

# Exam Note Sheet

Michael B. Nattinger

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## 1 Real Analysis

### 1.1 Theorems

**Bolzano-Weierstrass Theorem:** Every bounded real sequence contains at least one convergent subsequence.

**Monotone Convergence Theorem:** Every increasing sequence of real numbers that is bounded above converges. Every decreasing sequence of real numbers that is bounded below converges.

Every real sequence contains either a decreasing subsequence or increasing subsequence or possibly both.

A set is closed if and only if every convergent sequence contained in A has its limit in A.

$\lim_{x \rightarrow x_0} f(x) = y_0$  if and only if for any sequence  $\{x_n\} \in X$  such that  $x_n \rightarrow x_0$  and  $x_n \neq x_0$ , the sequence  $\{f(x_n)\}$  converges to  $y_0$ .

A function  $f$  is continuous at  $x_0$  if and only if one of the following equivalent statements is true:

- $f(x_0)$  is defined and either  $x_0$  is an isolated point or  $x_0$  is a limit point of X and  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ .
- For any sequence  $\{x_n\}$  s.t.  $x_n \rightarrow x_0$ , the sequence  $\{f(x_n)\}$  converges to  $f(x_0)$ .

Let  $(X, d)$  be a metric space.

- $\emptyset, X$  are simultaneously open and closed in X;
- the union of an arbitrary collection of open sets is open;
- the intersection of a finite collection of closed sets is closed;
- the union of a finite collection of closed sets is closed;
- the intersection of an arbitrary collection of closed sets is closed.

A function  $f$  is continuous iff for any closed set  $C$ , the set  $f^{-1}(C)$  is closed. A function  $f$  is continuous iff for any open set  $A$ , the set  $f^{-1}(A)$  is open.

**Supremum Property:** Every nonempty set of real numbers that is bounded above has a supremum, and the supremum is a real number. (Not generally the case for all numbers e.g. sets that would be bounded by irrational numbers in the reals do not have a supremum when they are instead defined in the rationals)

**Extreme Value Theorem:** Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. Then  $f$  attains its maximum and minimum on  $[a, b]$ .

**Intermediate Value Theorem:** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function. Then for any  $\gamma \in [f(a), f(b)]$  there exists  $c \in [a, b]$  s.t.  $f(c) = \gamma$ .

Let  $f$  be monotonically increasing. Then one-sided limits exist for all  $x$ . Moreover,  $\sup\{f(s) | a < s < x\} = f(x^-) \leq f(x) \leq f(x^+) = \inf\{f(s) | x < s < b\}$ .

**Contraction Mapping Theorem:** Let  $(X, d)$  be a nonempty **complete** metric space and  $T : X \rightarrow X$  a contraction with modulus  $\beta < 1$ . Then  $T$  has a unique fixed point  $x_0$ . Additionally,  $\forall x_0 \in X$  the sequence  $\{x_n\}$ , where  $x_n = T^n(x_0) = T(T(\dots T(x_0) \dots))$  converges to  $x_0$ .

Continuous dependence of the fixed point on parameters: Let  $(X, d)$  and  $(\Omega, \rho)$  be two metric spaces and  $T : X \times \Omega \rightarrow X$ . For each  $\omega \in \Omega$ , let  $T_\omega : X \rightarrow X$  be defined by  $T_\omega(x) = T(x, \omega)$ . Suppose  $(X, d)$  is complete,  $T$  is continuous in  $\omega$ , and  $\exists \beta < 1$  such that  $T_\omega$  is a contraction of modulus  $\beta$  for all  $\omega \in \Omega$ . Then the fixed point function  $x^* : \Omega \rightarrow X$  defined by  $x^*(\omega) = T_\omega(x^*(\omega))$  is continuous.

**Blackwell's Sufficient Conditions:** Let  $B(X)$  be the set of all bounded functions from  $X \rightarrow \mathbb{R}$  with metric  $d_\infty(f, g) = \sup_{x \in X} |f(x) - g(x)|$ . Let  $T : B(X) \rightarrow B(X)$  satisfy:

- (monotonicity) if  $f(x) \leq g(x) \forall x \in X$ , then  $(T(f))(x) \leq (T(g))(x) \forall x \in X$ .
- (discounting)  $\exists \beta \in (0, 1)$  s.t. for every  $a \geq 0$  and  $x \in X$ ,  $(T(f + a))(x) \leq (T(f))(x) + \beta a$ .

Then  $T$  is a contraction with modulus  $\beta$ .

Any closed subset of a compact space is compact. If  $A$  is a compact subset of a metric space, then  $A$  is closed and bounded.

