HW4

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1 Question 1

Let X, Y be two vector spaces such that dim $X = n$, dim $Y = m$. Then let $B = \{x_1, \dots, x_n\}$ be a basis for X and let $C = \{y_1, \dots, y_m\}$ be a basis for Y. For notational convenience define $A = \{1, \dots, n\} \times \{1, \dots, m\}$. For $(p, q) \in A$ consider the following linear transformation, $\mathcal{M}_{p,q} : X \to Y$, defined such that

$$\text{mtx}_{X,Y}(\mathcal{M}_{p,q}) = \begin{pmatrix} a_{1,1} & \dots & a_{1,q} & \dots & a_{1,m} \\ \dots & \dots & \dots & \dots & \dots \\ a_{p,1} & \dots & a_{p,q} & \dots & a_{p,m} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n,1} & \dots & a_{n,q} & \dots & a_{n,m} \end{pmatrix}$$

where $a_{i,j} = 1$ for $(i, j) = (p, q)$, and $a_{i,j} = 0$ for $(i, j) \neq (p, q)$. We will show that $\{\mathcal{M}_{p,q}\}_{(p,q)\in\mathbb{R}^2}$ is a basis of $L(X, Y)$.

<u>pf</u> Let $l \in L(X, Y)$. Then, l is a linear transformation from X to Y. Let $x \in X$ be arbitrary and define $y \in Y$ such that $l(x) = y$. Since B and C are bases for X and Y, we can find $\text{mtx}_{X,Y}(l) = \begin{pmatrix} b_{1,1} & \dots & b_{1,m} \\ \dots & \dots & \dots \\ b_{n,1} & \dots & b_{n,m} \end{pmatrix}$ and $\text{mtx}_{X,Y}(l)x = y$. Notice also that $\left(\sum_{(i,j)\in A} b_{i,j}\text{mtx}_{X,Y}(\mathcal{M}_{i,j})\right) x = \text{mtx}_{X,Y}(l)x = y \Rightarrow \sum_{(i,j)\in A} (b_{i,j}\mathcal{M}_{i,j}x) = y \Rightarrow \sum_{(i,j)\in A} (b_{i,j}\mathcal{M}_{i,j}(x)) = y$ so $\{\mathcal{M}_{p,q}\}_{(p,q)\in\mathbb{R}^2}$ spans $L(X, Y)$.

We will now show that $\{\mathcal{M}_{p,q}\}_{(p,q)\in\mathbb{R}^2}$ is independent. Let $l \in L(X, Y)$ such that $l(x) = \bar{0}\ \forall x \in X$. Then, $\text{mtx}_{X,Y}(l) = \begin{pmatrix} c_{1,1} & \dots & c_{1,m} \\ \dots & \dots & \dots \\ c_{n,1} & \dots & c_{n,m} \end{pmatrix}$ where $c_{i,j} = 0\ \forall (x, y) \in A$. Then, the corresponding $\sum_{(i,j)\in\mathbb{R}^2} c_{i,j}\mathcal{M}_{p,q} = \sum_{(i,j)\in\mathbb{R}^2} 0\mathcal{M}_{p,q}$ so $\{\mathcal{M}_{p,q}\}_{(p,q)\in\mathbb{R}^2}$ is independent.

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2 Question 2

Let $T \in L(X, X)$ and λ is T's eigenvalue. Let $A = \text{mtx}_X(T)$.

2.1 Prove that λ^k is an eigenvalue of T^k, $k \in \mathbb{N}$

pf We have that $Ax = \lambda x$ for some $x \in X$. Also note that $\text{mtx}_X(T^k) = A^k$. If $\lambda = 0$ then $Ax = 0x \Rightarrow A^k x = A^{k-1}0x = 0x = 0^k x = \lambda^k$ so λ^k is an eigenvalue of T^k. Now assume λ is nonzero. Then $Ax = \lambda x \Rightarrow \lambda^{-1}Ax = x \Rightarrow \lambda^{-1}A \ldots \lambda^{-1}Ax = x \Rightarrow (\lambda^{-1})^k A^k x = x \Rightarrow A^k x = \lambda^k x$ so λ^k is an eigenvalue of T^k.

