HW3

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August 28, 2020

1 Question 1

Let (X, d) be a nonempty complete metric space. Define $T : X \to X$ such that $d(T(x), T(y)) < d(x, y)\ \forall x \neq y, x, y \in X$. I will prove by counterexample that it is not necessarily the case that T has a fixed point.

pf Consider $(\mathbb{R}^+, d)$ with $d(x, y) = |x - y|$. Define $T : \mathbb{R}^+ \to \mathbb{R}^+$ as $T(x) = \sqrt{x^2 + 1}$. Let $x, y \in \mathbb{R}^+$ such that $x \neq y$. Then $d(T(x), T(y)) = |\sqrt{x^2 + 1} - \sqrt{y^2 + 1}| < |\sqrt{x^2} - \sqrt{y^2}| = |x - y| = d(x, y)$.[1] Furthermore, $\mathbb{R}^+ \subset \mathbb{R}$ is complete as $\mathbb{R}^+$ is closed and $\mathbb{R}$ is complete. We will show that T has no fixed points.

Assume for the purpose of contradiction that T has a fixed point, $x^* \in \mathbb{R}^+$. Then $x^* = T(x^*) = \sqrt{(x^*)^2 + 1} > \sqrt{(x^*)^2} = x^*$, so $x^* > x^*$, which is a contradiction. So, T has no fixed points.

2 Question 2

We will show that the following countable set, A, is compact in $\mathbb{R}$: $A := \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$.

pf We can map each element of A to an element of $\mathbb{N}$. We can do so by mapping 0 to 1, and by mapping $\frac{1}{n}$ to $n + 1$ for $n \in \mathbb{N}, n > 1$. Thus A is countable.

Given any open cover of A, we know that $\exists$ an open set S of the cover which contains 0. As S is open, and since the sequence $\{\frac{1}{n}\}_{n \in \mathbb{N}} \to 0$, S contains an infinite number of elements of A. Formally, $\exists N \in \mathbb{N}$ s.t. $\forall n > N, \{\frac{1}{n}\} \subset S$. We then know that there can be at most N elements of A not contained in S, so we can form a finite open cover of A through a union of S with at most N more open sets of the cover, one containing a cover of $\frac{1}{1}$, one containing a cover of $\frac{1}{2}$, ..., one containing a cover of $\frac{1}{N}$. Thus, we can form a finite subcover of A, so A is compact.

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*I worked on this assignment with my study group: Alex von Hafften, Andrew Smith, Ryan Mather, and Tyler Welch. I have also discussed problem(s) with Emily Case, Sarah Bass, and Danny Edgel.

[1] $|\sqrt{x^2 + 1} - \sqrt{y^2 + 1}| = \frac{|x^2 + y^2|}{|\sqrt{x^2+1} + \sqrt{y^2+1}|} = |\sqrt{x^2} - \sqrt{y^2}| \frac{|\sqrt{x^2} + \sqrt{y^2}|}{|\sqrt{x^2+1} + \sqrt{y^2+1}|} \leq |\sqrt{x^2} - \sqrt{y^2}|$, with equality iff $x = y = 0$. However, $x \neq y$, so $|\sqrt{x^2 + 1} - \sqrt{y^2 + 1}| < |\sqrt{x^2} - \sqrt{y^2}|$.

3 Question 3

Prove that the function $f(x) = \cos^2(x)e^{5-x-x^2}$ has a maximum on $\mathbb{R}$.

