Elliptic cocycle for $\mathrm{GL}_N(\mathbb{Z})$ and Hecke operators

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Period polynomial

Let f be a cusp form of weight k on $\mathrm{SL}_2(\mathbb{Z})$. The period polynomial of f is given by

$$r_f(x) = \int_0^{i\infty} f(\tau)(\tau - x)^{k-2}\, d\tau.$$

Period polynomial

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$$r_f(x) = \int_0^{i\infty} f(\tau)(\tau - x)^{k-2}\, d\tau.$$

For example, Let
$\Delta(\tau) = q\prod_{n\geq 1}(1-q^n)^{24} = q - 24q^2 + 252q^3 + \cdots$. be the Ramanujan Delta function. Shimura computed its period polynomial:

$$\begin{aligned} r_\Delta(x) =& \omega^+ \left(36(x^{10}-1) - 691(x^8 - 3x^6 + 3x^4 - x^2)\right) \\ &+ \omega^-(4x^9 - 25x^7 + 42x^5 - 25x^3 + 4x). \end{aligned}$$

where $\omega^+ = 0.0643382\ldots I$ and $\omega^- = 0.0092692\ldots$.

Period polynomial

Let A be a $\mathrm{GL}_2(\mathbb{Q})$-module. An A-valued modular symbol is a function:

$$\begin{aligned} r : \mathbb{Z}^2\backslash(0,0) \times \mathbb{Z}^2\backslash(0,0) &\to A \\ (\alpha, \beta) &\mapsto r\{\alpha, \beta\} \end{aligned}$$

satisfying

1. $r\{\alpha, \beta\} = 0$ for all $\alpha \in \mathbb{Q}\beta$,
2. $r\{\alpha, \beta\} + r\{\beta, \delta\} = r\{\alpha, \delta\}$ for all $\alpha, \beta, \delta \in \mathbb{Z}^2\backslash(0,0)$.

A modular symbol $r\{\alpha, \beta\}$ is said to be homogeneous if

$$r\{g\alpha, g\beta\} = g \cdot r\{\alpha, \beta\} \text{ for all } g \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } \alpha, \beta \in \mathbb{Z}^2\backslash(0,0).$$

Period polynomial

Let $\mathbb{C}[X_1, X_2]$ be the $\mathrm{GL}_2(\mathbb{Q})$-module with the action

$$(\gamma \cdot P)(X_1, X_2) = P((X_1, X_2)\gamma).$$

The period polynomial r_f can be extended to a $\mathbb{C}[X_1, X_2]$-valued modular symbol. Let $\alpha = \left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right), \beta = \left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right) \in \mathbb{Z}^2$ be two non-zero column vectors. Then we define

$$r_f\{\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right), \left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)\}(X_1, X_2) = \int_{\frac{b}{d}}^{\frac{a}{c}} f(\tau)(\tau X_1 + X_2)^{k-2}\, d\tau.$$

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Let $\alpha, \beta, \delta \in \mathbb{Z}^2$ be any non-zero vectors, then we have the cocycle relation

$$r_f\{\alpha, \beta\} + r_f\{\beta, \delta\} = r_f\{\alpha, \delta\}.$$

Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then r_f satisfies the homogeneous property:

$$r_f\{\gamma\alpha, \gamma\beta\} = \gamma \cdot r_f\{\alpha, \beta\}.$$

Hecke operator

Next, we recall two kinds of Hecke operators. The first one is the classical Hecke operator defined on modular forms:

Definition

Let f be a modular form of weight k.

$$T_m f(\tau) = m^{k-1} \sum_{\substack{ad=m \\ a,d>0}} \sum_{b=0}^{d-1} \frac{1}{d^k} f\left(\frac{a\tau + b}{d}\right)$$

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The second kind of Hecke operator is defined on modular symbols:

Definition

Let m be a positive integer,

$$\mathbb{T}_m r_f\{\alpha, \beta\} = \sum_{\gamma \in M_2(m)/\,\mathrm{SL}_2(\mathbb{Z})} \gamma \cdot r_f\{m\gamma^{-1}(\alpha, \beta)\}$$

Hecke operator

For example, when $m = 2$, $\{\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 2 \end{smallmatrix}\right)\}$ give a set of representatives of $M_2(2)/\operatorname{SL}_2(\mathbb{Z})$.

