# Micro HW2

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## 1 Question 1

### 1.1 Prove that if the production set $Y = \{(q, -z) : f(z) \geq q\} \subset \mathbb{R}^{m+1}$ is convex, the production function $f$ is concave.

Let $q_1 = f(z_1), q_2 = f(z_2)$. $(q_1, -z_1), (q_2, -z_2) \in Y$ by definition and by convexity $t(q_1, -z_1) + (1-t)(q_2, -z_2) \in Y, t \in (0, 1)$. By definition, $f(t(z_1) + (1-t)(z_2)) \geq tq_1 + (1-t)q_2 = tf(z_1) + (1-t)f(z_2)$ so f is concave.

### 1.2 Prove that if $f$ is concave, the cost function is convex in $q$.

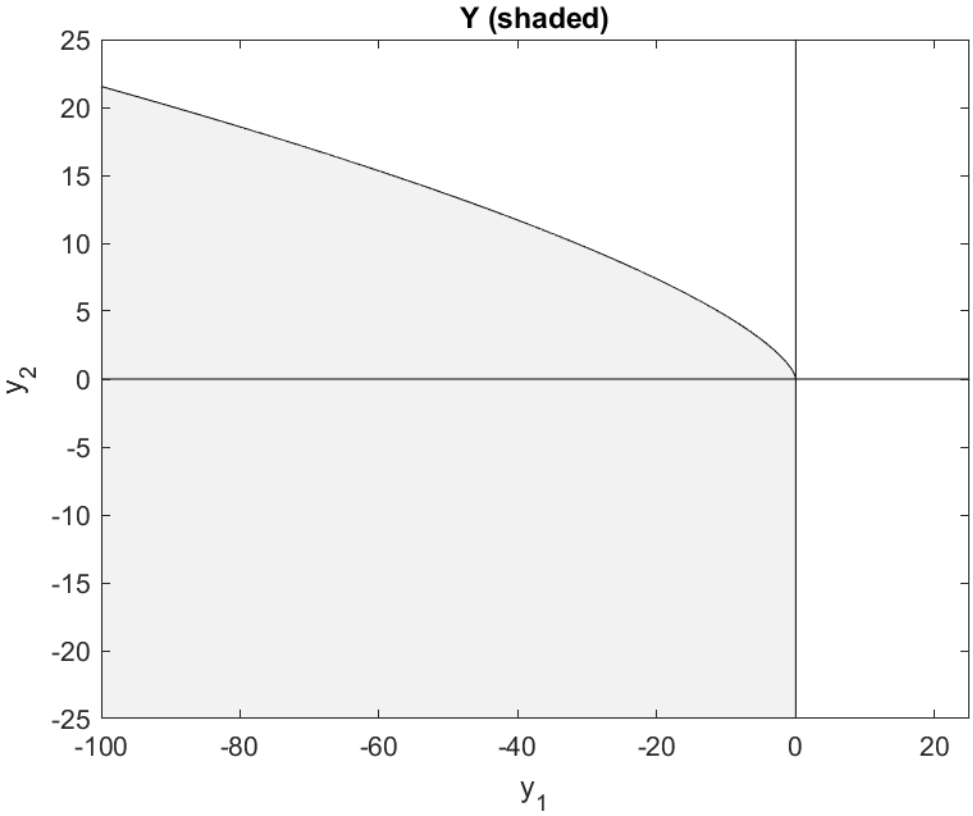
We can fix $w \in \mathbb{R}^k_+$ Let $q_1, q_2 \in \mathbb{R}_+$ Let $z_1 \in Z_1^*, z_2 \in Z_2^*$ where $Z_1^* = \underset{z:f(z)\geq q_1}{\arg\min}\, w \cdot z, Z_2^* = \underset{z:f(z)\geq q_2}{\arg\min}\, w \cdot z$.

By the concavity of $f$, for $t \in (0, 1)$ we have $f(tz_1 + (1-t)z_2) \geq tf(z_1) + (1-t)f(z_2) \geq tq_1 + (1-t)q_2$. Thus, we can produce at least $tq_1 + (1-t)q_2$ by using $tz_1 + (1-t)z_2$ inputs, so the minimum cost of producing $tq_1 + (1-t)q_2$ goods cannot be higher than the cost of those inputs, $w \cdot (tz_1 + (1-t)z_2) = t(w \cdot z_1) + (1-t)(w \cdot z_2)) = tc(q_1, w) + (1-t)c(q_2, w)$. Therefore, $tc(q_1, w) + (1-t)c(q_2, w) \geq c(tq_1 + (1-t)q_2, w)$.

