Ex0909A3: The generalized canonical transformation

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The problem: A system is described by the Hamiltonian H(q,p,t). The coordinate q is transformed q $\rightarrow$ Q=ψ(q,t)

1. Find the most general transformation p $\rightarrow$ P(q,p,t) so that (q,p) $\rightarrow$ (Q,P) will be canonical.
2. Calculate the appropriate generating function (explain why F_1(q,Q,t) isn't relevant).
3. Given the new Hamiltonian H'=0. Find the original Hamiltonian.
4. Prove that for ψ(q+ωt), the original Hamiltonian is a function that is linear to p.

The solution:

1. We will derive here the transformation for P:

$$[Q,P] = \frac{\partial Q}{\partial q}\cdot\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \frac{\partial Q}{\partial q}\cdot\frac{\partial P}{\partial p} = \frac{\partial \psi(q,t)}{\partial q}\cdot\frac{\partial P}{\partial p} = 1 \tag{1}$$

$$\frac{\partial P}{\partial p} = \frac{\partial q}{\partial \psi} = \frac{\partial q}{\partial Q} \tag{2}$$

$$\Rightarrow \boxed{P = p\cdot\frac{\partial q}{\partial Q} + f(q,t)} \tag{3}$$

2. Finding the generating function:
   We will choose the third class of generating function F_3(p,Q,t). For that we will define q as a function of Q and t, leading to q=g(Q,t), because it is given that Q=ψ(q,t). Consequently f can be expressed with Q and t.

$$-P = \frac{\partial F_3}{\partial Q} \tag{4}$$

$$-q = \frac{\partial F_3}{\partial p} \tag{5}$$

$$\frac{\partial F_3}{\partial Q} = -p\cdot\frac{\partial q}{\partial Q} - f(q,t) \tag{6}$$

$$F_3 = -pq - \int f(Q,t)dQ + h(p,t) \tag{7}$$

$$\frac{\partial F_3}{\partial p} = -q + \frac{\partial h}{\partial p} = -q \;\Rightarrow\; h = l(t) \tag{8}$$

$$F_3 = -pq - \int f(Q,t)dQ + l(t) \tag{9}$$

$$\Rightarrow \boxed{F_3 = -p\cdot g(Q,t) - \int f(Q,t)dQ} \tag{10}$$

We ignored the function that depends only on time because it can't affect the equation of motion.
The reason F_1(q,Q,t) isn't relevant is because it produces p and P from its partial derivatives and we want to use the given transformation q $\rightarrow$ Q=ψ(q,t). Also, we can see in equation (8) that if we change there the derivative to q (which will make the generating function be F_1) we will get an expression that contains p, but F_1 is not depended on p, so we will have a contradiction.

3. Here we will find the original Hamiltonian.
we know that:

$$0 = \grave{\mathcal{H}} = \mathcal{H} + \frac{\partial F_3}{\partial t} \tag{11}$$

$$\mathcal{H} = -\frac{\partial F_3}{\partial t} \tag{12}$$

$$\Rightarrow \boxed{\mathcal{H} = \frac{\partial(p \cdot g(Q,t))}{\partial t} + \frac{\partial(\int f(g(q,t),t)dQ)}{\partial t}} \tag{13}$$

4. Now it is given that q $\rightarrow$ Q=ψ(q+ωt). We can derive from that the new connection between q and Q:

$$q = u(Q) - \omega t \tag{14}$$

We shall find the original Hamiltonian.

$$\frac{\partial F_3}{\partial Q} = -p \cdot \frac{\partial q}{\partial Q} - f(Q,t) = -p \cdot \frac{\partial u}{\partial Q} - f(Q,t) \tag{15}$$

$$F_3 = -p \cdot u(Q) - \int f(Q,t)dQ + h(p,t) \tag{16}$$

$$\frac{\partial F_3}{\partial p} = -u(Q) + \frac{\partial h}{\partial p} = -q = -u(Q) + \omega t \;\Rightarrow\; h(p.t) = \omega t \cdot p \tag{17}$$

$$F_3 = -p \cdot u(Q) - \int f(Q,t)dQ + \omega t \cdot p \tag{18}$$

$$\mathcal{H} = -\frac{\partial F_3}{\partial t} \tag{19}$$

$$\Rightarrow \boxed{\mathcal{H} = \omega \cdot p + \frac{\partial(\int f(Q,t)dQ)}{\partial t}} \tag{20}$$

The original Hamiltonian is linear to p as requested.