Macro PS2

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1 Question 1

The planner solves the following maximization problem subject to the capital law of motion and the resource constraint:

$$\max_{\{C_t,I_t,K_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t logC_t$$
$$\text{s.t. } K_{t+1} = K_t^{1-\delta} I_t^{\delta}$$
$$\text{and } AK_t^{\alpha} = C_t + I_t$$

We can solve the resource constraint for I_t and plug it into the capital law of motion. Using this simplification, we can write down our Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t logC_t + \lambda_t \left(-K_{t+1} + K_t^{1-\delta} \left(AK_t^{\alpha} - C_t\right)^{\delta}\right)$$

Taking first order conditions with respect to C_t, K_{t+1} we find the following:

$$\begin{aligned}
\frac{\beta^t}{C_t} &= \lambda_t \delta K_t^{1-\delta}(AK_t^{\alpha} - C_t)^{\delta-1} \\
\lambda_t &= \lambda_{t+1}(K_{t+1}^{1-\delta}\delta(AK_{t+1}^{\alpha} - C_{t+1})^{\delta-1} A\alpha K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-\delta}\left(AK_{t+1}^{\alpha} - C_{t+1}\right)^{\delta}) \\
\Rightarrow \lambda_t &= \frac{\beta^t}{\delta C_t K_t^{1-\delta} I_t^{\delta-1}} \\
\Rightarrow \frac{1}{C_t K_t^{1-\delta} I_t^{\delta-1}} &= \frac{\beta}{C_{t+1} K_{t+1}^{1-\delta} I_{t+1}^{\delta-1}}(A\alpha\delta K_{t+1}^{\alpha-\delta} I_{t+1}^{\delta-1} + (1-\delta)K_{t+1}^{-\delta} I_{t+1}^{\delta})
\end{aligned}$$

$$\frac{1}{C_t K_t^{1-\delta} I_t^{\delta-1}} = \frac{\beta}{C_{t+1}}(A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-1} I_{t+1}) \tag{1}$$

The above equation forms our Euler equation.

Assume we are on the optimal trajectory at time t, and consider a one-period deviation in consumption by an amount D. Our resource constraint implies that this results in a decrease in I_t by an equal amount, D. Then, our K_{t+1} is reduced (to first order approximation) by $-\delta D K_t^{1-\delta} I_t^{\delta-1}$. Then, our consumption in the second equation is reduced by two effects: reduced K_{t+1} leads to less production at time $t+1$, and a larger gap to make up via I_{t+1} to get back onto the optimal trajectory at time $t+2$. The net effect of the first of these terms, to

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first order expansion, is $-(\delta D K_t^{1-\delta} I_t^{\delta-1})(A\alpha K_{t+1}^{\alpha-1})$, in other words, the reduction in C_{t+1} from the (first order approximation of the) decrease in production in period (t+1). Now we must address the second of these turns. $K_{t+2} = K_{t+1}^{1-\delta} I_{t+1}^{\delta}$ is fixed and we know the value of K_{t+1} so we can determine the value of I_{t+1}. To first order approximation, small deviations of capital and investment $(\Delta K_{t+1}), (\Delta I_{t+1})$ satisfy $(1-\delta)((\Delta K_{t+1}))(K_{t+1}^{-\delta} I_{t+1}^{\delta}) = -\delta(\Delta I_{t+1})(K_{t+1}^{1-\delta} I_{t+1}^{\delta-1}) \Rightarrow (\Delta I_{t+1}) = -\frac{1-\delta}{\delta}(I_{t+1} K_{t+1}^{-1})(\Delta K_{t+1})$. This is taken away from C_{t+1}. Therefore, our second effect of the reduction in K_{t+1} on C_{t+1} is $-(\delta \Delta K_t^{1-\delta} I_t^{\delta-1})\frac{1-\delta}{\delta}\frac{I_{t+1}}{K_{t+1}}$.

Our marginal utility by making this move is thus

$$dU = \beta^t C_t^{-1} D - \beta^{t+1} C_{t+1}^{-1} \left((\delta K_t^{1-\delta} I_t^{\delta-1}) \left(A\alpha K_{t+1}^{\alpha-1} + \frac{1-\delta}{\delta} \frac{I_{t+1}}{K_{t+1}} \right) \right) D$$

$$dU = 0 \Rightarrow C_t^{-1} = \beta C_{t+1}^{-1} (K_t^{1-\delta} I_t^{\delta-1}) \left(A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)\frac{I_{t+1}}{K_{t+1}} \right)$$

This yields (1), our euler condition. Therefore, the euler condition represents a no-profitable-deviation condition.

