LECTURE 1: BASIC OPERATIONS ON MATRICES:

Recall: A matrix is a rectangular array of numbers. With vertical entries known as columns and horizontal entries known as row.

$$A = [a_{ij}]_{(m,n)} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

In this case A is a matrix of order $m \times n$

Examples.

(i) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1+i & -5 \\ 3i & 2-i \end{bmatrix}$

(ii) $[-1]$ (iii) $\begin{bmatrix} \cos\theta & \sin\theta & 4 \\ -\tan\theta & \cos\theta & 6 \end{bmatrix}$

TYPES OF MATRICES

(i) Row Matrix; $1 \times p \rightarrow A = [1 \; 2 \; 3]$

Column Matrix; $P \times 1 \rightarrow B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(ii) Zero Matrix / Null Matrix;
all entries equal to zero.

(iii) Diagonal Matrix
Atleast one of the diagonal entries is non-zero and all the other entries are equal to zero

$$A = \begin{bmatrix} \times & & & 0 \\ & \times & & \\ & & \times & \\ 0 & & 0 & \times \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(iv) Upper triangular / lower triangular

All Entries below the main diagonal are equal zero. | Entries above the main diagonal are equal zero.

(v) Square Matrices. No of rows equal to number of columns.

$$A = \begin{bmatrix} \times & \times & \times & \times \\ & & \times & \times \\ & 0 & \times & \times \end{bmatrix} (n, n) \qquad \begin{bmatrix} & \times & & \\ \times & \times & \times & \\ \times & \times & & \end{bmatrix}$$

Matrix

(vi) Transpose of a A

Denoted by A^T; Obtained by interchanging the rows and columns

$A = [a_{ij}]$; $A^T = [a_{ji}]$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}; \quad A^T = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix}$$

(vii) Symmetric Matrices;

A matrix A is symmetric if

$$A = A^T$$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 0 & -6 \\ 1 & -6 & 1 \end{bmatrix}; \quad A^T = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 0 & -6 \\ 1 & -6 & 1 \end{bmatrix}$$

(viii) Skew-Symmetric Matrix

$A = -A^T$

$A \neq A^T$

$$A = \begin{bmatrix} 0 & 3 & -1 \\ 3 & 0 & 2 \\ & & \end{bmatrix}; \quad A^T = \begin{bmatrix} 0 & 3 & 1 \\ -3 & & -2 \\ & & \end{bmatrix}$$

Note: 1st order 2nd order ... A to ... and symmetric, it must square

$a_{ij} = -a_{ji}$; $a_{ii} = 0$

(ix) Hermitian Matrix;

If the entries of a matrix are complex, the conjugate of A denoted by $\bar{A}$, is a matrix whose entries correspond to the conjugate of the entries of A. A matrix A is said to be Hermitian if $A = (\bar{A})^T$.

The matrix $(\bar{A})^T$ is denoted by A^*.

Recall: Complex numbers are denoted by $\mathbb{C} = \{z : z = a + bi;\ a, b \in \mathbb{R}\}$

$a \in \mathbb{R}$; $a + bi \to \mathbb{R}$

Conjugate; $\bar{z}$

$\bar{z} = 1 - i$ → Real number

i - Imaginary component $i^2 = -1$

$z = 4i$; $\bar{z} = -4i$

$a = Re(z)$ → Real part $b = Im(z)$ → Imaginary part.

Absolute Value $x \in \mathbb{R}$; $|x|$

$x = 5$; $|x| = 5$; $|-5| = 5$ (of)

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Absolute value of $z = a+bi \in \mathbb{C}$; $|z|$

$|z| = \sqrt{a^2+b^2}$; $\Rightarrow z\bar{z} = |z|^2$

$z = a+bi$ $\quad \bar{z} = a-bi$; $z\bar{z} = a^2+b^2$

Example of a Hermitian Matrix

$$B = \begin{bmatrix} 3 & 2+i \\ 2-i & \sqrt{2} \end{bmatrix} ; \quad \bar{B} = \begin{bmatrix} 3 & 2-i \\ 2+i & \sqrt{2} \end{bmatrix} ;$$

$(\bar{B})^T = \begin{bmatrix} 3 & 2+i \\ 2-i & \sqrt{2} \end{bmatrix} = B^*$

(ii) $C = \begin{bmatrix} 2 & -3 & 2 \\ 0 & 3 & 2-i \\ & & 4 \end{bmatrix}$ (iii) $D = \begin{bmatrix} 0 & i & 2i \\ i & 0 & -4 \\ 2i & & \end{bmatrix}$

$\Rightarrow B = B^*$, Hence B is Hermitian

Def: Triangular Matrix

A square matrix is said to be

triangular if all entries above or below the main diagonal are zeros.

Def: ECHELON MATRIX

An Echelon Matrix is an m by n matrix having the following properties.

(i) The first entry in Row 1 is non-zero and is equal to 1. ($a_{11} = 1$), known as the pivot entry.

(ii) The first non-zero entry in row k, ($k \geq 2$) appears further to right of [illegible] the preceding row. [illegible] equal to one. This are called the leading 1's

(iii) If the leading 1 [illegible] (iv) Zero rows appear at the matrix.

Row-reduced Echelon form or Row canonical form.

Examples;

(i)

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Not in Echelon.

i

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(ii) $\begin{bmatrix} 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$; Not in Echelon Form.

(iii) $\begin{bmatrix} 1 & 2 & 4 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$; Not in Echelon Form.

(iv) $\begin{bmatrix} 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$; Not in Echelon Form.

(v) $\begin{bmatrix} 1 & 3 & 3 & 4 \\ 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; Echelon Form.

Elementary Row Operations:

Elementary row operations are used to transform a matrix into a special matrix. These operations are:

(i) Multiplying a row by a scalar

(ii) Adding scalar multiple of a

iii) rows to another
Interchanging rows

Example: Find a Matrix equivalent to A in Row Reduced Echelon Form (RREF)

$$A = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 3 \end{bmatrix}$$

$R_2 = R_2 + (-2)R_1$
$R_3 = R_3 + (-3)R_1$ $\Rightarrow A \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 5 & 3 \end{bmatrix}$

$R_2 = R_2 + (-1)R_3 \Rightarrow$

$R_3 = (-5)R_2 + 4(R_3) \Rightarrow A \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$R_1 = 4R_1 + 3R_2 \Rightarrow A \sim \begin{bmatrix} 4 & 8 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$R_1 = \frac{1}{4}R_1$

$R_2 = \frac{1}{4} R_2$
$R_3 = \frac{1}{3} R_3$
$\Rightarrow A \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Take Away: Find a matrix equivalent to A in Row Reduced Echelon Form

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

$R_2 = R_2 + (-2)R_1$
$R_4 = R_4 + (-2)R_1$
$\Rightarrow A \sim \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$

$$A \sim \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ & & & & & & \end{bmatrix}$$

$R_2 = -R_2 \Rightarrow$

$R_3 = \frac{1}{5} R_3$

$R_4 = \frac{1}{2} R_4$

$$\begin{matrix} 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 2 & 4 & 0 & 9 & 3 \end{matrix}$$

$R_3 = R_3 + (-1)R_2$

$R_4 = R_4 + (-2)R_2$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 \end{bmatrix}$$

$$R_4 \longleftrightarrow R_3 \Rightarrow A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 = R_1 + 2R_2 \Rightarrow A \sim$

$R_3 = \frac{1}{3} R_3$

$R_2 = R_2 + (-1)R_3 \Rightarrow A \sim$

RREF

LECTURE 2: SOLUTIONS TO SYSTEMS OF (LINEAR) EQUATIONS.

→ LINE; $y = mx + c$ — linear

$y^2 = ax$ → Quadratic

$xy = 4$

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$; $a_i \to$ constant
b : constant

$x_1, x_2, \ldots, x_n$

$2x_1 + 3x_2 = 10$

$x_1 = 2$; $x_2 = 2$ $(2, 2)$

$$\begin{cases} 2x_1 + 3x_2 = 10 \\ x_1 + x_2 = 5 \end{cases}$$

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32} + a_{33}z \end{cases}$$

m linear equations in n - unknowns

Consider the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \end{cases}$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

of m linear equations in n unknowns. This system can be expressed in the form $AX = B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}; \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}; \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Def: Any vector X that satisfies the preceding matrix equation i.e $AX = B$ is called a solution of the system.

Def: The system $AX = B$ is homogeneous if $B = 0$

Remark: We shall consider the following 3 cases

(i) When the system have no solution, in which it is called inconsistent

(ii) When the system has exactly one solution. (Unique solution)

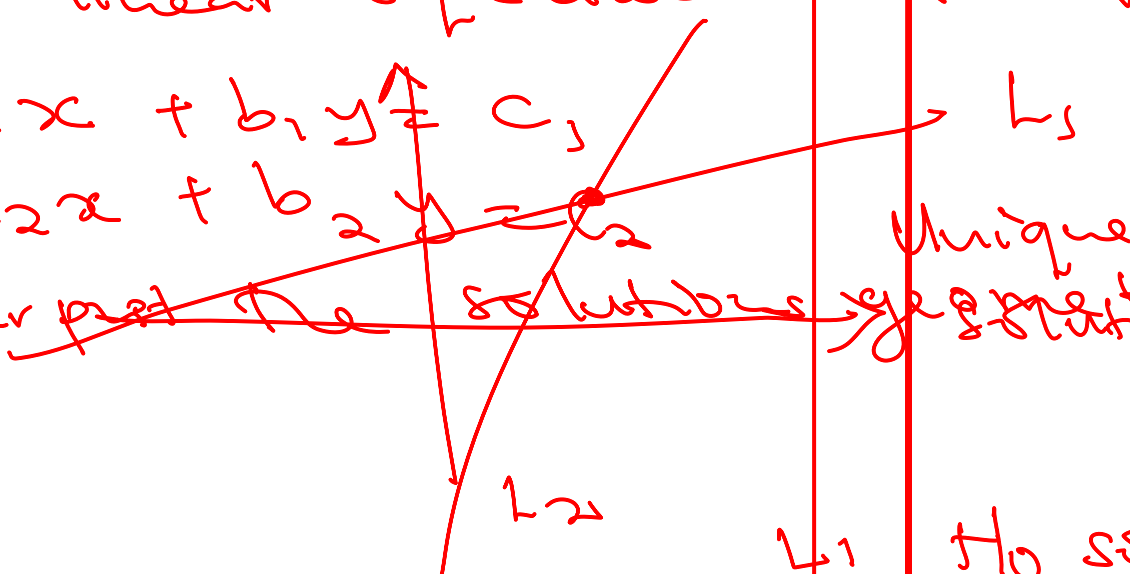
(iii) When the system has an infinite number of solutions

Example: Consider the system of two linear equations in two unknowns

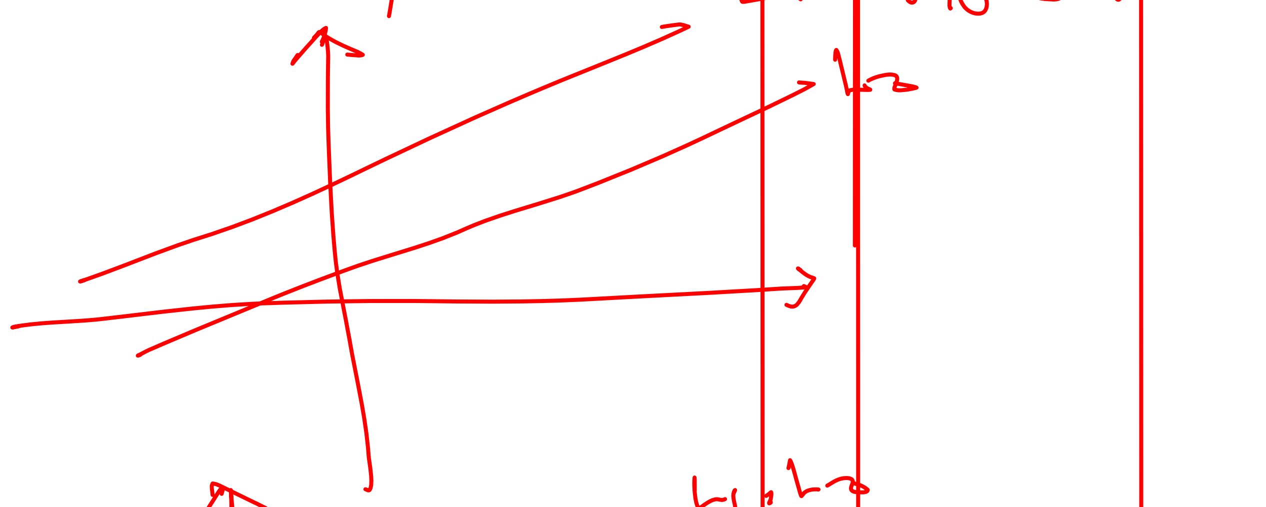
(i) $a_1x + b_1y = c_1$

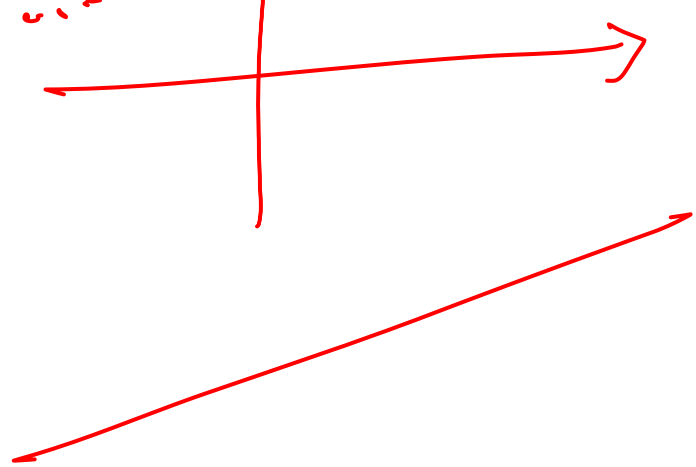
$a_2x + b_2y = c_2$

Interpret the solutions geometrically.



(ii)





(iii)

Infinite number of solutions

Methods of finding solutions to systems of linear Equations.

Def: Augmented matrix. rectangular array of numbers.

The system $AX=B$ can be abbreviated by writing $[A|B]$ or $[A:B]$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

This is called the augmented matrix of the system.

Example: $x_1 + x_2 + 2x_3 = 9$

$$4x_2 + 2x_1 - 3x_3 = 1$$
$$3x_1 + 6x_2 - 5x_3 = 0$$

The augmented matrix for the

System is

$$M = \begin{bmatrix} 1 & 1 & 2 & \vdots & 9 \\ 2 & 4 & -3 & \vdots & 1 \\ 3 & 6 & -5 & \vdots & 0 \end{bmatrix}$$

Remark: When constructing an augmented matrix, the unknowns must be written in the same order in each equation.

GAUSS ELIMINATION METHOD

The Gauss elimination method involves writing the augmented matrix for system in Echelon form, and a solution is obtained by back substitution.

Example 1: Solve the following system using the Gauss-Elimination method

$$x + y + 2z = 9$$
$$2x + 4y - 3z = 1$$
$$3x + 6y - 5z = 0$$

The augmented matrix for the system is

$$M = \begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 2 & 4 & -3 & : & 1 \\ 3 & 6 & -5 & : & 0 \end{bmatrix}$$

We form $\begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 2 & 4 & -3 & : & 1 \\ 3 & 6 & -5 & : & 0 \end{bmatrix}$ using elementary row operations

$R_2 = R_2 + (-2)R_1$, $R_3 = R_3 + (-3)R_1 \Rightarrow M \sim \begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 0 & 2 & -7 & : & -17 \\ 0 & 3 & -11 & : & -27 \end{bmatrix}$

$$R_3 = 3R_2 + (-2)R_3 \Rightarrow M \sim \begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 0 & 2 & -7 & : & -17 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

$$R_2 = \frac{1}{2}R_2 \Rightarrow M \sim \begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 0 & 1 & -7/2 & : & -17/2 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

$x + y + 2z = 9$

$y - 7/2\, z = -17/2$ $\quad y = 2;\ x = 1$

$z = 3$ $\quad x = 1;\ y = 2;\ z = 3$

II. GAUSS–JORDAN ELIMINATION METHOD

This method involves writing the augmented matrix in Row reduced Echelon or Row Canonical form: A Solution by back substitution.

Example: Method I

$R_2 = R_2 + 7R_3$

$R_1 = R_1 + (-2)R_3$ $\Rightarrow M \sim \left[\begin{array}{ccc:c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 0 & 1 & 3 \end{array}\right]$ $\left[\begin{array}{ccc:c} 1 & 1 & 0 & 3 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$R_1 = 2R_1 + (-1)R_2 \Rightarrow M \sim \left[\begin{array}{ccc:c} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$R_1 = \frac{1}{2}R_1$

$R_2 = \frac{1}{2}R_2 \Rightarrow M \sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$z = 3$

$y = 2$

$x = 1$

Example: Solve by Gauss-Jordan elimination

$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$

$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1$

$5x_3 + 10x_4 + 15x_6 = 5$

$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6$

Verify: RREF of M

$$M = \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right]$$

Remark:

(i) Variables that do not correspond to a 1 are called free variables and can be equal to any real number

(ii) The zero row implies that the variables in the equation represented can take any value: $0x_1 + 0x_2 + \dots + 0x_6 = 0$

In our example; x_2, x_4 and x_5 are free variables, we assign arbitrary real values: i.e let $x_2 = r$; $x_4 = s$;

$x_5 = t$, where $r, s, t \in \mathbb{R}$

$\therefore$ $x_6 = \frac{1}{3}$

The general solution is

$x_3 + 2x_4 = 1 \Rightarrow x_3 = 1 - 2s$

$x_1 + 3x_2 + 4x_4 + 2x_5 = 0 \Rightarrow x_1 = -3r - 4s - 2t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3r - 4s - 2t \\ r \\ 1 - 2s \\ s \\ t \\ 1/3 \end{bmatrix}$$

$$= r \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix}$$

A particular solution is obtained by assigning specific values to the free variables.