2.2 Prove that if T is invertible, then λ^{-1} is an eigenvalue of T^{-1}.

pf We have that $Ax = \lambda x$ for some $x \in X$. Then, since T is invertible, A is invertible and $\text{mtx}_X(T^{-1}) = A^{-1}$. Now assume $\lambda = 0$. Then $Ax = 0x = \bar{0} \Rightarrow A^{-1}A^{-1}Ax = A^{-1}A^{-1}\bar{0} = \bar{0} = 0x \Rightarrow A^{-1}x = 0x = 0^{-1}x = \lambda^{-1}x$ so λ^{-1} is an eigenvalue of T^{-1}.

Next, assume $\lambda \neq 0$. Then, $x = \lambda A^{-1}x \Rightarrow \lambda^{-1}x = A^{-1}x$ so λ^{-1} is an eigenvalue of T^{-1}.

2.3 Define an operator $S : X \to X$ such that $S(x) = T(x) - \lambda x \ \forall x \in X$. Is S linear? Prove that ker $S := \{x \in X | S(x) = \bar{0}\}$ is a vector space.

pf Let $a, b \in \mathbb{R}, x, y \in X$. $S(ax+by) = T(ax+by) - \lambda(ax+by) = aT(x) + bT(y) - \lambda ax - \lambda by = a(T(x) - \lambda x) + b(T(y) - \lambda y) = aS(x) + bS(y)$ so S is linear.

Note that, for $x \in \ker S$, $S(x) = \bar{0} \Rightarrow T(x) = \lambda x \Rightarrow Ax = \lambda x$ so x is an eigenvector for T, or $x = \bar{0}$. Let $x, y \in \ker S, a, b \in \mathbb{R}$. Then define $c := ax + bx$. Note that $S(c) = S(ax+by) = aS(x)+bS(y) = \bar{0}$ so $\ker S$ is closed under addition and scalar multiplication. Also, $0 \in \mathbb{R}$ so $0x = \bar{0} \in \ker S$, and note that for any $y \in \ker T$, $\bar{0} + y = y + \bar{0} = y$. We also have that, for $x \in \ker S$, $z := -x \in \ker S$ and $x + z = \bar{0}$. Therefore, $\ker S$ is a vector space.

3 Question 3

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (x - y, 2x + 3y)$. Let W be the standard basis of $\mathbb{R}^2$ and let V be another basis of $\mathbb{R}^2$, $V = \{(1, -4), (-2, 7)\}$ in the coordinate of W.

3.1 Find $\text{mtx}_W(T)$.

$T = (x - y)e_1 + (2x + 3y)e_2 = x(e_1 + 2e_2) + y(-e_1 + 3e_2)$. Thus, $\text{mtx}_W(T) = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$.

3.2 Find $\text{mtx}_V(T)$.

$\text{mtx}_V(T) = \text{mtx}_{W,V}(id)^{-1} \ \text{mtx}_W(T) \ \text{mtx}_{W,V}(id)$ so we first need to find $\text{mtx}_{W,V}(id)$.

To find $\text{mtx}_{W,V}(id)$ we find a matrix which maps $\{\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\}$ to $\{\begin{pmatrix}1\\-4\end{pmatrix}\begin{pmatrix}-2\\7\end{pmatrix}\}$. Note that this is trivially $\begin{pmatrix}1 & -2\\-4 & 7\end{pmatrix} = P$. We then have

$\text{mtx}_V(T) = \text{mtx}_{W,V}(id)^{-1}\ \text{mtx}_W(T)\ \text{mtx}_{W,V}(id) = P^{-1}\ \text{mtx}_W(T)P = \begin{pmatrix}-15 & 29\\-10 & 19\end{pmatrix}$.

