CS 107, Probability, Spring 2020
Lecture 10

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Content

- Conditional Probabilities
- Total Probability Formula

LZ

Medical Test gives a correct answer in 95% of cases: if the person is ill, it is saying that he/she is ill with 95% probability, and if the person is healthy, it is saying that he/she is healthy with 95% probability (i.e., in 95% of cases).
Now assume I am taking that test, and it says that I am ill. What is the probability that I am really ill?

Last Lecture ReCap

Last time we were talking about Geometric Probabilities:

- We assume our Experiment's Sample Space is $\Omega \subset \mathbb{R}^n$;
- For any Event $A \subset \Omega$, we define
  - if $n = 1$,
    $$\mathbb{P}(A) = \frac{Length(A)}{Length(\Omega)}$$
  - if $n = 2$,
    $$\mathbb{P}(A) = \frac{Area(A)}{Area(\Omega)}$$
  - if $n \geq 3$,
    $$\mathbb{P}(A) = \frac{Volume(A)}{Volume(\Omega)}$$

Conditional Probabilities

As we have talked earlier, we use Probability language in the situations when we do not have complete information. But in many cases, we have some *partial* information, and that knowledge, that partial information can change our assessment of the Probability, can change the Probability of an Event.

Conditional Probabilities

Example:

- What is your guess for the Probability that a Probability class (randomly chosen) student will get $> B$ grade?
- Now, what is your guess for the Probability that a Probability class student will get $> B$ grade, *if he/she is reading lecture notes and textbooks regularly, solving a lot of Probability problems, doing homework by him/herself, attending office hours and asking questions?*

Example:

- What is your guess for the Probability that a person will develop lung cancer in his lifetime?
- What is your guess for the same Probability, *if you know that a person is a heavy smoker?*

Conditional Probabilities: Example

Example: Assume we are rolling a fair die.

- What is the Probability that we will have 2 on the top face?
- What is the Probability that we will have 2 on the top face, if we know that Even number is rolled?
- What is the Probability that we will have 2 on the top face, if we know that Odd number is rolled?

Conditional Probabilities, Cont'd

Now, we want to formalize the notion of the Conditional Probability, i.e., Probability under some partial information. Assume Ω is our Experiment's Sample Space, and A, B are two events.

Conditional Probability

The conditional probability of A given B (or the probability of A under the condition of B) is defined to be

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Sometimes one defines $\mathbb{P}(A|B) = 0$, if $\mathbb{P}(B) = 0$. Some authors do not define the conditional probability in the case when $\mathbb{P}(B) = 0$.

Conditional Probabilities, Cont'd

Some interpretations for

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

- Counting Interpretation
- Geometric Interpretation

Example

Problem: We are drawing a card from the deck at random.

- What is the probability that the card will be a Queen?
- What is the probability that the card will be a Queen, if the card drawn is of Diamonds?

Example

Problem: The lifetime of some particular type of USB Flash disks is estimated to be 10 years max. What is the Probability that a USB stick of that type will serve more than 5.5 years, if it already served 3 years?

Solution: Here we assume Probabilities are Uniform (not a correct assumption?!). Later we will be solving this kind of problems with Exponential Distribution. The rest is OTB.

Properties of Conditional Probabilities

Assume $B \subset \Omega$ is a fixed event with $\mathbb{P}(B) > 0$. Then

a. For any event A,
$$\mathbb{P}(A|B) \geq 0;$$

b. $\mathbb{P}(B|B) = 1;$

c. If A is an event, then
$$\mathbb{P}(\overline{A}|B) = 1 - \mathbb{P}(A|B);$$

d. If $A_1, ..., A_n$ are some **mutually disjoint** events, then
$$\mathbb{P}(A_1 \cup A_2 \cup ... \cup A_n|B) = \mathbb{P}(A_1|B) + \mathbb{P}(A_2|B) + ... + \mathbb{P}(A_n|B);$$

e. If $A_1, ..., A_n, ...$ are some **mutually disjoint** events, then
$$\mathbb{P}(\cup_{n=1}^{\infty} A_n|B) = \sum_{n=1}^{\infty} \mathbb{P}(A_n|B);$$

