CS 107, Probability, Spring 2019
Lecture 06

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AUA

30 January 2019

Content

- Classical Probability Models: Finite Sample Spaces with Equally Likely Outcomes = Combinatorial Problems

Birthday Problem

We have 36 participants in our group of Probability class, including the instructor. What is the probability that at least two participants share the same birthday, i.e., they were born on the same day and month (but maybe not in the same year)?

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Answer: the Probability is larger than 83%.

Equally Likely Outcomes

Recall that when we have the Equally Likely Outcomes (or the uniform discrete distribution) model:

Outcome	ω_1	ω_2	ω_3	...	ω_n
Probability	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	...	$\frac{1}{n}$

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then for any Event A,

$$\mathbb{P}(A) = \frac{\text{number of elements favorable for the event } A}{\text{total number of possible outcomes}}$$

$$= \frac{\#A}{\#\Omega} = \frac{\#A}{n}$$

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Now, our Problem: We shuffle our deck and pick at random 2 cards. What is the probability that none of these cards will be hearts?

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$$\text{Probability} = \frac{\#\text{of favorable outcomes}}{\#\text{of all possible outcomes}} =$$

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- For the first case, we consider all **ordered** pairs:

$$\{(A\heartsuit, A\diamondsuit), (A\diamondsuit, A\heartsuit), (A\heartsuit, K\clubsuit), (K\clubsuit, A\heartsuit), \ldots\}$$

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- **Case 2:** We take cards **with replacement**

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Simplified Problem: We choose two letters from $\{a, b, c, d\}$. What is the probability of having no a and no d in the chosen pair?

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- If you consider an Experiment of drawing an item several times (say, 2 times) *without replacement*, then Unorderd/Ordered cases, if they are appropriate, will give the same Probability (if calculated correctly), because each outcome (a, b) in your Unordered Sample Space corresponds exactly to 2 outcomes in the Ordered Space: (a, b) and (b, a).

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Another Example:

Problem: Form all 2-digit numbers from $\{1, 2, 3\}$. Consider the Event when the obtained number contains an even digit. Calculate the Probability of that Event. Consider with/without replacement cases.

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  - $\mathbb{P}(\text{Contains an Even digit}) = \frac{2}{3} = \frac{4}{6}$.

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