3.3 Find $T(1,-2)$ in the basis V.

Note that $T(1,-2) = (3,-4)$ so we simply need to find $a, b \in \mathbb{R}$ such that $a(1,-4) + b(-2,7) = (3,-4)$. Note that if $a^* = -13$ and $b^* = -8$, then $a^*(1,-4) + b^*(-2,7) = (-13,52) + (16,-56) = (3,-4)$ so therefore $T(1,-2)$, in the basis of V, is $(-13,-8)$.

4 Question 4

We will solve the linear first order difference equations as described. We will specifically be solving the following system:

$X_t = \begin{pmatrix}1 & 4\\2 & -1\end{pmatrix} X_{t-1}, X_0 = \begin{pmatrix}1\\1\end{pmatrix}$.

4.1 Calculate eigenvalues and eigenvectors of A

For eigenvalues λ of A must satisfy $\det(A - \lambda I) = 0$. Then, $(1-\lambda)(-1-\lambda) - (4)(2) = 0 \Rightarrow \lambda^2 - 9 = 0 \Rightarrow \lambda = 3$ and $\lambda = -3$. Thus, $3, -3$ are eigenvalues of A. Now we must find their corresponding eigenvectors.

First let us find x such that $Ax = 3x$. Then, $x_1 + 4x_2 = 3x_1, 2x_1 - 1x_2 = 3x_2 \Rightarrow -2x_1 + 4x_2 = 0 = 2x_1 - 4x_2 \Rightarrow \begin{pmatrix}2\\1\end{pmatrix}$ is an eigenvector corresponding to an eigenvalue of 3. Similarly, we will find x such that $Ax = -3x \Rightarrow x_1 + 4x_2 = -3x_1, 2x_1 - 1x_2 = -3x_2 \Rightarrow x_1 + x_2 = 0 = 2x_1 + 2x_2 \Rightarrow \begin{pmatrix}1\\-1\end{pmatrix}$ is an eigenvector corresponding to an eigenvalue of -3.

4.2 Set $D = diag\{\lambda_1 \dots \lambda_n\}$ and $P = \{v_1, \dots, v_n\}$.

We define $D = \begin{pmatrix}3 & 0\\0 & -3\end{pmatrix}$. We also define $P = \begin{pmatrix}2 & 1\\1 & -1\end{pmatrix}$.

4.3 Calculate P^{-1} and $Pdiag\{\lambda_1^t, \dots, \lambda_n^t\}P^{-1}$.

We can calculate $P^{-1} = \frac{1}{-2-1}\begin{pmatrix}-1 & -1\\-1 & 2\end{pmatrix} = \begin{pmatrix}1/3 & 1/3\\1/3 & -2/3\end{pmatrix}$. $A^t = Pdiag\{\lambda_1^t, \dots, \lambda_n^t\}P^{-1} = \begin{pmatrix}2 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}3^t & 0\\0 & (-3)^t\end{pmatrix}\begin{pmatrix}1/3 & 1/3\\1/3 & -2/3\end{pmatrix}$

4.4 Plug A^t from step 3 to solve for X_t.

$X_t = A^t X_0 = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3^t & 0 \\ 0 & (-3)^t \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3^t & 0 \\ 0 & (-3)^t \end{pmatrix} \begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix} =$
$\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2(3^{t-1}) \\ (-3)^{t-1} \end{pmatrix} = \begin{pmatrix} 4(3^{t-1}) + (-3)^{t-1} \\ 2(3^{t-1}) - (-3)^{t-1} \end{pmatrix}$

5 Question 5

We want to find a sequence of real numbers $\{z_t\}_{t=1}^{\infty}$, which satisfies

$$z_t = a_1 z_{t-1} + \cdots + a_n z_{t-n} \tag{1}$$

where $a_1, \ldots, a_n \in \mathbb{R}$ and $z_0, \ldots, z_{-n+1} \in \mathbb{R}$ are given.

