Local Hecke algebras and Newforms

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Local Hecke algebras

G: locally compact p-adic group, S: compact subgroup, γ: character of S

$$H(G//S,\gamma) = \{f : G \longrightarrow \mathbb{C}, \text{ locally constant, compactly supported} : \\ f(kgk') = \overline{\gamma(k)}f(g)\overline{\gamma(k')} \text{ for all } k, k' \in S, g \in G\}$$

a $\mathbb{C}$-algebra under the convolution

$$f_1 * f_2(h) = \int_G f_1(g)f_2(g^{-1}h)dg = \int_G f_1(hg)f_2(g^{-1})dg$$

We want to describe such Hecke algebras with generators and relations for particular cases of G, S and γ.

Integral Weight: Fix a prime p.

Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $S = K_0(p^n) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) : c \in p^n\mathbb{Z}_p\}$, $\gamma = 1$

Denote by X_g the Characteristic function of double coset $K_0(p^n)gK_0(p^n)$.

$H(G//K_0(p^n)$, as a $\mathbb{C}$-vector space is spanned by X_g as g varies over the double coset representatives.

For $t \in \mathbb{Q}_p^*$, $d(t) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$, $w(t) = \begin{pmatrix} 0 & -1 \\ t & 0 \end{pmatrix}$ $z(t) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$;

for $s \in \mathbb{Q}_p$, $x(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$, $y(s) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$;

$$N = \{x(s) : s \in \mathbb{Q}_p\}, \qquad Z = \{z(t) : t \in \mathbb{Q}_p^*\}$$

$H(\mathrm{GL}_2(\mathbb{Q}_p)//K_0(p^n), 1)$

$n = 1$ **(Iwahori)** Using Iwahori-Bruhat decomposition we have

Lemma

Double coset representatives of G mod $K_0(p)$: $\;d(p^n)z(m),\; w(p^n)z(m)\;\; n,\; m \in \mathbb{Z}$.

Define $\mathcal{T}_n := X_{d(p^n)},\; \mathcal{U}_n := X_{w(p^n)},\; \mathcal{Z} := X_{z(p)} \in H(G//K_0(p)$.

Theorem (Iwahori)

$H(G//K_0(p)) = \langle \mathcal{U}_0, \mathcal{U}_1 \;:\; (\mathcal{U}_0 - p)(\mathcal{U}_0 + 1) = 0,\, \mathcal{U}_1^2 = \mathcal{Z},\, \mathcal{U}_0\mathcal{Z} = \mathcal{Z}\mathcal{U}_0,\; \mathcal{Z}\mathcal{U}_1 = \mathcal{U}_1\mathcal{Z}\rangle$.

Relations among $\mathcal{T}_n$ and $\mathcal{U}_n$: $\mathcal{U}_0 = \mathcal{T}_1\mathcal{U}_1$.

$n \geq 2$ Double coset representatives of $\mathrm{GL}_2(\mathbb{Z}_p)$ mod $K_0(p^n)$:

$$1,\; w(1),\; y(p),\; y(p^2),\; \ldots\; y(p^{n-1})$$

Define $\mathcal{U}_0 = X_{w(1)}$ and $\mathcal{V}_r = X_{y(p^r)}$ for $1 \leq r \leq n-1$ in $H(G//K_0(p^n))$.

Let $H(K//K_0(p^n))$ denotes the subspace of $H(G//K_0(p^n))$ consisting of functions supported on K.

Theorem

$H(K//K_0(p^n))$ is $n+1$ dimension commutative algebra with generators $\langle \mathcal{U}_0,\; \mathcal{Y}_1,\; \mathcal{Y}_2,\; \ldots \mathcal{Y}_n\rangle$ and defining relations:

Theorem

(a) $\mathcal{Y}_r^2 = p^{n-r}\mathcal{Y}_r$, $1 \le r \le n$

(b) $\mathcal{Y}_r\mathcal{Y}_l = p^{n-r}\mathcal{Y}_l = \mathcal{Y}_l\mathcal{Y}_r$, $l \le r$

(c) $\mathcal{U}_0\mathcal{Y}_r = p^{n-r}\mathcal{U}_0 = \mathcal{Y}_r\mathcal{U}_0$, $1 \le r \le n$

(d) $\mathcal{U}_0(\mathcal{U}_0 - p^n)(\mathcal{U}_0 + p^{n-1}) = 0$

Irreducible representations of $\mathrm{GL}_2(\mathbb{Z}_p)$ containing non-zero $K_0(p^n)$ fixed vector.