Remark: The system has an infinite number of solutions.

Take Away: Solve using the Gauss-Jordan elimination Method.

$x_1 + 2x_2 - x_3 + x_5 = 0$

$-x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0$

$x_1 + x_2 - 2x_3 - x_5 = 0$

$x_3 + x_4 + x_5 = 0$

The augmented matrix

$$M = \left[\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$R_2 = R_1 + 2(R_2)$

$R_3 = R_1 + (-2)R_3$ $\Rightarrow M \sim \left[\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 3 & -6 & 3 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$

$R_2 = \frac{1}{3}R_2$

$R_3 = \frac{1}{3}R_3$ $\Rightarrow M \sim \left[\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$

$R_3 = R_3 + (-1)R_2$

$R_4 = R_4 + (-1)R_2$ $\Rightarrow M \sim \left[\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \end{array}\right]$

$R_4 = 3R_2 + (-2)R_3$ $\Rightarrow M \sim \left[\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ & & 0 & & & \\ & & & & & 0 \end{array}\right]$

$R_1 = R_1 + R_2 \Rightarrow M \sim$
$R_2 = R_2 + R_3$
$R_3 = \tfrac{1}{2} R_3$

$$\Rightarrow M \sim \left[\begin{array}{ccccc|c} 2 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 = \tfrac{1}{2} R_1 \Rightarrow M \sim \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ x_2 and x_5 are free variables, let $x_2 = s$ and $x_5 = t$, where $s, t \in \mathbb{R}$

$x_4 = 0$ $\qquad$ $x_1 + x_2 + x_5 = 0$

$x_3 = -x_5 = -t$ $\qquad$ $\Rightarrow x_1 = -x_2 - x_5 = -s - t$

The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Exercise:

For which values of a will the following system have no solutions? Exactly one solution? Infinitely many solutions?

$$x + 2y - 3z = 4$$
$$3x - y + 5z = 2$$
$$4x + y + (a^2 - 14)z = a + 2$$

The augmented matrix for the system is

$$M = \left[\begin{array}{ccc:c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right]$$

$$M \sim \left[\begin{array}{ccc:c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-6 \end{array}\right]$$

$$M \sim \left[\begin{array}{ccc:c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right]$$

$\neq 0$

Row 3 (*): $(a^2-16)z = a-4$ or $0 = k$ have no solution

(*) $(a-4)(a+4)z = a-4 \Rightarrow$ exactly one solution ($\neq 0$, $\neq 0$)

$a = 4 \Rightarrow 0z = 0 \Rightarrow$ infinitely many solutions

$a = -4 \Rightarrow 0z = -8 \Rightarrow$ have no solution.

$a \neq \pm 4 \Rightarrow$ Unique solution.

II. CRAMER'S RULE:

Consider the system of two linear equations in two unknowns.

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases}$$

Solve by elimination

$$\sim \begin{cases} a_{11}a_{22}x + a_{12}a_{22}y = b_1 a_{22} \\ a_{12}a_{21}x + a_{12}a_{22}y = b_2 a_{12} \end{cases}$$

$$\Rightarrow (a_{11}a_{22} - a_{12}a_{21})x = b_1 a_{22} - b_2 a_{12}$$

$$\Rightarrow x = \frac{b_1 a_{22} - b_2 a_{12}}{}$$

$$\Rightarrow \quad y = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

Or. $\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases}$ coefficient matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Form the matrices A_i by substituting column i by $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

i.e. $A_1 = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$; $A_2 = \begin{bmatrix} a_{13} & b_1 \\ a_{22} & b_2 \end{bmatrix}$

$|A| = a_{11}a_{22} - a_{12}a_{21}$; $|A_1| = b_1a_{22} - b_2a_{12}$

$|A_2| = a_{12}b_2 - a_{22}b_1$ $AX = B$

It follows that $x = \dfrac{|A_1|}{|A|}$ and $y = \dfrac{|A_2|}{|A|}$

CRAMER'S RULE: If $AX = B$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has unique solution. This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \; \ldots, \; x_n = \frac{\det(A_n)}{\det(A)}$$

Where A_i is the matrix obtained by replacing the entries of the ith column of A by the matrix $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

Example: Use cramer's rule to solve

$$x_1 + 2x_3 = 6$$
$$-3x_1 + 4x_2 + 6x_3 = 30$$
$$-x_1 - 2x_2 + 3x_3 = 8$$

We write the system in the form of $AX = B$

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 6 \\ 30 \\ 8 \end{bmatrix}}_{B}$$

Find $|A| = \begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix}$

Recall: Determinants of order 3×3 and 2×2

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$; $|A| = a_{11}a_{22} - a_{12}a_{21}$

3×3: Cofactor Method: $|A| = \sum_{j=1}^{3} a_{ij} A_{ij} \to$ Cofactors

i fixed

$$A_{ij} = (-1)^{i+j} \underbrace{|M_{ij}|}_{\text{Minors}}$$

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix} \qquad |A| = \sum_{\substack{j=1 \\ i=1}}^{3} a_{ij} A_{ij} = 1 \cdot 24 + 0 \cdot 3 + 2 \cdot 10 = 44$$

$$A_{11} = (-1)^{1+1} |M_{11}| = (-1)^{1+1} \begin{vmatrix} 4 & 6 \\ -2 & 3 \end{vmatrix} = 24$$

$$A_{12} = (-1)^{1+2} |M_{12}| = (-1)^{1+2} \begin{vmatrix} -3 & 6 \\ -1 & 3 \end{vmatrix} = 3$$

$$A_{13} = (-1)^{1+3} |M_{13}| = (-1)^{1+3} \begin{vmatrix} -3 & 4 \\ -1 & -2 \end{vmatrix} = 10$$

(ii). Diagonalization technique.

$$\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 2 \\ -3 & 4 & 6 & -3 & 4 & 6 \\ -1 & -2 & 3 & -1 & -2 & 3 \end{array}$$

ONLY FOR 3×3 MATRICES

$$(12 + 0 + 12) - (-12 + 0 - 8) = 44$$

$|A| = 44 \neq 0 \Rightarrow$ Cramer's Rule applies.

$$A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}; \quad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

$|A_1| =$ $\qquad |A_2| =$ $\qquad |A_3| =$

$|A_1| = -40$ $\qquad |A_2| = 72$ $\qquad |A_3| = 152$

Take Away: Solve by Cramer's rule.

(i) $x + y + 2z = 1$ $\qquad$ (ii) $2x_1 - x_2 + x_3 = 8$

$2x - y + z = \ldots$ $\qquad$ $4x_1 + 3x_2 + x_3 = 7$

$$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|}, \quad x_3 = \frac{|A_3|}{|A|}$$

$x - 2y - 4z = -4$

$6x_1 + 2x_2 + 2x_3 = 15$

$$\det(A) = |A| = \begin{vmatrix} * & * & * \\ * & * & * \end{vmatrix}$$

(i)

$AX = B$

$n \times n$, $n \times 1$, $n \times 1$

$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & -2 & -4 \end{bmatrix}$; $A_1 = \begin{bmatrix} 1 & 1 & 3 \\ 2 & -1 & 1 \\ -4 & -2 & 4 \end{bmatrix}$

$|A| \neq 0$

$A_2 = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & -4 & 4 \end{bmatrix}$; $A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & -2 & -4 \end{bmatrix}$

$x = \frac{|A_1|}{|A|}$; $y = \frac{|A_2|}{|A|}$; $z = \frac{|A_3|}{|A|}$

INVERSES.

Def: Inverse: If for a given $n \times n$ matrix A, there is an $n \times n$ matrix A^{-1} such that $AA^{-1} = I = A^{-1}A$, where

I is the identity $n \times n$ matrix.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; ... $I_n = \begin{bmatrix} 1 & \cdots & 0 \\ 0 & 1 \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$

Then A^{-1} is an inverse of A with respect to matrix multiplication.

Remark: An $n \times n$ matrix A is said to be invertible if A^{-1} exist or non-singular.

Def: Cofactor Matrix.

Let $A = [a_{ij}]_{(n \times n)}$, $n \geq 2$. The cofactor matrix of A, denoted by Cof(A) is the $n \times n$ matrix whose entry in i row j column is $A_{ij} = (-1)^{i+j} |M_{ij}|$

For Example: Let $A = \begin{bmatrix} 2 & 1 & 0 \\ \cdots \end{bmatrix}$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$; $|A| = a_{11} a_{22} - a_{12} a_{21}$

$A_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = 4$; $A_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 1 \\ 3 & \end{vmatrix}$; $A_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 2 \\ 3 & 6 \end{vmatrix}$

$A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = -2$; A_{22}; $A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix}$

$A_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}$; $A_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}$

$\mathrm{Cof}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}_{(n \times n)}$

Def: Adjoint Matrix.

The adjoint Matrix, denoted by Adj(A) of an $n\times n$ matrix A is the transpose of Cof(A)

$$\operatorname{Adj}(A) = (\operatorname{Cof}(A))^T$$

For Example: Given $A = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -2 & -2 \\ -1 & -1 & 0 \end{bmatrix}$

Find $\operatorname{Adj}(A) = ?$ $\operatorname{Cof}(A) = ?$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix}$$

$(-1)(-1)$

$$\operatorname{Cof}(A) = \begin{bmatrix} 2 & -2 & -2 \\ 1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix};$$

$$\operatorname{Adj}(A) = (\operatorname{Cof}(A))^T = \begin{bmatrix} 2 & 1 & 0 \\ -2 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix}$$

Def: Classical definition of inverse.

An $n\times n$ matrix A is invertible if and only if $|A| \neq 0$. Moreover, for $n \geq 2$, if A^{-1} exists, then

$$A^{-1} = \frac{1}{|A|}\operatorname{Adj}(A)$$

Example: Given $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 3 & 0 & 2 \end{bmatrix}$. Find A^{-1}

$|A| = ?$

$$\begin{array}{ccc|cc} 2 & 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 0 & 2 \\ 3 & 0 & 2 & 3 & 0 \end{array}$$

$$|A| = \sum_{i=1, j=1}^{3} a_{ij} A_{ij} = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13} = 2 \cdot 4 + 1 \cdot 3 + 0 \cdot (-6) = 11$$

$$|A| = (8 + 3 + 0) - (0 + 0 + 0) = 11 \neq 0 \Rightarrow A^{-1} \text{ exist}$$

$$\operatorname{Cof}(A) = \begin{bmatrix} 4 & 3 & -6 \\ -2 & 4 & 3 \\ 1 & -2 & 4 \end{bmatrix}; \quad \operatorname{Adj}(A) = (\operatorname{Cof}(A))^T = \begin{bmatrix} 4 & -2 & 1 \\ 3 & 4 & -2 \\ -6 & 3 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \operatorname{Adj}(A) = \frac{1}{11} \begin{bmatrix} 4 & -2 & 1 \\ 3 & 4 & -2 \\ -6 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 4/11 & -2/11 & 1/11 \\ 3/11 & 4/11 & -2/11 \\ -6/11 & 3/11 & 4/11 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}; \quad A^{-1} = ?$$

$$A A^{-1} = I \qquad [A \,\vdots\, I] \sim [I \,\vdots\, A^{-1}]$$

Elementary Matrices and Method for finding A^{-1}

Def: Elementary Matrix.

An $n \times n$ matrix is called an elementary matrix if it can be obtained from the $n\times n$ identity I_n by performing elementary row operations

Remark: To find the inverse of a matrix A, we must find a sequence of elementary row operations that reduces A to the identity matrix. The procedure reduces the augmented matrix $[A \,\vdots\, I_n]$ to $[I_n \,\vdots\, A^{-1}]$

Example: Given $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 3 & 0 & 2 \end{bmatrix}$, find A^{-1}

$R_3 = 3R_1 + (-2)R_3$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 3 & -4 & 3 & 0 & -2 \end{array}\right]$$

$R_2 = 11R_2 + (-1)R_3 \sim$

$R_3 = 3R_2 + (-2)R_3 \sim$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 22 & 0 & 6 & 8 & -4 \\ 0 & 0 & 11 & -6 & 3 & 4 \end{array}\right]$$

$R_1 = 22R_1 + (-1)R_2 \sim$

$$\left[\begin{array}{ccc|ccc} 44 & 0 & 0 & 16 & -8 & 4 \\ 0 & 22 & 0 & 6 & 8 & -4 \\ 0 & 0 & 11 & -6 & 3 & 4 \end{array}\right]$$

$R_1 = \frac{1}{44}R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & & 4/11 & -2/11 & 1/11 \end{array}\right.$$

$$0 \quad :$$

$R_2 = \frac{1}{22} R_2$

$R_3 = \frac{1}{11} R_3$

$$\begin{matrix} 0 & 1 & 0 & : & 3/11 & 4/11 & -2/11 \\ 0 & 0 & 1 & : & -6/11 & & \end{matrix}$$

$$A^{-1} = \begin{bmatrix} 4/11 & -2/11 & 1/11 \\ 3/11 & 4/11 & -2/11 \\ -6/11 & 3/11 & 4/11 \end{bmatrix}$$

$A A^{-1} = I$

Take Away: Find the inverse of using elementary row operations

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Cryptography: → A → encoding A^{-1} → Inverse decoding

$B = B$ [2 27 * ; * Coded * Message 2 ; * * * * 27] ; $A =$ [−3 −3 ; 0 1 1 ; HATE 3 STOP]

a b c

BUY

Hill-Cipher 26 27 2 * * A 27 * * * 27 * * * *

CRYPTOGRAPHY: → Process of coding and decoding messages. One type of code makes use of a large invertible matrix, where the receiver of the message decodes it using the inverse of the matrix. The first matrix is called the encoding matrix, while its inverse is called the decoding matrix.

We illustrate the method for a 3x3 matrix.

Let the message be; BUY HSE STOCK

The encoding matrix be;

$$A = \begin{bmatrix} -3 & -3 & -4 \\ 0 & 1 & 1 \\ 4 & 3 & 4 \end{bmatrix}$$

We assign a number to each letter of the alphabet. For convenience let us associate each letter to its position in the alphabet and space 27 ... two words be denoted by 27.

The digital message is ...

We put our digital message in matrix form.

$$B = \begin{bmatrix} 2 & 27 & 5 & 20 & 11 \\ 21 & 14 & 27 & 15 & 27 \\ 25 & 19 & 19 & 3 & 27 \end{bmatrix}$$

The coded message is the matrix

$$C = AB = \begin{bmatrix} -3 & -3 & -4 \\ 0 & 1 & 1 \\ 4 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 27 & 5 & 20 & 4 \\ 21 & 14 & 27 & 15 & 27 \\ 25 & 19 & 19 & 3 & 27 \end{bmatrix}$$

encoded message
encoding

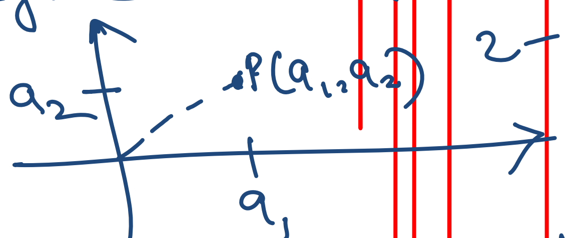
$$\underbrace{C}_{\text{encoded message}} = \underbrace{A}_{\text{encoding}} B$$

$$\Rightarrow A^{-1} C = \underbrace{A^{-1} A}_{I} B = B$$

original message

2. VECTOR SPACES

Recall: A vector is defined as a list of scalars $(a_1, a_2, ---, a_n)$ 2-d



Operations among vectors is defined as;

$$a + b = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

$$c\,a = (c a_1, c a_2, ---, c a_n)$$

Let $a = (1, -4, 3)$, $b = (2, 0, 6)$

$a+b = (1+2, -4+0, 3+6) = (3, -4, 9)$

$2 \cdot a = (2, -8, 6)$

The set of all vectors together with the set of real numbers and two operations possesses certain properties. All mathematical structures having these properties are called Vector Spaces.

Def 2.1 Vector Space

A set of elements V form a vector space over field

A_3: $a+(b+c) = (a+b)+c$ (Associativity Property)

A_1: $a+b \in V$ (Closure Property)

A_4: $\exists\, 0 \in V$ such that $a+0 = 0+a = a$ $\forall a \in V$

$\exists$ (There exist), $\in$ "belongs to"

$\forall$ (For all, for any, for every)

[Existence of additive identity]

A_5: $\forall\, v \in V$, $\exists\, -v \in V$ such that

$v + (-v) = 0$

[Existence of additive inverse]

M_1: $\forall\, a, b \in V,\ r \in K,\ r(a+b) = ra + rb$

M_2: $\forall\, a \in V,\ s, r \in K\ (rs)a = r(sa)$

(Distributive Property)

M_3: $\forall\, a \in V,\ s, r \in K\ (r+s)a = ra + sa$

M_4: $\exists\, 1 \in K$ s.t $\forall\, a \in V,\ 1 \cdot a = a$

(Multiplicative Identity)

Examples:

1. Let $V = \mathbb{N}$, $K = \mathbb{R}$

Natural numbers.

