Singular moduli for real quadratic fields

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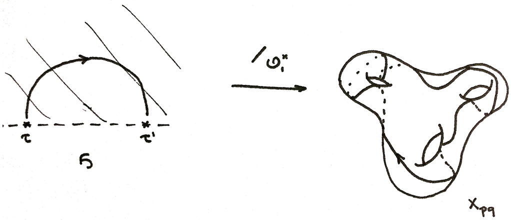


Joint work with Henri Darmon, Alice Pozzi, Yingkun Li

Outline

Outline

Consider the sine function

$$\sin \pi z = \pi z \prod_{n \in \mathbf{Z} \setminus \{0\}} \left(1 - \frac{z}{n}\right).$$

For any $z \in \mathbf{Q}$, the value $\sin \pi z$ is algebraic.

- $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$
- $\sin \frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$
- $\sin \frac{\pi}{16} = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}$

Theorem (Kronecker–Weber)

All finite abelian extensions of $\mathbf{Q}$ are (essentially) generated by combinations of

$$\sin(\pi z), \qquad z \in \mathbf{Q}$$

Consider the j-function

$$\begin{aligned} j(q) &= q^{-1} + 744 + 196884q + 21493760q^2 + \ldots \qquad q = e^{2\pi iz} \\ &= q^{-1}(1-q)^{-744}(1-q^2)^{80256}(1-q^3)^{-12288744}\cdots \\ &= \prod_{\gamma \in \mathrm{SL}_2(\mathbf{Z})}^{\mathrm{reg}} \frac{\zeta_3 - \gamma z}{\zeta_3 - \gamma \bar{z}} \end{aligned}$$

For any $z \in K$ imaginary quadratic, the value $j(z)$ is algebraic.

$$j(\sqrt{-14}) = 2^3\left(323 + 228\sqrt{2} + (231 + 161\sqrt{2})\sqrt{2\sqrt{2} - 1}\right)^3$$

Theorem (Kronecker–Weber)

All finite abelian extensions of K are (essentially) generated by combinations of

$$\begin{cases} \sin(\pi z) & z \in \mathbf{Q}, \\ j(z) & z \in K. \end{cases}$$

Singular moduli

Singular moduli, like Weber's example:

$$j(\sqrt{-14}) = 2^3\left(323 + 228\sqrt{2} + (231 + 161\sqrt{2})\sqrt{2\sqrt{2} - 1}\right)^3,$$

have several interesting features. In this case:

- It generates the Hilbert class field over $K = \mathbf{Q}(\sqrt{-14})$.
- It has an interesting prime factorisation (Gross–Zagier)

$$\mathrm{Nm}\, j(\sqrt{-14}) = 2^{24} \cdot 11^6 \cdot 17^3 \cdot 41^3$$

- It is closely related to the first derivative of the diagonal restriction of a real analytic family of Eisenstein series (Gross–Zagier).

Let τ_1, τ_2 be two CM points in $\mathcal{H}_\infty = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$.
Gross and Zagier (1985) find explicit formula for

$$\mathrm{Nm}\left(j(\tau_1) - j(\tau_2)\right) \quad \in \mathbf{Z}$$

- Algebraic proof: Its q-adic valuation is given in terms of arithmetic intersection of (oriented optimal) embeddings

$$\mathbf{Q}(\tau_1), \mathbf{Q}(\tau_2) \;\hookrightarrow\; B_{\infty q} = \text{Quat alg conductor } \infty q$$

- Analytic proof: Fourier coefficients of Hecke's real analytic Eisenstein series over F, attached to the genus character defined by $\mathbf{Q}(\tau_1, \tau_2)$.

$$\begin{tikzcd}
 & \mathbf{Q}(\tau_1,\tau_2) \arrow[dl, dash] \arrow[d, dash, "\chi"] \arrow[dr, dash] & \\
\mathbf{Q}(\tau_1) \arrow[dr, dash] & F \arrow[d, dash] & \mathbf{Q}(\tau_2) \arrow[dl, dash] \\
 & \mathbf{Q} &
\end{tikzcd}$$

