AUA CS 108, Statistics, Fall 2019

Lecture 33

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CI for μ, Normal Model, Notes

Note: To compare:

- If σ is known, $(1-\alpha)$-level CI for μ is

$$\overline{X} \pm z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}$$

- If σ is unknown, $(1-\alpha)$-level CI for μ is

$$\overline{X} \pm t_{n-1,1-\alpha/2} \cdot \frac{S}{\sqrt{n}}$$

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Note: If we will compare the quantiles of the same level of $\mathcal{N}(0,1)$ with $t(n-1)$, we will see that CIs for the case when σ is unknown are wider than for the case when σ is known. This is intuitive, of course - to compensate the uncertainty in σ, we need to take a wider interval:

```
c(qnorm(0.975), qt(0.975, df = 3), qt(0.975, df = 20))
```

```
## [1] 1.959964 3.182446 2.085963
```

Example

Example: Assume we want to estimate the average time μ our Stat students are spending for a homework. We collect information from 10 students for the last hw, and get the following responses (in hrs):

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[1] 2.17 1.42 2.13 0.56 1.21 2.22 1.35 2.37 1.47 1.70
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So $X \sim \mathcal{N}(\mu, \sigma^2)$ shows the time spent solving a hw for a (randomly chosen) student. Our unknown, μ is the average time to solve a hw. And we have an observation from a Random Sample

$$X_1, X_2, ..., X_{10} \sim \mathcal{N}(\mu, \sigma^2).$$

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$$X_1, X_2, ..., X_{10} \sim \mathcal{N}(\mu, \sigma^2).$$

We construct a 95% CI for μ, the average time to solve the hw, by the above formula:

Example

```r
n <- 10
cl <- 0.95; a = 1-cl
t <- qt(1-a/2, df = n-1)
smpl<-c(2.17,1.42,2.13,0.56,1.21,2.22,1.35,2.37,1.47,1.70)
s <- sd(smpl)
me <-  t*s/sqrt(n) #the Margin of Error
c(mean(smpl) - me, mean(smpl) + me)
```

```
## [1] 1.253748 2.066252
```

Example

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n <- 10
cl <- 0.95; a = 1-cl
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smpl<-c(2.17,1.42,2.13,0.56,1.21,2.22,1.35,2.37,1.47,1.70)
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me <-  t*s/sqrt(n) #the Margin of Error
c(mean(smpl) - me, mean(smpl) + me)
```

```
## [1] 1.253748 2.066252
```

Veeery unsatisfactory result, of course! Please spend **more time for your hw, read textbooks**!

Example, cont'd

Later, we will talk about the *t*-**Test**, let me now just do a *t*-Test for the hw solving hours:

```
smpl <- c(2.17,1.42,2.13,0.56,1.21,2.22,1.35,2.37,1.47,1.70)
t.test(smpl)
```

```
##
##  One Sample t-test
##
## data:  smpl
## t = 9.2435, df = 9, p-value = 6.86e-06
## alternative hypothesis: true mean is not equal to 0
## 95 percent confidence interval:
##  1.253748 2.066252
## sample estimates:
## mean of x
##      1.66
```

Example, cont'd

We can separate here the CI:

```r
smpl <- c(2.17,1.42,2.13,0.56,1.21,2.22,1.35,2.37,1.47,1.70)
tst <- t.test(smpl) #Keeping the test result in tst
tst$conf.int
```

