AUA CS108, Statistics, Fall 2020

Lecture 05

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4 Sep 2020

Contents

Empirical CDF

Let us recall the definition of the Empirical CDF:

Definition: The **Empirical Distribution Function, ECDF** or the **Cumulative Histogram** $ecdf(x)$ of our data $x_1, ..., x_n$ is defined by

$$ecdf(x) = \frac{\text{number of elements in our dataset} \leq x}{\text{the total number of elements in our dataset}} =$$

$$= \frac{\text{number of elements in our dataset} \leq x}{n}, \qquad \forall x \in \mathbb{R}.$$

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$$-1, 4, 7, 5, 4$$

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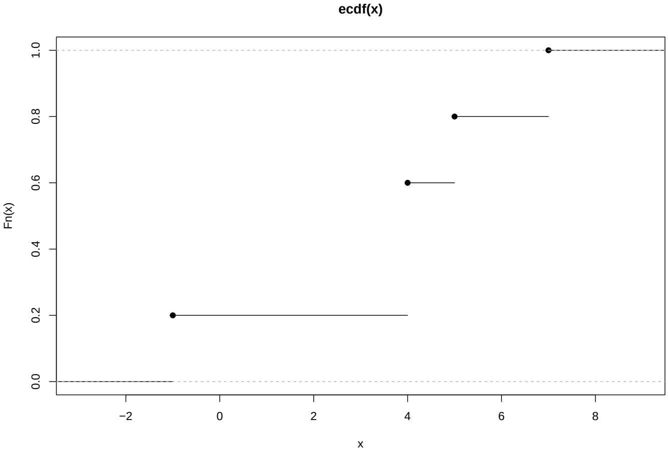
- Sort our Dataset from the lowest to the largest values
- Plot the Data points on the OX axis
- ECDF is 0 for values of x less that the smallest Datapoint, and is 1 for values of x bigger than the largest Datapoint
- For each Data point, calculate the Relative Frequency of that Datapoint (the number of times it occurs in our Dataset over the total number of Datapoints). At that Datapoint, do a Jump of the size of the Relative Frequency, and draw a horizontal line up to the next Datapoint

Example

Now, using **R**:

```
x <-c(-1, 4, 7, 5, 4)
f <- ecdf(x)
plot(f)
```



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$$-1, 4, 7, 5, 4$$

coincides with the CDF of a r.v.

X	-1	4	5	7
$\mathbb{P}(X = x)$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

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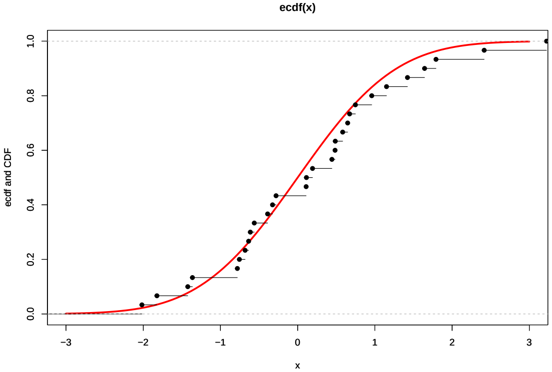
This Theorem says that if you will have enough datapoints from a Distribution, you can approximate the unknown CDF of your Distribution pretty well by using the ECDF.

Estimation of the CDF through ECDF

Let us check this theorem using **R**:

```
plot(pnorm, lwd = 3, col = 'red', xlim = c(-3,3),
     ylim = c(0,1), ylab = "ecdf and CDF")
n <- 30 ; x <- rnorm(n) #Taking a sample of size n from N(0,1)
f <- ecdf(x) #f will be the ECDF of our data x
par(new = TRUE) #this is to keep the previous graph
plot(f, xlim = c(-3,3), ylim = c(0,1), ylab = "ecdf and CDF")
```



Histograms

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To define the Histogram, first we divide the range of our Dataset into *class intervals* or *bins*:

