CS 107, Probability, Spring 2020
Lecture 33

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Content

- Independence of r.v.s

Independence of Random Variables

Independence of r.v.s, Intro

In our previous lectures, we have considered r.v.s X, and talked about their transform $Y = g(X)$. And, of course, having X and $Y = g(X)$, we can say for sure that Y depends explicitly on X, so X and Y are dependent. In this case, if I know the value of X, then I will know for sure the value for Y.
Now, if we have general Jointly distributed r.v.s X and Y, it can happen that, even, if I will know the value of X, Y can take different values, with different probabilities/distribution.

Independence of r.v.s, Intro

Say, if we consider the following Discrete case:

$Y \backslash X$	0	1
0	0.2	0.3
1	0.4	0.1

then, if $X = 0$, we can have both $Y = 0$ or $Y = 1$. Of course, in that case having $Y = 1$ is more probable than having $Y = 0$. So how we can measure if X and Y are dependent or not in this case?

Recall that, when talking about the Joint distribution of r.v.s X, Y, we noted that knowledge of their Joint PD(M)F is not only giving the individual (Marginal) distributions of X and Y, but also giving info about their relationship.

Independence of r.v.s, Intro

Pictorially,

We have the Joint Distribution of X and Y

$$=$$

We have the Distribution of X

$$+$$

We have the Distribution of Y

$$+$$

We have the Dependency/Relationship between X and Y

Independence of r.v.s, Intro

And also we have noted that having the individual distributions of X and Y is not enough to give their Joint Distribution, and, in particular, is not enough to calculate

$$\mathbb{P}\Big((X, Y) \in A\Big)$$

If we have the Distribution of X

$$+$$

We have the Distribution of Y

$$\neq$$

We have the Joint Distribution of X and Y.

Independence of r.v.s, Example

Example: Assume $X, Y \sim Bernoulli(0.5)$, defined on the same Sample Space, i.e.,

$$X, Y \sim \begin{pmatrix} 0 & 1 \\ 0.5 & 0.5 \end{pmatrix}.$$

Then we can have infinitely many Joint PMFs (Distributions) with these Marginals:

$Y \backslash X$	0	1
0	0.5	0
1	0	0.5

$Y \backslash X$	0	1
0	0	0.5
1	0.5	0

$Y \backslash X$	0	1
0	0.2	0.3
1	0.3	0.2

Independence of r.v.s, Intro

So, to completely describe the Joint PDF, we need to have

The Distribution of X

$$+$$

The Distribution of Y

$$+$$

The Dependency/Relationship between X **and** Y

$$=$$

The Joint Distribution of X **and** Y

Independence of r.v.s, Intro

In fact, describing the dependence, relationship between two (or more) r.v. is not an easy task at all. One needs to have the Joint Distribution to have a complete knowledge about the relationship between r.v.s, but one rarely have the Joint distribution in practice.

So usually people start from the basic type of dependence - very special case, but also very important case: when there is no dependence, when the r.v.s are independent. And in many studies people assume independence between measured quantities or approximate through independent quantities.

In this lecture we will give the notion of independence, and study properties of independent r.v.s, use independence to calculate probabilities. Later we will introduce two measures for dependency: covariance and correlation.

Independence of r.v.s, Intro

So here, in this lecture, we will consider Joint Distributions obtained in the following way:

The Distribution of X

$+$

The Distribution of Y

$+$

X **and** Y **are Independent**

$=$

The Joint Distribution of X and Y

But first some examples for getting the ideas.

Example: Assume the Joint PMF of X and Y is

$Y \backslash X$	0	1
7	0.3	0
9	0	0.7

What can be said about the relationship between X and Y? Well, if $X = 0$, then Y can be only 7, and if $X = 1$, then Y can be only 9. So this models the case when

$$X \sim \begin{pmatrix} 0 & 1 \\ 0.3 & 0.7 \end{pmatrix}$$

and Y is completely determined if knowing the values of X, say, we can write $Y = 2X + 7$ (or any other function of X, giving 7 and 9 at 0 and 1, respectively). So here Y depends on X, clearly.

Independence of r.v.s, Intro: Example

Example: Now, another pair of r.v.s: assume the Joint PMF of X and Y is

$Y \backslash X$	0	1	2
7	0.3	0.1	0.1
9	0.2	0.2	0.1

Let's think about the relationship between X and Y.

