

# New-Keynesian Model

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**Primitives** of the model:

1. preferences:  $U = \mathbb{E} \sum_{t=0}^{\infty} \beta^t \left[ \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\varphi}}{1+\varphi} \right]$ ,  $C_t = \left( \int C_{it}^{\frac{\theta-1}{\theta}} di \right)^{\frac{\theta}{\theta-1}}$ ,
2. technology:  $Y_{it} = A_t L_{it} + \text{i.i.d. shocks to costs of price adjustment } \xi_{it}$  (“Calvo fairy”)

$$\xi_{it} = \begin{cases} \infty & \text{w/p } \lambda \\ 0 & \text{w/p } 1 - \lambda \end{cases},$$

3. endowment:  $K_0$  is given.

**Households** choose consumption of each product, supply labor and purchase bonds:

$$\begin{aligned} \max \quad & \mathbb{E} \sum_{t=0}^{\infty} \beta^t \left[ \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\varphi}}{1+\varphi} \right], \quad C_t = \left( \int C_{it}^{\frac{\theta-1}{\theta}} di \right)^{\frac{\theta}{\theta-1}} \\ \text{s.t.} \quad & \int P_{it} C_{it} di + B_t = W_t L_t + \Pi_t + (1 + i_{t-1}) B_{t-1}. \end{aligned}$$

The resulting optimality conditions are the same as in the RBC and Dixit-Stiglitz model: demand for products

$$C_{it} = \left( \frac{P_{it}}{P_t} \right)^{-\theta} C_t, \tag{1}$$

the labor supply condition

$$C_t^\sigma L_t^\varphi = \frac{W_t}{P_t}, \tag{2}$$

and the Euler equation

$$\beta \mathbb{E}_t \left( \frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} (1 + i_t) = 1. \tag{3}$$

Also define the stochastic discount factor (SDF)  $\Theta_{t,t+j} = \frac{\beta^{t+j} C_{t+j}^{-\sigma} / P_{t+j}}{\beta^t C_t^{-\sigma} / P_t} = \beta^j \left( \frac{C_{t+j}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+j}}$ , which shows the price of one unit of money delivered in period  $t + j$  relative to the price of one unit of money in period  $t$ , i.e. the ratio of the Lagrange multipliers for periods  $t + j$  and  $t$  from household problem.

**Firms** are monopolistic competitors as in the DS economy, but are now subject to Calvo price friction making their problem dynamic. Firms that cannot adjust prices in period  $t$  keep them at the same level as in the previous period. Therefore, we focus on firms that can adjust prices and maximize expected discounted profits:

$$\begin{aligned} \max \quad & \mathbb{E}_t \sum_{j=0}^{\infty} \Theta_{t,t+j} \left( \tilde{P}_t C_{it+j} - W_{t+j} L_{it+j} \right) \\ \text{s.t.} \quad & C_{it} = \left( \frac{\tilde{P}_t}{P_t} \right)^{-\theta} C_t, \quad C_{it} = A_t L_{it}. \end{aligned}$$

Importantly, the expectation is taken with respect to both aggregate shocks and idiosyncratic Calvo shock. Substitute in the constraints and the SDF into the objective function and take the expectation with respect to the Calvo shock leaving only aggregate uncertainty :

$$\max_{\tilde{P}_t} \mathbb{E}_t \sum_{j=0}^{\infty} (\beta\lambda)^j \left( \frac{C_{t+j}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+j}} \left( \tilde{P}_t - \frac{W_{t+j}}{A_{t+j}} \right) \left( \frac{\tilde{P}_t}{P_{t+j}} \right)^{-\theta} C_{t+j}.$$

Drop constant exogenous terms with subscript  $t$ , take the FOC and rearrange:

$$\mathbb{E}_t \sum_{j=0}^{\infty} (\beta\lambda)^j P_{t+j}^{\theta-1} C_{t+j}^{1-\sigma} \left[ \tilde{P}_t - \frac{\theta}{\theta-1} \frac{W_{t+j}}{A_{t+j}} \right] = 0.$$

