CS 107 Section A - Probability

Spring 2020, AUA

Homework No. 10

Due time/date: 21 April, 2020

Note: Supplementary Problems will not be graded, but you are very advised to solve them and to discuss later with TA or Instructor.

Transformations of Random Variables

A Transformations, Transformations through CDFs

Problem 1. Which of the following r.v.s are not well-defined?

a. $Y = \sqrt{X+1}$, $X \sim Pois(1)$;

b. $Y = \frac{1}{X}$, $X \sim Unif([-1,1])$;

c. $Y = \frac{1}{X}$, $X \sim Bernoulli(0.2)$;

d. $Y = \ln(2-X)$, if X has the PDF $f(x) = C \cdot x^7$ if $x \in [0,1]$ and $f(x) = 0$, if $x \notin [0,1]$.

Problem 2. Assume X is a r.v. with the CDF $F_X(x)$. Express the CDF of the following r.v. in terms of F_X, if

a. $Y = 2X - 3$;

b. $Z = \ln(1 + e^X)$;

c. $T = |X|$.

B Transformations through PMFs (Functions of Discrete r.v.s)

Problem 3. Assume

$$X \sim \begin{pmatrix} -2 & 0 & 4 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}$$

Find the distribution of the following r.v.:

a. $Y = \dfrac{1}{X+1}$;

b. $Z = |X - 1|$.

Problem 4. Let $X \sim Pois(\lambda)$. Find the distribution of $Y = (X-1)^2$.

C Transformations through PDFs (Functions of Continuous r.v.s)

Problem 5. Assume $X \sim Unif[-2,2]$.

a. Find and plot the CDF of X;

b. Prove that $Y = \dfrac{X+2}{4}$ is a Standard Uniform r.v., i.e., $Y \sim Unif[0,1]$;

c. Find the PDF of $S = \sqrt{|X|}$.

Problem 6. Assume $X \sim Exp(3)$. Let

$$g(t) = \begin{cases} -2, & t < 1 \\ 7, & t \geq 1. \end{cases}$$

Find the distribution of $Y = g(X)$.

Problem 7. Assume $X \sim \mathcal{N}(0,1)$. Find the distribution of $Z = X^2$.

Note: The distribution of this Z is called the Chi-square distribution with 1 degrees of freedom, $\chi^2(1)$. The $\chi^2(n)$, $n \geq 1$, distribution is very important in Statistics.

Problem 8. Assume X is a r.v. with the following PDF:

$$f(x) = \begin{cases} C \cdot (1 - x^2), & x \in [-1,1]; \\ 0, & otherwise \end{cases}$$

Find the PDF of $Y = 1 - |X|$.

Problem 9. I am standing at the Baghramyan ave., and a car passes by the place I am standing at. I do not know the velocity of that car, but I can guess that it is 40 ± 5 km/h. I want to find the distance that that car will travel in 10 min after passing the place I am standing at. To that end, I assume that the velocity V of that car is constant, the car is doing a rectilinear (straight-line) motion, and V can be modeled as a Normal r.v., $V \sim \mathcal{N}(40, 5^2)$.

a. Let S be the r.v. measuring the distance from my standing point of that car after 10 min. Express S in terms of V;

b. What is the probability that $S > 8$km?

d. Describe the distribution of S, plot the PDF of S and give some explanation about the most possible places that car can be in 10 min.

Joint Distribution of r.v.s

D Joint CDF

Problem 10. Assume X and Y are Jointly Distributed r.v.s. Is it true, in general, that

$$\mathbb{P}(X > a, Y > b) = 1 - \mathbb{P}(X \leq a, Y \leq b) \ ?$$

Justify your answer.

E Joint PMF

Problem 11. Assume X, Y are jointly distributed discrete r.v.s with the following Joint PMF:

$Y \setminus X$	0	2	4
-1	0	0.1	0.1
0	0.2		0.15
1	0.2	0	0.15

a. Find the missing probability;
b. Find $\mathbb{P}(X > 2, Y < 1)$;
c. Find $\mathbb{P}(X \cdot Y > 0)$;
d. Find the PMF of X;
e. Find the PMF of Y;
f. Find the PMF of $X \cdot Y$.

Problem 12. Assume we have a box containing 5 white, 4 green and 7 black balls. We pick at random 3 balls. Let X be the number of white balls taken, and Y be the number of black balls taken.

a. Find the Joint PMF of X and Y, if the balls are taken with replacement, i.e., we return each time the taken ball into the box;
b. Calculate $\mathbb{P}(X \leq 2, Y \geq 2)$ and $\mathbb{P}(X - Y \leq 2)$;
c. (Supplementary) Find the Joint PMF of X and Y, and the above probabilities, if the balls are taken without replacement, i.e., we are not returning the taken ball into the box.

F Supplementary Problems

Problem 13. (Supplementary) Assume $X \sim Geom(p)$. Find the distribution of $Y = X \ (mod\ 3)$ (the reminder when dividing to 3).

Problem 14. (Supplementary) Assume $X \sim Pois(\lambda)$. Find the distribution of $Y = \cos(\pi \cdot X)$.

Problem 15. (Supplementary) Assume $X \sim Unif[-3, 3]$.

a. Describe the distribution of $T = \{X\}$, where $\{a\}$ means the Fractional Part of the number a.
b. Find a transformation (function) $g : \mathbb{R} \to \mathbb{R}$, such that the PDF of $W = g(X)$ will have the form
$$f_W(x) = \begin{cases} 4x^3, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$
c. Find a transformation (function) $g : \mathbb{R} \to \mathbb{R}$, such that $g(X) \sim Bernoulli(0.4)$.

Problem 16. (Supplementary) Assume $X \sim Exp(\lambda)$, and $Y = [X] + 1$. Show that Y is Geometric r.v., and find its parameter.

Problem 17. (Supplementary) Assume X is a continuous r.v. with PDF $f_X(x)$. Let $Y = \alpha X + \beta$ be a linear transform of X, $\alpha \neq 0$. Express the PDF of Y in terms of the PDF of X.

Problem 18. (Supplementary) Find a r.v. X and write a code to generate random numbers from it, if the PDF of X is

$$f(x) = \begin{cases} 0.2, & x \in [1,3] \\ 0.3, & x \in [5,6] \\ 0, & otherwise. \end{cases}$$

Check your program by drawing the relative frequency histogram and the theoretical pdf graph one over the other.