CS 107, Probability, Spring 2019
Lecture 16

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AUA

22 February 2019

Content

- Review Session

Experiment, Outcomes and the Sample Space

What is:

Experiment, Outcomes and the Sample Space

What is:

- An Experiment (Random Experiment)

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What is:

- An Experiment (Random Experiment)
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3. for any event $A \in \mathcal{F}$,
$$\mathbb{P}(\overline{A}) = 1 - \mathbb{P}(A);$$
Here $\overline{A} = A^c = \Omega \setminus A$.

Properties of the Probability Measure

4. If $A_1, A_2, \ldots, A_n \in \mathcal{F}$ are pairwise disjoint (mutually exclusive), i.e., if $A_i \cap A_j = \varnothing$ for $i \neq j$, then

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6. If $A_1, A_2, ..., A_n \in \mathcal{F}$ are events, not necessarily disjoint, then

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + ... + \mathbb{P}(A_n) -$$

$$-\mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_3) - ... - \mathbb{P}(A_{n-1} \cap A_n) +$$

$$+\mathbb{P}(A_1 \cap A_2 \cap A_3) + \mathbb{P}(A_1 \cap A_2 \cap A_4) + ... + \mathbb{P}(A_{n-2} \cap A_{n-1} \cap A_n) - ...$$

$$... + (-1)^{n-1}\mathbb{P}(A_1 \cap A_2 \cap ... \cap A_n).$$

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This is the general version of the previous property, and is called the inclusion-exclusion principle.

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9. if A and B are two events such that $A \subset B$, then

$$\mathbb{P}(A) \leq \mathbb{P}(B);$$

10. for any event $A \in \mathcal{F}$,

$$0 \leq \mathbb{P}(A) \leq 1;$$

Classical Probability Models: Finite Sample Spaces

The Finite Sample Space Probability Model is of the form:

Outcome	ω_1	ω_2	...	ω_n
$\mathbb{P}(\{\omega_k\})$	p_1	p_2	...	p_n

with $p_1 + p_2 + ... + p_n =$

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$$\mathbb{P}(A) = \sum_{\omega_i \in A} p_i,$$

and add $\mathbb{P}(\varnothing) = 0$.

Classical Probability Models: Countably Infinite Sample Spaces

This model looks like this:

Outcome	ω_1	ω_2	ω_3	...	ω_n	...
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where $p_k \geq 0$ for any k, and $\sum_{k=1}^{\infty} p_k = 1$.
Now, we define for any nonempty event $A \in \mathcal{F}$ (i.e., for any nonempty subset $A \subset \Omega$),

$$\mathbb{P}(A) = \sum_{\omega_k \in A} p_k,$$

and also $\mathbb{P}(\varnothing) = 0$.

Equally Likely Outcomes

Now assume we have a Discrete Model with finitely many outcomes:

$$\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$$

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In the table form:

Outcome	ω_1	ω_2	ω_3	...	ω_n
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In this case,

$$\mathbb{P}(A) = \frac{\text{number of elements favorable for the event } A}{\text{total number of possible outcomes}} = \frac{\#A}{\#\Omega}.$$

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Interpretation:

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e. If $A_1, ..., A_n, ...$ are some **mutually disjoint** events, then

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f. If A_1, A_2, B are some events and $\mathbb{P}(B) \neq 0$, then

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$$\cdot \mathbb{P}(A_n|A_1 \cap A_2 \cap ... \cap A_{n-1}).$$

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This property is called the Multiplication Rule.

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Remark: It is easy to see that the condition above is equivalent, except the cases when $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$, to

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- If A is independent of B and A is independent of C, and also $B \cap C = \varnothing$, then A is independent of $B \cup C$.

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Pairwise Independence

We will say that the events $A_1, A_2, ..., A_n$ are **Pairwise Independent**, if every pair A_i and A_j are Independent, for any $i \neq j$, i.e., if

$$\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i) \cdot \mathbb{P}(A_j), \qquad i \neq j.$$

Independence of more that two events, cont'd

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Mutual Independence

We say that $A_1, ..., A_n$ are **Mutually Independent** or just **Independent**, if for any subgroup of events $A_{i_1}, A_{i_2}, ..., A_{i_k}$,

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Good Luck for MT1!