AUA CS108, Statistics, Fall 2020

Lecture 20

Michael Poghosyan

12 Oct 2020

Contents

Convergence in Probability

Definition: We will say that $X_n \to X$ **in Probability**, and we will write $X_n \xrightarrow{\mathbb{P}} X$, if

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Equivalently, we can write

$$X_n \xrightarrow{\mathbb{P}} X \qquad \text{iff} \qquad \mathbb{P}\Big(|X_n - X| < \varepsilon\Big) \to 1 \text{ for any } \varepsilon > 0.$$

Convergence in the Mean Square Sence

Definition: We will say that $X_n \to X$ **in the Quadratic Mean Sense or in L^2 (or in the Mean Square Sense)**, and we will write $X_n \xrightarrow{L^2} X$ or $X_n \xrightarrow{qm} X$, if

$$MSE(X_n, X) = \mathbb{E}\Big((X_n - X)^2\Big) \to 0, \qquad \text{when} \quad n \to \infty.$$

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Here $MSE(X_n, X)$ is the *Mean Square Error* (of the approximation of X by X_n).

Convergence in Distributions

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Remark: This is equivalent to saying that for (almost) any subsets $A \subset \mathbb{R}$

$$\mathbb{P}(X_n \in A) \to \mathbb{P}(X \in A).$$

Convergence in Distributions

Remark on the notation: Usually, in the case of the Convergence in Distribution, we write the Distribution as the limit, e.g., we write

$$X_n \xrightarrow{D} \mathcal{N}(0,1)$$

instead of writing $X_n \xrightarrow{D} X$, $X \in \mathcal{N}(0,1)$.

Cauchy Principle for a.e, $\mathbb{P}$ and L^2 Convergence

Now, for checking the convergence of a sequence of r.v. X_n, we can use the following Theorem (Cauchy Principle):

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- If $X_n - X_m \to 0$ a.e. when $m, n \to +\infty$, then there exists a r.v. X such that $X_n \to X$ a.e.;

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- If $\mathbb{E}\Big((X_n - X_m)^2\Big) \to 0$ when $m, n \to +\infty$, then there exists a r.v. X such that $X_n \xrightarrow{L^2} X$.

Example

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- Is X_n convergent in the a.s. sense?