HW2

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1 Question 1

Consider the set $A = \{\frac{1}{n}\}_{n \in \mathbb{N}} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. I will prove that there does not exist $S \subset \mathbb{R}$ s.t. the set of S's limit points is A.

pf Assume for the purpose of contradiction the existence of $S \subset \mathbb{R}$, where the set of S's limit points are equal to A. We will show that 0 is a limit point. Let $\epsilon > 0$ be arbitrary. Let n be the smallest $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then, since $\frac{1}{2n} \in A$, $\frac{1}{2n}$ is a limit point of S so $\exists s \in S$ such that s is in a neighborhood of size $\frac{1}{4n}$ around $\frac{1}{2n}$. Then, $|0 - s| \leq |0 - (\frac{1}{2n} + \frac{1}{4n})| = \frac{1}{2n} + \frac{1}{4n} < \frac{1}{n} < \epsilon$. Additionally, we know that $|0 - s| \geq |0 - (\frac{1}{2n} - \frac{1}{4n})| = \frac{1}{4n} > 0$ so $s \neq 0$. Thus, every neighborhood of 0 contains a nonzero element of S, so 0 is a limit point of S, but $0 \notin A$, a contradiction.

2 Question 2

Prove that $f(x) = \cos x^2$ is not uniformly continuous on $\mathbb{R}$.

pf Let $\epsilon = 1$ and for the purpose of contradiction assume $f(x)$ is uniformly continuous on $\mathbb{R}$. Then $\exists \delta > 0$ s.t. $\forall x, y \in \mathbb{R}$ with $|x - y| < \delta$, $|\cos x^2 - \cos y^2| < 1$. Let $x_k = \sqrt{2\pi k}, y_k = \sqrt{2\pi k + \pi}$. Then, $\lim_{k \to \infty} |x_k - y_k| = 0$.[1] So, for any δ, we can find $\tilde{k}$ s.t. $|x_{\tilde{k}} - y_{\tilde{k}}| < \delta$, but for any k, $|\cos x_k^2 - \cos y_k^2| = |\cos(2\pi k) - \cos(2\pi k + \pi)| = |1 - (-1)| = |2| = 2 > 1$. This is a contradiction, so $f(x)$ is not uniformly continuous on $\mathbb{R}$.

3 Question 3

Let $f : \mathbb{R} \to \mathbb{R}_{++}$ be continuous on an interval $[a, b]$. We will prove that $(\frac{1}{f})$ is bounded on $[a, b]$.

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[1]$|x_k - y_k| = |\sqrt{2\pi k} - \sqrt{2\pi k + \pi}| = \frac{|2\pi k - (2\pi k + \pi)|}{|\sqrt{2\pi k} + \sqrt{2\pi k + \pi}|} = \frac{\pi}{|\sqrt{2\pi k} + \sqrt{2\pi k + \pi}|} \to 0$.

To prove the limit, take $\epsilon > 0$ and define N such that N is the smallest $n \in \mathbb{N}$ such that $n \geq \frac{\left(\frac{\pi}{2\epsilon}\right)^2}{2\pi}$. Then, for all $n \in \mathbb{N}$ with $n > N$, $\frac{\pi}{|\sqrt{2\pi n} + \sqrt{2\pi n + \pi}|} < \frac{\pi}{|2\sqrt{2\pi n}|} < \frac{\pi}{2\sqrt{2\pi N}} \leq \epsilon$.

pf As f is continuous, by the extreme value theorem $\exists \hat{x}, \tilde{x} \in [a, b]$ s.t. $f(\hat{x}) \geq f(x)$ $\forall x \in [a, b]$ and $f(\tilde{x}) \leq f(x)$ $\forall x \in [a, b]$. Since $f(\hat{x}), f(\tilde{x}) > 0$, $\exists m, M \in \mathbb{R}_{++}$ s.t. $m < f(\tilde{x})$ and $M > f(\hat{x})$. Assume for the purpose of contradiction that $(\frac{1}{f})$ is not bounded by $\frac{1}{m}$ from above. Then, $\exists c \in [a, b]$ s.t. $(\frac{1}{f})(c) > \frac{1}{m} \Rightarrow f(c) < m$ which is a contradiction. Next, assume for the purpose of contradiction that $(\frac{1}{f})$ is not bounded by $\frac{1}{M}$ from below. Then $\exists d \in [a, b]$ s.t. $(\frac{1}{f})(d) < \frac{1}{M} \Rightarrow f(d) > M$, a contradiction. Thus, $(\frac{1}{f})$ is bounded above by $\frac{1}{m}$ and below by $\frac{1}{M}$ on $[a, b]$. Thus, $(\frac{1}{f})$ is bounded on $[a, b]$.

4 Question 4

Let $f : [a, b] \to \mathbb{R}$ be a continuous function, $a < b$, $a, b \in \mathbb{R}$. Assume that $f(a) < 0 < f(b)$. We follow the construction of sequences $\{l_n\}$ and $\{u_n\}$ as described in the problem set.

4.1 Show that sequences $\{l_n\}$ and $\{u_n\}$ both converge.

pf For arbitrary $n^* \in \mathbb{N}$, $l_{n^*} \leq \frac{l_{n^*} + u_{n^*}}{2} \leq u_{n^*}$ [2] so $l_n \leq l_{n+1} \leq u_n$ $\forall n \in \mathbb{N}$ and $u_n \geq u_{n+1} \geq l_n$ $\forall n \in \mathbb{N}$. Thus, for any $n \in \mathbb{N}$, $l_1 \leq l_n \leq u_n \leq u_1$. Therefore, $\{l_n\}$ is monotone increasing and bounded above by u_1 and below by l_1, and $\{u_n\}$ is monotone decreasing and bounded below by l_1 and above by u_1. By the monotone convergence theorem, both $\{l_n\}$ and $\{u_n\}$ converge.

