Econometrics HW5

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1 Question 1

1.1 Part i

It is pretty simple to show that U_t, W_t, V_t are mean 0:

$$\begin{aligned} E[U_t] &= E[\epsilon_t \epsilon_{t-1}] \\ &= E[\epsilon_t]E[\epsilon_{t-1}] \\ &= 0. \end{aligned}$$

$$\begin{aligned} E[W_t] &= E[\epsilon_t \epsilon_0] \\ &= E[\epsilon_t]E[\epsilon_0] \\ &= 0. \end{aligned}$$

$$\begin{aligned} E[V_t] &= E[\epsilon_t^2 \epsilon_{t-1}] \\ &= E[\epsilon_t^2]E[\epsilon_{t-1}] \\ &= 0. \end{aligned}$$

We now will find autocovariance functions γ, ψ, η for U_t, W_t, V_t respectively.

$$\gamma(k) = Cov(U_t, U_{t-k}) = Cov(\epsilon_t \epsilon_{t-1}, \epsilon_{t-k}\epsilon_{t-k-1}) = \begin{cases} 0, k > 1 \\ Cov(\epsilon_t\epsilon_{t-1}, \epsilon_{t-1}\epsilon_{t-2}) = E[\epsilon_t \epsilon_{t-2}\epsilon_{t-1}^2] = 0, k = 1 \\ Cov(\epsilon_t\epsilon_{t-1}, \epsilon_t\epsilon_{t-1}) = E[\epsilon_t^2\epsilon_{t-1}^2] = \sigma^4, k = 0 \end{cases}$$

$$\psi(k) = Cov(W_t, W_{t-k}) = Cov(\epsilon_t\epsilon_0, \epsilon_{t-k}\epsilon_0) = \begin{cases} E[\epsilon_t\epsilon_0^2\epsilon_{t-k}] = 0, k > 0 \\ E[\epsilon_t^2\epsilon_0^2] = \sigma^4, k = 0 \end{cases}$$

$$\eta(k) = Cov(V_t, V_{t-k}) = Cov(\epsilon_t^2\epsilon_{t-1}, \epsilon_{t-k}^2\epsilon_{t-k-1}) = \begin{cases} 0, k > 1 \\ E[\epsilon_t^2\epsilon_{t-1}^3\epsilon_{t-2}] = 0, k = 1 \\ E[\epsilon_t^4\epsilon_{t-1}^2] = \sigma^2 E[\epsilon_t^4], k = 0 \end{cases}$$

Therefore, U_t, W_t, V_t all have autocovariance functions and are covariance stationary.

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, Katherine Kwok, and Danny Edgel.

1.2 Part ii

$$
\begin{aligned}
Var(\bar{U}) &= Var\left(\frac{1}{T}\sum_{t=1}^{T} U_t\right) \\
&= \frac{1}{T} Var(U_t) \\
&= \frac{\sigma^4}{T} \\
&\to 0
\end{aligned}
$$

The same is true for $Var(\bar{W}), Var(\bar{V})$ by the same logic. Therefore, $\bar{U}, \bar{V}, \bar{W}$ all converge in probability to their means (0).

1.3 Part iii

$$
\begin{aligned}
Var(\hat{\gamma}_U(0)) &= Var(\frac{1}{T}\sum_{t=1}^{T} U_t^2) \\
&= \frac{1}{T} Var(U_t^2) \\
&= \frac{1}{T} Var(\epsilon_t^2 \epsilon_{t-1}^2) \\
&= \frac{E[\epsilon_t^4 \epsilon_{t-1}^4] - E[\epsilon_t^2 \epsilon_{t-1}^2]^2}{T} \\
&= \frac{E[\epsilon_t^4] E[\epsilon_{t-1}^4] - E[\epsilon_t^2]^2 E[\epsilon_{t-1}^2]^2}{T} \\
&= \frac{E[\epsilon_t^4] E[\epsilon_{t-1}^4] - \sigma^8}{T} \\
&\to 0.
\end{aligned}
$$

$$
\begin{aligned}
Var(\hat{\gamma}_W(0)) &= Var(\frac{1}{T}\sum_{t=1}^{T} W_t^2) \\
&= \frac{1}{T} Var(W_t^2) \\
&= \frac{1}{T} Var(\epsilon_t^2 \epsilon_0^2) \\
&= \frac{E[\epsilon_t^4 \epsilon_0^4] - E[\epsilon_t^2 \epsilon_0^2]^2}{T} \\
&= \frac{E[\epsilon_t^4] E[\epsilon_0^4] - E[\epsilon_t^2]^2 E[\epsilon_0^2]^2}{T} \\
&= \frac{E[\epsilon_t^4] E[\epsilon_0^4] - \sigma^8}{T} \\
&\to 0.
\end{aligned}
$$