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, Ryan Mather, and Tyler Welch. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

# 2 Question 2

## 2.1 Draw Y




The shaded area in the above figure is $Y$ graphed in Matlab, for a sample value of $B = 1$.

## 2.2 Solve the firm's profit maximization problem to find $\pi(p)$ and $Y^*(p)$.

The firm chooses production to maximize profit: $\max_{-y_1, y_2 \in \mathbb{R}_+} p \cdot (y_1, y_2)'$ s.t. $y_2 \leq B(-y_1)^{2/3}$. Since profits are strictly increasing in $y_2$ the profit maximizing firm will set $y_2 = B(-y_1)^{2/3}$. We will also write $-y_1 = z$. Our optimization problem thus becomes: $\max_{q \in \mathbb{R}_+} p \cdot (-q, Bq^{2/3})'$. Taking the firm's first order conditions, we find that $0 = \frac{d\pi(q)}{dq} = 0 \Rightarrow \frac{d}{dq}\left(-p_1 q + p_2 B q^{2/3}\right) = 0 \Rightarrow -p_1 + (2/3)p_2 B q^{-1/3} = 0 \Rightarrow q = \left(\frac{Bp_2}{(3/2)p_1}\right)^3$. This production yields the maximum profits given $p$, which we can compute as:

$\pi(p) = -p_1 \left(\frac{Bp_2}{(3/2)p_1}\right)^3 + p_2 B \left(\frac{Bp_2}{(3/2)p_1}\right)^2 = \frac{4B^3 p_2^3}{27 p_1^2}$ and $Y^*(p) = \left(-\left(\frac{Bp_2}{(3/2)p_1}\right)^3, B\left(\frac{Bp_2}{(3/2)p_1}\right)^2\right)'$.

### 2.3 Verify that $\pi(p)$ is homogeneous of degree 1, and $y(p)$ is homogeneous of degree 0.

$\pi(\lambda p) = \lambda p_1 \left(\frac{B\lambda p_2}{(3/2)\lambda p_1}\right)^3 + \lambda p_2 B \left(\frac{B\lambda p_2}{(3/2)\lambda p_1}\right)^2 = \lambda(p_1 \left(\frac{Bp_2}{(3/2)p_1}\right)^3 + p_2 B \left(\frac{Bp_2}{(3/2)p_1}\right)^2) = \lambda\pi(p)$ so $\pi(p)$ is homogeneous of degree 1.

$y(\lambda p) = \left(-\left(\frac{B\lambda p_2}{(3/2)\lambda p_1}\right)^3, B\left(\frac{B\lambda p_2}{(3/2)\lambda p_1}\right)^2\right)' = \left(-\left(\frac{Bp_2}{(3/2)p_1}\right)^3, B\left(\frac{Bp_2}{(3/2)p_1}\right)^2\right)' = y(p)$ so $y(p)$ is homogeneous of degree 0.

### 2.4 Verify that $y_1(p) = \frac{\partial \pi}{\partial p_1}$ and $y_2(p) = \frac{\partial \pi}{\partial p_2}$.

$$\frac{\partial \pi}{\partial p_1} = \frac{\partial}{\partial p_1}\left(\frac{4B^3p_2^3}{27p_1^2}\right) = (-2)\left(\frac{4B^3p_2^3}{27p_1^3}\right) = -\left(\frac{Bp_2}{(3/2)p_1}\right)^3 = y_1(p)$$
$$\frac{\partial \pi}{\partial p_1} = \frac{\partial}{\partial p_1}\left(\frac{4B^3p_2^3}{27p_1^2}\right) = 3\frac{4B^3p_3^3}{27p_1^2} = B\left(\frac{Bp_2}{(3/2)p_1}\right)^2 = y_2(p)$$

