AUA CS108, Statistics, Fall 2020

Lecture 23

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19 Oct 2020

Contents

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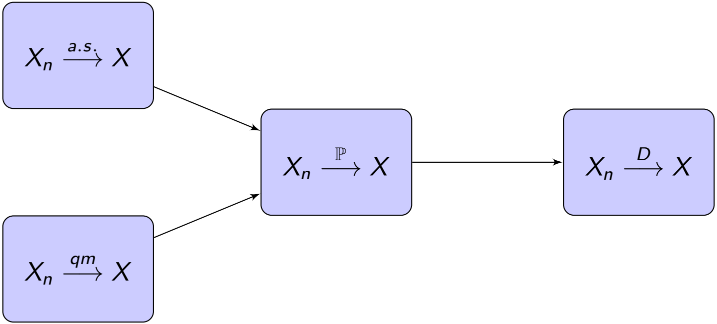
Relationship between Convergence Types

Theorem: (Convergence Relationship Diagram)



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Note: Inverse implications are not always correct. But, say, the following holds: If $X_n \xrightarrow{D} X$ and $X \equiv \textit{constant}$, then $X_n \xrightarrow{P} X$ (X_n and X are defined on the same Probability space).

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and to calculate the limit of this sequence $\overline{X}_1, \overline{X}_2, ..., \overline{X}_n, ...$, we will use our famous Limit Theorems: LLN and CLT.

Limit Theorems

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- X_n-s are independent. Say, in particular,

$$Var(X_1+X_2+...+X_n) = Var(X_1)+Var(X_2)+...+Var(X_n) = n\cdot Var(X_1).$$

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The interpretation of $\mathbb{E}(\overline{X}_n) = \mathbb{E}(X_1)$ and $Var(\overline{X}_n) = \frac{Var(X_1)}{n}$:the values of $\overline{X}_n$ are centered at $\mathbb{E}(X_1)$ and are becoming more and more concentrated around that number as n increases.

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The Weak Law of Large Numbers, WLLN:

If $X_1, X_2, ..., X_n$ are IID, with finite $\mathbb{E}(X_1)$ and Variance $Var(X_1)$, then

$$\overline{X}_n = \frac{X_1 + X_2 + ... + X_n}{n} \xrightarrow{\mathbb{P}} \mathbb{E}(X_1), \qquad n \to +\infty,$$

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i.e., for any $\varepsilon > 0$,

$$\mathbb{P}\left(\left|\frac{X_1 + X_2 + ... + X_n}{n} - \mathbb{E}(X_1)\right| \geq \varepsilon\right) \to 0, \qquad n \to +\infty.$$

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Note: This means that for any $\varepsilon > 0$, the chances that $\overline{X}_n$ is far from $\mathbb{E}(X_1)$ more than ε, is very small, if n is large.

The Strong LLN

The Strong Law of Large Numbers, SLLN, Kolmogorov

If $X_1, X_2, ..., X_n$ are IID, with finite $\mathbb{E}(|X_1|)$, then

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that is,

$$\mathbb{P}\left(\lim_{n\to+\infty} \frac{X_1 + X_2 + ... + X_n}{n} = \mathbb{E}(X_1)\right) = 1.$$

Visualization of the LLN

```r
set.seed(111); n <- 1000; expect <- 0.6
X <- rbinom(n, 1, expect)
S <- cumsum(X); p <- S/(1:n)
plot(p, type = "l")
abline(expect,0, col = "red", lwd = 2)
```