CS 107, Probability, Spring 2019
Lecture 19

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AUA

01 March 2019

Content

- CDF and its Properties
- Discrete r.v.

Laplace's Law of Succession

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LZ

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Laplace's answer was: $\dfrac{N+1}{N+2}$, where $N = 1,826,213$.

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- Having a Complete Information about a r.v. $=$ being able to calculate $\mathbb{P}(X \in A)$ for any $A \subset \mathbb{R}$;
- To be able to calculate $\mathbb{P}(X \in A)$ for any $A \subset \mathbb{R}$, it is enough to know the values of the CDF at any point:

$$F(x) = F_X(x) = \mathbb{P}(X \leq x), \qquad x \in \mathbb{R}.$$

Properties of the CDF

Assume $F(x)$ is the CDF of the r.v. X:

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Then there exists an Experiment with a Sample Space Ω, Probability Measure $\mathbb{P}$ and a r.v. $X : \Omega \to \mathbb{R}$ such that $F(x)$ is the CDF of X: $F(x) = F_X(x), \qquad x \in \mathbb{R}$.

Graphical Example of a CDF

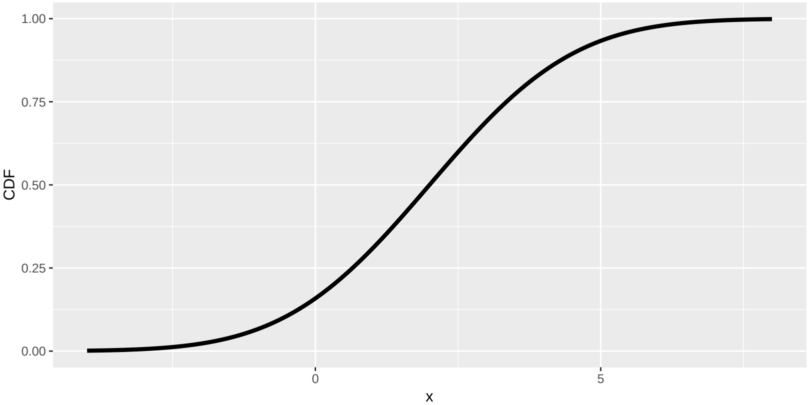

Figure: CDF of some r.v. X

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Here it is possible that $a = -\infty$ or/and $b = +\infty$

Reading Probabilities from the graph of CDF

Problem: Is the following a CDF of some r.v. X? If yes, calculate $\mathbb{P}(X = 2)$, $\mathbb{P}(2 < X \leq 3)$, $\mathbb{P}(X \leq 5)$, $\mathbb{P}(X > 2)$:

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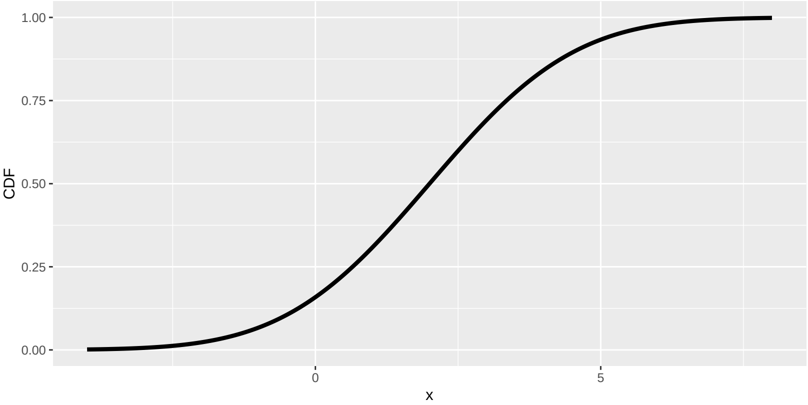

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Problem: Is the following a CDF of some r.v. X? If yes, calculate $\mathbb{P}(X = 1)$, $\mathbb{P}(0 < X \leq 3)$, $\mathbb{P}(X \leq 1)$, $\mathbb{P}(X > 0)$:

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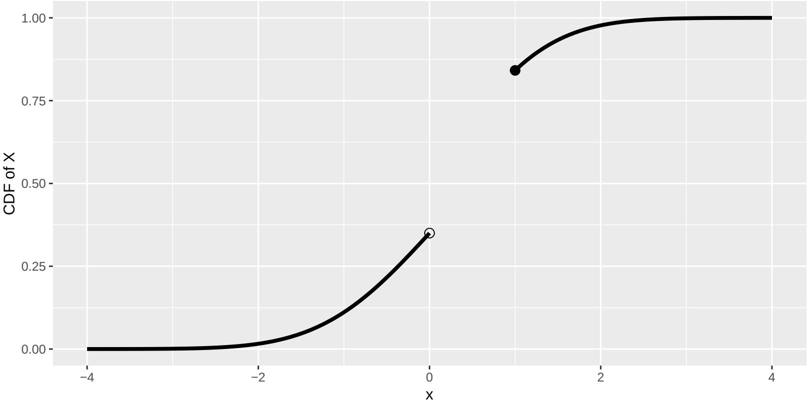

