HW7

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1 Question 1

Let $k = 1, x_1 \in X$. Then $\lambda_i = 1$ and $\lambda_1 x_1 = x_1 \in X$.

Assume that the statement holds for k, we will prove that it holds also for $k+1$.

Let $\lambda_1, \ldots, \lambda_{k+1} \geq 0$ with $\sum_{i=1}^{k+1} \lambda_i = 1$. Let $x_1, \ldots, x_{k+1} \in X$. If $S := \sum_{i=1}^{k} \lambda_i = 0$ then $\lambda_{k+1} = 1 \Rightarrow \sum_{i=1}^{k+1} \lambda_i x_i = x_{k+1} \in X$. Now assume $S > 0$. Define $\omega_i = \frac{\lambda_i}{S} \forall i \in \{1, \ldots, k\}$. Then note that $\sum_{i=1}^{k} \omega_i = 1$ so $x^* := \sum_{i=1}^{k} \omega_i x_i \in X$. Also, note that $S = 1 - \lambda_{k+1}$. Then,

$$\sum_{i=1}^{k+1} \lambda_i x_i = Sx^* + \lambda_{k+1} x_{k+1} = Sx^* + (1-S)x_{k+1} \in X$$

by the convexity of X.

2 Question 2

Let S be a set and let C be the set of all convex combinations of the elements of S.

Let $x \in C$. Then, for any convex set V which contains S, $x \in V$ by the statement proved in the previous question, so $x \in \text{co}(S)$. Thus, $C \subseteq \text{co}(S)$.

Let $x_1, x_2 \in C$. Then, x_1, x_2 are linear combinations of points in S, so for some $\lambda_1, \ldots, \lambda_n, \omega_1, \ldots, \omega_m$ and $y_1, \ldots, y_n, z_1, \ldots, z_m \in S$, $x_1 = \sum_{i=1}^{n} \lambda_i y_i, x_2 = \sum_{i=1}^{m} \omega_i z_i$. Then, for $0 \leq \tau_1, \tau_2 \leq 1, \tau_1 + \tau_2 = 0, \tau_1 x_1 + \tau_2 x_2 = \tau_1 \sum_{i=1}^{n} \lambda_i y_i + \tau_2 \sum_{i=1}^{m} \omega_i z_i$. Then, since $\sum_{i=1}^{n} \tau_1 \lambda_i = \tau_1 \sum_{i=1}^{n} \lambda_i = \tau_1$ and by identical logic $\sum_{i=1}^{m} \tau_2 \omega_i = \tau_2$, $\sum_{i=1}^{n} \tau_1 \lambda_i + \sum_{i=1}^{m} \tau_1 \omega_i = \tau_1 + \tau_2 = 1$ so $x_1 + x_2 \in C$ so C is convex. Clearly $S \subseteq C$ as for any $x \in S$, $x = 1x \in C$, so C is a convex set containing S so $\text{co}(S) \subseteq C \Rightarrow \text{co}(S) = C$.

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3 Question 3

Let X be a convex set and let $x, y \in \mathrm{cl}X, \lambda \in [0,1]$. Then, if x and y are both in X, by convexity $\lambda x+(1-\lambda)y \in X$ so $\lambda x+(1-\lambda)y \in \mathrm{cl}X$. Assume now that this is not the case. Consider the point $\lambda x+(1-\lambda)y$. Let $\epsilon > 0$. As x, y are either limit points or elements of X, $\exists x', y' \in X$ such that $d(x', x) < \epsilon, d(y', y) < \epsilon \Rightarrow d(\lambda x+(1-\lambda)y, \lambda x'+(1-\lambda)y') < \epsilon$, and $\lambda x + (1-\lambda)y \neq \lambda x' + (1-\lambda)y' \in X$ by convexity so $\lambda x + (1-\lambda)y$ is a limit point of X, so $\lambda x + (1-\lambda)y \in \mathrm{cl}X$.

