CS 107 Section A - Probability

Spring 2020, AUA

Homework No. 11

Due time/date: 28 April, 2020

Note: Supplementary Problems will not be graded, but you are very advised to solve them and to discuss later with TA or Instructor.

Joint Distribution of r.v.s

A Joint PDFs

Problem 1. Is the function $f(x, y)$ a Joint PDF for some r.v.'s X and Y? Explain your reasoning.

a. $f(x,y) = \begin{cases} 2, & (x,y) \in [0,1] \times [0,0.5]; \\ 1, & (x,y) \in [1,2] \times [0.5,1]; \\ 0, & \textit{otherwise}. \end{cases}$

b. $f(x,y) = \begin{cases} x \cdot y, & \text{if } 0 \leq x \leq 1,\ 0 \leq y \leq x; \\ 0, & \textit{otherwise}. \end{cases}$

Problem 2. Let X and Y be Jointly distributed r.v.s with the following Joint PDF:

$$f(x,y) = \begin{cases} 4xy \cdot e^{-x^2-y^2}, & \text{if } x \geq 0,\ y \geq 0; \\ 0, & \textit{otherwise}. \end{cases}$$

a. Calculate $\mathbb{P}(X < 3, Y > 2)$;
b. Calculate $\mathbb{P}(2X - 3Y = 1)$;
c. Calculate $\mathbb{P}(X^2 + Y^2 \leq 4)$;
d. Find the Marginal PDFs of X and Y;
e. Calculate $\mathbb{P}(X > 1)$ by using the Joint PDF or the Marginal PDF of X.

Problem 3. (Supplementary, but placed here) Assume the PDF of the random vector (X, Y) is

$$f(x,y) = \begin{cases} K \cdot (x + 2y), & \text{if } 0 \leq x \leq 1,\ 0 \leq y \leq 1; \\ 0, & \textit{otherwise}. \end{cases}$$

for some constant K.

a. Find K;

b. Calculate $\mathbb{P}((X,Y) \in D)$, where D is a trapezoid with vertices at $(0,0)$, $(0,1)$, $(1,1)$ and $(3,0)$;

c. Calculate $\mathbb{P}(X^2 + Y^2 \leq 1)$;

d. Find the Marginal PDF of Y.

B Some important Multivariate Distributions

Problem 4. Assume we are picking at random a point, uniformly, in $D \subset \mathbb{R}^2$, and let X and Y be the x- and y- coordinates of that point. We consider two cases, when

I. D is the triangle with vertices at $(-1,0)$, $(1,0)$ and $(0,1)$;

II. (Supplementary) $D = \{(x,y) : 1 \leq x^2 + y^2 \leq 4\}$.

For each case of D,

a. Find the Joint PDF of X and Y;

b. Find the (Marginal) PDFs of X and Y;

c. Calculate $\mathbb{P}(X \in [0,1], Y \in [0,1])$.

Problem 5. Assume

$$\mu = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}.$$

Let $(X,Y) \sim \mathcal{N}(\mu, \Sigma)$.

a. Construct the PDF of (X,Y);

b. Plot some level curves of the PDF of (X,Y).

c. (Supplementary) Calculate, using **R** or any other software, the probabilities

$$\mathbb{P}(X^2 + (Y-3)^2 < 3) \qquad \text{and} \qquad \mathbb{P}(X > 2);$$

d. (Supplementary) Plot, using some software, or your calculus knowledge, the graph of the Joint PDF of X and Y;

C Independence of r.v.s

Problem 6. a. Assume that X and Y are Discrete r.v.'s, and assume X and Y are Independent: $X \perp\!\!\!\perp Y$. Find the Joint PMF of X and Y, if

Y \ X	-2	1	2	PMF of Y
-10				$\frac{1}{4}$
10				$\frac{3}{4}$
PMF of X	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

b. Assume that the Joint PMF of Discrete r.v. X and Y is given by

Y \ X	2	3
-3	0.1	0.3
1	0.2	0.4

Are X and Y Independent? Prove your statement.

c. Assume X and Y are Discrete r.v. with the following PMFs:

X	0	2
$\mathbb{P}(X = x)$	0.3	0.7

and

Y	0	2
$\mathbb{P}(Y = y)$	0.3	0.7

Are X and Y Dependent? Explain.

Problem 7. Assume $X \sim \mathit{Binom}(4, 0.2)$ and $Y \sim \mathit{Pois}(1)$ and X and Y are Independent: $X \perp\!\!\!\perp Y$. Calculate

$$\mathbb{P}(X + Y \leq 2).$$

Problem 8. Assume $(X,Y) \sim \mathit{Unif}(D)$, where D is the square $D = \{(x,y) : |x| + |y| \leq 1\}$. Are X and Y Independent? Prove your statement.

Problem 9. Assume $X \sim \mathit{Unif}[-1, 2]$ and $Y \sim \mathit{Exp}(3)$, and X and Y are Independent.

a. Find the Joint PDF of X and Y;

b. Calculate $\mathbb{P}(X \in [1,2], Y \in [0,1])$.

