

Econometrics HW3

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1 Question 1

Let a random point be distributed uniformly on the square with vertices $(1, 1), (1, -1), (-1, 1), (-1, -1)$.

1.1 Determine $P(X^2 + Y^2 < 1)$.

$$\begin{aligned} P(X^2 + Y^2 < 1) &= \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{4} dx dy = \frac{1}{2} \int_{-1}^1 \sqrt{1-y^2} dy = \frac{1}{2} \left((1/2) \arcsin x + (1/2) x \sqrt{1-x^2} \right) \Big|_{-1}^1 \\ &= \frac{\pi}{4}. \end{aligned}$$

1.2 Determine $P(|X + Y| < 2)$

$$P(|X + Y| < 2) = \int_{-1}^1 \int_{-1}^1 \frac{1}{4} dx dy = \int_{-1}^1 \frac{1}{2} dy = 1.$$

2 Question 2

2.1 What conditions should a, b satisfy in order for $f(x, y)$ to be a bivariate PDF?

For $f(x, y)$ to be a bivariate PDF, it must integrate to one:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) h(y) dx dy = \int_{-\infty}^{\infty} g(x) dx \int_{-\infty}^{\infty} h(y) dy \\ &= ab. \end{aligned}$$

So, for $f(x, y)$ to be a bivariate PDF, $ab = 1$.

2.2 Find the marginal PDF of X and Y .

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} g(x) h(y) dy = g(x) \int_{-\infty}^{\infty} h(y) dy = bg(x), \\ f_Y(y) &= \int_{-\infty}^{\infty} g(x) h(y) dx = h(y) \int_{-\infty}^{\infty} g(x) dx = ah(y). \end{aligned}$$

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2.3 Show that X, Y are independent.

$f_{X,Y}(x, y) = g(x)h(y) = (ab)g(x)h(y) = bg(x)ah(y) = f_X(x)f_Y(y)$ so X, Y are independent.

3 Question 3

3.1 Find c such that $f(x, y)$ is a joint PDF.

We will find the value of c such that the integral of $f(x, y)$ on its support is 1.

$$\begin{aligned}\int_0^1 \int_0^{1-y} cxy dx dy &= c \int_0^1 y(1-y)^2/2 dy = (c/2) \int_0^1 y - 2y^2 + y^3 dy = (c/2)(y^2/2 - (2/3)y^3 + y^4/4|_0^1) \\ &= (c/2)((1/2) - (2/3) + (1/4)) = c/24.\end{aligned}$$

Thus, for the integral of $f(x, y)$ on its support to be equal to 1, $c = 24$.

3.2 Find the marginal distributions of X and Y .

$$\begin{aligned}f_X(x) &= \int_0^{1-x} 24xy dy = 24x(y^2/2|_0^{1-x}) = 12x(1-x)^2, \\ f_Y(y) &= \int_0^{1-y} 24xy dx = 24y(x^2/2|_0^{1-y}) = 12y(1-y)^2.\end{aligned}$$

Note: This is for $x, y \in [0, 1]$. For all other x, y , $f_X(x) = f_Y(y) = 0$.

3.3 Are X, Y independent?

X, Y are not independent. $f_{X,Y}(x, y) = 24xy \neq (12x(1-x)^2)(12y(1-y)^2) = f_X(x)f_Y(y)$. Note that our result from question 2 does not hold for this question because the region on which the joint PDF is nonzero is a function of X, Y . So, the joint distribution can not be separately factored into X, Y marginals as the support for the marginal for each random variable is a function of the realization of the other random variable.

4 Question 4

We will show that any random variable is uncorrelated with a constant.

$$P(k \leq y, X \leq x) = \begin{cases} 0, & y < k \\ P(X \leq x), & y \geq k \end{cases} = P(k \leq y)P(X \leq x)$$

so X, k independent and, therefore, X, k are uncorrelated.

5 Question 5

From the independence of X, Y , $E(XY) = EXEY = \mu_X\mu_Y$.

$$\begin{aligned}
 \sigma_{XY}^2 &= E[(XY)^2] - E[XY]^2 = E[X^2Y^2] - \mu_X^2\mu_Y^2 = E[X^2]E[Y^2] - \mu_X^2\mu_Y^2 \\
 &= (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_X^2\mu_Y^2 \\
 \text{Cov}(XY, Y) &= E((XY)Y) - E(XY)EY = E(XY^2) - \mu_X\mu_Y^2 = EXE(Y^2) - \mu_X\mu_Y^2 \\
 &= \mu_X(\sigma_Y^2 + \mu_Y^2) - \mu_X\mu_Y^2 = \mu_X\sigma_Y^2 \\
 \text{Corr}(XY, Y) &= \frac{\text{Cov}(XY, Y)}{\sqrt{\sigma_{XY}^2\sigma_Y^2}} = \frac{\mu_X\sigma_Y^2}{\sqrt{((\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_X^2\mu_Y^2)\sigma_Y^2}} \\
 &= \frac{\mu_X\sigma_Y}{\sqrt{\sigma_X^2\sigma_Y^2 + \mu_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2}}.
 \end{aligned}$$

6 Question 6

Let $(X_1, \dots, X_n)'$ be a random vector. We will prove via induction. First, let $n = 2$. Then, $\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2) + 2\text{Cov}(X_1, X_2)$.

