AUA CS108, Statistics, Fall 2020

Lecture 35

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16 Nov 2020

Contents

Examples, MLE

Example: Assume we have an observation

$$0, 1, 1, 2, 1, 0, 0, 1, 1$$

from the following Model:

X	0	1	2
$\mathbb{P}(X = x)$	$\frac{\theta}{10}$	$\frac{\theta}{5}$	$1 - \frac{3\theta}{10}$,

where $\theta \in [0, \frac{10}{3}]$.

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Solution: OTB

Example: Find the MLE Estimator for (μ, σ^2) in the $\mathcal{N}(\mu, \sigma^2)$ Model.

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Some Notes about MLE

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Note: It is remarkable, that ML Estimators, in general (if they exist, of course $\ddot\smile$), possess some nice properties. These properties make MLE one of the widely used methods of Estimation.

Properties of the MLE

It can be proven that, under some regularity conditions on the Parametric Family $\mathcal{F}_\theta$,

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- The MLE $\hat{\theta}_n^{MLE}$ is Asymptotically Normal and Efficient:

$$\hat{\theta}_n^{MLE} \overset{D}{\approx} \mathcal{N}\left(\theta, \frac{1}{n \cdot \mathcal{I}(\theta)}\right)$$

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So, MLE is **Consistent** and **Asymptotocally Efficient**. And this is why, for large Sample Size n, MLE is the Top 1 Choice, is (almost) unbeatable.

Fisher Information

in the above formulas,

$$\mathcal{I}(\theta) = -\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2}\ln f(X|\theta)\right) = \mathbb{E}\left[\left(\frac{\partial}{\partial\theta}\ln f(X|\theta)\right)^2\right],$$

is the **Fisher Information** for θ (X is a r.v. from the Distribution $\mathcal{F}_\theta$, and $f(x|\theta)$ is the corresponding PD(M)F).

Properties of the MLE, Cont'd

- Also,

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but, instead of $\mathcal{I}(\theta)$ we have $\mathcal{I}\left(\hat{\theta}_n^{MLE}\right)$.

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Note: We will use this later, to construct an (approximate) Confidence Interval for θ and for testing Hypotheses about θ.

Properties of the MLE, Cont'd

- If $\hat{\theta}$ is the MLE for θ, then for any function g, the MLE of $g(\theta)$ is $g(\hat{\theta})$, i.e.,
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Example Find the MLE for σ in $\mathcal{N}(\mu, \sigma^2)$ Model.

Solution: OTB

Other Methods to construct Point Estimators/Estimates

There are other important methods to construct Estimators: e.g.

- Bayesian Estimation: Maximum APosteriori (MAP) Estimators
- Bayesian Estimation: Bayes Estimators;
- OLS
- etc

Confidence Intervals

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i.e., we will (almost) **never** be correct in our guess. Sad news!

Prelude No. 2

But the good news is that even when we cannot exactly find the True value of our Parameter using $\hat{\theta}$, if $\hat{\theta}$ possesses some good properties, we believe that the Estimate obtained is a good approximation/Estimate for θ^*.

[1] Recall the $\widehat{SE}$, the Estimated Standard Error reporting story.

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And we can ask questions about how small is the *error* or how much sure are we in our Estimate (for the Unknown Parameter), and how large n needs to be to have a good estimate.

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Here we want to develop the theory of Confidence Intervals, which will contain answers to these questions.

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Let us state this in Mathematical terms. We will consider here only 1D case, i.e., we will assume $\theta \in \Theta \subset \mathbb{R}$.

Random Intervals and CI

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Example: Let $X_1, X_2, ..., X_n$ are IID r.v.s. Then

$$\left(\overline{X} - 0.1,\ \overline{X} + 0.1\right)$$

is a Random Interval.