Sparse Equidistribution of Hyperbolic Orbifolds

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May 13, 2020

Structure of the Talk

(1) A toy model: modular hyperbolæ
- Equidistribution
- Sparse equidistribution

(2) Heegner Points and Closed Geodesics
- Equidistribution
- Sparse equidistribution

(3) Hyperbolic Orbifolds
- Equidistribution
- Sparse equidistribution
- Level aspect

Modular Hyperbolæ

Let q be a positive integer. For each $d \in (\mathbb{Z}/q\mathbb{Z})^\times$, we let $\overline{d} \in (\mathbb{Z}/q\mathbb{Z})^\times$ denote its modular inverse, so that $d\overline{d} \equiv 1 \pmod{q}$.

Definition

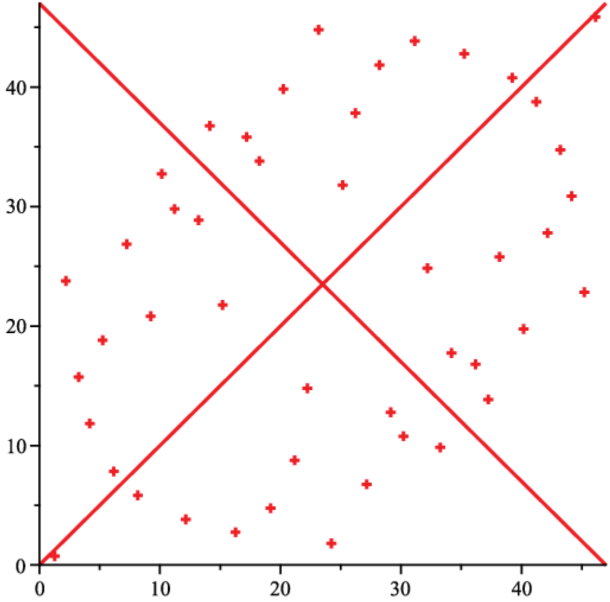
The modular hyperbola is the set

$$S_q := \left\{ \left(\frac{d}{q}, \frac{\overline{d}}{q} \right) \in \mathbb{T}^2 : d \in (\mathbb{Z}/q\mathbb{Z})^\times \right\}.$$

Here $\mathbb{T}^2 := (\mathbb{R}/\mathbb{Z})^2$ is the (2-)torus.

Example: $q = 47$ (unscaled)

(Image: Ford–Khan–Shparlinski)



Modular Hyperbolæ: Properties

Some properties of S_q:

- $\#S_q = \varphi(q)$,
- $(x_1, x_2) \in S_q$ iff $(x_2, x_1), (1 - x_1, 1 - x_2), (1 - x_2, 1 - x_1) \in S_q$,
- $\left(\frac{1}{q}, \frac{1}{q}\right) \in S_q$,
- $(x_1, x_2) \in S_q$ with $\frac{1}{q} < x_1 < \frac{1}{\sqrt{q}}$ only if $x_2 > \frac{1}{\sqrt{q}}$.

Modular Hyperbolæ: Limiting Behaviour

Question

What are the limiting statistical properties of $S_q \subset \mathbb{T}^2$ as $q \to \infty$?

The points $S_q \subset \mathbb{T}^2$ appear to behave just like *random* points on the torus.

Goal

Quantify the limiting behaviour of S_q as $q \to \infty$ in ways that are shared by *randomly chosen* points.

Equidistribution of Modular Hyperbolæ

Theorem (Beck–Khan (2002), Zhang (1996))

As $q \to \infty$, the modular hyperbolæ S_q equidistribute on $\mathbb{T}^2$.

Informally, the points S_q spread out randomly on $\mathbb{T}^2$.

Equidistribution

Let M be a topological space and μ a probability measure on M. Let μ_n be a sequence of probability measures on M.

Definition

The probability measures μ_n equidistribute on M w.r.t. μ if

$$\lim_{n \to \infty} \mu_n(B) = \mu(B)$$

for every continuity set $B \subset M$ (boundary has μ-measure zero).

Equidistribution

Let M be a topological space and μ a probability measure on M.
Let μ_n be a sequence of probability measures on M.

Definition

The probability measures μ_n equidistribute on M w.r.t. μ if

$$\lim_{n\to\infty} \int_M f(x)\, d\mu_n(x) = \int_M f(x)\, d\mu(x)$$

for all $f \in C_b(M)$ (continuous bounded).

Equidistribution

Definition

We define a probability measure μ_q on $\mathbb{T}^2$ by

$$\mu_q := \frac{1}{\varphi(q)} \sum_{x \in S_q} \delta_x.$$

So for $B \subset \mathbb{T}^2$ and $f : \mathbb{T}^2 \to \mathbb{C}$,

$$\mu_q(B) := \frac{\#(S_q \cap B)}{\varphi(q)},$$

$$\int_{\mathbb{T}^2} f(y)\, d\mu_q(y) := \frac{1}{\varphi(q)} \sum_{x \in S_q} f(x).$$

Equidistribution of Modular Hyperbolæ

Theorem (Beck–Khan (2002), Zhang (1996))

As $q \to \infty$, the probability measures μ_q equidistribute on $\mathbb{T}^2$ with respect to the Lebesgue measure μ on $\mathbb{T}^2$.

Equidistribution of Modular Hyperbolæ

Theorem (Beck–Khan (2002), Zhang (1996))

As $q \to \infty$,

$$\frac{\#(S_q \cap B)}{\varphi(q)} \to \mathrm{vol}(B)$$

for every continuity set $B \subset \mathbb{T}^2$.

Equidistribution of Modular Hyperbolæ

Theorem (Beck–Khan (2002), Zhang (1996))

As $q \to \infty$,

$$\frac{1}{\varphi(q)} \sum_{x \in S_q} f(x) \to \int_{\mathbb{T}^2} f(y)\, dy$$

for every continuous function f *on* $\mathbb{T}^2$.