pf Let $g(x) := \cos^2(x)$, $h(x) := e^{5-x-x^2}$. Thus $f(x) = g(x)h(x)\ \forall x \in \mathbb{R}$. cos and exp are continuous functions on $\mathbb{R}$ so $f(x), g(x), h(x)$ are all continuous on $\mathbb{R}$. Note that $0 \leq g(x) \leq 1\ \forall n \in \mathbb{N}$. Note also that $0 < h(x) \Rightarrow 0 \leq f(x)\ \forall x \in \mathbb{R}$. Furthermore, for $x \in (-\infty, -3) \cup (3, \infty)$, $h(x) < 1 \Rightarrow f(x) \leq 1$.[2] Notice that $f(0) = \cos^2(0)e^{5-0-0^2} = (1)(e^5) > 1$. Notice also that $[-3, 3]$ is compact in $\mathbb{R}$ by the Heine-Borel theorem, so by the extreme value theorem $\exists k \in [-3, 3]$ s.t. $f(k) \geq f(x)\ \forall x \in [-3, 3]$. Therefore, $f(k) \geq f(0) > 1 \geq f(x)\ \forall x \in (-\infty, -3) \cup (3, \infty)$, so $f(k) \geq f(x)\ \forall x \in \mathbb{R}$. Thus, $f(x)$ has a maximum on $\mathbb{R}$, at k.

4 Question 4

Suppose you have two maps of Wisconsin, one large and one small. We put the large one on top of the small one so that the small map is completely covered by the large one. Prove that a point on the small map is in the same location as it is on the large map.

pf Let the large (flat) map of Wisconsin $= W$ be a closed subset of $\mathbb{R}^2$. Since $\mathbb{R}^2$ is complete and W is closed, W is complete. We then have a strictly smaller map of Wisconsin, laid on top of the large map such that it is completely covered by the large map, and so we can define a mapping operator $T : W \to W$, described thusly:

Take an arbitrary $x \in W$. This is a point somewhere on the big map, and hence corresponds to some physical location in Wisconsin. There is a point on the small map of Wisconsin which corresponds to the same geographical location. The small map is covered by the large map, so this point on the small map is touching a point on the large map. The point on the large map that touches the point on the small map corresponding to the same location as x is $T(x)$.

Since by assumption the small map is smaller than the large map, $\exists \beta < 1$ such that $\forall x, y \in W$ we have $\beta d(x, y) \geq d(T(x), T(y))$.[3] Then T is a contraction of modulus β, so by the contraction mapping theorem T has a fixed point x^* where $x^* = T(x^*)$ so at x^* the big map and small map correspond to the same location in Wisconsin.

5 Question 5

Consider the set $X = \{-1, 0, 1\}$ and the space of all functions on X, $F_X = \{f : X \to \mathbb{R}\}$.

[2] For $x \in (-\infty, -3) \cup (3, \infty)$, $5 - x - x^2 < 0 \Rightarrow e^{5-x-x^2} = h(x) < 1$. Assume for the purpose of contradiction that $f(x^*) > 1$ for some $x^* \in (-\infty, -3) \cup (3, \infty)$. This implies $g(x^*)h(x^*) > 1$, so either $g(x^*) > 1$ or $h(x^*) > 1$. In either case we have a contradiction so $f(x) \leq 1\ \forall x \in (-\infty, -3) \cup (3, \infty)$.

[3] This is because the d(T(x),T(y)) will be the same as the distance on the small map between the points corresponding to the same locations x and y correspond to on the large map. The small map is shrunk by some $0 \leq \beta < 1$ relative to the big map, so $\beta d(x, y) \geq d(T(x), T(y))\ \forall x, y \in W$.

5.1 Show that F_X is a vector space.

- Let $a, b, c \in F_X$. $\forall x \in X$, $(a(x) + b(x)) + c(x) = a(x) + (b(x) + c(x))$.
- Let $a, b \in F_X$. $\forall x \in X$, $a(x) + b(x) = b(x) + a(x)$.
- Let $0(x) := 0 \; \forall x \in X$. $0(x) \in F_X$. Let $a \in F_X$. $\forall x \in X$, $a(x)+0(x) = 0(x)+a(x) = a(x)$.
- Let $a \in F_X$. Then $a^{'}(x) := -a(x) \; \forall x \in X$, and $a^{'} \in F_X$. $\forall x \in X$, $a(x)+a^{'}(x) = 0$.
- Let $a \in \mathbb{R}, b, c \in F_X$. $\forall x \in X$, $a(b(x) + c(x)) = ab(x) + ac(x)$.
- Let $a, b \in \mathbb{R}, c \in F_X$. $\forall x \in X$, $(a + b)c(x) = ac(x) + b(c(x))$.
- Let $a, b \in \mathbb{R}, c \in F_X$. $\forall x \in X$, $(ab)c(x) = a(bc(x))$.
- Let $a \in F_X$. $\forall x \in X$, $1a(x) = a(x)$.

