Economics 703 : Anwer Key to the Mid-Term Exam

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1. Let $\{K_n\}$ be a collection of non-empty closed subsets of $\mathbb{R}^N$ such that $K_{n+1} \subset K_n$ for all $n = 0, 1, ...,$ and such that diameter$(K_n) \to 0$ as $n \to \infty$. Give a complete characterization of the set $\cap_{n=0}^{\infty} K_n$.

   For any set $E \subset \mathbb{R}^N$ we have diameter$(E) = \sup\{d(p,q) : p \in E$ and $q \in E\}$. Setting δ_n =diameter(K_n), we are given that $\delta_n \to 0$ as $n \to \infty$. Now let $K = \cap_{n=0}^{\infty} K_n$, and select $x \in K$ and $y \in K$. Note that since x and y belong to K_n for every n, we have $d(x,y) \leq \delta_n$ for every n. Upon taking limits in this inequality as $n \to \infty$ we see that $d(x,y) = 0$. Therefore K contains at most one point.

   To establish that K is non-empty, for each n select $x_n \in K_n$. Note that for each n the sequence$\{x_n\}_{n=0}^{\infty} \subset K_0$, and that since the set K_0 is compact, it is also sequentially compact. Therefore $\{x_n\}$ has a subsequence that converges to some point $x_\infty \in K_0$. Without loss of generality we may assume that this subsequence is the original sequence, so that here exists a subsequence $x_n \to x_\infty$. We now claim that $x_\infty \in K$.

2. Find two subsets C and D of $\mathbb{R}^N$ and a point $x_0 \in \mathbb{R}^N$ such that $C \cup D$ is not connected but $C \cup D \cup \{x_0\}$ is connected.

   Let $C = B(0,r)$ and x_0 be such that $\|x\| = r$. Select $y = 2x$ and let $D = B(y,r)$. Then since $d(0,x_0) = r$ and $d(y,x_0) = r$ we have $x_0 \notin C$ and $x_0 \notin D$. Furthermore, C and D form a separation of $C \cup D$, as neither set contains a limit point of the other set. However, $C \cup D \cup \{x_0\}$ is connected, as x_0 is a limit point of both C and D. Thus $C \cup D \cup \{x_0\}$ cannot be separated.

3. Let $f : K \subset \mathbb{R}^N \to \mathbb{R}$ be continuous and K compact. Define

   $$M = \{x \in K : f(x) \geq f(y) \ \ \forall \ y \in K\}.$$

   Is M compact? Defend your answer.

Yes, M is compact. Indeed, let $m = \max_{x\in K} f(x)$, so that $M = \{y \in K : f(y) = m\} = f^{-1}(\{m\})$. Then since $\{m\}$ is a closed subset of $\mathbb{R}$, and since f is continuous, the set $f^{-1}(\{m\})$ is a closed subset of K. But since K is compact, and since closed subsets of compact sets are compact, it follows that M is compact.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be given by the rule

$$f(x) = \begin{cases} x^2\,, \text{ if } x \text{ is irrational} \\ 0, \text{ if } x \text{ is rational} \end{cases}.$$

Is $f(\cdot)$ continuous at 0? Is $f(\cdot)$ differentiable at 0? Defend your answer.

Let $\{x_n\}$ be a sequence such that $x_n \to 0$. Then $f(x_n) \leq x_n^2$ for every n, so $\limsup_{n\to\infty} f(x_n) \leq 0$. Furthermore, $f(x_n) \geq 0$ for every n, so $\liminf_{n\to\infty} f(x_n) \geq 0$. We conclude that $\limsup_{n\to\infty} f(x_n) = \liminf_{n\to\infty} f(x_n) = 0$. Thus $\lim_{n\to\infty} f(x_n) = 0$, i.e. $f(\cdot)$ is continuous at 0.

It is also differentiable at 0, for

$$0 \leq \frac{f(t) - f(0)}{t} \leq \frac{t^2}{t},$$

so using a similar argument as above we obtain

$$\lim_{n\to\infty} \frac{f(t) - f(0)}{t} = 0$$

i.e. $f(\cdot)$ is differentiable at 0 and $f'(0) = 0$.

5. Consider the equations

$$\begin{aligned} u(x,y) &= \frac{x^4 + y^4}{x}, \text{ if } x \neq 0 \\ &= 0, \text{ if } x = 0 \end{aligned}$$

and

$$v(x,y) = \sin x + \cos x.$$

Near which points $(x,y) \in \mathbb{R}^2$ can we solve for x, y in terms of u, v? Defend your answer.

Let $E = \{(x,y) \in \mathbb{R}^2 : x \neq 0\}$ and $f = (u,v)$. Then we may restrict the domain if f to the open set E, for f is clearly not invertible at any point (x,y) such that $x = 0$. Note that f is continuously differentiable on E, for

$$\begin{matrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{matrix} = \begin{matrix} \frac{3x^4 - y^4}{x^2} & \frac{4y^3}{x} \\ \cos x & -\sin y \end{matrix},$$

is continuous in (x,y) on the domain E. Furthermore whenever $\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \frac{y^4 - 3x^4}{x^2}\sin y - \frac{4y^3}{x}\cos x$ does not vanish, we can solve for x, y in terms u, v.

Since $x \neq 0$ this condition can be rewritten as $(y^4 - 3x^4)\sin y - 4xy^3 \cos x \neq 0$. Solving the equation

$$(y^4 - 3x^4)\sin y - 4xy^3 \cos x = 0$$

analytically is not feasible. However, we can exhibit some points where $\left|\frac{\partial(u,v)}{\partial(x,y)}\right| \neq 0$. For example, if $x_0 = \frac{\pi}{2}$ and $y_0 = \frac{\pi}{2}$ then $\sin y = 1$ and $\cos x = 0$, and so $(y^4 - 3x^4)\sin y - 4xy^3 \cos x = -2(\frac{\pi}{2})^4 \neq 0$.