CS 107, Probability, Spring 2019
Lecture 03

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AUA

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Content

- Probability Measure
- Properties of the Probability Measure

Linda is thirty-one years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.
Give your estimates for the probabilities that Linda is

(1) an elementary school teacher,
(2) active in the feminist movement,
(3) a bank teller,
(4) an insurance salesperson,
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This question is by Daniel Kahneman, psychologist, 2002 Nobel Prize receiver

Probability (Measure) Definition

Assume we have an Experiment with the Sample Space Ω and let $\mathcal{F}$ be the set of all Events in our Experiment.

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We want to be able to measure the Probability of *any Event*, so the Probability Measure is a function defined on all events:

Probability (Measure) Definition

Probability Measure Definition

A function $\mathbb{P} : \mathcal{F} \to \mathbb{R}$ is called a **Probability Measure** on $(\Omega, \mathcal{F})$, if it satisfies the following axioms:

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$$\mathbb{P}(A) \geq 0;$$

P2. $\mathbb{P}(\Omega) = 1$;

P3. For any sequence of pairwise mutually exclusive (disjoint) events $A_n \in \mathcal{F}$, i.e., for any sequence $A_n \in \mathcal{F}$ with $A_i \cap A_j = \varnothing$ for $i \neq j$, we have

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

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- Probability Measure is very similar (and shares the properties of) any other Measure - Cardinality (no. of elements), Length (in 1D), Area (in 2D), Volume (in 3D and moreD).
- The difference is only that the Probability of the Sample Space is 1, $\mathbb{P}(\Omega) = 1$.

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$$\Omega = \{\text{⚀}, \text{⚁}, \text{⚂}, \text{⚃}, \text{⚄}, \text{⚅}\}$$

- We can define (this is a definition!) the Probability Measure on this Experiment as:

$$\mathbb{P}(\Omega) = 1,$$

$$\mathbb{P}\left(\left\{\text{⚀}\right\}\right) = \mathbb{P}\left(\left\{\text{⚁}\right\}\right) = \mathbb{P}\left(\left\{\text{⚂}\right\}\right) = \mathbb{P}\left(\left\{\text{⚃}\right\}\right) =$$

$$= \mathbb{P}\left(\left\{\text{⚄}\right\}\right) = \mathbb{P}\left(\left\{\text{⚅}\right\}\right) = \frac{1}{6}$$

Examples, cont'd

- Note that I am not writing $\mathbb{P}\left(\text{⚀}\right)$, but $\mathbb{P}\left(\left\{\text{⚀}\right\}\right)$, since the Probability is defined on **Events**, but not Outcomes. But usually people use the notation $\mathbb{P}\left(\text{⚀}\right)$ for simplicity, and we will do that.

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- Another note: the definition above is not complete: we need to define the Probability of **any Event**, so we need to continue our definition by:

$$\mathbb{P}\left(\left\{\text{⚀},\text{⚁}\right\}\right) = \mathbb{P}\left(\left\{\text{⚀},\text{⚂}\right\}\right) = \ldots = \mathbb{P}\left(\left\{\text{⚄},\text{⚅}\right\}\right) = \frac{2}{6},$$

also

$$\mathbb{P}\left(\left\{\text{⚀},\text{⚄},\text{⚅}\right\}\right) = \frac{3}{6}$$

etc., and also $\mathbb{P}(\varnothing) = 0$.

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- Now we take any non-negative numbers $p_1, p_2, \dots, p_6$ with

$$p_1 + p_2 + \dots + p_6 = 1,$$

and define

$$\mathbb{P}\left(\left\{\text{⚀}\right\}\right) = p_1, \quad \mathbb{P}\left(\left\{\text{⚁}\right\}\right) = p_2, \quad \mathbb{P}\left(\left\{\text{⚂}\right\}\right) = p_3,$$

$$\mathbb{P}\left(\left\{\text{⚃}\right\}\right) = p_4, \quad \mathbb{P}\left(\left\{\text{⚄}\right\}\right) = p_5, \quad \mathbb{P}\left(\left\{\text{⚅}\right\}\right) = p_6.$$

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and also:

$$\mathbb{P}\left(\left\{\epsdice{1}, \epsdice{2}\right\}\right) = p_1 + p_2, \quad \mathbb{P}\left(\left\{\epsdice{3}, \epsdice{4}, \epsdice{6}\right\}\right) = p_3 + p_4 + p_6$$

etc, and $\mathbb{P}(\varnothing) = 0$.

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etc, and $\mathbb{P}(\varnothing) = 0$. In this way we will obtain a Probability Measure.

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Remarks and Facts:

As we have seen in the above examples, we need to give how to calculate the Probability of *each Event*. Even in the above examples, where the Sample Spaces are finite, listing of all values of Probability Measure is very long and boring.
Fortunately, in the case of Experiments with Discrete Sample Spaces (i.e., Sample Spaces, that are either Finite or Countably Infinite), one can just define the Probability of each Outcome[1]. Then, using the Properties of the Probability Measure, we will be able to calculate the Probability of each Event in that Experiment!

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Remarks and Facts: Cont'd

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$$\text{Sample Space} = \{H, T\},$$

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and then we can calculate $\mathbb{P}(\{H, T\})$ by using the property of a Probability Measure that we will give soon:

$$\mathbb{P}(\{H, T\}) = \mathbb{P}(\{H\}) + \mathbb{P}(\{T\}).$$

It remains only to add $\mathbb{P}(\varnothing) = 0$.

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- As an example, if we consider an Experiment of choosing a point (a real number) from $[0,1] = \Omega$ at random [2], then we can choose as Basic Events all intervals $[a,b] \subset [0,1]$, and define

$$\mathbb{P}([a,b]) = Length([a,b]) = b - a.$$

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  Using this, we will be able to calculate the Probability of any Event (there is another 18+ topic here!)

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3. for any event $A \in \mathcal{F}$,

$$\mathbb{P}(\overline{A}) = 1 - \mathbb{P}(A);$$

Here $\overline{A} = A^c = \Omega \setminus A$.

Properties of the Probability Measure

4. If $A_1, A_2, ..., A_n \in \mathcal{F}$ are pairwise disjoint (mutually exclusive), i.e., if $A_i \cap A_j = \varnothing$ for $i \neq j$, then

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$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i);$$

5. for any events $A, B \in \mathcal{F}$ (not necessarily disjoint),

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B);$$

Intermezzo

Recall the problem from the last lecture:

Assume our mobile phone Weather App says that there is a 50% chance of snow for this Saturday, and also 50% chance of snow this Sunday.
Is it true that it will snow for sure (i.e., with probability 1) this weekend?

What is your answer now?

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