AUA CS 108, Statistics, Fall 2019

Lecture 10

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Contents

- Sample Quantiles
- Theoretical Quantiles
- Q-Q Plots

Quiz

Quiz 01: For Density Histogram, **divide relative frequencies by the length of the corresponding interval!**

Last Lecture ReCap

- What is a BoxPlot?

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- What are the Quantiles?
- What is the difference between Quartiles and Quantiles?

Sample Quantiles, ReCap

Let $x: \ x_1, x_2, ..., x_n$ be our 1D numerical Dataset. Assume also that $\alpha \in (0, 1)$.

Definition: The Quantile of order α (or $100\alpha\%$ order, the α-Quantile) of x is defined by

$$q_\alpha = q_\alpha^x = x_{([\alpha \cdot n])}.$$

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Note: There are different definitions of the α-quantile in the literature and in software implementations. Say, **R** has 9 methods to calculate quantiles.

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Note: Sometimes Quantiles are called Percentiles.

Example

Example: Find the 20% and 60% quantiles of

$$x : -2, 3, 5, 7, 8, -3, 4, 5, 2$$

Solution: OTB

Example

Now, let us calculate Quantiles in **R**:

```r
x <- 1:15
quantile(x,0.21)
```

```
##  21%
## 3.94
```

```r
quantile(x, c(0.1,0.3,0.7))
```

```
##  10%  30%  70%
##  2.4  5.2 10.8
```

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If F has a Density, $f(x)$, then q_α can be calculated from

$$\int_{-\infty}^{q_\alpha} f(x)dx = \alpha.$$

Theoretical Quantiles, Geometrically, by CDF

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Theoretical Quantiles, Geometrically, by CDF

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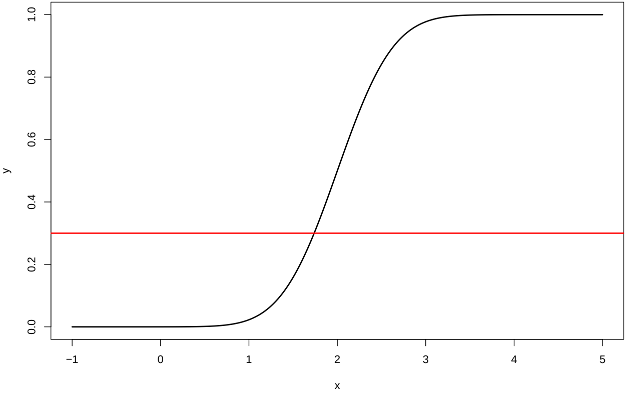
Theoretical Quantiles, Geometrically, by CDF

```
alpha <- 0.3
x <- seq(-1,5, by = 0.01)
y <- pnorm(x, mean = 2, sd = 0.5)
plot(x,y, type = "l", xlim = c(-1,5), lwd = 2)
abline(h = alpha, lwd = 2, col = "red")
```



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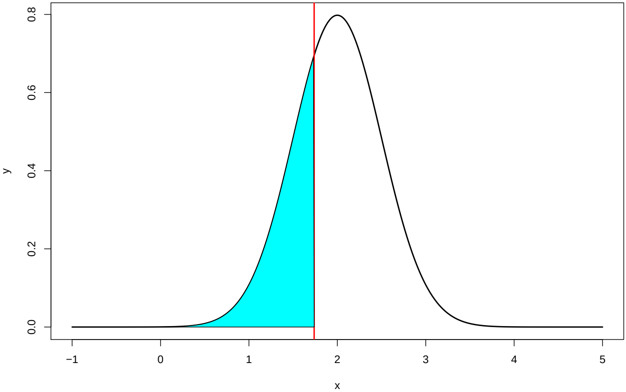
Now, assume our Distribution is continuous. We plot the graph of the PDF $y = f(x)$.

Theoretical Quantiles, Geometrically, by PDF

Now, assume our Distribution is continuous. We plot the graph of the PDF $y = f(x)$. We take q_α to be the smallest point such that the area under the PDF curve **left to** q_α is exactly α.

Theoretical Quantiles, Geometrically, by PDF

```
alpha <- 0.3; q.alpha <- qnorm(alpha, mean = 2, sd = 0.5)
x <- seq(-1,5, by = 0.01)
y <- dnorm(x, mean = 2, sd = 0.5)
plot(x,y, type = "l", xlim = c(-1,5), lwd = 2)
abline(v = q.alpha, lwd = 2, col = "red")
polygon(c(x[x<=q.alpha], q.alpha),c(y[x<=q.alpha],0),col="cyan")
```



Theoretical Quantiles, again

Now, if q_α is the α-quantile of some Distribution, and X is a r.v. from that Distribution, then

$$\mathbb{P}(X \leq q_\alpha) \geq \alpha \qquad \text{and} \qquad \mathbb{P}(X \geq q_\alpha) \geq 1 - \alpha.$$

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Note: Here we are taking inequalities, and not, say, $\mathbb{P}(X \leq q_\alpha) = \alpha$, since, in the Discrete r.v. case, we can have no q_α with exact equality. Say, if $X \sim \mathit{Bernoulli}(0.2)$, and $\alpha = 0.4$, then no q_α exists with $\mathbb{P}(X \leq q_\alpha) = \alpha$.

