## AUA CS108, Statistics, Fall 2020 Lecture 38

Lecture 36

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- Hypothesis Testing

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$$\left(\frac{\sum_{k=1}^{n}(X_{k}-\mu)^{2}}{\chi_{n,1-\frac{\alpha}{2}}^{2}}, \frac{\sum_{k=1}^{n}(X_{k}-\mu)^{2}}{\chi_{n,\frac{\alpha}{2}}^{2}}\right).$$

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Here  $\chi^2_{n,\beta}$  is the  $\beta$ -quantile of the  $\chi^2(n)$  Distribution.

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**Answer:** The following is an  $(1 - \alpha)$ -level CI for  $\sigma^2$ , when  $\mu$  is unknown:

$$\left(\frac{\sum_{k=1}^{n}(X_{k}-\overline{X})^{2}}{\chi_{n-1,1-\frac{\alpha}{2}}^{2}}, \frac{\sum_{k=1}^{n}(X_{k}-\overline{X})^{2}}{\chi_{n-1,\frac{\alpha}{2}}^{2}}\right).$$

Let me give the CI for  $\sigma^2$  again:

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Usually, you will see this in the following form:

$$\left(\frac{(n-1)\cdot S^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \frac{(n-1)\cdot S^2}{\chi^2_{n-1,\frac{\alpha}{2}}}\right),$$

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where S is the Sample Standard Deviation given by

$$S^{2} = \frac{\sum_{k=1}^{n} (X_{k} - X)^{2}}{n-1}.$$

**Example:** Assume we have scale of very high precision, and we want to measure the precision of our scale. Say, we weight a 50AMD coin, using that scale, 10 times, and we obtain the following measurements (in gramms):

[1] 3.449243 3.450802 3.453054 3.448778 3.452541 3.448835 3.448 [9] 3.450314 3.449047

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So now, using the above observations (weighting results), we will construct a 90% CI for  $\sigma^2$ .

Recall the  $(1 - \alpha)$ -level CI for  $\sigma^2$ :

$$\left(\frac{(n-1)\cdot S^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \frac{(n-1)\cdot S^2}{\chi^2_{n-1,\frac{\alpha}{2}}}\right).$$

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Now, we use **R** to do the rest: we want to construct 90% CI, so our  $\alpha = 0.1$ . We have 10 observations, so n = 10. We calculate  $S^2$ :

```
w <- c(3.449243, 3.450802, 3.453054, 3.448778, 3.452541, 3 s2 <- var(w)
```

s2

```
## [1] 4.605341e-06
```

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alpha <- 0.1
lq<-qchisq(alpha/2, df=9); uq<-qchisq(1-alpha/2, df=9)
c(lq,uq)</pre>
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Finally, we calculate our CI endpoints:

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n <- 10
c((n-1)*s2/uq, (n-1)*s2/lq)
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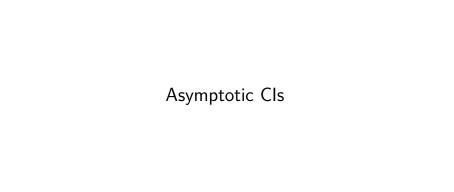
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**Note:** The actual value of sd I was using was: sd=0.002, so the true value of my  $\sigma^2$  was

$$\sigma^2 = 4 \cdot 10^{-6}$$
.



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**Asnwer:** The Random Interval (or, rather, the sequence of Intervals)

$$\left(\overline{X}_n - z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}}; \ \overline{X}_n + z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}}\right)$$

is a  $(1 - \alpha)$ -level Asymptotic CI for  $\mu$ .