CS 107, Probability, Spring 2020
Lecture 32

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Content

- Transformation of Random Vectors

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Transformation of Random Vectors

Recall that for the 1D case, the transform of a r.v. X is obtained by applying some function g on X: $Y = g(X)$ is the transformed r.v.. And our aim was to derive the distribution of Y, having g and the distribution of X.

Now, assume we have 2 Jointly Distributed r.v.s X and Y (defined on the same Experiment). Using X and Y, we can obtain new r.v.s, and there are many ways for obtaining new r.v.s:

- Transform 2 rvs into 1 rv;
- Transform 2 rvs into 2 rvs;
- Transform 2 rvs into 3 rvs; ...

Transformation of Random Vectors

Transforming 2 into 1: Say, we can use a Real-Valued function $g : \mathbb{R}^2 \to \mathbb{R}$ (it is enough to have our g defined on all possible pairs (X, Y)) to obtain a 1D r.v. $Z = g(X, Y)$. So, out of two r.v.s X and Y, we can obtain 1 new r.v. Z:

Examples:

- $Z = X + Y$;
- $Z = 2X - 3Y$;
- $Z = X^2 + Y^2$;
- $Z = (X^2 - 4\sqrt{Y}) \cdot \sin(X \cdot Y)$;
- ...

Problem: Find the distribution of $Z = g(X, Y)$, having the Joint Distribution of X and Y.

Transformation of Random Vectors

Transforming 2 into 2: We can use 2 Real-Valued functions $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ to transform (X, Y) onto new r.vector (U, V) by

$$\begin{cases} U = g_1(X, Y) \\ V = g_2(X, Y) \end{cases}$$

Examples:

$$\begin{cases} U = X + Y \\ V = X - Y \end{cases} \quad \text{or} \quad \begin{cases} U = X + 2Y \\ V = X \cdot Y \end{cases} \quad \dots$$

Problem: Find the Joint Distribution of U and V, having the Joint Distribution of X and Y.

Transformation of Random Vectors

In our lecture we will consider the case of transforming 2 r.v.s into 1 r.v., i.e., we will study the distribution of

$$Z = g(X, Y),$$

using the Joint Distribution of X and Y.

In particular, first we will consider the general case of g. Later, we will talk in more details about the distribution of

$$Z = X + Y.$$

Distribution of $Z = g(X, Y)$ for general g

CDF of $g(X, Y)$ through the Joint CDF of X, Y

First, let us consider the problem of expressing the CDF of $Z = g(X, Y)$ in terms of the Joint CDF of X and Y.

CDF of $g(X, Y)$ through the Joint CDF of X, Y

Assume that $F_{X,Y}(x, y)$ is the CDF of (X, Y). That is,

$$F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y), \qquad \forall x, y.$$

We want to find the CDF of Z, $F_Z(x)$. We know that

$$F_Z(x) = \mathbb{P}(Z \leq x) = \mathbb{P}(g(X, Y) \leq x).$$

Now, we need to express the inequality $g(X, Y) \leq x$ in the form $(X, Y) \in A_x$ for some set $A_x \subset \mathbb{R}^2$:

$$g(X, Y) \leq x \qquad \Leftrightarrow \qquad (X, Y) \in A_x.$$

Here, $A_x = \{(u, v) : g(u, v) \leq x\}$. Then we will have

$$F_Z(x) = \mathbb{P}(g(X, Y) \leq x) = \mathbb{P}\Big((X, Y) \in A_x\Big).$$

Unfortunately, it is not possible to express this in terms of $F_{X,Y}(x, y)$ in the general case.

CDF of $g(X, Y)$ through the Joint CDF of X, Y

Note: Btw, we have calculated this kind of things some lectures ago: say, we were calculating $\mathbb{P}(X^2 + Y^2 \le 1)$ etc. Here, we want to calculate, for any x the following probability: $\mathbb{P}(X^2 + Y^2 \le x)$.

So, in general case, working with CDFs is not so effective to find the distribution of

$$Z = g(X, Y).$$

So we will consider next how to find the

- PMF of Z through the Joint PMF of (X, Y), if (X, Y) is Jointly Discrete;
- PMF of Z through the Joint PDF of (X, Y), if (X, Y) is Jointly Continuous, but Z is Discrete;
- PDF of Z through the Joint PDF of (X, Y), if (X, Y) is Jointly Continuous, and Z is Continuous.

