# CS 107 Section A - Probability

## Spring 2020, AUA

# Homework No. 06

Due time/date: 23 March, 2020

**Note:** Supplementary Problems will not be graded, but you are very advised to solve them and to discuss later with TA or Instructor.

## A Random Variables

**Problem 1.** For each of the following Random Experiments, construct two different random variables - one with discrete values (finite or countably infinite) and the other one - non-discrete, if that is possible, defined on the Sample Space of that Experiment.

a. The experiment is tossing 3 fair coins;
b. The experiment is choosing at random a participant of our Probability course;
c. The experiment is choosing at random two participants of our Probability course;
d. The experiment is to consider the weather tomorrow;
e. The experiment is to consider the next football game of our national team.

**Problem 2.** Give examples of 5 different Random Variables concerning various aspects of our real life (say, $X$, the amount of tea, in ml, I will drink today, is a Random Variable, or $Y$, the no. of times I will visit the bathroom today, is another Random Variable ⌣̈).

**Problem 3.** Assume I am picking a card from a deck at random. Which of the following will be a Random Variable? Explain.

a. $X$ = the color of the card;
b. $Y$ = the suit of the card;
c. $Z$ = the denomination of the card (we assume Ace = 1, Jack = 11, Queen = 12, King = 13);
d. $T = -10$, if the card is Spades , $T = 0$, if Hearts, and $T = 7$ otherwise;
e. $U = T^2 + 22$;
f. $W =$ the blue component in the RGB form of $X$;
g. $K =$ the weight of the card.

## B Constructing CDFs

**Problem 4.** Assume I am tossing a biased coin, and let $X$ be $-3$, if Heads appears, and 30 otherwise (you can think as I will pay 3$ if Heads appears, and will get 30, if Tails will be shown). The odds are 3:1 for Heads.

a. Find the Probabilities of Heads and Tails;
b. Write down the values of $X$ and corresponding Probabilities in a table form;
c. Construct the CDF of $X$ analytically;
d. Draw the graph of that CDF.

**Problem 5.** Assume $X$ is my waiting time in the AUA cafeteria line for the day we will be back to our in-a-classroom classes again. As a rule, I am waiting no more than 10min, and $X$ is uniformly distributed. We can model this mathematically as: we are picking at random point $\omega$ from $\Omega = [0, 10]$, uniformly, and let $X$ be a r.v. defined by

$$X(\omega) = \omega, \qquad \forall \omega \in \Omega = [0, 10].$$

a. Find the CDF $F$ of $X$, analytically;
b. Draw the graph of $F$;
c. Calculate $\mathbb{P}(X > 7)$ by using the CDF;
d. Assume now I am interested in another r.v.,

$$Y(\omega) = \omega^2, \qquad \omega \in \Omega,$$

which measures my hungriness (the more I am waiting, the hungrier I am getting $\ddot\smile$). Find the CDF of $Y$, and calculate the Probability that $Y \leq 50$;

e. (Supplementary) Find the CDF of $Z(\omega) = 50 - \omega^4$.

**Problem 6.** Assume we are playing the Darts game. The board is a disk with a radius of 50 cm (we assume that we will hit the board for sure at some place). There are 2 smaller concentric circles inside that disk, with the same center as the board itself. The radius of the inner, smallest circle, is 10 cm, and the radius of the middle circle is 30cm. If we will hit the inner part of the smallest circle (the bull's eye), we will get 100$. If we will hit between the smallest and middle circles, then we will get 50$, otherwise we will get 10$. Let $X$ be our winning in one throw.

a. For each possible value of the r.v. $X$, calculate the Probability of taking that value;
b. Find the CDF of $X$ in an analytical form;
c. Draw the graph of that CDF.

