beta \mathbb{E}_t U_{Ct+1} \big(A_{t+1} F_{Kt+1} + (1 - \delta)(1 + \tau_{It+1}) \big),$$

where A_t is the efficiency wedge, G_t is the government consumption wedge, τ_{Lt} is the labor wedge, and τ_{It} is the investment wedge. Given $\{Y_t, C_t, L_t, K_t\}$ from the data, we can invert this system to estimate the wedges. By construction the system perfectly matches any empirical series or series generated from any model. CKM then do a wedge accounting exercise solving the model for endogenous variables including one wedge at a time. They find that the labor wedge played the central role in the Great Depression, while the investment wedge actually made the recession milder. Note also that the efficiency wedge alone can explain the dynamics of investment and to some extent of the output, but not the employment. CKM conclude that to understand the Great Depression, we need structural models that can generate labor wedge (models with sticky prices/wages, search friction), not investment wedge (models with financial frictions).