4.2 Both sequences converge to the same limit

pf We now define a new sequence $\{a_n\}$ with $a_n = u_n - l_n$ $\forall n \in \mathbb{N}$. During each n we are stepping midway in between u_n and l_n, or we are setting $u_n = l_n$, so $a_n \leq \frac{b-a}{2^{n-1}}$.[3] $\frac{b-a}{2^{n-1}} \to 0$[4] and $\frac{b-a}{2^{n-1}} \geq a_n \geq 0$ $\forall n \in \mathbb{N}$ so $a_n \to 0$ by the squeeze theorem. Thus, $\{u_n - l_n\} = \{a_n\} \to 0$, and it follows that $\lim_{n \to \infty} u_n = \lim_{n \to \infty} l_n$.

4.3 Define the common limit of two sequences c and show that $f(c) = 0$.

pf Since $\{u_n\}, \{l_n\}$ converge, define $u, l \in \mathbb{R}$ s.t. $u_n \to u$ and $l_n \to l$. Since by construction $f(u_n) \geq 0 \geq f(l_n)$ $\forall n \in \mathbb{N}$, since f is continuous and limits preserve weak inequalities $f(u) \geq 0 \geq f(l)$. Since we also know that $a_n \to 0$, $(u_n - l_n) \to 0$ so $u - l = 0 \Rightarrow u = l$. Thus, $f(u) \geq 0 \geq f(l) = f(u) \Rightarrow f(u) = f(l) = 0$.

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[2] This is because $u_n \geq l_n$ $\forall n \in \mathbb{N}$ which I will prove here by induction.
$u_1 = b \geq a = l_1$.
Assume $u_k \geq l_k$ for $k \in \mathbb{N}$. Then $u_k \geq \frac{u_k + l_k}{2} \geq l_k$ so $u_{k+1} \geq l_{k+1}$.

[3] I will prove $a_n = \frac{b-a}{2^{n-1}}$ or $a_n = 0$ formally here by induction. $a_1 = u_1 - l_1 = b - a \leq \frac{b-a}{2^{1-1}}$.
Assume $a_k = \frac{b-a}{2^{k-1}}$. Then, $a_{k+1} = u_{k+1} - l_{k+1} = u_k - l_k - \frac{u_k - l_k}{2} = \frac{u_k - l_k}{2} = \frac{a_k}{2} = \frac{b-a}{2*2^{k-1}} = \frac{b-a}{2^{(k+1)-1}}$, or $a_{k+1} = 0$. Now assume $a_k = 0$. Then, $a_{k+1} = 0$. Thus, $a_n \leq \frac{b-a}{2^n}$ $\forall n \in \mathbb{N}$.

[4] To prove $\lim_{n \to \infty} \frac{b-a}{2^{n-1}} = 0$, let $\epsilon > 0$. We can choose N to be the smallest $n \in \mathbb{N}$ such that $n \geq \log_2 \left(\frac{b-a}{\epsilon} \right) + 1$ and for $n > N$, $|\frac{b-a}{2^{n-1}} - 0| < |\frac{b-a}{2^N}| \leq \epsilon$.

5   Question 5

Prove that at any time there are two antipodal points on Earth that share the same temperature.

I will take as given that temperature is a continuous, natural phenomenon.

pf Take any great circle of the Earth, C. Starting at any point $c \in C$ we can define $t(x)$ as the temperature on the Earth at point $\tilde{x} \in C$, where x is the angle in radians between $\tilde{x}$ and c. We then define $T(x) = t(x) - t(x + \pi)$. Since $t(x)$ is continuous, $T(x)$ is continuous.[5]

We then have 3 possible cases:

1. $T(0) = 0$. Then $0 = t(0) - t(\pi) \Rightarrow t(0) = t(\pi)$.

2. $T(0) < 0$. Then $0 > t(0) - t(\pi) \Rightarrow 0 < t(\pi) - t(0) \Rightarrow T(\pi) > 0$.

3. $T(0) > 0$. Then $0 < t(0) - t(\pi) \Rightarrow 0 > t(\pi) - t(0) \Rightarrow T(\pi) < 0$.

Clearly in case (1) point c has an antipodal point of the same temperature. In cases (2) and (3), since $T(x)$ is continuous on $[0, \pi]$, by the intermediate value theorem $\exists k \in (0, \pi)$ s.t. $T(k) = 0 \Rightarrow t(k) - t(k + \pi) = 0 \Rightarrow t(k) = t(k + \pi)$. Thus, there exists some point $\tilde{k} \in C$, k radians from c, which has an antipodal point of the same temperature.

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[5]Here I will prove that the difference of two continuous functions is continuous.
Let $f(x), g(x) : \mathbb{R} \to \mathbb{R}$, and let $h(x) := f(x) - g(x)$. Let $x_0 \in \mathbb{R}$ and $\epsilon > 0$ be arbitrary. Since f, g are continuous, $\exists \delta_f, delta_g$ s.t. $\forall x \in \mathbb{R}$ where $|x - x_0| < \delta_f$ then $|f(x) - f(x_0)| < \frac{\epsilon}{2}$ and s.t. $\forall x \in \mathbb{R}$ where $|x - x_0 < \delta_g|$ then $|g(x) - g(x_0)| < \frac{\epsilon}{2}$. Define $d := \min\{d_f, d_g\}$. Then for $|x - x_0| < d$, $|h_x - h_{x_0}| = |f_x - g_x - f_{x_0} + g_{x_0}| \le |f(x) - f(x_0)| + |g(x) - g(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Thus, $h(x)$ is continuous.