CS 107 Section B - Probability

Spring 2019, AUA

Homework No. 09

Due time/date: 10:35AM, 05 April, 2019

Note: Supplementary Problems will not be graded, but you are very advised to solve them and to discuss later with TA or Instructor.

Note: Please provide your answers in the form of a decimal number, by calculating and simplifying fractions, with the accuracy of 2 digits after the period.

Problem 0. Add a Problem about the Transform of Discrete R.V. !

Problem 1. Assume $X \sim Unif[-4, 4]$.

a. Find and plot the CDF of X;

b. Prove that $Y = \dfrac{X+4}{8}$ is a Standard Uniform r.v., i.e., $Y \sim Unif[0, 1]$;

c. Assume $Z = -X$. Prove that $Z \sim Unif[-4, 4]$;

   Note: This means, particularly, that X and $-X$ are i.d. (identically distributed, have the same distribution), although different.

d. Calculate the probability $\mathbb{P}(X > 2)$ and $\mathbb{P}(|X| < 3)$;

e. Find the PDFs of $S = \sqrt{|X|}$ and $U = e^X$;

f. Describe the distribution of $T = [X]$ (i.e., find the CDF and PDF/PMF of T). Here $[a]$ means the Integer Part of the number a - the largest integer number not exceeding a;

g. (Supplementary) Describe the distribution of $T = \{X\}$, where $\{a\}$ means the Fractional Part of the number a.

h. (Supplementary) Find a transformation (function) $g : \mathbb{R} \to \mathbb{R}$, such that the PDF of $W = g(X)$ will have the form

$$f_W(x) = \begin{cases} 4x^3, & x \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

i. (Supplementary) Find a transformation (function) $g : \mathbb{R} \to \mathbb{R}$, such that $g(X) \sim Bernoulli(0.4)$.

Problem 2.

a. Assume X is a continuous r.v. with a strictly increasing CDF $F(x)$. Show that $Y = F(X) \sim Unif[0, 1]$;

b. Assume $Y \sim Unif[0, 1]$, and $F(x)$ is a continuous, strictly increasing function with $F(-\infty) = 0$ and $F(+\infty) = 1$. Show that the CDF of the r.v. $X = F^{-1}(Y)$ is F.

Note: The results obtained in this Problem are very important. This is one of the methods to generate random numbers from any distribution given by its CDF $F(x)$. Given any F, satisfying above conditions, first we generate uniformly distributed random numbers, then we plug that numbers into F^{-1}, and we obtain the required random numbers.

Problem 3. I am modeling the distribution of the final grades for our Probability course. I am assuming that the average grade X will be a Normal r.v., with mean 75, and standard deviation 8, i.e., $X \sim \mathcal{N}(75, 8^2)$.

a. Prove that the r.v. $Y = \dfrac{X - 75}{8}$ is a Standard Normal r.v., i.e., $Y \sim \mathcal{N}(0, 1)$;

Note: This transform is the *Standardization* of a Normal Variable.

b. Prove that for the any Standard Normal r.v. Y, $-Y$ is again Standard Normal r.v.;

Note: This means, particularly, that Y and $-Y$ are i.d. (identically distributed), although different.

c. It is known that for a Standard Normal r.v. Y, $\mathbb{P}(Y \leq 0) = \mathbb{P}(Y \geq 0) = \frac{1}{2}$. Prove that $\mathbb{P}(X \leq 75) = \frac{1}{2}$, and calculate the probability $\mathbb{P}(X \leq 85)$, using some Mathematical Software.

d. What is the probability that the average grade will be larger than 90?

e. Find the PDF of $Z = e^X$. Plot, using some software, the graph of that PDF.

Note: The distribution of Z is called the LogNormal Distribution[1], see https://en.wikipedia.org/wiki/Log-normal_distribution. This distribution is used widely in Financial Math to model Stock future prices, since it can assume only positive values (the price cannot be negative), but Normal r.v. can take negative values with positive probability.

f. (Supplementary) Prove that PDF of Normal r.v. with mean μ and standard deviation σ, i.e., the PDF of $Z \sim \mathcal{N}(\mu, \sigma^2)$, is a legitimate PDF, i.e., the integral of that function over $\mathbb{R}$ is 1.

g. (Supplementary) Calculate $\mathbb{P}(X \leq 85)$ using the Standard Normal tables you will find online, say, at https://en.wikipedia.org/wiki/Standard_normal_table or in most of the Probability and Statistics Textbooks.

Problem 4. I am standing at the Baghramyan ave., and a car passes by the place I am standing at. I do not know the velocity of that car, but I can guess that it is 40 ± 5 km/h. I want to find the distance that that car will travel in 10 min after passing the place I am standing at. To that end, I assume that the velocity V of that car is constant, the car is doing a rectilinear (straight-line) motion, and V can be modeled as a Normal r.v., $V \sim \mathcal{N}(40, 5^2)$.

[1]The logarithm of Z is Normal r.v..

a. Let S be the r.v. measuring the distance from my standing point of that car after 10 min. Express S in terms of V;

b. What is the probability that $S > 8$km?

d. Describe the distribution of S, plot the PDF of S and give some explanation about the most possible places that car can be in 10 min;

e. (Supplementary) Is my choice for the velocity model, $V \sim \mathcal{N}(40, 5^2)$, appropriate in this case? Explain your reasoning.

Problem 5. Assume $X \sim Exp(\lambda)$ for some $\lambda > 0$.

a. Prove the **memoryless** property: for any $t, s > 0$,

$$\mathbb{P}(X > t + s | X > s) = \mathbb{P}(X > t).$$

b. (Supplementary) Prove that the only Continuous Distribution possessing the Memoryless property is the Exponential one.

c. (Supplementary) Prove the Discrete Memoryless property for the Geometric Distribution, and prove that the only Discrete Distribution having this property is the Geometric one.

Problem 6. I am modeling daily phone call received by me during 10AM and 6PM. I am estimating that the average waiting time for the next call is 2 hours. Now, let X be my waiting time for the next call, and Y be the number of received calls.

a. Write a model for X. Explain your choice;

b. Calculate, in your model, the probability that I will wait for the next phone call more than 3 hours.

c. Write a model for Y. Explain your choice;

d. (Supplementary) Using **R** ot **Python**, give some possible scenario for the phone calls I will receive tomorrow. Then generate 100 possible scenarios, and calculate the average waiting time and number of calls.

Problem 7. (from [R]) Solve the Problem 5.11, page 212.

Problem 8. (from [R]) Solve the Problem 5.16, page 213.

Problem 9. (from [R]) Solve the Problem 5.17, page 213.

Problem 10. (from [R]) Solve the Problem 5.24, page 213.

Problem 11. (from [R]) Solve the Problem 5.33, page 214.

Problem 12. (Supplementary) Give a concrete example of a r.v. with a Uniform Distribution, i.e., give an example of a Probability Space $(\Omega, \mathcal{F}, \mathbb{P})$ and a function $X : \Omega \to \mathbb{R}$ such that $X \sim Unif[0, 1]$.

Problem 13. (Supplementary, **AI**[2]) Using **R**, **Python** or **MatLab**, solve the Problem 15 (How Long Is the Wait to Get the Potato Salad?) and Problem 17 (Waiting for Buses) of the book by Paul Nahin, *Digital Dice: Computational Solutions to Practical Probability Problems*.

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[2] = Absolutely Interesting!