CS 107, Probability, Spring 2019
Lecture 11

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AUA

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Content

- Total Probability Formula
- Bayes Formula

Medical Test gives a correct answer in 95% of cases: if the person is ill, it is saying that he/she is ill with 95% probability, and if the person is healthy, it is saying that he/she is healthy with 95% probability (i.e., in 95% of cases).

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Now assume I am taking that test, and it says that I am ill. What is the probability that I am really ill?

The Multiplication Rule

From the Conditional Probability definition, we obtain the following Multiplication (or Chain) Rule:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B) \cdot \mathbb{P}(B).$$

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Problem: We have 36 participants in our Prob class, from which 26 are male. Assume we know that only 15% of females love Jazz. We choose a participant from our class at random. What is the Probability that the chosen person will be a Jazz lover female student?

We know that

$$\mathbb{P}(\text{chosen person is a female}) = \frac{10}{36} = \frac{5}{18}$$

Also,

$$\mathbb{P}(\text{ person loves Jazz}|\text{ person is a female}) =$$

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$$\mathbb{P}(\text{chosen person is a female and she loves Jazz}) = 0.15 \cdot \frac{5}{18}.$$

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Now, let us give this in the tree form: look at the Board!

Jazz Problem, Cont'd

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Assume, that in the problem above we know additionally that only 20% of males love Jazz.

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Assume, that in the problem above we know additionally that only 20% of males love Jazz.

Then we can calculate the probability that the chosen person will be a Jazz lover male student.

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$$\mathbb{P}(\text{chosen person is a male and he loves Jazz}) =$$

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Important is that now we can calculate the probability that the chosen person will like Jazz:

Jazz Problem, Cont'd

$$\mathbb{P}(\text{chosen person likes Jazz}) =$$
$$= \mathbb{P}(\text{chosen person is a male and he loves Jazz}) +$$
$$+\mathbb{P}(\text{chosen person is a female and she loves Jazz}) =$$

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Now, let us generalize the formula:

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that is, **we take a partition of our Sample Space into disjoint Hypotheses (Events), calculate the probability of our event A under each Hypothesis, and then add the results**.

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Well, Probabilists love urns and balls, and dice and coins. The problem may seem very artificial. So let me give a problem of this type in a more realistic terms:

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Solution: OTB

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Hence,

Bayes Formula

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B) \cdot \mathbb{P}(B)}{\mathbb{P}(A)}$$

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- $\mathbb{P}(A|B)$ is called the **Likelihood** of A under B.

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$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(A|B_k)\cdot\mathbb{P}(B_k)}{\mathbb{P}(A)} =$$

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$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(A|B_k)\cdot\mathbb{P}(B_k)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|B_k)\cdot\mathbb{P}(B_k)}{\sum_{i=1}^{n}\mathbb{P}(A|B_i)\cdot\mathbb{P}(B_i)}.$$

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We know the solution. Now, the Inverse Probability question: Assume that the randomly chosen person is a smoker. What is the probability that he/she is of younger-age-group?

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Some Questions to Answer

- Give the TPF.

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- Give the idea of the Bayes Formula.