HW5

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1 Question 1

Let X, Y be normed vector spaces and $T \in L(X, Y)$.

1.1 Let there exist $m > 0$ s.t. $m||x|| \leq ||T(x)||$. Prove T is one-to-one.

Let $x \in \ker T$. Then $\exists m > 0$ s.t. $m||x|| \leq T(x) = 0 \Rightarrow x = \bar{0}$. Thus, $\ker T = \{\bar{0}\}$ so T is one-to-one, and invertible.

1.2 Show that T^{-1} is continuous on $T(X)$.

Let $a, b \in \text{Im}(T)$. Since T is invertible, $\exists x, y \in X$ s.t. $T(x) = a, T(y) = b$. Then $\exists m$ s.t. $m||x - y|| \leq ||T(x - y)|| = ||T(x) - T(y)|| \Rightarrow m||T^{-1}(a) - T^{-1}(b)|| \leq ||a - b|| \Rightarrow ||T^{-1}(a) - T^{-1}(b)|| \leq \frac{1}{m}||a - b||$. Thus, T^{-1} is lipschitz and therefore is continuous on $T(X)$.

1.3 Let T^{-1} be continuous on $T(X)$. Show that $\exists m > 0$ s.t. $m||x|| \leq ||T(x)||$.

We have that T^{-1} is continuous on $T(X)$. Then, since T^{-1} is linear, T^{-1} is lipschitz. Let $a \in \text{Im}(T)$. As T is lipschitz, there exists $M > 0$ s.t. $||T^{-1}(a) - T^{-1}(\bar{0})|| \leq M||a - \bar{0}|| \Rightarrow ||T^{-1}(a - \bar{0})|| \leq M||a|| \Rightarrow ||T^{-1}(a)|| \leq M||a|| \Rightarrow ||x|| \leq M||T(x)|| \Rightarrow \frac{1}{M}||x|| \leq ||T(x)||$.

2 Question 2

Consider linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(x, y) = (x + 5y, 8x + 7y)$. Since dim $T = 2$, T is bounded. Therefore, $\|T\| = \sup_{\|x\|_x = 1} \|T(x)\|$.

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2.1 Calculate $||T||$ using the norm $||(x,y)||_1 = |x| + |y|$ in $\mathbb{R}^2$.

$||T||$ is the supremum of $||T(x)||$ s.t. $||x|| = 1$. In this case, the supremum will be the most efficient normalized vector in terms of maximizing $|x + 5y| + |8x + 7y|$. In this case, y is more efficient in increasing $|x + 5y| + |8x + 7y|$ than x, so $(0, 1)$ will attain the supremum: $|5| + |7| = 12$.

2.2 Calculate $||T||$ using the norm $||(x,y)||_\infty = \max\{|x|, |y|\}$ in $\mathbb{R}^2$.

$||T||$ is the supremum of $T(x)$ s.t. $||x|| = 1$. In this case, unlike the previous problem, we can set both x and y to be 1 without penalty, but cannot set either higher than 1. Furthermore, T is strictly increasing in both x, y so $(1, 1)$ achieves the supremum: $\max\{|1 + 5|, |8 + 7|\} = 15$.

3 Question 3

Consider the standard basis in $\mathbb{R}^2$, W, and another orthonormal basis $V = \{(a_1, a_2), (b_1, b_2)\} = \{a, b\}$. Prove that the Euclidean norm of any vector $(x, y) \in \mathbb{R}^2$ is the same in W and V.

Let $(x,y)' \in \mathbb{R}^2$. $\exists v \in \mathbb{R}^2$ such that v is the representation of $(x, y)'$ in the basis of V. Since V is orthonormal and W is the standard basis in $\mathbb{R}^2$, $M := \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ is a mapping from V to W, and since V is orthonormal then M is an orthonormal matrix, so $M'M = I$. Then, Mv is v represented in the vector space W, so $Mv = (x,y)'$. Next, note that $\left\|(x,y)'\right\| = \sqrt{x^2 + y^2} = \sqrt{(x,y)(x,y)'} = \sqrt{(Mv)'(Mv)} = \sqrt{v'M'Mv} = \sqrt{v'Iv} = \sqrt{v'v} = \|v\|$, so the euclidean norm is the same for any standard basis.

4 Question 4

We will use the approach described to solve the following equation (with boundary condition) for $y(t) \in \mathbb{R}^2$:

$$\frac{d}{dt}y(t) = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} y_t, y(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

First we must find the eigenvalues of A. where A is the matrix in the differential equation. The eigenvalues satisfy $det(A - \lambda I) = 0 \Rightarrow (1 - \lambda)(-1 - \lambda) - 3 = 0 \Rightarrow -1 + \lambda - \lambda + \lambda^2 - 3 = 0 \Rightarrow \lambda^2 - 4 = 0 \Rightarrow \lambda = 2, -2$.

Next we will find our eigenvectors. $Ax = 2x \Rightarrow \begin{pmatrix} x_1 + x_2 - 2x_1 \\ 3x_1 - x_2 - 2x_2 \end{pmatrix} = \bar{0} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 2$. $Ax = -2x \Rightarrow \begin{pmatrix} x_1 + x_2 + 2x_1 \\ 3x_1 - x_2 + 2x_2 \end{pmatrix} = \bar{0} \Rightarrow \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -2$.

So, we have $P = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$, $D = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$. We can calculate $P^{-1} = \frac{1}{-3-1} \begin{pmatrix} -3 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & -1/4 \end{pmatrix}$.

Our solution is then the following.

$$
\begin{aligned}
y(t) = PDP^{-1}y(0) &= \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & -1/4 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} (3/2)e^{2t} \\ (-1/2)e^{-2t} \end{pmatrix} \\
&= \begin{pmatrix} (3/2)e^{2t} - (1/2)e^{-2t} \\ (3/2)e^{2t} + (3/2)e^{-2t} \end{pmatrix}
\end{aligned}
$$

5 Question 5

We will check the signs of our eigenvalues, and determine if our solution is stable.

Our eigenvalues were $2, -2$. Clearly $2 > 0$ so our solution is not stable. This is clear also from the form of $y(t)$. The e^{2t} term in the top and bottom of the vector goes to infinity.