Next, assume that $\text{Var}(\sum_{i=1}^n X_i) = (\sum_{i=1}^n \text{Var}(X_i)) + 2\left(\sum_{1 \leq i < j \leq n} \text{Cov}(X_i, X_j)\right)$. Then,

$$\begin{aligned}
 \text{Var}\left(\sum_{i=1}^{n+1} X_i\right) &= \text{Var}\left(\sum_{i=1}^n X_i\right) + \text{Var}(X_{n+1}) + 2\text{Cov}\left(\left(\sum_{i=1}^n X_i\right), X_{n+1}\right) \\
 &= \left(\sum_{i=1}^n \text{Var}(X_i)\right) + 2\left(\sum_{1 \leq i < j \leq n} \text{Cov}(X_i, X_j)\right) + \text{Var}(X_{n+1}) + 2\text{Cov}\left(\left(\sum_{i=1}^n X_i\right), X_{n+1}\right) \\
 &= \left(\sum_{i=1}^{n+1} \text{Var}(X_i)\right) + 2\left(\sum_{1 \leq i < j \leq n} \text{Cov}(X_i, X_j)\right) + 2\text{Cov}\left(\left(\sum_{i=1}^n X_i\right), X_{n+1}\right).
 \end{aligned}$$

Next, note that, for random variables X, Y, Z ,

$$\begin{aligned}
 \text{Cov}(X + Y, Z) &= E[(X + Y)Z] - E(X + Y)E(Z) = E[XZ + YZ] - (EX + EY)EZ \\
 &= E(XZ) - EXEZ + E(YZ) - EY EZ = \text{Cov}(X, Z) + \text{Cov}(Y, Z).
 \end{aligned}$$

We then have,

$$\begin{aligned}
 \text{Var}\left(\sum_{i=1}^{n+1} X_i\right) &= \left(\sum_{i=1}^{n+1} \text{Var}(X_i)\right) + 2\left(\sum_{1 \leq i < j \leq n} \text{Cov}(X_i, X_j)\right) + 2\sum_{i=1}^n \text{Cov}(X_i, X_{n+1}) \\
 &= \left(\sum_{i=1}^{n+1} \text{Var}(X_i)\right) + 2\left(\sum_{1 \leq i < j \leq n+1} \text{Cov}(X_i, X_j)\right).
 \end{aligned}$$

Therefore, by induction, $\text{Var}(\sum_{i=1}^n X_i) = (\sum_{i=1}^n \text{Var}(X_i)) + 2\left(\sum_{1 \leq i < j \leq n} \text{Cov}(X_i, X_j)\right)$.

7 Question 7

Let X, Y be jointly normal.

7.1 Derive the marginal distribution of X, Y and observe that both are normal distributions.

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x, y) dy \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-(2(1-\rho^2))^{-1}((x^2/\sigma_X^2) - 2\rho xy/(\sigma_X\sigma_Y) + (y^2/\sigma_Y^2))\right) dy \\
&= \frac{1}{\sqrt{2\pi}\sigma_X} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}\sigma_Y} \exp\left(-(2(1-\rho^2))^{-1}((x^2/\sigma_X^2) + (y/\sigma_Y - \rho x/\sigma_X)^2 - (\rho^2 x^2/\sigma_X^2))\right) dy \\
&= \frac{\exp\left(-(2(1-\rho^2))^{-1}((x^2/\sigma_X^2) - (\rho^2 x^2/\sigma_X^2))\right)}{\sqrt{2\pi}\sigma_X} \\
&\quad * \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \exp\left(-(2(1-\rho^2))^{-1}((y/\sigma_Y - \rho x/\sigma_X)^2)\right) dy \\
&= \frac{\exp\left(-(2(1-\rho^2))^{-1}((x^2/\sigma_X^2) - (\rho^2 x^2/\sigma_X^2))\right)}{\sqrt{2\pi}\sigma_X} = \frac{\exp\left(-(x^2/(2\sigma_X^2))\right)}{\sqrt{2\pi}\sigma_X}.
\end{aligned}$$

Note that this is the form of a normal distribution with mean 0 and variance σ_X^2 . By symmetry,

$f_Y(y) = \frac{\exp(-(y^2/(2\sigma_Y^2)))}{\sqrt{2\pi}\sigma_Y}$ is also a normal distribution with mean 0 and variance σ_Y^2 .