**Heine-Borel Theorem:** If  $A \subset \mathbb{R}^m$ , then  $A$  is compact if and only if  $A$  is closed and bounded.

Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. If  $f : X \rightarrow Y$  is continuous and  $C$  is a compact set in  $(X, d)$  then  $f(C)$  is compact in  $(Y, \rho)$ .

**Extreme Value Theorem:** If  $C$  is a compact set in a metric space  $(X, d)$  and  $f : C \rightarrow \mathbb{R}$  is continuous, then  $f$  is bounded on  $C$  and attains its maximum and minimum.

Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces,  $C \subset X$  compact,  $f : C \rightarrow Y$  continuous. Then  $f$  is uniformly continuous on  $C$ .

## 1.2 Space definitions

A **metric** on a set  $X$  is a function  $d : X \times X \rightarrow \mathbb{R}^+$  s.t.  $\forall x, y, z \in X$ ,

- $d(x, y) \geq 0$ , with equality  $\iff x = y$ ;
- $d(x, y) = d(y, x)$ ;
- $d(x, z) \leq d(x, y) + d(y, z)$ .

A **metric space** is a pair  $(X, d)$ , where  $X$  is a set and  $d$  is a metric on  $X$ . Examples include Euclidean space.

In a metric space,  $(X, d)$ , an **open ball** is  $B_\epsilon(x) = \{y \in X \mid d(x, y) < \epsilon\}$  and a **closed ball** is  $B_\epsilon[x] = \{y \in X \mid d(x, y) \leq \epsilon\}$ .

A **sequence** in a set  $X$  is a function  $s : \mathbb{N} \rightarrow X$ , denoted  $\{s_n\}$ , where  $s_n = s(n)$ . A sequence  $x_n$  in a metric space  $(X, d)$  **converges** to  $x \in X$  if  $\forall \epsilon > 0, \exists N(\epsilon) > 0$  s.t.  $\forall n > N(\epsilon) \ d(x_n, x) < \epsilon$ . We write  $x_n \rightarrow x$  or  $\lim_{n \rightarrow \infty} x_n = x$ .

A subset  $s \subset X$  in a metric space  $(X, d)$  is **bounded** if  $\exists x \in X, \beta \in \mathbb{R}$  s.t.  $\forall s \in S, d(x, s) < \beta$ . Every convergent sequence in a metric space is bounded.

Let  $(X, d)$  be a metric space. A set  $A \subset X$  is **open** if  $\forall x \in A \exists \epsilon > 0$  s.t.  $B_\epsilon(x) \subset A$ . A set  $C \subset X$  is **closed** if its complement is open. This depends on the metric space.  $[0, 1]$  is not open in  $(\mathbb{R}, d_E)$  but is open in  $([0, 1], d_E)$ .

Let  $(X, d)$ , and  $A \in X$ .  $x \in X$  is a **limit point** of  $A$  if  $\forall \epsilon > 0, (B_\epsilon(x) \setminus \{x\}) \cap A \neq \emptyset$ .

Let  $(X, d), (Y, \rho)$  be two metric spaces,  $A \subset X, f : A \rightarrow Y, x_0$  = limit point of  $A$ .  $f$  has a limit  $y_0$  as  $x$  approaches  $x_0$  if  $\forall \epsilon > 0 \exists \delta > 0$  s.t. if  $x \in A$  and  $0 < d(x, x_0) < \delta$ , then  $\rho(f(x), y_0) < \epsilon$ .

A function is **continuous** at  $x_0$  if  $\forall \epsilon > 0 \exists \delta > 0$  s.t. if  $d(x, x_0) < \delta$ , then  $\rho(f(x), f(x_0)) < \epsilon$ . ( $\delta$  can vary for different  $x_0$  and  $\epsilon$ )

A function  $f$  is continuous if it is continuous at every point of its domain.

A function is uniformly continuous if  $\forall \epsilon > 0 \exists \delta > 0$  s.t. if  $d(x, x_0) < \delta$ , then  $\rho(f(x), f(x_0)) < \epsilon$ . Note: delta depends only on epsilon!

Let  $(X, d), (Y, \rho)$  be two metric spaces,  $f : X \rightarrow Y, E \subset X$ . Then  $f$  is **Lipschitz** on  $E$  if  $\exists K > 0$  s.t.  $\rho(f(x), f(y)) \leq Kd(x, y) \forall x, y \in E$ .  $f$  is **locally Lipschitz** on  $E$  if  $\forall x \in E \exists \epsilon > 0$  s.t.  $f$  is Lipschitz on  $B_\epsilon(x) \cap E$ .