Express the optimal price as

$$\tilde{P}_t = \frac{\theta}{\theta-1} \frac{\mathbb{E}_t \sum_{j=0}^{\infty} (\beta\lambda)^j P_{t+j}^{\theta-1} C_{t+j}^{1-\sigma} \frac{W_{t+j}}{A_{t+j}}}{\mathbb{E}_t \sum_{j=0}^{\infty} (\beta\lambda)^j P_{t+j}^{\theta-1} C_{t+j}^{1-\sigma}} = \frac{\theta}{\theta-1} \mathbb{E}_t \sum_{j=0}^{\infty} \omega_{t,t+j} \frac{W_{t+j}}{A_{t+j}}, \quad (4)$$

where  $\omega_{t,t+j} = \frac{(\beta\lambda)^j P_{t+j}^{\theta-1} C_{t+j}^{1-\sigma}}{\mathbb{E}_t \sum_{\tau=0}^{\infty} (\beta\lambda)^\tau P_{t+\tau}^{\theta-1} C_{t+\tau}^{1-\sigma}}$ ,  $\mathbb{E}_t \sum_{j=0}^{\infty} \omega_{t,t+j} = 1$ .

**Aggregate conditions** include the market clearing for output  $Y_t = C_t$  and for labor

$$L_t = \int \frac{C_{it}}{A_t} di = \frac{C_t}{A_t} \int \left( \frac{P_{it}}{P_t} \right)^{-\theta} di. \quad (5)$$

The second aggregate condition is the law of motion for the ideal price index.

$$P_t^{1-\theta} = \int P_{it}^{1-\theta} di = (1-\lambda) \tilde{P}_t^{1-\theta} + \lambda P_{t-1}^{1-\theta}. \quad (6)$$

Finally, assume that the central bank follows the Taylor (1993) rule tightening the monetary stance in response to higher inflation. In log terms, it can be written as

$$i_t = \phi \pi_t + \varepsilon_t. \quad (7)$$

The decentralized equilibrium of the model is characterized by equations (1)-(7).

**Log-linearization** is required to make further progress in solving the model as it allows to convert the non-linear stochastic dynamic system (1)-(6) into two linear dynamic equations that can be then solved with the Blanchard-Kahn method:

1. Linearize the price-setting FOC before equation (4) using the fact that  $\bar{P} = \frac{\theta}{\theta-1} \frac{\bar{W}}{\bar{A}}$  in steady state:

$$\mathbb{E}_t \sum_{j=0}^{\infty} (\beta\lambda)^j \left[ 1 + (\theta - 1)p_{t+j} + (1 - \sigma)c_{t+j} \right] \left[ \bar{P}(1 + \tilde{p}_t) - \frac{\theta}{\theta - 1} \bar{M}C(1 + mc_{t+j}) \right] = 0,$$

$$\mathbb{E}_t \sum_{j=0}^{\infty} (\beta\lambda)^j (\tilde{p}_t - mc_{t+j}) = 0.$$

2. Rewrite the reset price in a recursive form:

$$\begin{aligned} \tilde{p}_t &= (1 - \beta\lambda) \mathbb{E}_t \sum_{j=0}^{\infty} (\beta\lambda)^j mc_{t+j} = (1 - \beta\lambda) mc_t + (1 - \beta\lambda) \mathbb{E}_t \sum_{j=1}^{\infty} (\beta\lambda)^j mc_{t+j} \\ &= (1 - \beta\lambda) mc_t + \beta\lambda \mathbb{E}_t (1 - \beta\lambda) \mathbb{E}_{t+1} \sum_{j=0}^{\infty} (\beta\lambda)^j mc_{t+1+j} = (1 - \beta\lambda) mc_t + \beta\lambda \mathbb{E}_t \tilde{p}_{t+1}. \end{aligned}$$

3. Linearize the aggregate price law of motion (6):

$$\bar{P}^{1-\theta} (1 + (1 - \theta)p_t) = (1 - \lambda) \bar{P}^{1-\theta} (1 + (1 - \theta)\tilde{p}_t) + \lambda \bar{P}^{1-\theta} (1 + (1 - \theta)p_{t-1}),$$

$$p_t = (1 - \lambda)\tilde{p}_t + \lambda p_{t-1}.$$

4. Rewrite the law of motion for  $\tilde{p}_t$  in terms of aggregate price. To this end, subtract  $p_t$  from both sides of the equation and multiply by it by  $1 - \lambda$ :

$$(1 - \lambda)(\tilde{p}_t - p_t) = (1 - \lambda)(1 - \beta\lambda)(mc_t - p_t) + \beta\lambda \mathbb{E}_t (1 - \lambda)(\tilde{p}_{t+1} - p_t)$$

and use the price index law of motion

$$\lambda(p_t - p_{t-1}) = (1 - \lambda)(1 - \beta\lambda)(mc_t - p_t) + \beta\lambda \mathbb{E}_t (p_{t+1} - p_t).$$