7.2 Derive the conditional distribution of Y given $X = x$. Observe that it is also a normal distribution.

$$\begin{aligned}
f_{Y|X}(y|x) &= \frac{f_{X,Y}(x, y)}{f_X(x)} \\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-(2(1-\rho^2))^{-1}((x^2/\sigma_X^2) - 2\rho xy/(\sigma_X\sigma_Y) + (y^2/\sigma_Y^2))\right) \\
&\quad * \left(\frac{\exp(-(x^2/(2\sigma_X^2)))}{\sqrt{2\pi}\sigma_X}\right)^{-1} \\
&= \frac{\exp(-(2(1-\rho^2))^{-1}((x^2/\sigma_X^2) - 2\rho xy/(\sigma_X\sigma_Y) + (y^2/\sigma_Y^2)) + (x^2/(\sigma_X^2)))}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \\
&= \frac{\exp(-(2(1-\rho^2))^{-1}((x^2/\sigma_X^2) - 2\rho xy/(\sigma_X\sigma_Y) + (y^2/\sigma_Y^2) - ((1-\rho^2)(x^2/\sigma_X^2))))}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \\
&= \frac{\exp(-(2(1-\rho^2))^{-1}((x^2/\sigma_X^2)(\rho^2) - 2\rho xy/(\sigma_X\sigma_Y) + (y^2/\sigma_Y^2)))}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \\
&= \frac{\exp(-(2(1-\rho^2))^{-1}((x^2/\sigma_X^2)(\rho^2) + (y/\sigma_Y - \rho x/\sigma_X)^2 - \rho^2 x^2/\sigma_X^2))}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \\
&= \frac{\exp(-(2(1-\rho^2))^{-1}(y/\sigma_Y - \rho x/\sigma_X)^2)}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \\
&= \frac{\exp(-(2)^{-1}(y/(\sqrt{1-\rho^2}\sigma_Y) - \rho x/(\sqrt{1-\rho^2}\sigma_X))^2)}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \\
&= \frac{\exp(-(2)^{-1}((y - (\rho x(\sigma_Y/\sigma_X)))/(\sqrt{1-\rho^2}\sigma_Y))^2)}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}}
\end{aligned}$$

This is also a normal distribution, with mean $\rho x(\sigma_Y/\sigma_X)$ and variance $\sigma_Y^2(1-\rho^2)$.

7.3 Derive the joint distribution of (X, Z) where $Z = (Y/\sigma_Y) - (\rho X/\sigma_X)$, and then show that X, Z are independent.

Note that our mapping from $\begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow \begin{pmatrix} X \\ Y/\sigma_Y - \rho X/\sigma_X \end{pmatrix}$ has an inverse mapping $\begin{pmatrix} X \\ Z \end{pmatrix} \rightarrow \begin{pmatrix} X \\ \sigma_Y Z + \sigma_Y(\rho X/\sigma_X) \end{pmatrix}$ with Jacobian determinant $\left| \begin{pmatrix} 1 & 0 \\ \rho\sigma_Y/\sigma_X & \sigma_Y \end{pmatrix} \right| = \sigma_Y$. Thus, the joint density of X, Z is $f_{X,Z}(x, z) = f_{X,Y}(x, \sigma_Y Z + \rho X\sigma_Y/\sigma_X)\sigma_Y$:

$$\begin{aligned} f_{X,Z}(x, z) &= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left(\frac{x^2}{\sigma_X^2} - 2\frac{x(z + \rho x/\sigma_X)\rho}{\sigma_X} + (z + (\rho x/\sigma_X))^2\right)\right) \\ &= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left(\frac{x^2}{\sigma_X^2} - 2\frac{xz\rho}{\sigma_X} - 2\frac{x^2\rho^2}{\sigma_X^2} + z^2 + 2z(\rho x/\sigma_X) + (\rho x/\sigma_X)^2\right)\right) \\ &= \frac{1}{2\pi\sigma_X\sqrt{1-\rho^2}} \exp\left(-\frac{x^2}{2\sigma_X^2} - \frac{z^2}{2(1-\rho^2)}\right) \\ &= \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{x^2}{2\sigma_X^2}\right) \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{z^2}{2(1-\rho^2)}\right). \end{aligned}$$

This clearly separates into separate distributions for X, Z so X, Z are independent.

8 Question 8

If X, Y are independent random variables, then $P(X \leq x \cap Y \leq y) = P(X \leq x)P(Y \leq y)$. Then, we have the following:

$$\begin{aligned} P(Z \leq z \cap W \leq w) &= P(g_1(X) \leq z \cap g_2(Y) \leq w) \\ &= P(X \leq g_1^{-1}(z) \cap Y \leq g_2^{-1}(w)) \\ &= P(X \leq g_1^{-1}(z))P(Y \leq g_2^{-1}(w)) \\ &= P(Z \leq z)P(W \leq w). \end{aligned}$$

Thus, Z, W are independent.