CS 107 Section A - Probability

Spring 2020, AUA

Homework No. 05

Due time/date: 09:35AM, 16 March, 2020

Note: Supplementary Problems will not be graded, but you are very advised to solve them and to discuss later with TA or Instructor.

A Independence of Events

Problem 1. Which of the following pairs of events are independent? And which of that pairs are mutually exclusive (disjoint)? Explain.

a. We are rolling a die. The first event is that an even number will be shown, the second event is that the number shown number will be 5;
b. We are rolling a die. The first event is that an even number will be shown, the second event is that the number shown number will be divisible by 3;
c. We are rolling 2 dice. The first event is that the maximum of two numbers will be exactly 4, and the second event is that one of the dice will show 3;
d. We are rolling 2 dice. The first event is that the maximum of two numbers will be exactly 4, and the second event is that one of the dice will show 5;
e. We are rolling 2 dice. The first event is that the maximum of two numbers will be exactly 4, and the second event is that one of the dice will show 4.

Problem 2. We are tossing two fair coins, independently, and consider the following Events:

$$A = \{\text{1st toss is a Head}\}, \quad B = \{\text{2nd toss is a Head}\},$$

$$C = \{\text{we have different results in two tosses}\}$$

Prove that A, B, C are Pairwise Independent, but not Independent.

Problem 3. Assume we are picking a point from $[0, 1]$ at random, uniformly. Are the following Events Independent:

a. $A = \{\text{the point is} < 0.2\}$, $B = \{\text{the point is} > 0.8\}$;
b. $A = \{\text{the point is} < 0.8\}$, $B = \{\text{the point is} > 0.2\}$;
c. $A = \{\text{the point is rational}\}$, $B = \{\text{the point is irrational}\}$?
d. For which x, y the Events $A = \{\text{the point is} < x\}$ and $B = \{\text{the point is} > y\}$ will be Independent?

Problem 4. Assume A, B, C are Independent (i.e., Mutually Independent), and A and D are disjoint. Moreover, assume that

$$\mathbb{P}(A) = 0.1, \mathbb{P}(B) = 0.1, \mathbb{P}(C) = 0.3, \mathbb{P}(D) = 0.5.$$

Calculate

a. $\mathbb{P}(A \cup B \cup C)$;

b. $\mathbb{P}(A|\overline{D})$;

c. $\mathbb{P}\Big((A \cup B) \cap C\Big)$;

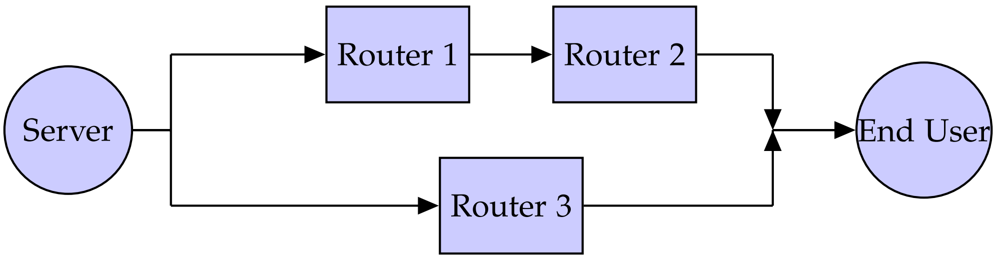
d. $\mathbb{P}(A|B \cap C)$;

e. $\mathbb{P}(A|B \cup C)$;

f. $\mathbb{P}(A \cup \overline{B}|C)$;

g. $\mathbb{P}(A \cup B|D)$, if $\mathbb{P}(B|D) = 0.7$.

Problem 5. Assume we have a Server, which is sending an information packet through 3 Routers given in the Figure below. The probability that Router 1 will go out of order is 0.2, that Router 2 will go out of order is 0.1, and for the Router 3 that probability is 0.12. What is the probability that the End User will get the information sent by the Server, if we know that the events that the Routers will go out of order are mutually independent?



B Binomial and Multinomial Probabilities

Problem 6. Assume I am rolling a fair die 5 times (independently), and let X be the number of prime numbers that will be shown during these 5 rolls.

a. Describe the Trial (Simple Experiment), and the *Success* Event in that Simple Experiment.

b. What is the number of Trials?

c. What is the Probability of Success (in one Trial)?

d. What it means that $X = 0$? $X = 1$?

e. Describe the Event $X = 4$ (you can do it in different ways!).

f. Clearly, X can be anything from $\{0, 1, 2, 3, 4, 5\}$. For any $k \in \{0, 1, 2, 3, 4, 5\}$, calculate

$$\mathbb{P}(X = k),$$

and give the results in the table form.

g. What is the most probable number of primes we can have in these 5 rolls?

Problem 7. Assume we are rolling a die with sides

{⚀, ⚀, ⚀, ⚀, ⚁, ⚂},

and the probability of rolling each side is the same. We repeat to roll this die 10 times. What is the probability that

a. We will have ⚀ shown exactly 7 times;

b. We will not have any ⚀ shown in 10 rolls;

c. We will have ⚀ shown at least 8 times;

d. We will have ⚀ shown 6 times, ⚁ shown 3 times and ⚂ shown 1 times.

e. We will not have ⚀ and ⚂ shown in these 10 rolls.

