# Permutation Fair Dice

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Prior Art

My Contributions
- Terminology and Notation
- Preliminary Results
- New Contstructions

Future Work

References

# Go First Dice

In 2010 Robert Ford and Eric Harshbarger discovered a set of four 12-sided dice that they called ”Go First Dice”.

These dice are non-standard dice and are numbered such that when rolled together:

- No two dice will ever show the same value.
- When sorted according to the values shown on the faces every permutation of the dice is equally likely.

# Go First Dice Details

The numbers on the faces of the Go First Dice are:

| Die | Faces | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | i | ii | iii | iv | v | vi | vii | viii | ix | x | xi | xii |
| A | 1 | 8 | 11 | 14 | 19 | 22 | 27 | 30 | 35 | 38 | 41 | 48 |
| B | 2 | 7 | 10 | 15 | 18 | 23 | 26 | 31 | 34 | 39 | 42 | 47 |
| C | 3 | 6 | 12 | 13 | 17 | 24 | 25 | 32 | 36 | 37 | 43 | 46 |
| D | 4 | 5 | 9 | 16 | 20 | 21 | 28 | 29 | 33 | 40 | 44 | 45 |

# Three-Player Go First Dice

It is also possible to construct a set of three 6-sided dice that are permutation fair.

The numbers on the faces of one such set are:

| Die | Faces | | | | | |
|---|---|---|---|---|---|---|
| | i | ii | iii | iv | v | vi |
| A | 1 | 5 | 10 | 11 | 13 | 17 |
| B | 3 | 4 | 7 | 12 | 15 | 16 |
| C | 2 | 6 | 8 | 9 | 14 | 18 |

# Five Player Go First Dice?

Notice that for a set $S$ of $n$ $m_s$-sided dice to be permutation fair, the number of possible outcomes must be divisible by the number of permutations on $n$ elements.

That is, we must have $n! \mid m_s^n$.

In particular, this means that for any set of five $m_s$-sided permutation fair dice we must have $30 \mid m_s$.

The smallest value of $m_S$ such that a set of five $m_s$-sided permutation fair dice exists is unknown.

# Larger Face Counts

Eric Harshbarger has discovered several sets of five-player permutation fair dice with more than thirty faces per die.

Most of those sets are comprised of dice with different numbers of faces. The non-homogeneous dice can be homogenized by duplicating the faces on each die to create a set of *m*-sided dice where *m* is the least common multiple of the number of faces on each of the original dice.

The best such set of dice that he has found can be realized as a set of five 180-sided dice.

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# Dice as Strings

A common convention in other works on non-standard dice is to represent each set of dice as a string.

A set of *n* dice is represented by a string comprised of *n* distinct characters.

The values on the faces of the dice are assigned in according to the indices of the corresponding symbols in the string.

# String Representation Example

For example, consider the three-player permutation fair dice with face values given by:

| Die | Faces | | | | | |
|---|---|---|---|---|---|---|
| | i | ii | iii | iv | v | vi |
| A | 1 | 6 | 8 | 11 | 15 | 16 |
| B | 2 | 5 | 9 | 10 | 13 | 18 |
| C | 3 | 4 | 7 | 12 | 14 | 17 |

This set of dice can also be represented by the string:

`abccba cabbac bcaacb`

# Go First String

The string representation of Go First Dice is:

`abcddcba dbaccabd cbaddabc cbaddabc dbaccabd abcddcba`

We'll call this string the *Go First String*.

Notice that:

- The Go First String is a palindrome.
- Both halves of the Go First String can be decomposed into three blocks, each of which is itself a palindrome.

# Strings as Dice

We can use the string representation of a set of dice to compute the probability of any outcome when the corresponding dice are rolled.

In particular, we can compute the number of ways that each pattern can occur.

We will write $(\mathbf{x})_s$ to denote the number of times that that the pattern **x** appears in the string s.