Transform of Joint Discrete R.V.s

Assume now X and Y are Discrete with values $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$, and with the Joint PMF

$Y \backslash X$	x_1	x_2	$\ldots$
y_1	$p_{1,1}$	$p_{2,1}$	$\ldots$
y_2	$p_{1,2}$	$p_{2,2}$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Assume also that $g : \mathbb{R}^2 \to \mathbb{R}$ is a given function, and $Z = g(X, Y)$. It is easy to see that Z is a Discrete r.v., and our aim is to find the PMF of Z.

Now, the values of Z will be $z_{ij} = g(x_i, y_j)$, with probabilities:

Z	$g(x_1, y_1)$	$g(x_1, y_2)$	$g(x_2, y_1)$	$g(x_2, y_2)$	$\ldots$
$\mathbb{P}(Z = z)$	$p_{1,1}$	$p_{1,2}$	$p_{2,1}$	$p_{2,2}$	$\ldots$

Transform of Joint Discrete R.V.s

In fact, like in 1D case, it is possible that two values $g(x_i, y_j)$ are coinciding: in that case we write in the PMF $g(x_i, y_j)$ just once, and we are adding the corresponding probabilities: mathematically, the Probability of Z being $z_{ij} = g(x_i, y_j)$ will be

$$\mathbb{P}(Z = z_{ij}) = \sum_{g(x_k, y_m) = z_{ij}} \mathbb{P}(X = x_k,\, Y = y_m).$$

Transform of Joint Discrete R.V.s: Example

Example 32.1: Assume X and Y are Jointly Discrete with the following Joint PMF:

$Y \backslash X$	0	1	2
-2	0.1	0	0.3
0	0.4	0.2	0

Find the Distribution of $Z = \dfrac{1}{4 + X + Y}$.

Transform of Joint Continuous R.V.s

Assume now X and Y are Jointly Continuous, with the Joint PDF $f(x, y)$, and assume that $Z = g(X, Y)$ is Discrete.
The method to find the PMF of Z is easier to explain on examples:

Transform of Joint Continuous R.V.s: Example

Example 32.2: Assume (X, Y) is Standard Bivariate Normal r. vector, and

$$Z = [X^2 + Y^2].$$

Find the distribution of Z.

Transform of Joint Continuous R.V.s: Example

Example 32.3: Assume $(X, Y) \sim Unif([-1, 1]^2)$. Find the Distribution of

$$Z = \begin{cases} 0, & X + Y \le 1 \\ 1, & \text{otherwise} \end{cases}$$

Transform of Joint Continuous R.V.s

Assume now X and Y are Jointly Continuous, with the Joint PDF $f(x, y)$, and assume that g is such that $Z = g(X, Y)$ is Continuous again. Then our problem becomes to find the PDF of Z in terms of $f(x, y)$.

Transform of Joint Continuous R.V.s

The general idea is to start by the CDF of Z:

$$F_Z(x) = \mathbb{P}(Z \le x) = \mathbb{P}(g(X, Y) \le x).$$

As above, we need to express the inequality $g(X, Y) \le x$ in the form

$$g(X, Y) \le x \qquad \Leftrightarrow \qquad (X, Y) \in A_x,$$

where $A_x = \{(u, v) : g(u, v) \le x\}$. Then,

$$F_Z(x) = \mathbb{P}(g(X, Y) \le x) = \mathbb{P}\Big((X, Y) \in A_x\Big).$$

Transform of Joint Continuous R.V.s

Now, since (X, Y) is Jointly continuous with the Joint PDF $f(x, y)$, we will have

$$F_Z(x) = \mathbb{P}(g(X, Y) \leq x) = \mathbb{P}\Big((X, Y) \in A_x\Big) = \iint_{A_x} f(u, v)\, du dv =$$

$$= \iint\limits_{g(u,v)\leq x} f(u, v)\, du dv.$$

And to find the PDF $f_Z(x)$ of Z, we just need to calculate the derivative of F_z:

$$f_Z(x) = \Big(F_Z(x)\Big)'.$$

Transform of Joint Continuous R.V.s: Example

Example 32.4: Assume (X, Y) are Jointly distributed with the Joint PDF

$$f(x, y) = \begin{cases} 6 \cdot e^{-2x-3y}, & x, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the distribution of $Z = \dfrac{X}{Y}$.

