CS 107 Section A - Probability

Spring 2020, AUA

Homework No. 12

Due time/date: 07 May, 2020

Note: Supplementary Problems will not be graded, but you are very advised to solve them and to discuss later with TA or Instructor.

Partial Numerical Characteristics of r.v.s

A Expectation and the Variance of a r.v.

Problem 1. Assume

$$X \sim \begin{pmatrix} -1 & 2 & 5 \\ 0.2 & 0.5 & 0.3 \end{pmatrix}$$

Calculate $\mathbb{E}(X)$, $\mathbb{E}(X^2)$, $Var(X)$ and $SD(X)$.

Problem 2. Assume X is a r.v. with the PDF

$$f(x) = \begin{cases} K \cdot x^2, & x \in [0,1]; \\ 0, & \text{otherwise.} \end{cases}$$

where K is a constant.

a. Calculate the value of K;
b. Calculate the expected value $\mathbb{E}(X)$;
c. Calculate the variance $Var(X)$;
d. Calculate $\mathbb{E}(X^2)$;
e. Calculate $\mathbb{E}(\sin(X))$;
f. (Supplementary) Calculate $Var(\sin(X))$.

Problem 3. I am playing the following game with my opponent: I am rolling 2 dice. If in the result I will have same number shown (say, I will have 5 shown on each die), then my opponent will pay me \$100. If the two numbers shown will be different, but they both will be primes, then my opponent will pay me \$50. In all other cases I need to pay \$40 to my opponent. What is my expected winning?

Problem 4. Assume our metro trains arrive with the interval exactly 10 min one after another. At the end of a nice working day at AUA, I am entering the Marshal Baghramyan metro station to take a train to home. What is my expected waiting time for the train? And what is the standard deviation for my waiting time? Explain your reasoning.

Problem 5. Let X and Y be two independent r.v.'s on the same probability space, and assume

$$\mathbb{E}(X) = -1, \quad Var(X) = 5 \qquad \text{and} \qquad \mathbb{E}(Y) = 3, \quad Var(Y) = 1.$$

Calculate

a. $\mathbb{E}\Big(3X - 2Y + 5\mathbb{E}(X) - 1\Big)$;

b. $\mathbb{E}(X^2)$;

c. $\mathbb{E}\Big((X - Y)^2\Big)$;

d. $Var\Big(3X + \mathbb{E}(Y)\Big)$;

e. $Var\Big(2X - 3Y\Big)$;

f. $SD(X + Y)$;

g. (Supplementary) Give 2 examples of a random variable X with the properties $\mathbb{E}(X) = -1$ and $Var(X) = 5$.

Problem 6. Assume $X \sim Pois(2)$, $Y \sim Exp(0.4)$ and $X \perp\!\!\!\perp Y$. Calculate

a. $\mathbb{E}(X^2)$ and $\mathbb{E}(Y^2)$;

b. $\mathbb{E}(3X - 2Y - 1)$;

c. $Var(3X - 2Y - 1)$.

B Covariance and Correlation Coefficient

Problem 7. Assume X and Y are Jointly Discrete with the following Joint PMF:

$Y \setminus X$	0	2	4
-1	0	0.1	0.1
0	0	0.2	0
1	0.2	0	0.4

Calculate

a. $\mathbb{E}(X)$ and $\mathbb{E}(Y)$;

b. $Cov(X, Y)$;

c. $Cor(X, Y)$.

Problem 8. Assume $(X, Y) \sim Unif(D)$, where D is the triangle with vertices at $(-1, 0)$, $(0, 1)$ and $(1, 0)$. Calculate

a. $\mathbb{E}(X)$;

b. $\mathbb{E}(X^2 + Y^2)$;

c. $Var(X)$;

d. $Cov(X, Y)$;

e. $Cor(X, Y)$.

Problem 9. Assume X and Y are r.v.s with

$$\mathbb{E}(X) = -1, \quad Var(X) = 5 \qquad \text{and} \qquad \mathbb{E}(Y) = 3, \quad Var(Y) = 1.$$

Calculate $Cov(2X - 4Y, 0.5Y + 2X - 1)$, if

a. $X \perp\!\!\!\perp Y$;

b. $Cov(X, Y) = -2$.

Limit Theorems: LLN and CLT

C Limit Theorems: LLN and CLT

Problem 10. Assume $X_k \sim Unif[0, 1]$, $k \in \mathbb{N}$, are IID.

a. Calculate[1] $\lim\limits_{n\to+\infty} \dfrac{X_1 + X_2 + ... + X_n}{n}$;

b. Assume $g \in C(\mathbb{R})$. Calculate the limit (in terms of g)

$$\lim_{n\to+\infty} \frac{g(X_1) + g(X_2) + ... + g(X_n)}{n}$$

c. Calculate the limit

$$\lim_{n\to+\infty} \frac{X_1^2 + X_2^2 + ... + X_n^2}{n}$$

d. (Supplementary) Calculate the limit

$$\lim_{n\to+\infty} \sqrt[n]{(1 + X_1) \cdot (1 + X_2) \cdot \ldots \cdot (1 + X_n)}$$

Problem 11. Assume X_k are IID r.v. with mean μ and variance σ^2. Calculate the limit

$$\lim_{n\to+\infty} \frac{(X_1 - \mu)^2 + (X_2 - \mu)^2 + ... + (X_n - \mu)^2}{n}.$$

Problem 12. Assume we are tossing a fair coin 10000 times. What is the (approximate) probability that the number of heads shown will be in between 4950 and 5150?