Real analytic Hilbert Eisenstein series $E_s(z_1, z_2)$ defined by Hecke:

$$\sum_{[\mathfrak{a}]\in \mathrm{Cl}(\Delta_1\Delta_2)} \chi(\mathfrak{a})\,\mathrm{Nm}(\mathfrak{a})^{1+2s} \sum_{(m,n)\in\mathfrak{a}^2/U}{}' \frac{y_1^s y_2^s}{(mz_1+n)(m'z_2+n')|mz_1+n|^{2s}|m'z_2+n'|^{2s}}$$

Gross–Zagier consider its diagonal restriction $E_s(z, z)$ and show

- When $s = 0$, have $E_s(z, z) = 0$,
- The holomorphic projection of the first derivative

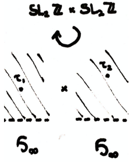
  $$\left(\frac{\partial}{\partial s} E_s(z, z)\right)\Bigg|_{s=0}^{\mathrm{hol}}$$

  has Fourier coefficients related to $\log \mathrm{Nm}\left(j(\tau_1) - j(\tau_2)\right)$.
- The holomorphic projection must vanish!
  $\Rightarrow$ formula for $\mathrm{Nm}\left(j(\tau_1) - j(\tau_2)\right)$.

Today: Analogues for real quadratic fields.

(Gross–Zagier) Let τ_1, τ_2 be CM points, consider

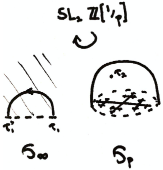
$$J_\infty(\tau_1, \tau_2) = j(\tau_1) - j(\tau_2) \ \in \ \overline{\mathbf{Q}}$$



- Related to real analytic Eisenstein family.
- $\mathrm{ord}_q\, J_\infty(\tau_1, \tau_2) =$ Intersection multiplicities
$$\mathbf{Q}(\tau_1), \mathbf{Q}(\tau_2) \ \hookrightarrow \ B_{\infty q}.$$

(With Darmon) Let τ_1, τ_2 be RM points, construct

$$J_p(\tau_1, \tau_2) \ \overset{?}{\in} \ \overline{\mathbf{Q}}$$

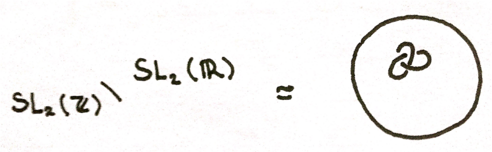


- Related to p-adic analytic families.
- $\mathrm{ord}_q\, J_p(\tau_1, \tau_2) \overset{?}{=}$ Intersection multiplicities
$$\mathbf{Q}(\tau_1), \mathbf{Q}(\tau_2) \ \hookrightarrow \ B_{pq}.$$

Outline

The work of Duke–Imamoḡlu–Tóth

Inspiration comes from work of Duke–Imamoḡlu–Tóth on linking numbers of modular geodesics.



If $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ is hyperbolic, get associated knot

$$\begin{aligned} \mathrm{Kn}(\gamma) &\hookrightarrow \mathrm{SL}_2(\mathbf{Z})\backslash \mathrm{SL}_2(\mathbf{R}) \\ t &\mapsto \mathrm{SL}_2(\mathbf{Z}) g \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}, \quad \text{where } g^{-1}\gamma g = \text{diagonal} \end{aligned}$$

- Linking $\mathrm{Kn}(\gamma)$ and trefoil $\leftrightarrow$ Dedekind–Rademacher cocycle (Ghys)
- Linking $\mathrm{Kn}(\gamma_1)$ and $\mathrm{Kn}(\gamma_2)$ $\leftrightarrow$ Knopp cocycle (DIT)

The map $\gamma \mapsto 12c/(cz+d)$ is a weight 2 cocycle, trivialised by the Eisenstein series $E_2(z)$. It lifts uniquely to a weight 0 cocycle:

$$0 \longrightarrow \mathrm{H}^1(\mathrm{SL}_2(\mathbf{Z}), \mathcal{O}) \xrightarrow{d} \mathrm{H}^1(\mathrm{SL}_2(\mathbf{Z}), \mathcal{O}_2) \longrightarrow 0$$