Transform of Joint Continuous R.V.s: Example

Example 32.5: Assume $(X, Y) \sim Unif([0,1]^2)$, and let

$$\begin{cases} U = X + Y \\ V = X - Y. \end{cases}$$

Find the Joint Distribution of (U, V).

Transform of Joint Continuous R.V.s: Example

Example 32.6: Assume $(X, Y) \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{\Sigma} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

Visualize, in **R**, the distributions of

a. $Z = 3X + 3Y$;

b. $Z = X^2 + Y^2$;

c. $Z = X^2 - Y^2$;

d. $Z = |X| + |Y|$.

Transform of Joint Continuous R.V.s: Example

Example 32.7: Assume $(X, Y) \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{\Sigma} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

Visualize, in **R**, the Joint Distributions of

$$\begin{cases} U = X + Y \\ V = X - Y. \end{cases}$$

Distribution of $Z = X + Y$

Distribution of the Sum

Of particular interest, in terms of applications, is the following special case: find the distribution of

$$Z = X + Y,$$

and, in general, the distribution of the sum of n r.v.s

$$Y = X_1 + X_2 + ... + X_n.$$

In particular, two main theorems of Probability Theory, the Law of Large Numbers (LLN) and the Central Limit Theorem (CLT), are all about the (asymptotic) properties of these kind of sums.

Distribution of the Sum: Examples

Some Examples:

- Say, my portfolio consists of 100 shares of Google (Alphabet Inc.) and 200 shares of Amazon. So I am a Rich Man!! $\ddot\smile$ Let Z be the price of my portfolio at the end of this year. Then
$$Z = 100 \cdot GOOG + 200 \cdot AMZN,$$
where $GOOG$ and $AMZN$ are r.v. showing the prices of Google and Amazon Stocks at the end of the year.
- My expenses at AUA cafeteria for some week can be written as
$$Z = X_1 + X_3 + X_5,$$
where X_i is the amount I will spend on the day i of the week. Of course, X_i-s are r.v.s depending on how much I will be hungry.

Distribution of the Sum: Examples

Some Examples:

- Usually, insurance companies are interested in the amount they will pay for some future time interval, say, for the upcoming week. This amount can be written as:

$$Y = X_1 + X_2 + ... + X_7,$$

  where X_i is the claim size for the day i of the week. X_i-s are r.v., so Y is a r.v.
- Going further, the same companies can also model they daily claims: If they will have M claims in a day, with sizes X_1, X_2, ..., X_M, then their daily claim size will be

$$Z = X_1 + X_2 + ... + X_M.$$

  Here important is that M **will be a r.v. itself (!)**, and the distribution of Z is an advanced topic. , Btw, can you give a Model for M?

Distribution of the Sum

Now, given two r.v.s X and Y, we will have the following cases for the distribution of the sum,

$$Z = X + Y.$$

- If X and Y are **Discrete**, then $Z = X + Y$ will be Discrete too, with a PMF

$$\mathbb{P}(Z = x) = \mathbb{P}(X + Y = x) = \sum_{x_i + y_j = x} \mathbb{P}(X = x_i, Y = y_j).$$

- If X and Y are **Jointly Continuous** with the Joint PDF $f_{X,Y}(x,y)$, then $Z = X + Y$ will be a Continuous r.v. with the PDF

$$f_Z(x) = f_{X+Y}(x) = \int_{-\infty}^{\infty} f_{X,Y}(t, x-t)\, dt \qquad \forall x \in \mathbb{R}.$$

Distribution of the Sum: Example

Example 32.8: Assume X and Y are Discrete r.v.s with the Joint PMF

$Y \backslash X$	-1	1
-1	0.1	0.2
0	0.3	0.1
1	0.1	0.2

Find the distribution of $Z = X + Y$.

Transform of Joint Continuous R.V.s: Example

Example 32.9: Assume (X, Y) are Jointly distributed with the Joint PDF

$$f(x, y) = \begin{cases} e^{-x-y}, & x, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the distribution of $Z = X + Y$.