Eisenstein Series, Dimension Formulae and Generalised Deep Holes of the Leech Lattice Vertex Operator Algebra

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Section 1

Why is the Monster?

Classification of finite, simple groups:

- cyclic groups $\mathbb{Z}_p$, p prime,
- alternating groups A_n, $n \geq 5$,
- 16 families of groups of Lie type (e.g. $\mathrm{PSL}_2(7)$),
- the Tits group,
- the 26 sporadic groups (e.g. the Monster group M).

"Groups are symmetries."

Knowing its properties, Griess constructed M as the automorphism group of the Griess algebra, a 196 884-dimensional commutative, nonassociative algebra [Gri82].

Question: Is there a more "primitive" construction (likely infinite or infinite-dimensional and "modular")?

Answer

The Moonshine module $V^\natural = \bigoplus_{n=0}^{\infty} V_n^\natural$ with the algebraic structure of a vertex operator algebra (2-dimensional conformal field theory) has automorphism group $\operatorname{Aut}(V^\natural) = M$ [FLM88].

The character (or graded dimension) of $V^\natural$

$$\mathrm{ch}_{V^\natural}(\tau) = \sum_{n=0}^{\infty} \dim(V_n^\natural) q^{n-1} = q^{-1} + 196\,884q + \ldots$$

is the j-function minus 744. Moreover, $V_2^\natural$ naturally carries the structure of the Griess algebra.

The construction of Moonshine module $V^\natural$ in a nutshell:

- rank-1 Heisenberg vertex operator algebra (free boson) $M(1,0)$,
- rank-24 Heisenberg vertex operator algebra $M_{\hat{\mathfrak{h}}}(1,0) \cong M(1,0)^{\otimes 24}$ associated with 24-dimensional $\mathbb{C}$-vector space $\mathfrak{h}$ equipped with bilinear form $\langle\cdot,\cdot\rangle$, irreducible modules $M_{\hat{\mathfrak{h}}}(1,\lambda)$, $\lambda \in \mathfrak{h}$ (plane wave with momentum λ),
- rank-24 lattice vertex operator algebra $V_L = \bigoplus_{\lambda\in L} M_{\hat{\mathfrak{h}}}(1,\lambda)$ associated with even (positive-definite) lattice $L \subseteq \mathfrak{h}$ (free boson on torus $\mathfrak{h}/L$), irreducible modules $V_{\lambda+L}$, $\lambda + L \in L'/L$.
- Choose $L = \Lambda$, the Leech lattice ("smallest" lattice CFT in central charge $c = 24$).
- Cyclic orbifold construction $V^\natural = V_\Lambda^{\mathrm{orb}(g)}$ where g is one of 39 fixed-point free automorphisms in $\mathrm{O}(\Lambda) \cong \mathrm{Co}_0$ ("smallest" CFT in $c = 24$) [FLM88, Car18].

Vertex Operator Algebras

- $\mathbb{C}$-vector space $V = \bigoplus_{n=0}^{\infty} V_n$, $\dim(V_n) < \infty$,
- vacuum vector $\mathbf{1} \in V_0$, Virasoro vector $\omega \in V_2$,
- algebra products $V \otimes V \to V$, $(a, b) \mapsto a_n b$ for $n \in \mathbb{Z}$,
- satisfying generalised associativity and commutativity constraints.

Vertex operator algebras and their representations (like conformal nets and Segal CFTs) give mathematically rigorous descriptions of 2-dimensional conformal field theories.

Recently, the discovery that some aspects of 4-dimensional superconformal field theories may also be captured by vertex operator algebras [BLL+15] has sparked renewed interest in vertex operator algebras from the physics community.

Section 2

A Tale of Two Classifications

In 1973 Niemeier showed [Nie73]:

Up to isomorphism there are exactly 24 *positive-definite, even, unimodular lattices of rank* 24 *and the isomorphism class of one of these lattices is uniquely determined by its root system.*

The Leech lattice Λ is the unique one amongst them without roots.

Niemeier applied Kneser's neighbourhood method to derive this classification. It can also be proved by means of harmonic theta series [Ven80] or the Minkowski-Siegel mass formula [CS99].

Conway, Parker and Sloane found a nice construction of the Niemeier lattices starting from the Leech lattice [CPS82, CS82]:

Up to equivalence there are exactly 23 *deep holes of the Leech lattice* Λ*, i.e. points in* $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ *which have maximal distance to* Λ*, and they are in bijection with the Niemeier lattices different from* Λ*.*

The construction is as follows: Let d be a deep hole corresponding to the Niemeier lattice N. Then the $\mathbb{Z}$-module in $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ generated by d and $\Lambda^d = \{x \in \Lambda \mid \langle x, d \rangle \in \mathbb{Z}\}$ is isomorphic to N.

