Arthurian Tales

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Joint work with N. Dummigan

*Sir not appearing in this talk.

Once upon a time there was a positive definite, even unimodular lattice $\mathcal{L} \subset \mathbb{Q}^n$ (with $8 \mid n$).

Attached to $\mathcal{L}$ is an Orthogonal group scheme, $O_{\mathcal{L}}$.

Fact

If $X_{\mathcal{L}}$ is the (finite) set of isometry classes of lattices in the genus of $\mathcal{L}$ then

$$O_{\mathcal{L}}(\mathbb{Q}) \backslash O_{\mathcal{L}}(\mathbb{A}_f) / \prod_p O_{\mathcal{L}}(\mathbb{Z}_p) \longleftrightarrow X_{\mathcal{L}} = \{[\mathcal{L}_1] = [\mathcal{L}], [\mathcal{L}_2], \ldots, [\mathcal{L}_h]\}.$$

The $\mathcal{L}_i$ are natural objects in the theory of quadratic forms.

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The $\mathcal{L}_i$ are natural objects in the theory of quadratic forms.

Compactness of $O_{\mathcal{L}}(\mathbb{R})$ implies that the space of automorphic forms of $O_{\mathcal{L}}$ of trivial weight and "level 1" can be viewed as the h dimensional space $M_{\mathcal{L}}$ of functions:

$$f : X_{\mathcal{L}} \longrightarrow \mathbb{C}.$$

Surely this space is uninteresting?! We need Hecke operators...

Definition

Let p be prime. A p-neighbour of $\mathcal{L}$ is a lattice $\mathcal{L}' \subset \mathbb{Q}^n$ such that $\mathcal{L}/(\mathcal{L} \cap \mathcal{L}') \cong \mathbb{Z}/p\mathbb{Z}$.

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Fact

Each p-neighbour of $\mathcal{L}$ is isometric to a unique $\mathcal{L}_i$ and we can compute all $N_p(n) = \frac{p^{n-1}-1}{p-1} + p^{\frac{n}{2}-1}$ of them.

Natural linear map on $M_{\mathcal{L}}$:

$$T_p(f)([\mathcal{L}_i]) = \sum_{\mathcal{L}'} f([\mathcal{L}']) = \sum_{i=1}^{h} N_p(\mathcal{L}_i, \mathcal{L}_j) f([\mathcal{L}_j]),$$

where $\mathcal{L}'$ are the p-neighbours of $\mathcal{L}_i$ and $N_p(\mathcal{L}_i, \mathcal{L}_j)$ is the number of p-neighbours of $\mathcal{L}_i$ that are isometric to $\mathcal{L}_j$ (natural numbers in the theory of quadratic forms).

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T_p can be thought of as one of a family of Hecke operators at p.
Don't we diagonalise Hecke operators?

Fact

There is a basis $v_1, v_2, ..., v_h \in M_{\mathcal{L}}$ of simultaneous eigenforms for the T_p, i.e. $T_p(v_i) = \lambda_p(v_i)v_i$ for all p.

Constant function $v_1([\mathcal{L}_i]) = 1$ satisfies $T_p(v_1) = N_p(n)v_1$.

This can be thought of as an Eisenstein series.
The other eigenforms/eigenvalues are much more mysterious.

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The v_i generate irreducible automorphic representations π_i of $O_{\mathcal{L}}(\mathbb{A}_{\mathbb{Q}})$ that are everywhere unramified and trivial at infinity.