The Dedekind–Rademacher symbol $\Phi(\gamma) \in \mathbf{Z}$ is defined by

$$\log \Delta(\gamma z) - \log \Delta(z) = 6 \log(-(cz+d)^2) + 2\pi i \Phi(\gamma).$$

so that the unique lift is given by the right hand side. When applied to a hyperbolic matrix γ with fixed point τ, we get

$$\begin{array}{ccc} 6\log(-(cz+d)^2) & + & 2\pi i \Phi(\gamma) \\ \Big\downarrow {\scriptstyle z=\tau} & & \Big\downarrow {\scriptstyle z=i\infty \quad (\text{+ homog.})} \\ 12\log(\varepsilon_\tau) & & \mathit{Link}(\mathrm{Kn}(\gamma), \text{trefoil}) \end{array}$$

Now upgrade the above to the setting

$$\begin{aligned}\Gamma &= \mathrm{SL}_2(\mathbf{Z}[1/p]) \\ \mathcal{O} &= \text{Analytic functions on } \mathcal{H}_p = \mathbf{P}^1(\mathbf{C}_p) \setminus \mathbf{P}^1(\mathbf{Q}_p)\end{aligned}$$

Want a multiplicative lift instead

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^1(\Gamma, \mathcal{O}^\times) \arrow[r] & \mathrm{H}^1(\Gamma, \mathcal{O}^\times/\mathbf{C}_p^\times) \arrow[r] \arrow[d, dotted, "\log"'] \arrow[dr, bend left, "\mathrm{dlog}"] & \mathrm{H}^1(\Gamma_0(p), \mathbf{C}_p^\times) \\ & & \mathrm{H}^1(\Gamma, \mathcal{O}) \arrow[r, "\mathrm{d}"] & \mathrm{H}^1(\Gamma, \mathcal{O}_2) \end{tikzcd}$$

From Euler system of Siegel units, construct $\Theta_{\mathrm{DR}} \in C^1(\Gamma, \mathcal{O}^\times)$ such that

$$\Theta_{\mathrm{DR}}(\gamma_1\gamma_2) = \Theta_{\mathrm{DR}}(\gamma_1)\Theta_{\mathrm{DR}}(\gamma_2)^{\gamma_1} \times p^{\Phi_p(\gamma_1,\gamma_2)}$$

where $\Phi_p : \Gamma_0(p) \to \mathbf{Z}$ is Dedekind–Rademacher morphism (viewed as an element in $\mathrm{H}^2(\Gamma, \mathbf{Z}) \supset \mathrm{H}^1(\Gamma_0(p), \mathbf{Z})$).

Let τ be RM point, then the *value*

$$\Theta_{\mathrm{DR}}[\tau] := \Theta_{\mathrm{DR}}(\gamma_\tau)(\tau)$$

can be computed easily, e.g. $p = 7$ and $\tau = \frac{-17+\sqrt{321}}{4}$ gives a root of

$$7^4x^6 - 20976x^5 - 270624x^4 + 526859689x^3 - 649768224x^2 - 120922465776x + 7^{16}$$

Conjecture (Darmon–Dasgupta)

Let $H/\mathbf{Q}(\tau)$ be the ring class field attached to τ, then

$$\Theta_{\mathrm{DR}}[\tau] \in \mathcal{O}_H[1/p]^\times$$

With Henri Darmon and Alice Pozzi, we prove this conjecture. Use p-adic Eisenstein family over $F = \mathbf{Q}(\tau_1)$ with an odd class character ψ

$$E_{k,\psi}^{(p)} := L_p(\psi, 1-k) + 4 \sum_{\nu \in \mathfrak{d}_+^{-1}} \sigma_{k-1,\psi}^{(p)}(\nu\,\mathfrak{d}) \exp(2\pi i(\nu_1 z_1 + \nu_2 z_2)),$$

Remark. Dasgupta–Kakde (forthcoming) give a different proof.

