CS 107 Section B - Probability

Spring 2019, AUA

Homework No. 12

Due time/date: 10:35AM, 10 May, 2019

Note: Supplementary Problems will not be graded, but you are very advised to solve them and to discuss later with TA or Instructor.

Note: Please provide your answers in the form of a decimal number, by calculating and simplifying fractions, with the accuracy of 2 digits after the period.

Problem 1. Let X and Y be two independent r.v.'s on the same probability space, and assume

$$\mathbb{E}(X) = -1, \quad Var(X) = 5 \qquad \text{and} \qquad \mathbb{E}(Y) = 3, \quad Var(Y) = 1.$$

Calculate

a. $\mathbb{E}\Big(3X - 2Y + 5\mathbb{E}(X) - 1\Big)$;

b. $\mathbb{E}(X^2)$;

c. $\mathbb{E}\Big((X - Y)^2\Big)$;

d. $SD(X + Y)$;

e. $Var\Big(3X + \mathbb{E}(Y)\Big)$;

f. $Var\Big(2X - 3Y\Big)$;

g. $Cov(2X - 4Y, 9Y + 2X - 1)$;

Problem 2. I am playing the following game with my opponent: I am rolling 2 dice. If in the result I will have same number shown (say, I will have 5 shown on each die), then my opponent will pay me \$100. If the two numbers shown will be different, but they both will be primes, then my opponent will pay me \$50. In all other cases I need to pay \$40 to my opponent. What is my expected winning?

Problem 3. Assume X is a r.v. with the PDF

$$f(x) = \begin{cases} K \cdot \sin(x), & x \in [0, \pi]; \\ 0, & \text{otherwise.} \end{cases}$$

where K is a constant.

a. Calculate the value of K;

b. Calculate the expected value $\mathbb{E}(X)$;

c. Calculate the variance $Var(X)$;

d. Calculate $\mathbb{E}(X^2)$;

e. Calculate $\mathbb{E}(\sin(X))$;

f. Calculate $Var(\sin(X))$.

Problem 4. Assume $X \sim Pois(2)$. Calculate $\mathbb{E}(X)$, $Var(X)$ and $\mathbb{E}(X^2)$.

Problem 5. Assume $X \sim Exp(1)$. Calculate $\mathbb{E}(X)$, $\mathbb{E}(X^3)$, $Var(X^2)$.

Problem 6. Assume $(X,Y) \sim Unif(D)$, where D is the triangle with vertices at $(-1,0)$, $(0,1)$ and $(1,0)$. Calculate

a. $\mathbb{E}(X)$;

b. $\mathbb{E}(X^2+Y^2)$;

c. $Var(X)$;

d. $Cov(X,Y)$.

Problem 7. Assume $X_1, X_2, ...$ is a sequence of IID random variables with the distribution

X_k	-1	1
$\mathbb{P}(X_k = x)$	0.5	0.5

Assume also $X_0 \equiv 0$, and denote

$$Y_n = X_0 + X_1 + X_2 + ... + X_n, \qquad n \in \mathbb{N} \cup \{0\}.$$

The sequence $Y_0, Y_1, Y_2, ...$ is called a **1D random walk**: imagine a drunk man standing at the point 0 at time $t = 0$ (the initial position, say, home). At the next time instant $t = 1$, he goes randomly either 1 units to the left or 1 units to the right with probabilities 0.5, and Y_1 is the position of our drunk man at time $t = 1$ (X_1 is $+1$ if he chooses to go right, and is -1, if he chooses to go to left, and $Y_1 = Y_0 + X_1$ is his new position). At time $t = 2$, he goes randomly to the left or right 1 units randomly, with equal probabilities, and his position on the real line at time $t = 2$ is Y_2, and so on. That is, Y_n is a r.v. showing possible positions of our drunk man at time n.

a. What is the set of all possible values of Y_n?

b. Give the PMF of Y_2;

c. Calculate the expected position of our drunk man at time $t = n$, and the variance of Y_n;

d. Approximate, for a large n, the probability that our drunk man will be between the points a and b, i.e., approximate $\mathbb{P}(a \le Y_n \le b)$;

e. (Supplementary) Calculate the probability that $Y_n = 0$, i.e., at the time $t = n$, our drunk man will return to the initial position (home).

f. (Supplementary) Prove that along the time, our drunk man will return to the initial position (home) infinitely many times.

Problem 8. Assume $X_k \sim Unif[0,1]$, $k \in \mathbb{N}$, are IID.

a. Calculate[1] $\lim_{n\to+\infty} \frac{X_1+X_2+...+X_n}{n}$;

b. Assume $g \in C(\mathbb{R})$. Calculate the limit (in terms of g)

$$\lim_{n\to+\infty} \frac{g(X_1)+g(X_2)+...+g(X_n)}{n}$$

c. Calculate the limit

$$\lim_{n\to+\infty} \frac{X_1^2+X_2^2+...+X_n^2}{n}$$

d. Calculate the limit

$$\lim_{n\to+\infty} \sqrt[n]{(1+X_1)\cdot(1+X_2)\cdot\ldots\cdot(1+X_n)}$$

Problem 9. Assume we are tossing a fair coin 10000 times. What is the (approximate) probability that the number of heads shown will be in between 4950 and 5150? You are free to calculate the Standard Normal r.v. probabilities with a computer software (i.e., if $Z \sim \mathcal{N}(0,1)$, you can calculate $\mathbb{P}(a < Z < b)$ using, say, MatLab or R).

Problem 10. (from [R]) Solve the Problem 8.2, page 390.

Problem 11. (from [R]) Solve the Problem 8.5, page 390.

Problem 12. (from [R]) Solve the Problem 8.15, page 391.

Problem 13. (Supplementary) Assume X is a r.v. with a finite variance. For $\alpha \in \mathbb{R}$, we define

$$f(\alpha) = \mathbb{E}\Big((X-\alpha)^2\Big).$$

Find the minimum of $f(\alpha)$ for $\alpha \in \mathbb{R}$.

Note: If you will use the derivative to find all critical points, do not forget to justify that the value obtained is actually the minimum of f.

Problem 14. (Supplementary, St. Petersburg paradox) Before giving the actual problem, let me explain what is a fair price for a game with uncertain outcomes. Say, I am playing a game, and I can have 3 outcomes: A, B or C. I know the probabilities of having A, B or C, p_1, p_2 and p_3, respectively ($p_1+p_2+p_3=1$), and I am winning \$$a$ in the A case, \$$b$ in the B case and \$$c$ in the case C. So if I will denote by X my possible winning, then X will be a r.v. with PMF

X	a	b	c
$\mathbb{P}(X=x)$	p_1	p_2	p_3

Now, what is the fair price to enter this game, i.e. how much I will pay to enter this game? The answer is that the fair price is $\mathbb{E}(X)$, the expected value of X. If I will play this game many-many times, then my average winning will be approximately $\mathbb{E}(X)$, the amount I have paid for playing that games (this is the Law of Large Numbers, and we will talk about that

[1] Here everywhere we use limits in the almost surely convergence sense, i.e., by writing $\lim Y_n = Y$ we mean that $Y_n \to Y$ almost surely.

soon, in one of our lectures). Say, if I will pay more than $\mathbb{E}(X)$ for each game, then, in the end, after playing a lot number of games, I will loose a fair amount of money. As another example, you can imagine a fair coin flipping game, with \$10 if heads appears, and \$20 if tails appears. What is the fair price for this game? The answer is that the fair price is $\mathbb{E}(Winning) = 0.5 \cdot \$10 + 0.5 \cdot \$20 = \$15$.

Now, about the St. Petersburg paradox. Assume I am playing a game against the casino. I am tossing a coin until it will turn up heads for the first time. If the first heads appears on the n-th toss, then my winning is 2^n USD. What is the fair price to enter this game? How much you will pay to play this game?

Problem 15. (Supplementary) We roll a fair die several times. What is the expected number of rolls until we will get ⚁ shown on the die?