CS 107, Probability, Spring 2019
Lecture 34

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Content

- Multivariate Uniform and Normal Distributions

Multivariate Uniform Distribution

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Bivariate Uniform Distribution

We will say that the r.vector (X, Y) has a Uniform Distribution over the region $D \subset \mathbb{R}^2$, and we will write $(X, Y) \sim Unif(D)$, if the Joint PDF of (X, Y) has the form

$$f(x, y) = \begin{cases} \dfrac{1}{Area(D)}, & (x, y) \in D \\ 0, & \text{otherwise.} \end{cases}$$

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Note: The n-dim case is: $\mathbf{X} = (X_1, ..., X_n) \sim Unif(D)$, for $D \subset \mathbb{R}^n$, if the Joint PDF of $\mathbf{X}$ is

$$f(\mathbf{x}) = \begin{cases} \dfrac{1}{Volume(D)}, & \mathbf{x} \in D \\ 0, & \text{otherwise.} \end{cases}$$

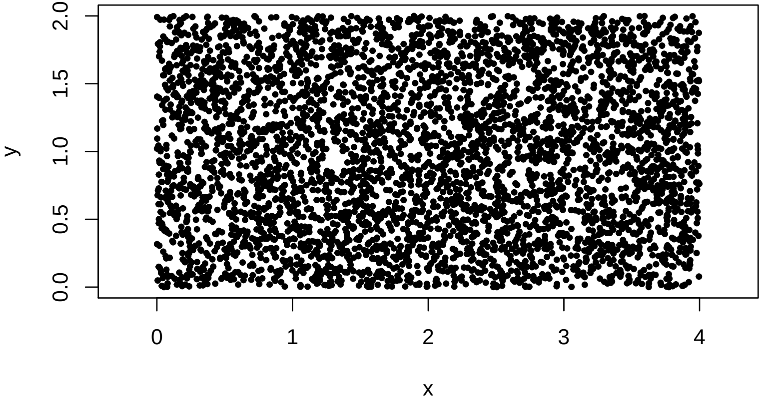

Figure: Random points (x,y) generated from $Unif([0,4] \times [0,2])$

R Code

R Code

```
#Generating and drawing n points from Unif[0,4]x[0,2]
n = 4500
x <- runif(n, min = 0, max = 4)
y <- runif(n, min = 0, max = 2)
plot(x,y,pch=20,xlim=c(0,4),ylim=c(0,2),asp=1,cex=0.8)
```

Example:

Example: Assume $(X, Y) \sim Unif(D)$, where D is the triangle with vertices at $(0,0)$, $(1,0)$ and $(0,1)$.

- Find the Joint PDF of (X, Y);
- Find the Marginal PDF of X and Y;
- Calculate the Probability $\mathbb{P}(Y < 0.5X)$;
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Exercise: (Not an easy one) Can you write a computer code to generate Uniform Random Numbers in the given (regular) Domains, say, in Triangles, Circles, Ellipses,... .

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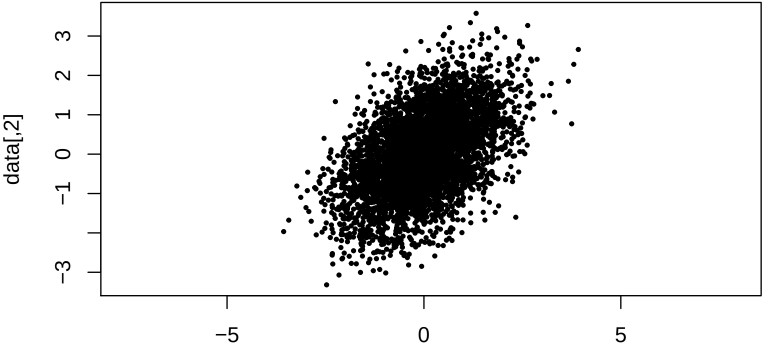

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$$\sigma_{12} = \sigma_{21}, \qquad \text{and} \qquad \sigma_{11} > 0, \qquad \det(\Sigma) > 0$$

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if the Joint PDF of (X, Y) is given by

$$f(\mathbf{x}) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} \cdot \exp\left\{-\frac{1}{2}(\mathbf{x}-\mu)^T \cdot \Sigma^{-1} \cdot (\mathbf{x}-\mu)\right\},$$

for any $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$

Example:

Example: Assume

$$\mu = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}.$$

- Check that Σ is PD (Positive Definite);
- Write the Joint PDF of a r.vector $(X, Y) \sim \mathcal{N}(\mu, \Sigma)$;
- Calculate the Probability $\mathbb{P}(X \in [-1, 3], Y > 2)$.

R Code

R Code for Bivariate Normal

```
mu <- c(0,0) # The Mean
Sigma <- matrix(c(1, .5, .5, 1), nrow = 2) #Cov Matrix
#Version 1
library(MASS)
data <- mvrnorm(5000, mu = mu, Sigma = Sigma )
plot(data, pch = 20, asp = 1, cex = 0.6)
#Version 2
#install.packages("mvtnorm")
library(mvtnorm)
data <- rmvnorm(1000, mean = mu, sigma = Sigma)
plot(data, pch = 20, asp = 1, xlim = c(-3,3))
```

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Also, an extract from a lecture on Machine Learning by Andrew Ng, co-founder of Google Brain and Coursera (Stanford University):

- https://www.youtube.com/watch?v=JjB58InuTqM