CS 107, Probability, Spring 2019
Lecture 25

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AUA

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Content

- Examples of Important Discrete R.V.s
- Examples of Important Continuous R.V.s

Do you think all digits are equal?

LZ

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Say, the chance that our dataset number will start by 1 is 30.1%, by 2 is 17.6% etc.

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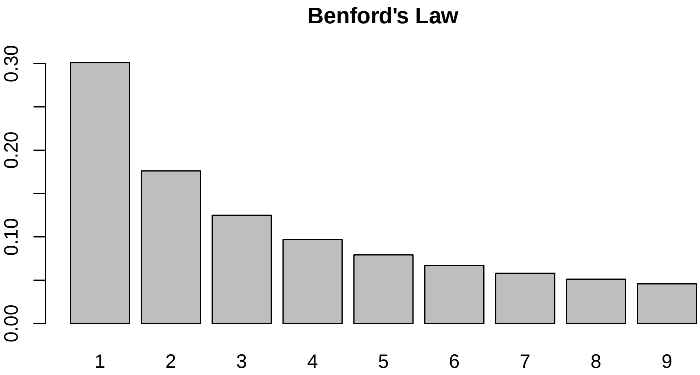

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Here are some examples: http://testingbenfordslaw.com/

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Values of X	$\sqrt{2}$	0	10
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- Give some possible result from one Experiment.
- Give some possible result from 10 Trials.

Prelude: Poisson Distribution

Let us recall the Poisson Distribution definition: $X \sim Poisson(\lambda)$ $(\lambda > 0)$, if the PMF of X is given by

$$\mathbb{P}(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \ldots$$

that is,

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Do not read as **Poison** ! $\ddot\smile$

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- Your turn!

Poisson Distribution: R Examples

R Code

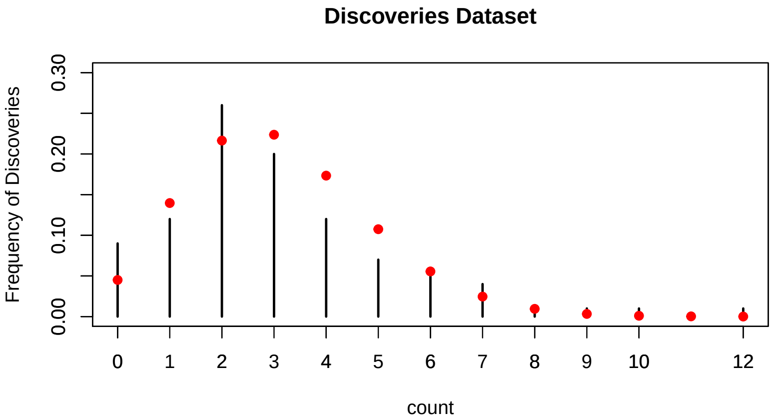
```
#Poisson Distribution
lambda <- 1.5
n <- 10
x <- 0:n
x.prob <- dpois(x,lambda) #PMF
#To have 2 plots side by side
par(mfrow=c(1,2))
#Plotting the PMF
plot(x, x.prob, type = "h", lwd = 5,
   main = "Poisson Distribution PMF")
#Now plotting the CDF
t <- seq(from = -1, to = n+1, by = 0.01)
y <- ppois(t,lambda)
plot(t,y, type = "s", lwd = 3,
   main = "Poisson Distribution CDF")
```

Poisson Distribution: R Examples

R Code

```
help("discoveries")
disc <- discoveries
plot(disc)
hist(disc, breaks = seq(0,13,1)) #histogram
#Fitting the data by the Poisson
lambda = mean(disc)
table(disc)
plot(table(disc)/length(disc), xlim = c(0,max(disc)),
   ylim = c(0,0.3), main = "Discoveries Dataset",
   ylab = "Frequency of Discoveries", xlab = "count")
n = 0:max(disc)
m = dpois(n, lambda)
par(new = T) #To keep the previous graph
plot(n,m, xlim = c(0,max(disc)), ylim = c(0,0.3),
   pch = 19, col="red", main = "", xlab = "", ylab = "")
```