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Note: If $\alpha = 0.5$, we call $q_\alpha = q_{0.5}$ to be the **Median of the Distribution**. So if we consider a Continuous r.v. and draw the PDF of that r.v., then the Median is the (leftmost) point dividing the area under the PDF curve into 50%-50% portions.

Theoretical Quantiles, again

Later we will use a lot quantiles. When constructing Confidence Intervals or Hypothesis Testing, we will use Quantiles of the Normal Distribution, t-Distribution, χ^2-Distribution.

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Say, we will take $\alpha \in (0,1)$ and find two points $a, b \in \mathbb{R}$ such that for $X \sim \mathcal{N}(0,1)$

$$\mathbb{P}(X \leq a) = \mathbb{P}(X \geq b) = \frac{\alpha}{2}.$$

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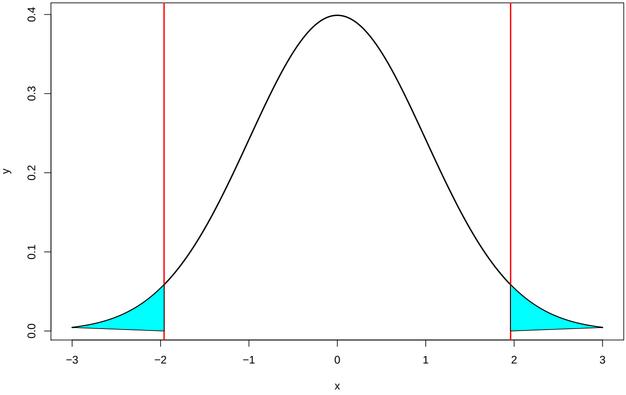
$$\mathbb{P}(X \le a) = \mathbb{P}(X \ge b) = \frac{\alpha}{2}.$$

The idea is to find a symmetric (in fact, the smallest length) interval $[a, b]$ such that for a Standard Normal r.v. X, the chances of $X \notin [a, b]$ are small, are exactly α.

Graphically

```
alpha <- 0.05; z.alpha <- qnorm(alpha/2, mean = 0, sd = 1)
x <- seq(-3,3, by = 0.01)
y <- dnorm(x, mean = 0, sd = 1)
plot(x,y, type = "l", xlim = c(-3,3), lwd = 2)
abline(v = z.alpha, lwd = 2, col = "red")
abline(v = -z.alpha, lwd = 2, col = "red")
polygon(c(x[x<=z.alpha], z.alpha),c(y[x<=z.alpha],0),col="cyan")
polygon(c(x[x>=-z.alpha], -z.alpha),c(y[x>=-z.alpha],0),col="cyan")
```



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Then, it is easy to see, if $\alpha \in (0, 0.5)$ because of the symmetry, that $b = -a$, and

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$$a = z_{\alpha/2}.$$

So

$$b = -z_{\alpha/2} = z_{1-\alpha/2}$$

Note: Please be careful when using Normal Tables. Usually, there is a picture above the table, on which you can find the explanation of the process. Just search "Normal tables" in Google Images.

Q-Q Plots

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- two given Datasets (possibly, of different sizes) are from the same Distribution;
- a given Dataset comes from a given Distribution;
- given two theoretical Distributions, check if one of them is a shifted-scaled version of the other one, or check if one has *fatter tails* than the other one

Q-Q Plots, Data vs Data

Now, assume we have two Datasets, not necessarily of the same size:

$$x: \ x_1, x_2, ..., x_n \qquad \text{and} \qquad y: \ y_1, y_2, ..., y_m$$

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Question: Are x and y coming from the same Distribution?

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Q-Q Plot helps to answer to this question visually. To draw the Q-Q Plot for Datasets, we take some levels of quantiles, say, for some n,

$$\alpha = \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}$$

and then draw the points (q_α^x, q_α^y).

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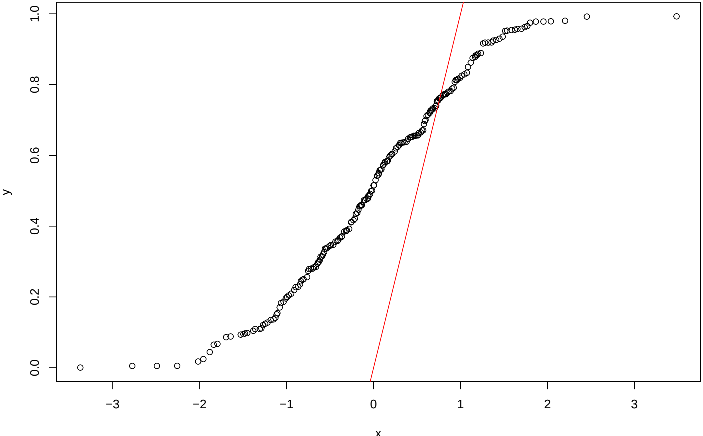
and then draw the points (q_α^x, q_α^y).

Idea: If x and y are coming from the same Distribution, then the Quantiles of x and y need to be approximately the same, $q_\alpha^x \approx q_\alpha^y$, so geometrically, the points (q_α^x, q_α^y) need to be close to the bisector line.

Example, Q-Q Plots, Data vs Data

```
x <- rnorm(1000)
y <- runif(200)
qqplot(x,y)
abline(0,1, col="red")
```



Example, Q-Q Plots, Data vs Data

```
x <- rnorm(1000)
y <- rnorm(500)
qqplot(x,y)
abline(0,1, col="red")
```