Permutation Fair Dice

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Permutation Fair Dice

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- Preliminary Results
- New Contstructions

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Go First Dice

In 2010 Robert Ford and Eric Harshberger discovered a set of four 12-sided dice that they called ”Go First DIce”.

These dice are non-standard dice and are numbered such that when rolled together:

- No two dice will ever show the same value.
- When sorted according to the values shown on the faces every permutation of the dice is equally likely.

Go First Dice Details

The numbers on the faces of the Go First Dice are:

Die	Faces											
	i	ii	iii	iv	v	vi	vii	viii	ix	x	xi	xii
A	1	8	11	14	19	22	27	30	35	38	41	48
B	2	7	10	15	18	23	26	31	34	39	42	47
C	3	6	12	13	17	24	25	32	36	37	43	46
D	4	5	9	16	20	21	28	29	33	40	44	45

Three-Player Go First Dice

It is also possible to construct a set of three 6-sided dice that are permutation fair.

The numbers on the faces of one such set are:

Die	Faces					
	i	ii	iii	iv	v	vi
A	1	5	10	11	13	17
B	3	4	7	12	15	16
C	2	6	8	9	14	18

Five Player Go First Dice?

Notice that for a set of n d-sided dice to be permutation fair, the number of possible outcomes must be divisible by the number of permutations on n elements.

That is, we must have $n! \mid d^n$.

So, for any set of five d-sided permutation fair dice we must have $30 \mid d$.

No one knows if such a set of dice exists!

Larger Face Counts

Eric Harshberger has discovered several sets of five-player permutation fair dice with significantly more than thirty faces per die.

Most of those sets are comprised of dice with different numbers of faces. The non-homogeneous dice can be homogenized by duplicating the faces on each die to create a set of s-sided dice where s is the least common multiple of the number of faces on each of the original dice.

The best such set of dice that he has found can be realized as a set of five 180-sided dice.

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Dice as Strings

A common convention in other works on non-standard dice is to represent each set of dice as a string.

A set of n dice is represented by a string comprised of n distinct characters.

The values on the faces of the dice are assigned in according to the indices of the corresponding symbols in the string.

String Representation Example

For example, consider the three-player permutation fair dice with face values given by:

Die	Faces					
	i	ii	iii	iv	v	vi
A	1	6	8	11	15	16
B	2	5	9	10	13	18
C	3	4	7	12	14	17

This set of dice can also be represented by the string:

`abccba cabbac bcaacb`

Go First String

The string representation of Go First Dice is:

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abcddcba dbaccabd cbaddabc cbaddabc dbaccabd abcddcba
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We'll call this string the *Go First String*.

Notice that:

- The Go First String is a palindrome.
- Both halves of the Go First String can be decomposed into three blocks, each of which is itself a palindrome.

m/n Permutation Fairness

Definition 1

A set of dice S is m/n permutation fair if $|S| = n$ and every subset $T \subset S$ with $|T| \leq m$ is permutation fair.

Lemma 2

Let S be a set of dice with string representation s. S is m/n permutation fair if and only if

$$(\mathbf{x})_s = \frac{d_s^m}{m!}.$$

for all $\mathbf{x} \subset s$ with $|\mathbf{x}| = m$.

Concatenating Dice

Definition 3

Let S and T be sets of dice with string representations s and t respectively. We define $S \| T$ to be the set of dice with string representation $s \| t$.

Palindromes

Proposition 4

Let S be a set dice with $|S| = n$ and let s be the string representation of S. If $s = t \| t'$ for some string t, then s is $2/n$ permutation fair.

Proof of Proposition 4

Because $s = t\|t'$ for some t, for all $x, y \in s$ we have

$$\begin{aligned}(xy)_s &= (xy)_{t\|t'} \\ &= (xy)_t + (xy)_{t'} + (x)_s(y)_{t'} \\ &= (xy)_t + \left(d_t^2 - (xy)_t\right) + d_t^2 \\ &= 2d_t^2.\end{aligned}$$

Furthermore, $s = t\|t'$ implies that $d_s = 2d_t$. Therefore,

$$(xy)_s = 2d_t^2 = \frac{d_s^2}{2}.$$

The result then follows from Lemma 2. $\square$

Concatenation Theorem

Lemma 5

Let S and T be sets of dice with string representations s and t respectively. If $\mathbf{x} \subset s$ *with* $|\mathbf{x}| = m$ *then*

$$(\mathbf{x})_{s\|t} = \sum_{i=0}^{m} (\mathbf{x}_{\mathbf{j} \leq \mathbf{i}})_s (\mathbf{x}_{j>i})_t$$

Theorem 6: Concatenation Theorem

Let S and T be sets of dice. If S and T are m/n *permutation fair then* $S\|T$ *is* m/n *permutation fair.*

Proof of Theorem 6

Because S and T are m/n permutation fair, Lemma 2 implies that for all $0 \leq i \leq m$ we have

$$(\mathbf{x}_{j \leq i})_s = \frac{d_s^i}{i!} \qquad \text{and} \qquad (\mathbf{x}_{j > i})_t = \frac{d_t^{m-i}}{(m-i)!}.$$

Therefore, Lemma 5 implies that

$$(\mathbf{x})_{s \| t} = \frac{1}{m!} \sum_{i=0}^{m} \binom{m}{i} d_s^i d_t^{m-i} = \frac{(d_s + d_t)^m}{m!}.$$

The result follows from another application of Lemma 2. $\square$

Relabelling Theorem

Theorem 7: Relabelling Theorem

Let S be a set of dice that is

Lifting Theorem

Theorem 8: Lifting Theorem

For all $1 \leq i \leq k$ let S_i be a set of m/n permutation fair dice with string representation s_i. If there exists a constant C such that $\sum(\mathbf{x})_{s_i} = C$ for all $\mathbf{x} \subset s$ with $|\mathbf{x}| = m+1$, then $S_1 \| S_2 \| \ldots \| S_k$ is $(m+1)/n$ permutation fair.