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$$\begin{aligned}
&\mathbb{T}_2 r_f\{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\} \\
=& r_f\{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right)\}(2X_1, X_2) + r_f\{\left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\}(X_1, 2X_2) \\
&+ r_f\{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 \\ 2 \end{smallmatrix}\right)\}(X_1 + X_2, 2X_2) \\
=& r_f\{e_1, e_2\}(2X_1, X_2) + r_f\{e_1, e_2\}(2X_1, X_1 + X_2) \\
&+ r_f\{e_1, e_2\}(X_1, 2X_2) + r_f\{e_1, e_2\}(X_1 + X_2, 2X_2)
\end{aligned}$$

where $e_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$.

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$$\begin{aligned}
&\mathbb{T}_2 r_f\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\} \\
=&r_f\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}\}(2X_1, X_2) + r_f\{\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}(X_1, 2X_2) \\
&+ r_f\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}\}(X_1 + X_2, 2X_2) \\
=&r_f\{e_1, e_2\}(2X_1, X_2) + r_f\{e_1, e_2\}(2X_1, X_1 + X_2) \\
&+ r_f\{e_1, e_2\}(X_1, 2X_2) + r_f\{e_1, e_2\}(X_1 + X_2, 2X_2)
\end{aligned}$$

where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
If $f = \Delta$, then we can check directly that

$$\mathbb{T}_2 r_\Delta\{e_1, e_2\}(X_1, X_2) = -24 r_\Delta\{e_1, e_2\}(X_1, X_2).$$

Hecke equivariance

The following theorem shows the equivariance between these two Hecke operators

Theorem (Eichler-Shimura-Manin)

Let f be a cusp form and m a positive integer. Then for any non-zero vectors $\alpha, \beta \in \mathbb{Z}^2$, we have

$$\mathbb{T}_m r_f\{\alpha, \beta\} = r_{T_m f}\{\alpha, \beta\}.$$

In particular, if we take f to be an eigenform under T_m, then $r_f\{\alpha, \beta\}$ is an eigenvector of $\mathbb{T}_m$ with the same eigenvalue.

Question

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In fact, we construct families of rational functions in N variables which share the Hecke equivariance property. To motivate the definition, we first introduce the Kronecker theta function and the elliptic cocycle.

Kronecker theta function

Let $\tau \in \mathbb{H}$ and $x \in \mathbb{C}$. The Jacobi theta function is given by

$$\theta_\tau(x) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \exp\left(\left(n + \frac{1}{2}\right) x\right).$$

Let $x, x' \in \mathbb{C}$. Then the Kronecker theta function is defined by

$$\mathcal{K}(\tau, x, x') = \frac{\theta'_\tau(0)\theta_\tau(x + x')}{\theta_\tau(x)\theta_\tau(x')}.$$

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$$\mathcal{K}(\tau, x, x') = \frac{\theta'_\tau(0)\theta_\tau(x+x')}{\theta_\tau(x)\theta_\tau(x')}.$$

It has the exponential expansion:

$$\mathcal{K}(\tau, x, x') = \frac{x+x'}{xx'} \exp\left(\sum_{\substack{k\geq 2 \\ k \text{ even}}} 2(x^k + x'^k - (x+x')^k)\, G_k(\tau) \frac{(2\pi i)^k}{k!} \right).$$

Zagier's result

Zagier proved that the Laurent expansion of $\mathcal{K}(\tau, xT, yT)\mathcal{K}(\tau, -xyT, T)$ in T encodes all the information of modular forms and its period polynomial. He proved that:

Theorem (Zagier)

$$\frac{1}{(2\pi i)^2}\mathcal{K}(\tau, xT, yT)\mathcal{K}(\tau, -xyT, T)$$
$$= \frac{(xy-1)(x+y)}{x^2y^2}(2\pi i T)^{-2} +$$
$$\sum_{k=4}^{\infty}\left(\sum_{\substack{f\in M_k\\ eigenform}} \frac{r_f^+(x)r_f^-(y)+r_f^-(x)r_f^+(y)}{(2i)^{k-3}(f,f)}f(\tau)\right)\frac{(2\pi i T)^{k-2}}{(k-2)!}.$$

Elliptic cocycle

From the theorem above, we know that the product of two Kronecker theta functions $\mathcal{K}$ contain the information of modular forms on $\mathrm{SL}_2(\mathbb{Z})$.