2 Question 2

The system of equations that pins down the law of motion for the system are the following:

$$\begin{aligned}
\frac{1}{C_t K_t^{1-\delta} I_t^{\delta-1}} &= \frac{\beta}{C_{t+1}} (A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta) K_{t+1}^{-1} I_{t+1}) \\
A K_t^{\alpha} &= C_t + I_t \\
K_{t+1} &= K_t^{1-\delta} I_t^{\delta}
\end{aligned}$$

We can use the resource constraint to rewrite the system of equations without I_t:

$$C_{t+1} = \beta C_t K_t^{1-\delta} (A K_t^{\alpha} - C_t)^{\delta-1} (A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta) K_{t+1}^{-1} (A K_{t+1}^{\alpha} - C_{t+1})) \tag{2}$$

$$K_{t+1} = K_t^{1-\delta} (A K_t^{\alpha} - C_t)^{\delta} \tag{3}$$

Equations (2) and (3) determine the law of motion of the system. We can use these equations and impose stationarity ($K_t = K_{t+1} = \bar{K}, C_t = C_{t+1} = \bar{C}$) to determine the steady state:

$$\begin{aligned}
1 &= \beta \bar{K}^{1-\delta} (A\bar{K}^{\alpha} - \bar{C})^{\delta-1} (A\alpha\delta \bar{K}^{\alpha-1} + (1-\delta) \bar{K}^{-1} (A\bar{K}^{\alpha} - \bar{C})) \\
1 &= \bar{K}^{-\delta} (A\bar{K}^{\alpha} - \bar{C})^{\delta}
\end{aligned}$$

The above equations pin down the steady state of the model.

3 Question 3

We will log linearize about the steady state defined in Question 2. We first will define $I = AK^{\alpha} - C$. Log linearizing I we get:

$$\begin{aligned}
\bar{I}(1 + i_t) &= A\bar{K}^{\alpha}(1 + \alpha k_t) - \bar{C}(1 + c_t) \\
\Rightarrow i_t &= A\frac{\bar{K}^{\alpha}}{\bar{I}} k_t - \frac{\bar{C}}{\bar{I}} c_t \\
\Rightarrow i_t &= A\alpha\bar{K}^{\alpha-1} k_t - \frac{\bar{C}}{\bar{I}} c_t,
\end{aligned}$$

where we have used the fact that equation (3) implies that $\bar{K} = \bar{I}$.

Using this we can log linearize equation (3):

$$\begin{aligned}
\bar{K}(1+k_{t+1}) &= \bar{K}^{1-\delta}(1+(1-\delta)k_t)\bar{I}^{\delta}(1+\delta i_t) \\
\Rightarrow k_{t+1} &= (1-\delta)k_t + \delta i_t \\
&= (1-\delta)k_t + \delta\left(A\alpha\bar{K}^{\alpha-1}k_t - \frac{\bar{C}}{\bar{I}}c_t\right) \\
&= (1-\delta + A\alpha\bar{K}^{\alpha-1}\delta)k_t - \delta\frac{\bar{C}}{\bar{I}}c_t
\end{aligned}$$

Now we can log linearize equation (2):

$$\begin{aligned}
(1+c_{t+1}) &= \beta(1+c_t)(1+(1-\delta)k_t)(1+(\delta-1)i_t) \\
&\quad * (A\alpha\delta\bar{K}^{\alpha-1}(1+(\alpha-1)k_{t+1}) + (1-\delta)(1-k_{t+1})(1+i_{t+1})) \\
c_{t+1} &= \beta(A\alpha\delta\bar{K}^{\alpha-1}(\alpha-1)k_{t+1} + (1-\delta)(i_{t+1}-k_{t+1}) \\
&\quad + (A\alpha\delta\bar{K}^{\alpha-1} + (1-\delta))(c_t + (1-\delta)k_t + (\delta-1)i_t)) \\
c_{t+1} &= \frac{A\alpha\delta\bar{K}^{\alpha-1}(\alpha-1)}{(A\alpha\delta\bar{K}^{\alpha-1}+(1-\delta))}k_{t+1} + \frac{(1-\delta)}{(A\alpha\delta\bar{K}^{\alpha-1}+(1-\delta))}i_{t+1} \\
&\quad - \frac{(1-\delta)}{(A\alpha\delta\bar{K}^{\alpha-1}+(1-\delta))}k_{t+1} + c_t + (1-\delta)k_t + (\delta-1)i_t
\end{aligned}$$

