CS 107, Probability, Spring 2019
Lecture 35

Michael Poghosyan

AUA

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Content

- Multivariate Normal (Gaussian) Distribution
- Transformation of Random Vectors
- Independent Random Variables

R Code

R Code for Bivariate Normal

```
mu <- c(0,0) # The Mean
Sigma <- matrix(c(1, .5, .5, 1), nrow = 2) #Cov Matrix
#Version 1
library(MASS)
data <- mvrnorm(5000, mu = mu, Sigma = Sigma )
plot(data, pch = 20, asp = 1, cex = 0.6)
#Version 2
#install.packages("mvtnorm")
library(mvtnorm)
data <- rmvnorm(1000, mean = mu, sigma = Sigma)
plot(data, pch = 20, asp = 1, xlim = c(-3,3))
```

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We assume we are given a vector in $\mathbb{R}^n$:

$$\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix},$$

and a Symmetric Positive Definite Matrix

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix}$$

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We say that the r. vector $\mathbf{X} = (X_1, X_2, ..., X_n)$ has a Multivariate Normal (or Gaussian) Distribution with the **mean** μ and the **covariance matrix** Σ, and we will write

$$\mathbf{X} \sim \mathcal{N}(\mu, \Sigma),$$

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$$\mathbf{X} \sim \mathcal{N}(\mu, \Sigma),$$

if the Joint PDF of $\mathbf{X}$ is given by

$$f(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \cdot \exp\left\{ -\frac{1}{2}(\mathbf{x}-\mu)^T \cdot \Sigma^{-1} \cdot (\mathbf{x}-\mu) \right\},$$

for any $\mathbf{x} \in \mathbb{R}^n$.

Multivariate Normal (Gaussian) Distribution

Another, equivalent definition of the Multivariate Normal Distribution is the following:

Multivariate Normal (Gaussian) Distribution

We say that the r. vector $\mathbf{X} = (X_1, X_2, ..., X_n)$ has a Multivariate Normal (or Gaussian) Distribution, if for any numbers[a] $a_1, a_2, ..., a_n \in \mathbb{R}$, the r.v.

$$Y = a_1 \cdot X_1 + a_2 \cdot X_2 + ... + a_n \cdot X_n$$

is Normally Distributed.

[a] We need to take care of the case $a_k = 0$ for all k. We can exclude this case, say, in the definition.

Marginals of Multivariate Normal Distribution

It is remarkable that the Marginal Distributions of Multivariate Normal Distribution are again Normal. In particular, if

$$(X, Y) \sim \mathcal{N}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \right)$$

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then

$$X \sim \mathcal{N}(\mu_1, \sigma_{11}) \qquad \text{and} \qquad Y \sim \mathcal{N}(\mu_2, \sigma_{22}).$$

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- We can use 2 Real-Valued functions $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ to transform (X, Y) onto new r.vector (U, V) by

$$U = g_1(X, Y), \qquad V = g_2(X, Y);$$

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Here we will consider the 1D case, when we transform our pair X, Y onto 1 r.v. Z.

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Now assume we have a r.vector (X, Y), and we form a new r.variable (1D r.v.!) from X and Y. Say, we take a function $g : \mathbb{R}^2 \to \mathbb{R}$, $g = g(x, y)$, and consider the r.v.

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The general problem here is:

Knowing the Joint Distribution of X and Y, find the Distribution of $Z = g(X, Y)$.

In particular, very important is the question of distribution of the sum $Z = X + Y$, and, in general, of the sum of n r.v.s

$$Y = X_1 + X_2 + ... + X_n.$$

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$$Z = 100 \cdot GOOG + 200 \cdot AMZN,$$

where $GOOG$ and $AMZN$ are r.v. showing the prices of Google and Amazon Stocks at the end of the year.

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  where X_i is the amount I will spend on the day i of the week. Of course, X_i-s are r.v.s depending on how much I will be hungry.

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- Going further, the same companies can also model they daily claims: If they will have M claims in a day, with sizes X_1, X_2, ..., X_M, then their daily claim size will be

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  Here important is that M **will be a r.v. itself (!)**, and the distribution of Z is an 18+ topic. , Btw, can you give a Model for M?

CDF of $g(X, Y)$ through the Joint CDF of X, Y

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Unfortunately, it is not possible to express this in terms of $F_{X,Y}(x, y)$ in the general case.

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Unfortunately, it is not possible to express this in terms of $F_{X,Y}(x, y)$ in the general case.
Btw, we have calculated this kind of thing some lecture ago: we were calculating $\mathbb{P}(Y - X \leq x)$, i.e., the CDF of $Z = Y - X$.

Transform of Joint Discrete R.V.s

Assume now X and Y are Discrete with values $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$, and with the Joint PMF

$Y \setminus X$	x_1	x_2	$\ldots$
y_1	$p_{1,1}$	$p_{2,1}$	$\ldots$
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$$\mathbb{P}(Z = z_{ij}) = \sum_{g(x_k, y_m) = z_{ij}} \mathbb{P}(X = x_k, Y = y_m).$$

Transform of Joint Discrete R.V.s, Example

And also in this case, we can calculate the CDF of Z:

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Example: Assume X and Y are given by

$Y \setminus X$	-1	1
3	0.1	0.2
2	0.3	0.1
4	0.1	0.2

- Find the PMF of $Z = X + Y$;
- Find the CDF of $Z = 2X - Y$.

Transform of Joint Continuous R.V.s

Now assume X and Y are Continuous r.v.s with Joint PDF $f_{X,Y}(x, y)$. Also, $g : \mathbb{R}^2 \to \mathbb{R}$ is a given function. Let $Z = g(X, Y)$. Again we are interested in the Distribution of Z.

Transform of Joint Continuous R.V.s

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Note: Second, we can find the CDF of Z by:

$$F_Z(x) = \mathbb{P}(g(X, Y) \leq x) = \iint_{g(u,v) \leq x} f_{X,Y}(u, v)\, dudv.$$

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- If X and Y are **Jointly Continuous** with the Joint PDF $f_{X,Y}(x, y)$, then $Z = X + Y$ will be a Continuous r.v. with the PDF
$$f_{X+Y}(x) = \int_{-\infty}^{\infty} f_{X,Y}(t, x - t)\, dt \qquad \forall x \in \mathbb{R}.$$