HW3

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1 Question 1

Let (X, d) be a nonempty complete metric space. Define $T : X \to X$ such that $d(T(x), T(y)) < d(x, y)\ \forall x \neq y, x, y \in X$. We will prove that T has a fixed point.

pf Let $x_0 \in X$ be arbitrary. We will then define $\{x_n\}$ s.t. $x_n = T(x_{n-1})\ \forall x \in \mathbb{N}$. Then $d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) < d(x_n, x_{n-1})\ \forall n \in \mathbb{N}$. So, by the monotone convergence theorem, $\{a_n\}$ defined as $a_n = d(x_n, x_{n-1})$ converges to its infimum. Its infimum is 0.[1] Thus, for $\epsilon > 0$ we can find $N \in \mathbb{N}$ such that $\forall n > N$, $|d(x_n, x_{n-1}) - 0| = d(x_n, x_{n-1}) < \epsilon$ so $\{x_n\}$ is Cauchy. Since X is complete, we conclude that $\{x_n\}$ converges to some limit and define $\lim_{n\to\infty} x_n = x$. Then, note $T(x) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x$.[2] Thus, x is a fixed point of T.

Assume for the purpose of contradiction that $\exists y \in X \setminus \{x\}$ s.t. y is a fixed point. Then $d(T(x), T(y)) < d(x, y)$ since $x \neq y$. However, $T(x) = x$ and, by assumption, $T(y) = y$, so $d(T(x), T(y)) = d(x, y)$, a contradiction. Thus, x is the unique fixed point of T.

2 Question 2

We will show that the following countable set, A, is compact in $\mathbb{R}$: $A := \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$.

pf We can map each element of A to an element of $\mathbb{N}$. We can do so by mapping 0 to 1, and by mapping $\frac{1}{n}$ to $n + 1$ for $n \in \mathbb{N}, n > 1$. Thus A is countable. Additionally A is bounded below by its smallest element, 0, and above by its largest element, 1. So A is bounded.

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[1] How do I prove this?

[2] This holds because T is continuous which I will prove here:
Let $x \in X$ be arbitrary. Let $\epsilon > 0$. Then for $\delta = \epsilon$, $|T(x) - T(y)| < |x - y|\ \forall y \in \mathbb{R}$ such that $|x - y| < \delta$. Thus, T is continuous.

We will now prove that A is closed. Let $\{a_n\}$ be a convergent subsequence, with $a_n \in A\ \forall n \in \mathbb{N}$. Let $\lim_{n\to\infty} a_n = a$. Assume for purpose of contradiction that $a \notin A$. We then have three possible cases:

1. Case 1: $a < 0$. By the definition of convergence, for $\epsilon = \frac{a}{2}$, $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $|a_n - a| < \epsilon$. However, all elements of A are nonnegative, and the convergence of $\{a_n\}$ implies the existence of an element of A, a_{n^*}, s.t. $a_{n^*} < \frac{a}{2} < 0$, a contradiction.

2. Case 2: $a > 1$. By the definition of convergence, for $\epsilon = \frac{a-1}{2}$, $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $|a_n - a| < \epsilon$. Note however that $\max A = 1$, and the convergence of $\{a_n\}$ implies the existence of an element of A, a_{n^*}, s.t. $1 < a - \epsilon < a_{n^*}$, a contradiction.

3. Case 3: $a \in [0, 1] \setminus A$. Then we can find $n^* := \arg\min_{n\in\mathbb{N}} |a - \frac{1}{n}|$. We can then define $\epsilon = \frac{|a-\frac{1}{n^*}|}{2}$, and by the definition of convergence $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $|a_n - a| < \epsilon$.[3] So, for some $\tilde{n} \in \mathbb{N}$, $|\frac{1}{\tilde{n}} - a| < \epsilon$, yet by construction $|a_{n^*} - a| > \frac{|a-\frac{1}{n^*}|}{2} = \epsilon$. This implies that $|a - \frac{1}{n^*}| = \min_{n\in\mathbb{N}} |a - \frac{1}{n}| > |a - \frac{1}{\tilde{n}}|$, which is a contradiction.

Thus, as all possible cases lead to contradictions, $a \in A$ so A is closed. Since A is closed and bounded it is compact, and as we showed earlier A is countable so A is an example of a countable set that is compact in $\mathbb{R}$.

