Econometrics HW1

Michael B. Nattinger

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1 Question 1

For two events $A, B \in S$, prove that $A \cup B = (A \cap B) \cup ((A \cap B^c) \cup (B \cap A^c))$.

pf Let $A, B \in S$. Then,

$$\begin{aligned}
A \cup B &= ((A \cap B) \cup (A \cap B^c)) \cup ((B \cap A) \cup (B \cap A^c)) \\
&= (A \cap B) \cup ((A \cap B^c) \cup (B \cap A) \cup (B \cap A^c)) \\
&= ((A \cap B) \cup (B \cap A)) \cup ((A \cap B^c) \cup (B \cap A^c)) \\
&= (A \cap B) \cup ((A \cap B^c) \cup (B \cap A^c)).
\end{aligned}$$

2 Question 2

Prove that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

pf

$$\begin{aligned}
P(A \cup B) &= P(((A \cap B) \cup (A \cap B^c)) \cup B) \\
&= P(((A \cap B) \cup (A \cap B^c)) \cup ((B \cap A) \cup (B \cap A^c))) \\
&= P(((A \cap B) \cup (B \cap A)) \cup ((A \cap B^c) \cup (B \cap A^c))) \\
&= P((A \cap B) \cup ((A \cap B^c) \cup (B \cap A^c))) \\
&= P(A \cap B) + P(A \cap B^c) + P(A^c \cap B) \\
&= P(A \cap B) + P(A \cap B^c) + P(A^c \cap B) + P(A \cap B) - P(A \cap B) \\
&= P((A \cap B) \cup (A \cap B^c)) + P((A^c \cap B) \cup (A \cap B)) - P(A \cap B) \\
&= P(A) + P(B) - P(A \cap B).
\end{aligned}$$

3 Question 3

Suppose that the unconditional probablity of a disease is 0.0025. A screening test for this disease has a detection rate of 0.9, and has a false positive rate of 0.01. Given that the screening test returns positive, what is the conditional probability of having the disease?

pf Let $A = \{\text{having the disease}\}, B = \{\text{testing positive}\}$. By (repeated iterations of) Bayes' Law, $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$. It is given to us in the question that $P(B|A) = 0.9, P(A) = 0.0025, P(B|A^c) = 0.01$. We can next calculate $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = P(B|A)P(A) + P(B|A^c)(1 - P(A)) = (0.9)(0.0025) + (0.01)(0.9975) = 0.012225$. Thus, $P(A|B) = \frac{(0.9)(0.0025)}{0.012225} = 0.184$.

4 Question 4

Suppose that a pair of events A and B are mutually exclusive, i.e. $A \cap B = \emptyset$, and that $P(A) > 0$ and $P(B) > 0$. Prove that A and B are not independent.

pf Let $A, B \in S$ with $P(A) > 0, P(B) > 0$. Assume for the purpose of contradiction A and B are independent. Then, $P(A \cap B) = P(A)P(B) > 0$. However, note that since $A \cap B = \emptyset$, $P(A \cap B) = 0$, which is a contradiction. Thus, A and B are not independent.

5 Question 5

Let A, B, C be three events with positive probabilities. Then A and B are independent given C if $P(A \cap B|C) = P(A|C)P(B|C)$. Consider the experiment of tossing two dice. Let $A = \{\text{First die is 6}\}, B = \{\text{Second die is 6}\}$, and $C = \{\text{Both dice are the same}\}$.

5.1 Show that A and B are independent (unconditionally), but A and B are dependent given C.

pf We have that $P(A) = P(D1 = 6, D2 = 1) + P(D1 = 6, D2 = 2) + P(D1 = 6, D2 = 3) + P(D1 = 6, D2 = 4) + P(D1 = 6, D2 = 5) + P(D1 = 6, D2 = 6) = \frac{1}{6} = P(B)$ by symmetry. Note also that $P(A \cap B) = P(D1 = 6, D2 = 6) = \frac{1}{36} = \frac{1}{6} * \frac{1}{6} = P(A)P(B)$ so A and B are independent (unconditionally).

Now note that $P(A \cap B|C) = P(D1 = 6, d2 = 6|\text{Both dice are the same}) = \frac{1}{6}$, but $P(A|C) = P(D1 = 6|\text{Both dice are the same}) = \frac{1}{6} = P(B|C)$ by symmetry. Then, $P(A|C)P(B|C) = \frac{1}{36} \neq \frac{1}{6} = P(A \cap B|C)$. Thus, A and B are dependent given C.