We define $X_t := \begin{pmatrix} z_t \\ z_{t-1} \\ \cdots \\ z_{t-n+1} \end{pmatrix}$. We now write $X_t = AX_{t-1}$ for some $n \times n$ matrix A.
Now, notice that we have:

$$\begin{pmatrix} z_t \\ z_{t-1} \\ \cdots \\ z_{t-n+1} \end{pmatrix} = A \begin{pmatrix} z_{t-1} \\ z_{t-2} \\ \cdots \\ z_{t-n} \end{pmatrix} = \begin{pmatrix} a_{1,1}z_{t-1} + a_{1,2}z_{t-2} + \cdots + a_{1,n}z_{t-n} \\ a_{2,1}z_{t-1} + a_{2,2}z_{t-2} + \cdots + a_{2,n}z_{t-n} \\ \cdots \\ a_{n,1}z_{t-1} + a_{n,2}z_{t-2} + \cdots + a_{n,n}z_{t-n} \end{pmatrix}.$$

From 1 we have that $a_{1,i} = a_i \ \forall i \in \{1, \ldots, n\}$. Notice also that $\forall j \in \{1, \ldots, n-1\}$, $a_{j+1,j} = 1$ and $a_{j+1,k} = 0$ for $k \neq j$. Thus,

$$A = \begin{pmatrix} a_1 & a_2 & \ldots & a_{n-1} & a_n \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

Next, we know that, for all t, $z_t = c_1\lambda_1^t + \cdots + c_n\lambda_n^t$ for coefficients $c_1, \ldots c_n$. We are given values for $z_0, \ldots z_{-n+1}$ so we can use our expression for z_t to set up a system which will identify our coefficients:

$$\begin{pmatrix} z_0 \\ \cdots \\ z_{-n+1} \end{pmatrix} = \begin{pmatrix} c_1\lambda_1^0 + \cdots + c_n\lambda_n^0 \\ \cdots \\ c_1\lambda_1^{-n+1} + \cdots + c_n\lambda_n^{-n+1} \end{pmatrix}$$

5.1 Applying this methodology

Let $n = 3, a_1 = 2, a_2 = 1, a_3 = -2, z_0 = 2, z_{-1} = 2, z_{-2} = 1$.

Then, $A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. We will now find our eigenvalues of A. These satisfy $det(A - \lambda I) = 0$

$$\Rightarrow (2-\lambda)(\lambda^2) + \lambda - 2 = (\lambda - 1)(\lambda - 2)(\lambda + 1)$$

so the eigenvalues of A are $2, 1, -1$. Now, we do not need to find eigenvectors. Rather, we can set up our initial value equations:

$$\begin{pmatrix} z_0 \\ z_{-1} \\ z_{-2} \end{pmatrix} = \begin{pmatrix} c_1\lambda_1^0 + c_2\lambda_2^0 + c_3\lambda_3^0 \\ c_1\lambda_1^{-1} + c_2\lambda_2^{-1} + c_3\lambda_3^{-1} \\ c_1\lambda_1^{-2} + c_2\lambda_2^{-2} + c_3\lambda_3^{-2} \end{pmatrix} = \begin{pmatrix} c_1 + c_2 + c_3 \\ c_1 2^{-1} + c_2 1^{-1} + c_3(-1)^{-1} \\ c_1 2^{-2} + c_2 1^{-2} + c_3(-1)^{-2} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1/2 & 1 & -1 \\ 1/4 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 1 \\ -1/3 \end{pmatrix}.$$

Now we can find our solution: $z_t = \frac{4}{3}(2)^t + 1^t - \frac{1}{3}(-1)^t = \frac{4}{3}(2)^t + 1 - \frac{1}{3}(-1)^t$.