$\mathbb{N} = \{1, 2, 3, 4, \dots\}$; A_2-fails $\therefore \mathbb{N}$ is not a vector space over $\mathbb{R}$.

2. $V = \mathbb{Z}$, $K = \mathbb{Z}$ - Integers $\rightarrow$ Vector space.

$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

$V = \mathbb{Z}$; $K = \mathbb{R} \rightarrow A_2 \rightarrow$ fails

$\mathbb{Q}$ $\qquad \mathbb{R}$; $\mathbb{Q} = \{a/b : a, b \in \mathbb{R},\ b \neq 0,\ (a,b) = 1\}$

over

→ Rational Numbers

$\mathbb{R}$ over $\mathbb{R}$?

③ The set $V = \mathbb{R}^n$ $(n \in \mathbb{N})$ (Euclidean Space)

$a + b = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$

$a = (a_i)$; $b = (b_i) = (b_1, b_2, \dots, b_n)$

④ Let V be the space of $m \times n$ real matrices with operations of matrix addition and scalar multiplication. $V = M_{(m,n)}$

⑤ Let V be the space of real-valued functions on the real line $V = \{f : \mathbb{R} \to \mathbb{R}\}$

$(f+g)(x) = f(x) + g(x)$ $\forall f, g \in V$,

$(kf)(x) = k f(x)$

⑦ Let V be the space of polynomials of degree $\leq n$ i.e. $p(x) \in V \Rightarrow$

$\exists\, a_0, a_1, a_2, \dots, a_n \in \mathbb{R}$ s.t.

$p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

$$V = P_n[x]$$

$$P_3[x] = \{p \in P_3[x] : p = a_0 + a_1x + a_2x^2 + a_3x^3\}$$

$$P_1 = 1 + 2x + 2x^2 + 3x^3$$

$$P_2 = 1 + 2x + 2x^2 \in P_3[x]$$

$$P_3 = a_0 + a_1 x \in P_3[x]$$

$$P_1 + P_2 = (1 + 2x + 2x^2 + 3x^3) + (1 + 2x + 2x^2) + 0x^3$$

$$= 2 + 4x + 4x^2 + 3x^3$$

$$3P_2 = 3(1 + 2x + 2x^2) = 3 + 6x + 6x^2$$

SUBSPACES

A subset W of a vector space V is called a subspace of V if W is a vector space under the addition and scalar multiplication defined on V.

Remark: If W is a set of one or more vectors from a vector space V, then W is a subspace of V if and only if the following conditions hold:

(i) If $u, v \in W$, $u + v \in W$ (Closed under $+$)

(ii) If $k \in K$, $u \in W$, $ku \in W$ (Closed under scalar Multiplication)

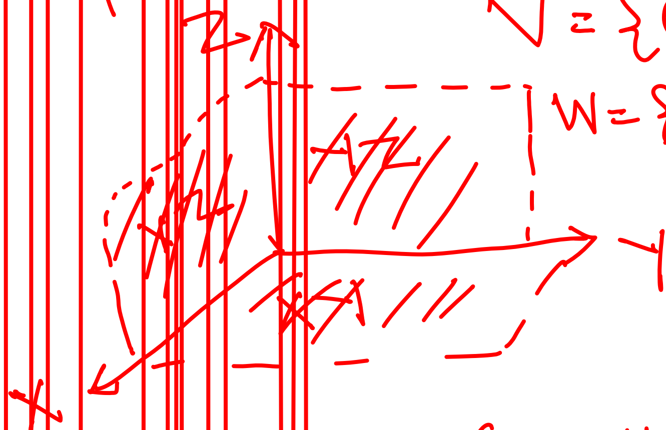
Examples: $\mathbb{R}^3$

xy-plane

1. Let V be the and W be the XY-plane. Then W is a subspace of V.

$V = \{(x, y, z),\ x, y, z \in \mathbb{R}\}$

$W = \{(x, y, 0);\ x, y \in \mathbb{R}\}$



Let $w_1 = (x_1, y_1, 0)$, $w_2 = (x_2, y_2, 0)$

Hence W is a subspace

$w_1 + w_2 = (x_1 + x_2,\ y_1 + y_2,\ 0)$

$x_1 + x_2 \in \mathbb{R}$, $y_1 + y_2 \in \mathbb{R}$

$\alpha w_1 = \alpha(x_1, y_1, 0) = (\alpha x_1, \alpha y_1, 0) \in W$

$\alpha x_1 \in \mathbb{R}$, $\alpha y_1 \in \mathbb{R}$

2. W 2×2 matrices have zero diagonal, the M

Let $A, B \in W$, then $A = \begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix}$; $B = \begin{bmatrix} 0 & b_{12} \\ b_{21} & 0 \end{bmatrix}$

$$A + B = \begin{bmatrix} 0 & a_{12} + b_{12} \\ a_{21} + b_{21} & 0 \end{bmatrix} \in W$$

Let $k \in K$, $kA = \begin{bmatrix} 0 & ka_{12} \\ ka_{21} & 0 \end{bmatrix} \in W$

3. Solution Space

the system of linear equations

Consider

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

A solution the system will be the 2,3 vector $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

If we consider the linear combination system of the AX ... $v_1, \ldots, v_n$ will be expressed in the form subspace W of $\mathbb{R}^n$

$W = k_1 v_1 + k_2 v_2 + \cdots + k_n v_n$, where $k_1, k_2, \ldots, k_n$ are scalars.

Examples. 1

1- Let $u = (1, 2, -1)$ and $v = (6, 4, 2) \in \mathbb{R}^3$

Show that $w = (9, 2, 7)$ is a linear combination of u and v

Let $k_1, k_2 \in \mathbb{R}$ such that

$w = k_1 u + k_2 v$

$(9, 2, 7) = k_1(1, 2, -1) + k_2(6, 4, 2)$

$(k_1, 2k_1, - \quad + (6k_2, 4k_2, 2k_2)$

$= (k_1 + 6k_2,\ 2k_1 + 4k_2,\ -k_1 + 2k_2)$

$$\Rightarrow \begin{cases} k_1 + 6k_2 = 9 \\ 2k_1 + 4k_2 = 2 \\ -k_1 + 2k_2 = 7 \end{cases} \qquad k_1 = ? \quad k_2 = ?$$

$u = (1, 2, -1)$, $v = (6, 4, 2)$, $w = (4, -1, 8)$

$$\begin{bmatrix} 1 & 6 & : & 9 \\ 2 & 4 & : & 2 \\ -1 & 2 & : & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & : & 9 \\ 0 & 8 & : & 16 \\ 0 & 8 & : & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & : & 9 \\ 0 & 1 & : & 2 \\ 0 & 0 & : & 0 \end{bmatrix}$$

$k_2 = 2$; $k_1 = -3$

w is a linear combination of u and v?

$(4, -1, 8) = \alpha_1 (1, 2, -1) + \alpha_2 (6, 4, 2)$

$$\Rightarrow \quad \begin{aligned} \alpha_1 + 6\alpha_2 &= 4 \\ 2\alpha_1 + 4\alpha_2 &= -1 \\ -\alpha_1 + 2\alpha_2 &= 8 \end{aligned}$$

$$M = \begin{bmatrix} 1 & 6 & : & 4 \\ 2 & 4 & : & -1 \\ -1 & 2 & : & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & : & 4 \\ 0 & 8 & : & 9 \\ 0 & 8 & : & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & : & 4 \\ 0 & 8 & : & 9 \\ 0 & 0 & : & 3 \end{bmatrix}$$

No solution $\Rightarrow$ w is not a linear combination of u and v

Def 2.4. Spanning Set

If $S = \{v_1, \dots, v_n\}$ set of vectors

$, v_2,$ is a

of a vector space V, such that every vector in V can be written as a linear combination of $v_1, v_2, \dots, v_n$ then we say S is the spanning set for V.

Example: 1. $V = \mathbb{R}^3$, $v \in \mathbb{R}^3$, $v = (a, b, c)$: $a, b, c \in \mathbb{R}$

$(a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1)$

$= a i + b j + c k$

$S = \{ i = (1, 0, 0), j = (0, 1, 0), k = (0, 0, 1) \}$

$\Rightarrow \mathbb{R}^3 = \text{Span } S$

2. $P_n[x]$, $p \in P_n(x) \Rightarrow$

$p = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

$P_n[x] = \text{Span} \{ 1, x, x^2, \dots, x^n \}$

Show that $S = \{ 1, 1-x, (1-x)^2 \}$ span $P_2[x]$. Let $p \in P_2[x]$

$\Rightarrow p = a_0 + a_1 x + a_2 x^2$, $a_0, a_1, a_2 \in K$

Suppose $p = k_1(1) + k_2(1-x) + k_3(1-x)^2$

$= k_1 + k_2 - k_2 x + k_3 - 2k_3 x + k_3 x^2$

$= (k_1 + k_2 + k_3) 1 + (-k_2 - 2k_3) x + k_3 x^2$

$$\Rightarrow \quad \left.\begin{aligned} k_1 + k_2 + k_3 &= a_0 \\ -k_2 - 2k_3 &= a_1 \\ k_3 &= a_2 \end{aligned}\right\}$$

The system is consistent $\therefore P_2[x] = \text{Span}\, S$

3. Let $S = \{v_1 = (1,1,2), v_2 = (1,0,1), v_3 = (2,1,3)\}$

Show that $\mathbb{R}^3 \neq \text{Span}\, S$

$v = k_1(1,1,2) + k_2(1,0,1) + k_3(2,1,3)$

$v \in \mathbb{R}^3$, $v = (a,b,c)$, $a,b,c \in \mathbb{R}$

$$\begin{cases} k_1 + k_2 + 2k_3 = a \\ k_1 + k_3 = b \\ 2k_1 + k_2 + 3k_3 = c \end{cases}$$

$$M = \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 3 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & 1 & a-b \\ 0 & 1 & 1 & 2a-c \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & 1 & a-b \\ 0 & 0 & \underline{0} & \underline{a+b-c} \end{array}\right] \Rightarrow$$

Inconsistent $\Rightarrow$ S does not span $\mathbb{R}^3$

$\mathbb{R}^3 \neq \text{Span}\, S$

Def.

2.4 LINEAR INDEPENDENCE

A vectors $S = \{v_1, v_2, \dots, v_n\}$ in V.

Set of

Then the vector equation

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \vec{0}$, has

~~Example~~: only solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

~~If~~ ~~Data~~ $V = \mathbb{R}^4$ let $v_1 = (2, -1, 0, 3)$, only solution, then

$v_2 = (1, 2, 5, -1)$ & $v_3 = (7, -1, 5, 8)$ linearly Independent

set. If there are other linearly dependent.

$S = \{v_1, v_2, v_3\}$ dependent set.

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. s.t

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\alpha_1 (2, -1, 0, 3) + \alpha_2 (1, 2, 5, -1) + \alpha_3 (7, -1, 5, 8) = (0, 0, 0, 0)$$

$$\Rightarrow \begin{aligned} 2\alpha_1 + \alpha_2 + 7\alpha_3 &= 0 \\ -\alpha_1 + 2\alpha_2 - \alpha_3 &= 0 \\ 5\alpha_2 + 5\alpha_3 &= 0 \\ 3\alpha_1 - \alpha_2 + 8\alpha_3 &= 0 \end{aligned}$$

$$M = \left[\begin{array}{ccc:c} 2 & 1 & 7 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 5 & 5 & 0 \\ 3 & -1 & 8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} 2 & 1 & 7 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 5 & 5 & 0 \end{array}\right]$$

Take Away: Let $V = P_2[x]$

Show that

$S = \{P_1 = 2 + 3x - x^2,\ P_2 = 6 + 9x - 3x^2\}$

is a linearly dependent set

Let $\alpha_1, \alpha_2 \in K$ such that

$$\alpha_1 P_1 + \alpha_2 P_2 = 0$$

$$\Rightarrow \alpha_1 (2 + 3x - x^2) + \alpha_2 (6 + 9x - 3x^2) = 0$$

$$2\alpha_1 + 3\alpha_1 x - \alpha_1 x^2 + 6\alpha_2 + 9\alpha_2 x - 3\alpha_2 x^2 = 0$$

$$(2\alpha_1 + 6\alpha_2)1 + (3\alpha_1 + 9\alpha_2)x + (-\alpha_1 - 3\alpha_2)x^2 = 0$$

$$= 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$

$$\Rightarrow \begin{cases} 2\alpha_1 + 6\alpha_2 = 0 \\ 3\alpha_1 + 9\alpha_2 = 0 \\ -1\alpha_1 - 3\alpha_2 = 0 \end{cases}$$

$$M = \left[\begin{array}{cc|c} 2 & 6 & 0 \\ 3 & 9 & 0 \\ -1 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 1 & 3 & 0 \\ -1 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] : \begin{cases} \alpha_3 \text{ is a free variable} \\ \alpha_2 = -\alpha_3 \\ \alpha_1 = -3\alpha_3 \end{cases} \text{ General solution}$$

Particular: $\alpha_3 = 1,\ \alpha_2 = -1,\ \alpha_1 = -3$

$$-3V_1 + V_2 + V_3 = 0$$

0 0 : 0

α_2 is a free variable $\Rightarrow \alpha_1 P_1 + \alpha_2 P_2 = 0$ has a non-zero, hence P_1, P_2 are linearly dependent.

2.5 BASES AND DIMENSION

Spanning set: $V = \text{Span}\, S$

linear independence

Def: A vector space is said to be finite dimensional or to be of dimension n, written $\dim(V) = n$. If there exist a linearly independent set $S = \{v_1, v_2, \ldots, v_n\}$, such that $V = \text{Span}\, S$

Example:

1. Let $V = \mathbb{R}^n$, let

$$S = \{e_1 = (1, 0, \ldots, 0),\ e_2 = (0, 1, 0 \ldots 0),\ \ldots\ e_n = (0, 0, \ldots, 1)\}$$

The S is a basis for $\mathbb{R}^n$, called the standard basis.

$v \in \mathbb{R}^n$, $v = (a_1, a_2, \ldots, a_n) \to n$-tuple

$$v = a_1 \overset{e_1}{(1, 0, \ldots 0)} + a_2 \overset{e_2}{(0, 1, \ldots 0)} + \ldots + a_n \overset{e_n}{(0, 0, \ldots 1)}$$

$$= a_1 e_1 + a_2 e_2 + \ldots + a_n e_n.$$

Let $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1(1,2)+\alpha_2(3,5)=(0,0)$

2. Let $V = \mathbb{R}^2$, $S = \{(1,2), (3,5)\}$ $\Rightarrow \alpha_1 + 3\alpha_2 = 0$, $2\alpha_1 + 5\alpha_2 = 0$, M

Show that S is a basis for $\mathbb{R}^2$

$$\sim \begin{bmatrix} 1 & 3 & : & 0 \\ 0 & 1 & : & 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} \alpha_2 = 0 \\ \alpha_1 = 0 \end{matrix}\right\} \Rightarrow S \text{ is a linearly independent set.}$$

Let $v \in \mathbb{R}^2$, $\Rightarrow v = (a,b)$

$$v = k_1(1,2) + k_2(3,5)$$

$$\Rightarrow (a,b) = (k_1 + 3k_2,\ 2k_1 + 5k_2)$$

$$\Rightarrow \begin{matrix} k_1 + 3k_2 = a \\ 2k_1 + 5k_2 = b \end{matrix} \qquad M = \begin{bmatrix} 1 & 3 & : & a \\ 2 & 5 & : & b \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & : & a \\ 0 & 1 & : & 2a-b \end{bmatrix} \Rightarrow$$ The system is consistent $\Rightarrow \mathbb{R}^2 = \operatorname{span} S$

Note: $S = \{(1,0), (0,1)\}$ is the standard for $\mathbb{R}^2$ and $S = \{(1,2)$ is another basis for $\mathbb{R}^2$. $\Rightarrow$ A V can have

Another basis vector space basis: a basis of $P[x]$ is more

3. $P_n[x] = \{$ space of polynomials of degree $\le n \}$

4. $S = \{1, x, x^2, \dots, x^n\}$ is the standard basis for $P_n[x]$, $\dim(P_n[x]) = n+1$

$$S = \left\{ E_{ij} : E_{ij} = \begin{cases} a_{ij} = 1, & i^{th} \text{ row } j^{th} \text{ column} \\ a_{ij} = 0, & \text{elsewhere} \end{cases} \right\}$$

is the standard basis for $M_{(m,n)}$

For example: $V = M_{(3,2)}$

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}; \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}; \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad E_{31} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}; \quad E_{32} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$S = \{E_{11}, E_{12}, E_{21}, E_{22}, E_{31}, E_{32}\}$ is a basis for $M_{(3,2)}$

2.6 Rank of a Matrix.

Let $A = [a_{ij}]_{(m,n)}$, then the Row space of A is a subspace of $\mathbb{R}^n$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

generated by rows of A, and the column space of A is a subspace of $\mathbb{R}^m$ generated by columns of A

Row space of $A = \text{Span}\{(a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn})\}$

Column space of $A = \text{Span}\{(a_{11}, \ldots, a_{m1}), \ldots, (a_{1n}, \ldots, a_{mn})\}$

For example $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 2 \end{bmatrix}_{(2\times 3)}$

$\text{Row sp}(A) = \text{Span}\{(2,1,3), (0,4,2)\} \subset \mathbb{R}^3$

$\text{Col sp}(A) = \text{Span}\{(2,0), (1,4), (3,2)\} \subset \mathbb{R}^2$

Remark: The dimensions of the row space and column space of a matrix are called, respectively, row rank and column rank

Def: The row rank of a matrix A is the number of non-zero rows in the echelon form of A, while form of A^T.

