HW3

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1 Question 1

Let (X, d) be a nonempty complete metric space. Define $T : X \to X$ such that $d(T(x), T(y)) < d(x, y)\ \forall x \neq y, x, y \in X$. We will prove that T has a fixed point.

pf Contraction mapping theorem. Sketch: $\exists \beta < 1$ such that $d(T(x), T(y)) \leq \beta d(x, y) < d(x, y)\ \forall x \neq y, x, y \in X$. Then T has a unique fixed point x^* by the contraction mapping theorem.

2 Question 2

We will show that the following countable set is compact in $\mathbb{R}$: $A := \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$.

pf We can map each element of A to an element of $\mathbb{N}$. We can do so by mapping 0 to 1, and by mapping $\frac{1}{n}$ to $n+1$ for $n \in \mathbb{N}, n > 1$. Thus A is countable. Additionally A is bounded below by its smallest element, 0, and above by its largest element, 1. So A is bounded.

We will now prove that A is closed. Let $\{a_n\}$ be a convergent subsequence, with $a_n \in A\ \forall n \in \mathbb{N}$. Let $\lim\limits_{n\to\infty} a_n = a$. Assume for purpose of contradiction that $a \notin A$. We then have three possible cases:

1. Case 1: $a < 0$. By the definition of convergence, for $\epsilon = \frac{a}{2}$, $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $|a_n - a| < \epsilon$. However, all elements of A are nonnegative, and the convergence of $\{a_n\}$ implies the existence of an element of A, a_{n^*}, s.t. $a_{n^*} \leq \frac{a}{2} < 0$, a contradiction.

2. Case 2: $a > 1$. By the definition of convergence, for $\epsilon = \frac{a-1}{2}$, $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $|a_n - a| < \epsilon$. Note however that $\max A = 1$, and the convergence of $\{a_n\}$ implies the existence of an element of A, a_{n^*}, s.t. $1 < a - \epsilon < a_{n^*}$, a contradiction.

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, Ryan Mather, and Tyler Welch. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

3. Case 3: $a \in [0,1] \setminus A$. Then we can find $n^* := \arg\min_{n \in \mathbb{N}} |a - \frac{1}{n}|$. We can then define $\epsilon = \frac{|a - \frac{1}{n^*}|}{2}$, and by the definition of convergence $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $|a_n - a| < \epsilon$.[1] So, for some $\tilde{n} \in \mathbb{N}, |\frac{1}{\tilde{n}} - a| < \epsilon$, yet by construction $|a_{n^*} - a| > \frac{|a - \frac{1}{n^*}|}{2} = \epsilon$. This implies that $|a - \frac{1}{n^*}| = \min_{n \in \mathbb{N}} |a - \frac{1}{n}| > |a - \frac{1}{\tilde{n}}|$, which is a contradiction.

Thus, as all possible cases lead to contradictions, $a \in A$ so A is closed. Since A is closed and bounded it is compact, and as we showed earlier A is countable so A is an example of a countable set that is compact in $\mathbb{R}$.

3 Question 3

Prove that the function $f(x) = \cos^2(x)e^{5-x-x^2}$ has a maximum on $\mathbb{R}$.

pf Maybe an extention of the extended case of the intermediate value theorem?

4 Question 4

Suppose you have two maps of Wisconsin, one large and one small. We put the large one on top of the small one so that the small map is completely covered by the large one. Prove that a point on the small map is in the same location as it is on the large map.

pf Definitely the contraction mapping theorem.

5 Question 5

Consider the set $X = \{-1, 0, 1\}$ and the space of all functions on X, $F_X = \{f : X \to \mathbb{R}\}$.

[1] Since $a \in [0,1] \setminus A$, and 0 is in A, the closest element of A to a must be nonzero as $a > 0$ so we can always find some sufficiently large $n \in \mathbb{N}$ with $0 < \frac{1}{n} \leq a$. Since we have defined this epsilon ball such that it is closer to a than the closest element of A is to a, 0 cannot be the $a_n \in A$ that satisfies this inequality.

5.1 Show that F_X is a vector space.

5.2 Show that the operator $T : F_X \to F_X$ defined by $T(f)(x) = f(x^2), x \in \{-1, 0, 1\}$ is linear.

5.3 Calculate kern T, Im T, and rank T.

6 Question 6

Consider the following system of linear equations:

$$\begin{cases} 0 = & x_1 + x_2 + 2x_3 + x_4, \\ 0 = & 3x_1 - x_2 + x_3 - x_4, \\ 0 = & 5x_1 - 3x_2 - 3x_4. \end{cases} \tag{1}$$

Let X be the set of $\{x_1, x_2, x_3, x_4\}$ which satisfy (1).

6.1 Show that X is a vector space.

6.2 Calculate dim X.