$$
\begin{aligned}
Var(\hat{\gamma}_V(0)) &= Var(\frac{1}{T}\sum_{t=1}^{T} V_t^2) \\
&= \frac{1}{T} Var(V_t^2) \\
&= \frac{1}{T} Var(\epsilon_t^4 \epsilon_{t-1}^2) \\
&= \frac{E[\epsilon_t^8 \epsilon_{t-1}^4] - E[\epsilon_t^4 \epsilon_{t-1}^2]^2}{T} \\
&= \frac{E[\epsilon_t^8] E[\epsilon_{t-1}^4] - E[\epsilon_t^4]^2 E[\epsilon_{t-1}^2]^2}{T} \\
&\to 0.
\end{aligned}
$$

So, these are converging to point masses. However, note that:

$$
\begin{aligned}
\hat{\gamma}_W(0) &= \frac{1}{T}\sum_{t=1}^{T} W_t^2 \\
&= \epsilon_0^2 \frac{1}{T}\sum_{t=1}^{T} \epsilon_t^2 \\
&\to_p \epsilon_0^2 \sigma^2.
\end{aligned}
$$

This is clearly not its expectation:

$$
\begin{aligned}
E[\hat{\gamma}_W(0)] &= E\left[\frac{1}{T}\sum_{t=1}^{T} W_t^2\right] \\
&= \frac{1}{T}\sum_{t=1}^{T} E\left[\epsilon_t^2 \epsilon_0^2\right] \\
&= E[\epsilon_t^2] E[\epsilon_0^2] \\
&= \sigma^4.
\end{aligned}
$$

The other estimators clearly do converge to their expectations.

1.4 Part iv

From the above, we know that $\bar{W}$ does not converge to its expectation, so we cannot use our Martingale CLT on $\sqrt{T}\bar{W}$. It is, in fact, not asymptotically normal:

$$
\sqrt{T}\bar{W} = \epsilon_0 \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \epsilon_t
$$

$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \epsilon_t$ is normal, but ϵ_0 is random and in general the product of a normal with another arbitrary random variable is not normal. Therefore, $\sqrt{T}\bar{W}$ is not normal.

We know that U_t is strictly stationary with finite second moment and convergence in probability of the second sample moment. Moreover,

$$
\begin{aligned}
E[U_t | U_{t-1}, \ldots, U_1] &= E[E[\epsilon_t \epsilon_{t-1} | \epsilon_{t-1}, \ldots, \epsilon_0] | U_{t-1}, \ldots, U_1] \\
&= E[\epsilon_{t-1}(0) | U_{t-1}, \ldots, U_1] \\
&= 0.
\end{aligned}
$$

Therefore, we can use the martingale CLT and $\sqrt{T}\bar{U} \to_d N(0, \sigma^4)$

We know that V_t is strictly stationary with finite second moment and convergence in probability of the second sample moment. Note that the conditioning trick that we used before will not work with V_t. Instead, we denote $\tilde{V}_t = V_{T-t+1}$ and work backwards:

$$\begin{aligned} E[\tilde{V}_t|\tilde{V}_{t-1}, \ldots, \tilde{V}_1] &= E[E[\epsilon^2_{T-t+1}\epsilon_{T-t}|\epsilon_T, \ldots, \epsilon_{T-t+1}]|\tilde{V}_{t-1}, \ldots, \tilde{V}_1] \\ &= E[\epsilon^2_{T-t+1}(0)|\tilde{V}_{t-1}, \ldots, \tilde{V}_1] \\ &= 0. \end{aligned}$$

Therefore, we can use the martingale CLT and prove that $\sqrt{T}V$ is asymptotically normal.

2 Question 2

2.1 Part i

The time series is easily simulated in Matlab. Code screenshots follow at the end of the problem set. Results from the first simulation are below:

