CS 107, Probability, Spring 2020
Lecture 07

Michael Poghosyan
mpoghosyan@aua.am

AUA

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Content

- Classical Probability Models: Finite Sample Spaces with Equally Likely (Equiprobable) Outcomes = Combinatorial Problems

LZ

Strange Mailman Problem

A Mailman needs to deliver 100 letters to given addresses. But he is putting letters at random to the mailboxes. What is the probability that at least one person will receive the correct letter?

Answer: the Probability is about 63%.

Last Lecture ReCap

Last time we were talking about Combinatorial Probability Problems.

Detailed Study of a Simpler Case

Last time we were solving a problem about taking 2 cards at random from a deck, and were calculating the chances of having no Hearts. Here is a simpler problem of the type:

Simplified Problem: We choose two letters from $\{a, b, c, d\}$. What is the probability of having no a and no d in the chosen pair?

Seems simple, but ...

Case 1: choosing without replacement

- Ordered Case
- Unordered Case

Case 2: choosing with replacement

- Ordered Case
- Unordered Case

Moral from the previous Example:

- When considering a combinatorial probability problem, try to think first what are your Outcomes and what is your Sample Space? Are the Outcomes Equiprobable?
- If your Sample Space consists of *unordered* pairs, then your Event under the interest also needs to consist of *unordered* pairs. And if your Sample Space consists of *ordered* pairs, then your Event under the interest also needs to consist of *ordered* pairs.

Moral from the previous Example:

- If you consider an Experiment of drawing an item several times (say, 2 times) *without replacement*, then Unordered/Ordered cases, if they are appropriate, will give the same Probability (if calculated correctly), because each outcome (a, b) in your Unordered Sample Space corresponds exactly to 2 outcomes in the Ordered Space: (a, b) and (b, a). That is, the no. of elements in the Favorable Event (Sample Space) in the Unordered case will be exactly the half of the Favorable Event (Sample Space) in the Ordered case. And, say, if you are drawing 3 times, you will have for each triple (a, b, c) in your Unordered SS 6 possible triples in the Ordered triples Sample Space: (a, b, c), (a, c, b), (b, a, c),... .

Moral from the previous Example:

- If you consider an Experiment of drawing an item several times (say, 2 times) *with replacement*, then the situation with Unordered/Ordered cases will be completely different. This is because not for each pair in the Unordered Space we will have 2 pairs in the Ordered Space. Say, for the pair of the form (a, a) (this is a possible outcome, since we draw with replacement), we will not have 2 pairs in the Ordered Space - we will have only the same pair (a, a).

Another Example:

Problem: We are picking at random 2 numbers from $\{1, 2, 3\}$, and write that numbers one after another. Consider the Event when the obtained 2-digit number contains an even digit. Calculate the Probability of that Event. Consider with/without replacement cases.

- **Case 1: Without replacement - we are not allowed to use the same digit again**
  - UnOrdered Sample Space $= \{12, 13, 23\}$
  - Our Event in The UnOrdered SS $= \{12, 23\}$
  - Ordered Sample Space $= \{12, 13, 21, 23, 31, 32\}$
  - Our Event in The Ordered SS $= \{12, 21, 23, 32\}$
  - $\mathbb{P}(\text{Contains an Even digit}) = \frac{2}{3} = \frac{4}{6}$.

Another Example:

Problem: We are picking at random 2 numbers from $\{1, 2, 3\}$, and write that numbers one after another. Consider the Event that at least one of the numbers picked is even. Calculate the Probability of that Event. Consider with/without replacement cases.

- **Case 2: With replacement - we are allowed to use the same digit again**
  - UnOrdered Sample Space $= \{11, 12, 13, 22, 23, 33\}$
  - Our Event in The UnOrdered SS $= \{12, 22, 23\}$
  - Ordered Sample Space $=$ $\{11, 12, 13, 21, 22, 23, 31, 32, 33\}$
  - Our Event in The Ordered SS $= \{12, 21, 22, 23, 32\}$
  - $\mathbb{P}(\text{Contains an Even digit}) =$

Example:

- **Problem:** We have a box containing 100 balls, from which 25 are white and the rest are black. We choose at random 12 balls. What is the probability of having exactly 4 white balls chosen?

- Clear assumption: Choosing each ball is equiprobable
- **Case 1:** We take balls without a replacement:
- **Case 2:** We take balls with a replacement:

Without Replacements Model, Solution 1:

Problem: We have a box containing 100 balls, from which 25 are white and the rest are black. We choose at random 12 balls. What is the probability of having exactly 4 white balls chosen?

Our Experiment is taking 12 balls at random. Let us fix, as an Outcome of our Experiment, the number of white and black balls in the chosen set of 12 balls. Say, if we will get 3 whites and 9 blacks, our Outcome will be $(3, 9)$, or if we will have 5 whites and 7 blacks, then our Outcome will be $(5, 7)$.

Without Replacements Model, Solution 1:

Problem: We have a box containing 100 balls, from which 25 are white and the rest are black. We choose at random 12 balls. What is the probability of having exactly 4 white balls chosen?

So our Sample Space is:

$$\Omega = \{(n, 12 - n) : n = 0, 1, ..., 12\}, \qquad \#\Omega = 13$$

The Event of our interest is:

$$A = \{(4, 8)\}, \qquad \#A = 1.$$

So the Probability of A is:

$$\mathbb{P}(A) = \frac{1}{13}.$$

Of course, not! The solution is incorrect :) Why?

Without Replacements Model, Solution 2:

Problem: We have a box containing 100 balls, from which 25 are white and the rest are black. We choose at random 12 balls. What is the probability of having exactly 4 white balls chosen?

Our Experiment is taking 12 balls at random. Let us record, as an Outcome of our Experiment, the color of each of 12 balls. Say, one of the outcomes can be

$$(w, b, w, w, b, w, w, w, w, w, b, b).$$

In this case, the Sample Space is:

$$\Omega = \{(x_1, x_2, \dots, x_{12}) : x_k \in \{w, b\}\}, \quad \text{and } \#\Omega = 2^{12}.$$

Without Replacements Model, Solution 2:

Problem: We have a box containing 100 balls, from which 25 are white and the rest are black. We choose at random 12 balls. What is the probability of having exactly 4 white balls chosen?

$$\Omega = \{(x_1, x_2, ..., x_{12}) : x_k \in \{w, b\}\}, \quad \text{and } \#\Omega = 2^{12}.$$

Now, the Event in which we are interested is

$A = \{(x_1, x_2, ..., x_{12})$: exactly 4 of x_k-s are w, and the rest are b$\}$

And $\#A = \binom{12}{4}$. So

$$\mathbb{P}(A) = \binom{12}{4} \cdot \frac{1}{2^{12}}.$$

Right? Not, of course! Why?

Without Replacements Model, Solution 3:

Problem: We have a box containing 100 balls, from which 25 are white and the rest are black. We choose at random 12 balls. What is the probability of having exactly 4 white balls chosen?

Now, the correct solution: OTB

Think about the with-replacements model by yourself!