HW6

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1 Question 1

Bob will travel along the road for some distance x, and then turn off the road and travel in the exact direction of "Happy Cow". Bob is minimizing his walking time to reach this point: $\min_{x\in[0,12]} x/5 + f(x)/3$ where $f(x)$ is the distance (in miles) through the woods that Bob must travel if Bob chooses to walk x miles on the main road. It can easily be shown via simple geometry that $f(x) = \sqrt{(12-x)^2+25}$. Thus, Bob solves the following:

$$\min_{x\in[0,12]} x/5 + \sqrt{(12-x)^2+25}/3.$$

We can take first order conditions of the objective function g with respect to x: $\frac{dg}{dx} = 1/5 - \frac{1}{6\sqrt{(12-x)^2+25}}(2(12-x)) = 0 \Rightarrow 1/5 = \frac{12-x}{3\sqrt{(12-x)^2+25}} \Rightarrow (9/25)((12-x)^2+25) = (12-x)^2 \Rightarrow (9*25)/16 = (12-x)^2 \Rightarrow (15/4) = (12-x), -(15/4) = (12-x)$. If $(12-x) < 0$ then $x > 12$ so $x \notin [0,12]$, so $(15/4) = (12-x) \Rightarrow x = 12-(15/4)$
$\Rightarrow x = (33/4)$ miles. Thus,

$$\begin{aligned}\min_{x\in[0,12]} x/5 + \sqrt{(12-x)^2+25}/3 &= (33/20) + \sqrt{(12-33/4)^2+25}/3 \\ &= 3.733 \text{ hours.}\end{aligned}$$

2 Question 2

Assume that x_0 is a local maximum of f. Then $\exists\delta \in (0,\epsilon]$ such that for any $x \in B_\delta(x_0)\setminus\{x_0\}, f(x_0) \geq f(x)$. Then, notice that $x_0 - \delta/2 \in B_\delta(x_0)$. Then, by the mean value theorem, $\exists c \in (x_0-\delta/2, x_0)$ such that $f'(c) = \frac{f(x_0)-f(x_0-\delta/2)}{\delta/2} \geq 0$ which is a contradiction, so x_0 is not a local maximum of f. Now assume that x_0 is a local minimum of f. Then $\exists\delta \in (0,\epsilon]$ such that for any $x \in B_\delta(x_0)\setminus\{x_0\}, f(x_0) \leq f(x)$. Then, notice that $x_0+\delta/2 \in B_\delta(x_0)$. Then, by the mean value theorem, $\exists c \in (x_0, x_0+\delta/2)$ such that $f'(c) = \frac{f(x_0+\delta/2)-f(x_0)}{\delta/2} \geq 0$ which is a contradiction, so x_0 is not a local minimum of f.

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3 Question 3

$$\begin{aligned}
\frac{\partial f}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\
&= (y^2 z)(1) + (2xyz)(2) + (xy^2)(1) \\
&= (2r+4s+t)^2(3r+s+t) + 4(t+2s+t)(2r+3s+t)(3r+s+t) + (t+2s+t)(2r+4s+t)^2, \\
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\
&= (y^2 z)(2) + (2xyz)(3) + (xy^2)(1) \\
&= 2(2r+4s+t)^2(3r+s+t) + 6(t+2s+t)(2r+3s+t)(3r+s+t) + (t+2s+t)(2r+4s+t)^2, \\
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} \\
&= (y^2 z)(3) + (2xyz)(1) + (xy^2)(1) \\
&= 3(2r+4s+t)^2(3r+s+t) + (t+2s+t)(2r+3s+t)(3r+s+t) + (t+2s+t)(2r+4s+t)^2.
\end{aligned}$$

4 Question 4

Let f be continuously differentiable on $X \subset \mathbb{R}^n$. Then, Df exists and is continuous on X. Let $x_0 \in X$ and let $B_\epsilon(x_0) \subset X$ be a closed epsilon ball around x_0. Since Df is continuous, it must be bounded on $B_\epsilon(x_0)$. Let $m_1^{i,j}, m_2^{i,j}$ be the upper and lower bounds of Df in dimension $i \in \{1, \ldots, n\}$ with respect to the input in dimension $j \in \{1, \ldots, n\}$ on $B_\epsilon(x_0)$, and let $M = \max_{i,j\in\{1,\ldots,n\},k\in\{1,2\}}\{|m_k^{i,j}|\}$. Then, $\|D_i f(x)\| \leq \left\|\vec{M}\right\|$ $\forall x \in B_\epsilon(x_0)$, for all dimensions $i \in \{1 \ldots n\}$, where $\vec{M}$ is the vector of size n containing M at every index. Let $x_1, x_2 \in B_\epsilon(x_0)$, then we will define $g(t) := f((1-t)x_1 + tx_2)$ for $t \in [0,1]$. Then, by the mean value theorem, there exists $t^* \in [0,1]$ such that $g'(t^*) = f(x_2) - f(x_1)$. However, note that $g'(t^*) = Df((1-t^*)x_1 + t^*x_2) \cdot (x_2 - x_1) = f(x_2) - f(x_1)$. By the Cauchy-Schwartz inequality in each dimension $i \in \{1 \ldots n\}$,

