CS 107, Probability, Spring 2019
Lecture 32

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AUA

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Content

- Joint Distribution of two R.V.s, Continuous Case

No LZ anymore, sorry :(

Joint PDF of a Continuous Random Vector

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Jointly Continuous R.V.s

We say the the r.v.s X and Y are **Jointly (Absolutely) Continuous**, if there exists a non-negative integrable function $f(u, v)$ defined on $\mathbb{R}^2$ such that

$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v)\, dudv, \qquad \forall (x, y) \in \mathbb{R}^2,$$

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The function f is called **the Joint PDF of X and Y**, and, if necessary, is denoted by

$$f(u, v) = f_{X,Y}(u, v).$$

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- Now, if the Joint CDF F is given, then

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}(x,y) \qquad \text{for almost all} \quad (x,y) \in \mathbb{R}^2.$$

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Geometric Interpretation: The Probability that $(X, Y) \in A$ is equal to the Volume of the body under the surface of the Joint PDF graph, over the region A.

Example

Example: Assume (X, Y) is a r.vector with the Joint PDF

$$f(x,y) = \begin{cases} K \cdot x \cdot y, & x, y \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

- Find K;
- Calculate $\mathbb{P}(Y < 1 - X)$

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Assume that (X, Y) is a continuous r.vector with a PDF $f(x, y)$. Then X and Y are both continuous r.v.s and for their PDFs $f_X(x)$ and $f_Y(x)$ we have

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\,dy, \qquad \forall x \in \mathbb{R},$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\,dx, \qquad \forall y \in \mathbb{R}.$$

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Note: In the last equality we have $f_Y(y)$, but, of course, you can replace y with x to have $f_Y(x)$.

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$$f(x,y) = \begin{cases} K \cdot (x+y), & x, y \in [0,2] \\ 0, & otherwise \end{cases}$$

- Find K
- Find the Marginal PDFs of X and Y
- Find the CDFs of (X, Y), and Marginal CDFs
- Calculate $\mathbb{P}(X^2 + Y^2 \leq 1)$
- Calculate the PDF of the r.v. $Z = Y - X$

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$$\mathbb{P}(X = x_i) = \sum_{j,k} \mathbb{P}(X = x_i, Y = y_j, Z = z_k)$$

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- And if X, Y, Z are Jointly Continuous, then their Joint PDF $f(x, y, z)$ and Joint CDF $F(x, y, z)$ satisfy

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Exercise: Can you express the Joint PDF of, say, X and Z, in terms of $f(x, y, z)$?

Important Multivariate Distributions

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Multinomial Distribution

Assume $n, m \in N$, and $p_k \in [0,1]$, $k = 1, 2, ..., m$ with $p_1 + ... + p_m = 1$. We say that the r.vector $\mathbf{X} = (X_1, X_2, ..., X_m)$ has a Multinomial Distribution with probabilities $\mathbf{p} = (p_1, p_2, ..., p_m)$, and we write

$$\mathbf{X} = (X_1, X_2, ..., X_m) \sim \mathit{Multinomial}(n, p_1, p_2, ..., p_m),$$

if its PMF is given by:

$$\mathbb{P}(X_1 = k_1, X_2 = k_2, ..., X_m = k_m) = \binom{n}{k_1, k_2, ..., k_m} \cdot p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_m^{k_m}$$

for any $k_1, ..., k_m \in \mathbb{N} \cup \{0\}$, with $k_1 + k_2 + ... + k_m = n$.

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Exercise: Find the Joint Distribution of, say, (X_1, X_2).

Example

Example: 10 AUA instructors are choosing (independently) at random one AUA student for some committee. We know that the relationship between the number of Bus/CSE/EC students is $8 : 3 : 1$. What is the Probability that among 10 chosen students, we will have exactly 3 Bus, 5 CSE and 2 EC students?