Problem 13. Assume that 21 year old person's height is a r.v. with the mean 160 cm and standard deviation 5cm. What is the probability that the average height of 100 persons of age 21 will be less than 150 cm?

[1] Here everywhere we use limits in the almost surely convergence sense, i.e., by writing $\lim Y_n = Y$ we mean that $Y_n \to Y$ almost surely.

D Supplementary Problems

Problem 14. (Supplementary) Construct 2 Jointly Continuous r.v.s X, Y with

$$Cov(X,Y) = 12.$$

Problem 15. (Supplementary) Assume X is a r.v. with a finite variance. For $\alpha \in \mathbb{R}$, we define

$$f(\alpha) = \mathbb{E}\Big((X-\alpha)^2\Big).$$

Find the minimum of $f(\alpha)$ for $\alpha \in \mathbb{R}$.

Note: If you will use the derivative to find all critical points, do not forget to justify that the value obtained is actually the minimum of f.

Problem 16. (Supplementary) We roll a fair die several times. What is the expected number of rolls until we will get ⚁ shown on the die?

Problem 17. (Supplementary, St. Petersburg paradox) Before giving the actual problem, let me explain what is a fair price for a game with uncertain outcomes. Say, I am playing a game, and I can have 3 outcomes: A, B or C. I know the probabilities of having A, B or C, p_1, p_2 and p_3, respectively ($p_1 + p_2 + p_3 = 1$), and I am winning \$$a$ in the A case, \$$b$ in the B case and \$$c$ in the case C. So if I will denote by X my possible winning, then X will be a r.v. with PMF

X	a	b	c
$\mathbb{P}(X = x)$	p_1	p_2	p_3

Now, what is the fair price to enter this game, i.e. how much I will pay to enter this game? The answer is that the fair price is $\mathbb{E}(X)$, the expected value of X. If I will play this game many-many times, then my average winning will be approximately $\mathbb{E}(X)$, the amount I have paid for playing that games (this is the Law of Large Numbers, and we will talk about that soon, in one of our lectures). Say, if I will pay more than $\mathbb{E}(X)$ for each game, then, in the end, after playing a lot number of games, I will loose a fair amount of money. As another example, you can imagine a fair coin flipping game, with \$10 if heads appears, and \$20 if tails appears. What is the fair price for this game? The answer is that the fair price is $\mathbb{E}(Winning) = 0.5 \cdot \$10 + 0.5 \cdot \$20 = \$15$.

Now, about the St. Petersburg paradox. Assume I am playing a game against the casino. I am tossing a coin until it will turn up heads for the first time. If the first heads appears on the n-th toss, then my winning is 2^n USD. What is the fair price to enter this game? How much you will pay to play this game?

Problem 18. (Supplementary) Assume $X_1, X_2, \ldots$ is a sequence of IID random variables with the distribution

X_k	-1	1
$\mathbb{P}(X_k = x)$	0.5	0.5

Assume also $X_0 \equiv 0$, and denote

$$Y_n = X_0 + X_1 + X_2 + \ldots + X_n, \qquad n \in \mathbb{N} \cup \{0\}.$$

The sequence $Y_0, Y_1, Y_2, ...$ is called a **1D random walk**: imagine a drunk man standing at the point 0 at time $t = 0$ (the initial position, say, home). At the next time instant $t = 1$, he goes randomly either 1 units to the left or 1 units to the right with probabilities 0.5, and Y_1 is the position of our drunk man at time $t = 1$ (X_1 is $+1$ if he chooses to go right, and is -1, if he chooses to go to left, and $Y_1 = Y_0 + X_1$ is his new position). At time $t = 2$, he goes randomly to the left or right 1 units randomly, with equal probabilities, and his position on the real line at time $t = 2$ is Y_2, and so on. That is, Y_n is a r.v. showing possible positions of our drunk man at time n.

a. What is the set of all possible values of Y_n?
b. Give the PMF of Y_2;
c. Calculate the expected position of our drunk man at time $t = n$, and the variance of Y_n;
d. Approximate, for a large n, the probability that our drunk man will be between the points a and b, i.e., approximate $\mathbb{P}(a \le Y_n \le b)$;
e. Calculate the probability that $Y_n = 0$, i.e., at the time $t = n$, our drunk man will return to the initial position (home).
f. Prove that along the time, our drunk man will return to the initial position (home) infinitely many times.

Problem 19. (Supplementary) Assume X_k are IID r.v. with $X_k \sim \mathcal{N}(\mu, \sigma^2)$. Calculate the limit

$$\lim_{n\to+\infty} \frac{X_1 + X_2 + ... + X_n}{X_1^2 + X_2^2 + ... + X_n^2}.$$

Problem 20. (Supplementary) Assume X and Y are randomly and independently chosen numbers from $[0, 1]$. Find

$$\mathbb{E}(|X - Y|).$$

Problem 21. (Supplementary) Write a simple **R** or **Python** code to simulate

a. daily claims for an insurance company;
b. daily number of customers and their spending in a day for some shop.