	$\hat{\alpha}_0$	$\hat{\alpha}_0$ LB	$\hat{\alpha}_0$ UB	$\hat{\delta}_0$	$\hat{\delta}_0$ LB	$\hat{\delta}_0$ UB	$\hat{\rho}_1$	$\hat{\rho}_1$ LB	$\hat{\rho}_1$ UB
$(T = 50, \rho_1 = 0.7)$	2.04	1.33	2.75	0.984	0.646	1.32	0.507	0.317	0.696
$(T = 50, \rho_1 = 0.9)$	1.68	0.762	2.6	0.835	0.598	1.07	0.883	0.787	0.98
$(T = 50, \rho_1 = 0.95)$	1.34	0.813	1.86	1.41	1.14	1.68	0.911	0.825	0.998
$(T = 150, \rho_1 = 0.7)$	0.956	0.605	1.31	1.08	0.943	1.23	0.67	0.591	0.749
$(T = 150, \rho_1 = 0.9)$	1.32	0.803	1.83	0.885	0.723	1.05	0.896	0.855	0.937
$(T = 150, \rho_1 = 0.95)$	0.918	0.462	1.38	0.853	0.696	1.01	0.95	0.925	0.974
$(T = 250, \rho_1 = 0.7)$	1.36	1.04	1.67	1.02	0.913	1.13	0.616	0.552	0.68
$(T = 250, \rho_1 = 0.9)$	0.873	0.608	1.14	0.96	0.837	1.08	0.909	0.879	0.938
$(T = 250, \rho_1 = 0.95)$	1.19	0.888	1.49	1.02	0.905	1.13	0.921	0.891	0.951

2.2 Part ii

Results from repeating the simulation 10000 times are below:

	$\hat{\alpha}_0$ Mean	$\hat{\alpha}_0$ Coverage	$\hat{\delta}_0$ Mean	$\hat{\delta}_0$ Coverage	$\hat{\rho}_1$ Mean	$\hat{\rho}_1$ Coverage
$(T = 50, \rho_1 = 0.7)$	1.09	0.901	1	0.916	0.659	0.89
$(T = 50, \rho_1 = 0.9)$	1.16	0.866	0.995	0.916	0.853	0.844
$(T = 50, \rho_1 = 0.95)$	1.13	0.843	0.991	0.915	0.897	0.802
$(T = 150, \rho_1 = 0.7)$	1.04	0.939	1	0.939	0.687	0.93
$(T = 150, \rho_1 = 0.9)$	1.09	0.921	1	0.938	0.887	0.913
$(T = 150, \rho_1 = 0.95)$	1.11	0.908	1	0.943	0.937	0.892
$(T = 250, \rho_1 = 0.7)$	1.03	0.939	1	0.945	0.692	0.937
$(T = 250, \rho_1 = 0.9)$	1.06	0.929	1	0.944	0.892	0.928
$(T = 250, \rho_1 = 0.95)$	1.08	0.922	1	0.943	0.942	0.914

