Exam Note Sheet

Michael B. Nattinger

August 29, 2020

1 Real Analysis

De Morgan's Laws: $(A \cap B)^c = A^c \cup B^c$; $(A \cup B)^c = A^c \cap B^c$.

The **cardinality** of a set is the size of the set. Two sets are **numerically equivalent** if they have the same cardinality. A set is **countably infinite** if it is numerically equivalent to $\mathbb{N}$.

A **metric** on a set X is a function $d : X \text{ x } X \to \mathbb{R}^+$ s.t. $\forall x, y, z \in X$,

- $d(x, y) \geq 0$, with equality $\iff x = y$;
- $d(x, y) = d(y, x)$;
- $d(x, z) \leq d(x, y) + d(y, z)$.

A **metric space** is a pair (X, d), where X is a set and d is a metric on X. Examples include Euclidean space.

In a metric space, (X, d), an **open ball** is $B_\epsilon(x) = \{y \in X | d(x, y) < \epsilon\}$ and a **closed ball** is $B_\epsilon[x] = \{y \in X | d(x, y) \leq \epsilon\}$.

A **sequence** in a set X is a function $s : \mathbb{N} \to X$, denoted $\{s_n\}$, where $s_n = s(n)$. A sequence x_n in a metric space (X, d) **converges** to $x \in X$ if $\forall \epsilon > 0, \exists N(\epsilon) > 0$ s.t. $\forall n > N(\epsilon)$ $d(x_n, x) < \epsilon$. We write $x_n \to x$ or $\lim\limits_{n\to\infty} x_n = x$.

A sequence in a metric space has **at most one limit**.

Consider a sequence $\{x_n\}$ and a rule that assigns to each $k \in \mathbb{N}$ a value $n_k \in \mathbb{N}$ such that $n_k < n_{k+1}$ for all k. Then $\{x_{n_k}\}$ is a **subsequence**. If $\{x_n\} \to x$ then any subsequence also converges to x.

A subset $s \subset X$ in a metric space (X, d) is **bounded** if $\exists x \in X, \beta \in \mathbb{R} s.t. \forall s \in S, d(x, s) < \beta$. Every convergent sequence in a metric space is bounded.

Limits preserve weak inequality.

If $x_n \to x, y_n \to y$, $x_n + y_n \to x + y$, $x_n y_n \to xy, x_n/y_n \to x/y$ so long as y_n, y nonzero. Same applies to $\mathbb{R}^n$ with operations taken coordinate-wise.

Bolzano-Weierstrass Theorem: Every bounded real sequence contains at least one convergent subsequence.

Monotone Convergence Theorem: Every increasing sequence of real numbers that is bounded above converges. Every decreasing sequence of real numbers that is bounded below converges.

Every real sequence contains either a decreasing subsequence or increasing subsequence or possible both.

Given a real sequence $\{x_n\}$, the infinite sum of its terms is well-defined if the sequence of partial sums converges.

Let (X, d) be a metric space. A set $A \subset X$ is **open** if $\forall x \in A \exists \epsilon > 0 s.t. B_\epsilon(x) \subset A$. A set $C \subset X$ is **closed** if its complement is open. This depends on the metric space. $[0, 1]$ is not open in $(\mathbb{R}, d_E)$ but is open in $([0, 1], d_E)$.

Let (X, d) be a metric space.

- $\emptyset, X$ are simultaneously open and closed in X;
- the union of an arbtrary collection of open sets is open;
- the intersection of a finite collection of closed sets is closed;
- the union of a finite collection of closed sets is closed;
- the intersection of an arbitrary collection of closed sets is closed.

$\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1 - \frac{1}{n}\right] = (0, 1)$, $\bigcap_{n=1}^{\infty} \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right) = \{1\}$

A set is closed if and only if every convergent sequence contained in A has its limit in A.

Let (X, d), and $A \in X$. $x \in X$ is a **limit point** of A if $\forall \epsilon > 0$, $(B_\epsilon(x) \setminus \{x\}) \cap A \neq \emptyset$.

Let $(X, d), (Y, \rho)$ be two metric spaces, $A \subset X, f : A \to Y, x_0$ =limit point of A. f has a limit y_0 as x approaches x_0 if $\forall \epsilon > 0 \exists \delta > 0 s.t.$ if $x \in A$ and $0 < d(x, x_0) < \delta$, then $\rho(f(x), y_0) < \epsilon$.

$\lim\limits_{x \to x_0} f(x) = y_0$ if and only if for any sequence $\{x_n\} \in X$ such that $x_n \to x_0$ and $x_n \neq x_0$, the sequence $\{f(x_n)\}$ converges to y_0.

