Economics 703 : Anwers to the Mid-Term Exam

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1. Let $f : A \subset R^n \to R$ be a differentiable function, where A is convex. Suppose that $\| Df(x) \| \leq M$ for all $x \in A$.

   (a) Prove that $| f(x) - f(y) | \leq M \| x - y \|$, for all $x, y \in A$.
   The proof of this result is essentially an application of the mean value theorem. In the case where $n = 1$ this follows immediately from the MVT, for then we have $f(y) - f(x) = f'(z)(y - x)$ for some $z \in (x, y)$. Thus $|f(x) - f(y)| \leq |f'(z)||x - y| \leq M|x - y|$. For the general case, we can use the one-dimensional idea on the line segment connecting x to y, as is shown below.
   Proof. Let x and y belong to A, and define the line segment connecting x and y by $z(t) = (1 - t)x + ty$, where $t \in [0, 1]$. Because A is convex, $z(t) \in A$ for every $t \in [0, 1]$. Let $h : [0, 1] \to R$ be given by $h(t) = f(z(t))$; note that $h(0) = f(x)$ and $h(1) = f(y)$. By the mean value theorem, there exists $r \in (0, 1)$ such that $h(1) - h(0) = (1 - 0)h'(r) = h'(r)$. Using the chain rule, we find that $h'(r) = Df(z(r))(y - x)$. Hence $f(y) - f(x) = Df(z(r))(y - x)$ for some $r \in (0, 1)$. We may now compute

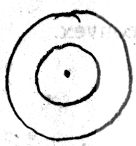
   $$|f(y) - f(x)| = |Df(z(r))(y - x)| \leq \|Df(z(r))\|\|y - x\| \leq M\|y - x\|$$

   ■

   (b) Does the above formula hold when A is not convex? Prove your claim.
   When A is not convex, the inequality need not hold. The simplest counterexample occurs when $n = 1$. Let $A = [0, 1] \cup [2, 3]$, and let $f(x) = 1$ for all $x \in [0, 1]$, and $f(x) = 4$ for all $x \in [2, 3]$. Then $f'(x) = 1$ for all $x \in A$, but letting $x = 5/2$ and $y = 1/2$ we see that $f(x) - f(y) = 4 - 1 = 3 > 2 = (x - y)$.
   Of course, when $n = 1$, connectedness is equivalent to convexity, while for $n > 1$ connected sets need not be convex. One might therefore be inclined to conjecture that the inequality still holds for connected sets, regardless of the value of n.
   This conjecture receives some support from the following argument. Since A is connected, suppose that there exists a differentiable path

$z : [0,1] \to A$ such that $z(0) = y$ and $z(1) = x$. Then by the mean value theorem applied to the composite function $f(z(t))$ we have $f(x) - f(y) = f(z(1)) - f(z(0)) = f'(z(t))z'(t)$ for some $t \in (0,1)$. We conclude that $\mid f(x) - f(y) \mid \leq M \max_{t \in [0,1]} |z'(t)|$. Thus, an inequality analogous to the MVT inequality still holds.

However, it is easy to construct examples where the path from x to y must traverse much more length than a direct line segment connecting these two points, causing $|z'(t)|$ to exceed $\| x - y \|$. One such example follows.



Let $A = \{(x,y) \in R^2 | x \geq 0 \ and \ 1 \leq x^2 + y^2 \leq 4\}$. Let $f(x,y) = \theta$, where $x = r\cos\theta$ and $y = r\sin\theta$. Thus f is linear along any semicircle of length r, and constant along any ray through the origin intersecting A. Expressing $\theta = \theta(x,y)$ as a function of x and y, we may compute the following partial derivatives:

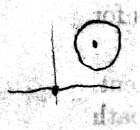
$$\theta_x = -\frac{y}{x^2+y^2}, \ \theta_y = \frac{x}{x^2+y^2}$$

Thus we have $f_x(x,y) = g'(\theta(x,y))\theta_x(x,y)$ and $f_y(x,y) = g'(\theta(x,y))\theta_y(x,y)$. This implies that $\|Df(x,y)\|^2 = g'(\theta)^2(\theta_x^2 + \theta_y^2) = g'(\theta)^2/(x^2+y^2) \leq g'(\theta)^2 = 1$. Now let $b = (0, -3/2)$ and $a = (0, 3/2)$. If the formula held, we would have $|f(a) - f(b)| \leq \|a - b\| = 3$. But using the formula $f(x,y) = g(\theta)$ we can directly compute that $f(b) = -\pi$ and $f(a) = \pi$, implying that $|f(a) - f(b)| = 2\pi > 3$. We conclude that the formula is not valid hold on connected but non-convex domains.

