HW5

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1 Question 1

Let X, Y be normed vector spaces and $T \in L(X, Y)$.

1.1 Let there exist $m > 0$ s.t. $m||x|| \leq ||T(x)||$. Prove T is one-to-one.

Let $x \in \ker T$. Then $\exists m > 0$ s.t. $m||x|| \leq T(x) = 0 \Rightarrow x = \bar{0}$. Thus, $\ker T = \{\bar{0}\}$ so T is one-to-one, and invertible.

1.2 Show that T^{-1} is continuous on $T(X)$.

Let $a, b \in \text{Im}(T)$. Since T is invertible, $\exists x, y \in X$ s.t. $T(x) = a, T(y) = b$. Then $\exists m$ s.t. $m||x - y|| \leq ||T(x - y)|| = ||T(x) - T(y)|| \Rightarrow m||T^{-1}(a) - T^{-1}(b)|| \leq ||a - b|| \Rightarrow ||T^{-1}(a) - T^{-1}(b)|| \leq \frac{1}{m}||a - b||$. Thus, T^{-1} is lipschitz and therefore is continuous on $T(X)$.

1.3 Let T^{-1} be continuous on $T(X)$. Show that $\exists m > 0$ s.t. $m||x|| \leq ||T(x)||$.

We have that T^{-1} is continuous on $T(X)$. Then, since T^{-1} is linear, T^{-1} is lipschitz. Let $a \in \text{Im}(T)$. As T is lipschitz, there exists $M > 0$ s.t. $||T^{-1}(a) - T^{-1}(\bar{0})|| \leq M||a - \bar{0}|| \Rightarrow ||T^{-1}(a - \bar{0})|| \leq M||a|| \Rightarrow ||T^{-1}(a)|| \leq M||a|| \Rightarrow ||x|| \leq M||T(x)|| \Rightarrow \frac{1}{M}||x|| \leq ||T(x)||$.

2 Question 2

Consider linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(x, y) = (x + 5y, 8x + 7y)$.

2.1 Calculate $||T||$ using the norm $||(x, y)||_1 = |x| + |y|$ in $\mathbb{R}^2$.

$||T||$ is the supremum of $||T(x)||$ s.t. $||x|| = 1$. In this case, the supremum will be the most efficient normalized vector in terms of maximizing $|x + 5y| + |8x + 7y|$. In this

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case, y is more efficient in increasing $|x+5y|+|8x+7y|$ than x, so $(0,1)$ will attain the supremum: $|5|+|7|=12$.

2.2 Calculate $||T||$ using the norm $||(x,y)||_\infty = \max\{|x|,|y|\}$ in $\mathbb{R}^2$.

$||T||$ is the supremum of $T(x)$ s.t. $||x||=1$. In this case, unlike the previous problem, we can set both x and y to be 1 without penalty, so $(1,1)$ achieves the supremum: $\max\{|1+5|,|8+7|\}=15$.

3 Question 3

Consider the standard basis in $\mathbb{R}^2$, W, and another orthonormal basis $V=\{(a_1,a_2),(b_1,b_2)\}=\{a,b\}$. Prove that the Euclidean norm of any vector $(x,y)\in\mathbb{R}^2$ is the same in W and V.

Let $x,y\in\mathbb{R}^2$. $\exists\alpha,\beta\in\mathbb{R}$ s.t. $x=\alpha a_1+\beta b_1, y=\alpha a_2+\beta b_2$. Then we can solve for α an β as follows:

$$\begin{aligned} x=\alpha a_1+\beta b_1 &\Rightarrow \alpha=\frac{x-\beta b_1}{a_1}\\ y=\alpha a_2+\beta b_2 &\Rightarrow \alpha=\frac{y-\beta b_2}{a_2}=\frac{x-\beta b_1}{a_1}\\ \Rightarrow \beta &= \frac{a_1y-a_2x}{a_1b_2-a_2b_1}, \alpha=\frac{b_2x-b_1y}{a_1b_2-a_2b_1}. \end{aligned}$$

Now, plugging into our euclidian norm, we have the following:

$$\begin{aligned} \sqrt{\alpha^2+\beta^2} &= \sqrt{\left(\frac{b_2x-b_1y}{a_1b_2-a_2b_1}\right)^2+\left(\frac{a_1y-a_2x}{a_1b_2-a_2b_1}\right)^2}\\ &= \sqrt{\frac{b_2^2x^2-2b_1yb_2x+b_1^2y^2+a_1^2y^2-2a_2xa_1y+a_2^2x^2}{(a_1b_2-a_2b_1)^2}}\\ &= \sqrt{\frac{(b_2^2+a_2^2)x^2-2(b_1b_2+a_1a_2)xy+(b_1^2+a_1^2)y^2}{(a_1b_2-a_2b_1)^2}} \end{aligned}$$

4 Question 4

We will use the approach described to solve the following equation (with boundary condition) for $y(t)\in\mathbb{R}^2$:

$$\frac{d}{dt}y(t)=\begin{pmatrix}1 & 1\\ 3 & -1\end{pmatrix}y_t, y(0)=\begin{pmatrix}1\\3\end{pmatrix}.$$

First we must find the eigenvalues of A. where A is the matrix in the differential equation. The eigenvalues satisfy $det(A-\lambda I)=0\Rightarrow(1-\lambda)(-1-\lambda)-3=0\Rightarrow -1+\lambda-\lambda+\lambda^2-3=0\Rightarrow\lambda^2-4=0\Rightarrow\lambda=2,-2$.

Next we will find our eigenvectors. $Ax = 2x \Rightarrow \begin{pmatrix} x_1 + x_2 - 2x_1 \\ 3x_1 - x_2 - 2x_2 \end{pmatrix} = \bar{0} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 2$. $Ax = -2x \Rightarrow \begin{pmatrix} x_1 + x_2 + 2x_1 \\ 3x_1 - x_2 + 2x_2 \end{pmatrix} = \bar{0} \Rightarrow \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -2$.

So, we have $P = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$, $D = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$. We can calculate $P^{-1} = \frac{1}{-3-1} \begin{pmatrix} -3 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & -1/4 \end{pmatrix}$.

Our solution is then the following.

$$\begin{aligned} y(t) = PDP^{-1}y(0) &= \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & -1/4 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} (3/2)e^{2t} \\ (-1/2)e^{-2t} \end{pmatrix} \\ &= \begin{pmatrix} (3/2)e^{2t} - (1/2)e^{-2t} \\ (3/2)e^{2t} + (3/2)e^{-2t} \end{pmatrix} \end{aligned}$$

5 Question 5

We will check the signs of our eigenvalues, and determine if our solution is stable.

Our eigenvalues were $2, -2$. Clearly $2 > 0$ so our solution is not stable. This is clear also from the form of $y(t)$. The e^{2t} term in the top and bottom of the vector goes to infinity.