Suppose  $X$  is bounded above. The supremum of  $X$ ,  $\sup X$ , is the smallest upper bound for  $X$ , i.e.  $\sup X$  satisfies

- $\sup X \geq x \ \forall x \in X$ ;
- $\forall y < \sup X \exists x \in X$  s.t.  $x > y$ .

And infimum is similarly defined.

A sequence is **Cauchy** if  $\forall \epsilon > 0 \exists N > 0$  s.t. if  $m, n > N$  then  $d(x_n, x_m) < \epsilon$ . Every convergent sequence in a metric space is Cauchy.

A metric space is **complete** if every Cauchy sequence contained in  $X$  converges to some point in  $X$ . Euclidean space is complete. If  $(X, d)$  is a complete metric space and  $Y \subset X$ , then  $(Y, d)$  is complete if and only if  $Y$  is closed.

A function from a metric space to itself is called an **operator**. An operator is a contraction of modulus  $\beta$  if  $\beta < 1$  and  $d(T(x), T(y)) \leq \beta d(x, y) \forall x, y \in X$ . Every contraction is uniformly continuous.

A collection of sets  $U = \{U_\lambda | \lambda \in \Lambda\}$  in a metric space is an open cover of the set  $A$  if  $U_\lambda$  is open for all  $\lambda \in \Lambda$  and  $A \subset \bigcup_{\lambda \in \Lambda} U_\lambda$ .

A set  $A$  in a metric space is **compact** if every open cover of  $A$  contains a finite subcover of  $A$ . To prove something is compact, find a way to reduce any general open cover into a finite subcover. To prove something is not compact, find an infinite subcover which cannot be reduced in this way.

## 2 Linear Algebra

A **vector space**  $V$  is a collection of vectors, which may be added together and multiplied by scalars, satisfying the following conditions:

- $\forall x, y, z \in V, (x + y) + z = x + (y + z)$ ;
- $\forall x, y \in V, x + y = y + x$ ;
- $\exists \vec{0} \in V$  s.t.  $\forall x \in V, x + \vec{0} = \vec{0} + x = x$ ;
- $\forall x \in V \exists (-x) \in V$  s.t.  $x + (-x) = \vec{0}$ ;
- $\forall \alpha \in \mathbb{R}, x, y \in V, \alpha(x + y) = \alpha x + \alpha y$ ;
- $\forall \alpha, \beta \in \mathbb{R}, x \in V, (\alpha + \beta)x = \alpha x + \beta x$ ;
- $\forall \alpha, \beta \in \mathbb{R}, x \in V, (\alpha\beta)x = \alpha(\beta x)$ ;
- $\forall x \in V, 1x = x$ .

A set spans  $V$  if  $V = \text{span}W$  where a span is a linear combination of a set of vectors.

A set is linearly independent if  $\nexists x_1, \dots, x_n \in X, a_1, \dots, a_n \in \mathbb{R}$  s.t.  $\sum a_i \neq 0$  and  $\sum a_i x_i = \vec{0}$ .

Alternatively, if  $a_i x_i = \vec{0}$  implies  $a_1 = \dots = a_n = 0$  then  $X$  is linearly independent.

A **basis** of  $V$  is a set of linearly independent vectors in  $V$  that spans  $V$ .

Let  $B$  be a basis for  $V$  and enumerate elements of  $B$  by a set  $\Lambda$  so that  $B = \{v_\lambda | \lambda \in \Lambda\}$ . Then every vector  $x \in V$  has a unique representation as a linear combination of elements of  $B$  with finitely many nonzero coefficients.

Every vector space has a basis. Any two bases of a vector space have the same cardinality.

If  $V$  is a vector space and  $W \subset V$  is linearly independent, then there exists a linearly independent set  $B$  such that  $W \subset B \subset \text{span}B = V$

Let  $V$  be a vector space. The dimension of  $V$  is the cardinality of any basis of  $V$ .  $\dim \mathbb{R}^n = n$ .

Suppose  $\dim V = n \in \mathbb{N}$ . If  $W \subset V$  and  $|W| > n$ , where  $|W|$  denotes the cardinality of  $W$ , then  $W$  is linearly dependent.

Suppose  $\dim V = n$  and  $W \subset V, |W| = n$ . Then, if  $W$  is linearly independent, then  $\text{span} W = V$ , so  $W$  is a basis of  $V$ ; if  $\text{span} W = V$ , then  $W$  is linearly independent, so  $W$  is a basis of  $V$ .

Let  $X$  and  $Y$  be two vector spaces. We say that  $T : X \rightarrow Y$  is a linear transformation if for all  $x_1, x_2 \in X, a_1, a_2 \in \mathbb{R}, T(a_1x_1 + a_2x_2) = a_1T(x_1) + a_2T(x_2)$ .