$I(n) := Ind_{K_0(p^n)}^{K} 1 = \{\phi : K \to \mathbb{C} \ : \ \phi(k_0 k) = \phi(k) \text{ for all } k_0 \in K_0(p^n), \ k \in K\}$

- right regular representation of K denoted by π_R, where $\pi_R(k)(\phi)(k') = \phi(k'k)$
- dimension $[K : K_0(p^n)] = p^{n-1}(p+1)$
- $I(n)^{K_0(p^n)} = H(K//K_0(p^n))$, so the dimension of $I(n)^{K_0(p^n)}$ is $n+1$.

By Frobenius Reciprocity an irreducible subrepresentation of $I(n)$ corresponds to an irreducible representation of K having a non zero $K_0(p^n)$ fixed vector.

Define $v_1 = \mathcal{U}_0 + \mathcal{Y}_1$, $v_2 = \mathcal{U}_0 - p\mathcal{Y}_1$ $w_k = \mathcal{Y}_k - p\mathcal{Y}_{k+1}$ for $1 \le k \le n-1$.

The representation $I(n)$ is a sum of $n+1$ irreducible subspaces given by:

$S_1 = \mathrm{Span}(\pi_R(K)v_1)$, $S_2 = \mathrm{Span}(\pi_R(K)v_2)$, $T_k = \mathrm{Span}(\pi_R(K)w_k)$, $1 \le k \le n-1$;

$\dim(S_1) = 1$, $\dim(S_2) = p$, $\dim(T_k) = p^{k-1}(p^2-1)$.

T_{n-1} is the unique irreducible representation of K such that T_{n-1} has a $K_0(p^n)$ fixed vector w_{n-1} but does not have $K_0(p^k)$ fixed vector for $k < n$;
w_{n-1} is the Casselman's new vector.

Full Hecke algebra $p = 2$, $n = 2$

Lemma

Double coset representatives of $\mathrm{GL}_2(\mathbb{Q}_2)$ *mod* $K_0(4)$ *(up to central elements* $z(t)$*):*

$$d(2^n),\ w(2^n) \quad \textit{for } n \in \mathbb{Z}, \quad d(2^n)y(2) \quad \textit{for } n \geq 0, \quad y(2)d(2^{-n}) \quad \textit{for } n \geq 1,$$

$$y(2)w(2^n),\ w(2^n)y(2),\ y(2)w(2^n)y(2) \quad \textit{for } n \geq 2.$$

Define

$$\mathcal{T}_n = X_{d(2^n)}, \quad \mathcal{U}_n = X_{w(2^n)}, \quad \mathcal{V} = X_{y(2)} \quad \text{and} \quad \mathcal{Z} = X_{z(2)}$$

be elements of the Hecke algebra $H(\mathrm{GL}_2(\mathbb{Q}_2)//K_0(4))$

Theorem

The Hecke algebra

$$H(G//K_0(4))/\langle\mathcal{Z}\rangle$$

is generated by $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{V}$ *with the defining relations:*

1. $\mathcal{U}_1^2 = 1 + \mathcal{V}$,
2. $\mathcal{U}_2^2 = 1$,
3. $\mathcal{U}_1\mathcal{V} = \mathcal{V}\mathcal{U}_1 = \mathcal{U}_1$,
4. $\mathcal{U}_2\mathcal{V}\mathcal{U}_2 = \mathcal{V}\mathcal{U}_2\mathcal{V}$.

Whittaker functions associated to new vectors

There is a bijection between finite dimensional irreducible representations of H and irreducible admissible representations of G with trivial central character and a $K_0(4)$ fixed vector.

Theorem

Every irreducible finite dimensional representation of Hecke algebra $H(G//K_0(4))$ modulo center is of dimension at most 3. These irreducible representations are given by one parameter family of three dimensional representation corresponding to the spherical representations, two 2-dimensional representations corresponding to the Steinberg representations, two characters corresponding to supercuspidal representations and characters corresponding 1 dimensional representation.

The proof uses action of two central elements of $H(G//K_0(4))$:
$C_1 = \mathcal{V} + \mathcal{U}_2 + \mathcal{V}\mathcal{U}_2\mathcal{V} - \mathcal{U}_2\mathcal{V} - \mathcal{V}\mathcal{U}_2$;
$C_2 =$
$\mathcal{U}_1\mathcal{U}_2 + \mathcal{U}_2\mathcal{U}_1 + \mathcal{U}_1\mathcal{U}_2\mathcal{V} + \mathcal{V}\mathcal{U}_2\mathcal{U}_1 - \mathcal{U}_1 - \mathcal{U}_2\mathcal{U}_1\mathcal{U}_2 - \mathcal{V}\mathcal{U}_2\mathcal{U}_1\mathcal{U}_2\mathcal{V} + \mathcal{U}_2\mathcal{U}_1\mathcal{U}_2\mathcal{V} + \mathcal{V}\mathcal{U}_2\mathcal{U}_1\mathcal{U}_2$.