4 Question 4

Define, for sake of convenience, $\phi = A\bar{K}^{\alpha-1}$. Note that the euler equation steady state yields $1-\delta+\phi\alpha\delta = 1/\beta$, and the investment steady state yields $\bar{C}/\bar{K} = \phi - 1$. Then, we can reduce the above equation to the following:

$$\begin{aligned}
c_{t+1} &= \beta(\phi\alpha\delta(\alpha-1) - 1 + \delta)k_{t+1} + \beta(1-\delta)i_{t+1} + c_t + (1-\delta)k_t - (1-\delta)i_t, \\
&= \beta(\phi\alpha\delta(\alpha-1) - 1 + \delta)k_{t+1} + \beta(1-\delta)\left(\phi\alpha k_{t+1} - (\phi-1)c_{t+1}\right) \\
&\quad + c_t + (1-\delta)k_t - (1-\delta)\left(\phi\alpha k_t - (\phi-1)c_t\right)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow c_{t+1}\left(1+\beta(1-\delta)(\phi-1)\right) &= \beta(\phi\alpha\delta(\alpha-1) + (1-\delta)(\phi\alpha-1))k_{t+1} \\
&\quad + (\delta + (1-\delta)\phi)c_t + (1-\delta)\left(1-\phi\alpha\right)k_t
\end{aligned}$$

$$\begin{aligned}
c_{t+1}\left(1+\beta(1-\delta)(\phi-1)\right) &= \beta(\phi\alpha\delta(\alpha-1) + (1-\delta)(\phi\alpha-1))\left(\beta^{-1}k_t - \delta(\phi-1)c_t\right) \\
&\quad + (\delta + (1-\delta)\phi)c_t + (1-\delta)\left(1-\phi\alpha\right)k_t \\
&= ((\phi\alpha\delta(\alpha-1) + (1-\delta)(\phi\alpha-1)) + (1-\delta) - (1-\delta)\phi\alpha)k_t \\
&\quad + ((\delta + (1-\delta)\phi) - \delta(\phi-1)\beta(\phi\alpha\delta(\alpha-1) + (1-\delta)(\phi\alpha-1)))c_t \\
&= \phi\alpha\delta(\alpha-1)k_t \\
&\quad + (\delta + \phi - \phi\delta + \delta\phi - \delta - \delta(\phi-1)\beta\phi\alpha(1+\delta(\alpha-1)))c_t
\end{aligned}$$

$$\begin{aligned}
c_{t+1} &= \frac{\phi\alpha\delta(\alpha-1)}{1+\beta(1-\delta)(\phi-1)}k_t \\
&\quad + \frac{\phi - \delta(\phi-1)\beta\phi\alpha(1+\delta(\alpha-1))}{1+\beta(1-\delta)(\phi-1)}c_t
\end{aligned}$$

Define $\theta := \phi - \delta(\phi-1)\beta\phi\alpha(1+\delta(\alpha-1))$. Then, we can write our log linearized law of motion as the following:

$$\begin{pmatrix} k_{t+1} \\ c_{t+1} \end{pmatrix} = X_{t+1} = \begin{pmatrix} \beta^{-1} & -\delta(\phi-1) \\ \frac{\phi\alpha\delta(\alpha-1)}{1+\beta(1-\delta)(\phi-1)} & \frac{\theta}{1+\beta(1-\delta)(\phi-1)} \end{pmatrix} \begin{pmatrix} k_t \\ c_t \end{pmatrix} = AX_t. \tag{4}$$

We now must decompose $A = \Gamma\Omega\Gamma^{-1}$. We will solve for the eigenvectors of A, which form the column vector $(1-\delta)$:

$$\begin{aligned}
det(A-\lambda I_2) &= det \begin{pmatrix} \beta^{-1}-\lambda & -\delta(\phi-1) \\ \frac{\phi\alpha\delta(\alpha-1)}{1+\beta(1-\delta)(\phi-1)} & \frac{\theta}{1+\beta(1-\delta)(\phi-1)} - \lambda \end{pmatrix} \\
&= (\beta^{-1}-\lambda)\left(\frac{\theta}{1+\beta(1-\delta)(\phi-1)} - \lambda\right) + \frac{\phi\alpha\delta(\alpha-1)\delta(\phi-1)}{1+\beta(1-\delta)(\phi-1)} \\
&= \frac{\theta\beta^{-1} + \phi\alpha\delta(\alpha-1)\delta(\phi-1)}{1+\beta(1-\delta)(\phi-1)} - \lambda\left(\beta^{-1} + \frac{\theta}{1+\beta(1-\delta)(\phi-1)}\right) + \lambda^2.
\end{aligned}$$

