CS 107 Section A - Probability

Spring 2020, AUA

Homework No. 07

Due time/date: 31 March, 2020

Note: Supplementary Problems will not be graded, but you are very advised to solve them and to discuss later with TA or Instructor.

A Discrete Random Variables, PMFs

Problem 1. Which of the followings are Discrete r.v.s:

a. $X =$ the number of webpages created and published this year;
b. $Y =$ the amount of rainfall in Yerevan this year;
c. $Z = \ln(GPA)$, where GPA is the GPA of a randomly selected student at AUA;
d. $U = 1$, if the daily change in the price of GE (General Electric) stock will be larger than 7%, and $U = 0$ otherwise;
e. $V =$ the highest temperature in Yerevan in Autumn;
f. $W =$ the circumference of the circle with radius 4.

Problem 2. Two persons are choosing an integer from $\{1, 2, 3, 4\}$, independently, with equal probabilities to choose each number. Let X be the r.v. showing the maximum of that two numbers, and Y be the r.v. showing the first person's number minus the second person's number.

a. What are the possible values of X?
b. Is X discrete?
c. If X is discrete, construct its PMF (in a table form and with the graph plot) and CDF (analytically and graphically). If it is not discrete, just give its CDF (analytically and graphically);
d. What are the possible values of Y?
e. Is Y discrete?
f. If Y is discrete, construct its PMF (in a table form and with the graph plot) and CDF (only graphically). If it is not discrete, just give its CDF (graphically).

Problem 3. Assume X is a discrete r.v. with the following PMF:

Values of X	-100	0	100
$\mathbb{P}(X = x)$	α	2α	$\alpha + 0.2$

a. Find all possible values of α;

b. Calculate $\mathbb{P}(X > -2)$;

c. Find the PMF of $Y = -X$;

d. (Supplementary) Design an Experiment such that X will be a r.v. on that Experiment;

e. (Supplementary) Design another Experiment such that X will be a r.v. on that Experiment.

Problem 4. Assume X is a discrete random variable given by its PMF:

Values of X	-3	2	7
$\mathbb{P}(X = x)$	0.2	0.5	0.3

Find the CDF $F(x)$ of X: write $F(x)$ in the analytic form, and give the graph of F.

Problem 5. Assume X is a random variable given by its CDF $F(x)$ (see Fig. 1). Find the PMF of X.