Proof of Equidistribution of Modular Hyperbolæ

Idea of proof: approximate $f \in C(\mathbb{T}^2)$ by linear combinations of exponentials.

Reduces problem to showing for each $(m, n) \in \mathbb{Z}^2$ that

$$\frac{1}{\varphi(q)} \sum_{(x_1,x_2) \in S_q} e\left(mx_1 + nx_2\right) \to \int_{\mathbb{T}^2} e\left(mx_1 + nx_2\right)\, dx_1\, dx_2.$$

Trivial if $(m, n) = (0, 0)$. RHS is zero if $(m, n) \neq (0, 0)$.

LHS is $S(m, n; q)/\varphi(q)$, where

$$S(m, n; q) := \sum_{d \in (\mathbb{Z}/q\mathbb{Z})^\times} e\left(\frac{md + n\overline{d}}{q}\right)$$

is a Kloosterman sum. Result then follows from the Weil bound

$$S(m, n; q) \ll_{m,n} \tau(q)\sqrt{q}.$$

Sparse Equidistribution

Which *subsets* of S_q equidistribute?

Given a subset H_q of $(\mathbb{Z}/q\mathbb{Z})^\times$ and letting

$$S_{H_q} := \left\{ \left(\frac{d}{q}, \frac{\overline{d}}{q} \right) \in \mathbb{T}^2 : d \in H_q \right\},$$

form the probability measure

$$\mu_{H_q} := \frac{1}{\# H_q} \sum_{x \in S_{H_q}} \delta_x.$$

Question

What conditions on H_q ensure that the probability measures μ_{H_q} equidistribute on $\mathbb{T}^2$?

Sparse Equidistribution

We cannot expect equidistribution for *arbitrary* subsets: if we take $H_q = \{a \in (\mathbb{Z}/q\mathbb{Z})^\times : a \leq q/2\}$, then

$$\mu_{H_q}\left(\left\{(x_1, x_2) \in \mathbb{T}^2 : x_2 > \frac{1}{2}\right\}\right) = 0 \neq \frac{1}{2}.$$

Conjecture

If H_q has an algebraic *structure, then μ_{H_q} equidistributes on $\mathbb{T}^2$ provided that $\#H_q > q^\delta$ for some fixed $\delta > 0$.*

Here *algebraic* structure is with respect to the group structure of $(\mathbb{Z}/q\mathbb{Z})^\times$, namely we only consider cosets aH_q with $a \in (\mathbb{Z}/q\mathbb{Z})^\times$ and H_q a subgroup of $(\mathbb{Z}/q\mathbb{Z})^\times$.

Sparse Equidistribution

Theorem (H. (2020))

Fix $\delta > 0$. For each positive cubefree integer q, pick a subgroup H_q of $(\mathbb{Z}/q\mathbb{Z})^\times$ and an associated coset $aH_q \subset (\mathbb{Z}/q\mathbb{Z})^\times$ for which $\#H_q \gg q^{\frac{1}{2}+\delta}$. Then the probability measures μ_{aH_q} equidistribute on $\mathbb{T}^2$ as q tends to infinity.

Sketch of proof.

By character orthogonality,

$$\int_{\mathbb{T}^2} e(mx_1 + nx_2)\, d\mu_{aH_q}(x_1, x_2)$$
$$= \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod{q} \\ \chi|_{H_q}=1}} \overline{\chi}(a) \sum_{d \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(d) e\left(\frac{md + n\overline{d}}{q}\right).$$

The sum over $d \in (\mathbb{Z}/q\mathbb{Z})^\times$ is the twisted Kloosterman sum $S_\chi(m, n; q)$, which satisfies the Weil bound $S_\chi(m, n; q) \leq \tau(q)\sqrt{(m, n, q)q}$.
The number of characters χ modulo q for which $\chi|_{H_q} = 1$ is $\varphi(q)/\#H_q$.
So this is $o(1)$ for fixed $(m, n) \neq (0, 0)$ if $\#H_q \gg q^{1/2+\delta}$. $\square$

Sparse Equidistribution

Theorem (H. (2020))

Fix $\delta > 0$. For each prime q, pick a subgroup H_q of $(\mathbb{Z}/q\mathbb{Z})^\times$ and an associated coset $aH_q \subset (\mathbb{Z}/q\mathbb{Z})^\times$ for which $\#H_q \gg q^\delta$. Then the probability measures μ_{aH_q} equidistribute on $\mathbb{T}^2$ as q tends to infinity.

The proof relies on deep bounds for exponential sums due to Bourgain, which require that q be prime.

Binary Quadratic Forms

Definition

An integral binary quadratic form Q is a homogeneous polynomial

$$Q(x, y) = ax^2 + bxy + cy^2$$

for which $a, b, c \in \mathbb{Z}$.

For brevity, we write $Q = [a, b, c]$.

- The discriminant of Q is $b^2 - 4ac$.
- Q is primitive if $(a, b, c) = 1$.
- Q is positive definite if $D < 0$ and $a, c > 0$.

Let D be a fundamental discriminant.
We let $\mathcal{Q}_D$ denote the set of primitive integral binary quadratic forms of discriminant D that are positive definite if $D < 0$.