The eigenvalues are the roots of the above expression. We can solve by applying the quadratic formula:

$$\begin{aligned}
\lambda = (1/2)&\left(\beta^{-1} + \frac{\theta}{1+\beta(1-\delta)(\phi-1)} \pm \sqrt{\left(\beta^{-1} + \frac{\theta}{1+\beta(1-\delta)(\phi-1)}\right)^2 - 4\frac{\theta\beta^{-1} + \phi\alpha\delta(\alpha-1)\delta(\phi-1)}{1+\beta(1-\delta)(\phi-1)}}\right) \\
= (1/2)&\left(\beta^{-1} + \frac{\phi - \delta(\phi-1)\beta\phi\alpha(1+\delta(\alpha-1))}{1+\beta(1-\delta)(\phi-1)}\right) \\
\pm (1/2)&\left(\sqrt{\left(\beta^{-1} + \frac{\phi - \delta(\phi-1)\beta\phi\alpha(1+\delta(\alpha-1))}{1+\beta(1-\delta)(\phi-1)}\right)^2 - 4\frac{\theta\beta^{-1} + \phi\alpha\delta(\alpha-1)\delta(\phi-1)}{1+\beta(1-\delta)(\phi-1)}}\right)
\end{aligned}$$

Now note the following:

$$\begin{aligned}
&\beta^{-1} + \frac{\phi - \delta(\phi-1)\beta\phi\alpha(1+\delta(\alpha-1))}{1+\beta(1-\delta)(\phi-1)} \\
&= \frac{\beta^{-1} + (1-\delta)(\phi-1)}{1+\beta(1-\delta)(\phi-1)} + \frac{\phi - \delta(\phi-1)\beta\phi\alpha(1+\delta(\alpha-1))}{1+\beta(1-\delta)(\phi-1)} \\
&= \frac{2\phi + \delta\phi(\alpha - 1 - \phi\beta\alpha + \beta\alpha - \phi\delta\beta(\alpha-1) + \delta\beta\alpha(\alpha-1))}{1+\beta(1-\delta)(\phi-1)} \\
&> \frac{2\phi}{1+\beta(1-\delta)(\phi-1)}
\end{aligned}$$

Furthermore, using the fact that $\beta^{-1} = 1 - \delta + \delta\alpha\phi \Rightarrow 1 - \beta + \beta\delta = \beta\delta\alpha\phi$,

$$\begin{aligned}
1+\beta(1-\delta)(\phi-1) &= \beta(1-\delta)\phi + 1 - \beta + \beta\delta \\
&= \beta(1-\delta)\phi + \beta\delta\phi\alpha \\
&= \phi(\beta - \beta\delta + \beta\delta\alpha) \\
&< \phi
\end{aligned}$$

Therefore, for the (+) eigenvalue (denoted λ_1)

$$\begin{aligned}
\lambda_1 &= (1/2)\left(\beta^{-1} + \frac{\theta}{1+\beta(1-\delta)(\phi-1)} + \sqrt{\left(\beta^{-1} + \frac{\theta}{1+\beta(1-\delta)(\phi-1)}\right)^2 - 4\frac{\theta\beta^{-1} + \phi\alpha\delta(\alpha-1)\delta(\phi-1)}{1+\beta(1-\delta)(\phi-1)}}\right) \\
&> (1/2)\left(\beta^{-1} + \frac{\phi - \delta(\phi-1)\beta\phi\alpha(1+\delta(\alpha-1))}{1+\beta(1-\delta)(\phi-1)}\right) \\
&> \frac{\phi}{1+\beta(1-\delta)(\phi-1)} \\
&> \frac{\phi}{\phi} \\
&= 1
\end{aligned}$$

To summarize, $\lambda_1 > 1$. This is the eigenvalue corresponding to the explosive eigenvector of our system. Working ahead slightly, in part (5) we solve for the exact formulation of the saddle path. This also proves the existence of the saddle path, and since λ_1 has a magnitude greater than one we know that λ_2 is guaranteed to have magnitude less than one.