2. Let $A \subset R^n$ and $x \in R^n$. Define $d(x,A) = \inf\{d(x,y) \mid y \in A\}$, where $d(x,y)$ is the Euclidean distance between x and y. Must there exist a $z \in A$ such that $d(x,A) = d(x,z)$? (If the answer is in the affirmative, provide a proof. If the answer is not in the affirmative, then delineate the different ways in which the statement can fail to hold).

   Suppose A is non-empty, so that the infimum is well defined, and suppose also that A is is compact. Define $f : A \to R_+$ by $f(y) = d(x,y)$. Observe that since the norm is a continuous function, the function f is continuous on A. It follows by the Weierstrass theorem that f attains a minimum on A, i.e. that there exists a $z \in A$ such that $d(x,z) = \inf\{d(x,y) \mid y \in A\}$.

   The only way the result can thus fail is if A is not compact. Since $A \subset R^n$ this means A must either not be closed or not be bounded. However boundedness of A does not present a problem, for if y is any point in A, then $0 \leq d(x,A) \leq d(x,y)$, so that the infimum is well defined as a real number. Here is an example in which A is not closed, and the result fails. Let $i = (1,1,...,1)$ be the following point in R^n, let $A = \{z \in R^n | d(z,i) < 1/2\}$, and let $x = 0$. Then $d(x,A) = 1/2$, but there is no point in A for which $d(x,y) = d(x,A)$.



3. Let $\{x_n\}$ be a sequence in R^n such that $x_n \to x$.

(a) Show that $x \in cl\{x_1, x_2, ...\}$.
Suppose to the contrary that x belongs to the complement of $cl\{x_1, x_2, ...\}$. Since $cl\{x_1, x_2, ...\}$ is closed, its complement is open in R^n. Hence there exists $r > 0$ such that $B(x, r) \cap cl\{x_1, x_2, ...\} = \phi$. This implies $d(x, x_n) > r > 0$ for all n, contradicting that x_n converges to x.

(b) Is x a limit point of $\{x_1, x_2, ...\}$?
If $x \neq x_n$ for any n, then the only way x can belong to $cl\{x_1, x_2, ...\}$ is for x to be a limit point of the set $\{x_1, x_2, ...\}$. However, it is possible for a sequence to converge to a limit, without the limit being a limit point of the sequence. Indeed, consider the sequence in R given by $x_n = 1$ for every n. Then $\lim_{n \to \infty} x_n = 1 = x$, but x is not a limit point of the set $\{1\}$.

4. Let $A \subset R^n$ be connected, and let $f : A \to R$ be continuous with $f(x) \neq 0$ for all $x \in A$. Prove or disprove the following claim : $f(x)f(y) > 0$ for all $x, y \in A$.

Since f is continuous, and A is connected, its image $f(A)$ is a connected set in R. If the claim did not hold, then there would exist x and y in A such that $f(x) > 0$ and $f(y) < 0$. Since connected sets in R are convex, this would imply $0 \in f(A)$, i.e. there exists a point $z \in A$ such that $f(z) = 0$, contradicting the hypothesis that $f(x) \neq 0$ for all $x \in A$.

5. Determine the second order Taylor formula of the function $f : R^2 \to R$ given by the rule $f(x, y) = (x + y)^2$ around the point $(x_0, y_0) = (0, 0)$.

A second order Taylor series expansion of f is the best second order polynomial in x and y approximating f. Now the function f itself is a second order polynomial in x and y, so the Taylor series expansion of f must coincide with f itself.

Formally, we can see this as follows. By Taylor's formula, $f(x, y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) + \frac{1}{2}(x, y)' \begin{bmatrix} f_{xx}(00) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{bmatrix} (x, y) + o(\|(x, y)\|^2)$

Since $f(x, y) = x^2 + 2xy + y^2$, we have $f(0,0) = 0$, $f_x(0,0) = 0$, $f_y(0,0) = 0$, $f_{xx}(0,0) = 2$, $f_{xy}(0,0) = 2$ and $f_{yy}(0,0) = 2$. Thus $f(x, y) = x^2 + 2xy + y^2$, and the remainder term is identical to zero.