Binary Quadratic Forms and Narrow Ideal Classes

The group $\Gamma := \mathrm{SL}_2(Z) \ni \gamma$ acts on $\mathcal{Q}_D$ via

$$(\gamma \cdot Q)(x, y) := Q\left(\gamma \begin{pmatrix} x \\ y \end{pmatrix}\right).$$

Proposition

The set $\Gamma \backslash \mathcal{Q}_D$ is isomorphic to the narrow class group Cl_D^+ of the quadratic field $\mathbb{Q}(\sqrt{D})$.

$$Q = [a, b, c] \mapsto \begin{cases} \dfrac{-b + \sqrt{D}}{2a}\mathbb{Z} + \mathbb{Z} & \text{if } a > 0, \\ \dfrac{b + \sqrt{D}}{-2a}\mathbb{Z} + \mathbb{Z} & \text{if } a < 0, \end{cases}$$

$$\mathfrak{a} = w\mathbb{Z} + \mathbb{Z} \mapsto \frac{\mathrm{N}(x - wy)}{\mathrm{N}(\mathfrak{a})}, \qquad w \in \mathbb{Q}(\sqrt{D}),\ w > \sigma(w).$$

Heegner Points

Let $D < 0$. For each $Q = [a, b, c] \in \mathcal{Q}_D$, define the point

$$z_Q := \frac{-b + i\sqrt{-D}}{2a} \in \mathbb{H}.$$

The orbit $\{\Gamma z_Q\}$ is a countable collection of points in $\mathbb{H}$ associated to the equivalence class $\Gamma Q \in \Gamma\backslash\mathcal{Q}_D$, or equivalently a single point on the modular surface $\Gamma\backslash\mathbb{H}$.

We call such a point a *Heegner point* or *CM point*. We let $z_A \in \Gamma\backslash\mathbb{H}$ denote such a point associated to an ideal class $A \in \mathrm{Cl}_D$, or equivalently an element ΓQ of $\Gamma\backslash\mathcal{Q}_D$.

For each $D < 0$, there are h_D such points, where $h_D := \# \mathrm{Cl}_D$ is the class number of $\mathbb{Q}(\sqrt{D})$. By the class number formula, the number of Heegner points is $\approx \sqrt{-D}$.

Closed Geodesics

Let $D > 0$. For each $Q = [a, b, c] \in \mathcal{Q}_D$, define the geodesic

$$\mathcal{C}_Q := \left\{ z \in \mathbb{H} : a|z|^2 + b\Re(z) + c = 0 \right\} \subset \mathbb{H}.$$

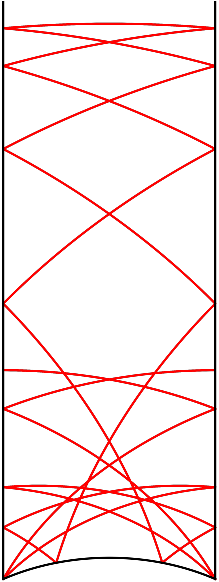
The orbit $\{\Gamma\mathcal{C}_Q\}$ is a countable collection of geodesics in $\mathbb{H}$ associated to the equivalence class $\Gamma Q \in \Gamma\backslash\mathcal{Q}_D$, or equivalently a single closed geodesic on the modular surface $\Gamma\backslash\mathbb{H}$.

We let $\mathcal{C}_A \in \Gamma\backslash\mathbb{H}$ denote such a closed geodesic associated to an ideal class $A \in \mathrm{Cl}_D^+$, or equivalently an element ΓQ of $\Gamma\backslash\mathcal{Q}_D$.

For each $D > 0$, there are h_D^+ such closed geodesics, where $h_D^+ := \# \mathrm{Cl}_D^+$ is the narrow class number of $\mathbb{Q}(\sqrt{D})$. Each has length $2 \log \epsilon_D$, where ϵ_D is the least totally positive unit in $\mathbb{Q}(\sqrt{D})$. By the class number formula, the sum of lengths of closed geodesics is $\approx \sqrt{D}$.

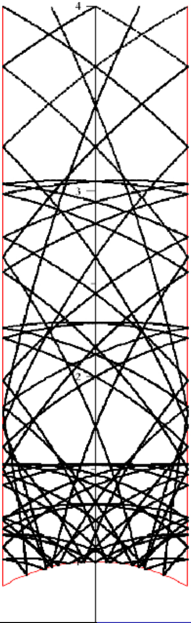
Example: $D = 19$

(Image: Constantin Kogler)



Example: $D = 377$

(Image: Einsiedler–Lindenstrauss–Michel–Venkatesh)



Heegner Points and Closed Geodesics: Limiting Behaviour

Question

What are the limiting statistical properties of Heegner points and closed geodesics as $D \to \pm\infty$?

Heegner points appear to behave just like *random* points on $\Gamma\backslash\mathbb{H}$.
Closed geodesics appear to behave just like *random* curves on $\Gamma\backslash\mathbb{H}$.

Goal

Quantify the limiting behaviour of Heegner points and closed geodesics as $D \to \pm\infty$ in ways that are shared by *randomly chosen* points and curves.

Equidistribution of Heegner Points

Definition

For $D < 0$, we define a probability measure μ_D on $\Gamma\backslash\mathbb{H}$ by

$$\mu_D(B) := \frac{\#\{A \in \mathrm{Cl}_D : z_A \in B\}}{h_D} \quad \text{for } B \subset \Gamma\backslash\mathbb{H},$$

$$\int_{\Gamma\backslash\mathbb{H}} f(z)\, d\mu_D(z) := \frac{1}{h_D} \sum_{A \in \mathrm{Cl}_D} f(z_A) \quad \text{for } f : \Gamma\backslash\mathbb{H} \to \mathbb{C}.$$

By the class number formula,

$$h_D = \frac{\omega_D}{2\pi}\sqrt{D}L(1, \chi_D).$$

Equidistribution of Closed Geodesics

Definition

For $D > 0$, we define a probability measure μ_D on $\Gamma\backslash\mathbb{H}$ by

$$\mu_D(B) := \frac{\sum_{A\in \mathrm{Cl}_D^+} \ell(\mathcal{C}_A \cap B)}{2h_D^+ \log \epsilon_D} \quad \text{for } B \subset \Gamma\backslash\mathbb{H},$$

$$\int_{\Gamma\backslash\mathbb{H}} f(z)\, d\mu_D(z) := \frac{1}{2h_D^+ \log \epsilon_D} \sum_{A\in \mathrm{Cl}_D^+} \int_{\mathcal{C}_A} f(z)\, ds \quad \text{for } f : \Gamma\backslash\mathbb{H} \to \mathbb{C}.$$

Here $\ell(\mathcal{C}) := \int_{\mathcal{C}} ds$ with $ds^2 = y^{-2}\, dx^2 + y^{-2}\, dy^2$ the length element on $\mathbb{H}$.