4 Question 4

Let f be concave. Then, let $(x_1, y_1), (x_2, y_2) \in \mathrm{hyp}f, \lambda \in [0,1]$. Then, for $x_3 := \lambda x_1 + (1-\lambda)x_2$, $y_3 = f(x_3) \geq \lambda f(x_1) + (1-\lambda)f(x_2) = \lambda y_1 + (1-\lambda)y_2 \to (x_3, y_3) \in \mathrm{hyp}f$ so $\mathrm{hyp}X$ is convex.

Let $\mathrm{hyp}X$ be convex. Then, let $x_1, x_2 \in X, \lambda \in [0,1]$ with $y_1 = f(x_1), y_2 = f(x_2)$. Then, $(x_1, y_1), (x_2, y_2) \in \mathrm{hyp}X$ so $(\lambda x_1+(1-\lambda)x_2, \lambda y_1+(1-\lambda)y_2) := (x_3, y_3) \in \mathrm{hyp}X \Rightarrow \lambda f(x_1) + (1-\lambda)f(x_2) = \lambda y_1 + (1-\lambda)y_2 = y_3 \leq f(x_3) = f(\lambda x_1 + (1-\lambda)x_2)$ so f is concave.

5 Question 5

Let X, Y be disjoint, convex, and closed subsets of $\mathbb{R}^n$, and let X be compact. Then define $A\{y-x : x \in X, y \in Y\}$. Let a_n be a sequence in A such that $a_n \to a \in \mathbb{R}^n$. There exist x_n, y_n such that $a_n = y_n - x_n \forall n$. Since X is compact, $\exists$ a sequence $\{x_{n_k}\} \to x \in X$. Since $a_{n_k} \to a, x_{n_k} \to x, \exists y \in \mathbb{R}^n$ such that $y_{n_k} \to y$,[1] and since Y is closed, $y \in Y$. Thus, $\{a_{n_k}\} \to y - x \in A$, and since $a_n \to a$, it must be that $y - x = a$, so $a \in A$. Now we apply the theorem from lecture, and strictly separate A from $\{0\}$: $\exists p$ such that $p'(y-x) > \beta > 0$ for some β, $\forall x \in X, y \in Y$. Since X is compact, note that $\exists x^*$ such that $p'x^* \geq p'x \ \forall x \in X$. Using the strict separation constant, we have that $p'x < \frac{\beta}{2} + p'x^*$. Now we also have that $p'y > \beta + p'x$, and thus it must be that $p'y > \beta + p'x^* since x^* \in X.$. Thus, $p'y > \frac{\beta}{2} + p'x^*$, and therefore $p'x < \frac{\beta}{2}p'x^* < p'y \ \forall x \in X, y \in Y$.

6 Question 6

Assume there exists an agreeable trade vector x. Then, $\inf_{\pi \in \Pi_A} \sum_{i=1}^n \pi_i x_i > 0$ and $\inf_{\pi \in \Pi_B} \pi(-x_i) > 0 \Rightarrow \sup_{\pi \in \Pi_B} \pi(x_i) < 0$. Thus, for any $\pi \in \Pi_A, \sum_{i=1}^n \pi_i x_i > 0$ so $\pi \notin \Pi_B$. Similarly, for any $\pi \in \Pi_B, \sum_{i=1}^n \pi_i x_i < 0$ so $\pi \notin \Pi_A$. Thus, $\Pi_A \cap \Pi_B = \emptyset$.

Assume that Π_A, Π_B are disjoint. Then we can apply the separating hyperplanes theorem and for some $a \in \mathbb{R}, x \in \mathbb{R}^n \setminus \{0\}$, $\sum_{i=1}^n \pi_{1,i} x_i \leq a \leq \sum_{i=1}^n \pi_{2,i} x_i \ \forall \pi_1 \in \Pi_A, \pi_2 \in \Pi_B$. We then know that there exists a trade y such that $y_i = x_i - a \forall i \in \{1 \dots n\}$, and then $\sum_{i=1}^n \pi_{1,i} y_i \leq 0 \leq \sum_{i=1}^n \pi_{2,i} y_i$ so y is an agreeable trade vector.

[1] If this were not the case then clearly a_{n_k}, x_{n_k} would not simultaneously be able to converge.