The Column rank A = the number of non-zero rows in Echelon form.

For example: Let $A = \begin{bmatrix} 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{bmatrix}$

(i) To find the row rank A, write A in Echelon form: $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(ii) To find the Column rank of A we write A^T in Echelon form.

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 10 \\ 0 & -3 & -6 \\ -1 & -3 & -5 \end{bmatrix}$$

Remark: (i) Linearly independent rows of A, form a basis for the row space, while linearly independent rows of A^T, form a basis for the Column space of A.

(ii) $\dim(\text{Row}\,sp(A))$ = Row rank of A

$\dim(\text{Column}\,sp(A))$ = Row rank of A^T (column rank of A)

Note: Rank of A = Row rank of A

= Row rank of A^T

Exercise: Let $A = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$ find

Rank of A

Application:

The dimension of the solution space W to the homogeneous system $Ax = 0$ is $n - r$, when n is the number of unknowns and r is the rank of the coefficient matrix A

Example: Find the dimension and a basis of the solution space W of the system of linear equations

$$x + 2y - 4z + 3r - s = 0$$
$$x + 2y - 2z + 2r + s = 0$$
$$2x + 4y - 2z + 3r + 4s = 0$$

$$A = \begin{bmatrix} 1 & 2 & -4 & 3 & -1 \\ 1 & 2 & -2 & 2 & 1 \\ 2 & 4 & -2 & 3 & 4 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 0 & 0 & 3 & -6 \\ 1 & 2 & -4 & 3 & -1 \\ 0 & 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & -4 & 3 & -1 \\ 0 & 0 & 1 & -1/2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Ra of $A = 2$, $\Rightarrow$ dimension of the solution space $W = 5-2=3$

Let $y=a$, $r=b$, $s=c$

$\Rightarrow$ $z = 1/2\, b - c$; $x = -2a + 4(1/2\, b - c) - 3b + c$

$x =$ $\qquad x = -2a - b - 3c$

General solution:

$$\begin{bmatrix} x \\ y \\ z \\ r \\ s \end{bmatrix} = \begin{bmatrix} -2a - b - 3c \\ a \\ 1/2\, b - c \\ b \\ c \end{bmatrix}$$

$\Rightarrow$ the set $S = \{v_1, v_2, v_3\}$ is a basis for the solution space W.

$$v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ 1/2 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$X = a\,v_1 + b\,v_2 + c\,v_3$

Take Away: Find a Homogeneous system

Whose solution set W is generated by

$\{(1,-2,0,3), (1,-1,-1,4), (1,0,-2,5)\}$

INNER PRODUCT SPACES

V – Vector space over a field

$\langle\ ,\ \rangle : V \times V \to K$ (Inner product)

Def. Inner Product

An inner product on a vector space V is a function that associates a number $\langle u,v\rangle$ with each pair vectors $u,v \in V$ in such a way that the following axioms are satisfied for all vectors $u,v,w \in V$ and all scalars $k \in K$.

(i) $\langle u,v\rangle = \overline{\langle v,u\rangle}$ (Conjugate property) is $\mathbb{C}$

$\langle u,v\rangle = \langle v,u\rangle$ in $\mathbb{R}$ (Symmetry)

(ii) $\langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle$ (additivity property)

(iii) $\langle ku,w\rangle = k\langle u,w\rangle$ (Homogeneity in the first variable)

(iv) $\langle v,v\rangle \geq 0$, $\langle v,v\rangle = 0$ iff $v = 0$ (Positivity axiom)

Example

(1) Let $V = \mathbb{R}^n$ over $\mathbb{R}$ then

$\langle u,v\rangle = u \cdot v = a_1b_1 + a_2b_2 + \cdots + a_nb_n$

$u = (a_i)$, $v = (b_i)$

Satisfies all inner product axioms and is called the euclidean or standard inner product on $\mathbb{R}^n$

Let $n=3$, $V=\mathbb{R}^n$, $u=(1,2,3)$, $v=(1,0,-4)$

$\langle u,v\rangle = \langle (1,2,3),(1,0,-4)\rangle$

$= 1\cdot 1 + 2\cdot 0 + 3(-4) = -11$

2. Let $V=\mathbb{R}^2$, $u=(u_1,u_2)$, $v=(v_1,v_2)\in\mathbb{R}^2$ then

$\langle u,v\rangle = 3u_1v_1 + 2u_2v_2$ defines an inner product.

(i) $\langle u,v\rangle = \langle (u_1,u_2),(v_1,v_2)\rangle = 3u_1v_1 + 2u_2v_2$

$\langle v,u\rangle = \langle (v_1,v_2),(u_1,u_2)\rangle = 3v_1u_1 + 2v_2u_2$

$\langle u,v\rangle = \langle v,u\rangle$ since multiplication in $\mathbb{R}$ is commutative.

(ii) Let $u=(u_1,u_2)$, $v=(v_1,v_2)$, $w=(w_1,w_2)$

$u+v = (u_1+v_1, u_2+v_2)$

$\langle u+v,w\rangle = \langle (u_1+v_1, u_2+v_2),(w_1,w_2)\rangle$

$= 3u_1w_1 + 2u_2w_2 + 3v_1w_1 + 2v_2w_2$

$= \langle u,w\rangle + \langle v,w\rangle$

Additivity property holds

(iii) Let $u=(u_1,u_2)$, $v=(v_1,v_2)$, $k\in\mathbb{R}$.

$\langle ku,v\rangle = \langle (ku_1,ku_2),(v_1,v_2)\rangle$

$<$

$= 3ku_1v_1 + 2ku_2v_2$

$= k(3u_1v_1 + 2u_2v_2) = k\langle u, v\rangle$

$\Rightarrow$ Homogeneity property holds.

(iv) $\langle v, v\rangle = \langle (v_1, v_2), (v_1, v_2)\rangle$

$= 3v_1^2 + 2v_2^2 > 0$

$\langle v, v\rangle = 0$ iff $v = (0, 0)$

Note: The inner product above is different from the standard inner product on $\mathbb{R}^2$. This shows that a vector space can have more than one inner product.

3. Let $V = \mathbb{C}^n$ be the Unitary space

Let $u, v \in \mathbb{C}^n$, the standard inner product ~~$\langle u, v\rangle$~~ $u_1\bar{v}_1 + u_2\bar{v}_2$ ~~in $\mathbb{C}^n$~~

$3 + 0i$

$3 - 0i = 3$

$\langle u, v\rangle = \sum_{i=1}^{n} u_i \bar{v}_i, \quad u = (u_i), \quad v = (v_i)$

For example. Let $V = \mathbb{M}(2, 2)$

Let $u = (1 + i, -3i), \quad v = (3, 1 - i)$

$U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \quad \langle u, v\rangle = (1 + i)3 + (-3i)(1 + i) = 6 \quad V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$

The formula $\langle u, v\rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$ defines an inner product on $M_{(2,2)}$

5. Let $V = P_2[x]$

Let $p = a_0 + a_1x + a_2x^2$; $q = b_0 + b_1x + b_2x^2$

define $\langle p, q\rangle = a_0b_0 + a_1b_1 + a_2b_2$ is an inner product on $P_2[x]$.

In general $V = P_n[x]$, then the standard inner product is defined as

$$\langle p, q\rangle = \int_a^b p(x)\, q(x)\, dx$$

LENGTH AND ANGLE IN INNER PRODUCT SPACES

Def: Norm.

If V is an inner product space, then the norm (or length) of a vector u is denoted by

$\|u\| = \langle u, u\rangle^{1/2}$

The distance between any two vectors $u, v \in V$ is $d(u,v) = \|u - v\|$

Example

1. Let $V = \mathbb{R}$ with standard inner product; then

$\sqrt{}$

$||u|| = u_1^2 + u_2^2 + \cdots + u_n^2$

2. Let $V = \mathbb{R}^2$ with inner product

$\langle u, v \rangle = 3u_1v_1 + 2u_2v_2$

Let $u = (1,0)$, $v = (0,1)$

$||u|| = \langle u, u \rangle^{1/2} = \langle (1,0), (1,0) \rangle^{1/2} = 3^{1/2} = \sqrt{3}$

$d(u,v) = ||u - v|| = \langle u-v, u-v \rangle^{1/2}$

$= \langle (1,-1), (1,-1) \rangle^{1/2}$

$= (3 + 2)^{1/2} = \sqrt{5}$

Properties of the Norm.

(i) $|\langle u, v \rangle|^2 \leq ||u||^2 \, ||v||^2$ (Cauchy-Schwarz Inequality)

Note: If V is the complex inner product space

then: $\langle u, kv \rangle = \bar{k} \langle u, v \rangle$ (Homogeneity in 2nd variable)

(ii) $||u|| \geq 0$

(iii) $||ku|| = |k| \, ||u||$

$||ku||^2 = \langle ku, ku \rangle$

(iv) $||u+v|| \leq ||u|| + ||v||$

$= k\bar{k} \langle u, u \rangle$

$= |k|^2 \, ||u||^2$

$\Rightarrow ||ku|| = |k| \, ||u||$

$k \in \mathbb{C} \Rightarrow k = a + bi$

$\bar{k} = a - bi$

$(a+bi)(a-bi) = a^2 + b^2$

$\Rightarrow k\bar{k} = |k|^2 =$

Let $k = 1 + i$

~~$|k| = \sqrt{\ }$~~

$$\|u+v\|^2 = \langle u+v, u+v\rangle$$
$$= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$$
$$= \|u\|^2 + \langle u,v\rangle + \overline{\langle u,v\rangle} + \|v\|^2$$

$$k+\bar{k} = (a+bi) + (a-bi) = 2a = 2\,\mathrm{Re}(k)$$
$$2\,\mathrm{Re}(k) < |k|$$

$$\Rightarrow \|u+v\|^2 \leq \|u\|^2 + 2|\langle u,v\rangle| + \|v\|^2$$
$$\leq \|u\|^2 + 2\|u\|\,\|v\| + \|v\|^2$$
$$= (\|u\| + \|v\|)^2$$
$$\Rightarrow \|u+v\| \leq \|u\| + \|v\|$$

Angle Between Vectors.

Suppose u and v are non-zero vectors in an ~~$\|u+v\|$~~ product space V, then the angle between u and v is given by $\langle u,v\rangle$

follows from the Cauchy Schwarz inequality: $|\langle u,v\rangle| \leq \|u\|\,\|v\|$

$$\Rightarrow \frac{|\langle u,v\rangle|}{\|u\|\,\|v\|} \leq 1$$

$\Rightarrow \quad -1 \le \dfrac{\langle u,v\rangle}{\|u\|\,\|v\|} \le 1$

For Example: Let $V = \mathbb{R}^4$ with standard inner product

Let $u = (4,3,1,-2)$ and $v = (-2,1,2,3)$

Find the cosine of the angle between u and v

$$\cos\theta = \frac{\langle u,v\rangle}{\|u\|\,\|v\|} = \frac{-9}{\sqrt{30}\,\sqrt{18}} \quad \checkmark$$

Remark: $\cos\theta = 0 \Rightarrow \theta = \pi/2$ orthogonal

- Orthogonal set: A set S is said to be orthogonal if elements ~~Def: S an inner product orthogonal~~ pairwise vectors are orthogonal if $\langle u,v\rangle = 0$

Def: Normal: A vector $v \in V$ is said to be normal if $\|v\| = 1$.

Note. For any $v \in V$, the vector $\dfrac{v}{\|v\|}$ is a normal vector. The process of dividing a vector by its norm is called normalization.

Def: Orthonormal Set.

A set S is said to be orthonormal, if it contains orthogonal vectors each of norm equal to 1.

Example:

1. Let $S = \{(1,0,\dots,0), (0,1,0,\dots,0), \dots (0,0,\dots,1)\}$ be the standard basis for $\mathbb{R}^n$ with ~~standard~~ inner product, then S is an orthonormal set.

2. (i) Let $V = P_2[-1,1]$ with inner product $\langle p,q\rangle = \int_{-1}^{1} p(x)\, q(x)\, dx$. Given $p = x$, $q = x^2$

(ii) Show that p and q are orthogonal.

General Pythagoras Theorem:

If u and v are orthogonal vectors in an inner product space V, then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

$\langle v, u\rangle +$

$\|u+v\|^2 = \langle u,u\rangle + \langle u,v\rangle^{\to 0} + \langle v,v\rangle$
$= \|u\|^2 + \|v\|^2$

Vector Space: $V \to$ Basis: Set of L.I vectors
- s.t $V =$ Span(S)

Inner Product Space V; Vector space with inner product defined on it.
$\to$; Orthogonal Set $\to$ Bases

GRAM-SCHMIDT ORTHOGONALIZATION PROCESS

$\langle u, v_1\rangle v_1 + \langle u, v_2\rangle v_2 + \cdots + \langle u, v_n\rangle v_n$ let $\{v_1, v_2, \ldots, v_n\}$

Remark: If S is an orthonormal basis ~~for~~ all ~~linear~~ inner product space and

and $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordinary basis for the vector space V, then an orthogonal basis $S' = \{\beta_1, \beta_2, \ldots, \beta_n\}$ can obtained from S, by taking:

$\alpha_1 = \beta_1$

$$\beta_2 = \alpha_2 - \frac{\langle \alpha_2, \beta_1 \rangle \beta_1}{\|\beta_1\|^2}$$

$$\beta_3 = \alpha_3 - \frac{\langle \alpha_3, \beta_1 \rangle \beta_1}{\|\beta_1\|^2} - \frac{\langle \alpha_3, \beta_2 \rangle \beta_2}{\|\beta_2\|^2}$$

$\vdots$

$$\beta_k = \alpha_k - \sum_{i=1}^{k-1} \frac{\langle \alpha_k, \beta_i \rangle \beta_i}{\|\beta_i\|^2}$$

Example: Consider the vector space $\mathbb{R}^3$ with Euclidean inner product. Apply the Gram-Schmidt process to transform the basis $S = \{u_1 = (1,1,1), u_2 = (0,1,1), u_3 = (0,0,1)\}$ into an orthonormal basis.

$v_1 = u_1$

$$V_2 = u_2 - \frac{\langle u_2, v_1 \rangle v_1}{\|v_1\|^2}$$

$\|v\| = \langle v, v \rangle^{1/2}$

$\|v\|^2 = \langle v, v \rangle$

$\langle u_2, v_1 \rangle = \langle (0,1,1), (1,1,1) \rangle = 2$

$\|v_1\|^2 = \langle (1,1,1), (1,1,1) \rangle = 3$

$$V_2 = (0,1,1) - \frac{2}{3}(1,1,1) = \left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)$$

$$V_3 = u_3 - \frac{\langle u_3, v_1\rangle v_1}{\|v_1\|^2} - \frac{\langle u_3, v_2\rangle v_2}{\|v_2\|^2}$$

$\langle u_3, v_1\rangle = \langle (0,0,1), (1,1,1)\rangle = 1$

$\langle u_3, v_2\rangle = \langle (0,0,1), (-2/3, 1/3, 1/3)\rangle = 1/3$

$\|v_2\|^2 = \langle(-2/3, 1/3, 1/3), (-2/3, 1/3, 1/3)\rangle = 2/3$

$V_3 = (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1}{2}(-2/3, 1/3, 1/3)$

$\Rightarrow$ S' is an orthogonal set.

An orthonormal set $(0, -1/2, 1/2)$; $\|v_3\|^2 = \langle (0, -1/2, 1/2), (0, -1/2, 1/2)\rangle = 1/2$ normalized

$S' = \{ v_1 = (1,1,1), v_2 = (-2/3, 1/3, 1/3), v_3 = (0, -1/2, 1/2) \}$

$\langle v_i, v_k \rangle = 0$

i.e. $S'' = \left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\}$

Exercise: Let $V = \mathbb{R}^3$ have inner product

$\langle u, v\rangle = \langle (u_1, u_2, u_3), (v_1, v_2, v_3)\rangle = u_1v_1 + 2u_2v_2 + 3u_3v_3$

Use the Gram Schmidt process to transform

$u_1 = (1,1,1)$, $u_2 = (1,1,0)$, $u_3 = (1,0,0)$

into an orthonormal basis

Let $v_1 = u_1 = (1,1,1)$

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle v_1}{\|v_1\|^2}$$

$\langle u_2, v_1\rangle = \langle (1,1,0),(1,1,1)\rangle = 1\cdot 1 + 2\cdot 1\cdot 1 + 3\cdot 0\cdot 1$
$= 3$

$\langle v_1, v_1\rangle = \langle (1,1,1),(1,1,1)\rangle = 1\cdot 1 + 2\cdot 1\cdot 1 + 3\cdot 1\cdot 1$
$\langle u_3, v_1\rangle = \langle (1,0,0),(1,1,1)\rangle = 1$ 6

$\langle u_3, v_2\rangle = \langle (1,0,0),(1/2,1/2,-1/2)\rangle = 1/2$

$v_3 = u_3 - \frac{\langle u_3, v_1\rangle v_1}{\|v_1\|^2} - \frac{\langle u_3, v_2\rangle v_2}{\|v_2\|^2}$

$\|v_2\|^2 = \frac{1}{4} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{4} = 3/2$

$v_3 = (1,0,0) - \frac{1}{6}(1,1,1) - \frac{1}{3}\left(1/2, 1/2, -1/2\right)$

$= \left(2/3, -1/3, 0\right)$, $\|v_3\|^2 = 2/3$

$S = \{\overset{v_1}{(1,1,1)}, \overset{v_2}{(1/2, 1/2, -1/2)}, \overset{v_3}{(2/3, -1/3, 0)}\}$

$\langle v_1, v_2\rangle = 1/2 + 2\cdot 1/2 - 3\cdot 1/2 = 0$
$\langle v_2, v_3\rangle$ $1/2\cdot 2/3 + 2\cdot(-1/6) =$

$\Rightarrow S$ is an orthogonal set

$=$ $\qquad$ 0

$\langle v_1, v_3 \rangle = 2/3 + 2\cdot(-1/3) = 0$

$\|v_3\|^2 = \langle (2/3, -1/3, 0), (2/3, -1/3, 0) \rangle$

$= 4/9 + 2\,(1/9) = 6/9 = 2/3$

Take $\|v_3\| = \sqrt{2/3}$:

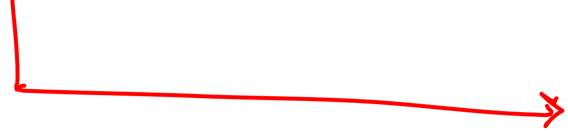
Let $\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\}$ with inner product $\langle p, q \rangle = \int_{-1}^{1} p(x)\, q(x)\, dx$ is an orthonormal basis for $\mathbb{R}^3$

Apply the Gram Schmidt process to transform the standard basis $\{1, x, x^2\}$ into an orthonormal basis.

