CS 107, Probability, Spring 2020
Lecture 03

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Content

- Probability Measure
- Properties of the Probability Measure

LZ

Linda is thirty-one years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.
Give your estimates for the probabilities that Linda is

(1) an elementary school teacher,
(2) active in the feminist movement,
(3) a bank teller,
(4) an insurance salesperson,
(5) a bank teller also active in the feminist movement.

This question is by Daniel Kahneman, psychologist, 2002 Nobel Prize receiver

Last Lecture ReCap

- A **Random Experiment** is an Experiment with unknown, more than one results;
- An **Outcome** of an Experiment is a possible result of that Experiment;
- The **Sample Space** is the set of all possible Outcomes;
- An **Event** is a subset of **Sample Space**.

Probability (Measure) Definition

Assume we have an Experiment with the Sample Space Ω and let $\mathcal{F}$ be the set of all Events in our Experiment.

So $A \in \mathcal{F}$ means that A is an Event in our Experiment. For example, $\Omega \in \mathcal{F}$ and $\varnothing \in \mathcal{F}$

We want to be able to measure the Probability of *any Event*, so the Probability Measure is a function defined on all events.

Probability (Measure) Definition

If A is an Event, i.e., if $A \in \mathcal{F}$, then we will denote by $\mathbb{P}(A)$ the Probability of A. The Probability (or the Chance) of an Event will be a real number between 0 and 1, i.e.,

$$0 \leq \mathbb{P}(A) \leq 1.$$

Sometimes we express it in the percentage form, i.e., if $\mathbb{P}(A) = 0.32$, we read it as:

- The Probability of A is 0.32 or
- The Probability of A is 32%.

Sometimes, we talk in terms of Odds: we say "The Odds for A are 4:1". This means that the Probability that A will happen is $\dfrac{4}{4+1} = \dfrac{4}{5}$.

Probability (Measure) Definition

Probability Measure Definition

A function $\mathbb{P} : \mathcal{F} \to \mathbb{R}$ is called a **Probability Measure** on $(\Omega, \mathcal{F})$, if it satisfies the following axioms:

P1. For any $A \in \mathcal{F}$,

$$\mathbb{P}(A) \geq 0;$$

P2. $\mathbb{P}(\Omega) = 1$;

P3. For any sequence of pairwise mutually exclusive (disjoint) events $A_n \in \mathcal{F}$, i.e., for any sequence $A_n \in \mathcal{F}$ with $A_i \cap A_j = \varnothing$ for $i \neq j$, we have

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

Some Remarks and Facts:

- This is a formal definition of **what is a Probability Measure**, what properties need to share the Probability measure, but we are not giving **how to define the Probability Measure**;
- One can define different Probability Measures on the Same Experiment;
- Probability Measure is very similar (and shares the properties of) any other Measure - Cardinality (no. of elements), Length (in 1D), Area (in 2D), Volume (in 3D and moreD).
- The difference is only that the Probability of the Sample Space is 1, $\mathbb{P}(\Omega) = 1$.

Examples

- Experiment: we are rolling a die.
- The Sample Space in this Example is

$$\Omega = \{\epsdice{1}, \epsdice{2}, \epsdice{3}, \epsdice{4}, \epsdice{5}, \epsdice{6}\}$$

- We can define (this is a definition!) the Probability Measure on this Experiment as:

$$\mathbb{P}(\Omega) = 1,$$

$$\mathbb{P}\left(\left\{\epsdice{1}\right\}\right) = \mathbb{P}\left(\left\{\epsdice{2}\right\}\right) = \mathbb{P}\left(\left\{\epsdice{3}\right\}\right) = \mathbb{P}\left(\left\{\epsdice{4}\right\}\right) =$$
$$= \mathbb{P}\left(\left\{\epsdice{5}\right\}\right) = \mathbb{P}\left(\left\{\epsdice{6}\right\}\right) = \frac{1}{6}$$