So what happens if we consider the product of N copies of Kronecker theta functions $\mathcal{K}$?

Elliptic cocycle

Recently, Charollois defined an $(N-1)$-cocycle for $\mathrm{GL}_N(\mathbb{Z})$ which consists of Kronecker theta function.

Definition

Let $\sigma \in M_N(\mathbb{Z})$, $x = (x_1, \cdots, x_N), x' = (x'_1, \cdots, x'_N)^t \in \mathbb{C}^N$ *and* $\tau \in \mathbb{H}$. *If* $\det(\sigma) \neq 0$,

$$\mathscr{E}_N(\tau, \sigma, x, x') = \frac{1}{\det \sigma} \sum_{y, y' \in \mathbb{Z}^N / \sigma \mathbb{Z}^N} \boldsymbol{e}(x \cdot y) \mathscr{K}(\tau, x\sigma, \sigma^{-1}(x' + y\tau + y')).$$

where $\boldsymbol{e}(a) = \exp(2\pi i a)$ *and* $\mathscr{K}(\tau, x, x') = \prod_{i=1}^N \mathcal{K}(\tau, x_i, x'_i)$ *be the multivariable Kronecker theta function.*

In particular, when $\sigma = \mathrm{Id}$,

$$\mathscr{E}_N(\tau, \mathrm{Id}, x, x') = \mathscr{K}(\tau, x, x') = \prod_{i=1}^N \mathcal{K}(\tau, x_i, x'_i).$$

Elliptic cocycle

We can view $\mathscr{E}_N$ as an $(N-1)$-cocycle for $\mathrm{GL}_N(\mathbb{Z})$. Let $\mathcal{A} = (A_1, A_2, \cdots, A_N) \in (\mathrm{GL}_N(\mathbb{Z}))^N$. Let σ_i be the first column of A_i and $\sigma(\mathcal{A}) = (\sigma_i)$. Charollois showed that

$$\begin{aligned}(\mathrm{GL}_N(\mathbb{Z}))^N &\to \mathcal{M}(x, x') \\ \mathcal{A} &\mapsto \mathscr{E}_N(\tau, \sigma(\mathcal{A}), x, x')\end{aligned}$$

gives a class in $H^{N-1}(\mathrm{GL}_N(\mathbb{Z}), \mathcal{M}(x, x'))$ where $\mathcal{M}(x, x')$ is the $\mathrm{GL}_N(\mathbb{Z})$-module of meromorphic functions in x, x'.

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Theorem (Charollois)

Let $\sigma_0, \sigma_1, \cdots, \sigma_N \in \mathbb{Z}^N$ be $N+1$ non-zero vectors. Then for any $x, x' \in \mathbb{C}^N$, we have

$$\sum_{j=0}^{N} (-1)^j \mathscr{E}_N(\tau, (\sigma_0, \cdots, \hat{\sigma}_j, \cdots, \sigma_N), x, x') = 0.$$

Hecke operator

Inspired by a recent paper of Bergeron-Charollois-Garcia where they introduced a differential form E_ψ that realizes an Eisenstein theta correspondence for the dual pair $(\mathrm{GL}_N; \mathrm{GL}_2)$, we can also define two Hecke operators for the pair $(\mathrm{GL}_N; \mathrm{GL}_2)$ on the elliptic cocycle $\mathscr{E}_N(\tau, \sigma, x, x')$.

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The first one corresponds to $\mathrm{GL}_2(\mathbb{Z})$ which is an analog of the classical Hecke operator on modular forms

$$T_m \mathscr{E}_N(\tau, \sigma, x, x') = m^{N-1} \sum_{\substack{a,d>0 \\ ad=m}} \frac{1}{d^N} \sum_{b=0}^{d-1} \mathscr{E}_N\left(\frac{a\tau + b}{d}, \sigma, ax, ax'\right),$$

and the second one corresponds to GL_N

$$\mathbb{T}_m \mathscr{E}_N(\tau, \sigma, x, x') = \sum_{\gamma \in M_N(m)/\mathrm{SL}_N(\mathbb{Z})} \mathscr{E}_N(\tau, m\gamma^{-1}\sigma, x\gamma, m\gamma^{-1}x').$$

Hecke operators T_m vs $\mathbb{T}_m$

Although these two Hecke operators are defined for different objects, they have an essential relation:

Theorem (Z.)