We consider its diagonal restriction $E^{(p)}_{k,\psi}(z,z)$, and show that

- When $k=1$, we get

$$\begin{array}{lcll} E^{(p)}_{1,\psi}(z,z) & = & L_p(\psi,0) - 4\sum_{n=1}^{\infty} \langle \{0\to\infty\}, T_n g_\psi\rangle\, q^n & \text{if } p \text{ split in } F, \\ E^{(p)}_{1,\psi}(z,z) & = & 0 & \text{if } p \text{ inert in } F. \end{array}$$

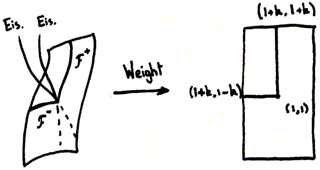
- When p is inert in F, the ordinary projection $\lim_{n\to\infty} U_p^{n!}$ of the first derivative equals

$$\left(\frac{\partial}{\partial k} E^{(p)}_{k,\psi}(z,z)\right)\Bigg|_{k=1}^{\mathrm{ord}} = \log(\mathrm{Nm}\,\Theta_{\mathrm{DR}}[\psi])\, E^{(p)}_2 + \sum_{f \text{ cusp}} \log(\mathrm{Nm}\, P^-_f(\psi))\, f$$

  where $P^-_f(\psi)$ = Stark–Heegner point.

Mirrors Gross–Zagier very closely! Can even be bootstrapped to a full *proof* of the conjecture on Θ_{DR}, via the (exclusively p-adic) connection with deformation theory of Galois representations.

With Henri Darmon and Alice Pozzi consider the eigenvariety around $E_{1,\psi}^{(p)}$, which was shown to be étale over weight space by Betina–Dimitrov–Shih.



The first derivatives of the three parallel weight deformations of the Artin representation $1 \oplus \psi$ are related, exhibiting $u_\psi \in \mathcal{O}_H[1/p]^\times$ such that

$$\begin{aligned} \log(\mathrm{Nm}\,\Theta_{\mathrm{DR}}[\psi]) &= L_p'(\psi, 0) && \text{(Diagonal restrictions)} \\ &= \log(\mathrm{Nm}\, u_\psi) && \text{(Galois deformations)} \end{aligned}$$

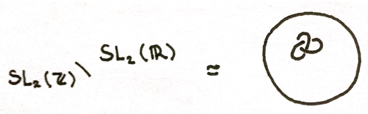
Remark 1. Working instead with the *anti-parallel* family $\mathcal{F}^-$ through the weight $(1, 1)$ form $E_{1,\psi}^{(p)}$, we prove the full conjecture, without the norm.

Remark 2. Charollois–Darmon define archimedean version of $\Theta_{\mathrm{DR}}[\tau]$, refining the Stark conjecture. (No Galois deformation, no proof!)

Outline

The work of Duke–Imamoḡlu–Tóth (Part II)

Let us return to the work of Duke–Imamoḡlu–Tóth on linking numbers of modular geodesics $\mathrm{Kn}(\gamma) \hookrightarrow \mathrm{SL}_2(\mathbf{Z}) \backslash \mathrm{SL}_2(\mathbf{R})$:



- Linking $\mathrm{Kn}(\gamma)$ and trefoil $\leftrightarrow$ Dedekind–Rademacher cocycle (Ghys)
- Linking $\mathrm{Kn}(\gamma_1)$ and $\mathrm{Kn}(\gamma_2)$ $\leftrightarrow$ Knopp cocycle (DIT)

The *Knopp cocycle* in $Z^1(\mathrm{SL}_2(\mathbf{Z}), \mathcal{O}_2)$ attached to an RM point τ is

$$\gamma \longmapsto \sum_{w \in \mathrm{SL}_2(\mathbf{Z})\tau} \frac{\{\infty \to \gamma\infty\} \cap \{w \to w'\}}{z - w}$$

Choie–Zagier: Those account for *all* elements in $Z^1_{\mathrm{par}}(\mathrm{SL}_2(\mathbf{Z}), \mathcal{O}_2)$.

