CS 107, Probability, Spring 2019
Lecture 27

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AUA

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Content

- Examples of Important Continuous R.V.s

The Galton Board

See https://www.youtube.com/watch?v=6YDHBFVIvIs

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Exponential Distribution

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Exercise: Check that this function is a PDF of some r.v.
Exercise: Prove that the CDF of $X \sim Exp(\lambda)$ is given by:

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0 \end{cases}$$

Exponential Distribution Examples

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The Rate Parameter; Exponential and Poisson Distributions

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Difference between the Poisson and Exponential Distribution is the following:
Exponential Distribution is calculating the **time between two events**, and the **time until the first event will happen**, which is continuous, and Poisson Distribution is calculating the **Number of events (occurrences)**, which is discrete.

Exponential Distributions: Example

Example: Assume $X \sim \mathit{Exp}(2.6)$. Calculate $\mathbb{P}(X = 2.6)$, $\mathbb{P}(X \leq 2)$, $\mathbb{P}(2 \leq X \leq 3.5)$, $\mathbb{P}(X > 3|X > 1)$.

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Analogously, the only Discrete Distribution sharing the Memoryless Property:

$$\mathbb{P}(X > m + n | X > m) = \mathbb{P}(X > n), \qquad \forall m, n \in \{0, 1, 2, ...\}$$

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$$\mathbb{P}(X > m + n | X > m) = \mathbb{P}(X > n), \qquad \forall m, n \in \{0, 1, 2, ...\}$$

is the Geometric Distribution (which can be described as the waiting time for the Bernoulli process).

Exponential Distribution: R Examples

R Code

```
### Exponential Distribution
lambda <- 2.4
x <- seq(from = -1, to = 5, by = 0.01)
y <- dexp(x, rate = lambda)
#plot the PDF
plot(x,y, type = "l", lwd = 3, main = "Exponential Distribu

z <- pexp(x, rate = lambda)
#plot the CDF
plot(x,z, type = "l", lwd = 3, main = "Exponential Distribu

##Comparing of 2 exponential PDFs
curve(dexp(x,rate = 1), -1, 5, ylim = c(0,2))
par(new = T)
curve(dexp(x,rate = 2), -1, 5, ylim = c(0,2), col = "red")
```

R Example: Moodle Page Visits Model

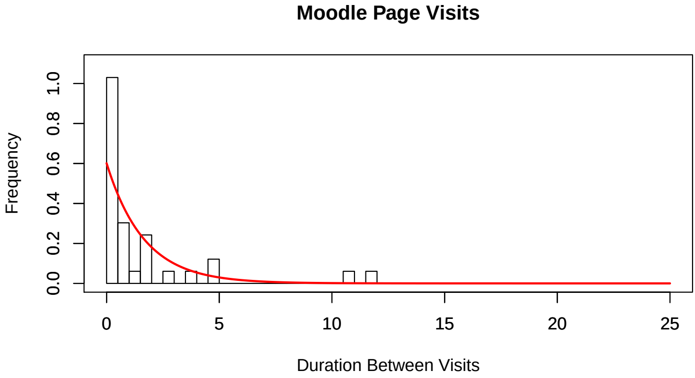

Figure: The histogram is for the time between our Probability Moodle visits by students in hours. Fitted red line is the Exponential Distribution with the $\lambda = \dfrac{1}{\text{average time between visits}}$

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The Normal (Gaussian) Distribution

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$$X \sim \mathcal{N}(\mu, \sigma^2),$$

if its PDF is given by

$$f(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad x \in \mathbb{R}.$$

Standard Normal Distribution

For example, if X is a Normal r.v. with the Mean 0 and Variance 1, i.e.

$$X \sim \mathcal{N}(0, 1),$$

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Exercise: Prove that $\varphi(x)$ is a PDF of some r.v.

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Normal Distribution: R Examples

```
#Standard Normal Distribution
t <- seq(from=-4,to=4, by=0.01)
y <- dnorm(t, mean = 0, sd = 1) #PDF
plot(t,y, type = "l", lwd = 2, main = "PDF of the Standard
z <- pnorm(t, mean = 0, sd = 1) #CDF
plot(t,z, type = "l", lwd = 2, main = "CDF of the Standard
#Other Mean and SD Normal PDFs
t <- seq(from=-2,to=6, by=0.01)
y <- dnorm(t, mean = 2, sd = 1)
plot(t,y, type = "l", lwd = 2, ylim = c(0,1), main = "PDF o
u <- dnorm(t, mean = 2, sd = 0.5)
par(new = T)
plot(t,u, type = "l", lwd = 2, col = "red", ylim = c(0,1),
w <- dnorm(t, mean = 2, sd = 1.5)
par(new = T)
plot(t,w, type = "l", lwd = 2, col = "green", ylim = c(0,1
```

Examples of Normal Distribution

You can find many examples of Normal Distribution in the real world.

Figure: Hallgrímskirkja, church in Reykjavík, Iceland

Other examples of Normal Distribution

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Figure: Normal Table

Other examples of Normal Distribution

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Surely you know what us **curving**. And, of course, you know that the Probability that we will do curving for Probability is 0 $\ddot\smile$.

But, what is curving, in fact? And what is the **curve** we are talking about? Of course, our new friend the Bell-Shaped Gaussian Curve!

See `https://www.youtube.com/watch?v=vqNExEhXHvc`

Now, Serious Things

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The importance of the Normal Distribution is justified by the Central Limit Theorem, which we will cover close to the end of our course, and you will use a lot in Statistics.

Now, Some Examples

Example: Assume $X \sim \mathcal{N}(0,1)$. Calculate $\mathbb{P}(X \in \{0,1,2\})$, $\mathbb{P}(-1 < X < 1)$, $\mathbb{P}(X < 0)$, $\mathbb{P}(X > 3)$