AUA CS 108, Statistics, Fall 2019

Lecture 31

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4 Nov 2019

Contents

Last Lecture ReCap

- What are the remarkable properties of MLE?

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- What is a Random Interval?

CI Problem Setting

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The usual values of the confidence level are $90\%, 95\%, 99\%$, so the usual values of α are $0.1, 0.05$ and 0.01.

CI

Definition: Assume $0 < \alpha < 1$, and let $L = L(x_1, ..., x_n, \alpha)$, $U = U(x_1, ..., x_n, \alpha)$ be two functions with $L(x_1, ..., x_n, \alpha) \leq U(x_1, ..., x_n, \alpha)$ for all $(x_1, ..., x_n, \alpha)$.

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$$(L, U) = \Big(L(X_1, ..., X_n, \alpha) \ , \quad U(X_1, ..., X_n, \alpha) \Big)$$

is called a **confidence interval (or confidence interval estimator) for θ of confidence level** $1 - \alpha$, if for any $\theta \in \Theta$,

$$\mathbb{P}(L < \theta < U) \geq 1 - \alpha.$$

CI

In the case we have a realization/observation of $X_1, ..., X_n$, say, $x_1, ..., x_n$, then the interval

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Going back to our CI, CI of the confidence level $1 - \alpha$ is a Random Interval that contains θ in more than $(1 - \alpha) \cdot 100\%$ of cases.

CI, Interpretation

Note: It is important to understand, that in the CI definition

$$\mathbb{P}(L < \theta < U) \geq 1 - \alpha$$

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So, if we will have/generate different observations, we will have different Intervals[1] (L, U), and we want to have that most of the time that interval contains our unknown Parameter value.

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CI, **R** Simulation

Example: Consider an example: our Model is $Exp(\lambda)$, and we have an observation from it. Let us take a Random Sample for the general case: $X_1, X_2, ..., X_n$ from $Exp(\lambda)$.

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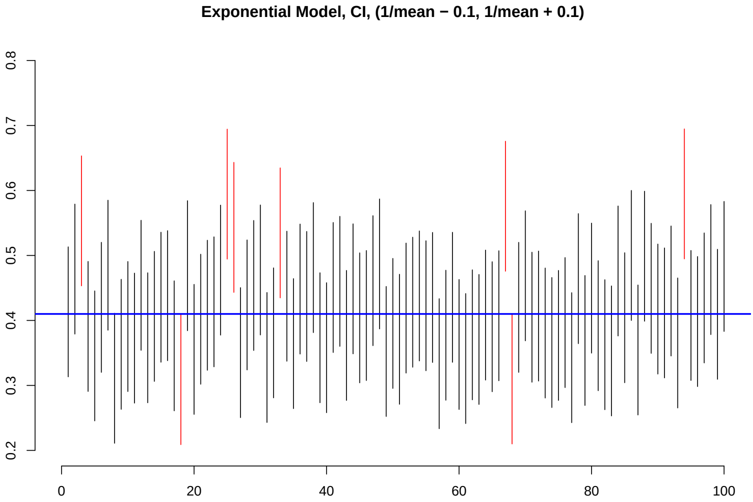
$$\left(\frac{1}{\overline{X}} - 0.1, \frac{1}{\overline{X}} + 0.1 \right)$$

and do some simulations:

CI, **R** Simulation



CI, **R** Simulation, Code

```r
#CI Idea, Exponential Model
lambda <- 0.41
conf.level <- 0.95; a = 1 - conf.level
sample.size <- 50; no.of.intervals <- 100
epsilon <- 0.1

plot.new()
plot.window(xlim = c(0,no.of.intervals), ylim = c(0.2,0.8))
axis(1); axis(2)
title("Exponential Model, CI, (1/mean - 0.1, 1/mean + 0.1)")
for(i in 1:no.of.intervals){
  x <- rexp(sample.size, rate = lambda)
  lo <- 1/mean(x) - epsilon; up <- 1/mean(x) + epsilon
  if(lo > lambda || up < lambda){
    segments(c(i), c(lo), c(i), c(up), col = "red")
  }
  else{
    segments(c(i), c(lo), c(i), c(up))
  }
}
abline(h = lambda, lwd = 2, col = "blue")
```

Methods to obtain Confidence Intervals

We will consider several methods to construct CIs:

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And then we will talk about Asymptotic CIs.

