CS 107, Probability, Spring 2019
Lecture 09

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AUA

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Content

- Classical Probability Models: Geometric Probabilities
- Conditional Probabilities

LZ

Air Passenger Problem

A fully-booked plane is about to be boarded by its hundred passengers. The first passenger loses his boarding card and sits down in a random place. From then on, a passenger, whenever finding his seat taken, very politely sits somewhere at random. What is the probability that the hundredth passenger sits in his own seat?

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$$\mathbb{P}(A) = \frac{measure(A)}{measure(\Omega)}.$$

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Problem: Assume we are throwing a darts missile into the quadratic darts board $\Omega = [-1, 1] \times [-1, 1]$. What is the probability that we will hit a point inside the unit disk?

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Some interpretations for

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- What is the probability that the card will be a Queen, if the card drawn is of Diamonds?

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d. If $A_1, ..., A_n$ are some **mutually disjoint** events, then
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e. If $A_1, ..., A_n, ...$ are some **mutually disjoint** events, then
$$\mathbb{P}(\cup_{n=1}^{\infty} A_n|B) = \sum_{n=1}^{\infty} \mathbb{P}(A_n|B);$$

Properties of Conditional Probabilities, Cont'd

f. If A_1, A_2, B are some events and $\mathbb{P}(B) \neq 0$, then

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h. (multiplication or chain rule) If $A_1, ..., A_n$ are some events, then

$$\mathbb{P}(A_1 \cap A_2 \cap ... \cap A_n) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2|A_1) \cdot \mathbb{P}(A_3|A_1 \cap A_2) \cdot ...$$

$$\cdot \mathbb{P}(A_n|A_1 \cap A_2 \cap ... \cap A_{n-1}).$$

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- Please note also that

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  but, in general,

$$\mathbb{P}(A|\overline{B}) \neq 1 - \mathbb{P}(A|B).$$

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King's Sister Problem: In the middle ages there was a story about a King. The parents of the King had 2 children. What is the probability that the other child is the sister of the King?

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