- If $X = 0$, then Y can be both 7 or 9, and the chances (odds) are $0.3 : 0.2 = 3 : 2$ for $Y = 7$ vs $Y = 9$.
- If $X = 1$, then again Y can be both 7 or 9, and the chances (odds) are $0.1 : 0.2 = 1 : 2$ for $Y = 7$ vs $Y = 9$.
- and, If $X = 2$, we will have the chances (odds) $0.1 : 0.1 = 1 : 1$ for $Y = 7$ vs $Y = 9$.

Independence of r.v.s, Intro: Example

Example: Now, another pair of r.v.s: assume the Joint PMF of X and Y is

$Y \backslash X$	0	1	2
7	0.3	0.1	0.1
9	0.2	0.2	0.1

So, say, if we will know the value of X that will change our information about the distribution of Y. For example, without any prior knowledge, $\mathbb{P}(Y = 7) = \mathbb{P}(Y = 9) = 0.5$, but if we learn that $X = 0$, then $\mathbb{P}(Y = 7|X = 0) = \frac{3}{5}$ and $\mathbb{P}(Y = 7|X = 0) = \frac{2}{5}$. This means that Y depends on X. Similarly, if we will fix the values of Y, and consider the corresponding distribution for X, we will see that X also depends on Y: X and Y are dependent.

Independence of r.v.s, Intro: Example

Example: Now, assume the Joint PMF of X and Y is

$Y \backslash X$	0	1	2
7	0.04	0.12	0.24
9	0.06	0.18	0.36

Let's think about the relationship between X and Y.

- If $X = 0$, then Y can be both 7 or 9, and the chances (odds) are $0.04 : 0.06 = 2 : 3$ for $Y = 7$ vs $Y = 9$.
- If $X = 1$, then again Y can be both 7 or 9, and the chances (odds) are $0.12 : 0.18 = 2 : 3$ for $Y = 7$ vs $Y = 9$.
- and, If $X = 2$, we will still have the same chances (odds) $0.24 : 0.36 = 2 : 3$ for $Y = 7$ vs $Y = 9$.

Independence of r.v.s, Intro: Example

Example: Now, assume the Joint PMF of X and Y is

$Y \backslash X$	0	1	2
7	0.04	0.12	0.24
9	0.06	0.18	0.36

Even without any prior knowledge about X, the distribution of Y is: $\mathbb{P}(Y = 7) = 0.4$ and $\mathbb{P}(Y = 9) = 0.6$. So, in fact, the information about (the value of) X is not making the chances of some values for Y to change. So Y doesn't depend on X. Similarly, knowing the value of Y will not give any new information about the distribution of X, so X and Y are independent.

Independence of Random Variables

Now, we will introduce the notion of Independence of r.v.s, using the ideas of Independence of Events.

Independence of Random Variables

Recall that for two Events A and B, we defined A and B to be independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B),$$

or, equivalently (for the case $\mathbb{P}(B) > 0$),

$$\mathbb{P}(A|B) = \mathbb{P}(A).$$

The idea was that knowledge about the Probability of happening one of A or B is not changing the Probability of happening of the other one. So if we knew that, say, the event A has a probability p, without any prior knowledge, then, after learning that B happened, we will still have the same probability p for A to happen.

Now, we define the Independence of 2 r.v. in a similar way.

Independence of Random Variables

Assume X and Y are two r.v. on the same Experiment:

Independence of R.V.

We say that r.v.s X and Y are (Statistically) **Independent**, and we write $X \perp\!\!\!\perp Y$ or $X \perp Y$, if

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A) \cdot \mathbb{P}(Y \in B)$$

for any $A, B \subset \mathbb{R}$, i.e., if the Events $\{X \in A\}$ and $\{Y \in B\}$ are Independent.

Equivalently, r.v.s X and Y are Independent, if

$$\mathbb{P}(X \in A \,|\, Y \in B) = \mathbb{P}(X \in A)$$

for all $A, B \subset \mathbb{R}$ such that $\mathbb{P}(Y \in B) \neq 0$.

The interpretation of Independence is that having information about Y (say, where is it lying), is not changing the distribution of X, is not saying anything new about X.

Independence notion is symmetric: we say X and Y are Independent, or X is independent of Y, or Y is independent of X interchangeably.

If X and Y are not Independent, then we say they are Dependent. And later we will talk about (linear) dependency measure, about covariance and corelation coefficient.