**Problem 7.** Assume that 80% of some shoe shop visitors are females. The chance that a female visitor will buy a pair of a shoe is 70%, and 3 out of 8 male visitors are buying a pair of a shoe in that shop (for simplicity, we assume a person will not buy more than a pair). Now, assume that during a quarter of an hour, only 3 persons are visiting that shoe shop, and let $X$ be the number of pairs of a shoe bought during that time interval (so $X$ can be anything from $\{0, 1, 2, 3\}$).

a. Find the probability that a visitor will buy a pair of a shoe;
b. For each $k \in \{0, 1, 2, 3\}$, calculate the Probability of $X = k$, and represent in the form of a table;
c. Find the CDF of $X$ in an analytical form;
d. Draw the graph of that CDF.

**Problem 8.** Let us construct three different r.v.s with the same CDF. We consider two different Experiments: the first one is tossing a fair coin. We take $X = 0$, if Heads appears, and $X = 1$ otherwise. Our next Experiment is picking a point at random from $\Omega = [0, 1]$. Let $Y$ and $Z$ be the following r.v.s on this Experiment:

$$Y = \begin{cases} 0, & if\ \omega \in [0, 0.5] \\ 1, & otherwise \end{cases} \qquad Z = \begin{cases} 0, & if\ \omega \in [0, 0.25] \cup [0.5, 0.75] \\ 1, & otherwise \end{cases}$$

Clearly, $Y$ and $Z$ are defined on the same Sample Space, Experiment, but $Y \not\equiv Z$. As to $X$, it is defined on completely other Sample Space, $\Omega = \{H, T\}$, so even we cannot check if $X = Y$ or not.

a. Write the values of $X, Y, Z$ and corresponding Probabilities in the Table form;
b. Find the CDFs of $X, Y, Z$ and check that they are coinciding.

**Remark:** This example is to stress that CDF is giving *how the values of a r.v. are distributed, which values are more probable, which are not*, but is not giving in which case the r.v. will take the particular value, is not giving the Sample Space. So you cannot recover exactly the r.v. from its CDF, you cannot give it in the analytical form $X(\omega) = ....$ The only thing you can, using the CDF, is to calculate Probabilities of the form $\mathbb{P}(X \in A)$, $A \subset \mathbb{R}$.

## C Properties of CDFs

**Problem 9.** Which of the following functions will be the CDF of some Random Variable $Y$? For those, which are CDFs, calculate the Probability $\mathbb{P}(-0.5 < Y < 0.7)$:

a. $F(x) \equiv 1$;
b. $F(x) = e^{-x}$;
c. $F(x) = \ln(1 + e^{-x})$;
d. $F(x) = 0$, if $x < -2$, and $F(x) = 1$, if $x \geq -2$;
e. $F(x) = 0$, if $x < 0$, $F(x) = x$, if $x \in [0, 1]$ and $F(x) = 1$, if $x > 1$.

**Problem 10.** Assume

$$F(x) = \begin{cases} 0, & x < -3; \\ 0.2, & -3 \leq x < 2; \\ 1, & x \geq 2. \end{cases} \qquad x \in \mathbb{R}.$$

a. Plot the graph of $F$;

b. Show that $F$ is the CDF of some r.v. $X$;
c. Calculate $\mathbb{P}(X=0)$;
d. Calculate $\mathbb{P}(X<0)$;
e. Calculate $\mathbb{P}(X=2)$;
f. Calculate $\mathbb{P}(X<2)$;
g. Calculate $\mathbb{P}(-3<X<3)$;
h. Calculate $\mathbb{P}(0<X<1)$;
i. Calculate $\mathbb{P}(X>3)$;
j. Describe the r.v. $X$.

**Problem 11.** We are given a function $F(x)$ by its graph, see Fig. 1.

a. Is this a legal CDF for some r.v. $X$? Explain. If yes, continue to next tasks, otherwise go to the next problem ◡̈
b. Calculate the probability $\mathbb{P}(X \leq 2)$.
c. Calculate the probability $\mathbb{P}(1.2 \leq X \leq 3)$.
d. What is the range of $X$ (here by the range of $X$ I mean the smallest closed set $A$ such that $\mathbb{P}(X \in A) = 1$, that is, $\mathbb{P}(X \in \overline{A}) = 0$)?