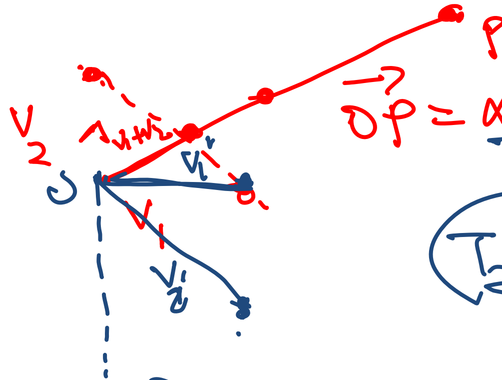
CHANGE OF BASIS.

Coordinates:

Example: Let $V = \mathbb{R}^2$ $\quad B = \{\overset{v_1}{(1,0)}, \overset{v_2}{(0,1)}\}$



$v_1 - v_2$



$\vec{OP} = \alpha v_1 + \beta v_2 = \alpha_1' v_1' + \alpha_2' v_2'$

Translation

way

Recall that $S = \{v_1, v_2, \dots, v_n\}$ is a basis for vector space V then every vector in V can be expressed in the form

$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

Def: The coordinate vector of v relative to S is a vector in $\mathbb{R}^n$ exactly defined by

$$[v]_S = [c_1, c_2, \dots, c_n]$$

Example 1. Let $S = \{\overset{v_1}{(1,2,1)}, \overset{v_2}{(2,9,0)}, \overset{v_3}{(3,3,4)}\}$ be a basis for $\mathbb{R}^3$. Find the coordinate vector of $v = (5,-1,9)$

Sol. Let $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

$\Rightarrow \quad (5,-1,9) = \alpha_1(1,2,1) + \alpha_2(2,9,0) + \alpha_3(3,3,4)$

$$\begin{cases} \alpha_1 + 2\alpha_2 + 3\alpha_3 = 5 \end{cases}$$

$\Rightarrow$

$$2\alpha_1 + 9\alpha_2 + 3\alpha_3 = -1$$
$$\alpha_1 + 4\alpha_3 = 9$$

$$M = \begin{bmatrix} 1 & 2 & 3 & : & 5 \\ 2 & 9 & 3 & : & -1 \\ 1 & & 4 & : & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & : & 5 \\ 0 & 5 & -3 & : & -11 \\ 0 & 2 & -1 & : & -4 \end{bmatrix}$$

$\Rightarrow \alpha_3 = 2, \; \alpha_2 = -1, \; \alpha_1 = 1$

$[v]_S = [3, -1, 2]$

2. Find the vector v in $\mathbb{R}^3$ whose coordinate vector with respect to S

is $[v]_S = [-1, 3, 2]$

Let $v = (x, y, z)$

$\Rightarrow (x, y, z) = -1(1, 2, 1) + 3(2, 9, 0) + 2(3, 3, 4)$

$\Rightarrow (x, y, z) = (11, 31, 7)$

3. Let $V = P_2[x]$.

Let $p = 4 - 3x + x^2$, find $[p]_S$

$S = \{p_1 = 1 + x, \; p_2 = (1 + x^2), \; p_3 = x + x^2\}$

Let $p = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3$

$$4-3x+x^2 = \alpha_1(1+x)+\alpha_2(1+x^2)+\alpha_3(x+x^2)$$

$$=(\alpha_1+\alpha_2)1+(\alpha_1+\alpha_3)x+(\alpha_2+\alpha_3)x^2$$

$$\Rightarrow \begin{cases} \alpha_1+\alpha_2 = 4 \\ \alpha_1+\alpha_3 = -3 \\ \alpha_2+\alpha_3 = 1 \end{cases} \qquad M = \left[\begin{array}{ccc:c} 1 & 1 & 0 & 4 \\ 1 & 0 & 1 & -3 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$[P] = [0, 4, 1, 3]$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & 7 \end{array}\right] \qquad \left[\begin{array}{ccc:c} 1 & 1 & 0 & 4 \\ & & -1 & 3 \\ 0 & 2 & 2 & \end{array}\right] \qquad \alpha_3 = -3 \quad \alpha_2 = 4$$

CHANGE OF BASIS:

If we change the basis for a vector space V from some old basis $B = \{u_1, u_2, \dots, u_n\}$ to some new basis $B' = \{v_1, v_2, \dots, v_n\}$, then the old coordinate matrix $[v]_B$ is related to the new coordinate matrix $[v]_{B'}$ by the equation

$$[v]_{B'} = P[v]_B,$$

where the columns of P are coordinate matrices of the old basis vectors relative to the new basis, that is column vectors of P are $[u_1]_{B'}, [u_2]_{B'}, \dots, [u_n]_{B'}$.

P is called the transition matrix from B to B'.

Example: Consider the bases

$B = \{u_1 = (1,0), (0,1)\}$, $B' = \{v_1 = (1,1), v_2 = (2,1)\}$

(i) Find the transition matrix P from B to B'

$\begin{cases} a_{21} + 2a_{22} = 0 \\ a_{21} + a_{22} = 1 \end{cases}$

Let $u_1 = a_{11} v_1 + a_{12} v_2 \Rightarrow (1,0) = a_{11}(1,1) + a_{12}(2,1)$

$u_2 = a_{21} v_1 + a_{22} v_2 \qquad (0,1) = a_{21}(1,1) + a_{22}(2,1)$

$$\left[\begin{array}{cc|c|c} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right] \Rightarrow [u_1]_{B'} = [-1, 1]$$

$[u_2]_{B'} = [2, -1]$

$\Rightarrow P = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$ $\quad R_1: -1 \;\; 2 \quad R_2: 1 \;\; -1$

(ii) Find the change of basis matrix Q, from B' to B

$Q = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

(iii) $PQ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Remark: If P is the transition matrix from B to B' then

(i) P is invertible

(ii) P^{-1} is the transition matrix from B' to B

$$\Rightarrow \underset{B\to B'}{P^{-1}} = \underset{B'\to B}{Q}$$

Take Away:

Consider the bases

$B = \{u_1 = (-3, 0, -3), u_2 = (-3, 2, 1), u_3 = (1, 6, -1)\}$

$B' = \{v_1 = (-6, -6, 0), v_2 = (-2, -6, 4), v_3 = (-2, -3, 7)\}$

(i) Find the transition matrix from B to B'

(ii) Compute $[w]_{B'}$, where $w = (-5, 8, 5)$

that: $[v]_{B'} = P[v]_B$

$f: V \to W$ → Vector space

Vector space

LINEAR TRANSFORMATIONS

Def If $f: V \to W$ is a function from a vector space V into the vector space W, then f is called a linear transformation if (i) $f(u+v) = f(u) + f(v)$ $\forall\ u, v \in V$

(ii) $f(kv) = k f(v), \quad v \in V, \quad k \in K = \mathbb{R}$ or $\mathbb{C}$

$f(u+v) = f((\cancel{y}\ \cancel{x_1+x_2}, \cancel{+y} y_1+y_2))$

Example, let $f: \mathbb{R}^2 \to \mathbb{R}^3$, $u = (x_1, y_1)$

and $v = (x_2, y_2)$

$= f(u+v) = f((x_1+x_2, y_1+y_2)) = f(x_1+x_2, (x_1+x_2)+(y_1+y_2), (x_1+x_2)-(y_1+y_2))$

$= f((x_1, y_1)) + f((x_2, y_2)) = f(u) + f(v)$

(ii) $f(kv) = f(k(x_2, y_2)) = f(kx_2, ky_2)$

$= (kx_2, kx_2+ky_2, kx_2-ky_2)$

$= k(x_2, x_2+y_2, x_2-y_2) = k f(v)$

② Let $f: \mathbb{R}^2 \to \mathbb{R}^2$

$f((x, y)) = (x^2, y)$

$f(1,2) = (1,2)$

$f(2,4) = (2^2, 4)$

e

Let $v = (x_1, y_1) \in \mathbb{R}$

$u = (x_2, y_2)$

$u+v = (x_1+x_2, y_1+y_2)$

$f(u+v) = ((x_1+x_2)^2, y_1+y_2)$

$(x_1^2+2x_1x_2+x_2^2, y_1+y_2)$

$f(u) + f(v) = (x_1^2, y_1) + (x_2^2, y_2) = (x_1^2 + x_2^2, y_1+y_2)$

$\Rightarrow f(u+v) \neq f(u) + f(v)$

is not a linear transformation.

Def: ~~Hence~~ Let A be a fixed $m \times n$ matrix. If we use matrix notation for vectors in $\overset{\text{input}}{\mathbb{R}^n}$ and $\overset{\text{output}}{\mathbb{R}^m}$, then we can define a function $T : \mathbb{R}^n \to \mathbb{R}^m$ by

$v \longrightarrow f(v)$

$$T(x) = Ax$$

$x \in \mathbb{R}^m$ $\qquad Ax \in \mathbb{R}^m$

The linear transformation T is called multiplication by A

(*) Matrix Representation of Linear Transformations

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation we can find an $m \times n$ matrix A such that T is multiplication by A

Let $B = \{e_1, e_2, \dots, e_n\}$ be a basis for $\mathbb{R}^n$ and $B' = \{e_1', e_2', \ldots, e_n'\}$ be a basis for $\mathbb{R}^m$, then the matrix

$$A = \left[[T(e_1)]_{B'} \; [T(e_2)]_{B'} \; \cdots \; [T(e_n)]_{B'} \right]$$

Where $[T(e_i)]_{B'}$ equals the coordinates of $(T(e_i))$ relative to B'

Example 1. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x_1, x_2) = (x_1 + 2x_2, x_1 - x_2)$

Find the matrix of T relative to the standard basis for $\mathbb{R}^2$

$$B = \{\overset{e_1}{(1,0)}, \overset{e_2}{(0,1)}\}$$

$$T(e_1) = T(1,0) = (1,1) = a_{11}(1,0) + a_{12}(0,1) = (a_{11}, a_{12})$$

$$T(e_2) = T(0,1) = (2,-1) = a_{21}(1,0) + a_{22}(0,1) = (a_{21}, a_{22})$$

$$\Rightarrow [T(e_1)]_B = [1,1] \; ; \; [T(e_2)]_B = [2,-1]$$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = [T]_B$$

Example Let $T: \mathbb{R}^3 \to \mathbb{R}^4$

$$T(x_1, x_2, x_3) = (x_1 + x_2, x_1 - x_2, x_3, x_1)$$

$B' = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$ (the vectors labelled e_1', e_2', e_3', e_4')

$\in \mathbb{R}^3$ $\qquad \in \mathbb{R}^4$

$T(e_1) = T(1,0,0) = (1,1,0,1) = a_{11}e_1' + a_{12}e_2' + a_{13}e_3' + a_{14}e_4'$

Let $\{e_1, e_2, e_3\}$ standard basis for $\mathbb{R}^3$

$T(e_2) = T(0,1,0) = (1,-1,0,0) = a_{21}e_1' + \cdots + a_{24}e_4'$

$T(e_3) = T(0,0,1) = (0,0,1,0) = a_{31}e_1' + \cdots + a_{34}e_4'$

$[T(e_1)]_{B'} = [1,1,0,1]$

$[T(e_2)]_{B'} = [1,-1,0,0]$

$[T(e_3)]_{B'} = [0,0,1,0]$

$$[T] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Example: 3 Let $T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(x_1, x_2) = (x_1 + 2x_2, x_1 - x_2)$

Find the matrix of T relative to

$B = \{(1,-1), (1,0)\}$ (labelled u_1, u_2)

$T(u_1) = T(1,-1) = (-1,2) = a_{11}(1,-1) + a_{12}(1,0)$

$(a_{11}, -a_{11}) + (a_{12}, 0)$

$T(u_2) = T(1,0) = (1,1) = a_{21}(1,-1) + a_{22}(1,0)$

$(a_{11} + a_{12}, -a_{11})$

$a_{11} = -2 \qquad a_{21} = -1$

$\Rightarrow (a_{11} + a_{12} 2 = -2) \Rightarrow (a_{11} + a_{12} 2 - a_{11})$
$(a_{21} + a_{22} = 1)$

$(a_{21} + a_{22} 2 = 1) = (a_{21} + a_{22} 2 \, 1 - a_{21})$

$$[T]_B = \begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix}$$

Example 4: Let $T: P_1[x] \to P_2[x]$
$T(p(x)) = x\,p(x)$. Find the standard matrix for T

$p(x) \in P_1[x] \Rightarrow p(x) = a_0 + a_1 x$
$p(x) \in P_2[x] \Rightarrow p(x) = a_0 + a_1 x + a_2 x^2$

Standard basis for $P_1[x]$, $B = \{1, x\}$
" " $P_2[x]$, $B' = \{1, x, x^2\}$

$T(1) = x = a_{11} 1 + a_{12} x + a_{13} x^2$
$T(x) = x^2 = a_{21} 1 + a_{22} x + a_{23} x^2$

$\Rightarrow 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2 \Rightarrow a_{11} = 0, a_{12} = 1; a_{13} = 0$
$0 \cdot 1 + 0 \cdot x + 1 \cdot x^2 \Rightarrow a_{21} = 0, a_{22} = 0; a_{23} = 1$

$$[T] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Remark: Let $T: V \to V$ be a linear transformation on a finite dimensional vector space V. If A is a matrix of T with respect to a basis B, and A' is the matrix of T with respect to B' then $A' = P^{-1} A P$, where P is the transition matrix from B' to B.

In Example 1, Standard matrix of T is $A = [T]_B = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ and $B = \{(1,0), (0,1)\}$

$A' = [T]_{B'} = \begin{bmatrix} -3 & -1 \\ 1 & 2 \end{bmatrix}$; $B' = \{(1,-1), (1,0)\}$

Change of basis matrix P from B' to B

$$P = \begin{bmatrix} & \\ & \end{bmatrix}$$

Take Away: $T: P_1[x] \to P_1[x]$

Note: $A' = P^{-1} A P \Rightarrow$ (similarity)

$T(a_0 + a_1 x) = a_0 + a_1(x+1)$

$B = \{6+3x, 10+2x\}$ A and A' are similar

Let

$B' = \{ 2, 3+2x \}$ $= 6 + 3(x+1)$

Find the matrix A of T relative to B

" A' of T relative to B'

Prove that matrices A and A' are similar

Remark:

When testing for similarity, it is easy to forget whether P is the transition matrix from B to B' (Incorrect) or from B' to B (correct). It may help to call B the old basis, B' the new basis, A the old matrix, and A' the new matrix.

Thus New matrix $= P^{-1}$ old Matrix P

$P \to$ New basis $\longrightarrow$ Old Basis

Let: $\underline{A} \Rightarrow \left[\begin{matrix} 3 & 0 \\ 8 & -1 \end{matrix} \right]$, n-dimensional $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ vector space $\underline{A}\underline{v} = \underline{3}(\underline{v})$

$A = [T]_B$ $n \times n$ matrix

$A(v) \; A(v) = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ Scalar

6. EIGENVALUES AND EIGENVECTORS

In many problems in science and mathematics, a linear operator $T: V \to V$ is given, and it is of importance to determine scalars λ for which the vector equation
$$Tv = \lambda v$$ has no zero solution.

Def 6.1 If A is an $n \times n$ matrix, then a non-zero vector $x \in \mathbb{R}^n$ is called an eigenvector of A if Ax is a scalar multiple of x, that is
$$Ax = \lambda x, \text{ for some scalar } \lambda.$$
The scalar λ is called an eigenvalue of A and x is said to be an eigenvector corresponding to λ.

