CS 107, Probability, Spring 2019
Lecture 04

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AUA

23 January 2019

Content

- Properties of the Probability Measure
- Classical Probability Models: Discrete Sample Spaces

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This is the famous Banach-Tarski Paradox. See `https://www.youtube.com/watch?v=s86-Z-CbaHA`

Properties of the Probability Measure (from our last lecture)

The Probability Measure satisfies the following properties:

1. $\mathbb{P}(\varnothing) = 0$;
2. If $A \cap B = \varnothing$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$;
3. $\mathbb{P}(\overline{A}) = 1 - \mathbb{P}(A)$ for any Event A;
4. If $A_i \cap A_j = \varnothing$ for $i \neq j$, then $\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \mathbb{P}(A_i)$;
5. for any events $A, B \in \mathcal{F}$,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B);$$

Properties of the Probability Measure

6. If $A_1, A_2, ..., A_n \in \mathcal{F}$ are events, not necessarily disjoint, then

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + ... + \mathbb{P}(A_n) -$$

$$-\mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_3) - ... - \mathbb{P}(A_{n-1} \cap A_n) +$$

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$$... + (-1)^{n-1}\mathbb{P}(A_1 \cap A_2 \cap ... \cap A_n).$$

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This is the general version of the previous property, and is called the inclusion-exclusion principle.

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10. for any event $A \in \mathcal{F}$,
$$0 \leq \mathbb{P}(A) \leq 1;$$

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$$Java = \text{Student knows Java programming}$$

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Then

$$\mathbb{P}(Java) = 0.5 \qquad \text{and} \qquad \mathbb{P}(C++) = 0.6.$$

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- Is it possible that no student with Java knowledge knows C++ programming? I.e., can $Java$ and $C++$ be mutually exclusive?

Example, cont'd

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- What is the probability that the student knows exactly one of these languages?

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- In our initial problem, what is the highest possible probability for a student to know Java and C++ programming?
- In our initial problem, what is the smallest possible probability for a student to know Java and C++ programming?

Classical Probability Models: Discrete Sample Spaces

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- The Probability Measure $\mathbb{P}$.

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- Every subset of Ω is an Event, i.e., $\mathcal{F} = 2^\Omega$, the power set of Ω.
- We take any real numbers $p_1, p_2, ..., p_n$ with

  $$p_1 \geq 0, ..., p_n \geq 0, \quad p_1 + p_2 + ... + p_n = 1,$$

  and define

  $$\mathbb{P}(\{\omega_1\}) = p_1, \quad \mathbb{P}(\{\omega_2\}) = p_2, \quad ..., \quad \mathbb{P}(\{\omega_n\}) = p_n.$$

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Outcome	ω_1	ω_2	...	ω_n
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- How many Probability Models we can have for the Experiment with the Sample Space $\Omega = \{a, b, c\}$? Describe them!