Econ 703 – Answer Key to the Midterm Exam

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1. The sequence can be written more explicitly as:

$$\{s_n\} = \{0, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{8}, \frac{7}{8}, \frac{7}{16}, \frac{15}{16}, ...\}.$$

Direct inspection of s_n reveals that there are only two subsequential limits of s_n, namely the limit of the even terms and the limit of the odd terms. Let $s_1 = \lim_{n\to\infty} s_{2n-1}$ and $s_2 = \lim_{n\to\infty} s_{2n}$, then we see that $s_1 = 1$ and $s_2 = \frac{1}{2}$.

Formally, we can show this as follows. From $s_{2n} = \frac{s_{2n-1}}{2}$ and $s_{2n+1} = s_{2n} + \frac{1}{2}$, upon taking limits as $n \to \infty$ it follows that $s_2 = \frac{s_1}{2}$ and $s_1 = s_2 + \frac{1}{2}$. Hence $s_2 = \frac{s_2}{2} + \frac{1}{4}$ or $s_2 = \frac{1}{2}$, and $s_1 = s_2 + \frac{1}{2} = 1$.

Since lim inf is the lowest subsequential limit of $\{s_n\}$, and lim sup is the highest subsequential limit of $\{s_n\}$, we conclude that

$$\liminf_{n\to\infty} s_n = \frac{1}{2} \quad \limsup_{n\to\infty} s_n = 1.$$

2. (a). Let $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

S is not open, for given any point $(x, y) \in S$, there exists no open ball $B((x, y), r)$ with positive radius contained in S.

S is closed, for if $\{(x_n, y_n)\}$ is a sequence in S such that $(x_n, y_n) \to (x, y)$, we have $x_n^2+y_n^2 = 1$ for every n; upon taking limits it follows that $x^2 + y^2 = 1$, i.e. $(x, y) \in S$.

S in not convex, because the point $(0, 0)$ is a convex combination (with weight $\frac{1}{2}$) of the points $(-1, 0)$ and $(1, 0)$ in S, but does not belong to S.

S is compact, because $S \subset B(0, 1 + \epsilon)$ for every $\epsilon > 0$, so it is bounded. We argued above that S is closed. Closed and bounded subsets of Euclidean spaces are compact.

S is connected, for if A and B are nonempty and disjoint, and $A \cup B = S$, then we necessarily have $A \cap \bar{B} \neq \emptyset$ and $\bar{A} \cap B \neq \emptyset$. Hence there exists no separation of S.

(b). Let $S = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, y = x\}$.

S is not open, for if $(x, y) \in S$ then there exists no $r > 0$ s.t. $B((x, y), r) \subset S$.

S is not closed, for it does not contain its limit points $(0, 0)$ and $(1, 1)$.

S is convex, for if $(x_1, y_1) \in S$ and $(x_2, y_2) \in S$, then $y_1 = x_1$ and $y_2 = x_2$, so $(x_{lambda}, y_{lambda}) \equiv \lambda(x_1, y_1)+(1-\lambda)(x_2, y_2) = (\lambda x_1+(1-\lambda)x_2, \lambda x_1+(1-\lambda)x_2)$. Since $x_1 \in (0, 1)$ and $x_2 \in (0, 1)$, we have $\lambda x_1 + (1 - \lambda)x_2 \in (0, 1)$ for all $\lambda \in [0, 1]$, implying $(x_{lambda}, y_{lambda}) \in S$.

S is not closed, so S cannot be compact.

S is connected, for if A and B are disjoint nonempty sets s.t. $A \cap B = S$ then $A \cap \bar{B} \neq \emptyset$ and $\bar{A} \cap B \neq \emptyset$. Hence there exists no separation of S.

(c). Let $S = \{(x,y) \in \mathbb{R}^2 : y = \sin\frac{1}{x}, x > 0\} \cup \{(0,0)\}$.
S is not open, for if $(x,y) \in S$ then there exists no $r > 0$ s.t. $B((x,y),r) \subset S$.
S is not closed, for all of the points in $T = \{(0,y) : -1 \leq y \leq 1\}$ are limits points of S, but only one of them namely $(0,0)$ belongs to S.
S is not convex, for if we take any two points at consecutive tops of the sinewave, none of the points on the interior of the line segment connecting them belong to S.
All compact sets must be closed. Since S is not closed, S cannot be compact. In addition compact sets must be bounded, and S is not bounded because the domain is unbounded.
S is connected, for the only conceivable separation is of the form $A = (0,0)$ and $B = S\backslash A$. However, then $A \cap \bar{B} = (0,0)$, so A and B do not separate S.

3. Let $T(x) = \{x \in \mathbb{R} : g(x) \leq y \leq h(x)\}$ where $g, h : \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying $h(x) \geq g(x)$ for all $x \in \mathbb{R}$.

   The correspondence $T(x)$ is u.h.c. To see this, let $W = (a,b)$ be a basic open set, and consider the upper inverse of W under T, i.e.

$$T_+^{-1}(W) = \{x \in \mathbb{R} : T(x) \subset W\}$$

We need to show that $T_+^{-1}(W)$ is an open subset in $\mathbb{R}$. Note that $x \in T_+^{-1}(W)$ if and only if $g(x) > a$ and $h(x) < b$. Thus we have

$$T_+^{-1}(W) = g^{-1}((a,\infty)) \cap h^{-1}((-\infty,b)).$$

Note that both (a,∞) and $(-\infty,b)$ are open sets. Since the inverse image of an open set under a continuous function is open, it follows that $g^{-1}((a,\infty))$ and $h^{-1}((-\infty,b))$ are open sets. Finite intersection of open sets are open, so $T_+^{-1}(W)$ is open.