Joint with Henri Darmon, we upgrade the above to the setting

$$\begin{array}{rcl} \Gamma & = & \mathrm{SL}_2(\mathbf{Z}[1/p]) \\ \mathcal{M} & = & \text{Meromorphic functions on } \mathcal{H}_p = \mathbf{P}^1(\mathbf{C}_p) \setminus \mathbf{P}^1(\mathbf{Q}_p) \end{array}$$

Construct multiplicative lift

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^1(\Gamma, \mathcal{M}^\times) \arrow[r] & \mathrm{H}^1(\Gamma, \mathcal{M}^\times/\mathbf{C}_p^\times) \arrow[r] \arrow[d, dotted, "\log"'] \arrow[dr, bend left, "\mathrm{dlog}"] & \mathrm{H}^1(\Gamma_0(p), \mathbf{C}_p^\times) \\ & & \mathrm{H}^1(\Gamma, \mathcal{M}) \arrow[r, "\mathrm{d}"] & \mathrm{H}^1(\Gamma, \mathcal{M}_2) \end{tikzcd}$$

Using explicit infinite products over Γ, construct $\Theta_\tau \in C^1(\Gamma, \mathcal{M}^\times)$. Its lifting obstruction is rich! When $X_0(p)$ has genus zero, we have

$$\Theta_\tau(\gamma_1\gamma_2) = \Theta_\tau(\gamma_1)\Theta_\tau(\gamma_2)^{\gamma_1} \times \varepsilon_\tau^{\Phi_p(\gamma_1,\gamma_2)}$$

where $\Phi_p : \Gamma_0(p) \to \mathbf{Z}$ is Dedekind–Rademacher morphism.

Assume that $X_0(p)$ has genus 0, i.e. $p = 2, 3, 5, 7, 13$ (for simplicity)

If τ_1, τ_2 are coprime RM points in $\mathcal{H}_p$, the quantity

$$J_p(\tau_1, \tau_2) := \Theta_{\tau_1}(\gamma_{\tau_2})(\tau_2)$$

is arithmetically very rich, and should be thought of as analogous to the quantity $J_\infty(\tau_1, \tau_2) := j(\tau_1) - j(\tau_2)$ considered by Gross–Zagier:

Conjecture (Darmon–V.)

Let H_i be the ring class field attached to τ_i, then

$$J_p(\tau_1, \tau_2) \in H_1 H_2$$

and is acted on in the obvious way by $\mathrm{Gal}(H_1 H_2/\mathbf{Q})$.

Remark 1. This implies complex conjugation always acts by inversion.
Remark 2. For general primes, lifting obstructions are killed by principal parts of weakly holomorphic forms of weight $1/2$, giving a map

$$M^{!!}_{1/2}(\Gamma_0(4p)) \longrightarrow \mathrm{H}^1(\Gamma, \mathcal{M}^\times).$$

Let $\Delta_1 = 5$, then for below choices of p and τ consider the quantity

$$J_p\left(\frac{1+\sqrt{5}}{2},\ \tau\right)$$

τ	$p = 11$	$p = 19$	$p = 59$
$2\sqrt{2}$	$\frac{3-4\sqrt{-1}}{5}$	$\frac{3-4\sqrt{-1}}{5}$	1
$3\sqrt{2}$	$\frac{11+21\sqrt{-3}}{2\cdot 19}$	$\frac{5-4\sqrt{-6}}{11}$	1
$4\sqrt{2}$	$\frac{57-176\sqrt{-1}}{5\cdot 37}$	$\frac{5-12\sqrt{-1}}{13}$	$\frac{3+4\sqrt{-1}}{5}$
$7\sqrt{2}$	$\frac{118393-8328\sqrt{-14}}{5^2\cdot 59\cdot 83}$	$\frac{93+95\sqrt{-7}}{2^2\cdot 67}$	$\frac{37+9\sqrt{-7}}{2^2\cdot 11}$
$8\sqrt{2}$	$\frac{1312-1425\sqrt{-1}}{13\cdot 149}$	$\frac{43+924\sqrt{-1}}{5^2\cdot 37}$	$\frac{3+4\sqrt{-1}}{5}$
$9\sqrt{2}$	$\frac{11387+12320\sqrt{-3}}{19^2\cdot 67}$	$\frac{43+4100\sqrt{-6}}{11^2\cdot 83}$	1
$11\sqrt{2}$	$-$	$\frac{209711-130467\sqrt{-11}}{2\cdot 5^2\cdot 59\cdot 163}$	$\frac{3+4\sqrt{-22}}{19}$

