# AUA CS 108, Statistics, Fall 2019

Lecture 32

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# Contents

# Last Lecture ReCap

- Give the definition of the $(1-\alpha)$-level CI.

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- Give the definition of the $(1-\alpha)$-level CI.
- Give the Chebyshev Inequality.
- What is the **R** command to generate 20 random numbers from the *Cauchy*(2) distribution?
- Give a $(1-\alpha)$-level CI for *p* in *Bernoulli*(*p*) Model.

# CI for the Proportion, Cheby Method

Recall that if we have a Random Sample

$$X_1, X_2, ..., X_n \sim Bernoulli(p),$$

then the interval

$$\left(\overline{X} - \frac{1}{2\sqrt{n \cdot \alpha}},\ \overline{X} + \frac{1}{2\sqrt{n \cdot \alpha}}\right)$$

is a CI for $p$ of level $1 - \alpha$.

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is a CI for $p$ of level $1 - \alpha$.

**Note:** Here

$$\frac{1}{2\sqrt{n \cdot \alpha}}$$

is called the **Margin of Error** (for the Interval Estimate of $p$).

# Examples

**Example:** Assume we are interested in the proportion of smokers in AUA. We ask 120 persons at AUA and learn that 55 of them are smokers. Construct a CI for the proportion of smokers in AUA of 95% confidence level.

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**Example:** Continuing the above Example: now assume we want to find that Proportion within the Error Margin 0.1, with the CL 95%. At least, how many persons at AUA we need to ask?

**Solution:** OTB

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Again, we want to construct a CI of CL $1-\alpha$ for $\theta$, using the Random Sample

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Then we call $g(X_1, ..., X_n, \theta)$ to be a **Pivot** for our model.

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This will give some inequality of the form $L < \theta < U$, and hence,
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so we can take $(L, U)$ as a CI.

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**Note:** Usually, we are solving
$$\mathbb{P}\Big(a < g(X_1, X_2, ..., X_n, \theta) < b\Big) = 1 - \alpha.$$

# CI for the Mean of $\mathcal{N}(\mu, \sigma^2)$, $\sigma$ is known, Pivotal Method

**Problem:** Assume we have a Random Sample

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But let us consider $\overline{X} - \mu$.

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The name of the above ratio is $Z$-**statistics**, and we will meet this again in Hypotheses testing part.

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$$\mathbb{P}(-b < Z < b) = 1 - \alpha.$$

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Because of the symmetry, we need to have that the area of both tails is $\alpha/2$:

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$$b = z_{1-\frac{\alpha}{2}},$$

where $z_{1-\alpha/2}$ is the $1 - \frac{\alpha}{2}$ quantile of the Standard Normal Distribution.

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So we obtained

$$\mathbb{P}\left(-z_{1-\frac{\alpha}{2}} < Z < z_{1-\frac{\alpha}{2}}\right) = 1 - \alpha.$$

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We plug here the value of $Z$:

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and solve for $\mu$:

$$\mathbb{P}\left(\overline{X} - z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha.$$

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Hence, the following interval:

$$\left(\overline{X} - z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}},\ \ \overline{X} + z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right)$$

is a $(1-\alpha)$-level CI for $\mu$.

# Example

**Example:** Assume we want to construct a 95% CI for $\mu$ in the $\mathcal{N}(\mu, \sigma^2)$ Model, when $\sigma$ is given, known.

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Since $1 - \alpha = 0.95$, then $\alpha = 0.05$.

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Since $1 - \alpha = 0.95$, then $\alpha = 0.05$. By the above formula, we need to calculate the Standard Normal quantile $z_{1-\alpha/2} = z_{0.975}$.

**R** gives:

```r
qnorm(0.975)
```

```
## [1] 1.959964
```

so our 95% CI will be

$$\left(\overline{X} - 1.96 \cdot \frac{\sigma}{\sqrt{n}}, \;\; \overline{X} + 1.96 \cdot \frac{\sigma}{\sqrt{n}}\right).$$

# Example

**Example with R:** We generate random numbers from $\mathcal{N}(2.31, 4)$ (so here we assume we know the true parameter value of $\mu$).

```
sigma <- 2
n <- 20
smpl <- rnorm(n, mean = 2.31, sd = sigma)
smpl
```

```
##  [1]  2.4067314  6.7984510  3.5772345  1.5963469 -1.894
##  [7]  3.0240236  2.9426280  3.7549903  4.4298303  3.125
## [13]  0.7173930  4.9879032  2.8020583  2.9063363  1.931
## [19]  3.4198754  1.0501713
```