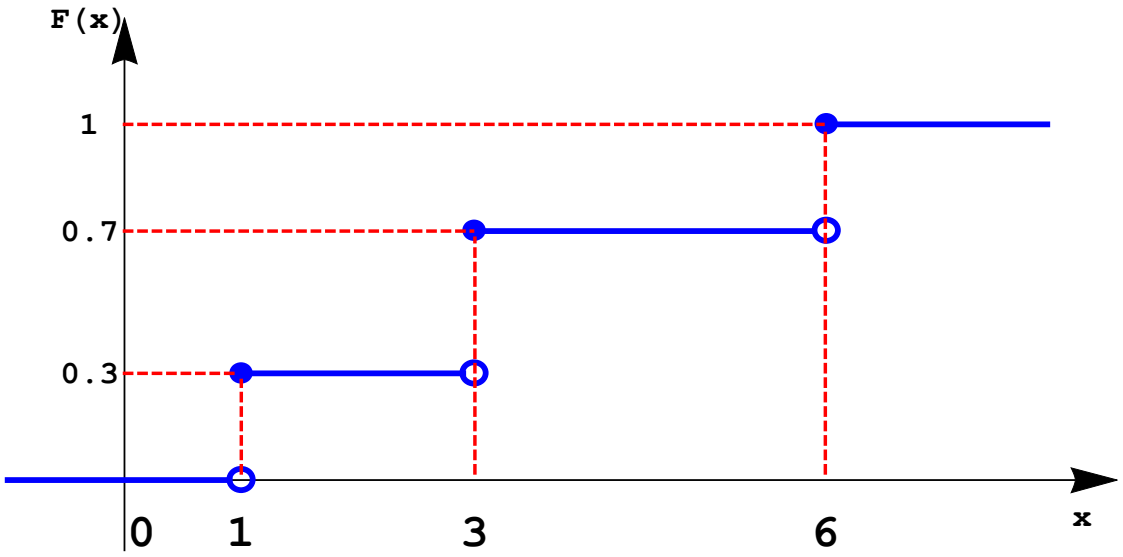

Figure 1: The CDF of X

B Continuous Random Variables, PDFs

Problem 6. Fig. 2 shows some function $f(x)$.

a. Is f a PDF for some r.v. X? Explain. If it is, proceed to next tasks, otherwise go to the next problem.

b. Calculate the probability $\mathbb{P}(X = 0.3 \text{ or } X = 7)$.

c. Calculate $\mathbb{P}(X < -2)$.

d. Calculate is the probability $\mathbb{P}(1 \leq X \leq 7)$?

e. What is the range of X?
Note: You need to find the largest set in the form of a union of intervals, such that the probability of taking values in its complement is 0.

f. Calculate the probability $\mathbb{P}(X \leq 5)$.

g. Which is more probable: $X \in [-1, 1]$ or $X \in [5, 7]$?

h. Find all values of $q \in \mathbb{R}$ such that[1] $\mathbb{P}(X \leq q) = 0.4$;

i. (Supplementary) Construct the CDF of X.

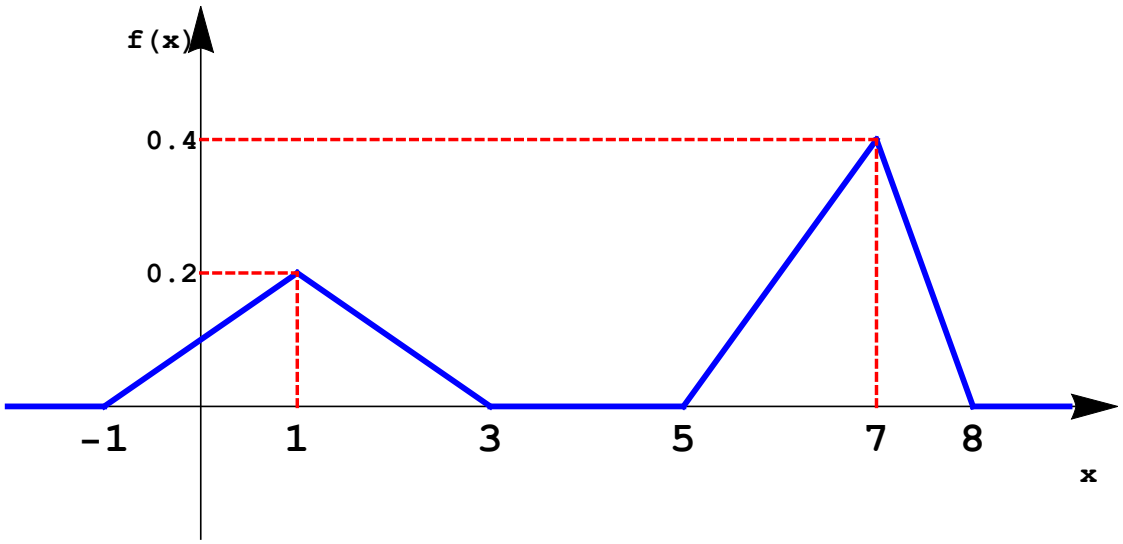

Figure 2: The graph of $f(x)$

Problem 7. We are given the CDF $F(x)$ of a r.v. X by its graph, see Fig. 3.

a. If X is discrete, calculate its PMF, and if X is continuous, find the PDF of X;

b. Find all values of q such that $\mathbb{P}(X \leq q) = 0.5$;

c. Find all values of q such that $\mathbb{P}(X \leq q) = 0.4$.