# $\ell/n$ Permutation Fairness

## Definition 1: $\ell/n$ Permutation Fairness

*A set of dice S is $\ell/n$ permutation fair if $|S| = n$ and every subset $T \subset S$ with $|T| \leq \ell$ is permutation fair.*

## Lemma 2

*Let S be a set of n $m_s$-sided dice with string representation s. S is $\ell/n$ permutation fair if and only if for all $\mathbf{x} \subset s$ with $|\mathbf{x}| = \ell$ we have*

$$(\mathbf{x})_s = \frac{m_s^\ell}{\ell!}.$$

# Concatenating Dice

## Definition 3

*Let $S$ and $T$ be sets of dice with string representations $s$ and $t$ respectively. We define $S \| T$ to be the set of dice with string representation $s \| t$.*

# Palindromes

## Proposition 4

*Let $S$ be a set of dice with $|S| = n$ and let $s$ be the string representation of $S$. If $s = t||t'$ for some string $t$, then $s$ is $2/n$ permutation fair.*

# Proof of Proposition 4

Because $s = t \| t'$ for some $t$, for all $x, y \in s$ we have

$$\begin{aligned}(xy)_s &= (xy)_{t\|t'} \\ &= (xy)_t + (xy)_{t'} + (x)_s(y)_{t'} \\ &= (xy)_t + \left(m_t^2 - (xy)_t\right) + m_t^2 \\ &= 2m_t^2.\end{aligned}$$

Furthermore, $s = t \| t'$ implies that $d_s = 2m_t$. Therefore,

$$(xy)_s = 2m_t^2 = \frac{m_s^2}{2}.$$

The result then follows from Lemma 2. $\square$

# Concatenation Theorem

## Lemma 5

*Let $S$ and $T$ be sets of dice with string representations $s$ and $t$ respectively. If $\mathbf{x} \subset s$ with $|\mathbf{x}| = \ell$ then*

$$(\mathbf{x})_{s||t} = \sum_{i=0}^{\ell} (\mathbf{x}_{\mathbf{j} \leq \mathbf{i}})_s (\mathbf{x}_{j>i})_t$$

## Theorem 6: Concatenation Theorem

*Let $S$ and $T$ be sets of dice. If $S$ and $T$ are $\ell/n$ permutation fair then $S||T$ is $\ell/n$ permutation fair.*

# Proof of Theorem 6

Because S and $T$ are $\ell/n$ permutation fair, Lemma 2 implies that for all $0 \leq i \leq \ell$ we have

$$(\mathbf{x}_{j \leq i})_s = \frac{m_s^i}{i!} \qquad \text{and} \qquad (\mathbf{x}_{j > i})_t = \frac{m_t^{\ell - i}}{(\ell - i)!}.$$

Therefore, Lemma 5 implies that

$$(\mathbf{x})_{s \| t} = \frac{1}{\ell!} \sum_{i=0}^{\ell} \binom{\ell}{i} m_s^i m_t^{\ell - i} = \frac{(m_s + m_t)^\ell}{\ell!}.$$

The result follows from another application of Lemma 2. $\square$

# Relabelling Theorem

## Theorem 7: Relabelling Theorem

*Let $S$ be a set of dice with string representation $s$ and $\sigma$ be a permutation on the characters of $s$. $S$ is $\ell/n$ permutation fair if and only if $\sigma(s)$ is $\ell/n$ permutation fair.*

# Lifting Theorem

## Theorem 8: Lifting Theorem

*For all* $1 \leq i \leq k$ *let* $S_i$ *be a set of* $\ell/n$ *permutation fair dice with string representation* $s_i$. *If there exists a constant* $C$ *such that* $\sum(\mathbf{x})_{s_i} = C$ *for all* $\mathbf{x} \subset s$ *with* $|\mathbf{x}| = \ell + 1$, *then* $S_1 \| S_2 \| \ldots \| S_k$ *is* $(\ell + 1)/n$ *permutation fair.*

# Revisiting $n = 3$

Consider the strings $r = \texttt{abccba}$, $s = \texttt{cabbac}$, and $t = \texttt{bcaacb}$.