Each π_i has local Langlands parameters (up to conjugation):

$$c_\infty(\pi_i) : W_{\mathbb{R}} \longrightarrow O_{\mathcal{L}}(\mathbb{C}),$$
$$c_p(\pi_i) : W_{\mathbb{Q}_p} \longrightarrow O_{\mathcal{L}}(\mathbb{C}).$$

- $c_\infty(\pi_i)(z) = \mathrm{diag}(w^{\frac{n}{2}-1}, w^{\frac{n}{2}-2}, \ldots, 1, w^{1-\frac{n}{2}}, w^{2-\frac{n}{2}}, \ldots, 1)$, where $z \in \mathbb{C}^\times = W_{\mathbb{C}} \hookrightarrow W_{\mathbb{R}}$ and $w = \frac{z}{\bar{z}}$,
- $t_p(\pi_i) = c_p(\pi_i)(\mathrm{Frob}_p)$ fully determines $c_p(\pi_i)$ and $\lambda_p(v_i) = p^{\frac{n}{2}-1}\mathrm{Tr}(t_p(\pi_i))$.

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Each π_i has a "global Arthur parameter", a formal unordered sum $\oplus \Pi_k[d_k]$ where:

- Π_k is a cuspidal automorphic representation of $\mathrm{GL}_{n_k}(\mathbb{A}_\mathbb{Q})$,
- $\sum n_k d_k = n$,
- $c_\infty(\pi_i) = \oplus c_\infty(\Pi_k[d_k])$ and $c_p(\pi_i) = \oplus c_p(\Pi_k[d_k])$.

- $c_\infty(\Pi_k[d_k])(z) =$
  $c_\infty(\Pi_k)(z) \otimes \mathrm{diag}(w^{\frac{d_k-1}{2}}, w^{\frac{d_k-3}{2}}, \ldots, w^{\frac{3-d_k}{2}}, w^{\frac{1-d_k}{2}})$,
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($c_\infty(\Pi_k)$, $c_p(\Pi_k)$ are $GL_{n_k}(\mathbb{A}_\mathbb{Q})$ Local Langlands parameters.)

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Phew...

Summary - Knowing the global Arthur parameter of π_i explicitly gives $\lambda_p(v_i)$ in terms of eigenvalues of automorphic representations of general linear groups (for all p simultaneously).

From now on we take $\mathcal{L} = E_n = D_n + \mathbb{Z}\mathbf{e}$, where $D_n = \{\mathbf{x} \in \mathbb{Z}^n \mid \sum x_i \equiv 0 \bmod 2\}$ and $\mathbf{e} = \frac{1}{2}(1, 1, ..., 1)$.

Interesting question

Fixing n, can we describe the global Arthur parameters of the π_i explicitly?

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n=8

$X_8 = \{[E_8]\}$, $M_8 = \mathbb{C}v_1$ with $v_1 = 1$.

Global Arthur parameter is $[7] \oplus [1]$:

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\begin{aligned}
c_\infty(z) &= \mathrm{diag}(w^3, w^2, w, 1, w^{-1}, w^{-2}, w^{-3}) \oplus (1) \\
&= \mathrm{diag}(w^3, w^2, w, 1, w^{-1}, w^{-2}, w^{-3}, 1), \\
p^3 \mathrm{Tr}(t_p) &= p^3 \mathrm{Tr}(\mathrm{diag}(p^3, p^2, p, 1, p^{-1}, p^{-2}, p^{-3}, 1)) \\
&= p^6 + p^5 + p^4 + ... + 1 + p^3 \\
&= \frac{p^7 - 1}{p - 1} + p^3 \\
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n=16

$X_{16} = \{[E_{16}], [E_8 \oplus E_8]\}$, $M_{16} = \mathbb{C}v_1 \oplus \mathbb{C}v_2$ with $v_1 = [1, 1]$ and $v_2 = [405, -286]$. We find $\lambda_2(v_2) = 1800$.

Global Arthur parameter of v_1 is $[15] \oplus [1]$...but what about v_2?

Guess - $\Delta_{11}[4] \oplus [7] \oplus [1]$ (with Δ_{11} being the GL_2 representation attached to $\Delta \in S_{12}(SL_2(\mathbb{Z}))$).

