Macro PS2

Michael B. Nattinger*

February 2, 2021

1 Question 1

The planner solves the following maximization problem subject to the capital law of motion and the resource constraint:

$$\max_{\{C_t, I_t, K_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t log C_t$$
$$\text{s.t. } K_{t+1} = K_t^{1-\delta} I_t^{\delta}$$
$$\text{and } AK_t^{\alpha} = C_t + I_t$$

We can solve the resource constraint for I_t and plug it into the capital law of motion. Using this simplification, we can write down our Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t log C_t + \lambda_t \left(-K_{t+1} + K_t^{1-\delta} \left(AK_t^{\alpha} - C_t \right)^{\delta} \right)$$

Taking first order conditions with respect to C_t, K_{t+1} we find the following:

$$\frac{\beta^t}{C_t} = \lambda_t \delta K_t^{1-\delta} (AK_t^{\alpha} - C_t)^{\delta - 1}$$
$$\lambda_t = \lambda_{t+1} (K_{t+1}^{1-\delta} \delta (AK_{t+1}^{\alpha} - C_{t+1})^{\delta-1} A\alpha K_{t+1}^{\alpha-1} + (1-\delta) K_{t+1}^{-\delta} \left(AK_{t+1}^{\alpha} - C_{t+1} \right)^{\delta})$$
$$\Rightarrow \lambda_t = \frac{\beta^t}{\delta C_t K_t^{1-\delta} I_t^{\delta-1}}$$
$$\Rightarrow \frac{1}{C_t K_t^{1-\delta} I_t^{\delta-1}} = \frac{\beta}{C_{t+1} K_{t+1}^{1-\delta} I_{t+1}^{\delta-1}} (A\alpha\delta K_{t+1}^{\alpha-\delta} I_{t+1}^{\delta-1} + (1-\delta) K_{t+1}^{-\delta} I_{t+1}^{\delta})$$

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, Katherine Kwok, and Danny Edgel.

$$\frac{1}{C_t K_t^{1-\delta} I_t^{\delta-1}} = \frac{\beta}{C_{t+1}}(A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-1} I_{t+1}) \tag{1}$$

The above equation forms our Euler equation.

Assume we are on the optimal trajectory at time t, and consider a one-period deviation in consumption by an amount D. Our resource constraint implies that this results in a decrease in I_t by an equal amount, D. Then, our K_{t+1} is reduced (to first order approximation) by $-\delta D K_t^{1-\delta} I_t^{\delta-1}$. Then, our consumption in the second equation is reduced by two effects: reduced K_{t+1} leads to less production at time $t+1$, and a larger gap to make up via I_{t+1} to get back onto the optimal trajectory at time $t+2$. The net effect of the first of these terms, to first order expansion, is $-(\delta D K_t^{1-\delta} I_t^{\delta-1})(A\alpha K_{t+1}^{\alpha-1})$, in other words, the reduction in C_{t+1} from the (first order approximation of the) decrease in production in period (t+1). Now we must address the second of these turns. $K_{t+2} = K_{t+1}^{1-\delta} I_{t+1}^{\delta}$ is fixed and we know the value of K_{t+1} so we can determine the value of I_{t+1}. To first order approximation, small deviations of capital and investment $(\Delta K_{t+1}), (\Delta I_{t+1})$ satisfy $(1-\delta)((\Delta K_{t+1}))(K_{t+1}^{-\delta} I_{t+1}^{\delta}) = -\delta(\Delta I_{t+1})(K_{t+1}^{1-\delta} I_{t+1}^{\delta-1}) \Rightarrow (\Delta I_{t+1}) = -\frac{1-\delta}{\delta}(I_{t+1} K_{t+1}^{-1})(\Delta K_{t+1})$. This is taken away from C_{t+1}. Therefore, our second effect of the reduction in K_{t+1} on C_{t+1} is $-(\delta \Delta K_t^{1-\delta} I_t^{\delta-1}) \frac{1-\delta}{\delta} \frac{I_{t+1}}{K_{t+1}}$.

Our marginal utility by making this move is thus

$$dU = \beta^t C_t^{-1} D - \beta^{t+1} C_{t+1}^{-1} \left((\delta K_t^{1-\delta} I_t^{\delta-1}) \left(A\alpha K_{t+1}^{\alpha-1} + \frac{1-\delta}{\delta} \frac{I_{t+1}}{K_{t+1}} \right) \right) D$$

$$dU = 0 \Rightarrow C_t^{-1} = \beta C_{t+1}^{-1} (K_t^{1-\delta} I_t^{\delta-1}) \left(A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)\frac{I_{t+1}}{K_{t+1}} \right)$$

This yields (1), our euler condition. Therefore, the euler condition represents a no-profitable-deviation condition.

2 Question 2

The system of equations that pins down the law of motion for the system are the following:

$$\begin{aligned} \frac{1}{C_t K_t^{1-\delta} I_t^{\delta-1}} &= \frac{\beta}{C_{t+1}}(A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-1} I_{t+1}) \\ A K_t^{\alpha} &= C_t + I_t \\ K_{t+1} &= K_t^{1-\delta} I_t^{\delta} \end{aligned}$$

We can use the resource constraint to rewrite the system of equations without I_t:

$$C_{t+1} = \beta C_t K_t^{1-\delta} (A K_t^{\alpha} - C_t)^{\delta-1} (A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta) K_{t+1}^{-1} (A K_{t+1}^{\alpha} - C_{t+1})) \tag{2}$$

$$K_{t+1} = K_t^{1-\delta} (A K_t^{\alpha} - C_t)^{\delta} \tag{3}$$

Equations (2) and (3) determine the law of motion of the system. We can use these equations and impose stationarity ($K_t = K_{t+1} = \bar{K}, C_t = C_{t+1} = \bar{C}$) to determine the steady state:

$$
\begin{aligned}
1 &= \beta \bar{K}^{1-\delta}(A\bar{K}^{\alpha} - \bar{C})^{\delta-1}(A\alpha\delta\bar{K}^{\alpha-1} + (1-\delta)\bar{K}^{-1}(A\bar{K}^{\alpha} - \bar{C})) \\
1 &= \bar{K}^{-\delta}(A\bar{K}^{\alpha} - \bar{C})^{\delta}
\end{aligned}
$$

The above equations pin down the steady state of the model.

3 Question 3