5.2 Let there be two urns, one with nine black balls and one white ball, and the other with one black ball and nine white balls. First randomly (with equal probability) select one urn. Then take two draws with replacement from the selected urn. Let A and B be drawing a black ball in the first and the second draw, respsctively, and let C be the event that urn 1 is selected. Show that A and B are not independent, but are conditionally independent given C.

pf $P(A \cap B|C) = P(\text{first draw is black and second draw is black}|C) = (0.9)(0.9)$ as we are picking with replacement. $P(A|C) = P(\text{first draw is black}|C) = 0.9 = P(B|C)$ as we are replacing in between draws. Then, $P(A \cap B|C) = 0.81 = (0.9)(0.9) = P(A|C)P(B|C)$ so A and B are conditionally independent given C. By the same logic, A and B are also conditionally independent given C^c.

Note that $P(A) = P(A|C)P(C) + P(A|C^c)P(C^c) = (0.9)(0.5) + (0.1)(0.5) = 0.5 = P(B)$, by symmetry. However, $P(A \cap B) = P(A \cap B|C)P(C) + P(A \cap B|C^c)P(C^c) = P(A|C)P(B|C)P(C) + P(A|C^c)P(B|C^c)P(C^c) = (0.9)(0.9)(0.5) + (0.1)(0.1)(0.5) = 0.41 \neq 0.25 = (0.5)(0.5) = P(A)P(B)$, so A and B are not independent.

6 Question 6

A CDF F_X is stochastically greater than a CDF F_Y if $F_X(t) \leq F_Y(t)$ for all t and $F_X(t) < F_Y(t)$ for some t. Prove that if $X \sim F_X$ and $Y \sim F_Y$, then $P(X > t) \geq P(Y > t)$ for every t and $P(X > t) > P(Y > t)$ for some t, that is, X tends to be bigger than Y.

pf Let $t^{'} \in \mathbb{R}$ be arbitrary. $P(X > t^{'}) = 1 - F_X(t^{'}) \geq 1 - F_Y(t^{'}) = P(Y > t^{'})$ so $P(X > t) \geq P(Y > t)$ for every t.

As stated in the question, $\exists t^* \in \mathbb{R}$ such that $F_X(t^*) < F_Y(t^*)$. Then, $P(X > t^*) = 1 - F_X(t^*) > 1 - F_Y(t^*) = P(Y > t^*)$ so $P(X > t) > P(Y > t)$ for some t.

7 Question 7

Show that the function

$$F_X(x) = \begin{cases} 0 \text{ if } x < 0 \\ 1 - e^{-x} \text{ if } x \geq 0 \end{cases}$$

is a CDF, and find $f_X(x)$ and $F_X^{-1}(y)$.

pf To prove $F_X(x)$ is a CDF we will show that it fulfills the three main properties of a CDF.

- $\lim_{x\to\infty} F_X(x) = lim_{x\to\infty} 1 - e^{-x} = 1 - 0 = 1$. $\lim_{x\to-\infty} = 0$.
- Clearly, for $x < 0$ $F_X(x)$ is constant so it is nondecreasing. For $x \geq 0$, $\frac{d}{dx}F_X(x) = \frac{d}{dx}1 - e^{-x} = e^{-x} > 0$ so $F_X(x)$ is nondecreasing.
- For $x < 0$, x is constant so clearly $F_X(x)$ is right-continuous. For $x \geq 0$, $F_X(x) = 1 - e^{-x}$. Since exp is continuous on $\mathbb{R}$, $F_X(x)$ is right-continuous.

Therefore, $F_X(x)$ is a CDF.

For $x < 0$, $F_X(x)$ is constant so $f_X(x) = 0$. For $x \geq 0$, $F_X(x) = 1 - e^{-x}$ so $f_X(x) = \frac{d}{dx}(1 - e^{-x}) = e^{-x}$. Thus,

$$f_X(x) = \begin{cases} 0 & , x < 0 \\ e^{-x} & , x \geq 0. \end{cases}$$

Define $G(y) := -ln(1 - y)$, $y > 0$. Then, for arbitrary $x \geq 0$, $G(F_x(x)) = G(1 - e^{-x}) = -ln(1 - (1 - e^{-x})) = -ln(e^{-x}) = x$, so $G(y) = F_X^{-1}(y)$.[1]

[1] Note that $F_X(x)$ is only invertible for $x \in [0, \infty) \Rightarrow 1 > F_X(x) > 0$ when $F_X(x)$ is invertible.