Ex0909A3: The generalized canonical transformation

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The problem: A system is described by the Hamiltonian H(q,p,t). The coordinate q is transformed

q $\rightarrow$ Q=ψ(q,t)

1. Find the most general transformation p $\rightarrow$ P(q,p,t) so that (q,p) $\rightarrow$ (Q,P) will be canonical.
2. Calculate the appropriate generating function (explain why F_1(q,Q,t) isn't relevant).
3. Given the new Hamiltonian H'=0. Find the original Hamiltonian.
4. Prove that for ψ(q+ωt), the original Hamiltonian is a function that is linear to p. What does it mean?

The solution:

1. We will derive here the transformation for P:

$$[Q,P] = \frac{\partial Q}{\partial q}\cdot\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \frac{\partial Q}{\partial q}\cdot\frac{\partial P}{\partial p} = \frac{\partial \psi(q,t)}{\partial q}\cdot\frac{\partial P}{\partial p} = 1 \tag{1}$$

$$\frac{\partial P}{\partial p} = \frac{\partial q}{\partial \psi} = \frac{\partial q}{\partial Q} \tag{2}$$

$$\Rightarrow \boxed{P = p\cdot\frac{\partial q}{\partial Q} + f(q,t)} \tag{3}$$

2. Finding the generating function:
We will choose the third class of generating function, and we will define q as a function of Q and t, leading to q=g(Q,t) because it is given that Q=ψ(q,t). Consequently f can be expressed with Q and t.

$$F_3(p,Q,t) \tag{4}$$

$$-P = \frac{\partial F_3}{\partial Q} \tag{5}$$

$$-q = \frac{\partial F_3}{\partial p} \tag{6}$$

$$\frac{\partial F_3}{\partial Q} = -p\cdot\frac{\partial q}{\partial Q} - f(q,t) \tag{7}$$

$$F_3 = -pq - \int f(Q,t)dQ + h(p,t) \tag{8}$$

$$\frac{\partial F_3}{\partial p} = -q + \frac{\partial h}{\partial p} = -q \;\Rightarrow\; h = l(t) \tag{9}$$

$$F_3 = -pq - \int f(Q,t)dQ + l(t) \tag{10}$$

$$\Rightarrow \boxed{F_3 = -p\cdot g(Q,t) - \int f(Q,t)dQ} \tag{11}$$

We ignored the function that depends only on time because it can't affect the equation of motion.

3. Here we will find the original Hamiltonian.
   we know that:

$$0 = \grave{\mathcal{H}} = \mathcal{H} + \frac{\partial F_3}{\partial t} \tag{12}$$

$$\mathcal{H} = -\frac{\partial F_3}{\partial t} \tag{13}$$

$$\Rightarrow \boxed{\mathcal{H} = \frac{\partial (p \cdot g(Q,t))}{\partial t} + \frac{\partial (\int f(g(q,t),t)dQ)}{\partial t}} \tag{14}$$

4. Now it is given that q $\rightarrow$ Q=ψ(q+ωt). We can derive from that the new connection between q and Q:

$$q = u(Q) - \omega t \tag{15}$$

   We shall find the original Hamiltonian.

$$\frac{\partial F_3}{\partial Q} = -p \cdot \frac{\partial q}{\partial Q} - f(Q,t) = -p \cdot \frac{\partial u}{\partial Q} - f(Q,t) \tag{16}$$

$$F_3 = -p \cdot u(Q) - \int f(Q,t)dQ + h(p,t) \tag{17}$$

$$\frac{\partial F_3}{\partial p} = -u(Q) + \frac{\partial h}{\partial p} = -q = -u(Q) + \omega t \;\Rightarrow\; h(p.t) = \omega t \cdot p \tag{18}$$

$$F_3 = -p \cdot u(Q) - \int f(Q,t)dQ + \omega t \cdot p \tag{19}$$

$$\mathcal{H} = -\frac{\partial F_3}{\partial t} \tag{20}$$

$$\Rightarrow \boxed{\mathcal{H} = \omega \cdot p + \frac{\partial (\int f(Q,t)dQ)}{\partial t}} \tag{21}$$

5. This is how you write an equation

$$\mathcal{I} = \int\limits_0^z d^4x\sqrt{-g}\left[\varphi^2\left(\mathcal{R} - 6sg^{\mu\nu}\kappa_\mu\kappa_\nu\right) + 4\omega g^{\mu\nu}D_\mu\varphi D_\nu\varphi + \lambda\varphi^4 + \frac{1}{4}g^{\mu\nu}g^{\lambda\sigma}X_{\mu\lambda}X_{\nu\sigma}\right]. \tag{22}$$

6. If you wish to add a set of equations (add the ampersands to align the two equations):

$$s = \frac{3+2\omega}{2\omega} \neq 1, \tag{23}$$

$$D_\mu\varphi = \varphi_{;\mu} + s\kappa_\mu\varphi, \tag{24}$$

   If you wish to have equations with no numbers add an asterisk

$$\begin{aligned} g^{\mu\nu}\left(\varphi^2_{;\mu} + 2\kappa_\mu\varphi^2\right)_{;\nu} &= \frac{\partial W_{eff}(\varphi^2)}{\partial\varphi^2}, \\ \frac{\partial W_{eff}(\varphi^2)}{\partial\varphi^2} &= \frac{1}{3+2}\omega\left(\frac{1}{2}\varphi V'(\varphi) - 2V(\varphi)\right). \end{aligned}$$

To write a vector

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \quad ; \quad \vec{r_i} = |\vec{r}|\,\hat{r} \tag{25}$$

To write a matrix

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{“\textbackslash qquad” makes space in equations and this is how you add text.} \tag{26}$$

You can also use “align” to tightly align the equal sign, i.e “=” is aligned: see Eq. (27) and Eq. (28), also note that **every end** of an equation needs to be punctuated, i.e : “,” or “.”, according to the sentence.

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \tag{27}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{pmatrix}, \tag{28}$$

and centered

$$\left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right] = i \sum_{k=1}^{n} f^{ijk} \frac{\lambda_k}{2},$$

$$f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2}.$$

7. Known arguments are **not** *written in italic mode*, some have special syntax in LaTeX otherwise just use “\mathrm”:

$$\cos(\omega t); \quad \sin(\omega t); \quad \mathrm{Tr}\left[\hat{\mathrm{A}}\right]; \quad \det\left[\hat{\mathcal{A}}\right]; \quad \tan(\omega t); \quad \mathrm{e}^{\frac{t}{\tau}}; \quad \exp\left(\frac{t}{\tau}\right); \quad \log\left(\frac{t}{\tau}\right). \tag{29}$$

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To include a picture use, note that the picture file has to be in the same folder as the *.tex file.