Describing the Independence of R.V.s

Checking independence using the above definition is practically impossible: we need to check the equality for any subsets $A, B \subset \mathbb{R}$. Fortunately, we have another way to do that: it can be proven that the followings are equivalent descriptions of independence:

- **In terms of CDFs, works for any r.v.s**

X and Y are Independent, $X \perp\!\!\!\perp Y$, if and only if

$$F_{X,Y}(x, y) = F_X(x) \cdot F_Y(y) \qquad \forall x, y \in \mathbb{R},$$

where $F_{X,Y}$, F_X, F_Y are the Joint CDF of X, Y and the Marginal CDFs of X and Y, respectively;

Describing the Independence of R.V.s

- **In terms of PMFs, for Discrete r.v.s**

If X and Y are Discrete, then X and Y are Independent if and only if

$$\mathbb{P}(X = x_i, Y = y_j) = \mathbb{P}(X = x_i) \cdot \mathbb{P}(Y = y_j) \qquad \forall i, j$$

- **In terms of PDFs, for Jointly Continuous r.v.s**

If X and Y are Jointly Continuous, then X and Y are Independent if and only if

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y) \qquad \text{for (almost) any } x, y \in \mathbb{R},$$

where $f_{X,Y}$, f_X, f_Y are the Joint PDF of X, Y and the Marginal PDFs of X and Y, respectively.

Describing the Independence of R.V.s

In short,

Independence means the Joint CDF/PDF/PMF is the product of individual (Marginal) CDF/PDF/PMF.

Now, if X and Y are Independent, and we are transforming them separately, then the results are also Independent:

Independence of Transformations

Assume X and Y are Independent, and $g, h : \mathbb{R} \to \mathbb{R}$ are any (measurable) functions. Then $g(X)$ and $h(Y)$ are Independent too.

Example: If X and Y are Independent, then

- $X^2 \perp\!\!\!\perp \cos(2Y + 1)$;
- $\sin(X) \perp\!\!\!\perp \ln\left(1 + \frac{1}{1+Y}\right)$, ...

Some Notes about Independence

Now, usually we deal with two types of problems concerning Independence:

Type 1: We know the Joint Distribution of X and Y, and we want to infer that X and Y are independent or dependent. Say, we know the Joint PDF of X and Y. Then we can study the Independence of X and Y by calculating the Marginal PDFs and checking that

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y), \qquad \forall x, y \in \mathbb{R}.$$

Some Notes about Independence

Type 2: We know, from the conditions of the problem, or from the very nature of X and Y, that they are Independent. Then we can construct the Joint Distribution of X and Y from the Distributions of X and Y separately and solve problems concerning the Joint Distribution of X, Y, say calculate $\mathbb{P}((X, Y) \in A)$. Say, if we will know that X and Y are independent (from the heart/nature of the problem, or because it is stated explicitly in the problem), and, say, if we will have their individual PDFs, we can find the Joint PDF of (X, Y):

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y), \qquad \forall x, y \in \mathbb{R}.$$

Independence of Discrete Random Variables

Independence of Discrete R.V.s

Assume X and Y are Discrete r.v.s with the PMFs

Values of X	x_1	x_2	$\dots$
$\mathbb{P}(X = x)$	p_1	p_2	$\dots$

Values of Y	y_1	y_2	$\dots$
$\mathbb{P}(Y = y)$	q_1	q_2	$\dots$

Then X and Y are independent, if and only if their Joint PMF has the following form:

$Y \backslash X$	x_1	x_2	$\dots$	$\mathbb{P}(Y = y)$
y_1	$p_1 \cdot q_1$	$p_2 \cdot q_1$	$\dots$	q_1
y_2	$p_1 \cdot q_2$	$p_2 \cdot q_2$	$\dots$	q_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\mathbb{P}(X = x)$	p_1	p_2	$\dots$	

Independence: Examples

Example 33.1: Assume X and Y are Discrete with the following PMFs:

Values of X	-2	0	1
$\mathbb{P}(X = x)$	0.1	0.6	0.3

Values of Y	10	20
$\mathbb{P}(Y = y)$	0.2	0.8

Are X and Y Independent?

Independence: Examples

Example 33.2: Assume X and Y are Discrete with the following PMFs:

Values of X	-2	0	1
$\mathbb{P}(X = x)$	0.1	0.6	0.3

Values of Y	10	20
$\mathbb{P}(Y = y)$	0.2	0.8

Assume also that $X \perp\!\!\!\perp Y$.

a. Find the Joint PMF of X and Y;

b. Calculate $\mathbb{P}(X \cdot Y < 10)$.