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p^7 \mathrm{Tr}(t_p(\Delta_{11}[4])) &= 2^7 \mathrm{Tr}(\mathrm{diag}(\alpha_p, \alpha_p^{-1}) \otimes \mathrm{diag}(p^{\frac{3}{2}}, p^{\frac{1}{2}}, p^{-\frac{1}{2}}, p^{-\frac{3}{2}})) \\
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&= p^7 (p^{\frac{3}{2}} + p^{\frac{1}{2}} + p^{-\frac{1}{2}} + p^{-\frac{3}{2}})(\alpha_p + \alpha_p^{-1}) \\
&= (p^3 + p^2 + p + 1)\tau(p), \\
p^7 \mathrm{Tr}(t_p) &= p^4 \left(\frac{p^7 - 1}{p - 1} \right) + p^7 + \tau(p) \left(\frac{p^4 - 1}{p - 1} \right).
\end{aligned}
$$

Plugging in $p = 2$ gives $1800 = \lambda_2(v_2)$.

We haven't proved that the parameter is correct, only that it works at ∞ and $p = 2$.

To prove it we use theta series. For each $m \geq 1$ there is a linear map:

$$\theta^{(m)} : M_n \to M_{\frac{n}{2}}(\mathrm{Sp}_{2m}(\mathbb{Z})),$$

$$[x_1, ..., x_h] \mapsto \sum_{i=1}^{h} \frac{x_i}{|\mathrm{Aut}(\mathcal{L}_i)|} \theta^{(m)}(\mathcal{L}_i).$$

Theorem (Ralllis)

- $\theta^{(m)}(v_i)$ is either 0 or an eigenform $F_i^{(m)}$.
- If $\frac{n}{2} \geq m$ and $\theta^{(m)}(v_i) = F_i^{(m)}$ then:

$$t_p(\pi_i) = \begin{cases} t_p(\pi_{F_i}) \cup \{p^{\pm(\frac{n}{2}-m-1)}, ..., p^{\pm 1}, 1\} & \text{if } \frac{n}{2} > m \\ t_p(\pi_{F_i}) \backslash \{1\} & \text{if } \frac{n}{2} = m \end{cases}$$

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If we can find a good m and the corresponding $F_i^{(m)}$ then we would be done. But this is infeasible.

Instead we can generate eigenforms $F \in M_{\frac{n}{2}}(\mathrm{Sp}_{2m}(\mathbb{Z}))$ that have the correct $t_p(\pi_F)$ and then show that $F \subset \mathrm{Im}(\theta^{(m)})$, by the following:

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If $\frac{n}{2} > m$ then $F \subset \mathrm{Im}(\theta^{(m)})$ if and only if $L(\mathrm{st}, F, \frac{n}{2} - m) \neq 0$.

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For $n = 16$ we can now explain why $\Delta_{11}[4] \oplus [7] \oplus [1]$ is the correct parameter.

- Ikeda Lift: $\Delta \to I^{(4)}(\Delta) \in S_8(\mathrm{Sp}_8(\mathbb{Z}))$ with:

$$L(\mathrm{st}, I^{(4)}(\Delta), s) = \zeta(s) \prod_{i=1}^{4} L(\Delta, s + 8 - i).$$

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Eisenstein congruences!

Recall the eigenforms $v_1 = [1, 1]$ and $v_2 = [405, -286]$.

$$286 v_1 + v_2 = [691, 0] \equiv [0, 0] \bmod 691.$$

Applying T_p to both sides gives (for all p):

$$\lambda_p(v_1) \equiv \lambda_p(v_2) \bmod 691$$
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n=24

$|X_{24}| = 24$ (Niemeier lattices).