It is well-known (Schmidt) that 2-adic representation appearing in the automorphic representation of a newform of level $4M$, M odd is supercuspidal. We can show that there are exactly two such supercuspidal the choice of which depends on the sign of Atkin-Lehner involution of the form. We do this by computing the whittaker function of the new vector for these supercuspidal representations. Our method does not require any realization of supercuspidal representations.

Translation - adelic to classical

$A_{2k}(N)$: Space of automorphic forms of weight $2k$ and level N

Gelbart: $A_{2k}(N) \xrightarrow{\sim} S_{2k}(N)$ inducing isomorphism of endomorphism algebras

$$q : \mathrm{End}_{\mathbb{C}}(A_{2k}(N)) \xrightarrow{\sim} \mathrm{End}_{\mathbb{C}}(S_{2k}(N))$$

If $N = p^n M,\ (p, M) = 1$ then $H(G//K_0(p^n), 1)$ is a subalgebra of $\mathrm{End}_{\mathbb{C}}(A_{2k}(N))$:

$$\text{for } \mathcal{T} \in H(G//K_0(p^n), 1) \text{ and } \Phi \in A_{2k}(N), \quad \mathcal{T}(\Phi)(g) = \int_G \mathcal{T}(x)\Phi(gx)dx$$

Proposition

For $f \in S_{2k}(N)$ and $1 \leq r \leq n-1$,

1. $\mathcal{T}_1 \longmapsto p^{1-k}U_p$
2. $\mathcal{U}_n \longmapsto W_{p^n}$ *(Atkin-Lehner)*
3. $q(\mathcal{V}_r)(f)(z) = \sum_{s \in \mathbb{Z}_p^*/1+p^{n-r}\mathbb{Z}_p} f|_{2k}A_s$ *where* $A_s \in \mathrm{SL}_2(\mathbb{Z})$ *is of the form* $\begin{pmatrix} a_s & b_s \\ p^r M & p^{n-r} - sM \end{pmatrix}$.
4. *If* $f \in S_{2k}(\Gamma_0(p^r M))$ *then* $q(\mathcal{V}_r)(f) = p^{n-r-1}(p-1)f$*, hence* $q(\mathcal{Y}_r)(f) = p^{n-r}f$.

New Space

$n = 1$: $Q_p := q(\mathcal{U}_0)$. Since $\mathcal{U}_0 = \mathcal{T}_1\mathcal{U}_1$,

$$Q_p = p^{1-k} U_p W_p, \qquad (Q_p - p)(Q_p + 1) = 0$$

$Q'_p :=$ conjugate of Q_p by W_p

$n \geq 2$: $R_{p^n} := q(\mathcal{V}_{n-1})$. Since $\mathcal{V}_{n-1} = \mathcal{Y}_{n-1} - 1$,

$$(R_{p^n} - (p-1))(R_{p^n} + 1) = 0$$

$R'_{p^n} :=$ conjugate of R_{p^n} by W_{p^n}

Theorem

Let $f \in S_{2k}(N)$. Then,

$$f \in S_{2k}^{\mathrm{new}}(N) \iff \begin{array}{l} Q_p(f) = -f = Q'_p f \text{ for all } p \| N \\ R_{q^n}(f) = -f = R'_{q^n} f \text{ for all } q^n \| N \text{ with } n \geq 2 \end{array}$$

Sketch: Let $N = p_1 p_2 \cdots p_r q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_s^{\alpha_s}$ with $p_i \neq q_j$ and $\alpha_j \geq 2$ for all j

- $S_{2k}^{\mathrm{new}}(N)$ is contained in the common -1-eigenspace of Q_{p_i}, Q'_{p_i}, $R_{{q_j}^{\alpha_j}}$, $R'_{{q_j}^{\alpha_j}}$
- $S_{2k}(N/p_i)$ is contained in the p_i eigenspace of Q_{p_i}
  $V(p_i)S_{2k}(N/p_i)$ is contained in the p_i eigenspace of Q'_{p_i}
  $S_{2k}(N/q_j)$ is contained in the $q_j - 1$ eigenspace of $R_{{q_j}^{\alpha_j}}$
  $V(q_j)S_{2k}(N/q_j)$ is contained in the $q_j - 1$ eigenspace of $R'_{{q_j}^{\alpha_j}}$
- Q_{p_i}, Q'_{p_i}, $R_{{q_j}^{\alpha_j}}$, $R'_{{q_j}^{\alpha_j}}$ are self-adjoint wrt Petersson inner product.