Example $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

The eigenvalue $\lambda = 3$. Since $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector

$$\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow Ax = 3x$$

To find eigenvalues of an $n \times n$ matrix A we rewrite $Ax = \lambda x$ as

$$Ax = \lambda I x \quad \rightarrow \text{identity } n\times n \text{ matrix}$$

$$\Rightarrow \quad Ax - \lambda I x = 0 \quad \rightarrow \text{zero } n\times n \text{ matrix}$$

$$\Rightarrow \quad (A - \lambda I)x = 0 \quad \text{---(1)}$$

For λ to be an eigenvalue, there must be a nonzero solution to the homogeneous system (1) i.e iff $\det(A - \lambda I) = 0$

Example 2. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 3 & 3 \\ -1 & 0 \end{bmatrix}$$

The characteristic equation of A, scalars satisfying this equation are called eigenvalues of A. $\det(A - \lambda I)$ is called the characteristic polynomial of A, denoted by $\Delta_A(\lambda)$ and $\Delta_A(\lambda) = 0$ is called

$$A = \begin{bmatrix} 3 & 3 \\ -1 & 0 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 3 & 3 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{bmatrix}$$

$$\Delta_A(\lambda) = \det(A - \lambda I) = -\lambda(3-\lambda) + 2$$
$$= \lambda^2 - 3\lambda + 2$$

$$\Delta_A(\lambda) = 0 \Rightarrow \lambda^2 - 3\lambda + 2 = 0$$
$$\Rightarrow (\lambda - 1)(\lambda - 2) = 0$$
$$\Rightarrow \lambda = 1 \text{ or } \lambda = 2$$

The solutions of the equation $\Delta_A(\lambda) = 0$ are $\lambda = 1$ or $\lambda = 2$, these are the eigenvalues of A.

Example 8. Find the eigenvalues of the matrix $A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$

$$\Delta_A(\lambda) = \det(A - \lambda I) = (-2-\lambda)(2-\lambda) + 5$$
$$= -4 + 2\lambda - 2\lambda + \lambda^2 + 5$$

$$A - \lambda I = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -2-\lambda & -1 \\ 5 & 2-\lambda \end{bmatrix}$$

$$\Delta_A(\lambda) = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = \sqrt{-1} = \pm i \notin \mathbb{R}, \text{ hence}$$

Note: If the field of scalars $K =$

A has no eigenvalues since $\pm i \in \mathbb{C}$

Example 4. Find the eigenvalues of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Remark: (i) If A is a 2×2 matrix, then

$$\Delta_A(\lambda) = \lambda^2 - \lambda\, tr(A) + \det(A)$$

(ii) If A is a 3×3 matrix, then

$$\Delta_A(\lambda) = \lambda^3 - \lambda^2\, tr(A) + (A_{11} + A_{22} + A_{33})\lambda - \det(A)$$

sum of diagonal entries

cofactors.

$$\Delta_A(\lambda) = \lambda^3 - 8\lambda^2 + 17\lambda - 4 = (\lambda - 4)(\lambda^2 - 4\lambda + 1)$$

Solution $\begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ $(\lambda)_1 = tr(A) = 8$ $4, \lambda = 2 \pm \sqrt{3}$

$|A| =$ $\lambda = 2 \pm \sqrt{3}$

$A_{33} =$

The eigenvectors of A corresponding

to an eigenvalue λ are the non-zero vectors that satisfy $Ax = \lambda x$.
Equivalently the eigenvectors corresponding to λ are the non-zero vectors in the solution $(A-\lambda I)x = 0$. We call this solution space the eigenspace of A corresponding to λ.

Example: Find bases for the eigenspaces of $A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$\Delta_A(\lambda) = \lambda^3 - 11\lambda^2 + 35\lambda - 25$

$\Delta_A(\lambda) = \lambda^3 - \lambda^2 \operatorname{tr}(A) + \lambda(A_{11} + A_{22} + A_{33}) - \det(A)$

$= (\lambda - 1)(\lambda - 5)^2$, $A_{ij} = (-1)^{i+j}|M_{ij}|$, $= (\lambda-1)(\lambda-5)(\lambda-5)$, $A_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 0 \\ 0 & 5 \end{vmatrix} = 15$

$\operatorname{tr}(A) = 11$, $A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix} = 5$, $A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & 0 \\ 0 & 5 \end{vmatrix} = 15$

$\Delta_A(\lambda) = 0$, $\lambda_1 = 1$, $\lambda_2 = 5$ (multiplicity 2)

Let $v_1 = (x_1, y_1, z_1)$ s.t. $(A - \lambda_1 I)v_1 = 0$

$A - \lambda_1 I = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$(A-\lambda_1 I)V_1 = 0 \Rightarrow \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$M = \left[\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$V_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow z_1 = 0,\ x_1 = y_1,\ y_1$ free variable. Let $y_1 = s \Rightarrow x_1 = s$

Let $V_2 = \begin{bmatrix} x_3 \\ y_3 \\ z_2 \end{bmatrix}$; $(A-\lambda_2 I)V_2 = 0$

$$A-\lambda_2 I = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A - 5I)v_2 = \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad M = \left[\begin{array}{ccc|c} -2 & -2 & 0 & 0 \\ -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

y_2 and z_2 are free variables, let $y_2 = t$, $z_2 = r$

$x_2 = -t$

$$v_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ r \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

~~Exercise~~: ~~let~~ ~~eigenspace~~ $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$, find a basis for the eigenspaces of A.

$\lambda_1 = 3$

$\lambda_2 = -1$

Take Away: Find a bases for the eigenspace of $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ & & \end{bmatrix}$

$-2 \quad 0 \quad 1$

Let $v_1 = (x_1, y_1)$

$(A - 3I)v_1 = 0 \Rightarrow \begin{bmatrix} 0 & 0 \\ 8 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$M = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 8 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

y_1 is a free variable, let $y_1 = 2s$

$\Rightarrow x_1 = s \Rightarrow v_1 = \begin{bmatrix} s \\ 2s \end{bmatrix} = s\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Next $v_2 = (x_2, y_2)$

$(A - (-1)I)v_2 = 0 \Rightarrow \begin{bmatrix} 4 & 0 \\ 8 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

y_2 free variable, let $y_2 = t$

$\Rightarrow x_2 = 0$

$\Rightarrow V_2 = \begin{bmatrix} 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Bases $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ $\quad A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

Form the matrix $P = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$; $P^{-1}AP = ?$

$\begin{bmatrix} 3 & 0 \\ & -1 \end{bmatrix}$

Application:

Def. Diagonalization:
A square matrix A is called diagonalizable if there is an invertible matrix P such that $P^{-1}AP$ is diagonal, the matrix P is said to diagonalize A.

Remark: Given a matrix A and a bases $B = \{P_1, P_2, \ldots, P_n\}$ for the ~~eigenspaces corresponding to eigenvalues of~~ A

P have P_1, P_2, P_3 as its column vectors diagonalizes A and $P^{-1}AP$ is a diagonal matrix with

Example: Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator given by

$$T(x_1, x_2, x_3) = (3x_1 - 2x_2, -2x_1 + 3x_2, 5x_3)$$

The standard matrix of T is

$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \text{ (Verify)}$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$ with the matrix

$P = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ diagonalizes A

i.e. $P^{-1} A P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$T: V \to V \to A = [T]_B$ $v \in V$ s.t.

$A v = \lambda v$

Takeaway: $D = P^{-1} A P$

diagonal matrix

$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$\underset{n\times n}{A}\, \underset{n\times 1}{x} = \lambda\, \underset{n\times 1}{x}$ eigenvalue

$\Leftrightarrow (A - \lambda I) x = 0$ eigenvector

$\Leftrightarrow (\lambda I - A) x = 0$

Characteristic equation for A

$$\Delta_A(\lambda) = \det(A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{vmatrix} = 0$$

$$\Delta_A(\lambda) = \lambda^3 - \lambda^2 \operatorname{tr}(A) + \lambda (A_{11} + A_{22} + A_{33}) - \det(A)$$

$tr(A) = 6$; $\det(A) = 6$; $A_{11} = 1$

$A_{22} = 6$

$A_{33} = 4$

$$\Delta_A(\lambda) = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

± 2 ; ± 3 , ± 6 , ± 1

$$\Delta_A(\lambda) = (\lambda - 1)\underbrace{(a\lambda^2 + b\lambda + c)}_{q_L(\lambda)} = \lambda^3 - 6\lambda^2 + 11\lambda - 6$$

$$q_L(\lambda) = \frac{\lambda^3 - 6\lambda^2 + 11\lambda - 6}{\lambda - 1} \qquad \frac{571}{3} = 190$$

$= \lambda^2 - 5\lambda + 6$

$$\Rightarrow \Delta_A(\lambda) = (\lambda - 1)(\lambda - 3)(\lambda - 2)$$

$1\;1\;1$

$0\;0\;0$

$$\begin{array}{r} 190 \\ 3\overline{)571} \\ -3 \\ \hline 27 \\ 27 \\ \hline 01 \end{array}$$

$$\begin{array}{l} \quad 100 + 90 \\ 3\overline{)500 + 70 + 1} \\ \quad 300 \\ \hline \quad 200 + 70 \\ \quad 200 + 70 \\ \hline \qquad 0 \quad 1 \end{array}$$

$$\Rightarrow \frac{571}{3} = 190 + \frac{1}{3}$$

Example:

$$\frac{x^3 - 1}{x - 1}$$

$$\begin{array}{r} x^2 + x + 1 \\ x - 1\overline{)x^3 - 1} \end{array}$$

$$\frac{x^3-1}{x-1} = (x^2+x+1)(x-1)$$

$$
\begin{array}{r}
-\ x^3 - x^2 \\
\hline
x^2 - 1 \\
-\ x^2 - x \\
\hline
x - 1 \\
\underline{x - 1} \\
0
\end{array}
$$

Eigenvalues $\lambda_1 = 1$; $\lambda_2 = 2$; $\lambda_3 = 3$

$\Rightarrow (A - \lambda_1 I)V_1 = 0$; $(A - \lambda_2 I)V_2 = 0$; $(A - \lambda_3 I)V_3 = 0$

Let $V_1 = (x_1, y_1, z_1)$; $V_2 = (x_2, y_2, z_2)$; $V_3 = (x_3, y_3, z_3)$

$(A - \lambda_1 I_3)V = 0$ ... be $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ eigenvectors corresponding to $\lambda_1 = 1$, ... respectively

$$M = \left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

y free

let $y = a$; $z = 0$;

, $\hat{z}$ a variable, $x_1 = 0$,

$$V_1 = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} = a \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$(A - \lambda_2 I)v_2 = 0 \Rightarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

z_2 is a free variable, let $z_2 = 2s$, $y_2 = 2s$

$$(A - \lambda_3 I)v_3 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\begin{bmatrix} -4 \\ 2 \end{bmatrix}$

z_3 is a free variable let $z_3 = t \Rightarrow x_3 = -t$, $y_3 = t$

$x_3 = -t$

$$V_3 = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$; diagonalizes A

Verify $P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

7. OPERATORS AND FORMS ON

(1, 2, 3)
$\Rightarrow 3$

INNER PRODUCT SPACES.

$\langle\, , \rangle : V \times V \to K$ → Inner product

$T : V \to V$ → i Operator

7.1 Types of Operators

7.1.1 Adjoint Operator

Let V be finite dimensional inner product space and $T : V \to V$ be a linear operator, then adjoint T^* is defined to be the linear operator such that $\langle Tu, v\rangle = \langle u, T^* v\rangle$

Remark: If $[T]_B$ is the matrix representation for T relative to a basis B, then the matrix of T^* relative to the same basis is

$$[T^*]_B = \left(\overline{[T]_B}\right)^T$$

Example: Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ with the standard matrix

$$[T] = \begin{bmatrix} 2 & 1-i & 0 \\ 3+2i & 0 & -4i \\ 2i & 4-3i & -3 \end{bmatrix}$$

Find an expression for T^*

$$[T]_B = \begin{bmatrix} 2 & 1+i & 0 \\ 3-2i & 0 & 4i \\ -2i & 4+3i & -3 \end{bmatrix}$$

The standard matrix for T^* is

$$[T^*]_B = \left(\overline{[T]_B}\right)$$

$$\Rightarrow [T^*]_B = \begin{bmatrix} 2 & 3-2i & -2i \\ 1+i & 0 & 4+3i \\ 0 & 4i & -3 \end{bmatrix}$$

$$T^*(x,y,z) = \begin{bmatrix} 2 & 3-2i & -2i \\ 1+i & 0 & 4+3i \\ 0 & 4i & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$T^*(x,y,z) = \big(2x+(3-2i)y-2iz,\ (1+i)x+z(4+3i),\ 4iy-3z\big)$$

defined

Example: Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ by

$T(x, y, z) = (2x + iy,\ y - 5iz,\ x + (1-i)y + 3z)$

Find the matrix of T relative to the basis $B = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$

Hence $[T(e_1)]_B = [2, 0, 1]$

$[T(0,1,0)]_B = (i, 1, 1-i)$

$[T(0,0,1)]_B = (0, -5i, 3)$

$[T(1,0,0)] = 2e_1 + 0 \cdot e_2 + 1 \cdot e_3$

$T(0,1,0) = i\, e_1 + 1 \cdot e_2 + (1-i) e_3$

$T(0,0,1) = (0, -5i, 3) = 0 e_1 + (-5i) e_2 + 3 e_3$

$$[T]_B = \begin{bmatrix} 2 & i & 0 \\ 0 & 1 & -5i \\ 1 & 1-i & 3 \end{bmatrix}$$

$$[T^*]_B = \overline{\left([T]_B\right)}^{T} = \begin{bmatrix} 2 & 0 & 1 \\ -i & 1 & 1+i \\ 0 & 5i & 3 \end{bmatrix}$$

$$T^*(x, y, z) = \begin{bmatrix} 2 & 0 & 1 \\ -i & 1 & 1+i \\ 0 & 5i & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x + z \\ -ix + y + (1+i)z \\ 5iy + 3z \end{bmatrix}$$

$\Rightarrow T^*(x, y, z) = (2x + z,\ -ix + y + (1+i)z,\ 5iy + 3z)$

T.1.2. Properties of the Adjoint T^*

Let S and T be linear operators on V and $k \in K$. Then

(i) $(S+T)^* = S^* + T^*$

(ii) $(kS)^* = \bar{k}\, S^*$

(iii) $(ST)^* = T^* S^*$

(iv) $(T^*)^* = T$

Def 7.1.3 Self Adjoint or Hermitian Operator

A linear operator T such that $T = T^*$ is called self adjoint operator

Example: Let T be the linear operator whose matrix is

$$A = \begin{bmatrix} 2 & 2+3i & 4-5i \\ 2-3i & 5 & 6+2i \\ 4+5i & 6-2i & -7 \end{bmatrix}$$

The matrix of T^* is $(\bar{A})^T$

$$(\bar{A})^T = \begin{bmatrix} 2 & 2+3i & 4-5i \\ 2-3i & 5 & 6+2i \\ 4+5i & 6-2i & -7 \end{bmatrix} = A$$

$\Rightarrow T^* = T$ Hence T is Hermitian.

Def 7.1.4 Normal Operator.

Let T be a linear operator on V, then T is said to be normal

If $TT^* = T^*T$ i.e. T and T^* commute.

Example: The linear T represented by $A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$

TT^*: $\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ normal

$T^*T = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, T^* represented by $(\bar{A})^T = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$

$TT^* = T^*T \Rightarrow$ Hence T is normal.

Def 7.1.4 Unitary Operator

An operator T defined on a vector space V is said to be unitary if

$$TT^* = T^*T = I = TT^{-1} = T^{-1}T$$

Remark: (i) Every unitary operator is normal.

(ii) If T is self adjoint, then $TT^* = T^2 \Rightarrow$ every self adjoint is normal.

(iii) T unitary $\Rightarrow T^* = T^{-1}$

7.2 FORMS ON VECTOR SPACES

Linear Operators ; $T : V \to V$

Linear Transformation $T : V \to W$

A form $f : V \times V \to$ Real or Complex

The Hermitian Form

A form $f : V \times V \to K$ on a vector space V is called Hermitian

if $f(\alpha, \beta) = \overline{f(\beta, \alpha)}$, $\alpha, \beta \in V$

Note: T is a linear operator on a finite dimensional inner product space V and f is a form on V, then

$$f(\alpha, \beta) = \langle T\alpha, \beta \rangle$$

If f is Hermitian, then $f(\alpha, \beta) = \overline{f(\beta, \alpha)}$

$$= \overline{\langle T\beta, \alpha \rangle} = \overline{\langle \beta, T^*\alpha \rangle} = \langle T^*\alpha, \beta \rangle$$

$$= \langle \alpha, T\beta \rangle$$

$$= \langle T^*\alpha, \beta \rangle$$

$$\Rightarrow \langle T\alpha, \beta \rangle = \langle T^*\alpha, \beta \rangle$$

$$\Rightarrow T = T^*$$

7.2.2. BILINEAR FORMS

Let V be a vector space over a field K.

A bilinear form on V is a function f

which assigns to each ordered pair of vectors α, β in V a scalar $f(\alpha, \beta) \in K$ and which satisfies

(i) $f(c\alpha_1 + \alpha_2, \beta) = c f(\alpha_1, \beta) + f(\alpha_2, \beta)$ $\forall\, \alpha_1, \alpha_2, \beta \in \mathbb{R}^2$

(ii) $f(\alpha, c\beta_1 + \beta_2) = c f(\alpha, \beta_1) + f(\alpha, \beta_2)$

Exercise: Check if

(i) $f(\alpha, \beta) = 1$ is a bilinear form on $\mathbb{R}^2$. Let $\alpha = (x_1, x_2)$, $\beta = (y_1, y_2)$

$f(c\alpha_1 + \alpha_2, \beta) = 1$

$c f(\alpha_1, \beta) = c \cdot 1$

$f(\alpha_2, \beta) = 1$

since $\Rightarrow f(c\alpha_1 + \alpha_2, \beta) \neq c f(\alpha_1, \beta) + f(\alpha_2, \beta)$

$\Rightarrow f$ is not a bilinear form on $\mathbb{R}^2$

(ii) $f(\alpha, \beta) = x_1 y_2 - x_2 y_1$

$f((x_1, x_2), (y_1, y_2)) = x_1 y_2 - x_2 y_1$

Let $\alpha_1 = (x_{11}, x_{12})$, $\alpha_2 = (x_{21}, x_{22})$

$f(c\alpha_1 + \alpha_2, \beta) = f((cx_{11} + x_{21}, cx_{12} + x_{22}), (y_1, y_2))$

$= (cx_{11} + x_{21}) y_2 - (cx_{12} + x_{22}) y_1$

$= cx_{11} y_2 - cx_{12} y_1 + x_{21} y_2 - x_{22} y_1$

$= c f(\alpha_1, \beta) + f(\alpha_2, \beta)$

Condition (i) is satisfied, similarly Condition (ii) is satisfied hence f is a bilinear

Matrix Representation of Bilinear form.

