AUA CS 108, Statistics, Fall 2019

Lecture 34

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Contents

Last Lecture ReCap

- Give the definition of the $\chi^2(n)$ Distribution.

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- Give the $(1-\alpha)$-level CI for σ^2 in the Normal Model.

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Recall that, for the Model $\mathcal{N}(\mu, \sigma^2)$, in the case when μ was unknown, we have obtained the following $(1-\alpha)$-level CI for σ^2:

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This is the same as

$$\left(\frac{(n-1)\cdot S^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \quad \frac{(n-1)\cdot S^2}{\chi^2_{n-1,\frac{\alpha}{2}}} \right),$$

where S is the Sample Standard Deviation given by

$$S^2 = \frac{\sum_{k=1}^{n}(X_k - \overline{X})^2}{n-1}.$$

Example

Example: Assume we have scale of very high precision, and we want to measure the precision of our scale. Say, we weight a 50AMD coin, using that scale, 10 times, and we obtain the following measurements (in gramms):

```
 [1] 3.449243 3.450802 3.453054 3.448778 3.452541 3.448835 3.448
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So now, using the above observations (weighting results), we will construct a 90% CI for σ^2.

Example, Cont'd

Recall the $(1-\alpha)$-level CI for σ^2:

$$\left(\frac{(n-1)\cdot S^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \quad \frac{(n-1)\cdot S^2}{\chi^2_{n-1,\frac{\alpha}{2}}} \right).$$

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Now, we use **R** to do the rest: we want to construct 90% CI, so our $\alpha = 0.1$. We have 10 observations, so $n = 10$. We calculate S^2:

```
w <- c(3.449243, 3.450802, 3.453054, 3.448778, 3.452541, 3.
s2 <- var(w)
s2
```