Problem 8. Assume the PDF of the r.v. X is given by:

$$f(x) = \begin{cases} c \cdot (x^2 + 1), & \text{if} \quad x \in [0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

a. Find the constant c;

b. Calculate the probability $\mathbb{P}(X \leq 0.2)$;

c. Let $F(x)$ be the CDF of X. Calculate $F(0)$, $F(0.5)$ and $F(1)$;

d. Find the CDF $F(x)$ of X, analytically;

e. Is it true that the most probable value of X is 1?

[1] The smallest such q is called the **quantile** of X (or of this Distribution) of the order 0.4.

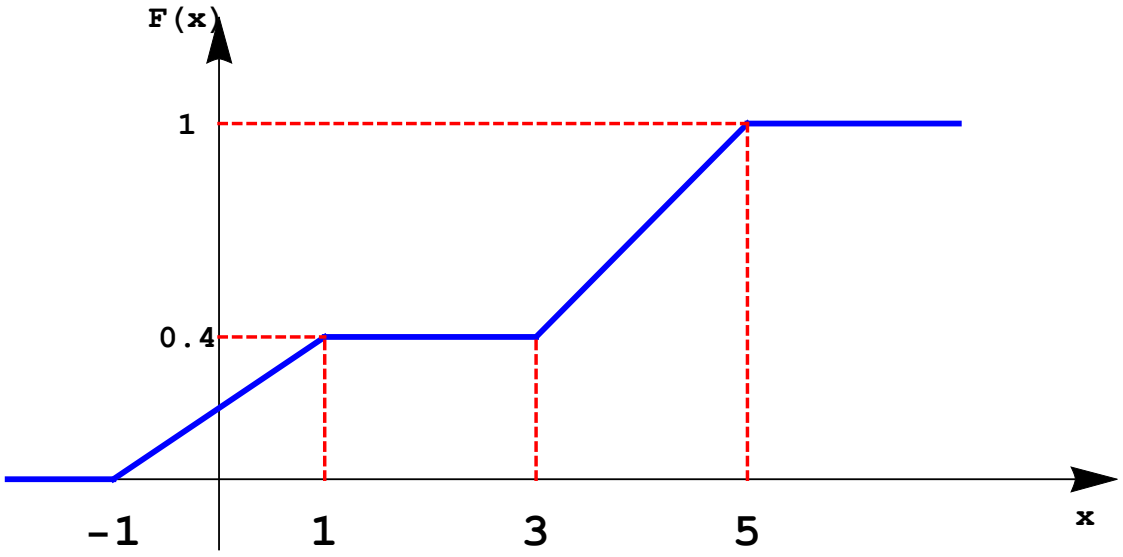

Figure 3: The graph of $F(x)$

f. Calculate $\mathbb{P}(X > 2)$;

g. Find q such that $\mathbb{P}(X \leq q) = 0.9$.
Note: You can use some numerical methods or a computer software to solve the obtained nonlinear equation.

Problem 9. Let X be a r.v. with the PDF

$$f(x) = \frac{K}{1+x^2}, \qquad x \in \mathbb{R}.$$

a. Find K;

b. Calculate $\mathbb{P}(X = 2.3)$;

c. Calculate $\mathbb{P}(X \in [0, 1])$;

d. Calculate $\mathbb{P}(X < 1 | X > 0)$;

e. Find the smallest symmetric (around the origin) interval containing the values of X with probability 95%, i.e., find the value of $a > 0$ such that

$$\mathbb{P}(-a \leq X \leq a) = 0.95;$$

f. Find the CDF $F(x)$ of X, analytically.

C Supplementary Problems

Problem 10. I have a Bounty Chocolate stick, which is 15cm long, and I will share it with you. I am breaking the chocolate stick at a random place (with uniform probabilities) along the length, and give the right-hand piece to you. My r.v. X is the calories I will get eating my piece, and it is calculated by $X(\omega) = 20 \cdot \omega^2$, where ω is the length of my share.

a. