CS 108 - Statistics, Sections B

Fall 2019, AUA

Homework No. 07

Due time/date: Section B: 10:32 AM, 01 November, 2019

Note: Please use **R** only in the case the statement of the problem contains (R) at the beginning. Otherwise, show your calculations on the paper. Supplementary Problems will not be graded, but you are very advised to solve them and to discuss later with TA or Instructor.

Problem 1: Fisher Information, MoM and MVUE

a. Fisher Information and MoM Estimator for the $Beta(\alpha, 1)$ Distribution

Consider a Parametric family of Distributions with PDF ($\alpha \in (0, +\infty)$)

$$f(x|\alpha) = \begin{cases} \alpha \cdot x^{\alpha-1}, & x \in (0,1) \\ 0, & otherwise, \end{cases}$$

- Calculate the Fisher Information $\mathcal{I}(\alpha)$;
- Find the MoM Estimator for α;
- Check the Consistency of $\hat{\alpha}^{MoM}$.

Note: This Distribution is a particular case of the **Beta**-Distribution[1] $Beta(\alpha, \beta)$, when $\beta = 1$.

b. MVUE for the Exponential Distribution

Sometimes, the Exponential Distribution is defined in the following way: we'll say that X is Exponentially distributed with the parameter β, if the PDF of X has the form

$$f(x|\beta) = \begin{cases} \frac{1}{\beta} \cdot e^{-x/\beta}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

In this case we will write $X \sim \widetilde{Exp}(\beta)$. Clearly, $X \sim \widetilde{Exp}(\beta)$ if and only if $X \sim Exp\left(\frac{1}{\beta}\right)$.

Clearly, if $X \sim \widetilde{Exp}(\beta)$, then

$$\mathbb{E}(X) = \beta \qquad \text{and} \qquad Var(X) = \beta^2.$$

Now,

[1] See, e.g., https://en.wikipedia.org/wiki/Beta_distribution

- Calculate the Fisher Information for the $\widetilde{Exp}(\beta)$ Model parameter β;
- Assuming we have a Random Sample

$$X_1, X_2, ..., X_n \sim \widetilde{Exp}(\beta),$$

  find the MoM Estimator for β;
- Prove that the MoM Estimator $\hat{\beta}^{MoM}$ is MVUE for β.
- (Supplementary) Calculate the MoM Estimator for our ordinary $Exp(\lambda)$ model parameter λ, and try to see why we are not doing the above for $\hat{\lambda}^{MoM}$.

Note: The above is equivalent of estimating $\frac{1}{\lambda}$ in the $Exp(\lambda)$ Model.

c. MVUE for the Normal Distribution

During our lecture, we have constructed the $\widehat{\sigma^2}^{MoM}$, and also, earlier, we have proved that this Estimator is a MVUE for the Variance σ^2. Now, we want to do this for the parameter μ of this model.

So we assume that σ^2 is known and fixed.

- Calculate the Fisher Information for the $\mathcal{N}(\mu, \sigma^2)$ Model parameter μ;
- Assuming we have a Random Sample

$$X_1, X_2, ..., X_n \sim \mathcal{N}(\mu, \sigma^2),$$

  find the MoM Estimator for μ;
- Prove that the MoM Estimator $\hat{\mu}^{MoM}$ is MVUE for μ.

d. MVUE for the Binomial Distribution

In fact, the Binomial Distribution is 2D-parametric Distribution, where the Parameters are m[2], the number of repetitions of a Trial, and p, the Probability of Success in one Trial. To simplify things, let us assume that m is known and fixed, and our aim is to Estimate p.

So we consider the Model $Binom(m, p)$, with m known and fixed.

- Calculate the Fisher Information for the $Binom(m, p)$ Model parameter p;
- Assuming we have a Random Sample

$$X_1, X_2, ..., X_n \sim Binom(m, p),$$

  find the MoM Estimator for p;
- Prove that the MoM Estimator $\hat{p}^{MoM}$ is MVUE for p.

[2] Here I am using m instead of n not to confuse with the Random Sample Size n.

Problem 2: The Method of Moments

a.

- Find the MoM Estimator for θ in the Parametric Model with the PDF ($\theta \in (0, +\infty)$)

$$f(x|\theta) = \begin{cases} 9 \cdot \theta^9 \cdot x^{-10}, & x \geq \theta \\ 0, & otherwise, \end{cases}$$

- Check the UnBiasedness and Consistency of the MoM Estimator.

b. MoM for Gamma Distribution

Gamma Distribution Family[3], is one of the important ones among all Distributions. It is a two-parametric Family of Distributions, denoted by $Gamma(\alpha, \beta)$. The aim of this problem is to Construct MoM Estimators for α and β, and, using observations, check how good are our Estimates.

- Find the formulae for the First and Second order Moments for this Distribution in Textbooks or Online (please be aware that there are two parametrizations, here, in ours, α is the *shape* parameter, and β is the *rate* parameter).
- Find the MoM Estimators for α and β
- (**R**) I have generated the following Dataset from Gamma Distribution, using **R**, and some parameters α and β:

```
##   [1]  1.8314358  3.5068022  1.9021544  1.9311683  2.4506474  1.3076168
##   [7]  3.2841308  2.1453588  6.3583929  0.9125459  1.0497036  3.5053462
##  [13]  5.2435455  1.8430702  3.1283204  2.1892085  0.8484110  2.5622832
##  [19]  1.7189725  2.7285723  2.3375566  2.9232977  5.3340547  1.2544215
##  [25]  8.7748185  4.3310692  0.8200883  4.2567581  0.3139263  3.9085270
##  [31]  4.9624075  1.9201225  3.3317165  2.7332516  1.3024626 10.8343385
##  [37]  4.6825457  2.1404540  0.7562919  3.6414628  2.1112214  3.4952583
##  [43]  0.5970462  3.5380416  5.2150621  1.3614640  2.6117846  1.8159541
##  [49]  0.5169138  2.5124691  3.5013392  3.3368801  0.8572007  6.4553792
##  [55]  8.6011775  1.4376073  0.4723840  2.8208109  1.4520227  1.2789404
##  [61]  0.7552889  1.1163648  0.7062065  1.1074022  6.8539689  2.0035534
##  [67]  3.5740196  5.7972806  5.1613411  7.2145042  7.1135339  4.2377204
##  [73]  1.1159071  4.5885815  3.5192624  0.4208828  1.0386598  2.3342677
##  [79]  1.3466932  5.5186891  1.5784883  3.9706863  5.6449191  8.1378222
##  [85]  4.2470504  3.8078827  2.0102668  1.7485158  2.2274422  2.4686840
##  [91]  2.1002233  4.9041384  2.2350189  0.3037267  1.7349850  8.2355609
##  [97]  4.4904943  3.3271395  1.8799883  1.7172326
```

Estimate α and β, using MoM, plot the Density Histogram of the Dataset over the interval $[0, 10]$, using 25 bins, and also plot over that the PDF of Gamma distribution with your calculated Estimates. Generate a next possible value for our Dataset.

[3]See https://en.wikipedia.org/wiki/Gamma_distribution