The limit of f as $x \to x_0$, when it exists, is unique.

A function is **continuous** at x_0 if $\forall \epsilon > 0 \exists \delta > 0$ s.t. if $d(x, x_0) < \delta$, then $\rho(f(x), f(x^0)) < \epsilon$. ($\delta$ can vary for different x^0 and ϵ)

A function f is continuous at x_0 if and only if one of the following equivalent statements is true:

- $f(x_0)$ is defined and either x_0 is an isolated point or x_0 is a limit point of X and $\lim\limits_{x \to x_0} f(x) = f(x_0)$.
- For any sequence $\{x_n\}$ s.t. $x_n \to x_0$, the sequence $\{f(x_n)\}$ converges to $f(x_0)$.

A function f is continuous if it is continuous at every point of its domain.

A function f is continuous iff for any closed set C, the set $f^{-1}(C)$ is closed. A function f is continuous iff for any open set A, the set $f^{-1}(A)$ is open.

A function is uniformly continuous if $\forall \epsilon > 0 \exists \delta > 0$ s.t. $if d(x, x_0) < \delta$, then $\rho(f(x), f(x_0)) < \epsilon$. Note: delta depends only on epsilon!

Uniform continuity implies continuity.

Let $(X,d),(Y,\rho)$ be two metric spaces, $f : X \to Y, E \subset X$. Then f is **Lipschitz** on E if $\exists K > 0$ s.t. $\rho(f(x), f(y)) \leq Kd(x,y) \forall x, y \in E$. f is **locally Lipschitz** on E if $\forall x \in E \exists \epsilon > 0$ s.t. f is Lipscitz on $B_\epsilon(x) \cap E$.

Lipschitz implies uniform continuity.

Let $X \subset \mathbb{R}$. Then $u \in \mathbb{R}$ is an upper bound for X if $x \leq u \forall x \in X$ (and opposite for lower bound). X is bounded above if there is an upper boound for X.

Suppose X is bounded above. The supremum of X, $supX$, is the smallest upper bound for X, i.e. $supX$ satisfies

- $supX \geq x \ \forall x \in X$;
- $\forall y < supX \exists x \in X$ s.t. $x > y$.

And infimum is similarly defined.

Supremum Property: Every nonempty set of real numbers that is bounded above has a supremum, and the supremum is a real number. (Not generally the case for all numbers e.g. sets that would be bounded by irrational numbers in the reals do not have a supremum when they are instead defined in the rationals)

Extreme Value Theorem: Let $f : [a,b] \to \mathbb{R}$ be continuous. Then f attains its maximum and minimum on $[a,b]$.

Intermediate Value Theorem: Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then for any $\gamma \in [f(a), f(b)]$ there exists $c \in [a,b]$ s.t. $f(c) = \gamma$.

Let f be monotonically increasing. Then one-sided limits exist for all x. Moreover, $sup\{f(s)|a < s < x\} = f(x^- \leq f(x) \leq f(x^+) = inf\{f(s)|x < s < b\}$.

A sequence is **Cauchy** if $\forall \epsilon > 0 \exists N > 0$ s.t. if $m, n > N$ then $d(x_n, x_m) < \epsilon$.

Every convergent sequence in a metric space is Cauchy.

A metric space is **complete** if every Cauchy sequence contained in X converges to some point in X. Euclidean space is complete. If (X,d) is a complete metric space and $Y \subset X$, then (Y,d) is complete if and only if Y is closed.

A function from a metric space to itself is called an **operator**. An operator is a contraction of modulus β if $\beta < 1$ and $d(T(x), T(y)) \leq \beta d(x,y) \ \forall x, y \in X$.

Every contraction is uniformly continuous.

A **fixed point** of an operator is an element $x_0 \in X$ s.t. $T(x_0) = x_0$.

Contraction Mapping Theorem: Let (X,d) be a nonempty **complete** metric space and $T : X \to X$ a contraction with modulus $\beta < 1$. Then T has a unique fixed point x_0. Additionally, $\forall x_0 \in X$ the sequence $\{x_n\}$, where $x_n = T^n(x_0) = T(T(\ldots T(x_0)\ldots))$ converges to x_0.

Continuous dependence of the fixed point on parameters: Let (X,d) and (Ω, ρ) be two metric spaces and $T : X \times \Omega \to X$. For each $\omega \in \Omega$, let $T_\omega : X \to X$ be defined by $T_\omega(x) = T(x, \omega)$. Suppose (X,d) is complete, T is continuous in ω, and $\exists \beta < 1$ such that T_ω is a contraction of modulus β for all $\omega \in \Omega$. Then the fixed point function $x^* : \Omega \to X$ defined by $x^*(\omega) = T_\omega(x^*(\omega))$ is continuous.