Problem 10.

a. Assume $X \sim \mathcal{N}(1, 2^2)$ and $Y \sim \mathcal{N}(2, 4^2)$. Are X and Y Independent? Explain.

b. Assume again that $X \sim \mathcal{N}(1, 2^2)$ and $Y \sim \mathcal{N}(2, 4^2)$, and now assume that X and Y are Independent. Calculate the Joint PDF of X and Y.

D Supplementary Problems

Problem 11. (Supplementary) Assume $X \sim \mathit{Unif}([0,2])$, $Y \sim \mathit{Bernoulli}(0.5)$ and $X \perp\!\!\!\perp Y$. Find the CDF of $X + Y$.

Problem 12. (Supplementary) Assume $X_1 \sim \mathit{Unif}([0,1])$, $X_2 \equiv 1$, $Y \sim \mathit{Bernoulli}(0.5)$ and Y is independent of X_1 and X_2. Consider the r.v.

$$Z = \begin{cases} X_1, & Y = 0 \\ X_2, & Y = 1. \end{cases}$$

Find and plot the CDF of Z.

Problem 13. (Supplementary) Let q_α be the α-level quantile of some distribution $\mathcal{D}$ with continuous and strictly increasing CDF F, i.e. , for $X \sim \mathcal{D}$,

$$\mathbb{P}(X \le q_\alpha) = \alpha.$$

Which one is larger: $q_{0.7}$ or $q_{0.8}$?

Problem 14. (Supplementary) Express the Joint CDF (PDF) of U, V in terms of the Joint CDF (PDF) of X, Y, if

a. $U = 3X + 2$, $V = 4Y - 2$;

b. $U = X + Y$, $V = X - Y$.

Problem 15. (Supplementary) Assume X and Y are Jointly Continuous with Joint CDF $F(x, y)$ and Joint PDF $f(x, y)$. Express (no proof is necessary):

a. $F(x, y)$ in terms of $f(x, y)$;

b. $f(x, y)$ in terms of $F(x, y)$;

c1. the (Marginal) CDF of X, $F_X(x)$ in terms of $F(x, y)$;

c2. the (Marginal) CDF of X, $F_X(x)$ in terms of $f(x, y)$;

d1. the (Marginal) PDF of Y, $f_Y(x)$ in terms of $F(x, y)$;

d2. the (Marginal) PDF of Y, $f_Y(x)$ in terms of $f(x, y)$;

e1. $\mathbb{P}(a \le X \le b, Y \ge c)$ in terms of $F(x, y)$;

e2. $\mathbb{P}(a \le X \le b, Y \ge c)$ in terms of $f(x, y)$;

f1. $\mathbb{P}(X \ge a, Y \le b)$ in terms of $F(x, y)$;

f2. $\mathbb{P}(X \ge a, Y \le b)$ in terms of $f(x, y)$;

g. $\mathbb{P}(X^4 + Y^4 \le 5)$ in terms of $f(x, y)$; write also the double integral in the iterated integrals form;

h. $\mathbb{P}(|X| + Y \le 5)$ in terms of $f(x, y)$; write also the double integral in the iterated integrals form;

i. $\mathbb{P}(X \in [0, 2], Y \le \sin(X))$ in terms of $f(x, y)$; write also the double integral in the iterated integrals form;

j. the CDF of the 1D random variable $Z = X + Y$ in terms of $f(x, y)$;

k. the CDF of the 1D random variable $Z = \max\{X, Y\}$ in terms of $F(x, y)$ and $f(x, y)$;

l. the CDF of the 1D random variable $Z = \min\{X, Y\}$ in terms of $F(x, y)$ and $f(x, y)$.

Problem 16. (Supplementary) Assume X and Y are discrete r.v.'s with values[1] $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$, respectively, and their Joint PMF is $\mathbb{P}(X = x_i, Y = y_j)$ for $i = 1, 2, \ldots$, $j = 1, 2, \ldots$. Express in terms of the Joint PMF:

a. Their Joint CDF $F(x, y)$;

[1] Finite or countably infinite, also not necessarily of the same size.

b. $\mathbb{P}(a \leq X \leq b, c \leq Y \leq d)$;

c. The (Marginal) CDF of X, $F_X(x)$;

d. $\mathbb{P}(X = x, Y \leq y)$;

e. $\mathbb{P}(a \leq X \leq b)$.

Problem 17. (Supplementary) Assume X and Y are Independent. Prove that $2X + 1$ and Y^3 are Independent too.

Problem 18. (Supplementary) Assume $(U, V) \sim \mathcal{N}(\mu, \Sigma)$, where

$$\mu = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}.$$

Prove that U and V are Independent.

Problem 19. (Supplementary) Assume X, Y and Z are IID, i.e., Independent and Identically Distributed, i.e., they all have the same CDF $F(x)$. Calculate the CDF of

$$U = \max\{X, Y, Z\} \qquad \text{and} \qquad V = \min\{X, Y, Z\}.$$

Generalize for n IID random variables.

Note: This result is important in Statistics.