Let V be a finite dimensional vector space and let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis. If f is a bilinear form on V, the matrix of f in the ordered basis B is the matrix A with entries $a_{ij} = f(\alpha_i, \alpha_j)$, denoted by $[f]_B$

$i, j \in \{1, 2\}$

Example: Let $V = \mathbb{R}^2$. Let f be the bilinear form defined on $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$ by

$f((x_1, x_2), (y_1, y_2))$

$f(\alpha, \beta) = x_1 y_1 + x_1 y_2 + x_2 y_1 + x_2 y_2$. Find $[f]_B$, $B = \{\underset{e_1}{(1,0)}, \underset{e_2}{(0,1)}\}$

By def. $[f]_B = [a_{ij}] = [f(e_i, e_j)]$

$a_{11} = f(e_1, e_1) = f((\overset{x_1}{1}, \overset{x_2}{0}), (\overset{y_1}{1}, \overset{y_2}{0})) = 1 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 = 1$

$a_{12} = f(e_1, e_2) = f((1,0), (0,1)) = 1 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 + 0 \cdot 1 = 1$

$a_{21} = f(e_2, e_1) = f((0,1), (1,0)) = 0 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 + 1 \cdot 0 = 1$

$a_{22} = f(e_2, e_2) = f((0,1), (0,1)) = 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 + 1 \cdot 1 = 1$

$f((x_1,x_2),(y_1,y_2)) = x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$

$B' \quad [f]_B \quad u_1 = (1,-1), \; u_2 = (1,1) \qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$[f]_{B'} = [b_{ij}] = [f(u_i, u_j)]$

$b_{11} = f(u_1, u_1) \qquad b_{21} = f(u_2, u_1)$

$b_{12} = f(u_1, u_2) \qquad b_{22} = f(u_2, u_2)$

$$[f]_{B'} = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

· Find the change of basis matrix P from B' to B; $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$

$B' = \{(1,-1), (1,1)\}$

$B = \{(1,0), (0,1)\}$ $\qquad P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$$[f]_{B'} = P^T [f]_B P$$

Exercise: Let f be the bilinear form on $\mathbb{R}^2$ defined by

$$f((x_1,x_2),(y_1,y_2)) = 2x_1y_1 - 3x_1y_2 + x_2y_2$$

(i) Find

$B' = \{v_1=(2,1),\ v_2=(1,-1)\}$

Find the matrix A of f in the basis $B = \{u_1=(1,0),\ u_2=(1,1)\}$

(iii) Find the change basis matrix from B to B'. Verify $B = P^T A P$

(iv) Find the matrix B of f in the basis

7.2.4 Types of Bilinear Forms

(i) Symmetric Bilinear form.

Let f be a bilinear form on the vector space V. We say that f is symmetric if $f(\alpha,\beta) = f(\beta,\alpha),\ \forall\ \alpha,\beta \in V$

i.e. If A is the matrix of f, then $A = A^T$

(ii) Skew-Symmetric bilinear form

A bilinear form on V is called skew symmetric if $f(\alpha,\beta) = -f(\beta,\alpha)$

i.e. if A is the matrix of f, then $A = -A^T$

(iii) Alternating Bilinear form.

A bilinear form on V is said to be alternating if: $f(v,v) = 0,\ \forall\ v \in V$

Note. $u+v \in V \Rightarrow f(u+v,\ u+v) = 0$

$\Rightarrow \overset{0}{\cancel{f(u,u)}} + f(u,v) + f(v,u) + \overset{0}{\cancel{f(v,v)}} = 0$

$\Rightarrow \quad f(u,v) = -f(v,u) \quad \forall \; u,v \in V$

Quadratic Form.

If f is a symmetric bilinear form, then the quadratic form associated with f is the function q from V into F defined by $q(\alpha) = f(\alpha, \alpha)$

By definition, the symmetric matrix representing $q(\underbrace{x_1, x_2, \ldots, x_n}_{\alpha})$ has diagonal entries a_{ii} equal to the coefficient of x_i^2 and has entries a_{ij} and a_{ji} each equal to half the coefficient of $x_i x_j$

Example 1. The quadratic form on $\mathbb{R}^3$

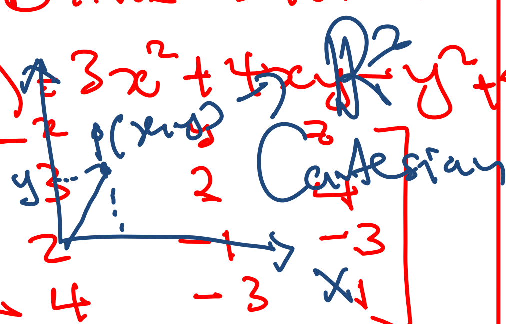
$q(x,y,z) = x^2 - 2yz + xz$, has matrix representation

$$A = \begin{array}{c} \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 0 & -1 \\ 1/2 & -1 & 0 \end{bmatrix} \end{array}$$

8. THREE DIMENSIONAL COORDINATE SYSTEM

2. $q(x,y,z) = 3x^2 + 4xy - y^2 + 8xz - 6yz + z^2$

$$A = \begin{array}{c} x \\ y \\ z \end{array} \begin{bmatrix} 3 & 2 & 4 \\ 2 & -1 & -3 \\ 4 & -3 & 1 \end{bmatrix}$$

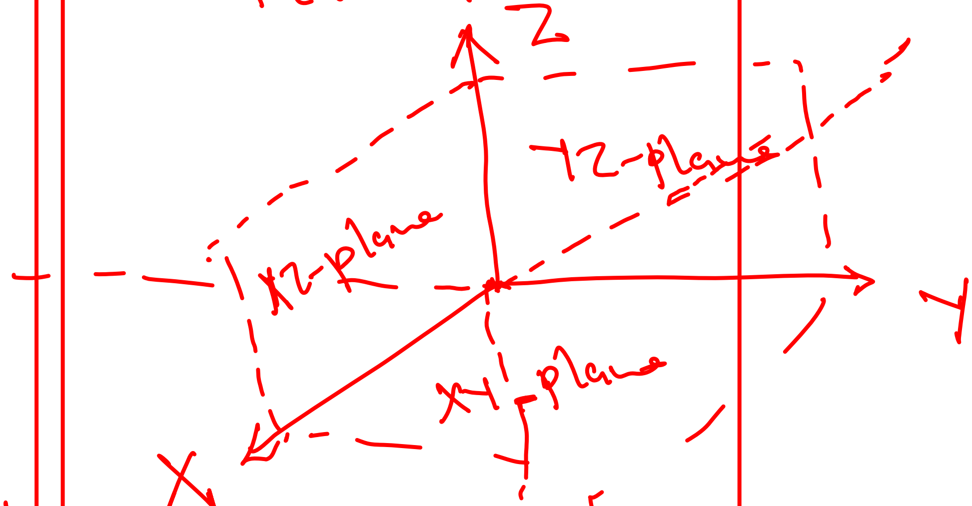


$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & -1 & -3 \\ 4 & -3 & 1 \end{bmatrix}$$

plane

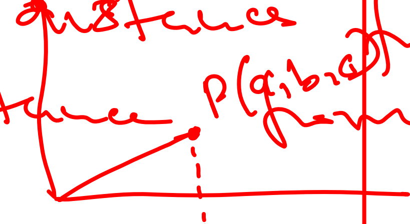
In order to represent points in space we first choose a fixed point O (the origin) and three directed lines through O that are perpendicular to each other called coordinate axes and labelled X-axis, Y-axis and Z-axis.

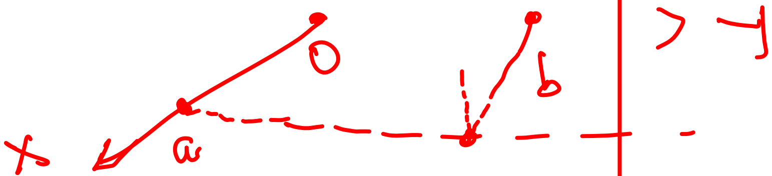
The three coordinate axes determine 3 coordinate planes; xy-plane, yz-plane and xz-plane.



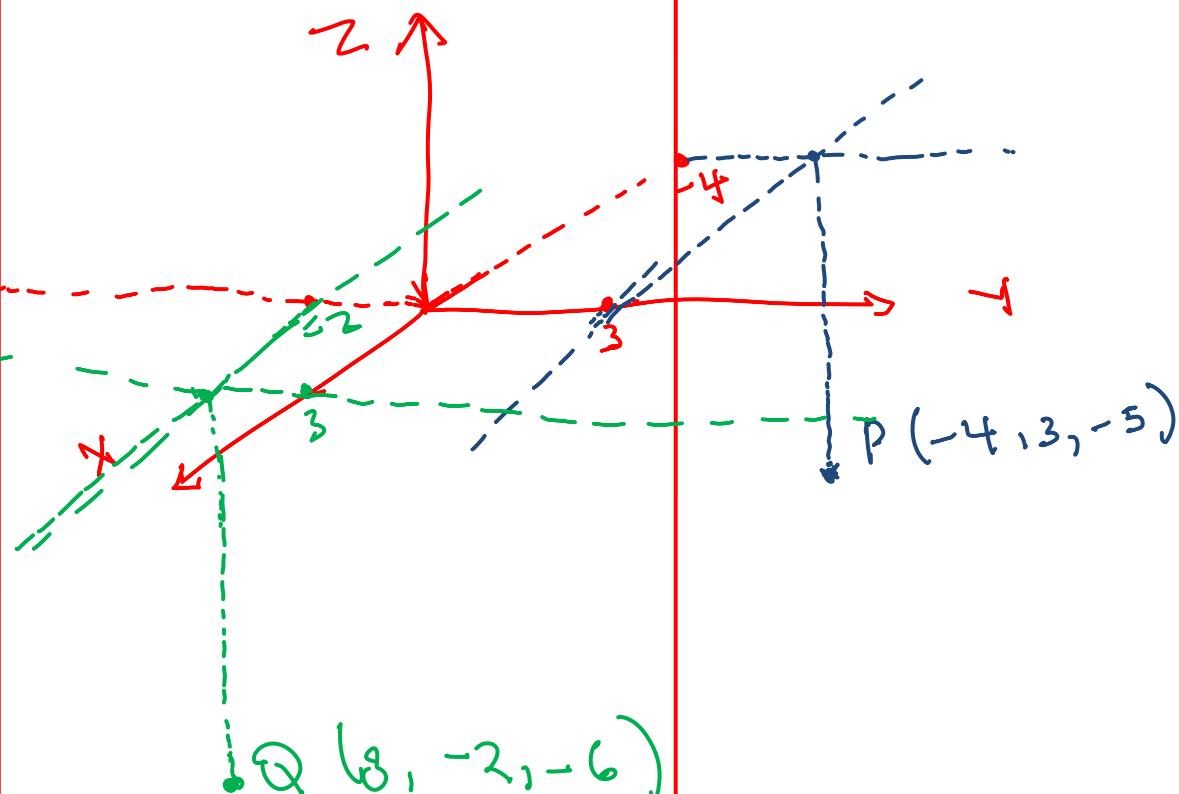
c is the distance from the xy-plane.

If P is a ~~point~~ point in space, let a be the directed distance from the yz-plane to P; b - distance from the xz-plane;





Examples: Locate the points P(-4, 3, -5) Q(3, -2, -6)



by an arrow or directed line segment

8.1 Operations on Vectors.

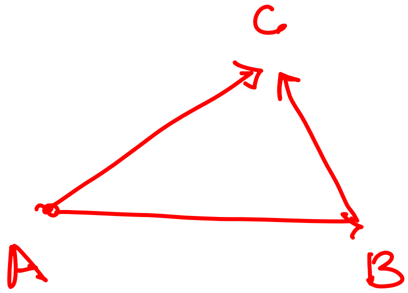
The length of the vector represents the magnitude of the vector and the

The term vector is used to indicate a quantity that has both magnitude and direction.

dotted line ... vector

Represented by $\vec{v}$

8.1.1 Combining vectors.



then: $\vec{AC} = \vec{AB} + \vec{BC}$

$\vec{AB} = \vec{AC} + \vec{CB}$

$\vec{BC} = -\vec{CB}$

8.1.2 Vectors Addition

Let $u, v \in \mathbb{R}^3$, then $u = (a_1, b_1, c_1)$

$v = (a_2, b_2, c_2)$

$u + v = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$

For example: $u = (-1, -3, 0)$, $v = (3, 2, -1)$

$= (2, -1, -1)$

8.1.3 Unit vectors:

Let $u \in \mathbb{R}^3$ s.t $u = (a, b, c)$ then $u = a(1,0,0) + b(0,1,0) + c(0,0,1)$

Let $i = (1,0,0)$, $j = (0,1,0)$, $k = (0,0,1)$ then

$u = ai + bj + ck$, the vectors i, j, k are called unit vectors

8.1.4. Dot Product:

If $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ then the dot product of $a, b \in \mathbb{R}$ is the number $a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$

For example: Let $u = (-1, 3, 4)$, $v = (2, 1, 3)$

then $u \cdot v = (-1)2 + 3(1) + 4(3) = 13$

Remark: If θ is the angle between u and v then

$$\cos\theta = \frac{u \cdot v}{|u| \cdot |v|}, \text{ where}$$

$$|u| = \sqrt{u \cdot u} \Rightarrow |u| = \sqrt{(-1)(-1) + 3\cdot 3 + 4\cdot 4} = \sqrt{26}$$

$$|v| = \sqrt{2\cdot 2 + 1\cdot 1 + 3\cdot 3} = \sqrt{14}$$

$$\cos\theta = \frac{u \cdot v}{|u||v|} = \frac{13}{\sqrt{26}\sqrt{14}}$$

Example: Show that $2i + 2j - k$ is perpendicular to $5i - 4j + 2k$.

$u(2, 2, -1)$

$$\cos\theta = \frac{2(5) + 2(-4) + (-1)2}{\sqrt{2^2 + 2^2 + (-1)^2}\sqrt{5^2 + (-4)^2 + 2^2}} = 0$$

$\theta = \pi/2$, hence $v(5, -4, 2)$

8.1.5 Cross Product

The cross product $u \times v$ of two vectors $u, v \in \mathbb{R}^3$, unlike the dot product, is a vector. For this reason it is also called the vector product. Note $u \times v$ is defined only when u and v are 3-D vectors.