Theorem (Chenevier/Lannes)

$$[1] \oplus [23]$$
$$\mathrm{Sym}^2\Delta \oplus [21]$$
$$\Delta_{21}[2] \oplus [1] \oplus [19]$$
$$\mathrm{Sym}^2\Delta \oplus \Delta_{19}[2] \oplus [17]$$
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$$\Delta_{21}[2] \oplus \Delta[8] \oplus [1] \oplus [3]$$
$$\Delta_{21,5}[2] \oplus \Delta_{17}[2] \oplus \Delta[4] \oplus [1] \oplus [3]$$
$$\mathrm{Sym}^2\Delta \oplus \Delta[10] \oplus [1]$$
$$\Delta[12]$$

- NOT easy to prove correctness (Arthur multiplicity formula).
- Old friends: Δ_{k-1} (weight k cuspform).
  New friends: $\mathrm{Sym}^2\Delta_{k-1}$ (symmetric square lifts),
  $\Delta_{j+2k-3,j+1}$ (genus 2 vector valued Siegel modular forms of weight (j,k)).
- $\lambda_p(v_{16}) \equiv \lambda_p(v_{22})$ mod 41 for all p, proving the congruence:

$$\tau_{4,10}(p) \equiv \tau_{22}(p) + p^{13} + p^8 \bmod 41,$$

  $\tau_{4,10}(p)$ eigenvalues of $F \in S_{4,10}(Sp_4(\mathbb{Z}))$,
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Idea

Why not work over a real quadratic field? Even unimodular lattices can then exist in much lower dimensions!

So that's what we did...for each of the fields $\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{5})$ we:

- Described X_n for all plausible dimensions,
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Fact

For $K = \mathbb{Q}(\sqrt{5})$ even unimodular lattices of rank n exist if and only if $4 \mid n$.

n=4 $|X_4| = 1$, $M_{\mathcal{L}} = \mathbb{C}v_1$ with $v_1 = 1$. Arthur parameter $[3] \oplus [1]$.

n=8 $|X_8| = 2$, $M_{\mathcal{L}} = \mathbb{C}v_1 \oplus \mathbb{C}v_2$ with $v_1 = [1, 1]$ and $v_2 = [-25, 42]$.

Arthur parameters: $[7] \oplus [1]$ and $\Delta_5[2] \oplus [1] \oplus [3]$ (where Δ_5 comes from $f \in S_6(SL_2(\mathcal{O}_K))$).

$25v_1 + v_2 \equiv [0, 0] \bmod 67$ implies $\lambda_{\mathfrak{p}}(v_1) = \lambda_{\mathfrak{p}}(v_2) \bmod 67$, which proves the (known) Eisenstein congruence:

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Arthur parameters: $[7] \oplus [1]$ and $\Delta_5[2] \oplus [1] \oplus [3]$ (where Δ_5 comes from $f \in S_6(SL_2(\mathcal{O}_K))$).

$25v_1 + v_2 \equiv [0, 0]$ mod 67 implies $\lambda_{\mathfrak{p}}(v_1) = \lambda_{\mathfrak{p}}(v_2)$ mod 67, which proves the (known) Eisenstein congruence:

$$\tau_6(\mathfrak{p}) \equiv 1 + N(\mathfrak{p})^5 \bmod 67.$$

n=12 $|X_{12}| = 15$

Theorem (Dummigan, F.)

$[1] \oplus [11]$	$?$
$\mathrm{Sym}^2\Delta_5 \oplus [9]$	$\Delta_{(9,5)}[2] \oplus \Delta_{(5,9)}[2] \oplus [1] \oplus [3]$
$\Delta_9^{(2)}[2] \oplus [1] \oplus [7]$	$\Delta_9^{(2)}[2] \oplus \Delta_5[2] \oplus [1] \oplus [3]$
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$\Delta_5[6]$	$\mathrm{Sym}^2\Delta_5 \oplus \Delta_5[4] \oplus [1]$
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New friends: $\Delta_{(k_1-1,k_2-1)}$
(non-parallel weight $f \in S_{k_1,k_2}(SL_2(\mathcal{O}_K))$).

New "Eisenstein" Congruences!

The $\lambda_{\mathfrak{p}}(v_i)$ corresponding to Arthur Parameters $\Delta_7[4] \oplus [1] \oplus [3]$ and $\Delta_{(9,5)}[2] \oplus \Delta_{(5,9)}[2] \oplus [1] \oplus [3]$ are congruent mod 29.