2.3 Part iii

It is clear from the simulation results that the estimated OLS coefficients are, on average, closer to the true value and are more often covered by the confidence interval as T increases. As the degree of persistence of Y_t is closer to 1, the bias of OLS tends to increase as measured by the coverage percentages falling. It seems likely, in my opinion, that the underperformance of the coverage ratios is caused by bias in the point estimates, as opposed to misidentification of the variance estimates.

```matlab
%% parameter definitions and preallocation
clear; close all; clc
rng(999); % for reproducibility
alpha0 = 1;%100; %scale down all coefficients by 100 to look nicer in table
delta0 = 1;%100;
beta0 = 0.01;%1;
T = [50 150 250];
rho1 = [0.7 0.9 0.95];
betas = zeros(4,3,3); % preallocation
CIl = zeros(4,3,3);
CIu = zeros(4,3,3);
scl = norminv(0.975); %~1.96
%% Part (i): generate results once
for i=1:3
    for j=1:3
        [Y,X] = gen_dat(T(i),rho1(j),alpha0,delta0,beta0);
        % calculate OLS coefficients, and h-r 95% CI
        tX = [ones(T(i),1) (1:T(i))' X(2:end) Y(1:end-1) ]; % rhs matrix
        betas(:,i,j) = tX\Y(2:end,:);
        r = Y(2:end,:) - tX*betas(:,i,j);
        Vhc0 = inv(tX'*tX)*(tX'*diag(r.^2)*tX)*inv(tX'*tX);
        SE = sqrt(diag(Vhc0));
        CIl(:,i,j) = betas(:,i,j)-scl*SE;
        CIu(:,i,j) = betas(:,i,j)+scl*SE;
    end
end
```

```matlab
%% Part (2): repeat and calculate simulation results
nsim = 10000;
betasN = zeros(4,3,3,nsim);
coverN = zeros(4,3,3,nsim);
for n=1:nsim
for i=1:3
    for j=1:3
        [Y,X] = gen_dat(T(i),rho1(j),alpha0,delta0,beta0);
        % calculate OLS coefficients, and h-r 95% CI
        tX = [ones(T(i),1) (1:T(i))' X(2:end) Y(1:end-1) ];
        betasN(:,i,j,n) = tX\Y(2:end,:);
        r = Y(2:end,:) - tX*betasN(:,i,j,n);
        Vhc0 = inv(tX'*tX)*(tX'*diag(r.^2)*tX)*inv(tX'*tX);
        SE = sqrt(diag(Vhc0));
        Cl = betasN(:,i,j,n)-scl*SE;
        Cu = betasN(:,i,j,n)+scl*SE;
        if alpha0>Cl(1) && alpha0<Cu(1) % covered true value?
            coverN(1,i,j,n) = 1;
        end
        if beta0>Cl(2) && beta0<Cu(2)
            coverN(2,i,j,n) = 1;
        end
        if delta0>Cl(3) && delta0<Cu(3)
            coverN(3,i,j,n) = 1;
        end
        if rho1(j)>Cl(4) && rho1(j)<Cu(4)
            coverN(4,i,j,n) = 1;
        end
    end
end
end
PcoverN = mean(coverN,4); %average over our simulations
MbetasN = mean(betasN,4);
```

```matlab
%% output tex tables
% construct table for single iteration results
mats = zeros(9,9);
for i = 1:3
    for j=1:3
        mats(3*(i-1)+j,[1 4 7]) = betas([1 3 4],i,j)';
        mats(3*(i-1)+j,1+[1 4 7]) = CIl([1 3 4],i,j)';
        mats(3*(i-1)+j,2+[1 4 7]) = CIu([1 3 4],i,j)';
    end
end
tabs = table(mats(:,1),mats(:,2),mats(:,3),mats(:,4),mats(:,5),mats(:,6),mats(:,7),mats(:,8),mats(:,9),...
    'RowNames',{'$(T=50,\rho_1=0.7)$' '$(T=50,\rho_1=0.9)$' '$(T=50,\rho_1=0.95)$' ...
    '$(T=150,\rho_1=0.7)$' '$(T=150,\rho_1=0.9)$' '$(T=150,\rho_1=0.95)$' ...
    '$(T=250,\rho_1=0.7)$' '$(T=250,\rho_1=0.9)$' '$(T=250,\rho_1=0.95)$'}, ...
    'VariableNames',{'$\hat{\alpha}_0$' '$\hat{\alpha}_0$ LB' '$\hat{\alpha}_0$ UB' ...
    '$\hat{\delta}_0$' '$\hat{\delta}_0$ LB' '$\hat{\delta}_0$ UB'  ...
    '$\hat{\rho}_1$' '$\hat{\rho}_1$ LB' '$\hat{\rho}_1$ UB'});
table2latex(tabs,'ps5s.tex')
% construct table for simulation results
mat = zeros(9,6);
for i = 1:3
    for j=1:3
        mat(3*(i-1)+j,[1 3 5]) = MbetasN([1 3 4],i,j)';
        mat(3*(i-1)+j,[2 4 6]) = PcoverN([1 3 4],i,j)';
    end
end
tab = table(mat(:,1),mat(:,2),mat(:,3),mat(:,4),mat(:,5),mat(:,6),...
    'RowNames',{'$(T=50,\rho_1=0.7)$' '$(T=50,\rho_1=0.9)$' '$(T=50,\rho_1=0.95)$' ...
    '$(T=150,\rho_1=0.7)$' '$(T=150,\rho_1=0.9)$' '$(T=150,\rho_1=0.95)$' ...
    '$(T=250,\rho_1=0.7)$' '$(T=250,\rho_1=0.9)$' '$(T=250,\rho_1=0.95)$'}, ...
    'VariableNames',{'$\hat{\alpha}_0$ Mean' '$\hat{\alpha}_0$ Coverage' ...
    '$\hat{\delta}_0$ Mean' '$\hat{\delta}_0$ Coverage' '$\hat{\rho}_1$ Mean' '$\hat{\rho}_1$ Coverage'});
table2latex(tab,'ps5.tex')
```

Data was generated using the following function:

```matlab
function [Y,X] = gen_dat(T,rho1,alpha0,delta0,beta0)
% generates time series data for HW5
V = normrnd(0,1,T,1);
U = normrnd(0,1,T,1);
Y = zeros(T+1,1);
X = Y;
Y(1) = normrnd(0,1,1,1);
X(1) = normrnd(0,1,1,1);
for i=1:T
    X(i+1) = X(i)*0.3 + V(i);
    Y(i+1) = alpha0 + i*beta0 + X(i+1)*delta0 + Y(i)*rho1 + U(i);
end
```