

HW6

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September 23, 2020

1 Question 1

Bob will travel along the road for some distance x , and then turn off the road and travel in the exact direction of "Happy Cow". Bob is minimizing his walking time to reach this point: $\min_{x \in [0,12]} x/5 + f(x)/3$ where $f(x)$ is the distance (in miles) through the woods that Bob must travel if Bob chooses to walk x miles on the main road. It can easily be shown via simple geometry that $f(x) = \sqrt{(12-x)^2 + 25}$. Thus, Bob solves the following:

$$\min_{x \in [0,12]} x/5 + \sqrt{(12-x)^2 + 25}/3.$$

We can take first order conditions of the objective function g with respect to x : $\frac{dg}{dx} = 1/5 - \frac{1}{6\sqrt{(12-x)^2 + 25}}(2(12-x)) = 0 \Rightarrow 1/5 = \frac{12-x}{3\sqrt{(12-x)^2 + 25}} \Rightarrow (9/25)((12-x)^2 + 25) = (12-x)^2 \Rightarrow (9 * 25)/16 = (12-x)^2 \Rightarrow (15/4) = (12-x), -(15/4) = (12-x)$. If $(12-x) < 0$ then $x > 12$ so $x \notin [0, 12]$, so $(15/4) = (12-x) \Rightarrow x = 12 - (15/4) \Rightarrow x = (33/4)L$.

2 Question 2

Assume that x_0 is a local maximum of f . Then $\exists \delta \in (0, \epsilon]$ such that for any $x \in B_\delta(x_0) \setminus \{x_0\}$, $f(x_0) \geq f(x)$. Then, notice that $x_0 - \delta/2 \in B_\delta(x_0)$. Then, by the mean value theorem, $\exists c \in (x_0 - \delta/2, x_0)$ such that $f'(c) = \frac{f(x_0) - f(x_0 - \delta/2)}{\delta/2} > 0$ which is a contradiction, so x_0 is not a local maximum of f . Now assume that x_0 is a local minimum of f . Then $\exists \delta \in (0, \epsilon]$ such that for any $x \in B_\delta(x_0) \setminus \{x_0\}$, $f(x_0) \leq f(x)$. Then, notice that $x_0 + \delta/2 \in B_\delta(x_0)$. Then, by the mean value theorem, $\exists c \in (x_0, x_0 + \delta/2)$ such that $f'(c) = \frac{f(x_0 + \delta/2) - f(x_0)}{\delta/2} > 0$ which is a contradiction, so x_0 is not a local minimum of f .

*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, and Ryan Mather. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

3 Question 3

$$\begin{aligned}
\frac{\partial f}{\partial r} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r} \\
&= (y^2 z)(1) + (2xyz)(2) + (xy^2)(1) \\
&= (2r + 4s + t)^2(3r + s + t) + 4(t + 2s + t)(2r + 3s + t)(3r + s + t) + (t + 2s + t)(2r + 4s + t)^2, \\
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s} \\
&= (y^2 z)(2) + (2xyz)(3) + (xy^2)(1) \\
&= 2(2r + 4s + t)^2(3r + s + t) + 6(t + 2s + t)(2r + 3s + t)(3r + s + t) + (t + 2s + t)(2r + 4s + t)^2, \\
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t} \\
&= (y^2 z)(3) + (2xyz)(1) + (xy^2)(1) \\
&= 3(2r + 4s + t)^2(3r + s + t) + (t + 2s + t)(2r + 3s + t)(3r + s + t) + (t + 2s + t)(2r + 4s + t)^2.
\end{aligned}$$

4 Question 4

Let f be continuously differentiable on $X \subset \mathbb{R}^n$. Then, f' exists and is continuous on X . Let $x_0 \in X$ and let $B_\epsilon(x_0) \subset X$ be a closed epsilon ball around x_0 . Since f' is continuous, it must be bounded on $B_\epsilon(x_0)$. Let m_1, m_2 be the upper and lower bounds of f' on $B_\epsilon(x_0)$, and let $M = \max\{|m_1|, |m_2|\}$. Then, $|f'(x)| \leq M \forall x \in B_\epsilon(x_0)$. Let $x_1, x_2 \in B_\epsilon(x_0)$ and assume for the purpose of contradiction that $|f(x_1) - f(x_2)| > M|x_1 - x_2|$. Then, by the mean value theorem $\exists c \in (x_1, x_2)$ such that $f'(c) = \frac{f(x_1) - f(x_2)}{x_2 - x_1}$, and note that this implies $|f'(c)| > M$. $c \in B_\epsilon(x_0)$ so this is a contradiction, and $|f(x_1) - f(x_2)| \leq M|x_1 - x_2|$, so f is locally lipschitz on X .

5 Question 5

$f(1, 1) = 0$. $\text{Det} D_X f = \text{Det}(5x^4 - 2x + 1) \Rightarrow \text{Det} D_X f(1, 1) = 5 - 2 + 1 \neq 0$. Then, by the implicit function theorem,

$$\begin{aligned}
\frac{\partial x(y)}{\partial y}|_{(1,1)} &= - \left(\frac{\partial f}{\partial x}|_{(1,1)} \right)^{-1} \left(\frac{\partial f}{\partial y}|_{(1,1)} \right) \\
&= -(5x^4 - 2x + 1)|_{(1,1)}^{-1} (-4y^2 - 2)|_{(1,1)} \\
&= -(4)^{-1}(-6) = \frac{3}{2}.
\end{aligned}$$

6 Question 6

$$Df(x, y) = \begin{pmatrix} 8x^3 - y \\ 2y - x \end{pmatrix} = \vec{0} \Rightarrow x = 2y, 64y^3 = y \Rightarrow y = 0, y = 1/8, y = -1/8 \\ \Rightarrow (x, y) = (0, 0), (1/4, 1/8), (-1/4, -1/8).$$

$$D^2f(x, y) = \begin{pmatrix} 24x^2 & -1 \\ -1 & 2 \end{pmatrix}.$$

$$D^2f(0, 0) = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}, D^2f(1/4, 1/8) = \begin{pmatrix} 3/2 & -1 \\ -1 & 2 \end{pmatrix} = D^2f(-1/4, -1/8).$$

First, we will investigate the point $(0, 0)$. $\text{Det}(D^2f(0, 0) - \lambda I) = 0 \Rightarrow -\lambda(2 - \lambda) - 1 = 0 \Rightarrow \lambda^2 - 2\lambda - 1 = 0 \Rightarrow \lambda = \frac{1}{2} - \frac{\sqrt{5}}{2}, \lambda = \frac{1}{2} + \frac{\sqrt{5}}{2}$. So, one eigenvalue is positive while the other is negative, so f has a saddle point at $(0, 0)$. Next we will investigate the point $(1/4, -1/4)$. $\text{Det}(D^2f(1/4, 1/8) - \lambda I) = 0 \Rightarrow (3/2 - \lambda)(2 - \lambda) - 1 = 0 \Rightarrow \lambda^2 - (7/2)\lambda + 2 = 0 \Rightarrow \lambda = \frac{7}{4} + \frac{\sqrt{17}}{4}, \lambda = \frac{7}{4} - \frac{\sqrt{17}}{4}$. Both eigenvalues are positive, so f has a local minimum at $(1/4, 1/2)$ and, since $\text{Det}(D^2f(1/4, 1/8)) = \text{Det}(D^2f(-1/4, -1/8))$, f has a local minimum at $(-1/4, -1/2)$.