Independence: Examples

Example 33.3: Assume $X \sim Binom(2, 0.3)$, $Y \sim Pois(2)$ and assume $X \perp\!\!\!\perp Y$.

a. Construct the Distribution of (X, Y);
b. Find $\mathbb{P}(X + Y < 2)$;
c. Find the Marginal PMFs of X and Y. (Silly question, of course $\ddot\smile$)

Independence: Examples

Example 33.4: Assume we are rolling two dice, independently. Let X be 0, if the first die is showing less than 3, be 1, if it is showing 3, 4, or 5, and be 2, is the first die is showing 6. And let Y be -10, if the second die is showing an odd number, and let Y be 12, if it is showing an even number.

a. Construct the Distribution of (X, Y);
b. Find $\mathbb{P}(X \cdot Y < 10)$.

Independence: Examples

Example 33.5: Assume the PMF of (X, Y) is given by

$Y \backslash X$	1	4	6
-2	0.1	0.2	0.1
6	0.1	0	0.1
7	0.2	0.1	0.1

Are X and Y Independent?

Independence: Examples

Example 33.6: Assume the PMF of (X, Y) is given by

$Y \backslash X$	1	4	6
-2	0.12	0.02	0.06
6	0.24	0.04	0.12
7	0.24	0.04	0.12

Are X and Y Independent?

Independence: Examples

Example 33.7: Assume I am picking at random 2 numbers from $\{1, 2, 3, 4\}$, uniformly, independently. Let Max be the maximum of that numbers, and Min be the minimum of that numbers. Are Max and Min independent?

Independence of Continuous Random Variables

Independence: Examples

Example 33.8: Assume $X \sim Unif[0,1]$ and $Y \sim Unif[2,4]$. Are X and Y Independent?

Independence: Examples

Example 33.9: Assume $X, Y \sim \mathcal{N}(0, 1)$. Are X and Y Independent?

Independence: Examples

Example 33.10: Assume $(X, Y) \sim \mathit{Unif}(T)$, where T is the triangle with the vertices at $(0, 0)$, $(0, 1)$ and $(1, 0)$. Are X and Y Independent?

Independence: Examples

Example 33.11: Assume $(X, Y) \sim Unif(D)$, where

$$D = \{(x, y) : x^2 + y^2 \leq 1\}.$$

Are X and Y Independent?

Independence: Examples

Example 33.12: Assume $(X, Y) \sim Unif(D)$, where

$$D = [-2, 2] \times [0, 1].$$

Are X and Y Independent?

Independence: Examples

Example 33.13: Assume X and Y are the height and weight of a (randomly chosen) person. Are X and Y Independent?

Independence: Examples

Example 33.14: Assume $X \sim \mathit{Unif}[0,3]$, $Y \sim \mathit{Exp}(2)$ and $X \perp\!\!\!\perp Y$.

a. Find $\mathbb{P}(|X| + |Y| \leq 1)$;

b. Calculate the same probability by using Monte Carlo Simulations, in **R**.

Independence: Examples

Example 33.15: Assume the lifetime of a desktop comp HDD is $Exp(0.2)$ distributed r.v., and the lifetime of its CPU is $Exp(0.1)$ distributed r.v. (in years). Assume that that lifetimes are Independent.

a. What is the Probability that computer HDD and CPU will work more than 3 years?
b. What is the Probability that either computer HDD or CPU will work more than 3 years?

Independence of more than 2 r.v.s

Independence of more than 2 r.v.s

Like in the case of Events, we are defining two notions of Independence of n r.v.s $X_1, X_2, ..., X_n$: Pairwise and Mutual Independence (or just Independence) .

We say that $X_1, ..., X_n$ are Pairwise Independent, if for any $i \neq j$, r.v.s X_i and X_j are Independent.

We say that $X_1, ..., X_n$ are Mutually Independent (or just Independent), if for any subsets $A_k \subset \mathbb{R}$,

$$\mathbb{P}(X_1 \in A_1, ..., X_n \in A_n) = \mathbb{P}(X_1 \in A_1) \cdot ... \cdot \mathbb{P}(X_n \in A_n).$$

Note: Mutual Independence implies Pairwise Independence, but the inverse implication is not correct, in general.

Independence of more than 2 r.v.s

Note: Later, and in Statistics, we will use a lot the statement $X_1, ..., X_n$ *are Independent, Identically Distributed (IID) r.v.s*, meaning that X_k-s are Mutually Independent, and they are Identically Distributed.

