CS 107, Probability, Spring 2020
Lecture 13

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Content

- Independence of Events

Last Lecture ReCap

Last time we were solving some Problems on TPF and Bayes Formula.

Independence of Events, Definition

The definition of the Independence is simple and intuitive: Assume we have two Events A and B in the same experiment.

Independence of Events

We say that the Events A and B are **Independent**, if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

Remark: It is easy to see that the condition above is equivalent to[1]

$$\mathbb{P}(A|B) = \mathbb{P}(A)$$

and is equivalent to

$$\mathbb{P}(B|A) = \mathbb{P}(B).$$

[1] Except the cases when $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

Independence of Events, Cont'd

We have equivalent definitions of the Independence for the case when $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$:

- $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$;
- $\mathbb{P}(A|B) = \mathbb{P}(A)$;
- $\mathbb{P}(B|A) = \mathbb{P}(B)$.

The last two are intuitive: say,

$$\mathbb{P}(A|B) = \mathbb{P}(A)$$

says that **the knowledge that B happened is not changing the probability that A will happen, it is not giving more chances for A to happen, or less chances for A to happen. So B is not affecting on the appearance chances of A.**

Remark: We say two Events A and B are **Dependent**, if they are not Independent.

Geometric/Counting Interpretation

Assume $A, B \subset \Omega$ are Events, and assume A and B are Independent. This means that

$$\mathbb{P}(A|B) = \mathbb{P}(A),$$

and, also,

$$\mathbb{P}(A|\overline{B}) = \mathbb{P}(A) \qquad \text{(i.e., } A \text{ and } \overline{B} \text{ are Independent too.)}$$

We can interpret this, either geometrically, or in a counting interpretation, that **the proportion of A in B is the same as the proportion of A in $\overline{B}$, and is the same as the proportion of A in Ω.**

Independence Use

Usually, we deal with the Independence notion in two ways:

- We calculate Events probabilities to **prove** that the Events are Independent;
- We suppose (from the intuition of the problem) that Events are Independent, and **use** Independence to calculate Probabilities.

Example

Problem 1: We are tossing two fair coins. Are the events "H on the first coin" and "H on the second coin" independent?
Solution:

- Approach 1: By using the definition, OTB
- Approach 2: By using Conditional Probabilities, OTB

Now, another problem:
Problem 2: We are tossing two fair coins. What is the Probability of having "H on the first coin" and "H on the second coin"?
Solution: Intuition suggests that two tosses (Events) are Independent. The rest is OTB

Example:

Problem: Assume we are choosing a real number from $[0, 4]$. Let A be the Event that our number is less than 3, and the Event $B = [1.5, 3.5]$. Are these Events Independent?
Solution: OTB
Solution: Again, do it using Conditional Probabilities

Example:

Problem: Assume we have a box with 60 red and 20 blue balls. We pick randomly a ball, then another one. Let A be the Event that the first ball was red, and B be the Event that the second one was red.

- Are A and B Independent, if we do not return the first ball into the box (without replacements model)?
- Are A and B Independent, if we return the first ball into the box (with replacements model)?

Exercise: What about the Events

$$A = \{\text{the first ball was red}\}, \quad B = \{\text{the second ball was blue}\}?$$

Are A and B Independent, in either of cases?

Important Remark

Assume now that the Events A and B are disjoint. Is it true that they are Independent ?

If the Events A and B are disjoint, then they are Dependent, unless $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

Interpretation: Knowing that, say, B happened, we will know for sure that A cannot happen, so knowing B changes the probability of having A!

Example:

Problem: Assume we are tossing a fair coin until H appears. What is the Probability that we will do exactly 10 tosses?
Solution: OTB

Problem: We randomly choose a family with 3 children. What is the Probability that the first child is a Girl, then Boy and again a Boy?
Solution: OTB

Problem: We roll a fair die 3 times. What is the Probability that the first roll will result in 5, the second roll in odd, and the third one in a prime number?
Solution: OTB