We have

| $\mathbf{x}$ | $(\mathbf{x})_r$ | $(\mathbf{x})_s$ | $(\mathbf{x})_t$ |
|---|---|---|---|
| abc | 2 | 2 | 0 |
| acb | 2 | 0 | 2 |
| bac | 0 | 2 | 2 |
| bca | 2 | 0 | 2 |
| cab | 0 | 2 | 2 |
| cba | 2 | 2 | 0 |

So, Theorem 8 implies that $r||s||t$ is 3/3 permutation fair.

# Revisiting $n = 4$

The Go First String can be written as $r \| s \| t \| t' \| s' \| r'$ where $r = \texttt{abcddcba}$, $s = \texttt{dbaccabd}$, and $t = \texttt{cbaddabc}$.

Observe that $r$, $s$ and $t$ are $2/4$ permutation fair. This is a consequence of both Proposition 4 and Theorem 7.

For all $\mathbf{x}$ with $|\mathbf{x}| = 3$ we have $(\mathbf{x})_r + (\mathbf{x})_s + (\mathbf{x})_t = 36$. Therefore, Theorem 8 implies that $r \| s \| t$ is $3/4$ permutation fair.

# Lifting from 3/$n$ to 4/$n$

If we let v = r||s||t then v is 3/4 permutation fair, v′ is 3/4 permutation fair, and v||v′ is the Go First String. The Go First String is 4/4 permutation fair.

In fact, this is an example of a more general phenomenon which we can use to construct a 4/$n$ permutation fair string out of any 3/$n$ permutation fair string.

## Theorem 9

*If $n \geq 4$, $t$ is a $3/n$ permutation-fair string, and $s = t||t'$, then $s$ is a $4/n$ permutation-fair string.*

# Proof of Theorem 9

Theorem 2 implies that it suffices to show that for all $\mathbf{x} \subset s$ with $|\mathbf{x}| = 4$ we have $(\mathbf{x})_s = m_s^4/24 = 8m_t^4/12$. Notice that we have

$$
\begin{aligned}
(\mathbf{x})_s &= \sum_{i=0}^{4} (\mathbf{x_{j\leq i}})_t (\mathbf{x_{j>i}})_{t'} \\
&= (\mathbf{x})_t + (\mathbf{x})_{t'} + \sum_{i=1}^{3} (\mathbf{x_{j\leq i}})_t (\mathbf{x_{j>i}})_{t'} \\
&= (\mathbf{x})_t + (\mathbf{x})_{t'} + \sum_{i=1}^{3} \frac{m_t^i}{i!} \frac{m_t^{4-i}}{(4-i)!} \\
&= (\mathbf{x})_t + (\mathbf{x})_{t'} + \frac{7m_t^4}{12}.
\end{aligned}
$$

So, it suffices to show that $(\mathbf{x})_t + (\mathbf{x})_{t'} = m_t^4/12$.

# Proof of Theorem 9 (Continued)

Let $\mathbf{x} = \mathtt{abcd}$ and $\mathbf{x}' = \mathtt{dcba}$. Observe that $(\mathbf{x})_{t'} = (\mathbf{x}')_t$.
Notice that we have

$$\begin{aligned}
m_t^4 \cdot \mathbb{P}\{\mathtt{a} < \mathtt{b} < \mathtt{c}\} &= (\mathtt{dabc})_t + (\mathtt{adbc})_t + (\mathtt{abdc})_t + (\mathtt{abcd})_t \\
m_t^4 \cdot \mathbb{P}\{\mathtt{a} < \mathtt{b}, \mathtt{d} < \mathtt{c}\} &= (\mathtt{dabc})_t + (\mathtt{adbc})_t + (\mathtt{abdc})_t + \\
&\quad (\mathtt{adcb})_t + (\mathtt{dacb})_t + (\mathtt{dcab})_t \\
m_t^4 \cdot \mathbb{P}\{\mathtt{d} < \mathtt{c} < \mathtt{b}\} &= (\mathtt{adcb})_t + (\mathtt{dacb})_t + (\mathtt{dcab})_t + (\mathtt{dcba})_t
\end{aligned}$$