Examples, cont'd

- Note that I am not writing $\mathbb{P}\left(\epsdice{1}\right)$, but $\mathbb{P}\left(\left\{\epsdice{1}\right\}\right)$, since the Probability is defined on **Events**, but not Outcomes. But usually people use the notation $\mathbb{P}\left(\epsdice{1}\right)$ for simplicity, and we will do that.
- Another note: the definition above is not complete: we need to define the Probability of **any Event**, so we need to continue our definition by:

$$\mathbb{P}\left(\left\{\epsdice{1},\epsdice{2}\right\}\right) = \mathbb{P}\left(\left\{\epsdice{1},\epsdice{3}\right\}\right) = \ldots = \mathbb{P}\left(\left\{\epsdice{5},\epsdice{6}\right\}\right) = \frac{2}{6},$$

also

$$\mathbb{P}\left(\left\{\epsdice{1},\epsdice{5},\epsdice{6}\right\}\right) = \frac{3}{6}$$

etc., and also $\mathbb{P}(\varnothing) = 0$.

Another Example

- Experiment: we are rolling a die, again.
- You know the Sample Space $\ddot\smile$
- Now we take any non-negative numbers $p_1, p_2, \ldots, p_6$ with
$$p_1 + p_2 + \ldots + p_6 = 1,$$
and define
$$\mathbb{P}\left(\left\{\epsdice{1}\right\}\right) = p_1, \quad \mathbb{P}\left(\left\{\epsdice{2}\right\}\right) = p_2, \quad \mathbb{P}\left(\left\{\epsdice{3}\right\}\right) = p_3,$$
$$\mathbb{P}\left(\left\{\epsdice{4}\right\}\right) = p_4, \quad \mathbb{P}\left(\left\{\epsdice{5}\right\}\right) = p_5, \quad \mathbb{P}\left(\left\{\epsdice{6}\right\}\right) = p_6.$$
and also:
$$\mathbb{P}\left(\left\{\epsdice{1}, \epsdice{2}\right\}\right) = p_1 + p_2, \quad \mathbb{P}\left(\left\{\epsdice{3}, \epsdice{4}, \epsdice{6}\right\}\right) = p_3 + p_4 + p_6$$
etc, and $\mathbb{P}(\varnothing) = 0$. In this way we will obtain a Probability Measure.

Remarks and Facts:

As we have seen in the above examples, we need to give how to calculate the Probability of *each Event*. Even in the above examples, where the Sample Spaces are finite, listing of all values of Probability Measure is very long and boring.
Fortunately, in the case of Experiments with Discrete Sample Spaces (i.e., Sample Spaces, that are either Finite or Countably Infinite), one can just define the Probability of each Outcome[1].
Then, using the Properties of the Probability Measure, we will be able to calculate the Probability of each Event in that Experiment!

[1] More precisely and correctly, the Probability of each Events containing one Outcome.

Remarks and Facts: Cont'd

- Say, in the Coin Tossing Experiment, with the

$$\text{Sample Space} = \{H, T\},$$

we can just give

$$\mathbb{P}(\{H\}) \qquad \textit{and} \qquad \mathbb{P}(\{T\}),$$

and then we can calculate $\mathbb{P}(\{H, T\})$ by using the property of a Probability Measure that we will give soon:

$$\mathbb{P}(\{H, T\}) = \mathbb{P}(\{H\}) + \mathbb{P}(\{T\}).$$

It remains only to add $\mathbb{P}(\varnothing) = 0$.

Remarks and Facts: Cont'd

- Unfortunately, the same is not the case for Experiments with Continuous Sample Spaces.
- But still, it will be possible to define the Probability of some Basic Events, and then to extend Probability calculations to any Event.
- This is an 18+ topic covered in advanced Probability courses with Measure Theory, and we will skip this part.
- As an example, if we consider an Experiment of choosing a point (a real number) from $[0, 1] = \Omega$ at random [2], then we can choose as Basic Events all intervals $[a, b] \subset [0, 1]$, and define

$$\mathbb{P}([a, b]) = Length([a, b]) = b - a.$$

  Using this, we will be able to calculate the Probability of any Event (there is another 18+ topic here!)

[2] in our example - uniformly

Properties of the Probability Measure

1. $\mathbb{P}(\varnothing) = 0;$