3 Question 3

Prove that the function $f(x) = \cos^2(x)e^{5-x-x^2}$ has a maximum on $\mathbb{R}$.

pf Let $g(x) := \cos^2(x)$, $h(x) := e^{5-x-x^2}$. Thus $f(x) = g(x)h(x)\ \forall x \in \mathbb{R}$. cos and exp are continuous functions so $f(x), g(x), h(x)$ are all continuous on $\mathbb{R}$. Note that $0 \le g(x) \le 1\ \forall n \in \mathbb{N}$. Note also that $0 < h(x) \Rightarrow 0 \le f(x)\ \forall x \in \mathbb{R}$. Furthermore, for $x \in (-\infty, -3) \cup (3, \infty)$, $h(x) < 1 \Rightarrow f(x) \le 1$.[4] Notice that $f(0) = \cos^2(0)e^{5-0-0^2} = (1)(e^5) > 1$. Notice also that $[-3, 3]$ is compact in $\mathbb{R}$ by the Heine-Borel theorem, so by the extreme value theorem $\exists k \in [-3, 3]$ s.t. $f(k) \ge f(x)\ \forall x \in [-3, 3]$. Therefore, $f(k) \ge f(0) > 1 \ge f(x)\ \forall x \in (-\infty, -3) \cup (3, \infty)$, so $f(k) \ge f(x)\ \forall x \in \mathbb{R}$.

4 Question 4

Suppose you have two maps of Wisconsin, one large and one small. We put the large one on top of the small one so that the small map is completely covered by the large

[3] Since $a \in [0, 1] \setminus A$, and 0 is in A, the closest element of A to a must be nonzero as $a > 0$ so we can always find some sufficiently large $n \in \mathbb{N}$ with $0 < \frac{1}{n} \le a$. Since we have defined this epsilon ball such that it is closer to a than the closest element of A is to a, 0 cannot be the $a_n \in A$ that satisfies this inequality.

[4] For $x \in (-\infty, -3) \cup (3, \infty)$, $5 - x - x^2 < 0 \Rightarrow e^{5-x-x^2} = h(x) < 1$. Assume for the purpose of contradiction that $f(x^*) > 1$ for some $x^* \in (-\infty, -3) \cup (3, \infty)$. This implies $g(x^*)h(x^*) > 1$, so either $g(x^*) > 1$ or $h(x^*) > 1$. In either case we have a contradiction so $f(x) \le 1\ \forall x \in (-\infty, -3) \cup (3, \infty)$.

one. Prove that a point on the small map is in the same location as it is on the large map.

pf Let the large (flat) map of Wisconsin $= W$ be a closed subset of $\mathbb{R}^2$. Since $\mathbb{R}^2$ is complete and W is closed, W is complete. We then have a strictly smaller map of Wisconsin, laid on top of the large map such that it is completely covered by the large map, and so we can define a mapping operator $T : W \to W$ s.t., for point $x \in W$, a point on the big map, $T(x)$ corresponds to the location on the big map on which the small map corresponds to the same geographical position as x. Since by assumption the small map is smaller than the large map, $\exists \beta < 1$ such that $\forall x, y \in W$ we have $\beta d(x, y) \geq d(T(x), T(y))$. Then T is a contraction of modulus β, so by the contraction mapping theorem T has a fixed point x^* where $x^* = T(x^*)$ so at x^* the big map and small map correspond to the same location in Wisconsin.

5 Question 5

Consider the set $X = \{-1, 0, 1\}$ and the space of all functions on X, $F_X = \{f : X \to \mathbb{R}\}$.

5.1 Show that F_X is a vector space.

5.2 Show that the operator $T : F_X \to F_X$ defined by $T(f)(x) = f(x^2), x \in \{-1, 0, 1\}$ is linear.

5.3 Calculate kern T, Im T, and rank T.

6 Question 6

Consider the following system of linear equations:

$$\begin{cases} 0 = & x_1 + x_2 + 2x_3 + x_4, \\ 0 = & 3x_1 - x_2 + x_3 - x_4, \\ 0 = & 5x_1 - 3x_2 - 3x_4. \end{cases} \tag{1}$$

Let X be the set of $\{x_1, x_2, x_3, x_4\}$ which satisfy (1).

6.1 Show that X is a vector space.

6.2 Calculate dim X.