For any $x, x' \in \mathbb{C}^N$ and $\sigma \in M_N(\mathbb{Z})$, we have the formula

$$\mathbb{T}_m \mathscr{E}_N(\tau, \sigma, x, x') = \sum_{d|m} A(N, d)\, T_{\frac{m}{d}} \mathscr{E}_N(\tau, \sigma, x, dx'),$$

where $\{A(N, d) \mid d = 1, 2, \cdots\}$ is a certain sequence of integers. For example, $A(2, 1) = 1$ and $A(2, d) = 0$ for all $d > 1$, $A(3, d) = d$ for all $d \geq 1$.

Hecke operators T_m vs $\mathbb{T}_m$

For example, in dimension 2, these two Hecke operators coincide, i.e.

$$\mathbb{T}_m \mathscr{E}_2(\tau, \sigma, x, x') = T_m \mathscr{E}_2(\tau, \sigma, x, x').$$

As a corollary, we will show later it covers the Eichler-Shimura-Manin's Hecke equivariance theorem.

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As a corollary, we will show later it covers the Eichler-Shimura-Manin's Hecke equivariance theorem. In dimension 3, we have

$$\mathbb{T}_m \mathscr{E}_3(\tau, \sigma, x, x') = \sum_{d|m} d T_{\frac{m}{d}} \mathscr{E}_3(\tau, \sigma, x, dx').$$

The proof of this Theorem was inspired by a method of Borisov-Gunnells that they used to prove the Hecke stability of the space of the toric modular forms under the operator T_m.

Construction of rational functions

Now we back to Zagier's result, we rewrite his result by

$$\mathscr{E}_2(\tau, \mathrm{Id}, (X_1 T, X_2 T), (-X_2 yT, X_1 yT)^t)$$
$$= \sum_{k=4}^{\infty} \left(\sum_f \frac{r_f^+\{e_1, e_2\}(X_1, X_2) r_f^-(y) + r_f^-\{e_1, e_2\}(X_1, X_2) r_f^+(y)}{(2i)^{k-3}(f,f)} f(\tau) \right) \frac{(2\pi i T)^{k-2}}{(k-2)!},$$

where $e_1 = \binom{1}{0}$, $e_2 = \binom{0}{1}$.

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where $e_1 = \binom{1}{0}$, $e_2 = \binom{0}{1}$. We can extend it by cocycle relation:

$$\mathscr{E}_2(\tau, \sigma, (X_1 T, X_2 T), (-X_2 yT, X_1 yT)^t)$$
$$= \sum_{k=4}^{\infty} \left(\sum_f \frac{r_f^+\{\sigma_1, \sigma_2\}(X_1, X_2) r_f^-(y) + r_f^-\{\sigma_1, \sigma_2\}(X_1, X_2) r_f^+(y)}{(2i)^{k-3}(f, f)} f(\tau) \right) \frac{(2\pi i T)^{k-2}}{(k-2)!},$$

where σ_1 is the first column of σ and σ_2 is the second column of σ.

Similar to the case of dimension 2, we can consider the Laurent expansion of the elliptic cocycle $\mathscr{E}_N$. For example, when $\sigma = \mathrm{Id}$, then $\mathscr{E}_N(\tau, \mathrm{Id}, XT, X'T)$ is a product of N Kronecker theta functions $\mathcal{K}$, where $X = (X_1, \cdots, X_N)$, $X' = (X'_1, \cdots, X'_N)^t$. We recall that

$$\mathcal{K}(\tau, x, x') = \frac{x + x'}{xx'} \exp\left(\sum_{\substack{k \geq 2 \\ k \text{ even}}} 2(x^k + x'^k - (x + x')^k) G_k(\tau) \frac{(2\pi i)^k}{k!} \right).$$

Hence Laurent expansion in T is given by

$$\mathscr{E}_N(\tau, \mathrm{Id}, XT, X'T) = \frac{F_{-N}(X, X')}{T^N} \exp\left(\sum_{\substack{k \geq 2 \\ k \text{ even}}} F_k(X, X') G_k(\tau) \frac{(2\pi i T)^k}{k!} \right).$$

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Obstruction: However, because of the term $G_2(\tau)$, we can only get the quasi-modular forms if we expand this exponential function.

