# Tullock-ian allocations
## (*alt title: More algebra than you care to see*)

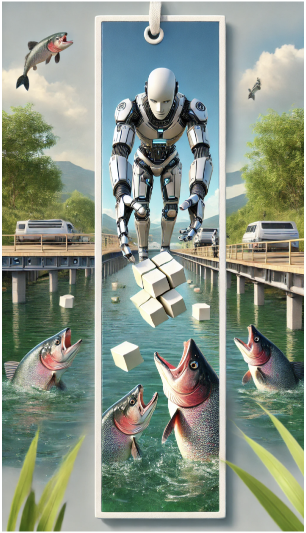

mike, quintus, geoff, kshitij, neha, dan robinson, anna

1$^{st}$ Annual Sonoma County Hackademethon

Wed – June 26, 2024

# Motivation

- Previously we explored the trade-off between "fairness" and "revenue" in block-space allocations.[1]
- The protocol may be willing to sacrifice revenue to attempt to encourage more builder decentralization.
- We also want to connect to the "fair-division" literature from social choice theory.[2]

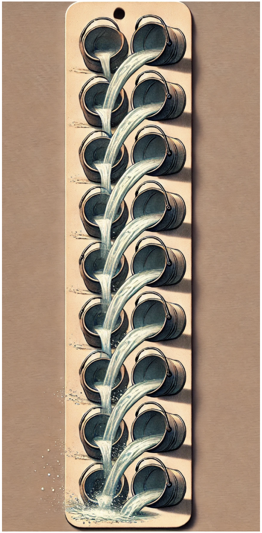


[1] https://ethresear.ch/t/on-block-space-distribution-mechanisms/19764

[2] https://en.wikipedia.org/wiki/Proportional-fair_rule

# Model

- Values - "*Everyone's got a price*"
  - $\mathbf{v} = [v_1, v_2, \ldots, v_n]$.
- Bids - "*Tell me something I don't know*"
  - $\mathbf{b} = [b_1, b_2, \ldots, b_n]$.
- Allocations - "*Gimme gimme gimme*"
  - $x_i = b_i / \sum_j b_j$.
- Utilities – "*Water Works & Electric Co.*"
  - $U_i = v_i x_i - b_i$



# Conjecture

- The proportional allocation rule
- The proportional allocation rule is the only proportionally-fair,
- The proportional allocation rule is the only proportionally-fair, Sybil-proof
- The proportional allocation rule is the only proportionally-fair, Sybil-proof Tullockian allocation
- The proportional allocation rule is the only proportionally-fair, Sybil-proof Tullockian allocation in the two-player
- The proportional allocation rule is the only proportionally-fair, Sybil-proof Tullockian allocation in the two-player, all-pay-as-bid auction.

# Warning

## ///soz in advance

- For the proportional allocation rule we know the equilibrium bids are

$$b_1^* = \frac{v_1^2 v_2}{(v_1+v_2)^2}, \quad b_2^* = \frac{v_1 v_2^2}{(v_1+v_2)^2}.$$

- E.g., if $v_1 = 2v_2$,

$$b_1^* = \frac{4v_2^3}{(v_2+v_2)^2} = v_2$$
$$b_2^* = \frac{2v_2^3}{(v_2+v_2)^2} = v_2/2$$

- So the utilities in equilibrium are (recall $U_i = v_i b_i / \sum_j b_j - b_i$ ),

$$U_1^* = v_1 \left(\frac{v_1^2 v_2}{(v_1+v_2)^2}\right) \Big/ \left[\left(\frac{v_1^2 v_2}{(v_1+v_2)^2}\right) + \left(\frac{v_1 v_2^2}{(v_1+v_2)^2}\right)\right] - \left(\frac{v_1^2 v_2}{(v_1+v_2)^2}\right)$$
$$= \frac{v_1^3}{(v_1+v_2)^2} \implies U_2^* = \frac{v_2^3}{(v_1+v_2)^2}.$$

- ... why do we care. Well, we now have an expression for $U_i^*(\mathbf{v})$.

$$U_i^* = \frac{v_i^3}{(v_1 + v_2)^2}.$$

- One way to claim fairness is, $U_i^*$ is *proportionally fair* if

$$\sum_i \frac{U_i - U_i^*}{U_i^*} \leq 0, \quad \forall U_i \neq U_i^*.$$

- Interpret as "the aggregate relative change in utility is negative."
- or "any other allocation that results in an increase in my utility has a larger corresponding decrease in someone else's utility."

# Approach

## Non-proportional alloc. rule

- So we need some set of candidates $U_i$ to test against

$$\sum_i \frac{U_i - U_i^*}{U_i^*} \leq 0, \quad \forall U_i \neq U_i^*.$$

- Recall our original statement: The proportional allocation rule is the only proportionally-fair, Sybil-proof Tullockian allocation in the two-player, all-pay-as-bid auction. (this means $\alpha > 1$, just trust me bro...)
- So now our statement is "are there any $\alpha > 1$ that have equilibrium $U_i$ such that proportional is not fair."
- So now we need a general form for "equilibria utility" of $x_i = b_i^\alpha / \sum_j b_j^\alpha$ mechanisms.

# Approach

## Non-proportional alloc. rule

- For other values of $\alpha$, we have the first order condition,

$$U_i = \frac{v_i b_i^\alpha}{b_1^\alpha + b_2^\alpha} - b_i. \tag{1}$$

- Taking the partial (just looking at `Player 1`),

$$\frac{\partial U_1}{\partial b_1} = \frac{\alpha v_1 b_1^{\alpha-1} b_2^\alpha}{(b_1^\alpha + b_2^\alpha)^2} - 1. \tag{2}$$

- Now things get truly hideous, but simplify rather nicely into,

$$U_1^*(\alpha) = v_1 \left( \frac{v_1^{\alpha+1}}{v_1^{\alpha+1} + v_2^{\alpha+1}} \right) - \frac{\alpha v_1^{\alpha+1} v_2^\alpha}{(v_1^\alpha + v_2^\alpha)^2} \tag{3}$$

# Next steps

- For all values of $\alpha$, show that

$$\sum_i \frac{U_i(\alpha) - U_i^*}{U_i^*} \leq 0, \quad \forall U_i \neq U_i^*.$$

- E.g., show that,

$$\sum_i v_i \left[ \left( \frac{v_i^{\alpha+1}}{v_1^{\alpha+1} + v_2^{\alpha+1}} \right) - \frac{\alpha v_i^{\alpha+1} v_{-i}^{\alpha}}{(v_1^{\alpha} + v_2^{\alpha})^2} - \frac{v_i^3}{(v_1 + v_2)^2} \right] \Bigg/ \frac{v_i^3}{(v_1 + v_2)^2} \leq 0$$

- ...not a today problem.

# thanks :)