Particular case:

$$\begin{aligned} S_{2k}(\Gamma_0(4)) =& (S_{2k}(\Gamma_0(1)) \oplus q(\mathcal{U}_1)S_{2k}(\Gamma_0(1)) \oplus q(\mathcal{U}_2)S_{2k}(\Gamma_0(1))) \\ & \oplus (S_{2k}^{\mathrm{new}}(\Gamma_0(2)) \oplus q(\mathcal{U}_2)S_{2k}^{\mathrm{new}}(\Gamma_0(2))) \oplus S_{2k}^{\mathrm{new}}(\Gamma_0(4)). \end{aligned}$$

Recall that $\mathcal{U}_1$, $\mathcal{U}_2$ are the characteristic functions of the double cosets of $\left(\begin{smallmatrix} 0 & -1 \\ 2 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & -1 \\ 4 & 0 \end{smallmatrix}\right)$ respectively.

Back to Whittaker function

Let W_f be a Whittaker function attached to a newform f of level $4M$, M odd. Then the 2-adic component $W = W_{f,2}$ satisfies

$$W(nzgk) = \psi(n)W(g),\ \ n \in N, z \in Z,\ \ g \in G, k \in K_0(4)$$

$$\mathcal{U}_0(W) = -W, \qquad \mathcal{U}_1(W) = \epsilon W;$$

here ψ is a character of N, and $\epsilon = \pm 1$.

Since $G = BK_0(4) \cup Bw_0K_0(4) \cup By(2)K_0(4)$,

Theorem

Upto a normalization by a scalar W is given by

$$W(g) = \begin{cases} 1 & g = d(t) \ \mathit{with}\ |t| = 1 \\ -1 & g = d(t)y(2) \ \mathit{with}\ |t| = 1 \\ \epsilon & g = d(t)w_0 \ \mathit{with}\ |t| = 4 \\ 0 & g = d(t) \ \mathit{with}\ |t| \neq 1, g = d(t)y(2) \ \mathit{with}\ |t| \neq 1, \\ & \mathit{or}\ g = d(t)w_0 \ \mathit{with}\ |t| \neq 4 \end{cases}$$

Thus, 2-adic representations corresponding to f are given by exactly two supercuspidals each of which is determined by the above Whittaker functions, which only depend on the sign of Atkin-Lehner involution of the form.

Half-integral weight setting

$S_{k+1/2}(4N)$: Space of cusp forms of weight $k+1/2$, level $4N$

- family of Hecke operators $\{T_{p^2}\}$, U_{p^2}
- $S_{k+1/2}(4N)$ has a basis of common eigenforms under $\{T_{p^2}\}$ for $(p,4N)=1$

Shimura(1973) Family of liftings: for a positive square-free integer t,

$$\mathrm{Sh}_t : S_{k+1/2}(4N) \longrightarrow M_{2k}(2N)$$

$$\sum_{n=1}^{\infty} a_n q^n \longmapsto \sum_{n=1}^{\infty} \left(\sum_{\substack{d|n, \\ (d,2M)=1}} \left(\frac{-1}{d}\right)^k \left(\frac{t}{d}\right) d^{k-1} a\left(t\frac{n^2}{d^2}\right) \right) q^n$$

- $T_p(\mathrm{Sh}_t(f)) = \mathrm{Sh}_t(T_{p^2}(f))$ for all primes p coprime to $2N$, (Hecke-equivariance)
  $U_p(\mathrm{Sh}_t(f)) = \mathrm{Sh}_t(U_{p^2}(f))$ for all primes p dividing $2N$.

Niwa(1977), Ueda(1988) Let M be odd, square-free. There exists an isomorphism

$$\psi : S_{k+1/2}(8M) \xrightarrow{\sim} S_{2k}(4M)$$

with $T_p(\psi(f)) = \psi(T_{p^2}(f))$ for all primes coprime to $2M$.
For $f \in S_{k+1/2}(4)$ further $U_2(\psi(f)) = \psi(U_4(f))$.