Revisiting $n = 3$

Consider the strings $r = \texttt{abccba}$, $s = \texttt{cabbac}$, and $t = \texttt{bcaacb}$.

We have

$\mathbf{x}$	$(\mathbf{x})_r$	$(\mathbf{x})_s$	$(\mathbf{x})_t$
$(\texttt{a}, \texttt{b}, \texttt{c})$	2	2	0
$(\texttt{a}, \texttt{c}, \texttt{b})$	2	0	2
$(\texttt{b}, \texttt{a}, \texttt{c})$	0	2	2
$(\texttt{b}, \texttt{c}, \texttt{a})$	2	0	2
$(\texttt{c}, \texttt{a}, \texttt{b})$	0	2	2
$(\texttt{c}, \texttt{b}, \texttt{a})$	2	2	0

So, Theorem 8 implies that $r \| s \| t$ is $3/3$ permutation fair.

Revisiting $n = 4$

The Go First String can be written as $r \| s \| t \| t' \| s' \| r'$ where $r = \texttt{abcddcba}$, $s = dbaccabd$, and $t = cbaddabc$.

Observe that r, s and t are $2/4$ permutation fair.

For all $\mathbf{x}$ with $|\mathbf{x}| = 3$ we have $(\mathbf{x})_r + (\mathbf{x})_s + (\mathbf{x})_t = 36$. Therefore, Theorem 8 implies that $r \| s \| t$ is $3/4$ permutation fair.

Lifting from $3/n$ to $4/n$

If we let v = r$\|$s$\|$t then v is $3/4$ permutation fair, then v′ is $3/4$ permutation fair and v$\|$v′ is the Go First String. Therefore, v$\|$v′ is $4/4$ permutation fair.

We've seen this phenomenon in other cases as well but haven't been able to turn it into a theorem. As such, we make the following conjecture.

Conjecture 9

If s is $3/n$ permutation fair and s is a palindrome, then s is $4/n$ permutation fair.

Tackling $n = 5$

We start with the string $\mathsf{s} = \texttt{abcde edcba}$. Notice that s is $2/5$ permutation fair.

Observe that if σ is a permutation on the characters $(\texttt{a}, \texttt{b}, \texttt{c}, \texttt{d}, \texttt{e})$, then $\sigma(\mathsf{s})$ is $2/5$ permutation fair as well.

If we can find a set of permutations $\{\sigma_i\}_{i=1}^{k}$ such that $\sum_{i=1}^{k}(\mathbf{x})_{\sigma_i(\mathsf{s})}$ is constant for all $\mathbf{x}$ with $|\mathbf{x}| = 3$, then Theorem 8 implies that $\sigma_1(\mathsf{s})\|\sigma_2(\mathsf{s})\|\ldots\|\sigma_k(\mathsf{s})$ is $3/5$ permutation fair.

A Family of Permutations

It turns out that we can find such a family!

$$\sigma_1 = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} & \mathtt{e} \\ \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} & \mathtt{e} \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} & \mathtt{e} \\ \mathtt{d} & \mathtt{c} & \mathtt{b} & \mathtt{a} & \mathtt{e} \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} & \mathtt{e} \\ \mathtt{e} & \mathtt{b} & \mathtt{c} & \mathtt{a} & \mathtt{d} \end{pmatrix} \qquad \sigma_4 = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} & \mathtt{e} \\ \mathtt{a} & \mathtt{c} & \mathtt{b} & \mathtt{d} & \mathtt{e} \end{pmatrix}$$

$$\sigma_5 = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} & \mathtt{e} \\ \mathtt{d} & \mathtt{b} & \mathtt{c} & \mathtt{a} & \mathtt{e} \end{pmatrix} \qquad \sigma_6 = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} & \mathtt{e} \\ \mathtt{e} & \mathtt{c} & \mathtt{b} & \mathtt{a} & \mathtt{d} \end{pmatrix}$$

Turning the Crank

If we let t = $\sigma_1(\mathsf{s})\|\sigma_2(\mathsf{s})\|\sigma_3(\mathsf{s})\|\sigma_4(\mathsf{s})\|\sigma_5(\mathsf{s})\|\sigma_6(\mathsf{s})$ then t is the string representation of a set of five 12-sided dice which are $3/5$ permutation fair.

Guided by Conjecture 9, we guess that $u = t\|t'$ might represent a set of five 24-sided dice that are $4/5$ permutation fair.

Indeed it is! We find that for all $\mathbf{x}$ with $|\mathbf{x}| = 4$ we have $(\mathbf{x})_u = 24^4/24 = 24^3 = 1384$ as per Lemma 2.

An Ugly Finish

We used a trick to lift our $3/5$ permutation fair dice to a set of $4/5$ permutation fair dice.

We haven't found a similar trick that we can apply to efficiently lift that solution to a set of $5/5$ permutation fair dice.

The best we've been able to do so far is to let

$$\mathsf{v} = \big\Vert_{\sigma \in \mathsf{S}_5} \sigma(\mathsf{u}).$$

This results in a string that represents a set of five 2880-sided dice that are 5/5 permutation fair.

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