Independence of more than 2 r.v.s

Now, if $X_1, X_2, ..., X_n$ are Discrete, then they will be Independent if and only if

$$\mathbb{P}(X_1 = x_1, X_2 = x_2, ..., X_n = x_n) = \mathbb{P}(X_1 = x_1) \cdot \mathbb{P}(X_2 = x_2) \cdot ... \cdot \mathbb{P}(X_n = x_n)$$

for any possible values of $x_1, x_2, ..., x_n$.
And if $X_1, X_2, ..., X_n$ are Jointly Continuous, then they will be Independent if and only if

$$f(x_1, x_2, ..., x_n) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) \cdot ... \cdot f_{X_n}(x_n).$$

for all $x_1, x_2, ..., x_n$, where f is the Joint PDF of $X_1, .., X_n$, and f_{X_k} is the Marginal PDF of X_k.

Independence: Examples

Example 33.16: Assume $X_1, X_2, ..., X_n$ are IID with a common CDF $F(x)$ and PDF $f(x)$.

a. Find the CDF and PDF of

$$U = \max\{X_1, X_2, ..., X_n\}$$

and

$$V = \min\{X_1, X_2, ..., X_n\}$$

b. Do **R** simulations to visualize the result.

Note: These two r.v.s are important in statistics and applications, and they are denoted by $U = X_{(1)}$ and $V = X_{(n)}$, and are called 1-st and n-th Order Statistics for $X_1, ..., X_n$.

Distribution of the Sum of Independent R.V.s

Distribution of the Sum of Independent R.V.s

Here we want to go back to the Distribution of the sum of r.v.s, but in the case when our r.v.s are Independent.
So now, assume $Z = X + Y$, and assume that $X \perp\!\!\!\perp Y$.

- If X and Y are **Discrete** and Independent, then

$$\mathbb{P}(Z = z) = \mathbb{P}(X + Y = z) = \sum_{x_i + y_j = z} \mathbb{P}(X = x_i, Y = y_j) =$$

$$= \sum_{x_i + y_j = z} \mathbb{P}(X = x_i) \cdot \mathbb{P}(Y = y_j) = \sum_{x_i} \mathbb{P}(X = x_i) \cdot \mathbb{P}(Y = z - x_i).$$

Note: This sum is called the Convolution of the sequences $\mathbb{P}(X = x_i)$ and $\mathbb{P}(Y = y_j)$.

Distribution of the Sum of Independent R.V.s

Again, $Z = X + Y$, and we have $X \perp\!\!\!\perp Y$.

- If X and Y are **Jointly Continuous** with the Joint PDF $f_{X,Y}(x,y)$ and Independent, then, for any $x \in \mathbb{R}$,

$$f_Z(z) = f_{X+Y}(z) = \int_{-\infty}^{\infty} f_{X,Y}(t, z-t)\,dt = \int_{-\infty}^{\infty} f_X(t)\cdot f_Y(z-t)\,dt.$$

Note: This integral is called the Convolution of the functions $f_X(x)$ and $f_Y(y)$.

Independence: Examples

Example 33.17: Assume $X \sim Bernoulli(0.3)$, $Y \sim Bernoulli(0.5)$ and $X \perp\!\!\!\perp Y$. Find the distribution of $Z = X + Y$.

Example 33.18:

a. Assume $X \sim Bernoulli(p)$, $Y \sim Bernoulli(p)$ and $X \perp\!\!\!\perp Y$. Find the distribution of $Z = X + Y$.

b. Assume $X_1, X_2, ..., X_n$ are IID with $X_k \sim Bernoulli(p)$. Find the Distribution of

$$X = X_1 + X_2 + ... + X_n.$$

Independence: Examples

Example 33.19: Assume $X \sim Binom(n, p)$, $Y \sim Binom(m, p)$ and $X \perp\!\!\!\perp Y$. Find the Distribution of $Z = X + Y$.

Independence: Examples

Example 33.20: Assume $X \sim Pois(\lambda_1)$, $Y \sim Pois(\lambda_2)$ and $X \perp\!\!\!\perp Y$. Find the Distribution of $Z = X + Y$.

Independence: Examples

Example 33.21: Assume $X, Y \sim Unif[0, 1]$ and $X \perp\!\!\!\perp Y$.

a. Find the distribution of $Z = X + Y$.
b. Obtain the shape of that distribution in **R**.

Independence: Examples

Example 33.22: Assume $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$ and $X \perp\!\!\!\perp Y$. Prove that

$$X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2).$$

Independence And Bivariate Normal

Assume

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \right)$$

Then

X and Y are Independent if and only if $\sigma_{1,2} = \sigma_{2,1} = 0$.

On Independence

Exercise: Think about the following: assume X is a r.v..

a. Is it true that X is dependent on X?

b. Is it true that X is dependent on $g(X)$?