# AUA CS 108, Statistics, Fall 2019

## Lecture 15

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# Contents

# Last Lecture ReCap

- Well, since this will be a Recap of a Recap, we can skip it $\ddot\smile$

# Additions to Important Distributions

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- Another page for the Relationship: L. Leemis Page

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I know, almost all examples are examples of *finite* sequences, but for theoretical (and practical) reasons we can assume they are infinite.

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Also, we will use a lot the CLT, say, to construct Confidence Intervals and design Tests for Hypotheses, so we need to know the CLT, and CLT is about the limit of a r.v. sequence.

# Convergence of a Sequence of r.v.

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Aha, that's the problem - it is not so easy to define the closedness $\ddot\smile$

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**Definition:** We will say that $X_n \to X$ **almost sure**, and we will write $X_n \to X$ a.s. or $X_n \xrightarrow{a.s.} X$, if

$$\mathbb{P}\Big(\omega \in \Omega : \lim_{n\to+\infty} X_n(\omega) = X(\omega)\Big) = 1,$$

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Equivalently, we can write

$$X_n \xrightarrow{a.s.} X \qquad \text{iff} \qquad \mathbb{P}\Big(X_n \not\to X\Big) = 0.$$

# Convergence in Probability

**Definition:** We will say that $X_n \to X$ **in Probability**, and we will write $X_n \xrightarrow{\mathbb{P}} X$, if

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Equivalently, we can write

$$X_n \xrightarrow{\mathbb{P}} X \qquad \text{iff} \qquad \mathbb{P}\Big(|X_n - X| < \varepsilon\Big) \to 1 \text{ for any } \varepsilon > 0.$$

# Convergence in the Mean Square Sence

**Definition:** We will say that $X_n \to X$ **in the Quadratic Mean Sense or in $L^2$ (or in the Mean Square Sense)**, and we will write $X_n \xrightarrow{L^2} X$ or $X_n \xrightarrow{qm} X$, if

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Here $MSE(X_n, X)$ is the *Mean Square Error* (of the approximation of $X$ by $X_n$).

# Convergence in Distributions

Now we assume that $X_n$ and $X$ are arbitrary r.v.'s, not necessarily defined on the same probability space, and $F_{X_n}(x)$ and $F_X(x)$ are their CDF's, respectively.

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**Remark:** This is equivalent to saying that for (almost) any subsets $A \subset \mathbb{R}$

$$\mathbb{P}(X_n \in A) \to \mathbb{P}(X \in A).$$

# Example

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- Is $X_n$ convergent in the sense of Distributions ?
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**Solution:** OTB. Not a good/correct example, impossible to answer to the questions except to the first one.

# Example

**Example:** Assume $X_n$ is a Discrete r.v. with the following PMF, defined on the same Probability Space:

| $X_n$ | $3 + \frac{1}{n^2}$ | $n$ |
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| $\mathbb{P}(X_n = x)$ | $1 - \dfrac{1}{n}$ | $\dfrac{1}{n}$. |

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Which of the followings are true (use only the definitions):

- $X_n \xrightarrow{\mathbb{P}} 3$;
- $X_n \xrightarrow{qm} 3$;
- $X_n \xrightarrow{D} 3$ ?

**Solution:** OTB