Use the definition of inflation and divide by  $\lambda$ :

$$\pi_t = \frac{(1 - \lambda)(1 - \beta\lambda)}{\lambda}(mc_t - p_t) + \beta\mathbb{E}_t\pi_{t+1}.$$

5. Linearize the labor supply condition

$$\sigma c_t + \varphi l_t = w_t - p_t$$

and the market clearing condition (5)

$$l_t + a_t = c_t - \theta \int (p_{it} - p_t) di = c_t = y_t,$$

where the last equality uses the fact that  $p_t = \int p_{it} di$ . Use these two equations to express the real marginal costs as follows:

$$mc_t - p_t = w_t - a_t - p_t = \sigma c_t + \varphi l_t - a_t = (\sigma + \varphi)y_t - (1 + \varphi)a_t.$$

6. Substitute the latter condition into the price-setting equation:

$$\pi_t = \frac{(1 - \lambda)(1 - \beta\lambda)(\sigma + \varphi)}{\lambda} \left( y_t - \frac{1 + \varphi}{\sigma + \varphi} a_t \right) + \beta\mathbb{E}_t\pi_{t+1}.$$

Denote the flexible price allocation with tilde and define the output gap as a log deviation of the output from the flexible price level  $x_t = y_t - \tilde{y}_t$ . Taking the limit  $\lambda = 0$ , we get the flexible-price output

$$\tilde{y}_t = \frac{1 + \varphi}{\sigma + \varphi} a_t$$

and hence, the NKPC is

$$\pi_t = \kappa x_t + \beta\mathbb{E}_t\pi_{t+1}, \quad \kappa \equiv \frac{(1 - \lambda)(1 - \beta\lambda)(\sigma + \varphi)}{\lambda}. \quad (8)$$

7. Finally, linearize the EE

$$\sigma\mathbb{E}_t\Delta c_{t+1} = i_t - \mathbb{E}_t\pi_{t+1}.$$

Define the “natural rate” as the real rate that prevails under flexible prices:

$$r_t^n = \sigma\mathbb{E}_t\Delta\tilde{y}_{t+1} = \sigma\mathbb{E}_t\frac{1 + \varphi}{\sigma + \varphi}\Delta a_{t+1}.$$

Express the EE in terms of the output gap to get the NKIS curve:

$$\sigma\mathbb{E}_t\Delta x_{t+1} = i_t - \mathbb{E}_t\pi_{t+1} - r_t^n. \quad (9)$$

**Dynamics** of the economy is described by the system (7)-(9). Substitute the Taylor rule into the NKIS equation and write the dynamic system in matrix form:

$$\mathbb{E}_t \begin{pmatrix} \pi_{t+1} \\ x_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} \\ \frac{\phi}{\sigma} - \frac{1}{\beta\sigma} & 1 + \frac{\kappa}{\beta\sigma} \end{pmatrix} \begin{pmatrix} \pi_t \\ x_t \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{\sigma} \end{pmatrix} \varepsilon_t.$$

Solving for the eigenvalues, we get

$$f(\lambda) = \begin{vmatrix} \frac{1}{\beta} - \lambda & -\frac{\kappa}{\beta} \\ \frac{\phi}{\sigma} - \frac{1}{\beta\sigma} & 1 + \frac{\kappa}{\beta\sigma} - \lambda \end{vmatrix} = \lambda^2 - \left(1 + \frac{\kappa}{\beta\sigma} + \frac{1}{\beta}\right) \lambda + \frac{1}{\beta} \left(1 + \frac{\phi\kappa}{\sigma}\right).$$

Given two control variables, one needs both eigenvalues greater than one for the system to have unique solution. For this to be the case, it is necessary and sufficient that  $f(0) > f(1) > 0$ . The first inequality is always true, while the second one holds iff  $\phi > 1$ . This condition is called the Taylor principle: the nominal rates have to increase more than one-to-one in response to an increase in inflation.