   **Note:** You do not need to prove that the set you are specifying is the smallest one. Just give your intuition behind your choice.
e. Calculate the probability $\mathbb{P}(X \in \{0, 1, 3, 4.5\})$.
f. Calculate the probability $\mathbb{P}(X \in [3, 5])$.
g. Calculate the probability $\mathbb{P}(X \in (0, 1))$.
h. Which is more probable: $X > 4$ or $X < 0$?

**Problem 12.** Fig. 2 shows the CDF of the r.v. $X$.

a. Calculate $\mathbb{P}(X = 1.5)$;
b. Calculate, as accurate as possible, the probability $\mathbb{P}(X \leq 5)$.
c. Calculate, as accurate as possible, the probability $\mathbb{P}(X < 5)$.
d. Calculate, as accurate as possible, the probability $\mathbb{P}(X \geq 11)$.
e. Which is more probable: $X \in [-5, 0]$ or $X \in [1, 2]$?

**Problem 13.** Assume $F(x)$ is the CDF of $X$, and assume also that $F$ is continuous everywhere, $F \in C(\mathbb{R})$. Express, in the terms of $F$, the following Probabilities:

a. $\mathbb{P}(X = x_0)$;
b. $\mathbb{P}(X \leq x_0)$;
c. $\mathbb{P}(X < x_0)$;
d. $\mathbb{P}(a < X < b)$;
e. $\mathbb{P}(a \leq X \leq b)$;
f. $\mathbb{P}(X \in [a, b] \cup [c, d])$, with $a < b < c < d$;
g. $\mathbb{P}(X \geq x_0)$;
h. $\mathbb{P}(|X| \geq a)$, where $a > 0$.

