AUA CS 108, Statistics, Fall 2019

Lecture 26

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Contents

Last Lecture ReCap

- Give the definition of the Fisher Information.

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- Give the Cramer-Rao Lower Bound

Cramer-Rao Inequality, C-R Lower Bound, Refresher

Recall that under some regularity conditions on the family of Distributions $\mathcal{F}_\theta$, the following holds:

Theorem (Cramer-Rao): Assume we have a Random Sample

$$X_1, X_2, ..., X_n \overset{IID}{\sim} \mathcal{F}_\theta$$

and the Fisher Information for the family $\mathcal{F}_\theta$ is $I(\theta)$. Assume also that $\hat{\theta}$ is an unbiased estimator for θ obtained from our Random Sample. Then, under the above mentioned regularity conditions,

$$Var(\hat{\theta}) \geq \frac{1}{n \cdot I(\theta)}.$$

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If there exists an **Unbiased Estimator** $\hat{\theta}$ with

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then that Estimator is a MVUE for θ. And recall that Unbiased Estimators satisfying this equality are called **Efficient Estimators**.

Example

Example: Show that in the Bernoulli Model, with a Random Sample

$$X_1, X_2, ..., X_n \sim Bernoulli(p), \qquad p \in [0, 1],$$

the Estimator

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Example: Show that in the Poisson Model, with a Random Sample

$$X_1, X_2, ..., X_n \sim Pois(\lambda), \qquad \lambda > 0,$$

the Estimator

$$\hat{\lambda} = \overline{X}$$

is the MVUE of λ.

Methods to find (good) Estimators

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Problem: The Problem is to find/construct a good Estimator for θ, using our Random Sample.

The Method of Moments

Theoretical Moments

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Note: Note that, in general, the Theoretical Moments of $\mathcal{F}_\theta$ are functions of θ.

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Note: The Empirical Moment is independent of the Parameter θ, it is just a Statistics, it is a function of $X_1, X_2, ..., X_n$.

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from the following Model:

X	0	1	2
$\mathbb{P}(X = x)$	$\frac{\theta}{10}$	$\frac{\theta}{5}$	$1 - \frac{3\theta}{10}$,

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Example: Find the MoM Estimator for λ in the *Exp*(λ) Model.

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Example: Find the MoM Estimator for θ in the *Unif* $[-\theta, \theta]$ Model.

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