- we take first the range: either $I = [\min_i\{x_i\}, \max_i\{x_i\}]$ or I is an interval containing $[\min_i\{x_i\}, \max_i\{x_i\}]$;

Histograms

- we take a finite partition of I: $I_1, I_2, ..., I_k$, i.e. I_j-s are disjoint, and their union is the interval I;

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- we calculate the number n_j of datapoints x_i lying in I_j:

$$n_j = \text{the number of data points in } I_j \qquad j = 1, 2, ..., k.$$

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Histograms

Definition: The **frequency histogram** of our continuous (or a grouped) data $x_1, ..., x_n$ is the piecewise constant function

$$h_{freq}(x) = n_j, \qquad \forall x \in I_j, \quad j = 1, 2, ..., k.$$

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Frequency histogram shows the number of observations in our dataset in each bin, in each class interval. One also defines $h_{freq}(x) = 0$ for all $x \notin I$.

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Here is the header:

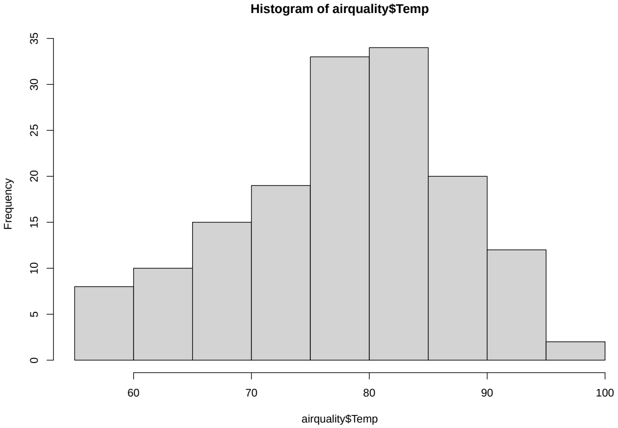
```r
head(airquality)
```

```
##   Ozone Solar.R Wind Temp Month Day
## 1    41     190  7.4   67     5   1
## 2    36     118  8.0   72     5   2
## 3    12     149 12.6   74     5   3
## 4    18     313 11.5   62     5   4
## 5    NA      NA 14.3   56     5   5
## 6    28      NA 14.9   66     5   6
```

Example

Let's Plot the histogram of the *Temp* (Temperature) Variable:

```
hist(airquality$Temp)
```



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- **R**, by default, is choosing some appropriate bins;
- **R**'s *hist* command default bins have equal lengths;
- **R** is adding the default *OX* axis name and the Figure Title.

Histograms

Next is the Relative Frequency Histogram definition:

Definition The **relative frequency histogram** of our continuous data $x_1, ..., x_n$ is the piecewise constant function

$$h_{relfreq}(x) = \frac{n_j}{n}, \qquad \forall x \in I_j, \quad j = 1, 2, ..., k.$$

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or, which is the same,

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The Default **R** package has no Relative Frequency Histogram Plotting command (or I do not know $\ddot\smile$). But you can use, say, the *lattice* library's *histogram* command:

```
library(lattice)
histogram(airquality$Temp)
```

The Density or Normalized Relative Frequency Histogram

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$$h_{dens}(x) = \frac{n_j}{n} \cdot \frac{1}{length(I_j)}, \qquad \forall x \in I_j.$$

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Here $length(I_j)$ is the length of the interval I_j. Also we define $h_{dens}(x) = 0$, if $x \notin I$.

Note

In the case (which is the mostly used one) when all intervals I_j have the same length:

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$$h_{dens}(x) = \frac{h_{relfreq}(x)}{h} = \frac{n_j}{n \cdot h}, \qquad \forall x \in I_j.$$

Idea of the Density Histogram

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The idea of dividing to the lenght of the corresponding interval, in the definition of the Density Histogram, is that in this case, the Total Area of all rectangles of our Histogram is 1.

Recall that all PDF functions integrate to 1. And the Density Histogram is approximating (estimating) the unknown PDF behind our Data!