AUA CS108, Statistics, Fall 2020

Lecture 12

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Now, let $\alpha \in (0, 1)$. We want to find a real number q_α dividing our (sorted) Dataset into the proportion $100\alpha\% - 100(1-\alpha)\%$, i.e., q_α is a point such that the α-portion of the Datapoints are to the left to q_α, and others are to the right.

Sample Quantiles

Let $x: \; x_1, x_2, ..., x_n$ be our 1D numerical Dataset. Assume also that $\alpha \in (0, 1)$.

Definition: The Quantile of order α (or $100\alpha\%$ order, the α-Quantile) of x is defined by

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Note: There are different definitions of the α-quantile in the literature and in software implementations. Say, **R** has 9 methods to calculate quantiles.

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Note: Sometimes Quantiles are called Percentiles.

Example

Example: Find the 20% and 60% quantiles of

$$x : -2, 3, 5, 7, 8, -3, 4, 5, 2$$

Solution: OTB

Example

Now, let us calculate Quantiles in **R**:

```r
x <- 1:15
quantile(x,0.21)
```

```
##  21%
## 3.94
```

```r
quantile(x, c(0.1,0.3,0.7))
```

```
##  10%  30%  70%
##  2.4  5.2 10.8
```

Theoretical Quantiles

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If F has a Density, $f(x)$, then q_α can be calculated from

$$\int_{-\infty}^{q_\alpha} f(x)dx = \alpha.$$

Theoretical Quantiles, Geometrically, by CDF

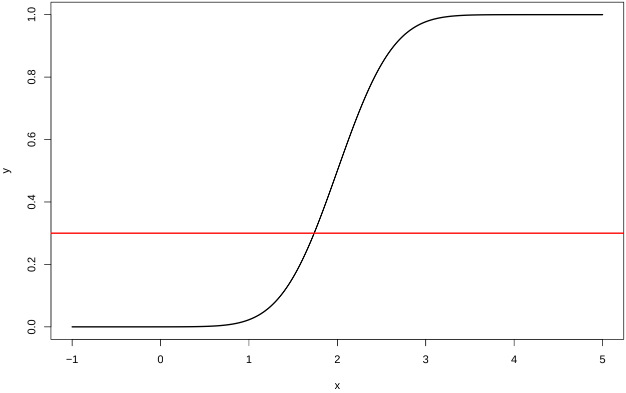
First we draw the CDF $y = F(x)$ graph, then draw the line $y = \alpha$.

Theoretical Quantiles, Geometrically, by CDF

First we draw the CDF $y = F(x)$ graph, then draw the line $y = \alpha$. Now, we keep the portion of the graph of $y = F(x)$ above (or on) the line $y = \alpha$. Then we take the leftmost point of the remaining part, and the x-coordinate of that point will be q_α.

Theoretical Quantiles, Geometrically, by CDF

```
alpha <- 0.3
x <- seq(-1,5, by = 0.01)
y <- pnorm(x, mean = 2, sd = 0.5)
plot(x,y, type = "l", xlim = c(-1,5), lwd = 2)
abline(h = alpha, lwd = 2, col = "red")
```



Theoretical Quantiles, Geometrically, by PDF

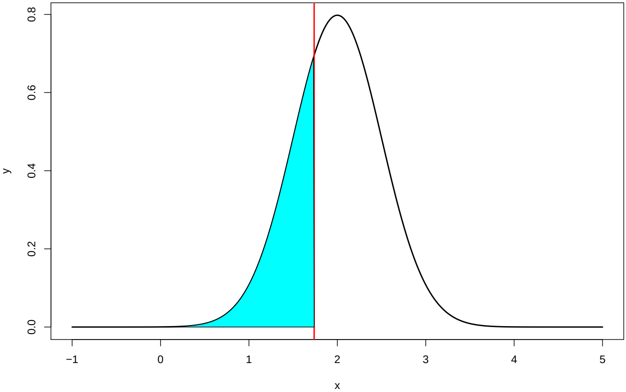
Now, assume our Distribution is continuous. We plot the graph of the PDF $y = f(x)$.

Theoretical Quantiles, Geometrically, by PDF

Now, assume our Distribution is continuous. We plot the graph of the PDF $y = f(x)$. We take q_α to be the smallest point such that the area under the PDF curve **left to** q_α is exactly α.

Theoretical Quantiles, Geometrically, by PDF

```
alpha <- 0.3; q.alpha <- qnorm(alpha, mean = 2, sd = 0.5)
x <- seq(-1,5, by = 0.01)
y <- dnorm(x, mean = 2, sd = 0.5)
plot(x,y, type = "l", xlim = c(-1,5), lwd = 2)
abline(v = q.alpha, lwd = 2, col = "red")
polygon(c(x[x<=q.alpha], q.alpha),c(y[x<=q.alpha],0),col="cyan")
```



Examples

Example: Find the 30% quantile of $\mathit{Unif}\,[3, 10]$

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Example: Find the 70% quantile of the Distribution with the PDF

$$f(x) = \begin{cases} 3x^2, & x \in [0, 1] \\ 0, & otherwise \end{cases}$$

Solution: OTB

Theoretical Quantiles, again

Now, if q_α is the α-quantile of some Distribution, and X is a r.v. from that Distribution, then

$$\mathbb{P}(X \leq q_\alpha) \geq \alpha \qquad \text{and} \qquad \mathbb{P}(X \geq q_\alpha) \geq 1 - \alpha.$$

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Note: Here we are taking inequalities, and not, say, $\mathbb{P}(X \leq q_\alpha) = \alpha$, since, in the Discrete r.v. case, we can have no q_α with exact equality. Say, if $X \sim \mathit{Bernoulli}(0.2)$, and $\alpha = 0.4$, then no q_α exists with $\mathbb{P}(X \leq q_\alpha) = \alpha$.

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Note: If $\alpha = 0.5$, we call $q_\alpha = q_{0.5}$ to be the **Median of the Distribution**.

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Note: If $\alpha = 0.5$, we call $q_\alpha = q_{0.5}$ to be the **Median of the Distribution**. So if we consider a Continuous r.v. and draw the PDF of that r.v., then the Median is the (leftmost) point dividing the area under the PDF curve into 50%-50% portions.