The classification of strongly rational, holomorphic vertex operator algebras bears similarities to the classification of positive-definite, even, unimodular lattices.

The weight-1 subspace V_1 of a strongly rational vertex operator algebra V is a reductive Lie algebra.

In 1993 Schellekens showed [Sch93] using arguments from the theory of modular forms:

Let V be strongly rational, holomorphic vertex operator algebra of central charge $c = 24$. Then there are at most 71 possibilities for the Lie algebra structure of V_1 (Schellekens' list).

He conjectured that all potential Lie algebras are realised and that the V_1-structure fixes the vertex operator algebra up to isomorphism.

Schellekens' list:

Rk. \ D.	0	24	36	48	60	72	84	96	108	120	132	144	156	168	192	216	240	264	288	300	312	336	360	384	408	456	552	624	744	1128
0	0																													
4			$C_{4,10}$																											
6			$A_{2,6}D_{4,12}$	$A_{6,7}$; $A_{1,2}D_{5,8}$	$A_{2,2}F_{4,6}$																									
8				$A_{1,2}A_{5,6}B_{2,3}$; $A_{4,5}^2$		$A_1^2D_{6,5}$; $A_1C_{5,3}G_{2,2}$																								
10				$A_{1,2}A_{3,4}^3$		$A_1^3A_{7,4}$; $A_1^2C_{3,2}D_{5,4}$		$A_2B_2E_{6,4}$		$A_3C_{7,2}$																				
12			$A_{1,4}^{12}$	$A_{2,3}^6$	$A_{2,2}^4D_{4,4}$; $B_{2,2}^6$	$A_1^3A_{5,3}D_{4,3}$	$A_{4,2}^2C_{4,2}$; $B_{3,2}^4$	$A_2^2A_{8,3}$	$B_{4,2}^3$	$A_3D_{7,3}G_2$; $E_{6,3}G_2^3$	$A_{8,2}F_{4,2}$		$B_{6,2}^2$	$A_5E_{7,3}$						$B_{12,2}$										
16				$A_{1,2}^{16}$		$A_1^4A_{3,2}^4$		$A_2^2A_{5,2}^2B_2$; $B_2^4D_{4,2}^2$		$A_3^2D_{5,2}^2$; $A_3A_{7,2}C_3^2$		$A_4A_{9,2}B_3$; $B_3^2C_4D_{6,2}$; C_4^4		$A_5C_5E_{6,2}$	$B_4^2D_{8,2}$; $B_4C_6^2$	$A_7D_{9,2}$	$B_5E_{7,2}F_4$; $C_8F_4^2$		B_6C_{10}					$B_8E_{8,2}$						
24		$\mathbb{C}^{24}$				A_1^{24}		A_2^{12}		A_3^8		A_4^6		$A_5^4D_4$; D_4^6	A_6^4	$A_7^2D_5^2$	A_8^3	$A_9^2D_6$; D_6^4			$A_{11}D_7E_6$; E_6^4	A_{12}^2	D_8^3		$A_{15}D_9$	$A_{17}E_7$; $D_{10}E_7^2$	D_{12}^2	A_{24}	E_8^3; $D_{16}E_8$	D_{24}

By the works of many authors the following result is now proved:

Theorem

Up to isomorphism there are exactly 70 strongly rational, holomorphic vertex operator algebra of central charge $c = 24$ with $V_1 \neq \{0\}$. Such a vertex operator algebra is uniquely determined by its V_1-structure.

The proof is based on a case-by-case analysis and uses a variety of methods, making the classification seem sporadic.

The Moonshine module $V^\natural$ is an example of a vertex operator algebra with $V_1 = \{0\}$, but the uniqueness is not known.

Recently, several *uniform* constructions were given:

- Höhn [Höh17],
- M.-Scheithauer [MS19],
- van Ekeren-Lam-M.-Shimakura [ELMS20],
- Höhn-M. [HM20].

These are all centred around certain 11 (+39) conjugacy classes in Co_0, the automorphism group of the Leech lattice Λ.

In the following we present the results of [MS19], a uniform proof of the existence part that generalises Conway, Parker and Sloane's construction of the Niemeier lattices from the Leech lattice Λ.

