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Solution Set to the Midterm Exam

1. (a) Let $g(n) = (-1)^{n-1}(1+1/n)$. Note that the absolute value of g(n), $|g(n)| = (1+1/n)$ is convergent, with limit equal to 1. Meanwhile, the sign of g(n) alternates between + 1 and -1. Consequently, all convergent subsequences of {g(n)} must have a limit equal to either + 1 or to -1. We conclude that liminf g(n)= -1 and limsup g(n)= + 1
(b) Let $g(n)= n(-1)^{n}$. Proceeding as above, we see that $|g(n)| = n$, i.e. the sequence |g(n)| diverges. Meanwhile, the sign of g(n) alternates between +1 and -1. Consequently, we have liminf $g(n) = -\infty$ and limsup $g(n)= +\infty$.

2. (a) Let $D=\{x \in \mathbb{R}^n: 1 \le \|x\| \le 2\}$. Let us start by examining the geometry of the set D. When n=1, D is the union of the intervals [-2,-1] and [+1,+2]. When n>1, D is a doughnut.

The set D is not open, for if we let x be any vector whose length is equal to 2, then there exists no r > 0 such that the open ball B(x,r) is entirely contained in D. Indeed, the vector $x'=(1+r/3)x$ belongs to B(0, r), since $\|x'-x\|=r\|x\|/3=(2/3)\,r<r$. But x' cannot belong to D, for $\|x'\|=(1+r/3)\|x\|>\|x\|=2$.

The set D is closed, for if $\{x_n\} \subset D$ is a convergent sequence, then its limit x must belong to D. To see this, observe that the norm is a continuous function on $\mathbb{R}^n$, so that we must have $\|x\|= \lim\|x_n\| \in [1,2]$.

The set D is bounded by B(0,r) for any r>2. Since it is also closed, it follows from the Heine Borel Theorem that D is compact.

For n=1, D is not connected, for the sets [-2,-1] and [1,2] form a separation of D. For n>1, we will show that D is path connected, and hence connected. Let x and z be two vectors in D. Then define $f:[0,1]\rightarrow D$ by the rule :

$$f(t) = ((1-t)\|x\|+t\|z\|)\left(\frac{x+t(z-x)}{\|x+t(z-x)\|}\right)$$

It is straightforward to check that f is continuous, and that f(0)= x and f(1)=z. Furthermore, $f(t) \in D$ for every t, since $\|f(t)\|=(1-t)\,\|x\| + t\,\|z\| \in [1,2]$ for all t.

(b) Let $D=\{x \in\mathbb{R}^n: \|x\| = 1\}$. That the set D is not open, closed, bounded, not connected when n=1, and connected for every n>1 is proven in the exact same manner as in (a) above.

(c) Let D be a finite non-empty set in $\mathbb{R}^n$. If $D = \phi$, then all statements about D are vacuously true, so let us assume that D consists of a finite collection of distinct points $\{x_1, x_2, \ldots, x_n\}$, where $n \geq 1$.

Observe that one point sets are trivially closed. Since the finite unions of closed sets are closed, D is closed. D is also bounded, for if we let $r = 1.5 \max\{\|x_i\|; i=1,\ldots,n\}$ then $D \subset B(0,r)$. By the Heine Borel Theorem, D is therefore compact.

To see that D is not open, let $r = \min\{d(x,y) | x \in D, y \in D \text{ and } x \neq y\}$. Since D is finite, the minimum is over a finite collection, so we know that $r > 0$. Then $B(x_1, r/2)$ does not intersect D in any point other than x_1, and hence cannot be contained in D.

Let #D denote the cardinality (the number of elements) of D. If #D=1, then D there exists no separation of D, so D is connected. If #D > 1, on the other hand, then we can separate D into the nonempty sets $\{x_1\}$ and $D \setminus \{x_1\}$.

3. Yes, the derivative of f exists at the point (x,y)=(0,0). Indeed, we have Df(0,0)=(0,0), for

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - Df(0,0)(x,y)}{\|(x,y) - (0,0)\|} = \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\|(x,y)\|}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2 y^2}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2} y^2 \leq \lim_{(x,y)\to(0,0)} y^2 = 0$$

4. Observe that $f(x,y)=x^2 + 2xy + y^2 + 6 = (x+y)^2.+6 \geq f(0,0) = 6$, so that f attains a global minimum at (0,0). At the same time, this same minimum is attained by all points (x,y) such that (x+y)=0, i.e. such that y = -x.

Just checking the second order conditions does not help here to uniquely identify the nature of the critical point. Indeed, we have

$$D^2 f(x,y) = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

This is a positive semidefinite matrix, since the first principal minor is equal to 2, and the second principal minor is equal to zero. The necessary condition for a local minimum is therefore satisfied at the point (0,0). We can rule out that the point (0,0) is a local maximum, for then the Hessian would have to be negative semidefinite (which fails since the sign of the first principal minor is not negative). However, as in the one-dimensional case, because the Hessian is not positive definite, we cannot conclude from this that f attains a local minimum (If the Hessian were positive definite, then (0,0) would be a strict local minimum, but as we saw above, locally there are many points that attain the same value). Note also that considering higher order derivatives of the function f, and using a Taylor expansion to analyze the nature of the critical point will not help here, for all higher order derivatives are zero.

5. Let $f(x,y)=(f_1(x,y),f_2(x,y)) = ((x^2 - y^2)/(x^2+y^2), xy/(x^2+y^2))$. Then we may compute :

$$\frac{\partial f_1}{\partial x} = \frac{2x(x^2+y^2)-(x^2-y^2)2x}{(x^2+y^2)^2} = \frac{4xy^2}{(x^2+y^2)^2}$$

$$\frac{\partial f_1}{\partial y} = -\frac{4x^2y}{(x^2+y^2)^2}$$

$$\frac{\partial f_2}{\partial x} = \frac{y(x^2+y^2)-xy(2x)}{(x^2+y^2)^2} = \frac{y^3-yx^2}{(x^2+y^2)^2}$$

$$\frac{\partial f_2}{\partial y} = \frac{x^3-y^2x}{(x^2+y^2)^2}$$

Near the point $(x,y)= (0,1)$, the denominator is strictly positive, so the partial derivatives of the component functions are continuous there. It follows from the inverse function theorem that we can locally invert f if the derivative of f at the point (0,1) is nonsingular. Because the partial derivatives exist and are continuous, the derivative in question exists and is given by

$$Df(0,1) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}_{x=0,y=1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

This matrix is singular, for its determinant is zero. Consequently, the inverse function theorem does not allow us to conclude that a local inverse exists. However, as we demonstrated in class with the example $f(x) = x^3$, it is still possible that a local inverse could exist, even though the conditions of the inverse function theorem do not hold (they are sufficient conditions for invertibility, not necessary conditions).

However, some reflection on the nature of the function f can help us resolve the issue. Indeed, let

$$u = \frac{x^2-y^2}{x^2+y^2} \quad v = \frac{xy}{x^2+y^2}$$

We see that at $x = 0$, any value of $y \neq 0$ yields $v = 0$. At the same time, at $x = 0$ any value of $y \neq 0$ yields $u = -1$. In any neighbourhood of the point (0,1) there are therefore a continuum of solutions in (x,y) to the equations $u = -1$ and $v = 0$. Thus f cannot be locally inverted.