By the class number formula,

$$\sum_{A\in \mathrm{Cl}_D^+} \ell(\mathcal{C}_A) = 2h_D^+ \log \epsilon_D = 2\sqrt{D} L(1, \chi_D).$$

Equidistribution of Heegner Points and Closed Geodesics

Theorem (Duke (1988))

(1) *As $D \to -\infty$ along negative fundamental discriminants, the probability measures μ_D equidistribute on $\Gamma\backslash\mathbb{H}$ with respect to the normalised Haar measure $\frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} d\mu = \frac{3}{\pi} \frac{dx\,dy}{y^2}$ on $\Gamma\backslash\mathbb{H}$.*

(2) *As $D \to \infty$ along positive fundamental discriminants, the probability measures μ_D equidistribute on $\Gamma\backslash\mathbb{H}$ with respect to the normalised Haar measure $\frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} d\mu = \frac{3}{\pi} \frac{dx\,dy}{y^2}$ on $\Gamma\backslash\mathbb{H}$.*

Theorem (Duke (1988))

(1) *As $D \to -\infty$ along negative fundamental discriminants,*

$$\frac{\#\{A \in \mathrm{Cl}_D : z_A \in B\}}{h_D} \to \frac{\mathrm{vol}(B)}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}$$

for every continuity set $B \subset \Gamma\backslash\mathbb{H}$.

(2) *As $D \to \infty$ along positive fundamental discriminants,*

$$\frac{\sum_{A \in \mathrm{Cl}_D^+} \ell(\mathcal{C}_A \cap B)}{2h_D^+ \log \epsilon_D} \to \frac{\mathrm{vol}(B)}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}$$

for every continuity set $B \subset \Gamma\backslash\mathbb{H}$.

Theorem (Duke (1988))

(1) *As $D \to -\infty$ along negative fundamental discriminants,*

$$\frac{1}{h_D} \sum_{A \in \mathrm{Cl}_D} f(z_A) \to \frac{1}{\mathrm{vol}(\Gamma \backslash \mathbb{H})} \int_{\Gamma \backslash \mathbb{H}} f(z) \, d\mu(z)$$

for every continuous bounded function f on $\Gamma \backslash \mathbb{H}$.

(2) *As $D \to \infty$ along positive fundamental discriminants,*

$$\frac{1}{2h_D^+ \log \epsilon_D} \sum_{A \in \mathrm{Cl}_D^+} \int_{\mathcal{C}_A} f(z) \, ds \to \frac{1}{\mathrm{vol}(\Gamma \backslash \mathbb{H})} \int_{\Gamma \backslash \mathbb{H}} f(z) \, d\mu(z)$$

for every continuous bounded function f on $\Gamma \backslash \mathbb{H}$.

Proof of Equidistribution: Reduction to Weyl Sums

Idea of proof: approximate $f \in C_b(\Gamma\backslash\mathbb{H})$ by linear combinations of the constant function, Maaß cusp forms, and (direct integrals of) Eisenstein series.

Reduces problem to showing for each Maaß cusp form f and each $t \in \mathbb{R}$ that the following Weyl sums are $O(D^{1/2-\delta})$ for some $\delta > 0$:

$$W_{D,f} := \begin{cases} \displaystyle\sum_{A \in \mathrm{Cl}_D} f(z_A) & \text{for } D < 0, \\ \displaystyle\sum_{A \in \mathrm{Cl}_D^+} \int_{\mathcal{C}_A} f(z)\, ds & \text{for } D > 0, \end{cases}$$

$$W_{D,t} := \begin{cases} \displaystyle\sum_{A \in \mathrm{Cl}_D} E\left(z_A, \frac{1}{2} + it\right) & \text{for } D < 0, \\ \displaystyle\sum_{A \in \mathrm{Cl}_D^+} \int_{\mathcal{C}_A} E\left(z, \frac{1}{2} + it\right) ds & \text{for } D > 0. \end{cases}$$

Proof of Equidistribution: Period Formulæ

Proposition (Waldspurger (1985))

We have that

$$|W_{D,f}|^2 \approx \sqrt{D}\frac{L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \otimes \chi_D\right)}{L(1, \operatorname{sym}^2 f)},$$

$$|W_{D,t}|^2 \approx \sqrt{D}\left|\frac{\zeta\left(\frac{1}{2}+it\right) L\left(\frac{1}{2}+it, \chi_D\right)}{\zeta(1+2it)}\right|^2.$$

Proof of Equidistribution: Subconvexity

Proposition (Iwaniec (1987))

There exists some $\delta > 0$ such that

$$L\left(\frac{1}{2}, f \otimes \chi_D\right) \ll_f |D|^{\frac{1}{2}-\delta},$$

$$L\left(\frac{1}{2} + it, \chi_D\right) \ll_t |D|^{\frac{1}{4}-\frac{\delta}{2}}.$$

Subconvex bound: trivial bounds $O_\varepsilon(|D|^{1/2+\varepsilon})$ and $O_\varepsilon(|D|^{1/4+\varepsilon})$ are the analogue of the bound $\zeta(1/2+it) \ll_\varepsilon (|t|+1)^{1/4+\varepsilon}$ due to the Phragmen–Lindelöf convexity principle.

Conrey–Iwaniec (2000): one can take any $\delta < \frac{1}{6}$.

Generalised Lindelöf hypothesis: one can take any $\delta < \frac{1}{2}$.

Sparse Equidistribution

Which *subsets* of Heegner points and closed geodesics equidistribute?

