HW3

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1 Real Analysis

De Morgan's Laws: $(A \cap B)^c = A^c \cup B^c$; $(A \cup B)^c = A^c \cap B^c$.

The **cardinality** of a set is the size of the set. Two sets are **numerically equivalent** if they have the same cardinality. A set is **countably infinite** if it is numerically equivalent to $\mathbb{N}$.

A **metric** on a set X is a function $d : X \text{ x } X \to \mathbb{R}^+$ s.t. $\forall x, y, z \in X$,

- $d(x, y) \geq 0$, with equality $\iff x = y$;
- $d(x, y) = d(y, x)$;
- $d(x, z) \leq d(x, y) + d(y, z)$.

A **metric space** is a pair (X, d), where X is a set and d is a metric on X. Examples include Euclidean space.

In a metric space, (X, d), an **open ball** is $B_\epsilon(x) = \{y \in X | d(x, y) < \epsilon\}$ and a **closed ball** is $B_\epsilon[x] = \{y \in X | d(x, y) \leq \epsilon\}$.

A **sequence** in a set X is a function $s : \mathbb{N} \to X$, denoted $\{s_n\}$, where $s_n = s(n)$. A sequence x_n in a metric space (X, d) **converges** to $x \in X$ if $\forall \epsilon > 0, \exists N(\epsilon) > 0$ s.t. $\forall n > N(\epsilon)$ $d(x_n, x) < \epsilon$. We write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

A sequence in a metric space has **at most one limit**.

Consider a sequence $\{x_n\}$ and a rule that assigns to each $k \in \mathbb{N}$ a value $n_k \in \mathbb{N}$ such that $n_k < n_{k+1}$ for all k. Then $\{x_{n_k}\}$ is a **subsequence**. If $\{x_n\} \to x$ then any subsequence also converges to x.

A subset $s \subset X$ in a metric space (X, d) is **bounded** if $\exists x \in X, \beta \in \mathbb{R} s.t. \forall s \in S, d(x, s) < \beta$. Every convergent sequence in a metric space is bounded.

Limits preserve weak inequality.

If $x_n \to x, y_n \to y$, $x_n + y_n \to x + y$, $x_n y_n \to xy, x_n/y_n \to x/y$ so long as y_n, y nonzero. Same applies to $\mathbb{R}^n$ with operations taken coordinate-wise.

Bolzano-Weierstrass Theorem: Every bounded real sequence contains at least one convergent subsequence.

Monotone Convergence Theorem: Every increasing sequence of real numbers that is bounded above converges. Every decreasing sequence of real numbers that is bounded below converges.

Every real sequence contains either a decreasing subsequence or increasing subsequence or possible both.

Given a real sequence $\{x_n\}$, the infinite sum of its terms is well-defined if the sequence of partial sums converges.

Let (X, d) be a metric space. A set $A \subset X$ is **open** if $\forall x \in A \exists \epsilon > 0 s.t. B_\epsilon(x) \subset A$. A set $C \subset X$ is **closed** if its complement is open. This depends on the metric space. $[0, 1]$ is not open in $(\mathbb{R}, d_E)$ but is open in $([0, 1], d_E)$.

Let $X, d)$ be a metric space.

- $\emptyset, X$ are simultaneously open and closed in X;
- the union of an arbtrary collection of open sets is open;
- the intersection of a finite collection of closed sets is closed;
- the union of a finite collection of closed sets is closed;
- the intersection of an arbitrary collection of closed sets is closed.

$\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1 - \frac{1}{n}\right] = (0, 1)$, $\bigcap_{n=1}^{\infty} \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right) = \{1\}$

A set is closed if and only if every convergent sequence contained in A has its limit in A.

Let (X, d), and $A \in X$. $x \in X$ is a **limit point** of A if $\forall \epsilon > 0$, $(B_\epsilon(x) \setminus \{x\}) \cap A \neq \emptyset$.

Let $(X, d), (Y, \rho)$ be two metric spaces, $A \subset X, f : A \to Y, x_0$ =limit point of A. f has a limit y_0 as x approaches x_0 if $\forall \epsilon > 0 \exists \delta > 0 s.t.$ if $x \in A$ and $0 < d(x, x_0) < \delta$, then $\rho(f(x), y_0) < \epsilon$.

$\lim\limits_{x \to x_0} f(x) = y_0$ if and only if for any sequence $\{x_n\} \in X$ such that $x_n \to x_0$ and $x_n \neq x_0$, the sequence $\{f(x_n)\}$ converges to y_0.

The limit of f as $x \to x_0$, when it exists, is unique.

A function is **continuous** at x_0 if $\forall \epsilon > 0 \exists \delta > 0$ s.t. if $d(x, x_0) < \delta$, then $\rho(f(x), f(x^0)) < \epsilon$. ($\delta$ can vary for different x^0 and ϵ)

A function f is continuous at x_0 if and only if one of the following equivalent statements is true:

- $f(x_0)$ is defined and either x_0 is an isolated point or x_0 is a limit point of X and $\lim\limits_{x \to x_0} f(x) = f(x_0)$.
- For any sequence $\{x_n\}$ s.t. $x_n \to x_0$, the sequence $\{f(x_n)\}$ converges to $f(x_0)$.

A function f is continuous if it is continuous at every point of its domain.

A function f is continuous iff for any closed set C, the set $f^{-1}(C)$ is closed. A function f is continuous iff for any open set A, the set $f^{-1}(A)$ is open.

A function is uniformly continuous if $\forall \epsilon > 0 \exists \delta > 0$ s.t. $if d(x, x_0) < \delta$, then $\rho(f(x), f(x_0)) < \epsilon$. Note: delta depends only on epsilon!

Uniform continuity implies continuity.

Let $(X,d),(Y,\rho)$ be two metric spaces, $f:X\to Y, E\subset X$. Then f is **Lipschitz** on E if $\exists K>0$ s.t. $\rho(f(x),f(y))\leq Kd(x,y)\forall x,y\in E$. f is **locally Lipschitz** on E if $\forall x\in E\exists\epsilon>0$ s.t. f is Lipscitz on $B_\epsilon(x)\cap E$.

Lipschitz implies uniform continuity.

Let $X\subset\mathbb{R}$. Then $u\in\mathbb{R}$ is an upper bound for X if $x\leq u\forall x\in X$ (and opposite for lower bound). X is bounded above if there is an upper boound for X.

Suppose X is bounded above. The supremum of X, $supX$, is the smallest upper bound for X, i.e. $supX$ satisfies

- $supX\geq x\ \forall x\in X$;
- $\forall y<supX\exists x\in X$ s.t. $x>y$.

And infimum is similarly defined.

Supremum Property: Every nonempty set of real numbers that is bounded above has a supremum, and the supremum is a real number. (Not generally the case for all numbers e.g. sets that would be bounded by irrational numbers in the reals do not have a supremum when they are instead defined in the rationals)

Extreme Value Theorem: Let $f:[a,b]\to\mathbb{R}$ be continuous. Then f attains its maximum and minimum on $[a,b]$.

Intermediate Value Theorem: Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Then for any $\gamma\in[f(a),f(b)]$ there exists $c\in[a,b]$ s.t. $f(c)=\gamma$.

Let f be monotonically increasing. Then one-sided limits exist for all x. Moreover, $sup\{f(s)|a<s<x\}=f(x^-\leq f(x)\leq f(x^+)=inf\{f(s)|x<s<b\}$.

2 Linear Algebra