From the above, F_X meets the definition of a vector space.

5.2 Show that the operator $T : F_X \to F_X$ defined by $T(f)(x) = f(x^2), x \in \{-1, 0, 1\}$ is linear.

Let $a, b \in \mathbb{R}, c, d \in F_X$. Then, $\forall x \in X$, $T(ac(x) + bd(x)) = ac(x^2) + bd(x^2) = aT(c(x)) + bT(d(x))$. Thus, T is a linear transformation.

5.3 Calculate ker T, Im T, and rank T.

- For $a \in F_X$ such that $T(a(x)) = 0 \; \forall x \in X \Rightarrow a(x^2) = 0 \; \forall x \in X$ so $a(0) = 0$ and $a(1) = 0$. Thus, $\ker T = \{f | X \to \mathbb{R}, f(0) = f(1) = 0\}$.
- Let $a, b \in \mathbb{R}$. $\exists f \in F_X$ s.t. $f(0) = a, f(1) = b$. We then have the following:

  $$T(f(x)) = f(x^2) = \begin{cases} a, & \text{if } x \in \{0\} \\ b, & \text{if } x \in \{-1, 1\} \end{cases},$$

  so the image of T is $\{f | f(1) = f(-1) = a, f(0) = b, a, b \in \mathbb{R}\}$.
- To find the rank of T, we will first find a basis. Define $f_0, f_1 \in F_X$ s.t. $f_0(-1) = 0, f_0(0) = 1, f_0(1) = 0, f_1(-1) = 1, f_1(0) = 0, f_1(1) = 1$.

  Let $f \in F_X$. Then, $\exists a, b \in \mathbb{R}$ s.t. $f(0) = a, f(1) = b$. Then, $T(f(0)) = a, T(f(1)) = T(f(-1)) = b$. Also, note that $af_0(0) + bf_1(0) = a = T(f(0)), af_0(1) + bf_1(1) = b = T(f(1)), af_0(-1) + bf_1(-1) = b = T(f(-1))$ so $\{f_0, f_1\}$ spans T.

  Let $a, b \in \mathbb{R}$ s.t. $af_0(x) + bf_1(x) = 0(x) \; \forall x \in \{-1, 0, 1\}$, with $0(x)$ defined as in 5.1. Then, $af_0(0) + bf_1(0) = a = 0$ and $af_0(1) + bf_1(1) = b = 0$ so $a = b = 0$. Thus, $\{f_0, f_1\}$ are independent and span T so they form a basis for T. Therefore, since the cardinality of $\{f_0, f_1\}$ is 2, rank $T = 2$.

6 Question 6

Consider the following system of linear equations:

$$\begin{cases} 0 = & x_1 + x_2 + 2x_3 + x_4, \\ 0 = & 3x_1 - x_2 + x_3 - x_4, \\ 0 = & 5x_1 - 3x_2 - 3x_4. \end{cases} \tag{1}$$

Let X be the set of $\{x_1, x_2, x_3, x_4\}$ which satisfy (1).