- Δ_7 is base change of "dihedral" $g \in S_8(\Gamma_0(5), \chi_5)$
- $\Delta_{(9,5)}$ corresponds to $h \in S_{[10,6]}(\mathrm{SL}_2(\mathcal{O}_K))$.
- Congruence implies (at split prime p):

$$a_g(p)(1+p^2) \equiv a_h(\mathfrak{p}) + a_h(\overline{\mathfrak{p}}) \bmod \mathfrak{q}_{29}.$$

- LHS corresponds to a reducible $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$-rep and so RHS must too (residually). Indeed we observe:

$$a_h(\mathfrak{p}) \equiv \overline{\alpha}^7 + \alpha^7 N(\mathfrak{p})^2 \bmod \mathfrak{q}'_{29},$$

for any $\mathfrak{p} \nmid 29$ and totally positive α generating $\mathfrak{p}$.

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We conjecture this congruence exists in more generality.

- Modulus 29 comes from the fact that $g \equiv \overline{g} \bmod \langle\sqrt{-29}\rangle$.
- Weights are $8 = 4 + 4$ and $[10, 6] = [4 + 2(4) - 2, 4 + 2]$.

Theoretical justification:

- Can lift h to a vector valued paramodular F such that $\lambda_F(p) = a_h(\mathfrak{p}) + a_h(\overline{\mathfrak{p}})$ at split primes. We get (conjectural) congruence of Klingen-Eisenstein type:

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- In general these congruences should link weights $j + k$ and $[j + 2k - 2, j + 2]$ (if F is a lift). The "dihedral" prime can then be shown to appear in the Deligne period for $L_{\{5\}}(\mathrm{ad}^0(g), k - 1)$.

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The following is made more precise in the paper...and is stated in more generality.

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Suppose:

- $g \in S_{j+k}(\Gamma_0(5), \chi_5)$, eigenform with $j \geq 0$ even and $k \geq 4$.
- $g \equiv \overline{g} \bmod \mathfrak{q}$, with "dihedral" $\mathfrak{q} \mid q, q > 2(j+k), q \neq 5$.
- g ordinary at $\mathfrak{q}$ and $\overline{\rho}_{g,\mathfrak{q}}$ absolutely irreducible.
- $q \nmid (5^{k-1} - 1)$ (local obstruction to F being a lift).

Then there exists $h \in S_{[j+2k-2,j+2]}(\mathrm{SL}_2(\mathcal{O}_K))$ such that:

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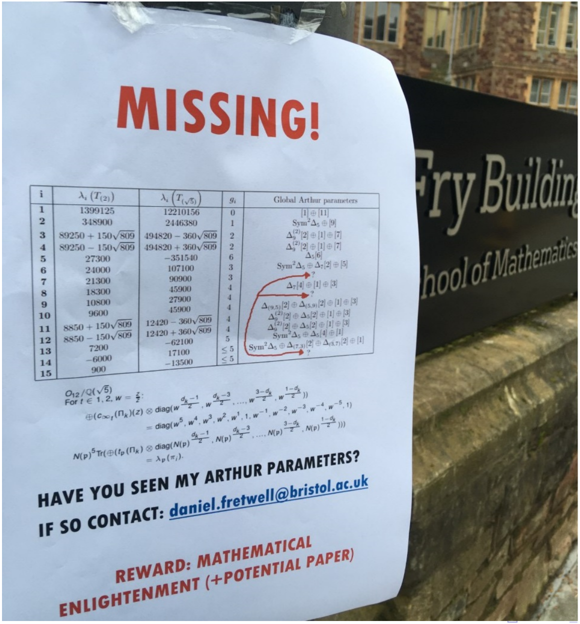
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HAVE YOU SEEN MY ARTHUR PARAMETERS?
IF SO CONTACT: daniel.fretwell@bristol.ac.uk
REWARD: MATHEMATICAL
ENLIGHTENMENT (+POTENTIAL PAPER)