Construction of rational functions

keypoint: To overcome this problem, we put $X' = MX^t$ where $M \in M_N(\mathbb{C})$ with $M + M^t = 0$. Then we can show that the function $F_2(X, MX^t) = 0$!

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keypoint: To overcome this problem, we put $X' = MX^t$ where $M \in M_N(\mathbb{C})$ with $M + M^t = 0$. Then we can show that the function $F_2(X, MX^t) = 0$!
We consider the function $\mathscr{E}_N(\tau, \sigma, XT, MX^t yT)$. It has the following Laurent expansion in T:

Theorem (Z.)

Let $\sigma \in M_N(\mathbb{Z})$, $X \in \mathbb{C}^N$ and $y \in \mathbb{C}$. $M \in M_N(\mathbb{C})$ with $M + M^t = 0$.

$$\frac{1}{(2\pi i)^N}\mathscr{E}_N(\tau,\sigma, XT, MX^t yT) = P_{-N}(\sigma, M, X, y)(2\pi i T)^{-N} + \sum_{k \geq 4} \sum_{\substack{f \text{ eigenform} \\ \text{weight } k}} P_f(\sigma, M, X, y) f(\tau) \frac{(2\pi i T)^{k-N}}{(k-N)!}.$$

where P_{-N} and P_f are certain rational functions in X and P_f/P_{-N} are polynomials.

Hecke equivariance

We can extend the definition of P_f to all modular forms by linearity. Then we have the following Hecke equivariance:

Corollary

For any modular form $f(\tau)$, we have the following formula:

$$\mathbb{T}_m P_f(\sigma, M, X, y) = \sum_{d|m} A(N, d) P_{T_{\frac{m}{d}} f}(\sigma, M, X, dy).$$

But because of the factor d, the function $P_f(\sigma, M, X, y)$ is still not an eigenvector even we take f to be an eigenform. To deal with it, we consider the Laurent expansion in y.

Hecke equivariance

Corollary

If we write the Laurent expansion

$$P_f(\sigma, M, X, y) = \sum_{t \geq -N} P_f^{(t)}(\sigma, M, X) y^t,$$

and take f to be a normalized eigenform, then if the rational function $P_f^{(t)}(\sigma, M, X)$ is non-zero, it is an eigenvector of $\mathbb{T}_m$

$$\mathbb{T}_m P_f^{(t)}(\sigma, M, X) = \left(\sum_{d|m} A(N, d) a_f\left(\frac{m}{d}\right) d^t \right) P_f^{(t)}(\sigma, M, X).$$

L-series

For each rational function $P_f^{(t)}(\sigma, M, X)$, it corresponds a family of eigenvalues, hence we can consider the L-series

$$L_f^{(t)}(s) = \sum_{m \geq 1} \left(\sum_{d|m} A(N, d) a_f\Big(\frac{m}{d}\Big)\, d^t \right) m^{-s}.$$

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Theorem (Z.)

Let $f(\tau)$ be an eigenform of weight k. Then for $\mathrm{Re}(s) > \max\{k, N+t\}$, $L_f^{(t)}(s)$ *converges absolutely. It has the decomposition*

$$L_f^{(t)}(s) = L(f, s) \prod_{j=1}^{N-2} \zeta(s - j - t),$$

where $L(f, s)$ is the L-function associated to the eigenform f.

Examples

There are only finitely many t such that $P_f^{(t)}$ is non-zero, and for every eigenform f, we can always find at least one t such that $P_f^{(t)}$ is non-zero.

For example, when $N = 2$, the rational function $P_f^{(t)}$ is non-zero for $0 \leq t \leq k-2$, and it is in fact the period polynomial:

$$P_f^{(t)}(\sigma, M, (X_1, X_2)) = r_f^{\pm}\{\sigma_1, \sigma_2\}(X_1, X_2)$$

up to the parity of t, where σ_1, σ_2 are the first and second column of σ respectively and $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Examples

When $N = 3$, let $f = G_k$ be the Eisenstein series of weight k and $M = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$ with a, b, c algebraic independent over $\mathbb{Q}$.