Properties of Conditional Probabilities, Cont'd

f. If A_1, A_2, B are some events and $\mathbb{P}(B) \neq 0$, then

$$\mathbb{P}(A_1 \cup A_2|B) = \mathbb{P}(A_1|B) + \mathbb{P}(A_2|B) - \mathbb{P}(A_1 \cap A_2|B);$$

g. If A is an event with $\mathbb{P}(A) \neq 0$, then

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B) \cdot \mathbb{P}(B) = \mathbb{P}(B|A) \cdot \mathbb{P}(A);$$

h. (multiplication or chain rule) If $A_1, ..., A_n$ are some events, then

$$\mathbb{P}(A_1 \cap A_2 \cap ... \cap A_n) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2|A_1) \cdot \mathbb{P}(A_3|A_1 \cap A_2) \cdot ...$$

$$\cdot \mathbb{P}(A_n|A_1 \cap A_2 \cap ... \cap A_{n-1}).$$

Remarks:

- Please note that, in general,

$$\mathbb{P}(A|B) \neq \mathbb{P}(B|A).$$

The correct relation between these two conditional probabilities is:

$$\mathbb{P}(A|B) \cdot \mathbb{P}(B) = \mathbb{P}(B|A) \cdot \mathbb{P}(A),$$

This means that we can have $\mathbb{P}(A|B) = \mathbb{P}(B|A)$ only in the case $\mathbb{P}(A) = \mathbb{P}(B)$.

- Please note also that

$$\mathbb{P}(\overline{A}|B) = 1 - \mathbb{P}(A|B),$$

but, in general,

$$\mathbb{P}(A|\overline{B}) \neq 1 - \mathbb{P}(A|B).$$

Example: 2 Children Problem

Problem: The family has 2 children. What is the probability that one of them is a girl, and the other is a boy?
Implicit Assumption: In this type of problems we assume that the birth probability of a girl and a boy is the same.
The answer is:

Now, some modification of the problem.
King's Sister Problem: In the middle ages there was a story about a King. The parents of the King had 2 children. What is the probability that the other child is the sister of the King?

Difference between $\mathbb{P}(A \cap B)$ and $\mathbb{P}(A|B)$

Remark: Please note that $\mathbb{P}(A \cap B)$ and $\mathbb{P}(A|B)$ are different things, in general:

- $\mathbb{P}(A \cap B)$ is the Probability that A and B will happen;
- $\mathbb{P}(A|B)$ is the Probability that A will happen, given that B already happened.

You can think like this:

- $\mathbb{P}(A \cap B)$ is the Proportion of $A \cap B$ in Ω;
- $\mathbb{P}(A|B)$ is the Proportion of $A \cap B$ in B.

So if the Sample Space was Ω when calculating Unconditional Probabilities, now, to calculate Conditional Probability of $A|B$, the Sample Space will be now $B \subset \Omega$.

Example

Example: This is again our Example with the fair die.

- What is the Probability that we will have 2 on the top face?

$$\mathbb{P}(2) =$$

- What is the Probability that we will have 2 on the top face, if we know that Even number is rolled?

$$\mathbb{P}(2|\textit{Even}) =$$

- What is the Probability that we will have 2 on the top face and an Even number rolled?

$$\mathbb{P}(2 \cap \textit{Even}) = \mathbb{P}(2)$$

Interpretation of the Conditional Probability

Going back to the previous slide, assume our initial Sample Space is Ω, and we have a Probability Space $(\Omega, \mathcal{F}, \mathbb{P})$. In particular, $\mathbb{P}(\Omega) = 1$.
We fix the Condition: $B \subset \Omega$, with $\mathbb{P}(B) \neq 0$. Let also $A \subset \Omega$ be an Event. Now, if we want to talk about the Event $A|B$, we think as changing (rescaling) our Sample Space to B: we want to have a new Probability Space on B, using the previous one. So we want to have a Probability Space $(B, \mathcal{F}_B, \mathbb{P}_B)$, where

- B is our new Sample Space;
- $\mathcal{F}_B$ is our new set of Events;
- $\mathbb{P}_B$ is a Probability (Measure) on B.