Economics 703 : Answers to Mid-Term Exam

Raymond Deneckere

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1. Which of the following sets are compact? Which are connected? Substantiate your claim.

   (a) A finite set in $\mathbb{R}^n$.
   Let $C = \{x_1, ..., x_k\}$ denote the set in question, and let $\{O_\alpha\}_{\alpha \in A}$ be an arbitrary open cover of C. For each $j = 1, ..., k$ pick O_{α_j} such that $x_j \in O_{\alpha_j}$. This is possible because $\cup_{\alpha \in A} O_\alpha \supset C$. Then $\{O_{\alpha_j}\}_{j=1}^k$ is a finite subcover of C. Hence C is compact.
   If C consists of a single point, then C is connected. This is because there exists no separation of C : if A and B are two nonempty sets whose union is C, then it must be the case that $A = B$, i.e. that A and B are not disjoint. If, on the other hand C contains more than one element then C is not connected. Indeed, if A and B are disjoint nonempty sets whose union is C, say $A = \{x_1, ..., x_l\}$ and $B = \{x_{l+1}, ..., x_k\}$, then neither one can contain a limit point of the other. This is because all the points in C are isolated.

   (b) The rationals in $[0, 1]$.
   Let $C = Q \cap [0, 1]$. Then C is not closed, for all irrational numbers in the unit interval are limit points of a sequence of rationals converging to them (consider the decimal expansion). Hence by the Heine-Borel theorem C is not compact. C is also not connected because the connected subsets of $\mathbb{R}$ are convex.

   (c) $C = \{(x, y) \in \mathbb{R}^2 | xy \geq 1\} \cap \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 5\}$.
   The set C is not closed, for its northeast boundary is defined by a strict inequality. Hence by the Heine-Borel Theorem C is not compact.
   The set C is not connected. Consider $C_1 = \{(x, y) \in \mathbb{R}^2_+ | xy \geq 1\} \cap \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 5\}$ and $C_2 = \{(x, y) \in \mathbb{R}^2_- | xy \geq 1\} \cap \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 5\}$. Then $C_1 \neq \phi$, $C_2 \neq \phi$, $C_1 \cup C_2 = C$. Furthermore, since $d(C_1, C_2) > 0$ neither these sets contains a limit point of the other. We conclude that C_1 and C_2 form a separation of C.

2. Let $A \subset \mathbb{R}$ and $M = \sup A$. Prove of disprove the following claim : M is a limit point of A.

The statement is false, as the following simple counterexample demonstrates. Let $A = [0,1] \cup \{2\}$. Then $\sup A = 2$, but 2 is an isolated point of A.

3. Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$ satisfying $||x_k - x_l|| \leq \frac{1}{k} + \frac{1}{l}$. Does x_k converge? Why or why not?

   The sequence $\{x_k\}$ is a Cauchy sequence. Since $\mathbb{R}$ is complete, the sequence converges to a limit in $\mathbb{R}$. To see that the sequence is Cauchy, pick $\varepsilon > 0$, and let $N > \frac{2}{\varepsilon}$. Then for $k \geq N$ and $l \geq N$ we have $||x_k - x_l|| \leq \frac{1}{k} + \frac{1}{l} \leq \frac{2}{N} < \epsilon$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be given by the rule $f(x) = ax^3 + bx^2 + cx + d$, where $a > 0$. Show that f has a real root, i.e. there exists $x_0 \in \mathbb{R}$ such that $f(x_0) = 0$.

   Observe that the function f is continuous (as a polynomial, it is differentiable everywhere, and hence continuous). Since $a > 0$ we have $\lim_{x\to\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$. Hence there exists $M < \infty$ such that $f(x) \geq K$ for all $x \geq M$ and $f(x) \leq -K$ for all $x \leq -M$. Since $f(-M) < 0 < f(M)$, it follows from the intermediate value theorem that there exists $x_0 \in (-M, M)$ such that $f(x_0) = 0$.

5. Determine whether the "curve" described by the equation $x^2+y+\sin(xy) = 0$ can be written in the form $y = g(x)$ in a neighbourhood of $(0,0)$. Can the equation be written in the form $x = h(y)$ in a neighbourhood of $(0,0)$? Prove your claim.

   Let $f(x,y) = x^2 + y + \sin(xy)$. Then $\frac{\partial f}{\partial x} = 2x + y\cos(xy)$ and $\frac{\partial f}{\partial y} = 1 + x\cos(xy)$. These derivatives are continuous, so f is a C^1function. Hence the conditions for the implicit function theorem to be applicable are satisfied. Since $\cos(0) = 1$ we have $\frac{\partial f}{\partial x}(0,0) = 0$ and $\frac{\partial f}{\partial x}(0,0) = 1$. The condition of the implicit function theorem for y to be expressable as a function of x in a neighborhood of $(0,0)$, namely that $f_y \neq 0$, is therefore satisfied. Indeed, we have $g'(0) = -\frac{f_x}{f_y}(0,0) = 0$.

   However, the condition for x to be expressible as a function of y in a neighbourhood of $(0,0)$, namely that $f_x(0,0) \neq 0$, is not satisfied. Hence we cannot conclude that x can be expressed as a function of y in a neighbourhood of $(0,0)$.

   To see whether or not this can be done, let us suppose that $y = y_0 > 0$, and see if there exists a solution in x to the equation $f(x, y_0) = 0$. First note that $x = 0$ is not a solution, for $f(0, y_0) = y_0 > 0$. Next, using a first-order Taylor series expansion of $\sin xy_0$ around $x = 0$, we have $\sin xy_0 = y_0 \sin(x'y_0)$, for some $x' \in (0,x)$. Hence $f(x,y_0) = x^2 + y_0 + y_0\sin(x'y_0)$. Now since $\sin(x'y_0) \geq -1$, we see that $f(x,y_0) \geq x^2 > 0$ for all $x \neq 0$. Hence the equation $f(x,y_0)$ does not have a solution in x for any $y_0 > 0$.

   At the same time, for $y_0 < 0$, we have $f(0,y_0) = y_0 < 0$. Furthermore, since $|\sin(xy_0)| \leq 1$, and since $x^2 \to \infty$ for $x \to \pm\infty$, there exists $x_0 < 0$

and $x_1 > 0$ such that $f(x_0, y_0) > 0$ and $f(x_1, y_0) > 0$. It then follows from the intermediate value theorem that there exists $x'_0 \in (x_0, 0)$ and $x'_1 \in (0, x_1)$ such that $f(x'_0, y_0) = f(x'_1, y_1) = 0$. Thus for $y_0 > 0$ there are always two solutions in x for the equation $f(x, y_0) = 0$.

A somewhat quicker (but less insightful) way to see the the same thing goes as follows. Observe that since f is a C^2 function, the function $g(x)$ is actually be a C^2 function. More precisely, since $g'(x) = -\frac{f_x}{f_y}(x, g(x))$ we have $g''(x) = -\frac{(f_{xx}+f_{xy}g')f_y-(f_{yx}+f_{yy}g')f_x}{f_y^2}(x, g(x))$. Using the facts that $f_x(0,0) = 0$, $f_y(0,0) = 1$, $f_{xx}(0,0) = 2$, and $g'(0) = 0$, we obtain that $g''(0) = -2$. This means that near $x = 0$, $g(x)$ behaves like the quadratic $-x^2$. Thus for any $y_0 < 0$ near zero there exists $x'_0 < 0$ and $x'_1 > 0$ close to zero so that $g(x'_0) = g(x'_1) = y_0$, and for $y_0 > 0$ near zero there exists no x near zero so that $g(x) = y_0$.