For a subset H of Cl_D^+, define the probability measure μ_H by

$$\mu_H(B) := \begin{cases} \dfrac{\#\{A \in H : z_A \in B\}}{\#H} & \text{for } D < 0, \\ \dfrac{\sum_{A \in H} \ell(\mathcal{C}_A \cap B)}{2\#H \log \epsilon_D} & \text{for } D > 0. \end{cases}$$

Question

What conditions on $H \subset \mathrm{Cl}_D^+$ ensure that the probability measures μ_H equidistribute on $\Gamma\backslash\mathbb{H}$?

Sparse Equidistribution

Proposition (Aka–Einsiedler (2016))

If $D > 0$ and $\#H \log D / h_D^+ \to \infty$, then the probability measures μ_H equidistribute on $\Gamma\backslash\mathbb{H}$.

Proposition (Bourgain–Kontorovich (2018))

If $D > 0$, then for each $\delta > 0$, there exist subsets H of Cl_D^+ for which $\#H\, D^{-\delta} / h_D^+ \to \infty$ but the probability measures μ_H do not *equidistribute on $\Gamma\backslash\mathbb{H}$.*

We cannot expect sparse equidistribution for arbitrary subsets; we need H to have an *algebraic* structure, so that we consider cosets CH with $C \in \mathrm{Cl}_D^+$ and H a subgroup of Cl_D^+.

Conjecture (Michel–Venkatesh (2006))

Fix $0 \leq \delta < \frac{1}{2}$.

(1) *For each negative fundamental discriminant* $D < 0$, *let* CH *be a coset of* Cl_D *with* $\#H \gg (-D)^{-\delta} h_D$. *Then the probability measures* μ_H *equidistribute on* $\Gamma\backslash\mathbb{H}$.

(2) *For each positive fundamental discriminant* $D > 0$, *let* CH *be a coset of* Cl_D^+ *with* $\#H \gg D^{-\delta} h_D^+$. *Then the probability measures* μ_H *equidistribute on* $\Gamma\backslash\mathbb{H}$.

Sparse Equidistribution

Theorem (Michel–Venkatesh (2006))

The conjecture is true in the range $\delta < \frac{1}{4}$ under the assumption of the generalised Lindelöf hypothesis.

Theorem (Harcos–Michel (2006))

The conjecture is true unconditionally in the range $\delta < \frac{1}{23042} \approx 0.00004$.

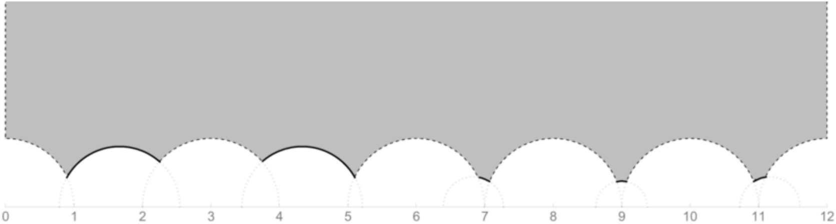
When $D > 0$ is a fundamental discriminant, there is another geometric invariant associated to each narrow ideal class $A \in \mathrm{Cl}_D^+$, or equivalently each equivalence class of binary quadratic forms $\Gamma Q \in \Gamma \backslash \mathcal{Q}_D$.

Theorem (Duke–Imamoḡlu–Tóth (2016))

- *Associated to each $A \in \mathrm{Cl}_D^+$ is a thin subgroup Γ_A of Γ, a Fuchsian group of the second kind.*
- *Let $\mathcal{N}_A \subset \mathbb{H}$ be the Nielsen region of Γ_A: the smallest nonempty Γ_A-invariant open convex subset of $\mathbb{H}$. Then $\Gamma_A \backslash \mathcal{N}_A$ is a hyperbolic orbifold (hyperbolic Riemann surface with signature).*
- *The boundary of $\Gamma_A \backslash \mathcal{N}_A$ is a simple closed geodesic whose image in $\Gamma \backslash \mathbb{H}$ is the closed geodesic $\mathcal{C}_A$.*

Example: $D = 28$ and $A = (\sqrt{D})$

(Image: Duke–Imamoḡlu–Tóth)



Hyperbolic Orbifolds: Limiting Behaviour

Question

What are the limiting statistical properties of hyperbolic orbifolds as $D \to \infty$?

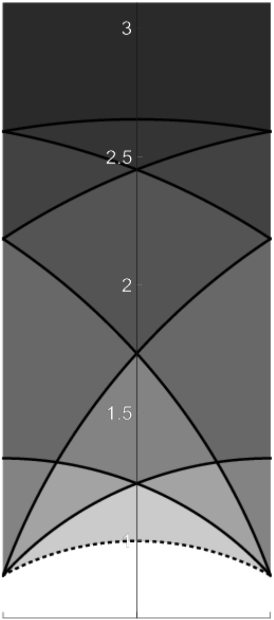
Hyperbolic orbifolds appear to behave just like *random* covers of $\Gamma \backslash \mathbb{H}$.

Goal

Quantify the limiting behaviour of hyperbolic orbifolds as $D \to \infty$ in ways that are shared by *randomly chosen* covers of $\Gamma \backslash \mathbb{H}$.

