AUA CS108, Statistics, Fall 2020

Lecture 38

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Contents

- Confidence Intervals
- Hypothesis Testing

CI for σ^2, $\mathcal{N}(\mu, \sigma^2)$ Model, μ is **known**

Problem: Assume we have a Random Sample

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Answer: The $(1-\alpha)$-level CI for σ^2, when μ is known, is

$$\left(\frac{\sum_{k=1}^{n}(X_k-\mu)^2}{\chi^2_{n,1-\frac{\alpha}{2}}}, \quad \frac{\sum_{k=1}^{n}(X_k-\mu)^2}{\chi^2_{n,\frac{\alpha}{2}}} \right).$$

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Here $\chi^2_{n,\beta}$ is the β-quantile of the $\chi^2(n)$ Distribution.

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Answer: The following is an $(1 - \alpha)$-level CI for σ^2, when μ is unknown:

$$\left(\frac{\sum_{k=1}^{n} (X_k - \overline{X})^2}{\chi^2_{n-1, 1-\frac{\alpha}{2}}}, \quad \frac{\sum_{k=1}^{n} (X_k - \overline{X})^2}{\chi^2_{n-1, \frac{\alpha}{2}}} \right).$$

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Let me give the CI for σ^2 again:

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Usually, you will see this in the following form:

$$\left(\frac{(n-1)\cdot S^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \quad \frac{(n-1)\cdot S^2}{\chi^2_{n-1,\frac{\alpha}{2}}} \right),$$

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where S is the Sample Standard Deviation given by

$$S^2 = \frac{\sum_{k=1}^n (X_k - \overline{X})^2}{n-1}.$$

Example

Example: Assume we have scale of very high precision, and we want to measure the precision of our scale. Say, we weight a 50AMD coin, using that scale, 10 times, and we obtain the following measurements (in gramms):

```
[1] 3.449243 3.450802 3.453054 3.448778 3.452541 3.448835 3.448
[9] 3.450314 3.449047
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So now, using the above observations (weighting results), we will construct a 90% CI for σ^2.

Example, Cont'd

Recall the $(1-\alpha)$-level CI for σ^2:

$$\left(\frac{(n-1)\cdot S^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}, \quad \frac{(n-1)\cdot S^2}{\chi^2_{n-1,\frac{\alpha}{2}}} \right).$$

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Now, we use **R** to do the rest: we want to construct 90% CI, so our $\alpha = 0.1$. We have 10 observations, so $n = 10$. We calculate S^2:

```
w <- c(3.449243, 3.450802, 3.453054, 3.448778, 3.452541, 3.
s2 <- var(w)
s2
```

```
## [1] 4.605341e-06
```

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alpha <- 0.1
lq<-qchisq(alpha/2, df=9); uq<-qchisq(1-alpha/2, df=9)
c(lq,uq)
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Finally, we calculate our CI endpoints:

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n <- 10
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Note: The actual value of `sd` I was using was: $sd=0.002$, so the true value of my σ^2 was

$$\sigma^2 = 4 \cdot 10^{-6}.$$

Asymptotic CIs

Asymptotic CI for the Mean of General Distribution

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Asnwer: The Random Interval (or, rather, the sequence of Intervals)

$$\left(\overline{X}_n - z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}};\ \ \overline{X}_n + z_{1-\alpha/2} \cdot \frac{S_n}{\sqrt{n}}\right)$$

is a $(1-\alpha)$-level Asymptotic CI for μ.