Schellekens' list (revisited):

Rk. \ D.	0	24	36	48	60	72	84	96	108	120	132	144	156	168	192	216	240	264	288	300	312	336	360	384	408	456	552	624	744	1128
0	0																													
4			$C_{4,10}$																											
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8				$A_{1,2}A_{5,6}B_{2,3}$ $A_{4,5}^2$		$A_1^2D_{6,5}$ $A_1C_{5,3}G_{2,2}$																								
10				$A_{1,2}A_{3,4}^3$		$A_1^3A_{7,4}$ $A_1^2C_{3,2}D_{5,4}$		$A_2B_2E_{6,4}$		$A_3C_{7,2}$																				
12			$A_{1,4}^{12}$	$A_{2,3}^6$	$A_{2,2}^4D_{4,4}$ $B_{2,2}^6$	$A_1^3A_{5,3}D_{4,3}$	$A_{4,2}^2C_{4,2}$ $B_{3,2}^4$	$A_2^2A_{8,3}$	$B_{4,2}^3$	$A_3D_{7,3}G_2$ $E_{6,3}G_2^3$	$A_{8,2}F_{4,2}$		$B_{6,2}^2$	$A_5E_{7,3}$						$B_{12,2}$										
16				$A_{1,2}^{16}$		$A_1^4A_{3,2}^4$		$A_2^2A_{5,2}^2B_2$ $B_2^4D_{4,2}^2$		$A_3^2D_{5,2}^2$ $A_3A_{7,2}C_3^2$		$A_4A_{9,2}B_3$ $B_3^2C_4D_{6,2}$ C_4^4		$A_5C_5E_{6,2}$	$B_4^2D_{8,2}$ $B_4C_6^2$	$A_7D_{9,2}$	$B_5E_{7,2}F_4$ $C_8F_4^2$		B_6C_{10}					$B_8E_{8,2}$						
24		$\mathbb{C}^{24}$				A_1^{24}		A_2^{12}		A_3^8		A_4^6		$A_5^4D_4$ D_4^6	A_6^4	$A_7^2D_5^2$	A_8^3	$A_9^2D_6$ D_6^4			$A_{11}D_7E_6$ E_6^4	A_{12}^2	D_8^3		$A_{15}D_9$	$A_{17}E_7$ $D_{10}E_7^2$	D_{12}^2	A_{24}	E_8^3 $D_{16}E_8$	D_{24}

Section 3

Modular Forms and Dimension Formulae

An important method to construct vertex algebras is the cyclic orbifold construction [EMS20a]:

Let V be a strongly rational, holomorphic vertex operator algebra and g an automorphism of V of finite order n and type 0. Then the fixed-point subalgebra V^g is a strongly rational vertex operator algebra. It has exactly n^2 non-isomorphic irreducible modules, which can be realised as the eigenspaces of g acting on the twisted modules $V(g^j)$ of V.

Cyclic Orbifold Construction

The sum $V^{\mathrm{orb}(g)} := \bigoplus_{j\in\mathbb{Z}/n\mathbb{Z}} V(g^j)^g$ is a strongly rational, holomorphic vertex operator algebra.

We can use the deep holes of the Leech lattice Λ to construct the vertex operator algebras corresponding to the Niemeier lattices:

Theorem

Let $d \in \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ be a deep hole of the Leech lattice Λ corresponding to the Niemeier lattice N. Then $g = \mathrm{e}^{-2\pi \mathrm{i} d_0}$ is an automorphism of the vertex operator algebra V_Λ associated with Λ of order equal to the Coxeter number of N and type 0.

The corresponding cyclic orbifold construction $V_\Lambda^{\mathrm{orb}(g)}$ is isomorphic to the vertex operator algebra V_N associated with N.

We shall see that the other elements on Schellekens' list can be constructed in a similar way.

Vector-valued modular forms for the Weil representation play an important role in many areas of mathematics.

The simplest examples are theta series. Let L be a positive-definite, even lattice of even rank $2k$ and $D = L'/L$ its discriminant form. Then $\theta(\tau) = \sum_{\gamma\in D} \theta_\gamma(\tau)e^\gamma$ with $\theta_\gamma(\tau) = \sum_{\alpha\in\gamma} q^{\langle\alpha,\alpha\rangle/2}$ is a modular form of weight k for the Weil representation of D.

Another example comes from cyclic orbifold theory [EMS20a]:

Under the same conditions as above the n^2 characters of the irreducible modules of V^g combine to a vector-valued modular form χ of weight 0 for the Weil representation of the hyperbolic lattice $I\!I_{1,1}(n)$.