   The correspondence $T(x)$ is l.h.c. To see this, consider the lower inverse of W under T, i.e.

$$T_-^{-1}(W) = \{x \in \mathbb{R} : T(x) \cap W \neq \emptyset\}$$

We need to show that $T_-^{-1}(W)$ is an open subset in $\mathbb{R}$. Note that $x \in T_-^{-1}(W)$ if and only if $g(x) > a$ or $h(x) < b$. Thus we have

$$T_-^{-1}(W) = g^{-1}((a,\infty)) \cup h^{-1}((-\infty,b)).$$

Each of the sets on the right-hand side of this equality was argued above to be open. Since the union of open sets are open, we conclude $T_-^{-1}(W)$ is open.

   T is a continuous correspondence, for its is both u.h.c. and l.h.c.

4. Direct computation shows that

$$f'(x) = ax^{a-1}\sin\frac{1}{x} - x^{a-2}\cos\frac{1}{x} \qquad (*)$$

whenever $x \neq 0$. For any $x \in [-1,1]$ s.t. $x \neq 0$, $f'(x)$ is continuous at x, and hence f is certainly differentiable and continuous at such x. We therefore need only investigate the behavior of f and f' at $x = 0$.

(a) Continuity of f at $x = 0$ requires that $f(x) = x^a \sin(\frac{1}{x}) \to f(0) = 0$. Since $\sin(\frac{1}{x})$ takes on the value $+1$ infinitely often as $x \to 0$, we must have $x^a \to 0$ as $x \to 0$. This is true iff $a > 0$. Thus f is continuous iff $a > 0$.

(b) The derivative of f at $x = 0$, if it exists, is defined as the limit of the difference quotient

$$\boxed{\frac{f(x) - f(0)}{x} = x^{a-1} \sin(\frac{1}{x})}$$

as $x \to 0$. If $a \leq 1$, the difference quotient does not tend to any limit as $x \to 0$, so for f to be differentiable we must have $a > 1$. In that case $x^{a-1} \to 0$ as $x \to 0$, so we have

$$|\frac{f(x) - f(0)}{x}| = |x^{a-1} \sin(\frac{1}{x})| \leq |x^{a-1}||\sin(\frac{1}{x})| \leq |x^{a-1}|,$$

i.e. $\frac{f(x)-f(0)}{x} \to 0$ as $x \to 0$. We conclude that f is differentiable iff $a > 1$, in which case $f'(0) = 0$.

(c) For f to be continuously differentiable, we must have $\lim_{x \to 0} f'(x) = f'(0) = 0$. We have already shown that $\lim_{x \to 0} x^{a-1} \sin(\frac{1}{x}) = 0$ whenever $f'(0)$ is defined (i.e. $a > 1$). It follows from (*) that f is continuously differentiable iff

$$x^{a-2} \cos(\frac{1}{x}) \to 0$$

as $x \to 0$. Reasoning analogous to part (b) establishes that this happens iff $a - 2 > 0$, i.e. $a > 2$.

5. If $f : \mathbb{R} \to \mathbb{R}^n$, then $f(x)$ can be written as $(f_1(x), ..., f_n(x))$, where $f_i : \mathbb{R} \to \mathbb{R}$ are the component functions. Note that if all of the component functions are differentiable, then we have for each i:

$$\lim_{n \to 0} \frac{f_i(x + h) - f_i(x) - f_i'(x)h}{h} = 0.$$

Consequently, upon letting $A = (f_1'(x), ..., f_n'(x))$, we have

$$\|\frac{f(x + h) - f(x) - Ah}{h}\| \leq \max_i |\frac{f_i(x + h) - f_i(x) - f_i'(x)h}{h}| \to 0$$

as $h \to 0$, proving that $A = (f_1'(x), ..., f_n'(x))$ is the derivative of f.

In our case, when $n = 2$, we see that $f_1'(t) = 1$ and $f_2'(t) = 2t$; both of these are well defined at $t = 0$, so f is differentiable at $t = 0$.

6. We proved in class that compact subsets of a metric space (X, d) are closed. Thus if X is compact, it is also closed.

Compact metric space are also bounded, as we will now demonstrate.

Proof: If X is compact, then the open cover $\{B(x, 1); x \in X\}$ must have a finite subcover, that is, there exists a finite collection $\{x_1, ..., x_N\}$ s.t. $X \subset \cup_{i=1}^{N} B(x_i, 1)$. Now if $x \in B(x_i, 1)$, then by the triangle inequality

$$d(0, x) \leq d(0, x_i) + d(x, x_i) \leq m_i + 1$$

where $m_i = d(0, x_i)$. Hence for arbitrary $x \in X$, we have

$$d(0, x) \leq \max_{i=1,...n} m_i + 1 \equiv M.$$

It follows that $B(0, M)$ covers X, i.e. X is bounded.

The converse is not true: there exist metric spaces that are closed and bounded but not compact. We encountered one such example in Homework 4 : the space $E = \{p \in \mathbb{Q} : 2 < p^2 < 3\}$ is closed and bounded in $\mathbb{Q}$, but not compact. A simpler example is $X = (0, 1)$. The entire space X is always closed (it contains all of its limit points), and is clearly bounded, but not compact.