Example: $D = 28$ and $A = (\sqrt{D})$

(Image: Duke–Imamoḡlu–Tóth)



Definition

For $D > 0$, we define a probability measure μ_D on $\Gamma\backslash\mathbb{H}$ by

$$\mu_D(B) := \frac{\sum_{A \in \mathrm{Cl}_D^+} \mathrm{vol}(\Gamma_A \backslash \mathcal{N}_A \cap B)}{\sum_{A \in \mathrm{Cl}_D^+} \mathrm{vol}(\Gamma_A \backslash \mathcal{N}_A)} \quad \text{for } B \subset \Gamma\backslash\mathbb{H},$$

$$\int_{\Gamma\backslash\mathbb{H}} f(z)\, d\mu_D(z) := \frac{1}{\sum_{A \in \mathrm{Cl}_D^+} \mathrm{vol}(\Gamma_A \backslash \mathcal{N}_A)} \sum_{A \in \mathrm{Cl}_D^+} \int_{\Gamma_A \backslash \mathcal{N}_A} f(z)\, d\mu(z)$$

$$\text{for } f : \Gamma\backslash\mathbb{H} \to \mathbb{C}.$$

We have the bounds

$$\frac{\log \epsilon_D}{\log D} \ll \mathrm{vol}(\Gamma_A \backslash \mathcal{N}_A) \ll \log \epsilon_D,$$

so that by the class number formula,

$$D^{\frac{1}{2} - \varepsilon} \ll_{\varepsilon} \sum_{A \in \mathrm{Cl}_D^+} \mathrm{vol}(\Gamma_A \backslash \mathcal{N}_A) \ll \sqrt{D} \log D.$$

Equidistribution of Hyperbolic Orbifolds

Theorem (Duke–Imamoḡlu–Tóth (2016))

As $D \to \infty$ along positive fundamental discriminants, the probability measures μ_D equidistribute on $\Gamma \backslash \mathbb{H}$ with respect to the normalised Haar measure $\frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} d\mu = \frac{3}{\pi} \frac{dx\, dy}{y^2}$ on $\Gamma \backslash \mathbb{H}$.

Equidistribution of Hyperbolic Orbifolds

Theorem (Duke–Imamoḡlu–Tóth (2016))

As $D \to \infty$ along positive fundamental discriminants,

$$\frac{\sum_{A \in \mathrm{Cl}_D^+} \mathrm{vol}(\Gamma_A \backslash \mathcal{N}_A \cap B)}{\sum_{A \in \mathrm{Cl}_D^+} \mathrm{vol}(\Gamma_A \backslash \mathcal{N}_A)} \to \frac{\mathrm{vol}(B)}{\mathrm{vol}(\Gamma \backslash \mathbb{H})}$$

for every continuity set $B \subset \Gamma \backslash \mathbb{H}$.

Theorem (Duke–Imamoḡlu–Tóth (2016))

As $D \to \infty$ along positive fundamental discriminants,

$$\frac{1}{\sum_{A \in \mathrm{Cl}_D^+} \mathrm{vol}(\Gamma_A \backslash \mathcal{N}_A)} \sum_{A \in \mathrm{Cl}_D^+} \int_{\Gamma_A \backslash \mathcal{N}_A} f(z) \, d\mu(z) \to \frac{1}{\mathrm{vol}(\Gamma \backslash \mathbb{H})} \int_{\Gamma \backslash \mathbb{H}} f(z) \, d\mu(z)$$

for every continuous bounded function $f : \Gamma \backslash \mathbb{H} \to \mathbb{C}$.

Proof of Equidistribution of Hyperbolic Orbifolds

Idea of proof: approximate $f \in C_b(\Gamma\backslash\mathbb{H})$ by linear combinations of the constant function, Maaß cusp forms, and (direct integrals of) Eisenstein series.

Reduces problem to showing for each Maaß cusp form f and each $t \in \mathbb{R}$ that the following Weyl sums are $O(D^{1/2-\delta})$ for some $\delta > 0$:

$$W_{D,f} := \sum_{A \in \mathrm{Cl}_D^+} \int_{\Gamma_A \backslash \mathcal{N}_A} f(z)\, d\mu(z),$$

$$W_{D,t} := \sum_{A \in \mathrm{Cl}_D^+} \int_{\Gamma_A \backslash \mathcal{N}_A} E\left(z, \frac{1}{2} + it\right)\, d\mu(z).$$

Proof of Equidistribution of Hyperbolic Orbifolds

Lemma

For each Maaß cusp form f and each $t \in \mathbb{R}$,

$$W_{D,f} = W_{D,t} = 0.$$

Idea of proof.

The pair of hyperbolic orbifolds $\Gamma_A \backslash \mathcal{N}_A$ and $\Gamma_{JA^{-1}} \backslash \mathcal{N}_{JA^{-1}}$ cover $\Gamma \backslash \mathbb{H}$ evenly, where $J = (\sqrt{D}) \in \mathrm{Cl}_D^+$. $\square$

Corollary

Equidistribution is trivial!

Which *subsets* of hyperbolic orbifolds equidistribute?

For an element C of Cl_D^+ and a subgroup H of Cl_D^+, define the probability measure μ_{CH} on $\Gamma\backslash\mathbb{H}$ by

$$\mu_{CH}(B) := \frac{\sum_{A \in CH} \mathrm{vol}(\Gamma_A\backslash\mathcal{N}_A \cap B)}{\sum_{A \in CH} \mathrm{vol}(\Gamma_A\backslash\mathcal{N}_A)} \quad \text{for } B \subset \Gamma\backslash\mathbb{H},$$

$$\int_{\Gamma\backslash\mathbb{H}} f(z)\, d\mu_{CH}(z) := \frac{1}{\sum_{A \in CH} \mathrm{vol}(\Gamma_A\backslash\mathcal{N}_A)} \sum_{A \in CH} \int_{\Gamma_A\backslash\mathcal{N}_A} f(z)\, d\mu(z) \quad \text{for } f : \Gamma\backslash\mathbb{H} \to \mathbb{C}.$$

We have the bounds

$$\frac{\#H}{h_D^+} D^{\frac{1}{2}-\varepsilon} \ll_\varepsilon \sum_{A \in \mathrm{Cl}_D^+} \mathrm{vol}(\Gamma_A\backslash\mathcal{N}_A) \ll \frac{\#H}{h_D^+}\sqrt{D}\log D.$$

Take $H = (\mathrm{Cl}_D^+)^2$ to be the subgroup of Cl_D^+ for which every narrow ideal class is a square. A coset CH is a genus in the group of genera $\mathrm{Gen}_D := \mathrm{Cl}_D^+ / (\mathrm{Cl}_D^+)^2$.

