Notation

- $\Gamma_0(q) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) ;\ a, b, c, d \in \mathbb{Z},\ q|c,\ \ ad - bc = 1 \right\}$

- Action of $\Gamma_0(q)$ on the upper half-plane $\mathfrak{H}$ given by:

$$\gamma z := \frac{az+b}{cz+d} \quad \text{for } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$$

- $$f(z) = \sum_{n=0}^{\infty} a(n) e^{2\pi i n z} \in S_2(q)$$

normalised newform of weight 2 *for level* q.

Notation (cont.)

- $\langle r \rangle := 2\pi i \int_{i\infty}^{r} \mathrm{Re}(f(z)dz);$ $\qquad r \in \mathbb{Q}$

"raw" modular symbol

- For $x \in [0,1]$, $M \in \mathbb{N}$,

$$G_M(x) := \frac{1}{M} \sum_{0 \le a \le Mx} \langle \frac{a}{M} \rangle$$

Conjecture (Mazur, Rubin, Stein)

Conjecture A [Mazur, Rubin, Stein (2016)] For all $x \in [0,1]$,

$$\lim_{M \to \infty} G_M(x) = \frac{1}{2\pi i} \sum_{n \geq 1} \frac{a(n)\left(\cos(2\pi nx) - 1\right)}{n^2}.$$

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Mazur, Rubin's Programme: Investigating the growth of the Mordell-Weil group of abelian varieties over large abelian extensions A over $\mathbb{Q}$ via the study of vanishing of values of L-functions and statistics of modular symbols.
Question: As L runs through (infinite) cyclic extensions of $\mathbb{Q}$, how often is $\text{rank}E(L) > \text{rank}E(\mathbb{Q})$?

Mazur-Rubin's approach

a. By Birch and Swinnerton-Dyer, *Question* can be reformulated in terms of vanishing of twisted L-functions at 1.

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b. Special values of L-functions can be expressed in terms of modular symbols (Birch-Stevens formula):

$$\tau(\chi)L(E,\chi,1) = \sum_{a \in (\mathbb{Z}/m\mathbb{Z})^*} \chi(a)\langle \frac{a}{m} \rangle; \qquad \chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}$$

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c. Study the growth behaviour of various types of combinations of modular symbols, e.g. $G_M(x)$, second moments, Theta coefficients, to shed light on the frequency of vanishing of $L(E,\bar{\chi},1)$ and, via a., b. , to rank questions. *Conjectures for such combinations numerically tested (Mazur, Rubin, Stein)*

Other work on Conjecture A

-Blomer, Fouvry, Kowalski, Michel, Milicevic, Sawin: Special case $x = 1$ and as M runs to infinity through primes.

-Average version of Conjecture A by Petridis-Risager using an example of higher-order modular forms.

-Recently, H-S. Sun has given a simplified proof in the case of square-free q.

Outline of the proof

Our approach (D., Hoffstein, Kıral, Lee)

(i) Let $\mathbf{1}_{[0,x]}$ be the characterstic function of $[0,x]$. Then

$$G_M(x) = \frac{1}{M} \sum_{0 \le a \le M} \langle \frac{a}{M} \rangle \mathbf{1}_{[0,x]} \left(\frac{a}{M} \right).$$

If $\{h\}$ is a family of smooth periodic function "converging" to $\mathbf{1}_{[0,x]}$, then

$$\frac{1}{M} \sum_{0 \le a \le M} \langle \frac{a}{M} \rangle h \left(\frac{a}{M} \right) " \to " G_M(x).$$

Specifically,

$$G_M(x) = \frac{1}{M} \sum_{0 \le a \le M} \langle \frac{a}{M} \rangle h \left(\frac{a}{M} \right) + O_{h,q}(M^{-1/4+\epsilon}).$$

Outline of the proof (cont.)

(ii) Modular symbols can be interpreted in terms of *(additively) twisted L-functions:* Let, for $\mathrm{Re}(s) > 3/2$,

$$L(f, s; a/c) = \sum_{n \geq 1} \frac{a(n) e^{2\pi i n \frac{a}{c}}}{n^s}$$

and

$$\Lambda(f, s; a/c) = \left(\frac{c}{2\pi}\right)^s \Gamma(s) L(f, s; a/c).$$

This function has a analytic continuation to the entire complex plane. Then,

$$\langle \frac{a}{M} \rangle = \frac{\pi}{c} \left(\Lambda(f, 1; a/c) - \Lambda(f, 1; -a/c)\right)$$

Outline of the proof (cont.)

