AUA CS 108, Statistics, Fall 2019

Lecture 27

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Last Lecture ReCap

- Give a good Estimator for p in $Bernoulli(p)$ Model and justify your choice.

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- Give a good Estimator for λ in $Pois(\lambda)$ Model and justify your choice.

Examples, MoM

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Solution: OTB

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$$\begin{cases} \textbf{1-st order Theoretical Moment} = \textbf{1-st order Empirical Moment} \\ \textbf{2-nd order Theoretical Moment} = \textbf{2-nd order Empirical Moment} \end{cases}$$

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Example

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Example: Find the MoM Estimator for (a, b) in the $Unif[a, b]$ Model.

Solution: OTB

Example

Example: Let us do an experiment in **R**, concerning the last example:

```
a <- 2.5; b <- 3.24
x <- runif(10, min = a, max = b)
x.bar <- mean(x)
z <- sqrt(3* (mean(x^2) - x.bar^2))
a.hat.MoM<- x.bar - z
b.hat.MoM <- x.bar + z
c(a.hat.MoM, b.hat.MoM)
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## [1] 2.529413 3.219274
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```

Of course, we can just take $\hat{a} = X_{(1)}$ and $\hat{b} = X_{(n)}$:

```
c(min(x), max(x))
```

```
## [1] 2.520091 3.170063
```

Notes

Note: If you do not want to obtain the *MoM Estimator*, but just the *MoM Estimate*, just plug the Observations into the Empirical Moments, and then solve the Moments Equality Equations.

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Note: Sometimes we are interested not in the Estimate of the Parameter itself, but in the Estimate for some function of that Parameter. Say, we want to Estimate $h(\theta)$, where θ is our unknown Parameter. Then one of the approaches is the Plug-in Principle: find an Estimator $\hat{\theta}$ for θ, say, using the MoM, and then plug that in h, to obtain $h(\hat{\theta})$ as an Estimator for $h(\theta)$.

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the solution of which we denote by $\hat{\theta}_n$. This gives

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Now, by the WLLN, $\overline{X}_n \xrightarrow{\mathbb{P}} \mathbb{E}(X_1) = e(\theta)$, so

$$\hat{\theta}_n = e^{-1}(\overline{X}_n) \xrightarrow{\mathbb{P}} e^{-1}(e(\theta)) = \theta.$$

The Maximum Likelihood Method

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Ok, let's do some calculations.

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So our guess was to **select the value of p giving the highest likelihood to our outcome**.