Main result (Half-integral weight)

Kohnen(1980, 1981)

$$S^+_{k+1/2}(4M) := \{f = \sum_{n=1}^{\infty} a_n q^n \in S_{k+1/2}(4M) : a_n = 0 \text{ for } (-1)^k n \equiv 2, 3 \pmod 4\}$$

$$\begin{array}{l} S^+_{k+1/2}(4M) \\ \quad \cup\!\shortmid \\ S^{+,\mathrm{new}}_{k+1/2}(4M) \xrightarrow{\sim} S^{\mathrm{new}}_{2k}(M) \end{array}$$

Ueda-Yamana(2010) defined $S^+_{k+1/2}(8M)$ to consist of $f = \sum_{n=1}^{\infty} a_n q^n \in S_{k+1/2}(\Gamma_0(8M))$ such that $a_n = 0$ for $(-1)^k n \equiv 2, 3 \pmod 4$ and proved that $S^{+,\mathrm{new}}_{k+1/2}(8M) \xrightarrow{\sim} S^{\mathrm{new}}_{2k}(2M)$

Theorem

Let M be odd, square-free. Let $S^-_{k+1/2}(8M) \subseteq S_{k+1/2}(8M)$ be the common -1-eigenspace of operators $\widetilde{Q}_p$, $\widetilde{Q}'_p$ for all $p \mid M$ and that of operators $\widetilde{V}_4$, $\widetilde{V}'_4$.

- *$S^-_{k+1/2}(8M)$ has a basis of eigenforms under T_{q^2}, $(q, 2M) = 1$ and U_{p^2}, $p \mid 2M$*
- $S^-_{k+1/2}(8M) \xrightarrow{\sim} S^{\mathrm{new}}_{2k}(4M)$
- *$S^-_{k+1/2}(8M)$ has multiplicity-one in the full space $S_{k+1/2}(8M)$*
- *If $f = \sum_{n=1}^{\infty} a_n q^n \in S^-_{k+1/2}(8M)$ then $a_n = 0$ for $(-1)^k n \equiv 0, 1 \pmod 4$*

The operators $\widetilde{Q}_p$, $\widetilde{Q}'_p$, $\widetilde{V}_4$, $\widetilde{V}'_4$ come from local Hecke algebras of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$.

$\tilde{G} = \widetilde{\mathrm{SL}_2}(\mathbb{Q}_p)$: non-trivial central extension of $\mathrm{SL}_2(\mathbb{Q}_p)$ by $\{\pm 1\}$ (Kubota 2-cocycle)

Let $\overline{K_0(p^n)}$ be the subgroup of $\tilde{G}$ which is the inverse image of $K_0(p^n) \cap \mathrm{SL}_2(\mathbb{Q}_p)$ under the natural map from $\tilde{G}$ to $\mathrm{SL}_2(\mathbb{Q}_p)$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \tilde{G} & \longrightarrow & \mathrm{SL}_2(\mathbb{Q}_p) & \longrightarrow & 1 \\ & & \{(I, \pm 1)\} & & (g, \pm 1) & \longmapsto & g & & \end{array}$$

For $t \in \mathbb{Q}_p^\times$, $\quad w(t) = \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}$, $h(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$

$T = \{(h(t), \epsilon) : t \in \mathbb{Q}_p^\times,\ \epsilon = \pm 1\}$, $\qquad N_{\widetilde{\mathrm{SL}_2}(\mathbb{Q}_p)}(T) = \{(h(t), \epsilon),\ (w(t), \epsilon)\}$

$\overline{g} := (g, 1)$

p odd, $n = 1$:

$\overline{K_0(p)} = K_0(p) \times \{\pm 1\}$, $\gamma : (A, \epsilon) \longmapsto \tilde{\gamma}(A)\epsilon$ where $\tilde{\gamma}$ is a character of $K_0(p)/K_1(p)$
Double coset representatives of $\tilde{G}$ mod $\overline{K_0(p)}$: $(h(p^n), 1),\ (w(p^{-n}), 1),\ n \in \mathbb{Z}$.

Proposition

If γ is quadratic, $H(G//\overline{K_0(p)},\ \gamma)$ is supported on $\overline{h(p^n)}$, $\overline{w(p^{-n})}$, $n \in \mathbb{Z}$

Extend γ to $N_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)}(T)$: for $t = p^n u \in \mathbb{Q}_p^\times$,

$$\gamma(\overline{h(t)}) = \gamma(\overline{h(u)}) \begin{cases} 1 & \text{if } n \text{ is even} \\ \varepsilon_p \left(\frac{u}{p}\right) & \text{if } n \text{ is odd;} \end{cases} \quad \gamma(\overline{w(t)}) = \gamma(\overline{h(u)}) \begin{cases} 1 & \text{if } n \text{ is even} \\ \varepsilon_p \left(\frac{-u}{p}\right) & \text{if } n \text{ is odd.} \end{cases}$$

Here $\epsilon_p = 1$ if $p \equiv 1 \pmod 4$ else $\epsilon_p = i$, so $\epsilon_p^2 = \left(\frac{-1}{p}\right)$.