Blackwell's Sufficient Conditions: Let $B(X)$ be the set of all bounded functions from $X \to \mathbb{R}$ with metric $d_\infty(f,g) = sup_{x \in X}|f(x) - g(x)|$. Let $T : B(X) \to B(X)$ satisfy:

- (monotonicity) if $f(x) \leq g(x)\ \forall x \in X$, then $(T(f))(x) \leq (T(g))(x)\ \forall x \in X$.

- (discounting) $\exists \beta \in (0,1)$ s.t. for every $a \geq 0$ and $x \in X$, $(T(f+a))(x) \leq (T(f))(x) + \beta a$.

Then T is a contraction with modulus β.

A collection of sets $U = \{u_\lambda | \lambda \in \Lambda\}$ in a metric space is an open cover of the set A if U_λ is open for all $\lambda \in \Lambda$ and $A \subset \bigcup_{\lambda \in \Lambda} U_\lambda$.

A set A in a metric space is **compact** if every open cover of A contains a finite subcover of A.

To prove something is compact, find a way to reduce any general open cover into a finite subcover. To prove something is not compact, find an infinite subcover which cannot be reduced in this way.

Any closed subset of a compact space is compact. If A is a compact subset of a metric space, then A is closed and bounded.

Heine-Borel Theorem: If $A \subset \mathbb{R}^m$, then A is compact if and only if A is closed and bounded.

Closed intervel $[a,b]$ is compact in (R^m, d_E) for any $a, b \in \mathbb{R}^m$.

Let (X,d) and (Y,ρ) be metric spaces. If $f : X \to Y$ is continuous and C is a compact set in (X,d) then $f(C)$ is compact in (Y,ρ).

Extreme Value Theorem: If C is a compact set in a metric space (X,d) and $f : C \to \mathbb{R}$ is continuous, then f is bounded on C and attains its maximum and minimum.

Let (X,d) and (Y,ρ) be metric spaces, $C \subset X$ compact, $f : C \to Y$ continuous. Then f is uniformly continuous on C.

2 Linear Algebra

A **vector space** V is a collection of vectors, which may be added together and multiplied by scalars, satisfying the following conditions:

- $\forall x, y, z \in V, (x+y)+z = x+(y+z)$;

- $\forall x, y \in V, x+y = y+x$;

- $\exists \vec{0} \in V$ s.t. $\forall x \in V, x + \vec{0} = \vec{0} + x = x$;

- $\forall x \in V \exists (-x) \in V$ s.t. $x + (-x) = \vec{0}$;

- $\forall \alpha \in \mathbb{R}, x, y \in V, \alpha(x+y) = \alpha x + \alpha y$;

- $\forall \alpha, \beta \in \mathbb{R}, x \in V, (\alpha+\beta)x = \alpha x + \beta x$;

- $\forall \alpha, \beta \in \mathbb{R}, x \in V, (\alpha\beta)x = \alpha(\beta x)$;

- $\forall x \in V, 1x = x$.

A set spans V if $V = \text{span}W$ where a span is a linear combination of a set of vectors.

A set is linearly independent if $\nexists x_1, \ldots, x_n \in X, a_1, \ldots, a_n \in \mathbb{R}$ s.t. $\sum a_i \neq 0$ and $\sum a_i x_i = \vec{0}$.

Alternatively, if $a_i x_i = \vec{0}$ implies $a_1 = \cdots = a_n = 0$ then X is linearly independent. A **basis** is of V is a set of linearly independent vectors in V that spans V.

Let B be a basis for V and enumerate elements of B by a set Λ so that $B = \{v_\lambda | \lambda \in \Lambda\}$. Then every vector $x \in V$ has a unique representation as a linear combination of elements of B with finitely many nonzero coefficients.

Every vector space has a basis. Any two bases of a vector space have the same cardinality.

If V is a vector space and $W \subset V$ is linearly independent, then there exists a linearly independent set B such that $W \subset B \subset \text{span}B = V$

Let V be a vector space. The dimension of V is the cardinality of any basis of V. $\text{dim}R^n = n$.

Suppose $\text{dim}V = n \in \mathbb{N}. If W \subset V and |W| > n$, where $|W|$ denotes the cardinality of W, then W is linearly dependent.

Suppose $\text{dim}V = n$ and $W \subset V, |W| = n$. Then, if W is linearly independent, then $\text{span}W = V$, so W is a basis of V; if $\text{span}W = V$, then W is linearly independent, so W is a basis of V.

