CS 107 Section A - Probability

Spring 2020, AUA

Homework No. 09

Due time/date: 14 April, 2020

Note: Supplementary Problems will not be graded, but you are very advised to solve them and to discuss later with TA or Instructor.

Important Discrete Random Variables

A Negative Binomial Distribution

Problem 1. Assume $X \sim NBinom(5, 0.7)$.

a. What is the range of X?

b. Write down the PMF of X;

c. Calculate $\mathbb{P}(X = 7)$;

d. (Supplementary) Assume also $Y \sim NBinom(5, 0.9)$. Which one the following probabilities are larger:

$$\mathbb{P}(X < 10) \qquad \text{or} \qquad \mathbb{P}(Y < 10)?$$

Explain!

Problem 2. V is ready to go to the local besedka to play a chess with one of his neighbors, either S or W. He will play until he will win 3 times (and then he will return back home; also assume that there are no draws, for simplicity). The probability that he will win S in one game is 0.25, and the probability that he will win W in one game is 0.8.

a. Assume his opponent is S, waiting at the besedka. What is the probability that V will play 10 games?

b. Assume his opponent is W. What is the probability that W will have 2 wins when the match (tournament) will over?

c. Assume his opponent is W. What is the probability that W will have more wins than V, when the match (tournament) will over?

d. Now, assume that the odds are 2:3 that S will be his opponent (vs W will be). What is the probability that V will play 10 games?

e. (Supplementary) Again assume that the odds are 2:3 that S will be his opponent (vs W will be). The length of a game is 10min. What is the probability that V will return back home in 2 hours?

B HyperGeometric Distribution

Problem 3. Assume $U \sim Hyper(15, 7, 5)$.

a. Give some explanation on what is calculating/counting U;
b. Write down the PMF of U;
c. Calculate $\mathbb{P}(U = 3)$.

Problem 4. The dealer, from a well shuffled ordinary deck of 52 cards, is dealing my hand, 7 cards. Assume D is the number of Diamonds suit cards in my hand, and Ace is the number of Aces in my hand.

a. Write Appropriate models for the r.v.s D and Ace;
b. Write down the PMFs of D and Ace;
c. Which one is larger: $\mathbb{P}(D = 3)$ or $\mathbb{P}(Ace = 2)$?.

Problem 5. Yerjanik has a lot of friends and relatives, and has a strange habit of inviting them to her birthday randomly. Last year, during her birthday party, he invited 35 of them. This year, again for her birthday, 50 persons were participating at the party, and it turned out that 11 of them were participating last year also. We assume that the number of friends and relatives has not changed during that one year. Estimate the number of friends and relatives of Yerjanik.

Important Continuous Random Variables

C Uniform Distribution

Problem 6. Assume $X \sim Unif[-3, 4]$.

a. Write down and plot the PDF of X;
b. Write down and plot the CDF of X;
c. Calculate $\mathbb{P}(|X - 1| < 2)$;
d. Find $q \in \mathbb{R}$ such that $\mathbb{P}(X \leq q) = 0.6$;
e. Let $Y = -X$. Find the distribution of Y;
f. (Supplementary) Calculate $\mathbb{P}(X \leq x | X > 0)$, for any $x \in \mathbb{R}$.

Problem 7. In one of the local fruit and vegetables store, the storekeeper is usually rounding down the actual weight of the customer basket, to the nearest $\frac{n}{2}$ Kilograms (n is an integer). Say, if the actual weight is 2.671Kg, she is counting the weight as 2.5Kg, and if the actual weight is 3.423479Kg, she is charging for 3Kg. What is the probability that, for a basket, the excess weight between the actual and rounded one will be more than 300gr, if we know that

a. that basket's actual weight is between 4.1Kg and 4.4 Kgs?
b. (Supplementary) that basket's actual weight is between 4.1Kg and 4.85 Kgs?

Note: Here you need to do some reasonable probabilistic assumption about the excess weight of a basket.

