Economics 703 : Mid-Term Exam

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1. Let (X, d) be a compact metric space, and let $f : X \to X$ be an *isometry*, i.e. $d(f(x), f(y)) = d(x, y)$, for all $x, y \in X$. Show that $f(X)$ is compact.

   First, let us show that f is a continuous function. To show this, we will argue that f is continuous at every point $x \in X$. For arbitrary $x \in X$, note that $d(f(x), f(y)) = d(x, y) \to 0$ as $y \to x$. Therefore $f(y) \to f(x)$ as $y \to x$, showing that $f(\cdot)$ is continuous at x.

   Next, we argue that the set $f(X)$ is a compact subset of X. Let $\{V_\alpha\}_{\alpha \in A}$ be an arbitrary open cover of $f(X)$. Since f is continuous, each set $O_\alpha = f^{-1}(V_\alpha)$ is an open subset of X. Furthermore, $\{O_\alpha\}_{\alpha \in A}$ must cover X, since for every $x \in X$, there must exist some $\alpha \in A$ s.t. $V_\alpha \ni f(x)$. Since X is compact, there exists a finite subset B of A, such that $\cup_{\alpha \in B} O_\alpha \supset X$. This implies that $\cup_{\alpha \in B} V_\alpha \supset f(X)$, so that $\{V_\alpha\}_{\alpha \in B}$ is a finite subcover of $f(X)$. Therefore $f(X)$ is compact.

   An alternative argument runs as follows. Let $\{y_n\}$ be a sequence in $f(X)$. Then for each n there exists $x_n \in X$ such that $f(x_n) = y_n$. Since X is compact, it is sequentially compact, implying that the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Let us denote the limit by x^*, i.e. $x^* = \lim_{k\to\infty} x_{n_k}$. Since f is continuous, we know that $f(x_{n_k}) \to f(x^*) \in f(X)$. But then the sequence $\{y_{n_k}\}$ is a convergent subsequence of $\{y_n\}$ in $f(X)$. Thus $f(X)$ is sequentially compact, and hence compact.

   Finally, one could also prove the result as follows. Let $\{y_n\}$ be a sequence in $f(X)$, that converges to a point $y^* \in X$. We will show that $y^* \in f(X)$, implying that $f(X)$ is a closed subset of X. Since X is compact, and since closed subsets of compact sets are compact, it then follows that $f(X)$ is compact.

   To show closedness of $f(X)$, let $x_n \in X$ be such that $y_n = f(x_n)$. Since X is compact, it is sequentially compact, implying that the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Let us denote the limit by x^*, i.e. $x^* = \lim_{k\to\infty} x_{n_k}$. Since f is continuous, we know that $f(x_{n_k}) \to f(x^*)$. Because $\lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f(x_n) = y^*$, we conclude that $y^* = f(x^*)$, and therefore that $y^* \in f(X)$.

2. Let $B(0, r)$ be an open ball of radius $r > 0$ in $\mathbb{R}^n$.

   (a) Define a closed hypercube in $\mathbb{R}^n$.

   (b) Prove that $B(0, r)$ contains a closed hypercube. (A picture is *not* a proof!)

   Hypercube is a horror film directed by Andrzej Sekula, the sequel to the 1999 cult hit "Cube".

   All joking aside, a hypercube in $\mathbb{R}^n$ is an n-dimensional generalization of a square in $\mathbb{R}^2$, or a cube in $\mathbb{R}^3$. Thus a hypercube is a special version of a n-cell, in which all the edge-lengths are equal. We can describe a (closed) n-cell by its endpoints $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$, as follows:

$$\{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i \text{ for all } i = 1, ..., n\} = \prod_{i=1}^{n} [a_i, b_i].$$

   The diagonal of an n-cell is the length of the line segment connecting a to b. An n-cell is a hypercube if and only if $b_i - a_i$ is the same for all i.

   Thus a closed n-cell centered at the origin, with edge length equal to $z > 0$, is:

$$C(0, z) = \{x \in \mathbb{R}^n : -\frac{z}{2} \leq x_i \leq \frac{z}{2}, \text{ for all } i = 1, ..., n\}.$$

   The diagonal of $C(0, z)$ equals $\sqrt{nz^2} = z\sqrt{n}$. Since any point in $C(0, z)$ is a distance of at most half the diameter of $C(0, z)$ away from the origin, we will have $C(0, z) \subset B(0, r)$ if and only if $\frac{\sqrt{n}}{2} z < r$.

