AUA CS108, Statistics, Fall 2020

Lecture 24

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Contents

Supplements, LLN

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$$\lim_{n \to +\infty} \frac{g(X_1) + g(X_2) + ... + g(X_n)}{n}$$

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To give the general idea of the CLT, let us use the following transform: for a r.v. X, let us denote

$$Z = Standardize(X) = \frac{X - \mathbb{E}(X)}{\sqrt{Var(X)}} = \frac{X - \mathbb{E}(X)}{SD(X)},$$

the Standardization (normalization, scaling) of X.

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$$\mathbb{E}(Z) = 0 \qquad \text{and} \qquad Var(Z) = 1.$$

Basic Idea of the CLT

The basic idea of the CLT is the following: if we have a sequence of IID r.v. X_n, and we consider their sum S_n or their average $\overline{X}_n$, then

$$Standardize(S_n) \xrightarrow{D} \mathcal{N}(0,1)$$

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Easy and beautiful, isn't it?

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The CLT states:

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{D} \mathcal{N}(0, 1).$$

Two forms of CLT

Of course, these two forms of the CLT are the same: we have

$$Standardize(S_n) = \frac{S_n - \mathbb{E}(S_n)}{\sqrt{Var(S_n)}} = \frac{S_n - n \cdot \mu}{\sqrt{n} \cdot \sigma}$$

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Now,

$$\frac{S_n - n \cdot \mu}{\sqrt{n} \cdot \sigma} = \frac{n \cdot (\frac{S_n}{n} - \mu)}{\sqrt{n} \cdot \sigma} = \frac{\frac{S_n}{n} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}},$$

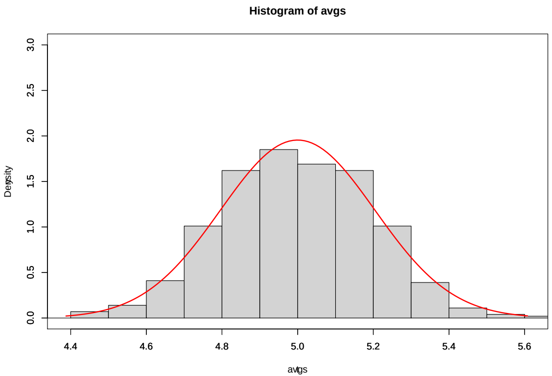
so

$$Standardize(S_n) = Standardize(\overline{X}_n).$$

Hence, the above two versions of CLT are the same, just one is in terms of S_n, the other one is in terms of $\overline{X}_n$.

CLT, Visually

```r
n <- 600 # Sample Size
m <- 1000 # no of Samples
rate <- 0.2
x <- rexp(n*m, rate = rate)
theo.mean <- 1/rate #theoretical mean
theo.sd <- 1/rate #theoretical SD
m <- matrix(x, ncol = m); d <- data.frame(m)
avgs <- sapply(d, mean)
a = theo.mean-3*theo.sd/sqrt(n); b = theo.mean+3*theo.sd/sqrt(n)
hist(avgs, freq = F, xlim = c(a, b), ylim=c(0,3))
par(new = T)
t <- seq(a,b, 0.01)
y <- dnorm(t, mean = theo.mean, sd = theo.sd/sqrt(n))
plot(t,y, type = "l", col="red", lwd = 2, xlim = c(a,b), ylim=c(0,3))
```



CLT, Visually, v2

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m <- matrix(x, ncol = m); d <- data.frame(m)
avgs <- sapply(d, mean)
qqnorm(avgs, ylab = "Averages"); qqline(avgs)
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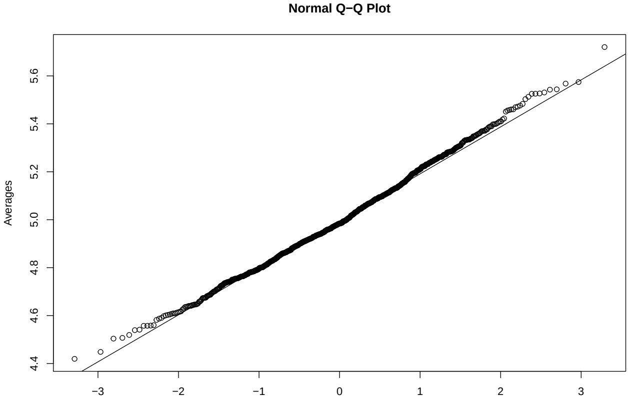

CLT, Roughly

In a non-rigorous way, we can write, for large n (here $\approx$ means approximately distributed as):

$$\frac{S_n - n \cdot \mu}{\sqrt{n} \cdot \sigma} \approx \mathcal{N}(0,1) \qquad \textit{and} \qquad \frac{\overline{X}_n - \mu}{\sigma / \sqrt{n}} \approx \mathcal{N}(0,1).$$

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or

$$S_n \approx \mathcal{N}(n\mu, n\sigma^2) \qquad \textit{and} \qquad \overline{X}_n \approx \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right).$$

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- If X_k-s are independent, have the Mean $\mathbb{E}(X_k) = \mu$ and $Var(X_K) = \sigma^2$, and **are Normally Distributed**, i.e., $X_k \sim \mathcal{N}(\mu, \sigma^2)$, then

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$$S_n \sim \mathcal{N}(n\mu, n\sigma^2) \qquad \textit{and} \qquad \overline{X}_n \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right);$$

so we know the **exact Distributions** of S_n and $\overline{X}_n$.

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- If X_k-s are independent, have the Mean $\mathbb{E}(X_k) = \mu$ and $Var(X_K) = \sigma^2$, and **from any Distribution** (but the same Distribution), then

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- If X_k-s are independent, have the Mean $\mathbb{E}(X_k) = \mu$ and $Var(X_K) = \sigma^2$, and **from any Distribution** (but the same Distribution), then

$$S_n \approx \mathcal{N}(n\mu, n\sigma^2) \qquad \text{and} \qquad \overline{X}_n \approx \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right);$$

  and we know the **asymptotic Distributions** (approximate Distributions for large n) of S_n and $\overline{X}_n$.