6.1 Show that X is a vector space.

Adding the first two equations together we find $4x_1 + 3x_3 = 0 \Rightarrow x_3 = -\frac{4}{3}x_1$. We then simplify the third equation as $0 = 5x_1 - 3(x_2 + x_4) \Rightarrow x_1 = \frac{3}{5}(x_2 + x_4)$. Substituting this back into the equation for x_3, we find $x_3 = -\frac{4}{3}\frac{3}{5}(x_2 + x_4) = -\frac{4}{5}(x_2 + x_4)$. So, for any x_1, x_2, x_3, x_4 satisfying (1), $\exists a, b \in \mathbb{R}$, with $a = x_2$ and $b = x_4$, such that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{3}{5}(a+b) \\ a \\ -\frac{4}{5}(a+b) \\ b \end{pmatrix} = \begin{pmatrix} \frac{3}{5}a \\ a \\ -\frac{4}{5}a \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{3}{5}b \\ 0 \\ -\frac{4}{5}b \\ b \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ 1 \\ -\frac{4}{5} \\ 0 \end{pmatrix} a + \begin{pmatrix} \frac{3}{5} \\ 0 \\ -\frac{4}{5} \\ 1 \end{pmatrix} b = \vec{u}a + \vec{v}b.$$

For $c, d \in X$, $\exists m, n, o, p \in \mathbb{R}$ such that $c = m\vec{u} + n\vec{v}, d = o\vec{u} + p\vec{v}$. Then, $c + d = m\vec{u} + n\vec{v} + o\vec{u} + p\vec{v} = (m + o)\vec{u} + (n + p)\vec{v} \in X$. Additionally, for $c \in X$ $\exists m, n \in \mathbb{R}$ such that $c = m\vec{u} + n\vec{v}$. For $\alpha \in \mathbb{R}$, $\alpha c = \alpha(m\vec{u} + n\vec{v}) = (\alpha m)\vec{u} + (\alpha n)\vec{v} \in X$. Thus, properties 1,2,5,6,7 of the definition of a vector space are trivially satisfied for X by the properties of $\mathbb{R}^4$.

Furthermore, note that $\vec{0} = 0\vec{u} + 0\vec{v} \in X$. For $c \in X$, $\exists a, b \in \mathbb{R}$ such that $c = a\vec{u} + b\vec{v}$ and $c + \vec{0} = (a + 0)\vec{u} + (b + 0)\vec{v} = (0 + a)\vec{u} + (0 + b)\vec{v} = c + \vec{0} = a\vec{u} + b\vec{v} = c$. This satisfies property 3 of the definition of a vector space.

For $c \in X$ $\exists m, n \in \mathbb{R}$ s.t. $c = m\vec{u} + n\vec{v}$, then $1 * c = 1 * (m\vec{u} + n\vec{v}) = 1 * m\vec{u} + 1 * n\vec{v} = m\vec{u} + n\vec{v} = c$, and $\exists d \in X$ such that $d = (-m)\vec{u} + (-n)\vec{v}$, and $c + d = m\vec{u} + n\vec{v} + (-m)\vec{u} + (-n)\vec{v} = (m - m)\vec{u} + (n - n)\vec{v} = 0\vec{u} + 0\vec{v} = \vec{0}$. So, properties 4 and 8 of the definition of a vector space are satisfied.

As X satisfies all properties $1 - 8$ of the definition of a vector space, X is a vector space.

6.2 Calculate dim X.

From the above, any $x \in X$ can be written as a linear combination of $\vec{u}, \vec{v}$. Thus, $\vec{u}, \vec{v}$ span X. We will now check $\vec{u}, \vec{v}$ for linear dependence.

Let $a, b \in \mathbb{R}$ such that $a\vec{u} + b\vec{v} = \vec{0}$. Then,

$$\begin{pmatrix} \frac{3}{5} \\ 1 \\ -\frac{4}{5} \\ 0 \end{pmatrix} a + \begin{pmatrix} \frac{3}{5} \\ 0 \\ -\frac{4}{5} \\ 1 \end{pmatrix} b = \vec{0} \Rightarrow \begin{pmatrix} \frac{3}{5}(a+b) \\ a \\ -\frac{4}{5}(a+b) \\ b \end{pmatrix} = \vec{0} \Rightarrow a = b = 0.$$

So, $\vec{u}, \vec{v}$ are independent and span X. Thus, $\{\vec{u}, \vec{v}\}$ is a basis for X. Since $\{\vec{u}, \vec{v}\}$ is a basis for X, the cardinality of X is 2 so dim $X = 2$.