Let X and Y be two vector spaces. We say that $T : X \to Y$ is a linear transformation if for all $x_1, x_2 \in X, a_1, a_2 \in \mathbb{R}$, $T(a_1x_1 + a_2x_2) = a_1T(x_1) + a_2T(x_2)$.

If $L(X, Y)$ is the set of all linear transformations from X to Y, then $L(X, Y)$ is a vector space.

If $R : X \to Y, S : Y \to Z$ then $R \circ S : X \to Y$ is a linear transformation.

Let $T \in L(X, Y)$. $\text{Im}T : \{T(x)|x$
$in X\}$ and $\text{ker}T := \{x \in X | T(x) = \vec{0}\}$, and the rank of T is $\text{rank}T := \text{dim}(\text{Im}(T)$.

If $T \in L(X, Y)$, then $\text{Im}T$ and $\text{ker}T$ are vector subspaces of Y and X, respectively.

Let X be a finite-dimensional vector space and $T \in L(X, Y)$.
Then $\text{dim}X = \text{dim}(\text{ker}T)+\text{rank}T = \text{dim}(\text{ker}T)+\text{dim}(\text{Im}T)$.

$T \in L(X, Y)$ is invertible if there exists a function $S : Y \to X$ such that $S(T(x)) = x \forall x \in X, T(S(y)) = y \forall y \in Y$. The transformation S is called the inverse of T and is denoted T^{-1}.

T is invertible means:

- T is one-to-one: $\forall x_1 \neq x_2, T(x_1) \neq T(x_2)$.
- T is onto: $\forall y \in Y \exists x \in X$ s.t. $T(x) = u$.

If $T \in L(X, Y)$ is invertible, then $T^{-1} \in L(Y, X)$

$T \in L(X, Y)$ is one-to-one if and only if $\text{ker}T = \vec{0}$.

Two vector spaces X and Y are isomorphic if there exists an invertible linear function from X to Y. A function with these properties is called an isomorphism.

Let X and Y be two vector spaces, and let V be a basis for X. Then a linear transformation $T : X \to Y$ is completely defined by its value on V:

- Given any set $\{y_\lambda | \lambda \in \Lambda\} \subset Y$, $\exists T \in L(X, Y)$ s.t. $T(v_\lambda) = y_\lambda$ for all $\lambda \in \Lambda$.
- If $S, T \in L(X, Y)$ and $S(v_\lambda) = T(v_\lambda)$ for all $\lambda \in \Lambda$, then $S = T$.

Two vectors spaces X and Y are isomorphic iff $\dim X = \dim Y$. If $\dim X = n$, then X is isomorphic to $\mathbb{R}^n$.

Let $V = \{v_1, \ldots, v_n\} \in X$ is a basis of X. Then $\forall x \in X$ has a unique representation $x = \sum_{i=1}^n a_i v_i$. Then $crd_V(x) = \begin{pmatrix} a_1 \\ \ldots \\ a_n \end{pmatrix} \in \mathbb{R}^n$ is an isomorphism from X to $\mathbb{R}^n$.

Let V, W be bases of X, Y. Then $\forall y \in Y$ has a unique representation $y = \sum_{i=1}^m a_i w_i$, e.g. $T(v_1) = \sum_{i=1}^m a_{i1} w_i, \ldots, T(v_n) = \sum_{i=1}^m a_{in} w_i$. Then,

$\mathrm{mtx}_{W,V}(T) = \begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} \end{pmatrix}$ is an isomorphism from $L(X, Y)$ to $M_{m \times n}$.

Example: Let $X = Y = \mathbb{R}^2, V = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}, W = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}\}, T(x) = x$.

Then, $\mathrm{mtx}_{W,V} = \begin{pmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{pmatrix} = (W^{-1})T(V)$.

$\mathrm{mtx}_{W,V}(T)\ \mathrm{mtx}_{V,U}(S) = \mathrm{mtx}\ _{W,U}(T \circ S)$

$\dim X = n, T \in L(X, X)$. $\mathrm{mtx}_V(T) = \mathrm{mtx}_{V,V}(T)$.

$\mathrm{mtx}_V(T) = P^{-1} \mathrm{mtx}_W(T) P$, where $P = \mathrm{mtx}_{W,V}(id)$.

$A, B \in M_{n \times n}$ are similar if $A = P^{-1} B P$ for some invertible matrix P.

If $\dim X = n$, then

- if $T \in L(X, X)$, then any two matrix representations of T are similar.
- two similar matrices represent the same linear transformation T, relative to suitable bases.