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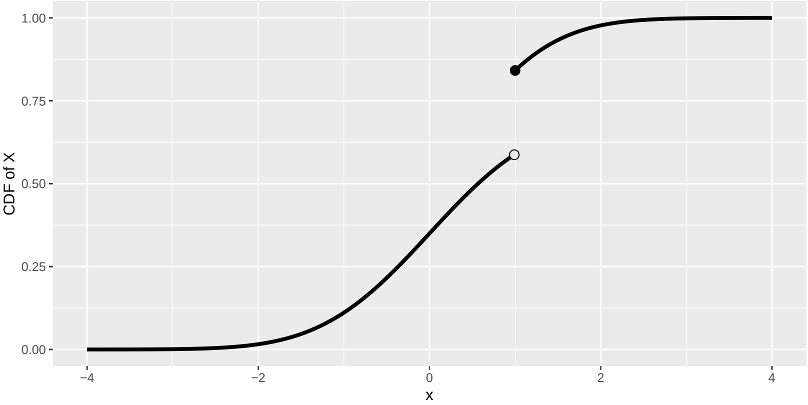

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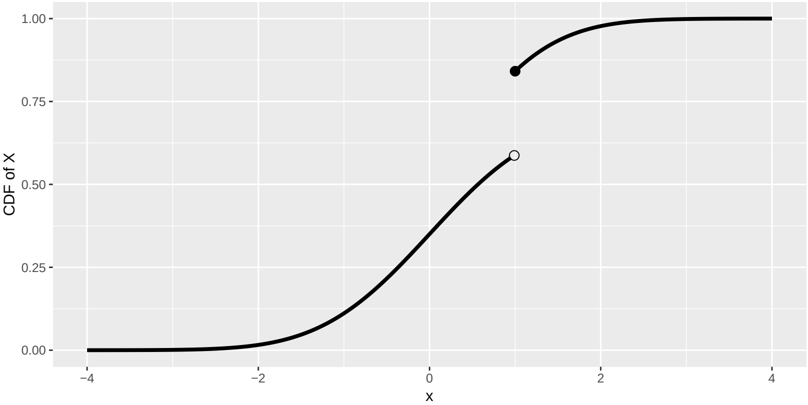

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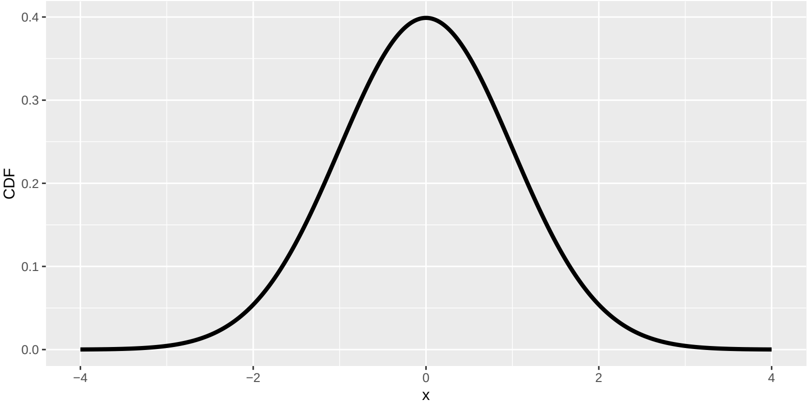

Figure: CDF of some r.v. X

Reading Probabilities from the graph of CDF

Problem: Below you can find graphs of 2 CDFs: Red is for r.v. X, and Black is for Y. Which one is larger: $\mathbb{P}(2 < X < 4)$ or $\mathbb{P}(2 < Y < 4)$?

Reading Probabilities from the graph of CDF

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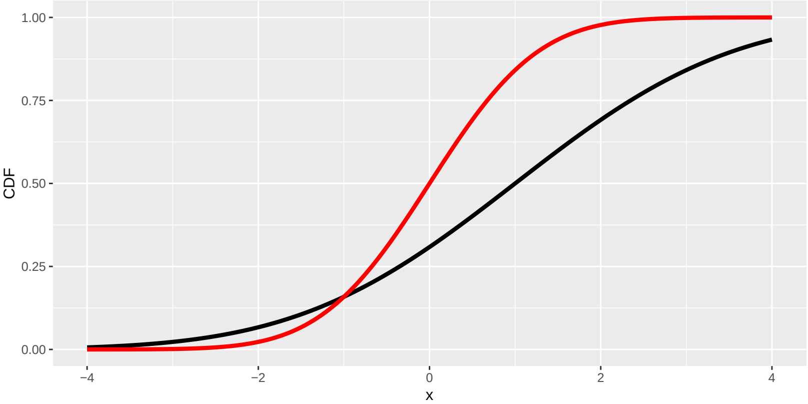

Figure: CDFs of r.v. X and Y

Discrete R.V.s

Discrete Random Variables

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We say that the r.v. X is **Discrete**, if the Range of X,

$$Range(X) = \{X(\omega) : \omega \in \Omega\} = \text{The set of all possible values of } X$$

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So if X is Discrete, then the Range of X can be written as

$$Range(X) = \{x_1, x_2, x_3, ...\},$$

where the set on the RHS[1] can be also finite.

[1] Right Hand Side

Example:

- Let X be the number of Heads when tossing 5 coins. What is the range of X? Is X discrete?

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- If Ω is Discrete, then X will be Discrete;
- If Ω is not Discrete, then X CAN BE Discrete.