We now make the following changes to our system:

$$\Gamma^{-1}X_{t+1} = Y_{t+1} = \Omega\Gamma^{-1}X_t = \Omega Y_t$$

As Ω is diagonal with the explosive eigenvalue in the upper left entry, we know that $Y_{1,t} = 0\forall t$. This defines our saddle path. Equivalently, the saddle path is determined by the second column of the eigenvector matrix Γ. We do not explicitly solve for Γ, but these facts lead directly to the observation that the solution yields some z such that $c_t = zk_t$,[1] which defines the Blanchard-Kahn first order approximation to the saddle path. In Part 5 we explicitly show that the true solution is $c_t = \alpha k_t$, and since the Blanchard-Kahn approximation is a first-order taylor approximation of a linear relationship, the Blanchard-Kahn approximation yields exactly the true solution, i.e. $z = \alpha$.

5 Question 5

We will guess that the solution to the euler equation is of the form $C_t = ZK_t^z$:

$$\begin{aligned}
ZK_{t+1}^z &= \beta ZK_t^z K_t^{1-\delta}(AK_t^\alpha - ZK_t^z)^{\delta-1}(A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-1}(AK_{t+1}^\alpha - ZK_{t+1}^z)) \\
K_{t+1} &= K_t^{1-\delta}(AK_t^\alpha - ZK_t^z)^\delta
\end{aligned}$$

The above system has several possible solutions. The first such solution is the "eat everything" option where $I_t = 0 \Rightarrow AK_t^\alpha = C_t \Rightarrow K_{t+1} = 0$. This is a possible solution but not the only solution, and in general it is not the solution corresponding to the saddle path.

To find the other solutions for Z, z we simplify the above expressions:

$$\begin{aligned}
Z(K_t^{1-\delta}(AK_t^\alpha - ZK_t^z)^\delta)^z &= \beta ZK_t^z K_t^{1-\delta}(AK_t^\alpha - ZK_t^z)^{\delta-1}(A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-1}(AK_{t+1}^\alpha - ZK_{t+1}^z)) \\
K_t^z(AK_t^{\alpha-1} - ZK_t^{z-1})^{z\delta} &= \beta K_t^z K_t^{1-\delta}(AK_t^\alpha - ZK_t^z)^{\delta-1}(A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-1}(AK_{t+1}^\alpha - ZK_{t+1}^z)) \\
(AK_t^{\alpha-1} - ZK_t^{z-1})^{z\delta} &= \beta(AK_t^{\alpha-1} - ZK_t^{z-1})^{\delta-1}(A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)(AK_{t+1}^{\alpha-1} - ZK_{t+1}^{z-1}))
\end{aligned}$$

[1] Let $\Gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for some a, b, c, d. $Y_{1,t} = 0$ implies that the first row of $\Gamma^{-1}X_t = 0$. As eigenvectors are determined only up to scale, without loss of generality we can define Γ such that $det(\Gamma) = 1 \Rightarrow \Gamma^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Therefore, $Y_{1,t} = 0 \Rightarrow k_t d = c_t b \Rightarrow c_t = zk_t$, for $z = d/b$. We do not have to worry about $b = 0$ in this case because our exact solution in part (5) is exactly of the form of $c_t = zk_t$ for some $-\infty < z < \infty$ so $b \neq 0$.

Note that if $z = \alpha$ this collapses to the following:

$$\begin{aligned}
((A-Z)K_t^{\alpha-1})^{\alpha\delta} &= \beta((A-Z)K_t^{\alpha-1})^{\delta-1}(A\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)((A-Z)K_{t+1}^{\alpha-1})) \\
((A-Z)K_t^{\alpha-1})^{\alpha\delta} &= \beta((A-Z)K_t^{\alpha-1})^{\delta-1}(A\alpha\delta + (1-\delta)((A-Z)))K_{t+1}^{\alpha-1} \\
((A-Z)K_t^{\alpha-1})^{\alpha\delta} &= \beta((A-Z)K_t^{\alpha-1})^{\delta-1}(A\alpha\delta + (1-\delta)((A-Z)))K_t^{\alpha-1}((A-Z)K_t^{\alpha-1})^{\delta(\alpha-1)} \\
((A-Z)K_t^{\alpha-1})^{\delta} &= \beta((A-Z)K_t^{\alpha-1})^{\delta-1}(A\alpha\delta + (1-\delta)(A-Z))K_t^{\alpha-1} \\
((A-Z)K_t^{\alpha-1}) &= \beta(A\alpha\delta + (1-\delta)(A-Z))K_t^{\alpha-1} \\
(A-Z) &= \beta(A\alpha\delta + (1-\delta)(A-Z)) \\
A-Z &= \beta A\alpha\delta + (1-\delta)\beta A - (1-\delta)\beta Z \\
Z(1-\beta+\beta\delta) &= A - \beta A\alpha\delta - (1-\delta)\beta A \\
\Rightarrow Z &= \frac{A(1-\beta\alpha\delta-(1-\delta)\beta)}{(1-\beta+\beta\delta)}.
\end{aligned}$$