Define $\mathcal{T}_n$, $\mathcal{U}_n \in H(G//\overline{K_0(p)},\ \gamma)$ resp. precisely supported on $\overline{h(p^n)}$, $\overline{w(p^{-n})}$:

$$\mathcal{T}_n(\overline{h(p^n)}) = \overline{\gamma(\overline{h(p^n)})}, \qquad \mathcal{U}_n(\overline{w(p^{-n})}) = \overline{\gamma(\overline{w(p^{-n})})}$$

Theorem

$$H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)//\overline{K_0(p)},\ 1) = \langle \mathcal{U}_0,\ \mathcal{U}_1\ :\ (\mathcal{U}_0 - p)(\mathcal{U}_0 + 1) = 0, \quad \mathcal{U}_1^2 = p\rangle$$

$$H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)//\overline{K_0(p)},\ \left(\frac{\cdot}{p}\right)) = \langle \mathcal{U}_0,\ \mathcal{U}_1\ :\ (\mathcal{U}_1 - \epsilon_p p)(\mathcal{U}_1 + \epsilon_p) = 0, \quad \mathcal{U}_0^2 = \left(\frac{-1}{p}\right) p\rangle$$

Thus,

$$H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)//\overline{K_0(p)}, 1) \cong H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)//\overline{K_0(p)}, \left(\frac{\cdot}{p}\right)) \cong \textit{Iwahori Hecke alg. of } \mathrm{PGL}_2(\mathbb{Q}_p)$$

$p = 2,\ n = 2,\ n = 3$:

$M_2 :=$ center of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2) = \langle(-I, 1)\rangle$, cyclic of order 4

$n = 2$ (**Loke-Savin**):

$G = \widetilde{\mathrm{SL}}_2(\mathbb{Q}_2), \quad S = \overline{K_0(4)} = K_1(4) \times M_2, \qquad \gamma :\ (-I, 1) \longmapsto \zeta_4;\ K_1(4) \longmapsto 1$

Extend γ to $N_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)}(T)$: $\quad \overline{h(2^n)} \mapsto 1 \qquad \overline{w(1)} \mapsto \frac{1+\gamma(-I,1)}{\sqrt{2}}$

Define $\mathcal{T}_n,\ \mathcal{U}_n \in H(G//\overline{K_0(4)},\ \gamma)$ as before.

Theorem (Loke-Savin, 2010)

$H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(4)},\ \gamma) = \langle \mathcal{U}_0,\ \mathcal{U}_1\ :\ (\mathcal{U}_0 - 2\sqrt{2})(\mathcal{U}_0 + \sqrt{2}) = 0, \quad \mathcal{U}_1^2 = 1\rangle$

$H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(4)},\ \gamma) \cong$ *Iwahori Hecke alg. of* $\mathrm{PGL}_2(\mathbb{Q}_2)$

$n=3$: $G=\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$, $\quad S=\overline{K_0(8)}$; $\quad S/(K_1(8)\times M_2)=\langle\overline{h(5)}\rangle$ order 2

$\chi_1,\ \chi_2:\quad (-I,1)\mapsto\zeta_4;\quad K_1(8)\mapsto 1;\quad \chi_1(\overline{h(5)})=1,\quad \chi_2(\overline{h(5)})=-1$

Extend $\chi_1,\ \chi_2$, define $\mathcal{T}_n,\ \mathcal{U}_n\in H(G//\overline{K_0(8)},\ \chi_i)$ as before, and define

$$\mathcal{V}\in H(G//\overline{K_0(8)},\ \chi_1) \text{ supported on } \overline{\left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right)} \text{ such that } \mathcal{V}(\overline{\left(\begin{smallmatrix}1&0\\4&1\end{smallmatrix}\right)})=1,$$

$$\mathcal{Z}\in H(G//\overline{K_0(8)},\ \chi_2) \text{ supported on } \overline{\begin{pmatrix}1&1/2\\0&1\end{pmatrix}} \text{ such that } \mathcal{Z}(\overline{\begin{pmatrix}1&1/2\\0&1\end{pmatrix}})=1.$$

Theorem

Let $\hat{\mathcal{U}}_1=\mathcal{U}_1/\sqrt{2},\quad \hat{\mathcal{U}}_2=\mathcal{U}_2/\sqrt{2}$ *then*

$$H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)},\ \chi_1)=\langle\hat{\mathcal{U}}_1,\ \hat{\mathcal{U}}_2,\ \mathcal{V}:\quad \hat{\mathcal{U}}_1^2=1+\mathcal{V},\quad \hat{\mathcal{U}}_2^2=1$$
$$\hat{\mathcal{U}}_2\mathcal{V}\hat{\mathcal{U}}_2=\mathcal{V}\hat{\mathcal{U}}_2\mathcal{V},\quad \hat{\mathcal{U}}_1\mathcal{V}=\mathcal{V}\hat{\mathcal{U}}_1=\hat{\mathcal{U}}_1\rangle$$

$$H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)},\ \chi_2)=\langle\hat{\mathcal{U}}_1,\ \hat{\mathcal{U}}_2,\ \mathcal{Z}:\quad \textit{relations as above with roles of}$$
$$\hat{\mathcal{U}}_1,\ \hat{\mathcal{U}}_2 \textit{ interchanged and } \mathcal{Z} \textit{ replaces } \mathcal{V}\rangle$$