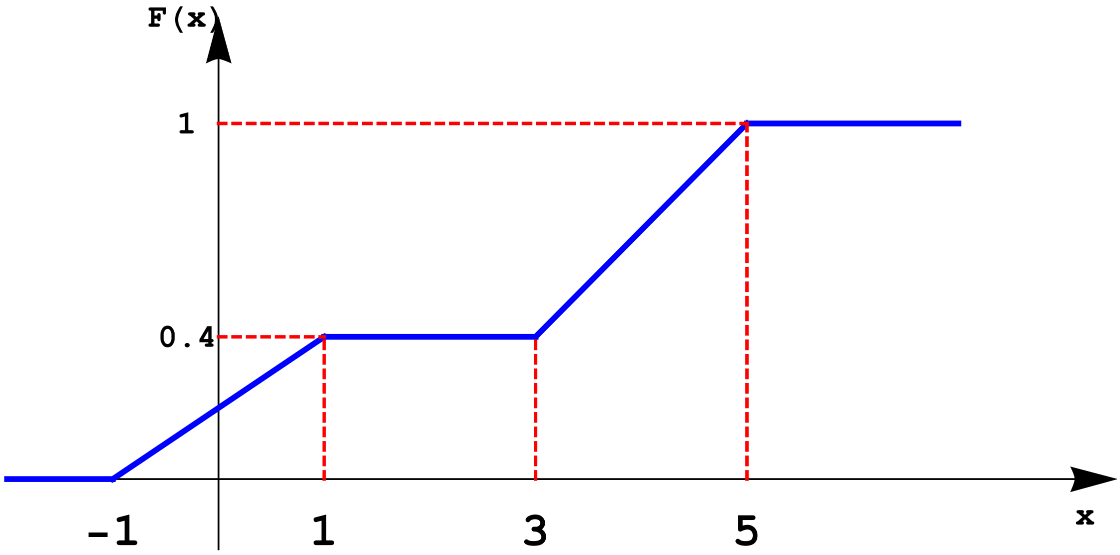


Figure 1: The graph of $F(x)$

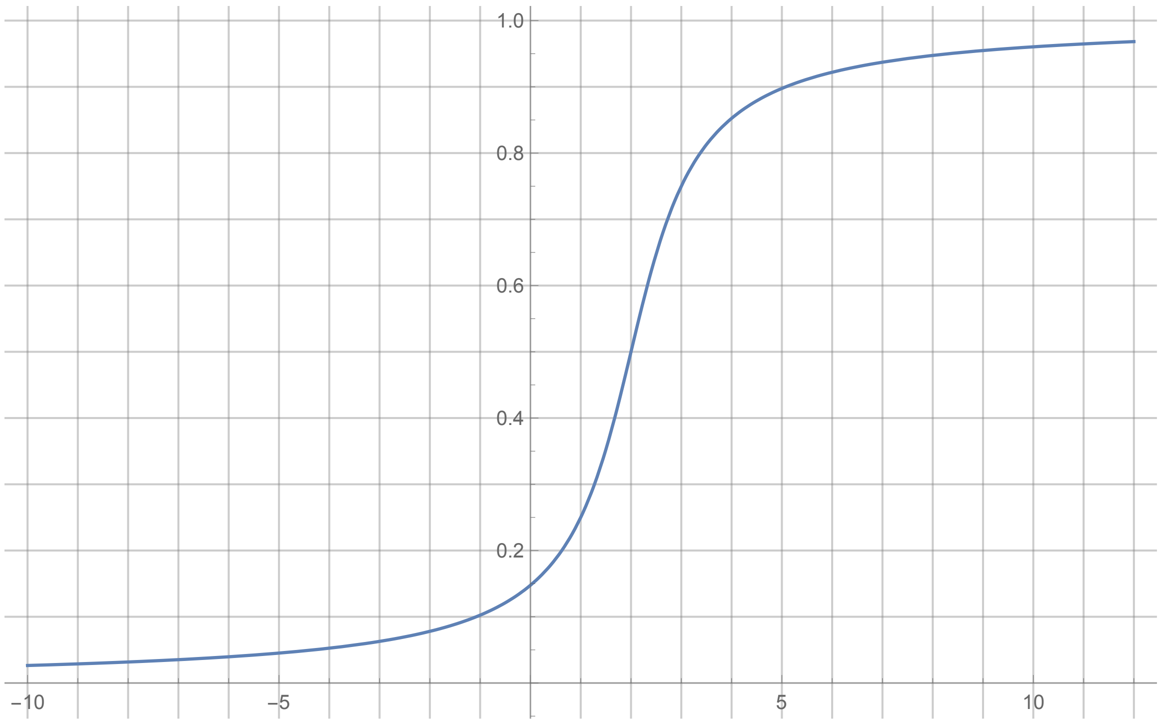


Figure 2: The graph of $F(x)$

## D Supplementary Problems

**Problem 14.** (Supplementary) Assume $F$ is the CDF of some r.v. $X$, and assume $x_1, x_2, ..., x_n$ are all discontinuity points of $F$. Prove that

$$\sum_{k=1}^{n} \Big( F(x_k) - F(x_k-) \Big) \leq 1.$$

**Problem 15.** (Supplementary) Assume $F$ is the CDF of some r.v. $X$. Let $DP$ be the set of Discontinuity Points of $F$. Prove that $DP$ is at most countably infinite (i.e., either it is empty, or finite, or countably infinite).

**Problem 16.** (Supplementary) Assume we have two r.v.s on $\Omega$, and let the CDF of $X$ be $F_X$, and the CDF of $Y$ be $F_Y$. We consider another r.v. $Z$, the mixture of $X$ and $Y$, in the following way: we toss a biased coin, with the Heads Probability equal to $p \in [0,1]$. If Heads appears, then $Z(\omega) = X(\omega)$, for any $\omega$, otherwise, $Z(\omega) = Y(\omega)$ for any $\omega$. Express the CDF of $Z$ in terms of the CDFs of $X$ and $Y$.