Therefore, $C_t = ZK_t^z$ satisfies the euler conditions for $z = \alpha, Z = \frac{A(1-\beta\alpha\delta+(1-\delta)\beta)}{(1-\beta+\beta\delta)}$. It defines, therefore, the saddle path. Note that $C_t = ZK_t^z \Rightarrow \frac{C_t}{C} = \frac{ZK_t^z}{ZK^z} \Rightarrow c_t = zk_t$. Therefore, the saddle path, written in terms of log-deviation from steady state, is exactly linear. The Blanchard-Kahn approximation is, therefore, a linear approximation of a linear function, and thus it must yield the exact solution to the social planner's problem.

6 Question 6

Let us first write the planner's problem. We will jump immediately to the lagrangian formulation:

$$\mathcal{L} = E_t \sum_{t=0}^{\infty} \beta^t logC_t + \lambda_t \left(-K_{t+1} + K_t^{1-\delta}\left(A_t K_t^\alpha - C_t\right)^\delta\right)$$

This yields the following first order conditions:

$$\begin{aligned}
\frac{\beta^t}{C_t} &= \lambda_t \delta K_t^{1-\delta}(A_t K_t^\alpha - C_t)^{\delta-1} \\
\lambda_t &= E_t \lambda_{t+1}(K_{t+1}^{1-\delta}\delta(A_{t+1}K_{t+1}^\alpha - C_{t+1})^{\delta-1}A_{t+1}\alpha K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-\delta}\left(A_{t+1}K_{t+1}^\alpha - C_{t+1}\right)^\delta) \\
\Rightarrow \lambda_t &= \frac{\beta^t}{\delta C_t K_t^{1-\delta} I_t^{\delta-1}} \\
\Rightarrow \frac{1}{C_t K_t^{1-\delta} I_t^{\delta-1}} &= E_t\left[\frac{\beta}{C_{t+1}}(A_{t+1}\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-1} I_{t+1})\right]
\end{aligned}$$

The above expression forms our euler condition for the stochastic case. Inspired by our solution to question (5) we will guess the solution to the euler equation in the stochastic case takes the form $C_t = ZK_t^z$, and solve for Z_t, z:

$$\begin{aligned}
E_t[Z_{t+1}K_{t+1}^z] &= \beta E_t[Z_t K_t^z K_t^{1-\delta}(A_t K_t^\alpha - Z_t K_t^z)^{\delta-1}(A_{t+1}\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-1}(A_{t+1}K_{t+1}^\alpha - Z_{t+1}K_{t+1}^z))] \\
K_{t+1} &= K_t^{1-\delta}(A_t K_t^\alpha - Z_t K_t^z)^\delta
\end{aligned}$$

As before we still have the 'eat everything' solution which will trivially satisfy the euler condition. We will solve for an additional solution:

$$\begin{aligned}
E_t[Z_{t+1}(K_t^{1-\delta}(A_tK_t^\alpha - Z_tK_t^z)^\delta)^z] &= \beta E_t[Z_tK_t^zK_t^{1-\delta}(A_tK_t^\alpha - Z_tK_t^z)^{\delta-1} \\
&\quad * (A_{t+1}\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)K_{t+1}^{-1}(A_{t+1}K_{t+1}^\alpha - Z_{t+1}K_{t+1}^z))] \\
E_t[Z_{t+1}](A_tK_t^{\alpha-1} - ZK_t^{z-1})^{z\delta} &= \beta Z_t(A_tK_t^{\alpha-1} - Z_tK_t^{z-1})^{\delta-1} \\
&\quad * E_t[(A_{t+1}\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)(A_{t+1}K_{t+1}^{\alpha-1} - Z_{t+1}K_{t+1}^{z-1}))]
\end{aligned}$$