D Exponential Distribution

Problem 8. Assume $X \sim Exp(0.9)$.

a. Write down the PDF of X;
b. Write down the CDF of X;
c. Calculate $\mathbb{P}(X \geq 2)$;
d. Calculate $\mathbb{P}(1 < X < 2)$;
e. Calculate $\mathbb{P}(X \geq 2 \,|\, X < 3)$;
f. (Supplementary) Assume also $Y \sim Exp(10)$. Which of the probabilities is larger:

$$\mathbb{P}(X > \alpha) \qquad \text{or} \qquad \mathbb{P}(Y > \alpha).$$

Interpret X and Y as waiting times and explain.
g. (Supplementary) Find $\mathbb{P}(X \geq t \,|\, X > 1)$, for any t.

Problem 9. At some food delivery service, there are, in average, 25 orders in 2 hours. Let X be the number of orders in an hour, and let Y be the time between the orders, in minutes.

a. What is the average time between orders (average waiting time for an order)?
b. Give appropriate models for X and Y;
c. Calculate the probability that there will be more than 10 orders in an hour ;
d. Calculate the probability that there will be exactly 3 orders in 10 min,
e. Calculate the probability that the time until the next order will be less than 3 min.

Problem 10. (Supplementary) Assume $X \sim Exp(\lambda)$, and $Y = \lambda \cdot X$. Prove that $Y \sim Exp(1)$.
Hint: Calculate the CDF of Y.

E Normal Distribution

Problem 11. Assume $X \sim \mathcal{N}(1, 3^2)$.

a. Write down the PDF of X;
b. Find $\mathbb{P}(X > 1)$. Explain.
c. Show geometrically, as an area, and calculate (using tables or some software) $\mathbb{P}(-1 < X < 2.5)$.

Problem 12. The weight of a Marilla ttvaser (sour cream $\ddot\smile$) package (box) is announced to be 350gr. Of course, it is not believable that the **exact weight** of the package will be 350gr, but will be around that. We assume that the actual weight W (in grams) is Normally distributed with the mean 350 and standard deviation 10. What is the probability that the actual weight will be

a. between 340 and 360 grams?
b. between 345 and 355 grams?
c. larger than 370gr?

F Supplementary Problems

Problem 13. (Supplementary) Show that the PMF of $X \sim Hyper(N, m, n)$ is satisfying

$$\sum_{k=0}^{n} \mathbb{P}(X = k) = 1.$$

Problem 14. (Supplementary) Construct, explicitly, two different examples of a Probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a r.v. $X(\omega)$ on that space such that $X \sim Unif([0, 1])$.

Problem 15. (Supplementary) Prove that the only continuous r.v. sharing the Memoryless property is the Exponential one, and the only discrete r.v. sharing that property is the Geometric one.

Problem 16. (Supplementary) Here is the famous Banach's matchbox problem[1]: A mathematician carries two matchboxes at all times: one in his left pocket and one in his right. Each time he needs a match, he is equally likely to take it from either pocket. Suppose he reaches into his pocket and discovers for the first time that the box picked is empty. If it is assumed that each of the matchboxes originally contained N matches, what is the probability that there are exactly k matches in the other box?

Problem 17. (Supplementary) Assume $X \sim NBinom(m, p)$ and $Y \sim Binom(n, p)$. Show that

$$\mathbb{P}(Y \geq m) = \mathbb{P}(X \leq n).$$

Problem 18. (Supplementary) Assume $X \sim NBinom(m, p)$. For which k we will have the maximum probability $\mathbb{P}(X = k)$?

Problem 19. (Supplementary) Assume that a person is adding a post to her FB wall with the rate 5 posts in 3 hours. And assume that, with probability 0.7, the post is her selfie photo. What is the distribution of the number of her selfie photos in a day?

Problem 20. (Supplementary) Assume A and B are playing some game against each other, several times (several rounds). The probability that A will win in one round is p (we assume that there are no draws). What is the probability that A will win n rounds before B will win m rounds?

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[1] https://en.wikipedia.org/wiki/Banach's_matchbox_problem.