Discoveries Dataset Model Result

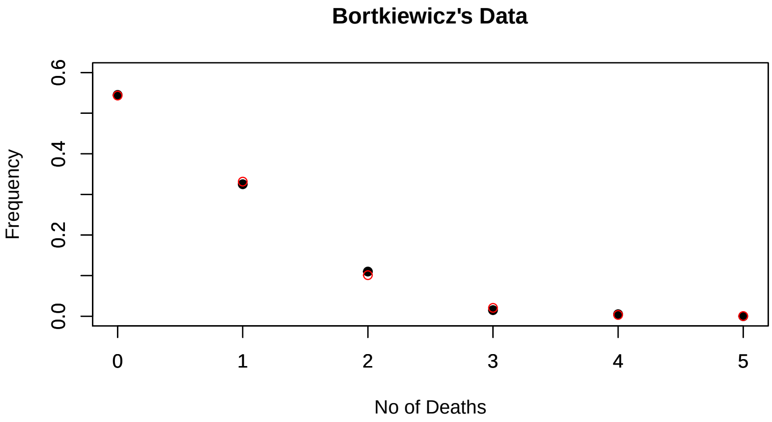


Poisson Distribution: R Examples

R Code

```
#Bortkiewicz's Data
#Deaths by horse-kick in Prussian Army cavalry corps
#(Bortkiewicz 1898). N is the number of units
#corresponding to each number of deaths.
hk<-data.frame(D = c(0,1,2,3,4,5),
   N = c(109,65,22,3,1,0))
plot(hk)
lambda <- sum(hk$D*hk$N)/sum(hk$N)
n = 0:5
m = dpois(n, lambda)
plot(n,m, xlim = c(0,5), ylim = c(0,0.6),
    col="red", type = "l", lwd = 3)
par(new = T)
plot(hk$D, hk$N/sum(hk$N) , xlim = c(0,5),
    ylim = c(0,0.6), pch=19)
```

Bortkiewicz Dataset Model Result



Approximation of Binomial Distribution through Poisson

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Now, assume n is large enough, but also p is small enough, and $\lambda = n \cdot p$. Then,

$$\mathbb{P}(X=k)=\binom{n}{k}p^k(1-p)^{n-k} \approx e^{-\lambda}\cdot\frac{\lambda^k}{k!}$$

Poisson Distribution: R Examples

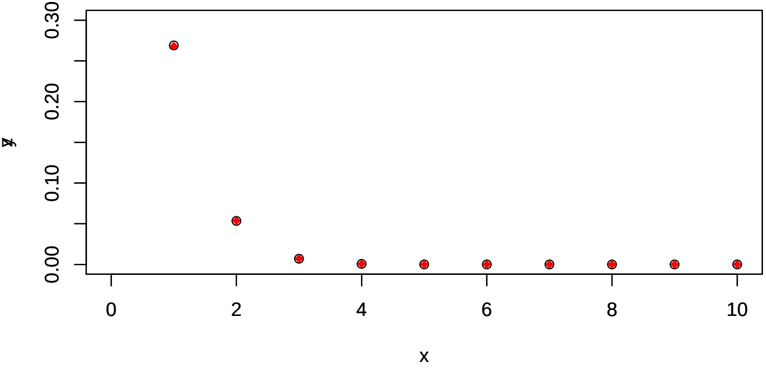
R Code

```
# Approximation of Binomial by Poisson
n <- 100
p <- 0.004
lambda <- n*p

x <- 0:20
y <- dbinom(x, size = n, prob = p)
plot(x,y, ylim = c(0,0.4))
z <- dpois(x, lambda = lambda)
par(new=T)
plot(x,z, pch = 18, col = "red", ylim = c(0,0.4))

#Plotting the difference
plot(x, abs(y-z))
```

Poisson approximation of Binomial Distribution



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