### 2.5 Calculate $D_p y(p)$ and verify it is symmetric, positive semidefinite, and $[D_p y]p = 0$

$$D_p y(p) = \begin{pmatrix} \frac{\partial y_1(p)}{\partial p_1} & \frac{\partial y_2(p)}{\partial p_1} \\ \frac{\partial y_1(p)}{\partial p_2} & \frac{\partial y_2(p)}{\partial p_2} \end{pmatrix} = \begin{pmatrix} \frac{8B^3p_2^3}{9p_1^4} & -\frac{8B^3p_2^2}{9p_1^3} \\ -\frac{8B^3p_2^2}{9p_1^3} & \frac{8B^3p_2}{9p_1^2} \end{pmatrix}$$

Upon observation. it is clear that $D_p y(p)$ is symmetric. Furthermore, the first element of $D_p y(p)$, $\frac{8B^3p_2^3}{9p_1^4}$, is positive because $B, p_1, p_2$ are all positive. Next we will check the matrix's determinant:

$$\det D_p y(p) = \frac{8B^3p_2^3}{9p_1^4}\frac{8B^3p_2}{9p_1^2} - \left(-\frac{8B^3p_2^2}{9p_1^3}\right)\left(-\frac{8B^3p_2^2}{9p_1^3}\right) = \frac{128B^6p_2^4}{81p_1^6}.$$

The determinant is positive, so $D_p y(p)$ is positive semidefinite.

$$[D_p y(p)]p = \begin{pmatrix} \frac{8B^3p_2^3}{9p_1^4} & -\frac{8B^3p_2^2}{9p_1^3} \\ -\frac{8B^3p_2^2}{9p_1^3} & \frac{8B^3p_2}{9p_1^2} \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \frac{8B^3p_2^3}{9p_1^3} - \frac{8B^3p_2^3}{9p_1^3} \\ -\frac{8B^3p_2^2}{9p_1^2} + \frac{8B^3p_2^2}{9p_1^2} \end{pmatrix} = \vec{0}.$$

# 3 Question 3

### 3.1 What conditions must hold for this profit function to be rationalizable?

From lecture, the profit function is rationalizable if and only if it is homogenous of degree 1 and convex.

### 3.2 Show that $y \in Y^O$ must have $y_1 \leq 0$.

Let $(y_1, y_2) \in Y^O$. Then $p_1 y_1 + p_2 y_2 \leq A p_1^{-2} p_2^3 \; \forall p_1, p_2 \in \mathbb{R}$. Assume for the purpose of contradiction that $y_1 > 0$. Now fix $p_2 > 0$, we can make $p_1 y_1 + p_2 y_2$ unboundedly large by increasing $p_1$ while $A p_1^{-2} p_2^3$ goes to 0, so we can force a contradiction. Thus $y_1 \leq 0$.

### 3.3 Solve the minimization problem and describe the production set $Y^O$.

We want to solve the following problem: $\min_{r>0} Ar^2 - \frac{y_1}{r}$. Taking first order conditions, $2Ar + \frac{y_1}{r^2} = 0 \Rightarrow r = \left(\frac{-y_1}{2A}\right)^{1/3}$. So, $\min_{r>0} Ar^2 - \frac{y_1}{r} = A\left(\frac{-y_1}{2A}\right)^{2/3} - \frac{y_1}{\left(\frac{-y_1}{2A}\right)^{1/3}} = 2^{-2/3} A^{1/3} (-y_1)^{2/3} + 2^{1/3} A^{1/3} (-y_1)^{2/3} = (2^{-2/3} + 2^{1/3}) A^{1/3} (-y_1)^{2/3}$ .

Therefore, $Y^O = \{(y_1, y_2) : y_2 \leq (2^{-2/3} + 2^{1/3}) A^{1/3} (-y_1)^{2/3}\}$.

### 3.4 Verify that $Y = Y^O$ would generate the data we started with.

Let $p_1, p_2 \in \mathbb{R}_{++}$. We want to find $\max_{(y_1, y_2) \in Y} p_1 y_1 + p_2 y_2$. We know that profits are increasing in $y_2$ so $y_2 = (2^{-2/3} + 2^{1/3}) A^{1/3} (-y_1)^{2/3}$. From the previous question we know that $\pi(p) = \frac{4B^3 p_2^3}{27 p_1^2}$ where $B = (2^{-2/3} + 2^{1/3}) A^{1/3}$. Simplifying,

$$\pi(p) = \frac{4(27/4) A p_2^3}{27 p_1^2} = A p_1^{-2} p_2^3$$

Thus, $Y$ generates $\pi(p) = A p_1^{-2} p_2^3$.