

# HW6

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September 29, 2020

## 1 Question 1

Bob will travel along the road for some distance  $x$ , and then turn off the road and travel in the exact direction of "Happy Cow". Bob is minimizing his walking time to reach this point:  $\min_{x \in [0,12]} x/5 + f(x)/3$  where  $f(x)$  is the distance (in miles) through the woods that Bob must travel if Bob chooses to walk  $x$  miles on the main road. It can easily be shown via simple geometry that  $f(x) = \sqrt{(12-x)^2 + 25}$ . Thus, Bob solves the following:

$$\min_{x \in [0,12]} x/5 + \sqrt{(12-x)^2 + 25}/3.$$

We can take first order conditions of the objective function  $g$  with respect to  $x$ :  $\frac{dg}{dx} = 1/5 - \frac{1}{6\sqrt{(12-x)^2 + 25}}(2(12-x)) = 0 \Rightarrow 1/5 = \frac{12-x}{3\sqrt{(12-x)^2 + 25}} \Rightarrow (9/25)((12-x)^2 + 25) = (12-x)^2 \Rightarrow (9 * 25)/16 = (12-x)^2 \Rightarrow (15/4) = (12-x), -(15/4) = (12-x)$ . If  $(12-x) < 0$  then  $x > 12$  so  $x \notin [0, 12]$ , so  $(15/4) = (12-x) \Rightarrow x = 12 - (15/4) \Rightarrow x = (33/4)$  miles. Thus,

$$\begin{aligned} \min_{x \in [0,12]} x/5 + \sqrt{(12-x)^2 + 25}/3 &= (33/20) + \sqrt{(12-33/4)^2 + 25}/3 \\ &= 3.733 \text{ hours.} \end{aligned}$$

## 2 Question 2

Assume that  $x_0$  is a local maximum of  $f$ . Then  $\exists \delta \in (0, \epsilon]$  such that for any  $x \in B_\delta(x_0) \setminus \{x_0\}$ ,  $f(x_0) \geq f(x)$ . Then, notice that  $x_0 - \delta/2 \in B_\delta(x_0)$ . Then, by the mean value theorem,  $\exists c \in (x_0 - \delta/2, x_0)$  such that  $f'(c) = \frac{f(x_0) - f(x_0 - \delta/2)}{\delta/2} \geq 0$  which is a contradiction, so  $x_0$  is not a local maximum of  $f$ . Now assume that  $x_0$  is a local minimum of  $f$ . Then  $\exists \delta \in (0, \epsilon]$  such that for any  $x \in B_\delta(x_0) \setminus \{x_0\}$ ,  $f(x_0) \leq f(x)$ . Then, notice that  $x_0 + \delta/2 \in B_\delta(x_0)$ . Then, by the mean value theorem,  $\exists c \in (x_0, x_0 + \delta/2)$  such that  $f'(c) = \frac{f(x_0 + \delta/2) - f(x_0)}{\delta/2} \geq 0$  which is a contradiction, so  $x_0$  is not a local minimum of  $f$ .

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\*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

### 3 Question 3

$$\begin{aligned}
\frac{\partial f}{\partial r} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r} \\
&= (y^2 z)(1) + (2xyz)(2) + (xy^2)(1) \\
&= (2r + 4s + t)^2(3r + s + t) + 4(t + 2s + t)(2r + 3s + t)(3r + s + t) + (t + 2s + t)(2r + 4s + t)^2, \\
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s} \\
&= (y^2 z)(2) + (2xyz)(3) + (xy^2)(1) \\
&= 2(2r + 4s + t)^2(3r + s + t) + 6(t + 2s + t)(2r + 3s + t)(3r + s + t) + (t + 2s + t)(2r + 4s + t)^2, \\
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t} \\
&= (y^2 z)(3) + (2xyz)(1) + (xy^2)(1) \\
&= 3(2r + 4s + t)^2(3r + s + t) + (t + 2s + t)(2r + 3s + t)(3r + s + t) + (t + 2s + t)(2r + 4s + t)^2.
\end{aligned}$$

### 4 Question 4

Let  $f$  be continuously differentiable on  $X \subset \mathbb{R}^n$ . Then,  $Df$  exists and is continuous on  $X$ . Let  $x_0 \in X$  and let  $B_\epsilon(x_0) \subset X$  be a closed epsilon ball around  $x_0$ . Since  $Df$  is continuous, it must be bounded on  $B_\epsilon(x_0)$ . Let  $m_1^{i,j}, m_2^{i,j}$  be the upper and lower bounds of  $Df$  in dimension  $i \in \{1, \dots, n\}$  with respect to the input in dimension  $j \in \{1, \dots, n\}$  on  $B_\epsilon(x_0)$ , and let  $M = \max_{i,j \in \{1, \dots, n\}, k \in \{1, 2\}} \{|m_k^{i,j}|\}$ . Then,  $\|D_i f(x)\| \leq \|\vec{M}\| \forall x \in B_\epsilon(x_0)$ , for all dimensions  $i \in \{1 \dots n\}$ , where  $\vec{M}$  is the vector of size  $n$  containing  $M$  at every index. Let  $x_1, x_2 \in B_\epsilon(x_0)$ , then we will define  $g(t) := f((1-t)x_1 + tx_2)$  for  $t \in [0, 1]$ . Then, by the mean value theorem, there exists  $t^* \in [0, 1]$  such that  $g'(t^*) = f(x_2) - f(x_1)$ . However, note that  $g'(t^*) = Df((1-t^*)x_1 + t^*x_2)(x_2 - x_1) = f(x_2) - f(x_1)$ . By the Cauchy-Schwartz inequality in each dimension  $i \in \{1 \dots n\}$ ,