Pairing the character χ of V^g with a certain Eisenstein series of weight 2 for the dual Weil representation we obtain:

Theorem (Dimension Formula)

Let V be a strongly rational, holomorphic vertex operator algebra of central charge $c = 24$ and g an automorphism of V of finite order n and type 0. Then

$$\dim(V_1^{\mathrm{orb}(g)}) = 24 + \sum_{m|n} c_n(m) \dim(V_1^{g^m}) - R(g)$$

where the $c_n(m)$ are defined by $\sum_{m|n} c_n(m)(t, m) = n/t$ for all $t \mid n$ and the rest term $R(g)$ is non-negative.

In particular

$$\dim(V_1^{\mathrm{orb}(g)}) \leq 24 + \sum_{m|n} c_n(m) \dim(V_1^{g^m}).$$

We give an explicit formula for the rest term $R(g)$.

The dimension formula was first proved by Montague for $n = 2, 3$ [Mon94], then generalised to $n = 5, 7, 13$ [Möl16] and finally to all n such that $\Gamma_0(n)$ has genus 0 in [EMS20b].

The previous proofs all used explicit formulae of Hauptmoduln. We show here that the dimension formula is really an obstruction coming from the Eisenstein space.

Proof of Dimension Formula.

- Unique Eisenstein series $f \in \mathcal{E}_2(\Gamma_0(n))$ such that $[f|_{M_s}](0) = 1$ for all cusps s with $[s] \neq [\infty]$. Then $[f|_{M_\infty}](0) = 1 - \psi(n)$.
- For $L = II_{1,1}(n)$ and $D = L'/L$ lift f to vector-valued modular form

$$F = \sum_{M \in \Gamma_0(n) \backslash \Gamma} f|_M \, \bar{\rho}_D(M^{-1}) e^0,$$

  of weight 2 for the dual Weil representation $\bar{\rho}_D$.
- The pairing $(\chi, \bar{F}) = \sum_{\gamma \in D} \chi_\gamma F_\gamma$ is a (scalar-valued) modular form for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ of weight 2 which is holomorphic on $\mathbb{H}$ and meromorphic at ∞.
- By the residue theorem, the constant term of $(\chi, \bar{F})$ must be 0. Using explicit formulae for the Weil representation in [Sch09] this yields the dimension formula. $\square$

Section 4

Generalised Deep Holes

The above upper bound is our motivation for the definition of a generalised deep hole:

Definition (Generalised Deep Hole)

Let V be a strongly rational, holomorphic vertex operator algebra of central charge $c = 24$ and g an automorphism of V of finite order n. Then g is called a *generalised deep hole* of V if

1. g has type 0,
2. the upper bound in the dimension formula is attained,
3. $\operatorname{rk}(V_1^g) = \operatorname{rk}(V_1^{\operatorname{orb}(g)})$.

For example, let $d \in \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ be a deep hole of the Leech lattice Λ. Then $g = \mathrm{e}^{-2\pi\mathrm{i}d_0}$ is a generalised deep hole of the vertex operator algebra V_Λ.

Moreover, the 51 (38) constructions of the Moonshine module $V^\natural$ are generalised deep holes of V_Λ [Car18, FLM88].

The existence of generalised deep holes is restricted by Deligne's bound on the Fourier coefficients of cusp forms:

Theorem

In the above situation suppose that the order of g is a prime p such that $\Gamma_0(p)$ has positive genus. Then $R(g) \geq 24$. In particular g is not a generalised deep hole.

We prove this result by pairing the character of V^g with a certain cusp form.

Finally, we give a uniform construction of the Lie structures on Schellekens' list:

Theorem (Uniform Construction)

Let $\mathfrak{g}$ *be one of the* 71 *Lie algebras on Schellekens' list. Then there is a generalised deep hole* $g \in \mathrm{Aut}(V_\Lambda)$ *such that* $(V_\Lambda^{\mathrm{orb}(g)})_1 \cong \mathfrak{g}$.

We believe that this correspondence is injective for automorphisms with non-trivial fixed-point sublattice:

Conjecture

The cyclic orbifold construction $g \mapsto V_\Lambda^{\mathrm{orb}(g)}$ *defines a bijection between the algebraic conjugacy classes of generalised deep holes* $g \in \mathrm{Aut}(V_\Lambda)$ *with* $\mathrm{rk}((V_{\Lambda^g})_1) > 0$ *and the isomorphism classes of strongly rational, holomorphic vertex operator algebras* V *of central charge* $c = 24$ *with* $V_1 \neq \{0\}$.

Thank you for your attention!

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