Further,

$$H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)},\chi_1)\cong H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)},\chi_2)\cong H(\mathrm{GL}_2(\mathbb{Q}_2)//H_0(4))$$

The last statements of the theorems are instances of “local Shimura correspondence”.

Gelbart-Waldspurger: $q : \mathrm{End}_{\mathbb{C}}(A_{k+1/2}(4N, \left(\frac{-1}{\cdot}\right)^k)) \xrightarrow{\sim} \mathrm{End}_{\mathbb{C}}(S_{k+1/2}(4N))$

For $N = 2^\nu M$, M odd, $\nu \geq 2$ we obtain the following operators on $S_{k+1/2}(N)$:

- For M square-free

$H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)//\overline{K_0(p)}, 1)$:

$$\mathcal{U}_0 \longmapsto \widetilde{Q}_p, \qquad \mathcal{U}_1/\sqrt{p} \longmapsto \widetilde{W}_{p^2}$$

$$(\widetilde{Q}_p - p)(\widetilde{Q}_p + 1) = 0, \qquad (\widetilde{W}_{p^2})^2 = 1$$

$\widetilde{Q}'_p$: conjugate of $\widetilde{Q}_p$ by $\widetilde{W}_{p^2}$

- On $S_{k+1/2}(4M)$, M odd

$H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(4)}, \gamma)$:

$$\mathcal{U}_0/\sqrt{2} \longmapsto \widetilde{Q}_2, \qquad \mathcal{U}_1 \longmapsto \widetilde{W}_4$$

$$(\widetilde{Q}_2 - 2)(\widetilde{Q}_2 + 1) = 0, \qquad (\widetilde{W}_4)^2 = 1$$

$\widetilde{Q}'_2$: conjugate of $\widetilde{Q}_2$ by $\widetilde{W}_4$

- On $S_{k+1/2}(8M)$, M odd

$H(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)//\overline{K_0(8)}, \chi_1)$:

$$\mathcal{V} \longmapsto \widetilde{V}_4, \qquad \hat{\mathcal{U}}_2 \longmapsto \widetilde{W}_8$$

$$(\widetilde{V}_4)^2 = 1, \qquad (\widetilde{W}_8)^2 = 1$$

$\widetilde{V}'_4$: conjugate of $\mathcal{V}$ by $\widetilde{W}_8$

- The operators defined above are self-adjoint wrt the Petersson inner product.
- Study the corresponding eigenspaces

Theorem

Let M be odd, square-free.

- *Let $S^-_{k+1/2}(4M) \subseteq S_{k+1/2}(4M)$ be the common -1-eigenspace of operators $\widetilde{Q}_p$, $\widetilde{Q}'_p$ for all $p \mid 2M$. Then*

$$S^-_{k+1/2}(4M) \xrightarrow{\sim} S^{\mathrm{new}}_{2k}(2M).$$

- *Let $S^-_{k+1/2}(8M) \subseteq S_{k+1/2}(8M)$ be the common -1-eigenspace of operators $\widetilde{Q}_p$, $\widetilde{Q}'_p$ for $p \mid M$ and $\widetilde{V}_4$, $\widetilde{V}'_4$. Then*

$$S^-_{k+1/2}(8M) \xrightarrow{\sim} S^{\mathrm{new}}_{2k}(4M).$$

Fourier coefficient condition

Kohnen defined function

$$P_8(f) = f|[\xi + \xi^{-1}]_{k+1/2} \ \text{ where } \xi = \left(\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, e^{\pi i/4} \right).$$

We have

$$P_8\left(\sum_n a_n q^n\right) = \sqrt{2}\left(\frac{2}{2k+1}\right)\left(\sum_n^{(1)} a_n q^n - \sum_n^{(2)} a_n q^n\right)$$

where $\sum_n^{(1)}$, respectively $\sum_n^{(2)}$, runs over n with $(-1)^k n \equiv 0, 1 \pmod 4$, respectively $(-1)^k n \equiv 2, 3 \pmod 4$.