We guess and verify that $z = \alpha$ is a solution.

$$\begin{aligned}
E_t[Z_{t+1}]((A_t - Z_t)K_t^{\alpha-1})^{\alpha\delta} &= \beta Z_t((A_t - Z_t)K_t^{\alpha-1})^{\delta-1}E_t[(A_{t+1}\alpha\delta K_{t+1}^{\alpha-1} + (1-\delta)((A_{t+1} - Z_{t+1})K_{t+1}^{\alpha-1}))] \\
E_t[Z_{t+1}]((A_t - Z_t)K_t^{\alpha-1})^{\alpha\delta} &= \beta Z_t((A_t - Z_t)K_t^{\alpha-1})^{\delta-1}E_t[(A_{t+1}\alpha\delta + (1-\delta)((A_{t+1} - Z_{t+1})))K_{t+1}^{\alpha-1}] \\
E_t[Z_{t+1}]((A_t - Z_t)K_t^{\alpha-1})^{\alpha\delta} &= \beta Z_t((A_t - Z)K_t^{\alpha-1})^{\delta-1} \\
&\quad * E_t[(A_{t+1}\alpha\delta + (1-\delta)((A_{t+1} - Z_{t+1})))K_t^{\alpha-1}((A_t - Z_t)K_t^{\alpha-1})^{\delta(\alpha-1)}] \\
E_t[Z_{t+1}]((A_t - Z_t)K_t^{\alpha-1})^{\delta} &= \beta Z_t[((A_t - Z_t)K_t^{\alpha-1})^{\delta-1}E_t[(A_{t+1}\alpha\delta + (1-\delta)(A_{t+1} - Z_{t+1}))]K_t^{\alpha-1} \\
E_t[Z_{t+1}]((A_t - Z_t)K_t^{\alpha-1}) &= \beta Z_tE_t[(A_{t+1}\alpha\delta + (1-\delta)(A_{t+1} - Z_{t+1}))]K_t^{\alpha-1} \\
E_t[Z_{t+1}](A_t - Z_t) &= \beta E_t[(A_{t+1}\alpha\delta + (1-\delta)(A_{t+1} - Z_{t+1}))] \\
A_t - Z_t &= \beta\frac{E_t[A_{t+1}\alpha\delta + (1-\delta)\beta A_{t+1} - (1-\delta)\beta Z_{t+1}]}{E_t[Z_{t+1}]} \\
Z_t &= A_t - \beta\frac{E_t[A_{t+1}\alpha\delta + (1-\delta)\beta A_{t+1} - (1-\delta)\beta Z_{t+1}]}{E_t[Z_{t+1}]}
\end{aligned}$$

Therefore, for the recursive function Z_t which satisfies the expression,

$$Z_t = A_t - \beta\frac{E_t[A_{t+1}\alpha\delta + (1-\delta)\beta A_{t+1} - (1-\delta)\beta Z_{t+1}]}{E_t[Z_{t+1}]},$$

$C_t = Z_tK_t^\alpha$ satisfies the euler equation.

7 Question 7

Our solution for Question 5 shows that consumption and capital deviations from the steady state are perfectly correlated. This comes from the formulation for capital law of motion, which ensures that investment is perfectly correlated with capital deviations, and therefore consumption will also be perfectly correlated with capital levels via the resource constraint. In the real world, consumption and capital deviations are highly correlated, as we saw in lecture 4. This implies that the model may do a decent job of explaining fluctuations of consumption and capital levels.

8 Question 8 (bonus task)

A simple way of introducing labor follows:

$$\begin{aligned}
U(C_t, L_t) &= logC_t - L_t^2/2, \\
F(K_t, L_t) &= AK_t^\alpha L_t^{1-\alpha}
\end{aligned}$$

In solving this model, almost everything will be the same, however there will be a labor-leisure tradeoff. After log linearlizing, the main difference will be an intratemporal relationship between labor and consumption. This can be used to solve for labor deviations from the steady state in terms of consumption deviations within the same period, and the rest of the Blanchard-Kahn method (eigenvector decomposition, solving for explosive roots, etc.) will be very similar to those solved for in part 5. Labor deviations can then be backed out. In other words, by introducing labor in this way we have not sacrificed analytical tractability.