```
## [1] 4.605341e-06
```

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The Degrees of Freedom in our case, since we do not know μ, is $n-1=9$, and we calculate the cooresponding quantiles for the $\chi^2(9)$:

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alpha <- 0.1
lq<-qchisq(alpha/2, df=9); uq<-qchisq(1-alpha/2, df=9)
c(lq,uq)
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## [1]  3.325113 16.918978
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Finally, we calculate our CI endpoints:

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n <- 10
c((n-1)*s2/uq, (n-1)*s2/lq)
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## [1] 2.449797e-06 1.246516e-05
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Note: The actual value of `sd` I was using was: $sd = 0.002$, so the true value of my σ^2 was

$$\sigma^2 = 4 \cdot 10^{-6}.$$

CI for σ^2, $\mathcal{N}(\mu, \sigma^2)$ Model, Summary

Again, as above, let us summarize what we have obtained for this model. The problem is: given a Random Sample

$$X_1, X_2, ..., X_n \overset{IID}{\sim} \mathcal{N}(\mu, \sigma^2),$$

and $\alpha \in (0, 1)$, we want to construct an $1 - \alpha$-level CI for the unknown parameter σ^2.

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To construct an Asymptotic CI for θ, we take $\alpha \in (0, 1)$.

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Definition: Assume, for any n, $L_n = L_n(x_1, ..., x_n, \alpha)$, $U_n = U_n(x_1, ..., x_n, \alpha)$ be two functions with $L_n(x_1, ..., x_n, \alpha) \leq U_n(x_1, ..., x_n, \alpha)$ for all $(x_1, ..., x_n, \alpha)$. The sequence of Random Intervals

$$(L_n, U_n) = \big(L_n(X_1, ..., X_n, \alpha); \; U_n(X_1, ..., X_n, \alpha)\big)$$

is called an **Asymptotic Confidence Interval sequence** (or just an Asymptotic Confidence Interval for θ of (Asymptotic) level $1 - \alpha$, if for any $\theta \in \Theta$,

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Usually, we will have that the limit above exists, so we will use

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Asymptotic CI for the Mean of General Distribution

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We consider the following t-Statistics (or, rather, a sequence of Statistics):

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$$t_n \xrightarrow{D} \mathcal{N}(0,1).$$

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The resti is standard: first we find numbers a, b such that

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$$\mathbb{P}(-z_{1-\alpha/2} < Z < z_{1-\alpha/2}) = 1 - \alpha.$$

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$$t_n = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \cdot \frac{\sigma}{S_n}.$$

Now, by the CLT, the first factor tends to $\mathcal{N}(0,1)$ in Distributions, and the second one, as can be proved, tends to 1 in Probability. From this we can obtain that

$$t_n \xrightarrow{D} \mathcal{N}(0,1).$$

So the Asymptotic Distribution of t_n is independent of μ.

The resti is standard: first we find numbers a, b such that

$$\mathbb{P}(a < Z < b) = 1 - \alpha, \qquad \textit{where} \quad Z \sim \mathcal{N}(0,1).$$

We can take, as previously, $-a = b = z_{1-\alpha/2}$. Then

$$\mathbb{P}(-z_{1-\alpha/2} < Z < z_{1-\alpha/2}) = 1 - \alpha.$$

Since $t_n \xrightarrow{D} \mathcal{N}(0,1)$, then

$$\mathbb{P}(-z_{1-\alpha/2} < t_n < z_{1-\alpha/2}) \to \mathbb{P}(-z_{1-\alpha/2} < Z < z_{1-\alpha/2}) = 1 - \alpha.$$

Asymptotic CI for the Mean of General Distribution

We plug the value of t_n here and solve for μ to obtain

$$\mathbb{P}\left(\overline{X}_n - z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}} < \mu < \overline{X}_n + z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}}\right) \to 1 - \alpha$$

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so the Random Interval (or, rather, the sequence of Intervals)

$$\left(\overline{X}_n - z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}}; \;\; \overline{X}_n + z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}}\right)$$

is a $(1 - \alpha)$-level Asymptotic CI for μ.

Asymptotic CI for the Mean of General Distribution

Note: We have obtained the following $(1-\alpha)$-level Asymptotic CI for μ:

$$\left(\overline{X}_n - z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}};\ \ \overline{X}_n + z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}}\right)$$

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- the same form of CI was obtained for the Normal Model, when σ^2 was unkown;

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This is because

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- this interval is a little bit larger than the previous one, so it is also an AsympCI for μ of level $1-\alpha$;

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- when $n \geq 30$, these two almost coincide;

Asymptotic CI for the Mean of General Distribution

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This is because

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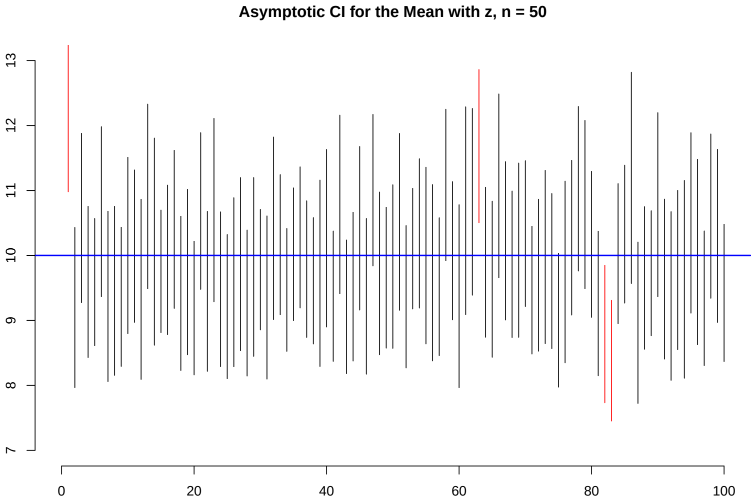
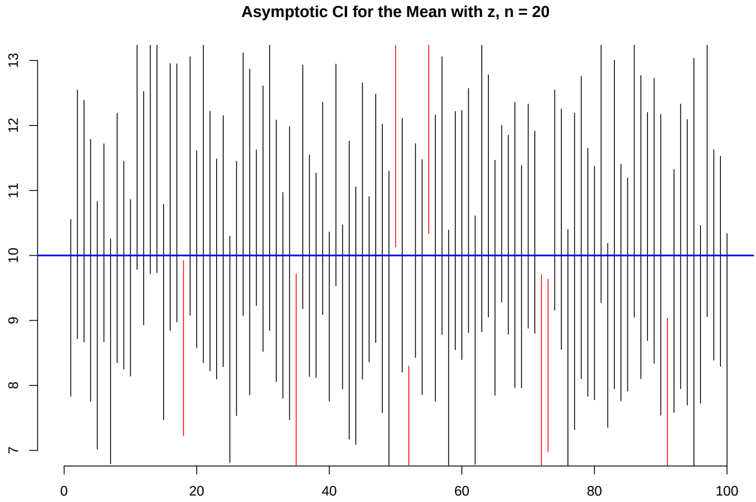
- this interval is a little bit larger than the previous one, so it is also an AsympCI for μ of level $1-\alpha$;
- when $n \geq 30$, these two almost coincide;
- although in the theory these intervals work for large n, but, in practice, the latter one works also for small n

Example

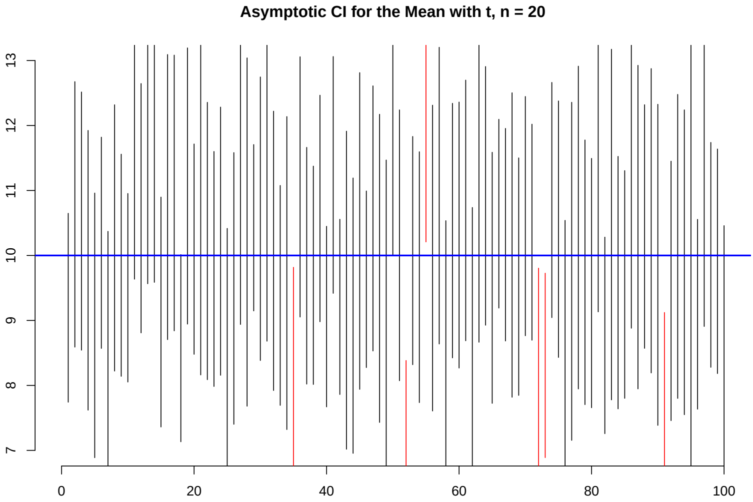

Asymptotic CI for the Mean with z, n = 50
7
8
9
10
11
12
13
0
20
40
60
80
100

Example



Asymptotic CI for the Mean with z, n = 20
7
8
9
10
11
12
13
0
20
40
60
80
100

Example


Asymptotic CI for the Mean with t, n = 20
7
8
9
10
11
12
13
0
20
40
60
80
100