Prob Refresher, Chebyshev Inequality

Recall the Cheby Inequality: If X is a r.v. with finite Mean $\mathbb{E}(X)$ and Variance $Var(X)$, then for any $\varepsilon > 0$,

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or, which is the same,

$$\mathbb{P}\Big(|X - \mathbb{E}(X)| < \varepsilon\Big) \geq 1 - \frac{Var(X)}{\varepsilon^2}.$$

CI for the Mean, Variance is given, Cheby Method

Example: Assume $X_1, X_2, ..., X_n$ are Independent r.v. with the same Mean $\mathbb{E}(X_k) = \mu$ and the same Variance $Var(X_k) = \sigma^2$.

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so if we will plug $\overline{X}$ in the Cheby Inequality, we will obtain

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for any $\varepsilon > 0$.

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for any $\varepsilon > 0$. Now, take $\dfrac{\sigma^2}{n \cdot \varepsilon^2} = \alpha$. Here, σ, n and α are known, so this equality will give us the value for ε:

$$\varepsilon = \frac{\sigma}{\sqrt{n \cdot \alpha}}.$$

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Plugging this value into the above Cheby Inequality, we will get

$$\mathbb{P}\Big(|\overline{X} - \mu| < \frac{\sigma}{\sqrt{n \cdot \alpha}}\Big) \geq 1 - \alpha,$$

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The CI length obtained above is

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Note: If we increase the Confidence Level, i.e., if we decrease α, then the length of CI increases. This is intuitive too: if we want to be more sure where our unknown Parameter is lying, we will get a larger interval.

CI for the Proportion, Cheby Method

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Here, on the RHS, we have the unknown parameter value p, which is not desirable. To get rid of that, we use the estimate $p(1-p) \leq \dfrac{1}{4}$, so $\mathbb{P}(|\overline{X}-p| < \varepsilon) \geq 1 - \dfrac{1}{4n \cdot \varepsilon^2}$.

CI for the Proportion, Cheby Method, Cont'd

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$$\frac{1}{4n \cdot \varepsilon^2} = \alpha, \qquad i.e. \qquad \varepsilon = \frac{1}{2\sqrt{n \cdot \alpha}}.$$

Plugging into the inequality above, this will give

$$\mathbb{P}(|\overline{X} - p| < \frac{1}{2\sqrt{n \cdot \alpha}}) \geq 1 - \alpha,$$

or,

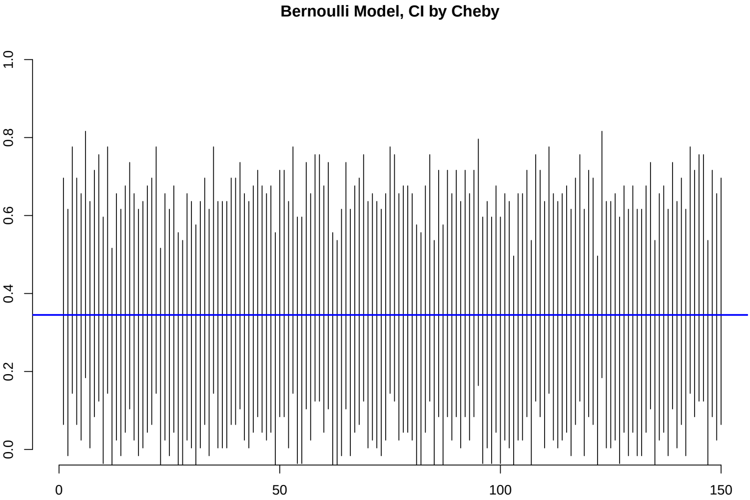
$$\mathbb{P}\left(\overline{X} - \frac{1}{2\sqrt{n \cdot \alpha}} < p < \overline{X} + \frac{1}{2\sqrt{n \cdot \alpha}}\right) \geq 1 - \alpha.$$

This means that the interval

$$\left(\overline{X} - \frac{1}{2\sqrt{n \cdot \alpha}}, \ \overline{X} + \frac{1}{2\sqrt{n \cdot \alpha}}\right)$$

