CS 107, Probability, Spring 2020
Lecture 15

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Content

- Repeated, Independent Trials: Binomial Distribution
- Repeated, Independent Trials: Multinomial Distribution

Last Lecture ReCap

Last time we were talking about Independence of several Events, and also about Repeated Trials Model:

- We learned that if A and B are independent, then so are A and $\overline{B}$, and also $\overline{A}$ and B, also $\overline{A}$ and $\overline{B}$;
- We call Events $A_1, A_2, ..., A_n$ Pairwise Independent, if $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i) \cdot \mathbb{P}(A_j), \qquad i \neq j.$
- We call Events $A_1, ..., A_n$ Mutually Independent or just Independent, if for any subgroup of events $A_{i_1}, A_{i_2}, ..., A_{i_k}$,

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_k}) = \mathbb{P}(A_{i_1}) \cdot \mathbb{P}(A_{i_2}) \cdot ... \cdot \mathbb{P}(A_{i_k}),$$

Last Lecture ReCap

Last time we were talking about Independence of several Events, and also about Repeated Trials Model:

- We talked about the Repeated Trials Model: we have a Simple Experiment (Trial), and an Event in that Experiment, called a Success. Let p be the Probability of that Event in a Trial. We repeat our Trial n times, independently, and we are interested in the Probability that we will have exactly k Successes: if X is the number of Successes in n Trials, then

$$\mathbb{P}(X = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}.$$

Repeated Indep Trials: Binomial Probabilities

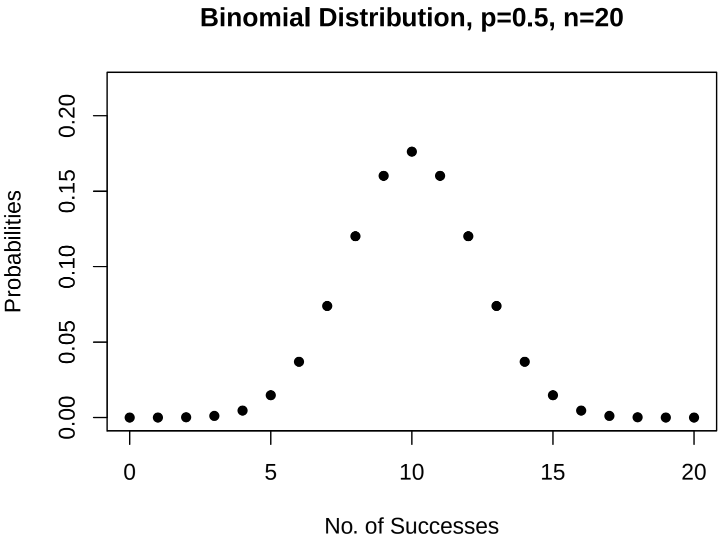
We can represent this in the form of a Table: Let X be the number of Successes in n Trials. Then the *Distribution* of X is given by:

X	0	1	2	...	n
$\mathbb{P}(X=k)$	$\binom{n}{0}p^0(1-p)^n$	$\binom{n}{1}p^1(1-p)^{n-1}$	$\binom{n}{2}p^2(1-p)^{n-2}$	...	$\binom{n}{n}p^n(1-p)^0$

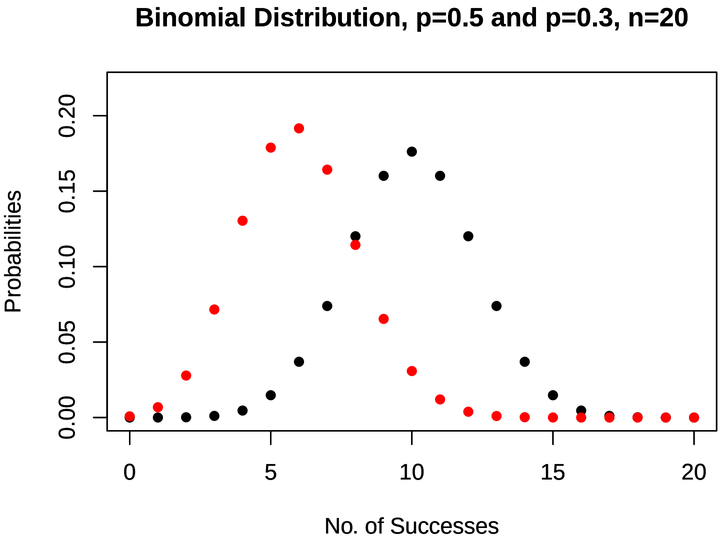
We read this as, say: for $X=2$,

$$\mathbb{P}(X=2)=\binom{n}{2}p^2(1-p)^{n-2}.$$

Binomial Distribution



Binomial Distribution



Example:

Problem: Assume that 20 times we are picking a card at random from the deck, with a replacement (in order to have independent trials !). What is the probability that exactly 12 times we will have Hearts?