Finally, because $\mathbb{P}\{\mathtt{a} < \mathtt{b} < \mathtt{c}\} = \mathbb{P}\{\mathtt{d} < \mathtt{c} < \mathtt{b}\} = 1/6$ and $\mathbb{P}\{\mathtt{a} < \mathtt{b}, \mathtt{d} < \mathtt{c}\} = \mathbb{P}\{\mathtt{a} < \mathtt{b}\}\mathbb{P}\{\mathtt{d} < \mathtt{c}\} = 1/4$ we conclude that

$$(\mathbf{x})_t + (\mathbf{x})_{t'} = (\mathtt{abcd})_t + (\mathtt{dcba})_t = 2\frac{m_t^4}{6} - \frac{m_t^4}{4} = \frac{m_t^4}{12}$$

as required. $\square$

# Proof of Theorem 9 (Continued)

Let $\mathbf{x} = \mathtt{abcd}$ and $\mathbf{x}' = \mathtt{dcba}$. Observe that $(\mathbf{x})_{t'} = (\mathbf{x}')_t$.
Notice that we have

$$m_t^4 \cdot \mathbb{P}\{\mathtt{a} < \mathtt{b} < \mathtt{c}\} = \cancel{(\mathtt{dabc})_t} + \cancel{(\mathtt{adbc})_t} + \cancel{(\mathtt{abdc})_t} + (\mathtt{abcd})_t$$

$$m_t^4 \cdot \mathbb{P}\{\mathtt{a} < \mathtt{b}, \mathtt{d} < \mathtt{c}\} = \cancel{(\mathtt{dabc})_t} + \cancel{(\mathtt{adbc})_t} + \cancel{(\mathtt{abdc})_t} + \cancel{(\mathtt{adcb})_t} + \cancel{(\mathtt{dacb})_t} + \cancel{(\mathtt{dcab})_t}$$

$$m_t^4 \cdot \mathbb{P}\{\mathtt{d} < \mathtt{c} < \mathtt{b}\} = \cancel{(\mathtt{adcb})_t} + \cancel{(\mathtt{dacb})_t} + \cancel{(\mathtt{dcab})_t} + (\mathtt{dcba})_t$$

Finally, because $\mathbb{P}\{\mathtt{a} < \mathtt{b} < \mathtt{c}\} = \mathbb{P}\{\mathtt{d} < \mathtt{c} < \mathtt{b}\} = 1/6$ and $\mathbb{P}\{\mathtt{a} < \mathtt{b}, \mathtt{d} < \mathtt{c}\} = \mathbb{P}\{\mathtt{a} < \mathtt{b}\}\mathbb{P}\{\mathtt{d} < \mathtt{c}\} = 1/4$ we conclude that

$$(\mathbf{x})_t + (\mathbf{x})_{t'} = (\mathtt{abcd})_t + (\mathtt{dcba})_t = 2\frac{m_t^4}{6} - \frac{m_t^4}{4} = \frac{m_t^4}{12}$$

as required. $\square$

# Tackling $n = 5$

We start with the string

$$s_5 = \texttt{abcdeedcba}.$$

Notice that s is 2/5 permutation fair.

If we can find a family of permutations $\{\sigma_i\}_{i=1}^{k}$ such that $\sum_{i=1}^{k}(\mathbf{x})_{\sigma_i(s_5)}$ is constant for all $\mathbf{x}$ with $|\mathbf{x}| = 3$, then Theorem 8 implies that if

$$t_5 = \sigma_1(s_5)\|\sigma_2(s_5)\|\ldots\|\sigma_k(s_5)$$

is 3/5 permutation fair.