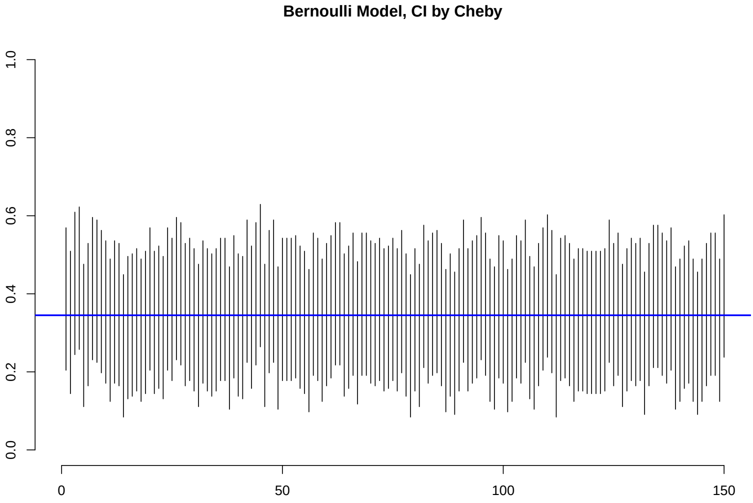
is a CI for p of level $1 - \alpha$.

CI for Bernoulli, **R** Simulation



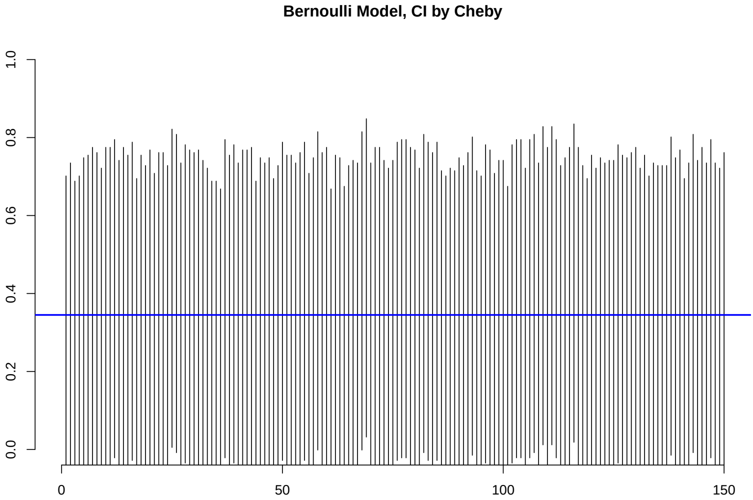
Sample Size $= 50$, $CL = 95\%$

CI for Bernoulli, **R** Simulation



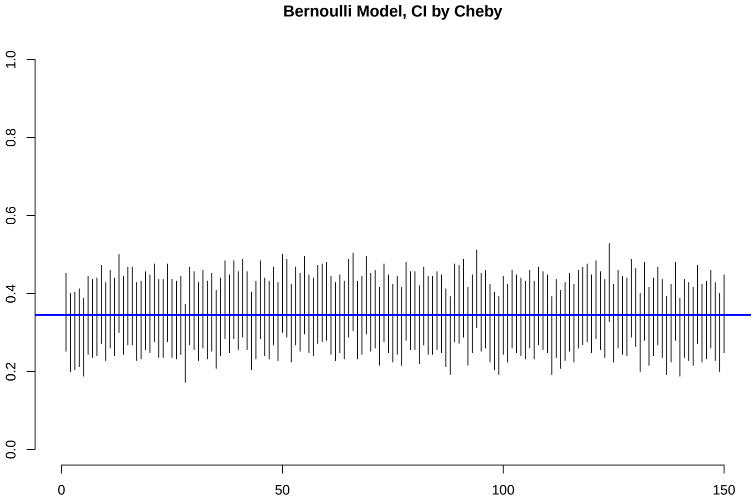
Sample Size $= 150$, $CL = 95\%$

CI for Bernoulli, **R** Simulation



Sample Size $= 150$, $CL = 99\%$

CI for Bernoulli, **R** Simulation



Sample Size $= 250$, $CL = 90\%$

CI, **R** Simulation, Code

```
#CI Idea, Bernoulli Model
p <- 0.345
conf.level <- 0.9; a = 1 - conf.level
sample.size <- 250; no.of.intervals <- 150
ME <- 1/(2*sqrt(sample.size*a)) #Margin of Error
plot.new()
plot.window(xlim = c(0,no.of.intervals), ylim = c(0,1))
axis(1); axis(2)
title("Bernoulli Model, CI by Cheby")
for(i in 1:no.of.intervals){
  x <- rbinom(sample.size, size = 1, prob = p)
  lo <- mean(x) - ME
  up <- mean(x) + ME
  if(lo > p || up < p){
    segments(c(i), c(lo), c(i), c(up), col = "red")
  }
  else{
    segments(c(i), c(lo), c(i), c(up))
  }
}
abline(h = p, lwd = 2, col = "blue")
```