Observe that for any pair of primes p, q there seems to be an equality

$$\text{“}\mathrm{ord}_p\text{”}(q\text{-adic invariant}) = \text{“}\mathrm{ord}_q\text{”}(p\text{-adic invariant}).$$

Let $p = 2$, then the RM values of

$$J_2\left(\frac{1+\sqrt{13}}{2}, -\right)$$

at the six RM points of discriminant 621 are all roots of

$$\begin{array}{rl}
53266281197421626898704636823062295969007036119297599934916 & x^6 \\
-27836752624445107255550537796183532261306810430217742390746 & x^5 \\
-29297701627429700833818885363891546270240998098759334148135 & x^4 \\
+87958269550388100260309855891207245711288562805656560629805 & x^3 \\
-29297701627429700833818885363891546270240998098759334148135 & x^2 \\
-27836752624445107255550537796183532261306810430217742390746 & x \\
+53266281197421626898704636823062295969007036119297599934916 & = 0
\end{array}$$

This generates ring class field of conductor 621 over $K = \mathbf{Q}(\sqrt{13})$.

The q-adic valuation of $J_p(\tau_1, \tau_2)$ is conjecturally given by arithmetic intersection numbers of geodesics on Shimura curve X_{pq} attached to quaternion algebra B_{pq}. In this example, the constant term is

$$2^2 \cdot 7^7 \cdot 19^4 \cdot 37^5 \cdot 47^2 \cdot 59^2 \cdot 67^3 \cdot 97^2 \cdot 109^2 \cdot 229 \cdot 241 \cdot 379 \cdot 631 \cdot 709 \cdot 733 \cdot 1009$$

James Rickards developed algorithms to compute these intersection numbers. He finds that the only q for which some geodesic of discriminant 13 intersects some geodesic of discriminant 621 on $X_{2,q}$ are:

q	7	19	37	47	59	67	97	109
$\langle -, - \rangle_{q,\infty}$	7	4	5	2	2	3	2	2
q	229	241	379	631	709	733	1009	
$\langle -, - \rangle_{q,\infty}$	1	1	1	1	1	1	1	

Towards a proof?

The connection between differences of singular moduli and derivatives of real analytic families of modular forms place singular moduli in the context of the *Kudla programme*, which seeks connections between algebraic cycles and Fourier coefficients of families of automorphic forms.

For real quadratic singular moduli, it is reasonable to try to establish similar connections in the context of an emerging *p-adic Kudla programme.* It has the additional connection

$$\begin{gathered}p\text{-Adic families of modular forms}\\ \updownarrow\\ \text{Deformations of Galois representations}\end{gathered}$$

Ongoing work with Henri Darmon and Yingkun Li adapts this strategy to meromorphic cocycles. Replaces Eisenstein family by a family $\mathcal{F}_{k+1/2}$ obtained from a Hida family of theta series via the functorialities

$$M_k \xrightarrow{\mathrm{DN}} M_{k,k} \xrightarrow{\mathrm{res}} M_{2k} \xleftarrow{\mathrm{Shim}} M_{k+1/2}$$

Again there is vanishing at $k = 1$, and

$$\left(\frac{\partial}{\partial k}\mathcal{F}_{k+1/2}\right)\Bigg|_{k=1}^{\mathrm{ord}} = \sum_D \log(\mathrm{Nm}\, J_p(\chi, D))q^D + (\mathrm{Alg.terms})$$

where the algebraic terms are determined by the deformation theory of the Hida family around weight $k = 1$ worked out by Darmon–Lauder–Rotger.

When $X_0(p)$ has genus zero, this ordinary projection must vanish! Flow of information is reversed from the CM setting of Gross–Zagier.

Conclusion

We hope that this may lead to a tentative 'RM theory' which provides rich arithmetic invariants that play a role in an emerging p-adic Kudla programme similar to that of CM singular moduli.

The lack of a geometric substitute for 'RM elliptic curve' is offset by many advantages of the p-adic setting, especially the connection with the arithmetic of number fields via the theory of Galois deformations.

Thanks for having me!