Proposition

$\frac{1}{\sqrt{2}}\left(\frac{2}{2k+1}\right) P_8 = \widetilde{V}_4 \widetilde{W}_8 \widetilde{V}_4 = \widetilde{W}_8 \widetilde{V}_4 \widetilde{W}_8 = \widetilde{V}'_4.$

Consequently, $S^+_{k+1/2}(8M)$ is the $+1$ eigenspace of $\widetilde{V}'_4$. The -1 eigenspace of $\widetilde{V}'_4$ consists of f such that $a_n = 0$ for $(-1)^k n \equiv 0, 1 \pmod 4$.

Particular case:

$$\begin{aligned} S_{k+1/2}(\Gamma_0(8)) =& (A^+(4) \oplus q(\mathcal{U}_1)A^+(4) \oplus q(\mathcal{U}_2)A^+(4)) \\ & \oplus (S^-(4) \oplus q(\mathcal{U}_2)S^-(4)) \oplus S^-(8). \end{aligned}$$

Here $\mathcal{U}_1$, $\mathcal{U}_2$ are elements in the Hecke algebra $H(\tilde{G}, \overline{K_0(8)}, \chi_1)$ coming from $w(2^{-1})$, $w(2^{-2})$ respectively and

$$A^+(4) = \widetilde{W}_4 S^+(4) = q(\mathcal{U}_1)S^+(4).$$

Example

The space $S_{3/2}(\Gamma_0(152))$ is 8-dimensional and there are four primitive Hecke eigenforms of weight 2 and level dividing 76, namely F_{19} of level 19, G_{38}, H_{38} of level 38 and K_{76} of level 76. We have

$$S_{3/2}(\Gamma_0(152)) = S_{3/2}(152, F_{19}) \oplus S_{3/2}(152, G_{38}) \oplus S_{3/2}(152, H_{38}) \oplus S_{3/2}(152, K_{76}).$$

We compute the Shimura decomposition. $S_{3/2}(152, F_{19})$ is 3-dimensional space and is spanned by

$$\begin{aligned}
f_1 &= q + q^5 - 2q^6 - q^9 - q^{17} + 2q^{25} + 2q^{30} + 2q^{42} - 3q^{45} + O(q^{50}),\\
f_2 &= q^4 - 2q^{11} - 2q^{16} + 2q^{19} + q^{20} - 2q^{24} + 3q^{28} + 2q^{35} - q^{36} + O(q^{40}),\\
f_3 &= q^7 - q^{11} - 2q^{16} + q^{19} + 2q^{28} + q^{35} - 2q^{39} - q^{43} + 2q^{44} - q^{47} + O(q^{50}),
\end{aligned}$$

$S_{3/2}(152, G_{38})$ is 2-dimensional space and is spanned by

$$\begin{aligned}
g_1 &= q - 2q^5 + q^6 + 2q^9 - q^{17} - q^{25} - 3q^{26} - 4q^{30} + 3q^{38} + 5q^{42} + O(q^{50}),\\
g_2 &= q^4 + q^7 - q^{16} - 2q^{20} - 3q^{23} + q^{24} - q^{28} + 2q^{36} + q^{39} + 2q^{47} + O(q^{50}),
\end{aligned}$$

$S_{3/2}(152, H_{38})$ is 2-dimensional space and is spanned by

$$\begin{aligned}
h_1 &= q^2 + 2q^{10} - 3q^{13} - q^{14} - 2q^{18} - q^{21} + 2q^{22} + q^{29} + O(q^{30})\\
h_2 &= q^3 - q^8 + q^{12} - q^{19} - q^{27} - q^{32} - 2q^{40} + q^{48} + O(q^{50})
\end{aligned}$$

Example

$S_{3/2}(152, K_{76})$ is 1-dimensional space and is spanned by

$$k_1 = q^2 - q^{10} - q^{14} + q^{18} + 2q^{21} - q^{22} - 2q^{29} - 2q^{33} - q^{34} + 2q^{37} + q^{38} - 2q^{41} + O(q^{50}).$$

The Kohnen plus space $S^+_{3/2}(152)$ is 4-dimensional and it is spanned by $\{f_2, f_3, g_2, h_2\}$. We further note that $S_{3/2}(76, F_{19})$ is 2-dimensional and spanned by $\{f_1 + f_3, f_2 - f_3\}$ and $S^-_{3/2}(76)$ is 2-dimensional and spanned by $\{g_1 - g_2, h_1 - h_2\}$. The minus space at level 152, $S^-_{3/2}(152)$, is 1-dimensional and spanned by k_1, and is Shimura equivalent to K_{76}. Note that k_1 satisfies the Fourier coefficient condition as noted in Corollary .

Thank you very much!