CS 107, Probability, Spring 2019
Lecture 29

Michael Poghosyan

AUA

32 March 2019

Content

- Functions of Random Variables (aka Transformations of Random Variables)

LZ

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See, for example, `https://czep.net/weblog/52cards.html`

Question: What is the probability that two random shuffles coincide?

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See `https://www.youtube.com/watch?v=AxJubaijQbI` and the NY Times article at this link.

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See also: `https://en.wikipedia.org/wiki/Shuffling#Suffici`

Do not tell anybody that I was teaching you card shuffling during the classes $\ddot\smile$

Describing Transformations of X through CDFs

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Example: Assume $X \sim Exp(\lambda)$, and $Y = X^2$. Find the CDF of Y.
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Example: Find the PMF of $Y = X^2 + 2$, if X is a Discrete r.v. given by

Values of X	-2	0	1	2
$\mathbb{P}(X = x)$	0.1	0.3	0.2	0.4

Solution: OTB

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$$f_Y(x) = F'_Y(x).$$

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Exercise: Do the same for strictly decreasing and differentiable g.

Example:

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$$f_X(x) = C \cdot x^5, \quad x \in [0, 1], \qquad f(x) = 0, \text{ otherwise}$$

and assume $Y = X^4 + 2$. Find the PDF of Y.
Solution: OTB