AUA CS 108, Statistics, Fall 2019

Lecture 13

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Contents

About my OH: Is it OK to split the OH into 1h on Monday, 11:30 - 12:30 and 1h on Wednesday, 11:30 - 12:30 ?

Last Lecture ReCap

- Give the definition of the Sample Covariance

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- Give the definition of the Sample Correlation Coefficient

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- For any Dataset x,

$$cov(x, x) = var(x)$$

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$$-1 \leq \rho_{xy} \leq 1;$$

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Say, if x is a Dataset of heights of some persons, in centimeters, y their weights in grams, and if x' will be the same heights Dataset using meters as units, and y' will be the weights in Kg-s, then $cov(x, y)$ and $cov(x', y')$ will not be the same, but $cor(x, y) = cor(x', y')$.

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So it is not easy to interpret the magnitude of the covariance, but the magnitude of the correlation coefficient is the strength of the linear relationship.

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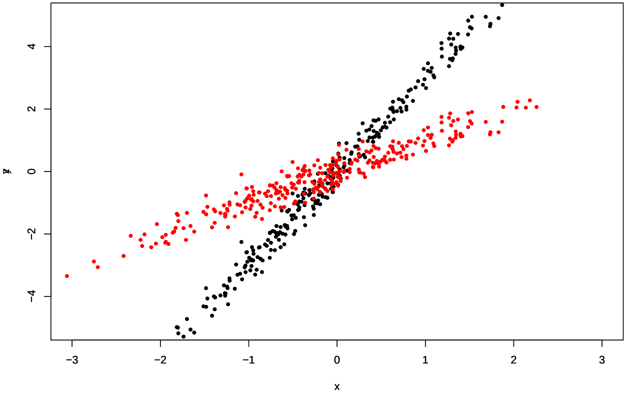
  then if x is increasing, then y tends to be smaller.
- The magnitude of the Correlation Coefficient shows the strength of the Linear Relationship.

Example

For which of the following pairs the Correlation is higher?



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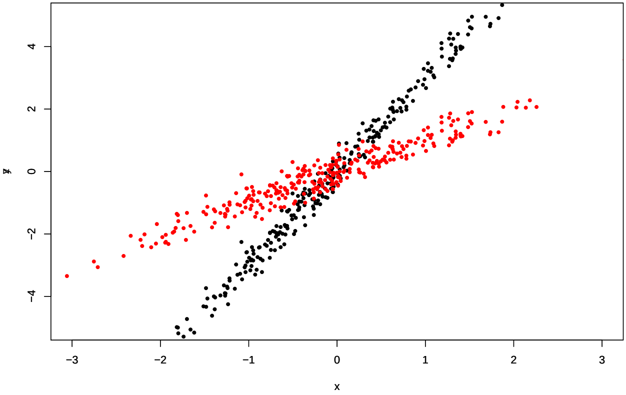


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## [1] 0.9954556 0.9610155
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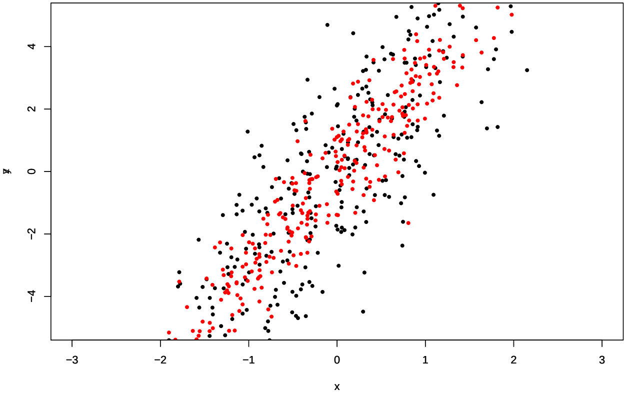

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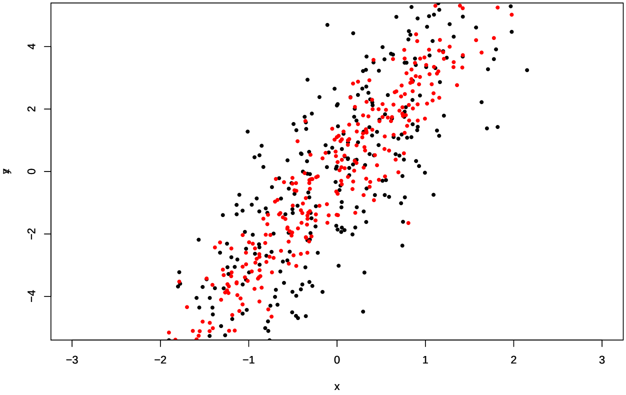


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Correlation is a Measure of Linear Relationship

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x <- runif(2000, -2,2)
y <- x^2 + 0.3*rnorm(2000)
plot(x,y, pch = 20)
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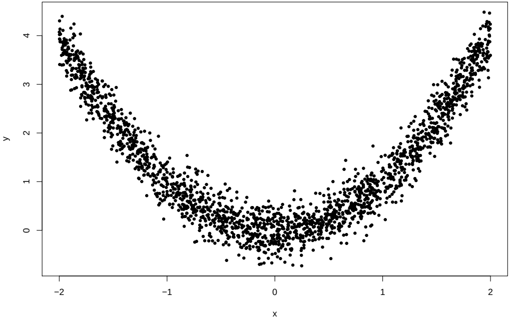


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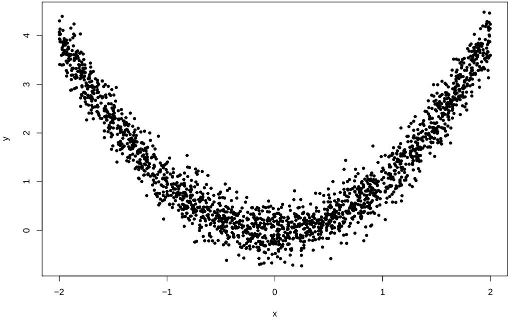


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See more at Wiki

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- If working with multiple variables, one can calculate the Multiple correlation
- One can interpret the Correlation Coefficient as a Cosine of the angle between the r.v.s (or observations), see Wiki
- There are other measures of Association between variables, such as Rank Correlations, say, Kendal's τ

Correlation is not Causation

- Some Examples: Spurious Correlations

Anscombe Quartet

Wiki

Reminder on Random Variables and Distributions

Random Variables

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So $X = X(\omega)$, but usually we forget about ω, and use X.

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So for a Continuous r.v., another complete characteristic, besides the CDF, is its PDF.

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or, in a table form,

Values of X	x_1	x_2	...
$\mathbb{P}(X = x)$	p_1	p_2	...

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- The Variance

$$Var(X) = \mathbb{E}\Big((X - \mathbb{E}(X))^2\Big) = \mathbb{E}(X^2) - \Big[\mathbb{E}(X)\Big]^2.$$