Then we can check that $P_{G_k}^{(t)}$ is non-zero for $-2 \leq t \leq 11$ by computer. For example:

$$P_{G_k}^{(-2)}(M, X) = \frac{2((-aX_2 - bX_3)X_1^{k-1} + (aX_2^{k-1} + bX_3^{k-1})X_1 + (-cX_3X_2^{k-1} + cX_3^{k-1}X_2))}{(k-1)!(aX_2 + bX_3)(-aX_1 + cX_3)(bX_1 + cX_2)}.$$

Examples

When $N = 3$, let $f = \Delta$, we can check that $P_\Delta^{(t)}$ is non-zero for $-1 \leq t \leq 10$ by computer. For example, let $M = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$ with a, b, c algebraic independent over $\mathbb{Q}$. Then

$$P_\Delta^{(-1)}(M, X) = \frac{1}{2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 691 (aX_2 + bX_3)(bX_1 + cX_2)(aX_1 - cX_3)} \Big((4a^2X_2^2 + 8baX_3X_2 + 4b^2X_3^2)X_1^{10} + (-25a^2X_2^4 - 25baX_3X_2^3 - 25baX_3^3X_2 - 25b^2X_3^4)X_1^8 + (25caX_3X_2^4 + 25cbX_3^2X_2^3 - 25caX_3^3X_2^2 - 25cbX_3^4X_2)X_1^7 + (42a^2X_2^6 + 42baX_3X_2^5 + 42baX_3^5X_2 + 42b^2X_3^6)X_1^6 + (-42caX_3X_2^6 - 42cbX_3^2X_2^5 + 42caX_3^5X_2^2 + 42cbX_3^6X_2)X_1^5 + (-25a^2X_2^8 - 25baX_3X_2^7 - 25baX_3^7X_2 - 25b^2X_3^8)X_1^4 + (25caX_3X_2^8 + 25cbX_3^2X_2^7 - 25caX_3^7X_2^2 - 25cbX_3^8X_2)X_1^3 + (4a^2X_2^{10} + 25baX_3^3X_2^7 - 42baX_3^5X_2^5 + 25baX_3^7X_2^3 + 4b^2X_3^{10})X_1^2 + (-8caX_3X_2^{10} + 25caX_3^3X_2^8 - 25cbX_3^4X_2^7 - 42caX_3^5X_2^6 + 42cbX_3^6X_2^5 + 25caX_3^7X_2^4 - 25cbX_3^8X_2^3 + 8cbX_3^{10}X_2)X_1 + (4c^2X_3^2X_2^{10} - 25c^2X_3^4X_2^8 + 42c^2X_3^6X_2^6 - 25c^2X_3^8X_2^4 + 4c^2X_3^{10}X_2^2) \Big).$$

Sketch of proof

Finally, I would like to give a sketch of the proof of the following theorem:

Theorem (Z.)

For any $x, x' \in \mathbb{C}^N$ and $\sigma \in M_N(\mathbb{Z})$, we have the formula

$$\mathbb{T}_m \mathscr{E}_N(\tau, \sigma, x, x') = \sum_{d|m} A(N, d)\, T_{\frac{m}{d}} \mathscr{E}_N(\tau, \sigma, x, dx'),$$

where $\{A(N, d) \mid d = 1, 2, \cdots\}$ is a certain sequence of integers. For example, $A(2, 1) = 1$ and $A(2, d) = 0$ for all $d > 1$, $A(3, d) = d$ for all $d \geq 1$.

Sketch of proof

We note that the Kronecker theta function $\mathcal{K}$ has the following Fourier expansion:

$$\begin{aligned}\mathcal{K}(\tau, x_0, x_0') =& 2\pi i\Bigg(1 - \frac{1}{1-\mathbf{e}(x_0)} - \frac{1}{1-\mathbf{e}(x_0')} \\ &- \sum_{m,n\geq 1} q^{mn}\big(\mathbf{e}(nx_0 + mx_0') - \mathbf{e}(-nx_0 - mx_0')\big)\Bigg).\end{aligned}$$