$$\begin{aligned}
|f_i(x_2) - f_i(x_1)| &\leq \|D_i f((1-t^*)x_1 + t^*x_2)\| |x_{2,i} - x_{1,i}| \\
&\leq \|\vec{M}\| |x_{2,i} - x_{1,i}| \\
\Rightarrow \sqrt{\sum_{i=1}^n (f_i(x_2) - f_i(x_1))^2} &\leq \sqrt{\sum_{i=1}^n \|\vec{M}\|^2 (x_{2,i} - x_{1,i})^2} \\
\Rightarrow \|f(x_2) - f(x_1)\| &\leq \|\vec{M}\| \|x_2 - x_1\|
\end{aligned}$$

so  $f$  is locally lipschitz on  $X$ .

## 5 Question 5

$f(1, 1) = 0$ .  $\text{Det} D_X f = \text{Det}(5x^4 - 2x + 1) \Rightarrow \text{Det} D_X f(1, 1) = 5 - 2 + 1 \neq 0$ . Then, by the implicit function theorem,

$$\begin{aligned} \frac{\partial x(y)}{\partial y}|_{(1,1)} &= - \left( \frac{\partial f}{\partial x}|_{(1,1)} \right)^{-1} \left( \frac{\partial f}{\partial y}|_{(1,1)} \right) \\ &= -(5x^4 - 2x + 1)|_{(1,1)}^{-1} (-3y^2 - 2)|_{(1,1)} \\ &= -(4)^{-1}(-5) = \frac{5}{4}. \end{aligned}$$

## 6 Question 6

$$\begin{aligned} Df(x, y) &= \begin{pmatrix} 8x^3 - y \\ 2y - x \end{pmatrix} = \vec{0} \Rightarrow x = 2y, 64y^3 = y \Rightarrow y = 0, y = 1/8, y = -1/8 \\ &\Rightarrow (x, y) = (0, 0), (1/4, 1/8), (-1/4, -1/8). \end{aligned}$$

$$D^2 f(x, y) = \begin{pmatrix} 24x^2 & -1 \\ -1 & 2 \end{pmatrix}.$$

$$D^2 f(0, 0) = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}, D^2 f(1/4, 1/8) = \begin{pmatrix} 3/2 & -1 \\ -1 & 2 \end{pmatrix} = D^2 f(-1/4, -1/8).$$

First, we will investigate the point  $(0, 0)$ .  $\text{Det}(D^2 f(0, 0) - \lambda I) = 0 \Rightarrow -\lambda(2 - \lambda) - 1 = 0 \Rightarrow \lambda^2 - 2\lambda - 1 = 0 \Rightarrow \lambda = \frac{1}{2} - \frac{\sqrt{5}}{2}, \lambda = \frac{1}{2} + \frac{\sqrt{5}}{2}$ . So, one eigenvalue is positive while the other is negative, so  $f$  has a saddle point at  $(0, 0)$ . Next we will investigate the point  $(1/4, 1/8)$ .  $\text{Det}(D^2 f(1/4, 1/8) - \lambda I) = 0 \Rightarrow (3/2 - \lambda)(2 - \lambda) - 1 = 0 \Rightarrow \lambda^2 - (7/2)\lambda + 2 = 0 \Rightarrow \lambda = \frac{7}{4} + \frac{\sqrt{17}}{4}, \lambda = \frac{7}{4} - \frac{\sqrt{17}}{4}$ . Both eigenvalues are positive, so  $f$  has a local minimum at  $(1/4, 1/8)$  and, since  $\text{Det}(D^2 f(1/4, 1/8)) = \text{Det}(D^2 f(-1/4, -1/8))$ ,  $f$  has a local minimum at  $(-1/4, -1/8)$ .

Assume  $\exists x, y \in \mathbb{R}$  such that  $f(x, y) < f(1/4, 1/8) = f(-1/4, -1/8)$ . Then,  $2x^4 + y^2 - xy + 1 > 127/128 \Rightarrow 2x^4 + y^2 - xy + 1/128 < 0$  so  $x, y \notin \mathbb{R}$  which is a contradiction. Thus,  $(1/4, 1/8), (-1/4, -1/8)$  are global minima. It is clear that no global maxima exists as  $f(x, y) \rightarrow \infty$  as  $x \rightarrow \pm\infty, y \rightarrow \pm\infty$ .