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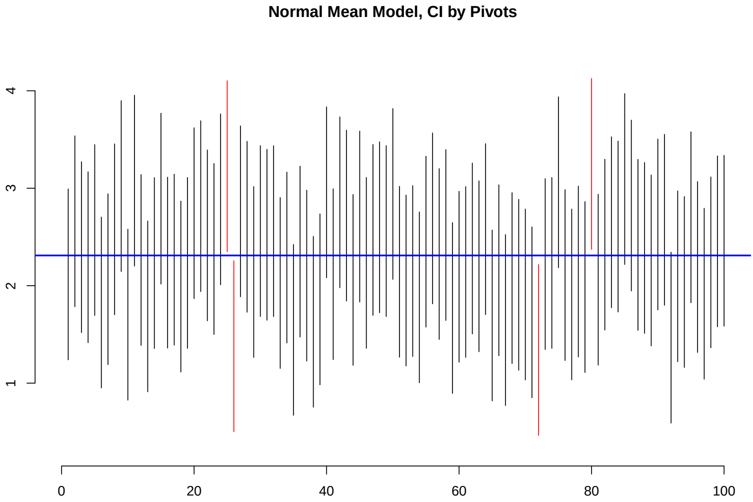
Now we construct the 95% CI by the above formula:

```
me <- 1.96* sigma/sqrt(n) #the Margin of Error
c(mean(smpl) - me, mean(smpl) + me)
```

```
## [1] 1.764134 3.517211
```

# Example, Simulation



# Example, Simulation, Code

```
mu <- 2.31; sigma <- 2
conf.level <- 0.95; a = 1 - conf.level
sample.size <- 20; no.of.intervals <- 100
z <- qnorm(1-a/2) ## our quantile
ME <- z*sigma/sqrt(sample.size) #Margin of Error

plot.new()
plot.window(xlim = c(0,no.of.intervals), ylim = c(mu-2,mu+2))
axis(1); axis(2)
title("Normal Mean Model, CI by Pivots")
for(i in 1:no.of.intervals){
  x <- rnorm(sample.size, mean = mu, sd = sigma)
  lo <- mean(x) - ME; up <- mean(x) + ME
  if(lo > mu || up < mu){
    segments(c(i), c(lo), c(i), c(up), col = "red")
  }
  else{
    segments(c(i), c(lo), c(i), c(up))
  }
}
abline(h = mu, lwd = 2, col = "blue")
```

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We know some good Estimators for $\sigma$: let us take, in this case, the following version of Sample SD:

$$S^2 = \frac{\sum_{k=1}^{n}(X_k - \overline{X})^2}{n-1}, \qquad i.e., \qquad S = \sqrt{\frac{\sum_{k=1}^{n}(X_k - \overline{X})^2}{n-1}}.$$

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# $t$-Distribution

It turns out that the Distribution of above

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is the famous $t$-Distribution with $n - 1$ degrees of freedom:

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**Definition:** If $X_1, X_2, ..., X_n \sim \mathcal{N}(\mu, \sigma^2)$ are IID and

$$S^2 = \frac{\sum_{k=1}^{n}(X_k - \overline{X})^2}{n-1},$$

then the Distribution of

$$t = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

is called the **Student's $t$-Distribution with $n - 1$ degrees of freedom**[1], and is denoted by $t(n-1)$.

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# Student's Paper

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VOLUME VI MARCH, 1908 No. 1

BIOMETRIKA.

## THE PROBABLE ERROR OF A MEAN.

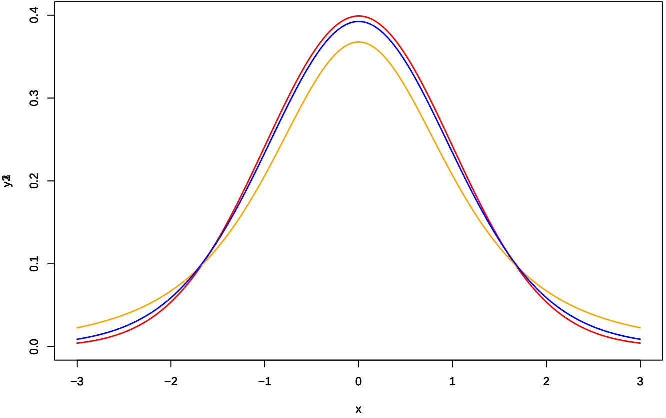
BY STUDENT.

*Introduction.*

ANY experiment may be regarded as forming an individual of a "population" of experiments which might be performed under the same conditions. A series of experiments is a sample drawn from this population.

# t-Distribution

```
x <- seq(-3,3, 0.01)
y1 <- dnorm(x); y2 <- dt(x, df = 3); y3 <- dt(x, df = 15)
plot(x,y1, type = "l", col = "red", lwd = 2, ylim = c(0, 0.4))
par(new = T)
plot(x,y2, type = "l", col = "orange", lwd = 2, ylim = c(0, 0.4))
par(new = T)
plot(x,y3, type = "l", col = "blue", lwd = 2, ylim = c(0, 0.4))
```



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Again, because of the symmetry, we need to have that the area of both tails is $\alpha/2$:

$$\mathbb{P}(t \leq -b) = \mathbb{P}(t \geq b) = \frac{\alpha}{2}.$$

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Hence,

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$$b = t_{1-\frac{\alpha}{2}}(n-1) = t_{n-1,1-\frac{\alpha}{2}},$$

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Hence, the following interval:

$$\left(\overline{X} - t_{n-1,1-\alpha/2}\cdot\frac{S}{\sqrt{n}},\ \ \overline{X} + t_{n-1,1-\alpha/2}\cdot\frac{S}{\sqrt{n}}\right)$$

is a $(1-\alpha)$-level CI for $\mu$.