# A Family of Permutations

It turns out that we can find such a family:

$$\sigma_1 = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} & \mathtt{e} \\ \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} & \mathtt{e} \end{pmatrix} \qquad \sigma_4 = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} & \mathtt{e} \\ \mathtt{a} & \mathtt{c} & \mathtt{b} & \mathtt{d} & \mathtt{e} \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} & \mathtt{e} \\ \mathtt{d} & \mathtt{c} & \mathtt{b} & \mathtt{a} & \mathtt{e} \end{pmatrix} \qquad \sigma_5 = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} & \mathtt{e} \\ \mathtt{d} & \mathtt{b} & \mathtt{c} & \mathtt{a} & \mathtt{e} \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} & \mathtt{e} \\ \mathtt{e} & \mathtt{b} & \mathtt{c} & \mathtt{a} & \mathtt{d} \end{pmatrix} \qquad \sigma_6 = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} & \mathtt{e} \\ \mathtt{e} & \mathtt{c} & \mathtt{b} & \mathtt{a} & \mathtt{d} \end{pmatrix}$$

# Turning the Crank

If we let $t_5 = \sigma_1(s_5)\|\sigma_2(s_5)\|\ldots\|\sigma_6(s_5)$ then $t_5$ is the string representation of a set of five 12-sided dice which are 3/5 permutation fair.

Guided by Conjecture 9, we guess that $u = t_5\|t'_5$ represents a set of five 24-sided dice that are 4/5 permutation fair.

It does! We find that for all $\mathbf{x}$ with $|\mathbf{x}| = 4$ we have $(\mathbf{x})_u = 24^4/24 = 24^3 = 1384$ as per Lemma 2.

# An Ugly Finish

We used a trick to lift our 3/5 permutation fair dice to a set of 4/5 permutation fair dice.

We haven't found a similar trick that we can apply to efficiently lift that solution to a set of 5/5 permutation fair dice.

The best we've been able to do so far is to let

$$v = \bigsqcup_{\sigma \in S_5} \sigma(u).$$

This results in a string that represents a set of five 2880-sided dice that are 5/5 permutation fair.

# Tackling $n = 6$

We start with the string

$$s_6 = \texttt{abcdeffedcba}.$$

Notice that $s_6$ is 2/6 permutation fair.

If we can find a family of permutations $\{\sigma_i\}_{i=1}^k$ such that $\sum_{i=1}^k (\mathbf{x})_{\sigma_i(s_6)}$ is constant for all $\mathbf{x}$ with $|\mathbf{x}| = 3$, then Theorem 8 implies that if

$$t_6 = \sigma_1(s_6) \| \sigma_2(s_6) \| \dots \| \sigma_k(s_6)$$

then $t_6$ is 3/6 permutation fair.

# Another Family of Permutations

It turns out that we can find such a family:

$$\sigma_1 = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} & \mathtt{e} & \mathtt{f} \\ \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} & \mathtt{e} & \mathtt{f} \end{pmatrix} \qquad \sigma_4 = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} & \mathtt{e} & \mathtt{f} \\ \mathtt{a} & \mathtt{d} & \mathtt{c} & \mathtt{b} & \mathtt{f} & \mathtt{e} \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} & \mathtt{e} & \mathtt{f} \\ \mathtt{e} & \mathtt{c} & \mathtt{b} & \mathtt{d} & \mathtt{f} & \mathtt{a} \end{pmatrix} \qquad \sigma_5 = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} & \mathtt{e} & \mathtt{f} \\ \mathtt{e} & \mathtt{d} & \mathtt{b} & \mathtt{c} & \mathtt{a} & \mathtt{f} \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} & \mathtt{e} & \mathtt{f} \\ \mathtt{f} & \mathtt{b} & \mathtt{d} & \mathtt{c} & \mathtt{e} & \mathtt{a} \end{pmatrix} \qquad \sigma_6 = \begin{pmatrix} \mathtt{a} & \mathtt{b} & \mathtt{c} & \mathtt{d} & \mathtt{e} & \mathtt{f} \\ \mathtt{f} & \mathtt{c} & \mathtt{d} & \mathtt{b} & \mathtt{a} & \mathtt{e} \end{pmatrix}$$

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