Problem 8. It is known that 8% of Probability Students get the A+ grade. We are choosing randomly a sample of 27 Probability Students[1], and let X be the number of such students in that sample. What is the Probability that $X > 2$?

Problem 9. We have 29 Probability course participants, including the students, TA and the instructor. Assuming that all week days are equiprobable for the birth of a person, find the probability that

a. Exactly 12 participants have their birthdays on Sunday this year;

b. Less than 4 participants have their birthday on Monday this year;

c. 4 participants have their birthday on Monday this year, 5 on Tuesday, 6 on Wednesday, 2 on Thursday, 4 on Friday and the rest on Saturday.

d. 7 participants have their birthday on Saturday, 5 on Sunday, and all others have their birthdays on other (i.e., working) days.

Problem 10. Assume we want to test the recent elections results. The election was between 3 candidates, A, B and C, and the results of the election was:

Candidate A - 45% of votes;

Candidate B - 30% of votes;

Candidate C - 25% of votes.

To test the results, we open the phonebook, pick by random 11 times a phone number (with replacements, so it is possible to choose the same phone number several times), call by that number and ask to report anonymously for whom the vote was given. What is the probability to have that 4 persons called gave their votes for the candidate A, 5 persons voted for B and 2 persons voted for C?

Problem 11. We have a box with 10 white, 5 red and 7 green balls. We draw a ball at random, fix the color and then return to the box. We repeat this experiment 10 times.

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[1] Let us assume, for simplicity, that the choice is with replacements.

a. What is the probability that we will have exactly 6 white balls in 10 experiments?
b. What is the probability that the number of the white balls in 10 experiments will be between 5 and 7 (including 5 and 7)?
c. What is the probability that we will have exactly 3 white, 6 red and 1 green balls drawn?
d. What is the probability that we will have at least 7 white and at most 2 red balls drawn?

C Applications of Conditional Probabilities

Problem 12. We want to build a toy language model. To that end, we assume that our language consists of only the following sentences:

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My name is Michael. And what is your name? My name is Gohar.
I am a Mathematician. I like talking about Mathematics.

The name of this street is Baghramyan.

You need to know Probability formulas and notions. You studied Probability
for almost two months. How good do you know Probability Theory? I know
Probability pretty well, and I know how solve this Probability homework.

I prefer doing homework than watching a football game. I know how to play
a piano.
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a. Construct the Vocabulary of our Language
b. We consider the Unigram Model. Calculate the probability of having the sentence

   `I like Probability.`

c. Now we consider the Bigram Model. We want to construct a 3-word sequence in this model. To that end, choose the most probable first word, then "autocomplete" to 3-word sentence (i.e., after choosing the first word, choose the most probable successor, then the most probable successor of the second word).

Problem 13. We have the following ranking of courses (a fake one, of course, just a random selection of 0-s and 1-s ◡̈) - ranks are 0-1, $1 = \textit{Like}$, $0 = \textit{Didn't Like}$

Student \ Course	Calc 1	Calc 2	Calc 3	I2CS	LA	RA	OOP	Prob
S1	1	1	1	1	0	1	0	0
S2	0	1	1	0	1	1	0	1
S3	1	1	0	1	1	0	0	1
S4	1	0	1	0	1	1	1	0
S5	1	0	0	0	1	1	1	0
S6	1	0	1	1	0	1	0	0
S7	0	1	0	0	1	0	1	0
S8	0	0	0	1	0	1	0	1
S9	1	1	1	1	1	1	1	0
S10	0	0	0	1	0	1	1	0

a. Student S is taking (and liking!) the course Calc3. Recommend her the next course she will like;

b. Student S is taking (and liking!) the course Calc3. Recommend her 2 next courses she will like most;

c. (Supplementary) Now, Student S is taking (and liking!) the courses $Calc3$ and $I2CS$. Recommend her 2 other course that she would probably like.

d. (Supplementary) Now assume rankings are from 0 to 5, integers (5 = "great course"). Fill randomly (or by asking your friends to rank) the table above, and think about how to recommend the next course to Student S, who ranked 5 her recent $Calc3$ class.

Problem 14. Another toy example, for Naive Bayes Classification (NBC). In the attached .csv file you will find some partial data from one of my exams at YSU. Data provides two midterms grades (0-5 points), and the final result (Pass/Fail), and the Final Exam results are missing. Now assume we have a new Student, who has 2 points from the Midterm 1 and 1 points from the Midterm 2. What is the decision of the NBC, will this Student pass my exam?

D Supplementary Problems

Problem 15. (Supplementary) When preparing for a birthday party, the host person invited 12 friends. She knows that for each friend there is a 5% chance that he/she will not be able to join the party.

a. What is the probability that exactly 10 persons will attend the party?

b. The host wants to buy a birthday cake. She wants to be sure by 80% that everybody will have his/her piece of cake. For how many persons at least she needs to order the cake?

Remark: You need to interpret what it means that "She wants to be sure by 80% that everybody will have his/her piece of cake".

Problem 16. (Supplementary) Assume that the probability of Success is p in one Trial. We make n Independent Trials, and denote by X the number of Successes we will have. What is the most probable value of X?

Problem 17. (Supplementary) The box is full of 10 white and 1 golden balls. A and B are playing a game: they are taking at random a ball from the box, without replacements, until the golden ball will be drawn. The person taking the golden ball will win. They are taking balls alternating each other, in the order A-B-A-B-A-B-... . Whose chances are higher to win?