```
## [1] 1.253748 2.066252
## attr(,"conf.level")
## [1] 0.95
```

CI for μ, Normal Model, Summary

Let us summarize what we have obtained for this model.

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$$X_1, X_2, ..., X_n \overset{IID}{\sim} \mathcal{N}(\mu, \sigma^2),$$

and $\alpha \in (0, 1)$, we want to construct an $1 - \alpha$-level CI for the unknown parameter μ.

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We have considered 2 cases: when σ^2 was known and unknown. Here we give the summary:

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	σ^2 is known	σ^2 is unknown
Pivot:		

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Pivot:	$Z = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$	$t = \frac{\overline{X}-\mu}{S/\sqrt{n}}$
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Here S is the Sample Standard Deviation given by

$$S^2 = \frac{\sum_{k=1}^{n}(X_k - \overline{X})^2}{n-1}.$$

CI for σ^2, $\mathcal{N}(\mu, \sigma^2)$ Model, μ is **known**

Problem: Assume we have a Random Sample

$$X_1, X_2, ..., X_n \sim \mathcal{N}(\mu, \sigma^2),$$

where μ is **known**.

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$$\frac{X_k - \mu}{\sigma} \sim$$

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$$\left(\frac{X_k - \mu}{\sigma}\right)^2.$$

Is this a Pivot?

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Is this a Pivot? Yes, of course. What is the Distirbution of this r.v.? No, of course $\ddot\smile$. Well, we can use our Prob knowledge, but let's keep this to you. The fact is that this Pivot will not give a good result, as we are not using all the information we have.

CI for σ^2, $\mathcal{N}(\mu, \sigma^2)$ Model, μ is **known**

So we will use the following as a Pivot:

$$Y = \sum_{k=1}^{n} \left(\frac{X_k - \mu}{\sigma} \right)^2 .$$

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$$Y = \sum_{k=1}^{n} \left(\frac{X_k - \mu}{\sigma} \right)^2 \sim \chi^2(n)$$

$\chi^2(n)$ Distribution

Definition: Assume

$$Z_1, Z_2, ..., Z_n \sim \mathcal{N}(0,1)$$

and Z_k-s are Independent (so they are IID). The Distribution of

$$Y = Z_1^2 + Z_2^2 + ... + Z_n^2$$

is called the **Chi-Squared Distribution with n Degrees of Freedom**, and is denoted by $\chi^2(n)$:

$$Y = Z_1^2 + Z_2^2 + ... + Z_n^2 \sim \chi^2(n).$$

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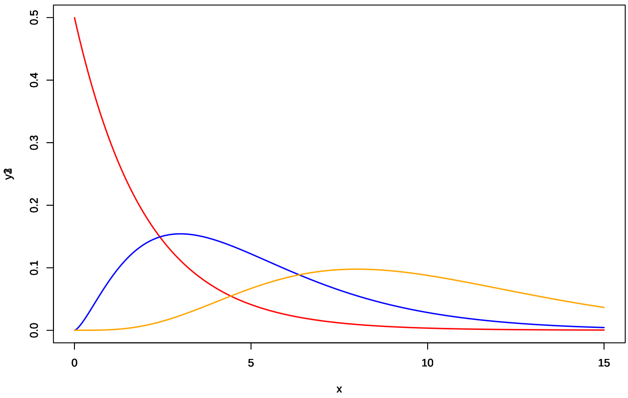
Exercise: What is $\mathbb{E}(Y)$?

$\chi^2(n)$ Distribution, PDF graphs

```
x <- seq(from = 0, to = 15, by = 0.01)
y1<-dchisq(x, df=2); y2<-dchisq(x, df=5); y3<-dchisq(x, df=10)
plot(x,y1,type="l",lwd=2,col="red", ylim=c(0,0.5)); par(new=T)
plot(x,y2,type="l",lwd=2,col="blue", ylim=c(0,0.5)); par(new=T)
plot(x,y3,type="l",lwd=2,col="orange", ylim = c(0,0.5))
```



CI for σ^2, $\mathcal{N}(\mu, \sigma^2)$ Model, μ is **known**

Back to our CI for σ^2 Problem: We wanted to use the following as a Pivot:

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Now, as the general idea of Pivot CI construction was suggesting, we want to find a, b such that

$$\mathbb{P}(a < Y < b) = 1 - \alpha.$$

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$$\mathbb{P}(a < Y < b) = 1 - \alpha.$$

The Distribution of Y, $\chi^2(n)$, is not symmetric, so it is not so easy to find the *minimum length CI*.

CI for σ^2, $\mathcal{N}(\mu, \sigma^2)$ Model, μ is **known**

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This means that we found a $(1 - \alpha)$-level CI for σ^2:

$$\left(\frac{\sum_{k=1}^{n}(X_k - \mu)^2}{\chi^2_{n,1-\frac{\alpha}{2}}}, \quad \frac{\sum_{k=1}^{n}(X_k - \mu)^2}{\chi^2_{n,\frac{\alpha}{2}}}\right).$$

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$$X_1, X_2, ..., X_n \sim \mathcal{N}(\mu, \sigma^2),$$

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Well, we cannot use this now, since μ is unknown to us. So we will adjust Y a little bit. Can you give some suggestion? Of course, thanks:

$$\chi^2 = \sum_{k=1}^{n} \left(\frac{X_k - \overline{X}}{\sigma} \right)^2 .$$

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Great thing is that this is a Pivot: the Distribution of

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Now, doing the same calculations as above, we will arrive at the following $(1-\alpha)$-level CI for σ^2 :

$$\left(\frac{\sum_{k=1}^{n}(X_k - \overline{X})^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \quad \frac{\sum_{k=1}^{n}(X_k - \overline{X})^2}{\chi^2_{n-1,\frac{\alpha}{2}}} \right).$$

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Usually, you will see this in the following form:

$$\left(\frac{(n-1)\cdot S^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \quad \frac{(n-1)\cdot S^2}{\chi^2_{n-1,\frac{\alpha}{2}}} \right),$$

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where S is the Sample Standard Deviation given by

$$S^2 = \frac{\sum_{k=1}^{n}(X_k - \overline{X})^2}{n-1}.$$

Example

Example: Assume we have scale of very high precision, and we want to measure the precision of our scale. Say, we weight a 50AMD coin, using that scale, 10 times, and we obtain the following measurements (in gramms):

```
 [1] 3.449243 3.450802 3.453054 3.448778 3.452541 3.448835 3.448
 [9] 3.450314 3.449047
```

Our aim is to Estimate the Precision of the Scale.