$$\begin{aligned}
|f_i(x_2) - f_i(x_1)| &\leq \|D_i f((1-t^*)x_1 + t^*x_2)\| \, |x_{2,i} - x_{1,i}| \\
&\leq \left\|\vec{M}\right\| |x_{2,i} - x_{1,i}| \\
\Rightarrow \sqrt{\sum_{i=1}^{n}(f_i(x_2) - f_i(x_1))^2} &\leq \sqrt{\sum_{i=1}^{n}\left\|\vec{M}\right\|^2 (x_{2,i} - x_{1,i})^2} \\
\Rightarrow \|f(x_2) - f(x_1)\| &\leq \left\|\vec{M}\right\| \|x_2 - x_1\|
\end{aligned}$$

so f is locally lipschitz on X.

5 Question 5

$f(1,1) = 0$. $\text{Det}D_X f = \text{Det}(5x^4 - 2x + 1) \Rightarrow \text{Det}D_X f(1,1) = 5 - 2 + 1 \neq 0$. Then, by the implicit function theorem,

$$\begin{aligned}
\frac{\partial x(y)}{\partial y}|_{(1,1)} &= -\left(\frac{\partial f}{\partial x}|_{(1,1)}\right)^{-1}\left(\frac{\partial f}{\partial y}|_{(1,1)}\right) \\
&= -(5x^4 - 2x + 1|_{(1,1)})^{-1}(-3y^2 - 2|_{(1,1)}) \\
&= -(4)^{-1}(-5) = \frac{5}{4}.
\end{aligned}$$

6 Question 6

$$\begin{aligned}
Df(x,y) &= \begin{pmatrix} 8x^3 - y \\ 2y - x \end{pmatrix} = \vec{0} \Rightarrow x = 2y, 64y^3 = y \Rightarrow y = 0, y = 1/8, y = -1/8 \\
&\Rightarrow (x,y) = (0,0), (1/4, 1/8), (-1/4, -1/8). \\
D^2 f(x,y) &= \begin{pmatrix} 24x^2 & -1 \\ -1 & 2 \end{pmatrix}. \\
D^2 f(0,0) &= \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}, D^2 f(1/4, 1/8) = \begin{pmatrix} 3/2 & -1 \\ -1 & 2 \end{pmatrix} = D^2 f(-1/4, -1/8).
\end{aligned}$$

First, we will investigate the point $(0,0)$. $\text{Det}(D^2 f(0,0) - \lambda I) = 0 \Rightarrow -\lambda(2 - \lambda) - 1 = 0 \Rightarrow \lambda^2 - 2\lambda - 1 = 0 \Rightarrow \lambda = \frac{1}{2} - \frac{\sqrt{5}}{2}, \lambda = \frac{1}{2} + \frac{\sqrt{5}}{2}$. So, one eigenvalue is positive while the other is negative, so f has a saddle point at $(0,0)$. Next we will investigate the point $(1/4, 1/8)$. $\text{Det}(D^2 f(1/4, 1/8) - \lambda I) = 0 \Rightarrow (3/2 - \lambda)(2 - \lambda) - 1 = 0 \Rightarrow \lambda^2 - (7/2)\lambda + 2 = 0 \Rightarrow \lambda = \frac{7}{4} + \frac{\sqrt{17}}{4}, \lambda = \frac{7}{4} - \frac{\sqrt{17}}{4}$. Both eigenvalues are positive, so f has a local minimum at $(1/4, 1/2)$ and, since $\text{Det}(D^2 f(1/4, 1/8)) = \text{Det}(D^2 f(-1/4, -1/8))$, f has a local minimum at $(-1/4, -1/2)$.

Assume $\exists x, y \in \mathbb{R}$ such that $f(x,y) < f(1/4, 1/8) = f(-1/4, -1/8)$. Then, $2x^4 + y^2 - xy + 1 > 127/128 \Rightarrow 2x^4 + y^2 - xy + 1/128 < 0$ so $x, y \notin \mathbb{R}$ which is a contradiction. Thus, $(1/4, 1/8), (-1/4, -1/8)$ are global minima. It is clear that no global maxima exists as $f(x,y) \to \infty$ as $x \to \pm\infty, y \to \pm\infty$.