Sketch of proof

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Since the elliptic cocycle is a product of N copies of the Kronecker theta function $\mathcal{K}$, by expanding the product, we see that it consists of the series:

$$\sum_{\substack{m\in S_1\\ n\in S_2}} q^{m\cdot n}\mathbf{e}(x\cdot n + m\cdot x') \tag{1}$$

where the summation over certain subsets $S_1, S_2 \subset \mathbb{Z}^N$

Sketch of proof

More precisely, let L be a lattice in $\mathbb{R}^N$ and C a polyhedral cone respect to L generated by N linear independent vectors. C^* is the dual of C. Let C_1 be a face of C^* and C_2 be a face of C. We define the following series:

$$f_{L,C_1,C_2}(\tau, x, x') = \sum_{\substack{m \in L^* \cap C_1 \\ n \in L \cap C_2}} q^{m \cdot n} \mathbf{e}(x \cdot n + m \cdot x').$$

Sketch of proof

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Similar to the elliptic cocycle $\mathscr{E}_N$, we introduce the parameter σ by

$$f_{L,C_1,C_2}(\tau, \sigma, x, x') = \frac{\operatorname{sign}(D)}{|D|^n} \sum_{\substack{z\in L^*/L^*D\sigma^{-1}\\ z'\in L/\sigma L}} f_{L,C_1,C_2}\left(\frac{\tau}{|D|}, \frac{(x+z)\sigma}{|D|}, \sigma^{-1}(x'+z')\right).$$

Sketch of proof

By comparing the Fourier expansions of $\mathscr{E}_N(\tau,\sigma,x,x')$ and $f_{\mathbb{Z}^N,C_1,C_2}(\tau,\sigma,x,x')$, we can show that $\mathscr{E}_N(\tau,\sigma,x,x')$ is a linear combination of $f_{\mathbb{Z}^N,C_1,C_2}(\tau,\sigma,x,x')$:

$$\mathscr{E}_N(\tau,\sigma,x,x') = \sum_{(C_1,C_2)} f_{\mathbb{Z}^N,C_1,C_2}(\tau,\sigma,x,x'),$$

where the summation runs through certain pairs of faces (C_1, C_2) of (C^*, C), $C = e_1\mathbb{R}_{\geq 0} + \cdots + e_N\mathbb{R}_{\geq 0}$. Hence it is enough to prove the following formula for the function f_{L,C_1,C_2}

$$\begin{aligned} &\sum_{\gamma\in M_N(m)/\operatorname{SL}_N(\mathbb{Z})} f_{L,C_1,C_2}(\tau, m\gamma^{-1}\sigma, x\gamma, m\gamma^{-1}x') \\ =&\sum_{d|m} A(N,d)\, T_{\frac{m}{d}} f_{L,C_1,C_2}(\tau,\sigma,x,dx'). \end{aligned}$$

Sketch of proof

Consider the sum

$$\sum_{S} f_{S,C_1,C_2}(\tau, p^k x, x')$$

where S runs through the lattices such that $L \subset S \subset \frac{1}{p^k} L$ and $[S : L] = p^{k(N-1)}$. By using a method of Borisov-Gunnells, we show that it is equal to

$$\sum_{t=0}^{k} A(N, p^t)\, T_{p^{k-t}} f_{L,C_1,C_2}(\tau, x, p^t x').$$

Sketch of proof

Now we turn to the cocycle part. Let $\sigma \in M_N(\mathbb{Z})$ with $\det(\sigma) \neq 0$, then we can show that

$$f_{L,C_1,C_2}(\tau,\sigma,x,x') = f_{\sigma^{-1}L,C_1,C_2}(\tau,\mathrm{Id},x\sigma,\sigma^{-1}x').$$

Since all the sublattices S of L/p^k of index p^k which contain L are given by $\frac{1}{p}\gamma L$ where $\gamma \in M_N(p^k)/\operatorname{SL}_N(\mathbb{Z})$, then above formula gives

$$\begin{aligned}
&\sum_{\gamma \in M_N(p^k)/\operatorname{SL}_N(\mathbb{Z})} f_{L,C_1,C_2}(\tau,p^k\gamma^{-1}\sigma,x\gamma,p^k\gamma^{-1}x') \\
&= \sum_S f_{S,C_1,C_2}(\tau,px,x') \\
&= \sum_{t=0}^{k} A(N,p^t)\,T_{p^{k-t}} f_{L,C_1,C_2}(\tau,\sigma,x,p^t x').
\end{aligned}$$

Thank you!