Economics 703 : Answer Key to the Mid-Term Exam

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Answer three out of four questions. Each question is worth 33 points; the remaining point is free. Be sure to substantiate your answers by citing the proper definitions, and by proving your assertions.

1. Consider the set $Z_+ = \{0, 1, 2, ...\}$ of nonnegative integers as a subset of $\mathbb{R}$. Is Z_+ closed, open or neither?

   Z_+ is closed. To see this, note that the complement of Z_+ in $\mathbb{R}$ consists of $(-\infty, 0) \cup_{i=1}^{\infty} (i-1, i)$, which is a union of open intervals. Since arbitrary unions of open sets are open, and since open intervals are open, the complement of Z_+ in $\mathbb{R}$ is open. Hence Z_+ is a closed subset of of $\mathbb{R}$.

   Z_+ is not an open subset of $\mathbb{R}$. This is because every open ball around a point $n \in Z_+$ intersects Z_+ in at most a finite number of points, and hence is never entirely contained in Z_+.

2. Define $f : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$\begin{aligned} f(x,y) &= \tfrac{|y|}{x^2} 2^{-|y|/x^2} \quad && \text{if } x \neq 0 \\ &= 0 && \text{if } x = 0 \end{aligned}$$

   Determine wether or not f is continuous.

   The function f is not continuous. To see this consider a sequence $\{x_n, y_n\}$ on the surface $y = \lambda x^2$, where $x_n \to 0$ as $n \to \infty$, and where $\lambda > 0$. Then $f(x_n, y_n) = \lambda 2^{-\lambda}$ for all n, and so $\lim_{n\to\infty} f(x_n, y_n) = \lambda 2^{-\lambda} > 0$. It follows that there exists a sequence $\{x_n, y_n\}$ with $(x_n, y_n) \to (0, 0)$ such that $\lim_{n\to\infty} f(x_n, y_n) \neq f(0, 0)$. Hence f is not continuous at $(0, 0)$, and therefore not a continuous function.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by the rule $f(x, y) = x^3 - 7x^2 + y^2$. Solve the problem of maximizing and minimizing f.

   The conditions of the Weierstrass Theorem are not satisfied, for the domain of the function $f(\cdot, \cdot)$ is unbounded (and hence, by the Heine-Borel Theorem, not compact), so a maximizer and minimizer are not guaranteed to exist.

   Worse yet, there exists no $(x_0, y_0) \in \mathbb{R}^2$ such that $f(x, y) \geq f(x_0, y_0)$ for all $(x, y) \in \mathbb{R}^2$, or $(x_1, y_1) \in \mathbb{R}^2$such that $f(x_1, y_1) \geq f(x, y)$ for all $(x, y) \in \mathbb{R}^2$. Indeed, fix $x \in \mathbb{R}$. Then $\lim_{y\to\infty} f(x, y) = +\infty$. Furthermore, since $x^3 - 7x^2 = x(x^2 - 7) \to -\infty$ as $x \to -\infty$, we see that for any fixed $y \in \mathbb{R}$, we have $\lim_{x\to-\infty} f(x, y) = -\infty$. Hence for any $(x_1, y_1) \in \mathbb{R}^2$ there always exists (x, y)such that $f(x, y) > f(x_1, y_1)$, and for any (x_0, y_0) $\in \mathbb{R}^2$ there always exists $(x, y) \in \mathbb{R}^2$ such that $f(x, y) < f(x_0, y_0)$, and so neither the maximization problem nor the minimization problem have a solution.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be given by the rule $f(x) = ax^3 + bx^2 + cx + d$, where $a > 0$. Show that f has a real root, i.e. there exists $x_0 \in \mathbb{R}$ such that $f(x_0) = 0$.

   Observe that the function f is continuous (as a polynomial, it is differentiable everywhere, and hence continuous). Since $a > 0$ we have $\lim_{x\to\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$. Hence there exists $M < \infty$ such that $f(x) \geq K$ for all $x \geq M$ and $f(x) \leq -K$ for all $x \leq -M$. Since $f(-M) < 0 < f(M)$, it follows from the intermediate value theorem that there exists $x_0 \in (-M, M)$ such that $f(x_0) = 0$.