Def: If $u = (a_1, a_2, a_3)$ and $v = (b_1, b_2, b_3)$ then the cross product of u and v is the vector

$$u \times v = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$ (determinant)

For example: $i \times j = k$ ($i = (1,0,0)$, $j(0,1,0)$)

$j \times i = -k$, $k \times j = -i$

$$i \times j = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = k = (0,0,1)$$

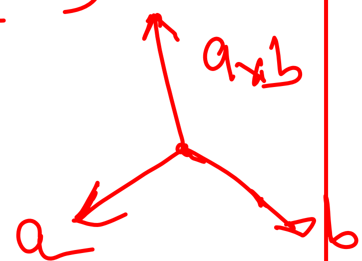
$j \times k = i$, $i \times k = j$, $k \times i = -j$

Example 2. $a = (1,3,4)$, $b = (2,7,-5)$

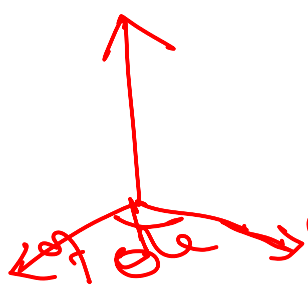
$$a \times b = \begin{vmatrix} i & j & k \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = -43i + 13j + k = (-43, 13, 1)$$

Remarks:

(i) The vector $a\times b$ is orthogonal to both a and b i.e. $(a\times b)\cdot a = 0 = (a\times b)\cdot b$



(ii) If θ is the angle between a and b, then $|a\times b| = |a||b|\sin\theta$



(iii) The length of the cross product $a\times b$ is equal to the area of the parallelogram determined by a and b. (i.e $|a\times b|$)

Exercise:

1. Find a vector perpendicular to the plane that passes through the points

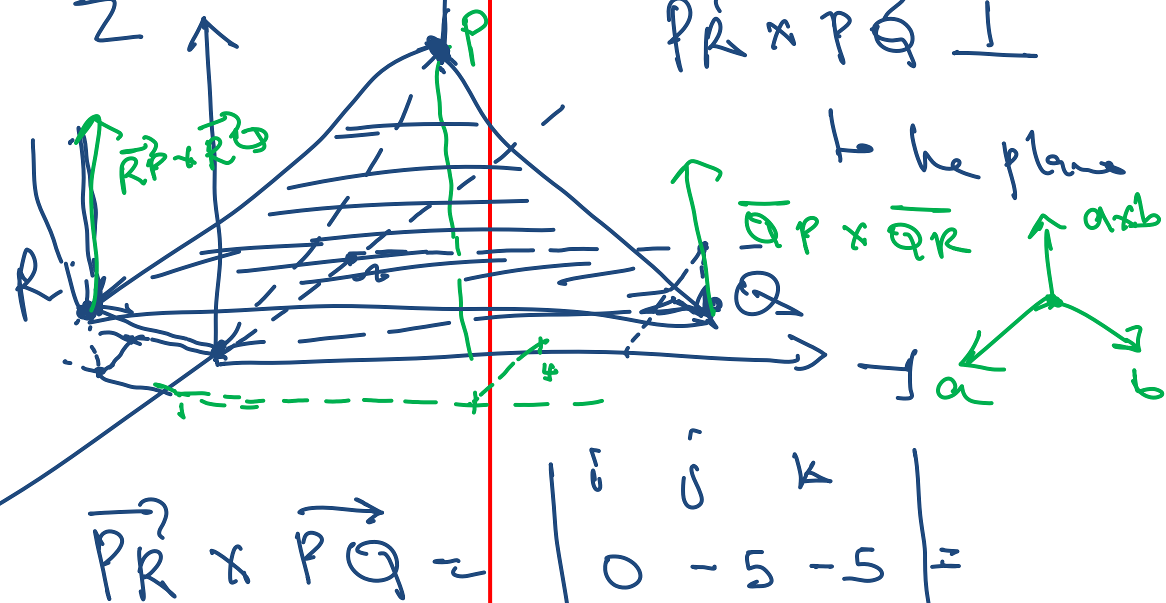
P, $Q(-2,5,-1)$, $R = (1,-1,1)$

$\vec{PR} = \vec{PO} + \vec{OR}$

$= (0,-5,-5)$

$\vec{PQ} = \vec{PO} + \vec{OQ}$

$= (-3,1,-7)$



$\vec{PR} \times \vec{PQ}$ $\perp$ to the plane

$$\vec{PR} \times \vec{PQ} = \begin{vmatrix} i & j & k \\ 0 & -5 & -5 \\ -3 & 1 & -7 \end{vmatrix} =$$

$-40i - 15j + 15k$

$(-40, -15, 15)$

2. Find the area of the triangle with vertices $P(1,4,6)$, $Q(-2,5,-1)$, $R = (1,-1,1)$

The area of the parallelogram with adjacent sides PQ and PR is the length of the cross product

$|PQ \times PR|$

$\frac{1}{2} |PQ \times PR|$

The

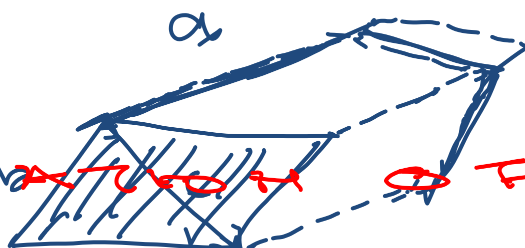
area of ΔPQR is

$\vec{PQ} \times \vec{PR}$

$\frac{1}{2}|PQ \times PR| = \frac{5}{2}\sqrt{82}$

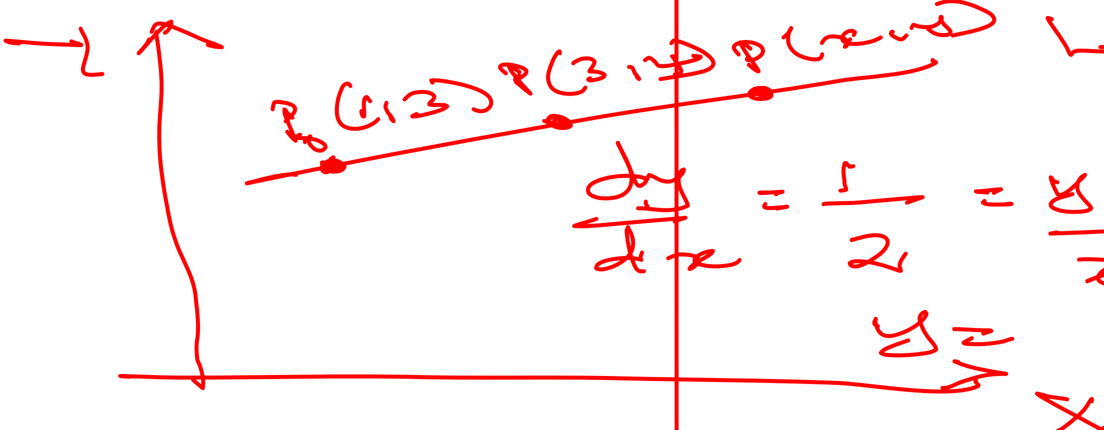
$\vec{PQ} \times \vec{PR} = (40 + 15j - 15k)$; $|PQ \times PR| = \sqrt{40^2 + 15^2 + 15^2}$

4. The volume of the parallelpiped determined by vectors a, b and c is the magnitude of the scalar triple product $V = |\underline{a} \cdot (\underline{b \times c})|$



8.2 EQUATION OF A LINE IN 3-D PLANE

2-D $y = mx + c$



$P_0(1,3)$ $P(3,4)$ $P(x,y)$ L

$\frac{dy}{dx} = \frac{1}{2} = \frac{y-3}{x-1}$

$y =$

In 3-D Space

Consider a point $P_0(x_0, y_0, z_0)$ on a line L. In 3-D the direction of the line is conveniently described by a vector. Let v be a vector parallel to L. Let $P(x, y, z)$ be an arbitrary point on L and let r_0 and r be the position vectors of P_0 and P. If a is the vector P_0P, then $r = a + r_0$.

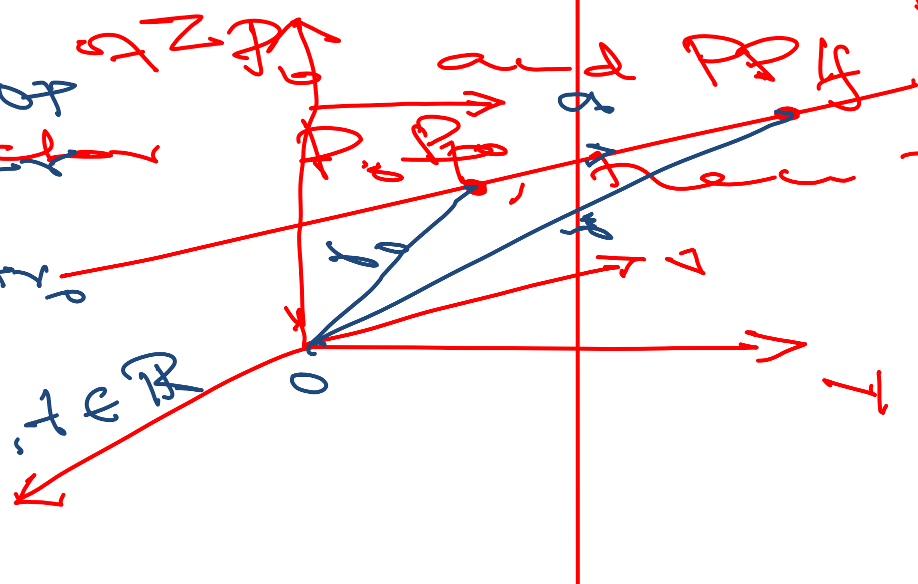
$P_0P = P_0O + OP$

$a = -r_0 + r$

$\Rightarrow r = a + r_0$

$P_0P \parallel v$

$\Rightarrow a = tv, \ t \in \mathbb{R}$



Since a and v are parallel there is a scalar $t \in \mathbb{R}$ such that $a = tv$

$\Rightarrow r = r_0 + tv$ is the vector equation of L

If $v = (a, b, c)$, then

$\Rightarrow (x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$

$\Rightarrow (x, y, z) = (x_0 + ta, y_0 + tb, z_0 + tc)$

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \end{cases}$$

This is the ~~parametric~~ parametric equation of the line through $P_0(x_0, y_0, z_0)$

$\Rightarrow \dfrac{x - x_0}{} = \dfrac{y - y_0}{} = \dfrac{z - z_0}{}$

parallel to the vector $v = (a, b, c)$

Example:

Find a vector equation and parametric equation of the line that passes through point $(5, 1, 3)$ and is parallel to the vector $i + 4j - 2k$

$r_0 = (5, 1, 3)$, $r = (x, y, z)$

$v = (1, 4, -2)$

Vector equation; $r = r_0 + tv$

$(x, y, z) = (5, 1, 3) + t(1, 4, -2)$

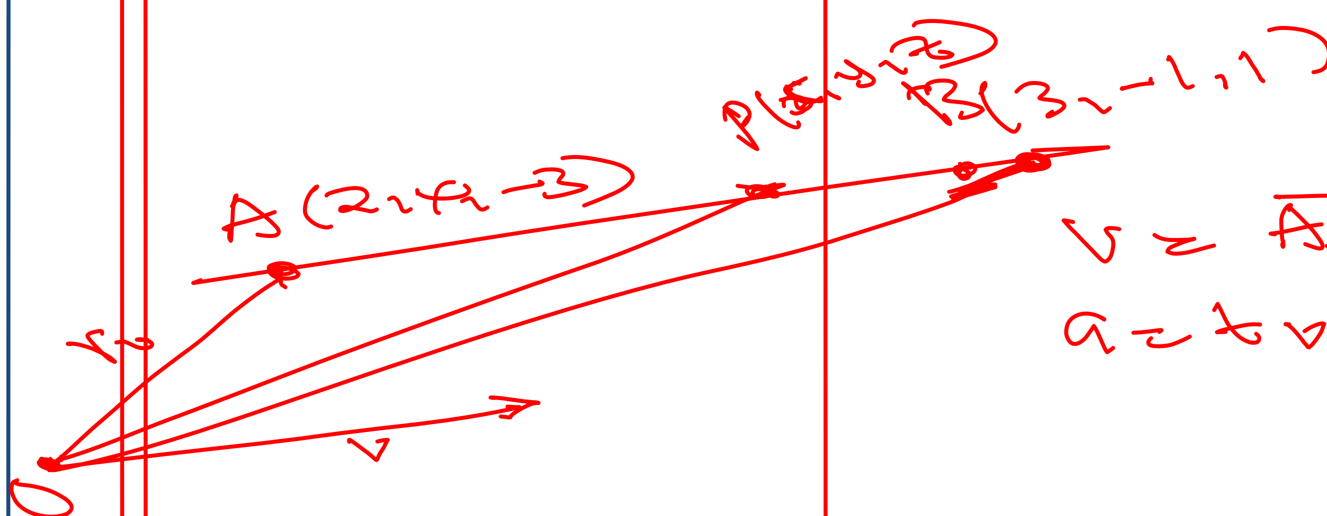
Parametric equation;

$(x, y, z) = (5 + t, 1 + 4t, 3 - 2t)$ parametric equation

$\begin{cases} x = 5 + t \\ y = 1 + 4t \\ z = 3 - 2t \end{cases}$

$\frac{x-5}{1} = \frac{y-1}{4} = \frac{z-3}{-2}$

passes A $(2, 4, -3)$, B$(3, -1, 1)$



$v = \overrightarrow{AB}$

$a = tv$

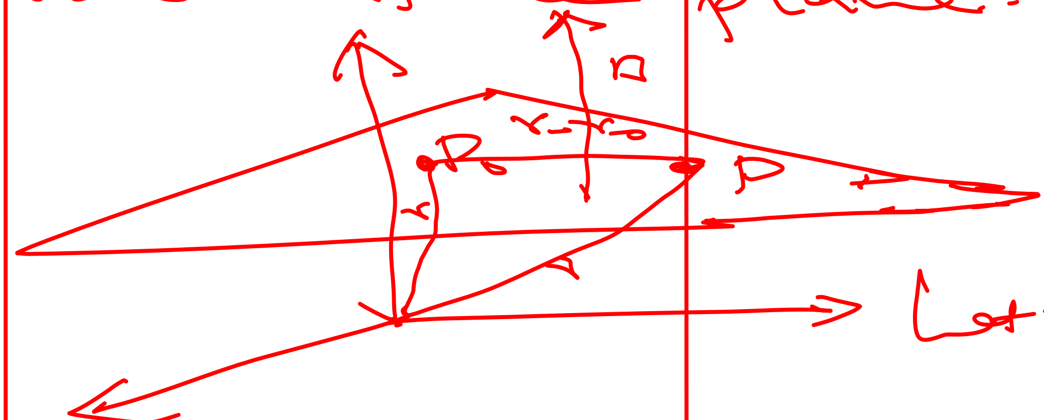
$\Rightarrow r = r_0 + tv$

$\Rightarrow (x,y,z) = (2,4,-3) + t(1,-5,4)$

$$\Rightarrow \frac{x-2}{1} = \frac{y-4}{-5} = \frac{z+3}{4}$$

8.3 EQUATION OF A PLANE

A plane in space is determined by a point $P_0(x_0,y_0,z_0)$ in the plane and a vector n that is orthogonal to the plane. Let $P(x,y,z)$ be an arbitrary point in the plane and r_0 and r be the position vectors for P_0 and P. Then the vector $r-r_0$ represents $\overrightarrow{P_0P}$ and n is orthogonal to every vector on the plane. Hence $n\cdot(r-r_0)=0$. This is called the vector equation of the plane.



Let $n=(a,b,c)$

$n\cdot(r-r_0) = (a,b,c)\cdot(x-x_0, y-y_0, z-z_0) = 0$

$\Rightarrow a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

the scalar equation of the plane through $P_0(x_0,y_0,z_0)$ with normal vector $n=(a,b,c)$

Example: Find the intercepts and the equation of the plane through the point $(2,4,-1)$ with normal $n=(2,3,4)$.

$(2,3,4)\cdot(x-2, y-4, z+1) = 0$

or $n\cdot r = n\cdot r_0$

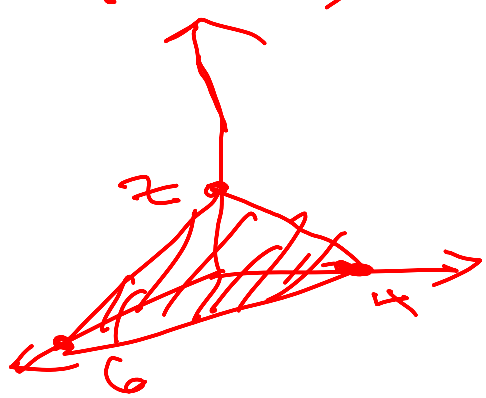
$\Rightarrow 2(x-2)+3(y-4)+4(z+1)=0$

$2x+3y+4z=12$

x-intercept $x=6$

y " $y=4$

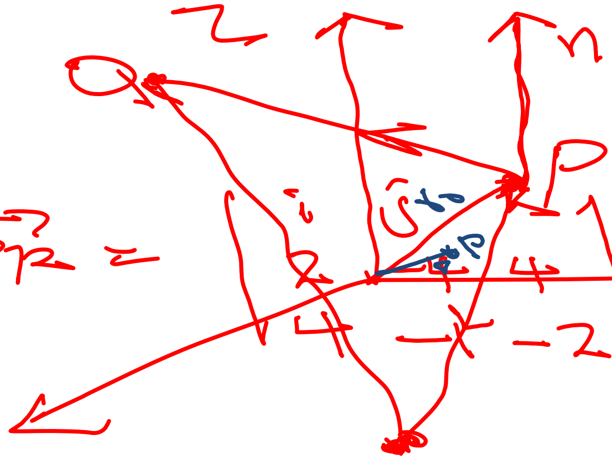
z " $z=3$



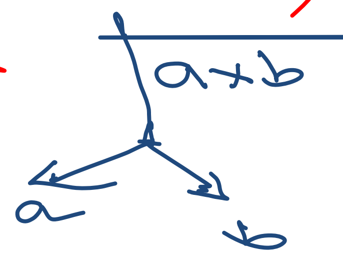
Example 2. Find an equation

Find the plane that passes through the points

$P(1,3,2)$, $Q(3,-1,6)$, $R=(5,2,0)$



$n = \vec{PQ} \times \vec{PR}$



$$n = \vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12i + 20j + 14k$$

$r_0 = (1,3,2)$, $\vec{r} = (x,y,z)$

$$n \cdot (r - r_0) = (12, 20, 14) \cdot (x-1, y-3, z-2) = 0$$

$$\Rightarrow \quad 6x + 10y + 7z = 50$$

Remark:

① Two planes are parallel if their normal vectors are parallel

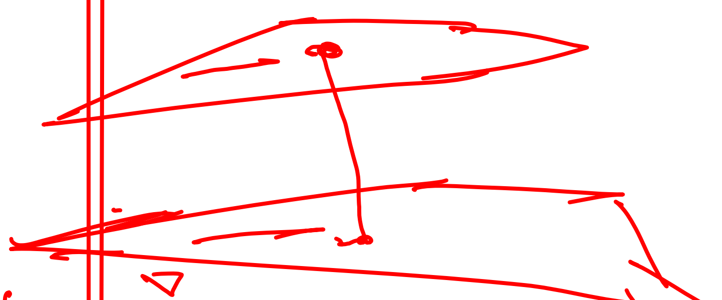
② The angle between two

planes is such that

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|}$$

(D) Find distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is written as

Example: Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$



$(10, 2, -2)$ and $(5, 1, -1)$ are parallel

Let $P_1(10, 2, -2)$, then

$$D = \frac{|5(10) + 2 + 2|}{\sqrt{5^2 + 1^2 + (-1)^2}}$$

the plane

~~Thanksgiving Activity: Find the~~ ... 5y ... point ... = 18

at which the line with

parametric equations

$x = 2 + 3t, \; y = 4t, \; z = 5 + t$