(iii) Because of (i) we look at

$$\frac{1}{M}\sum_{0\le a\le M}\langle\frac{a}{M}\rangle h\left(\frac{a}{M}\right).$$

If $\hat{h}(n)$ is the n-th Fourier coefficient of h, set

$$\alpha_{n,M}(s) := \frac{1}{M}\left(\sum_{a \mod M} e^{-\frac{2\pi ian}{M}} L(f,t;a/M)\right) = \sum_{r\equiv m \mod M}\frac{a(m)}{m^t}$$

With (ii)

$$\frac{1}{M}\sum_{0\le a\le M}\langle\frac{a}{M}\rangle h\left(\frac{a}{M}\right) = \frac{1}{2}\sum_{n\in\mathbb{Z}}\hat{h}(n)(\alpha_{-n,M}(1)-\alpha_{n,M}(1))$$

Outline of the proof (cont.)

Asymptotics of $\alpha_{n,M}$

Key tool: Functional equation of $L(f, s, a/M)$ and its implications

The approximate functional equation for $L(f, t, a/M)$ implies

Proposition 0.1 *For* $X > 1$,

$$\alpha_{n,M}(1) = \sum_{r \equiv n \mod M} \frac{a(r)}{r} V\left(\frac{X}{2\pi r}\right) + O_q(X^{-\frac{1}{2}-\epsilon} M^{-\frac{1}{2}+\epsilon})$$

where

$$V(y) := \frac{1}{2\pi i} \int_{(2)} (2\pi y)^u \Gamma(u) G(u) du$$

for a G, even, EBV, of pol. decay as $|\mathrm{Im}(u)| \to \infty$, such that $G(0) = 1$.

Outline of the proof (cont.)

Setting

$$X = M^{3/2} \prod_{\substack{p|(q,M) \\ p^2|q}} p^{\frac{1}{2}\mathrm{ord}_p(q)+1}$$

we can deduce

Proposition 0.2

$$\sum_{n\in\mathbb{Z}} \hat{h}(n)\alpha_{n,M}(1) = \sum_{n\in\mathbb{Z}} \hat{h}(n)\frac{a(n)}{n} + O_q(M^{-\frac{1}{4}+\epsilon}).$$

Thus

$$\frac{1}{M}\sum_{0\leq a\leq M} \langle \frac{a}{M}\rangle h\left(\frac{a}{M}\right) = \frac{1}{2}\sum_{n\geq 1}(\hat{h}(-n)-\hat{h}(n))\frac{a(n)}{n} + O_q(M^{-\frac{1}{4}+\epsilon}).$$

Outline of the proof (cont.)

Theorem 0.3 (D., Hoffstein, Kıral, Lee) *For each* $x \in [0,1]$,

$$G_M(x) = \frac{1}{2\pi i} \sum_{n\geq 1} \frac{a(n)\,(\cos(2\pi nx) - 1)}{n^2} + O((Mq)^{\epsilon} M^{-\frac{1}{4}} \prod_{\substack{p|(q,M)\\ p^2|q}} p^{\frac{1}{4} ord_p(q) + \frac{1}{2}}).$$

Proof By Prop. 0.2,

$$\frac{1}{M}\sum_{0\le a\le M}\langle\frac{a}{M}\rangle h\left(\frac{a}{M}\right)=\frac{1}{2}\sum_{n\ge1}(\hat{h}(-n)-\hat{h}(n))\frac{a(n)}{n}+O_q(M^{-\frac{1}{4}+\epsilon}).$$

As $h" \to " \mathbf{1}_{[0,x]}$,

$$LHS" \to " G_M(x)$$

and

$$RHS" \to " \frac{1}{2}\sum_{n\ge1}\left(\frac{1-e^{2\pi inx}}{-2\pi inx}-\frac{1-e^{-2\pi inx}}{2\pi in}\right)+O_q(M^{-\frac{1}{4}+\epsilon})$$
$$=\frac{1}{2\pi i}\sum_{n\ge1}\frac{a(n)\left(\cos(2\pi nx)-1\right)}{n^2}+O_q(M^{-\frac{1}{4}+\epsilon})$$

Theorem 0.4 (D., Hoffstein, Kıral, Lee) *(In the special case of $k = 2$)*
$f \in S_2$, normalised newform of level q
$a, d \in \mathbb{Z}$; $(a, d) = 1$.
$M_d = \prod_{\substack{p|d \\ ord_p(d) \geq ord_p(q)}} p^{ord_p(d)}$
$r_d = d/M_d$
$R_d = r_d$-primary factor of q.
Set

$$\tilde{L}(s, f; a/d) := (lcm(q, d^2))^{s/2}\Gamma(s)(2\pi)^{-s}L(s, f, a/d).$$

Then,

$$\tilde{L}(s, f; a/d) \doteq \sum_{\rho|r_d} \sum_{\substack{\chi \mod \rho \\ \chi \text{ primitive}}} \tilde{L}(2 - s, f^{\chi}|_2 W_{lcm(R_d, r_d^2)}; \widetilde{a/d})$$

where $f^{\chi}(z) = \sum_{n \geq 1} a(n)\chi(n)e^{2\pi i n z}$, W_n is an Atkin-Lehner operator and $\widetilde{a/d}$ an explicit fraction with denominator a divisor of M_d.