Theorem (Duke–Imamoḡlu–Tóth (2016))

As $D \to \infty$ along positive fundamental discriminants, the probability measures μ_{CH} equidistribute on $\Gamma\backslash\mathbb{H}$ with respect to the normalised Haar measure $\frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} d\mu = \frac{3}{\pi}\frac{dx\,dy}{y^2}$ on $\Gamma\backslash\mathbb{H}$.

Nontrivial!

Proof of Sparse Equidistribution

Need to show for each Maaß cusp form f that

$$\sum_{A \in CH} \int_{\Gamma_A \backslash \mathcal{N}_A} f(z)\, d\mu(z) = o_f\left(\sum_{A \in CH} \mathrm{vol}(\Gamma_A \backslash \mathcal{N}_A)\right).$$

(Similarly for Eisenstein series.)

By character orthogonality, LHS is

$$\frac{\#H}{h_D^+} \sum_{\substack{\chi \in \widehat{\mathrm{Cl}_D^+} \\ \chi|_H = 1}} \overline{\chi}(C) \sum_{A \in \mathrm{Cl}_D^+} \chi(A) \int_{\Gamma_A \backslash \mathcal{N}_A} f(z)\, d\mu(z),$$

while RHS is $\approx \frac{\#H}{h_D^+}\sqrt{D}$.

Number of characters $\chi \in \widehat{\mathrm{Cl}_D^+}$ for which $\chi|_H = 1$ is $h_D^+/\#H$.

Proof of Sparse Equidistribution

Corollary

Equidistribution follows from the bound

$$W_{\chi,f} := \sum_{A \in \mathrm{Cl}_D^+} \chi(A) \int_{\Gamma_A \backslash \mathcal{N}_A} f(z)\, d\mu(z) = O_f\left(\frac{\#H}{h_D^+} D^{\frac{1}{2}-\delta}\right)$$

uniformly for $\chi \in \widehat{\mathrm{Cl}_D^+}$ *for which* $\chi|_H = 1$.

Problem is hardest when $\#H$ is very small compared to h_D^+.
For $H = (\mathrm{Cl}_D^+)^2$, we have that $\#H/h_D^+ = 2^{1-\omega(D)} \gg_\varepsilon D^{-\varepsilon}$.
Not very sparse!

So we need to show that there exists $\delta > 0$ such that $|W_{\chi,f}|^2 \ll_f D^{1-\delta}$.

Proof of Sparse Equidistribution

For $H = (\mathrm{Cl}_D^+)^2$, each $\chi \in \widehat{\mathrm{Cl}_D^+}$ for which $\chi|_H = 1$ is a *genus character* associated to a factorisation $D = D_1 D_2$ and a pair of quadratic Dirichlet characters χ_1, χ_2 modulo D_1, D_2.

Lemma (Duke–Imamoḡlu–Tóth (2016))

For each $\chi \in \widehat{\mathrm{Cl}_D^+}$ for which $\chi|_H = 1$, we have that

$$|W_{\chi,f}|^2 \approx \sqrt{D}\frac{L\left(\frac{1}{2}, f \otimes \chi_1\right) L\left(\frac{1}{2}, f \otimes \chi_2\right)}{L(1, \mathrm{sym}^2 f)}.$$

Proof of sparse equidistribution.

Input Iwaniec's subconvex bound

$$L\left(\frac{1}{2}, f \otimes \chi_1\right) \ll_f D_1^{\frac{1}{2}-\delta}, \qquad L\left(\frac{1}{2}, f \otimes \chi_2\right) \ll_f D_2^{\frac{1}{2}-\delta}. \qquad \square$$

Sparser Equidistribution

Question

Can one prove sparse equidistribution for cosets other than genera?

Conjecture (à la Michel–Venkatesh)

As $D \to \infty$ along positive fundamental discriminants, the probability measures μ_{CH} equidistribute on $\Gamma\backslash\mathbb{H}$ with respect to the normalised Haar measure $\frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} d\mu = \frac{3}{\pi} \frac{dx\,dy}{y^2}$ on $\Gamma\backslash\mathbb{H}$ provided that $\#H \gg D^{-\delta} h_D^+$ for some fixed $\delta < \frac{1}{2}$.

Theorem (H.–Nordentoft (2020+))

The conjecture holds in the range

(1) $\delta < \frac{625}{3309568} \approx 0.0001888$ *unconditionally,*

(2) $\delta < \frac{1}{4}$ *assuming the generalised Lindelöf hypothesis.*

Proof of Sparser Equidistribution

We need to show

$$W_{\chi,f} := \sum_{A \in \mathrm{Cl}_D^+} \chi(A) \int_{\Gamma_A \backslash \mathcal{N}_A} f(z)\, d\mu(z) = O_f\left(\frac{\#H}{h_D^+} D^{\frac{1}{2}-\delta}\right)$$

uniformly for $\chi \in \widehat{\mathrm{Cl}_D^+}$ for which $\chi|_H = 1$.

The method of Duke–Imamoḡlu–Tóth's proof for $H = (\mathrm{Cl}_D^+)^2$ relies on the fact that each χ is a genus character. For such χ,

$$W_{\chi,f} \approx D^{\frac{3}{4}} b(D_1) b(D_2)$$

where $b(n)$ denotes the n-th Fourier coefficient of the Katok–Sarnak lift of f, a Maaß form of weight $1/2$.
Then work of Waldspurger in an explicit form due to Baruch and Mao shows that $|b(D_j)|^2 \approx |D_j|^{-1} L(1/2, f \otimes \chi_j)$.

Proof of Sparser Equidistribution

Method fails when χ is not a genus character. Nonetheless...