Solution:

- Simple Experiment $=$ Trial $=$ Picking a Card at random;
- Event in the Trial $= A =$Card is a Hearts;
- $\mathbb{P}(\text{Event}) = \frac{1}{4}$
- Number of Trials (Repetitions) $= 20$

So

$$\mathbb{P}(\text{Exactly 12 Hearts}) = \binom{20}{12} \cdot \left(\frac{1}{4}\right)^{12} \cdot \left(1 - \frac{1}{4}\right)^{20-12}.$$

Example: Cont'd

Problem: In the problem above, what is the probability that we will have Hearts in more than 17 cases? In not more than 17 cases?
Solution: OTB

Example:

Problem: Some entrance exam test consists of 30 multiple choice problems. Every correct choice gives 1 points. Each problem has suggested 5 choices for the answer, from which only one is correct. If the person will choose randomly the answers, what is the probability that

- He/she will get 25 points?
- He/she will get more than 25 points?
- He/she will get less than 29 points?

Repeated, Independent Trials: Multinomial Case

Now we consider an Experiment consisting of repetition of a Simple Experiment, which can result in 2 or more than 2 events. More precisely,

- We have a Simple Experiment (Trial);
- The Simple Experiment can result in one of the mutually exclusive Events $A_1, A_2, ..., A_m$;
- The probability of having A_k in one Trial is p_k, i.e., $\mathbb{P}(A_k) = p_k$, and $p_1 + p_2 + ... + p_m = 1$;
- We repeat our Trial n times, independently;
- We are interested in the probability that we will have exactly k_1 outcomes from A_1, exactly k_2 outcomes from A_2,..., and exactly k_m outcomes from A_m.
- Clearly, we need to have (we are doing n trials)

$$k_1 + k_2 + ... + k_m = n.$$

Multinomial Case, Remark

Note that the Binomial Case is the particular case of the Multinomial: in that case

- Each trial can result either in A (Success) or in $\overline{A}$ (Failure);
- So we have $m = 2$, $A_1 = A$, $A_2 = \overline{A}$;
- We have $p_1 = \mathbb{P}(A_1) = \mathbb{P}(A) = p$, so $p_2 = \mathbb{P}(A_2) = \mathbb{P}(\overline{A}) = 1 - p$
- We are interested in having exactly k Successes in n Trials, that is, we are interested in having exactly $k_1 = k$ times A_1 (Success) and exactly $k_2 = n - k_1$ times A_2 (Failure)

Repeated Indep Trials: Multinomial Probabilities

Now, for our Multinomial Model, let us denote by X_k the number of occurrences of A_k in n Trials, $k = 1, 2, ..., m$. So we are interested in the Probability of the Event

$$X_1 = k_1, \; X_2 = k_2, \quad ..., \quad X_m = k_m.$$

Remark: Usually, Mathematicians use vector notations: they denote

$$\mathbf{X} = [X_1, X_2, ..., X_m]^T,$$

and write

$$\mathbf{X} = \mathbf{k}$$

as the vector-form of the above Event.

Intermezzo: Partitions

Problem: How many different words can be obtained from the word

$$TRALALA$$

by rearranging its letters?

Problem: Assume we have n elements, and nonnegative integers $k_1, ..., k_m$ with

$$k_1 + k_2 + ... + k_m = n.$$

In how many ways we can divide our n elements into disjoint groups, where the 1st group has k_1 elements, 2nd group has k_2 elements, ..., m-th group has k_m elements?

Repeated Indep Trials: Multinomial Probabilities

Multinomial Probabilities

The probability that exactly k_1 times we will have the event A_1, exactly k_2 times we will have the event A_2, ... , exactly k_m times we will have the event A_m in the above described n trials, is

$$\mathbb{P}(X_1 = k_1, X_2 = k_2, ..., X_m = k_m) = \binom{n}{k_1, k_2, ..., k_m} \cdot p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_m^{k_m},$$

where $p_k = \mathbb{P}(A_k)$, $k = 1, ..., m$.

Here

$$\binom{n}{k_1, k_2, ..., k_m} = \frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_m!}$$

Note that in the binomial case, i.e., when $m = 2$, we get the Binomial Probabilities formula.