3. Prove or disprove the following statement: Every normed linear vectorspace is connected.

   The intuition for this result is that noremed linear vectorspaces are convex sets, and that convex sets are connected. We can prove this as follows.

   Let X be a normed linear vectorspace. Since X is a linear vectorspace, given any two points x, y in X, and any $\lambda \in [0, 1]$, we know that the points λx and $(1 - \lambda)y$ belong to X, and so also their sum $\lambda x + (1 - \lambda)y$. Now consider the function $f : [0, 1] \to X$ given by the rule $f(\lambda) = \lambda x + (1 - \lambda)y$. Since

$$\| f(\lambda) - f(\lambda') \| \quad = \| (\lambda - \lambda')(x - y) \| \quad = |\lambda - \lambda'| \ \| x - y \|$$

   it follows that $\| f(\lambda) - f(\lambda') \| \to 0$ as $\lambda' \to \lambda$, and hence that $f(\cdot)$ is a continuous function. We conclude that X is path-connected, and hence connected.

4. Let $f, g : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be continuous functions satisfying

$$f(x)g(x) > 0, \text{ for all } x \in [a, b].$$

Does the equation

$$(x - a)f(x) + (x - b)g(x) = 0$$

have a solution on the interval $[a, b]$? Either prove or disprove the statement.

Since $f(x)g(x) > 0$, for all $x \in [a, b]$, we must either have $f(x) > 0$ and $g(x) > 0$ for all $x \in [a, b]$, or $f(x) < 0$ and $g(x) < 0$ for all $x \in [a, b]$. To see why, suppose that $f(a) > 0$ but $f(c) < 0$ for some $c \in (a, b]$. Then by the intermediate value theorem, there would exist $d \in (a, c)$ such that $f(d) = 0$, contradicting that $f(d)g(d) > 0$.

Define the function $h : \mathbb{R} \to \mathbb{R}$ by the rule $h(x) = (x - a)f(x) + (x - b)g(x)$. Then h is a continuous function, since by the continuity of the functions f and g we have $\lim_{y \to x} h(y) = (x-a)f(x)+(x-b)g(x) = h(x)$. Furthermore, note that $h(a) = -(b - a)g(a)$ and $h(b) = (b - a)f(b)$. By the result from the previous paragraph, we therefore either have $h(a) < 0$ and $h(b) > 0$, or $h(a) > 0$ and $h(b) < 0$. Thus $h(a) < 0 < h(b)$ or $h(b) < 0 < h(a)$. In either case, it follows from the intermediate value theorem that there exists $x \in (a, b)$ such that $h(x) = 0$.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function. We say that such a function is harmonic if $\frac{\partial^2 f}{\partial x^2}(x, y) + \frac{\partial^2 f}{\partial y^2}(x, y) = 0$ holds for all $(x, y) \in \mathbb{R}^2$. Suppose that the point (x_0, y_0) is a strict local maximum of f. Characterize the second derivative of f at the point (x_0, y_0). (Be as specific as you can!)

Since (x_0, y_0) is a local maximum of f, and f is a C^2 function, the second derivative of f, $D^2 f(x_0, y_0)$, must be a negative semidefinite matrix. Recall that the second derivative of f is the (2×2) matrix containing the second cross-partials:

$$D^2 f(x_0, y_0) = \begin{array}{cc} \frac{\partial^2 f}{\partial x^2}(x_0, y_0) & \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \\ \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) & \frac{\partial^2 f}{\partial y^2}(x_0, y_0) \end{array} ,$$

and that because f is C^2, Young's Theorem implies $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$. Negative semi-definiteness of the matrix $D^2 f(x_0, y_0)$ then means that:

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) \leq 0 \quad \frac{\partial^2 f}{\partial y^2}(x_0, y_0) \leq 0 \quad \frac{\partial^2 f}{\partial x^2}(x_0, y_0)\frac{\partial^2 f}{\partial y^2}(x_0, y_0) - \left(\frac{\partial^2 f}{\partial y^2}(x_0, y_0)\right)^2 \geq 0 \ . \tag{1}$$

Because the function f is harmonic, if we had $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0$, it would follow that $\frac{\partial^2 f}{\partial y^2}(x_0, y_0) > 0$, violating (1). Thus we must have $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) = 0$. A parallel argument establishes that $\frac{\partial^2 f}{\partial y^2}(x_0, y_0) = 0$. But then the third inequality in (1) implies

$$-\left(\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)\right)^2 \geq 0,$$

which can only hold if $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = 0$. We conclude that $D^2 f(x_0, y_0) = 0$.