Lemma (H.–Nordentoft (2020+))

We have that

$$|W_{\chi,f}|^2 \approx \sqrt{D}\frac{L\left(\frac{1}{2}, f \otimes \Theta_\chi\right)}{L(1, \mathrm{sym}^2 f)}.$$

Here Θ_χ is the theta series associated to χ: it is a Maaß form of weight 0, Laplacian eigenvalue $1/4$, level D, and nebentypus χ_D.

Proof of sparser equidistribution.

Input the Michel–Harcos subconvex bound

$$L\left(\frac{1}{2}, f \otimes \Theta_\chi\right) \ll_{f,\varepsilon} D^{1-\frac{625}{3309568}+\varepsilon}.$$

Assuming Lindelöf, we instead have $O_{f,\varepsilon}(D^\varepsilon)$. $\square$

Weyl Sums

The identity for $|W_{\chi,f}|^2$ proceeds in several steps:

(1) We show that

$$W_{\chi,f} \approx \sum_{A \in \mathrm{Cl}_D^+} \chi(A) \int_{\mathcal{C}_A} (R_0 f)(z) \, \frac{dz}{\Im(z)},$$

where the weight k raising operator is

$$R_k := \frac{k}{2} + (z - \overline{z}) \frac{\partial}{\partial z}.$$

The identity for $|W_{\chi,f}|^2$ proceeds in several steps:

(2) We express this cycle integral adèlically as a period integral,

$$\mathscr{P}_{\Omega}(\phi) := \int\limits_{\mathbb{A}_{\mathbb{Q}}^{\times} E^{\times} \backslash \mathbb{A}_{E}^{\times}} \phi(x) \Omega^{-1}(x) \, dx,$$

where $E = \mathbb{Q}(\sqrt{D})$, $\Omega : \mathbb{A}_{E}^{\times} \to \mathbb{C}^{\times}$ is the idèlic lift of the character χ, and $\phi : \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}$ is the adèlic lift of the weight 2 automorphic form $R_0 f$.

The identity for $|W_{\chi,f}|^2$ proceeds in several steps:

(3) We use Waldspurger's formula, in an explicit form due to Martin and Whitehouse (and with additional modifications of our own) to express $|\mathscr{P}_{\Omega}(\phi)|^2$ in terms of $L(1/2, f \otimes \Theta_{\chi})$.

Level-Aspect Equidistribution

In ongoing work, we are investigating a related problem: sparse equidistribution in the level aspect.

Theorem? (H.–Nordentoft (2020++))

Let D be a positive fundamental discriminant. For each positive squarefree integer q for which every prime dividing q splits in $\mathbb{Q}(\sqrt{D})$, there exists a hyperbolic orbifold of level q, $\Gamma_A(q)\backslash\mathcal{N}_A(q)$, associated to each ideal class $A \in \mathrm{Cl}_D^+$.

Previously, Heegner points of level q and closed geodesics of level q have been constructed; one replaces Γ with $\Gamma_0(q)$.

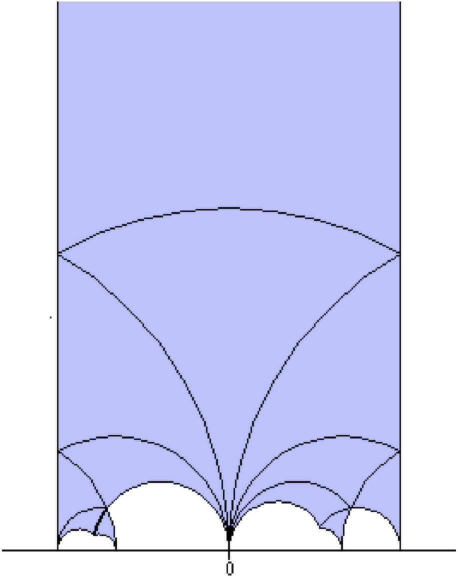
Question

Do these hyperbolic orbifolds equidistribute in the level aspect on $\Gamma_0(q)\backslash\mathbb{H}$?

Hybrid problem: q and D both varying!

Example: $\Gamma_0(6)\backslash\mathbb{H}$

(Image: Angelantonj–Florakis–Pioline)



Level-Aspect Equidistribution

We partition $\Gamma_0(q)\backslash\mathbb{H}$ into $q\prod_{p|q}(1+p^{-1})$ translates of $\Gamma\backslash\mathbb{H}$:

$$\Gamma_0(q)\backslash\mathbb{H} = \bigcup_{\omega_q\in\Gamma/\Gamma_0(q)} \omega_q^{-1}\Gamma\backslash\mathbb{H}.$$

As q grows, the volume of each translate is constant, whereas $\mathrm{vol}(\Gamma_0(q)\backslash\mathbb{H}) \gg q$: sparse equidistribution in the level aspect.

Level-Aspect Equidistribution

For simplicity, take $H = (\mathrm{Cl}_D^+)$, so that CH is a genus.

Theorem? (H.–Nordentoft (2020++))

Fix $\delta \geq 0$. Let D be a positive fundamental discriminant and let q be a positive squarefree integer with $q \leq D^{\delta}$ and such that every prime dividing q splits in E. For each such q, choose $\omega_q \in \Gamma/\Gamma_0(q)$. Then as $qD \to \infty$,

$$\frac{\mathrm{vol}(\Gamma_0(q)\backslash\mathbb{H})}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} \frac{\sum_{A \in CH} \mathrm{vol}(\Gamma_A(q)\backslash\mathcal{N}_A(q) \cap \Gamma_0(q)\omega_q^{-1}\Gamma\backslash\mathbb{H})}{\sum_{A \in CH} \mathrm{vol}(\Gamma_A(q)\backslash\mathcal{N}_A(q))} \to 1$$

for $\delta < \frac{1}{12}$ unconditionally and for $\delta < \frac{1}{4}$ assuming the generalised Lindelöf hypothesis.

Thank you!