If  $L(X, Y)$  is the set of all linear transformations from  $X$  to  $Y$ , then  $L(X, Y)$  is a vector space.

If  $R : X \rightarrow Y, S : Y \rightarrow Z$  then  $R \circ S : X \rightarrow Z$  is a linear transformation.

Let  $T \in L(X, Y)$ .  $\text{Im} T : \{T(x) | x \in X\}$  and  $\ker T := \{x \in X | T(x) = \vec{0}\}$ , and the rank of  $T$  is  $\text{rank} T := \dim(\text{Im}(T))$ .

If  $T \in L(X, Y)$ , then  $\text{Im} T$  and  $\ker T$  are vector subspaces of  $Y$  and  $X$ , respectively.

Let  $X$  be a finite-dimensional vector space and  $T \in L(X, Y)$ .

Then  $\dim X = \dim(\ker T) + \text{rank} T = \dim(\ker T) + \dim(\text{Im} T)$ .

$T \in L(X, Y)$  is invertible if there exists a function  $S : Y \rightarrow X$  such that  $S(T(x)) = x \forall x \in X, T(S(y)) = y \forall y \in Y$ . The transformation  $S$  is called the inverse of  $T$  and is denoted  $T^{-1}$ .

$T$  is invertible means:

- $T$  is one-to-one:  $\forall x_1 \neq x_2, T(x_1) \neq T(x_2)$ .
- $T$  is onto:  $\forall y \in Y \exists x \in X$  s.t.  $T(x) = y$ .

If  $T \in L(X, Y)$  is invertible, then  $T^{-1} \in L(Y, X)$

$T \in L(X, Y)$  is one-to-one if and only if  $\ker T = \{\vec{0}\}$ .

Two vector spaces  $X$  and  $Y$  are isomorphic if there exists an invertible linear function from  $X$  to  $Y$ . A function with these properties is called an isomorphism.

Let  $X$  and  $Y$  be two vector spaces, and let  $V$  be a basis for  $X$ . Then a linear transformation  $T : X \rightarrow Y$  is completely defined by its value on  $V$ :

- Given any set  $\{y_\lambda | \lambda \in \Lambda\} \subset Y, \exists T \in L(X, Y)$  s.t.  $T(v_\lambda) = y_\lambda$  for all  $\lambda \in \Lambda$ .
- If  $S, T \in L(X, Y)$  and  $S(v_\lambda) = T(v_\lambda)$  for all  $\lambda \in \Lambda$ , then  $S = T$ .

Two vector spaces  $X$  and  $Y$  are isomorphic iff  $\dim X = \dim Y$ . If  $\dim X = n$ , then  $X$  is isomorphic to  $\mathbb{R}^n$ .

Let  $V = \{v_1, \dots, v_n\} \in X$  is a basis of  $X$ . Then  $\forall x \in X$  has a unique representation

$x = \sum_{i=1}^n a_i v_i$ . Then  $\text{crd}_V(x) = \begin{pmatrix} a_1 \\ \dots \\ a_n \end{pmatrix} \in \mathbb{R}^n$  is an isomorphism from  $X$  to  $\mathbb{R}^n$ .

Let  $V, W$  be bases of  $X, Y$ . Then  $\forall y \in Y$  has a unique representation  $y = \sum_{i=1}^m a_i w_i$ , e.g.  $T(v_1) = \sum_{i=1}^m a_{i1} w_i, \dots, T(v_n) = \sum_{i=1}^m a_{in} w_i$ . Then,

$\text{mtx}_{W,V}(T) = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ \dots & \dots & \dots & \dots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix}$  is an isomorphism from  $L(X, Y)$  to  $M_{m \times n}$ .

Example: Let  $X = Y = \mathbb{R}^2$ ,  $V = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ ,  $W = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ ,  $T(x) = x$ .

Then,  $\text{mtx}_{W,V} = \begin{pmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{pmatrix} = (W^{-1})T(V)$ .

$$\text{mtx}_{W,V}(T) \text{ mtx}_{V,U}(S) = \text{mtx}_{W,U}(T \circ S)$$

$$\dim X = n, T \in L(X, X). \text{ mtx}_V(T) = \text{mtx}_{V,V}(T).$$

$$\text{mtx}_V(T) = P^{-1} \text{mtx}_W(T) P, \text{ where } P = \text{mtx}_{W,V}(id).$$

$$A, B \in M_{n \times n} \text{ are similar if } A = P^{-1}BP \text{ for some invertible matrix } P.$$

If  $\dim X = n$ , then

- if  $T \in L(X, X)$ , then any two matrix representations of  $T$  are similar.
- two similar matrices represent the same linear transformation  $T$ , relative to suitable bases.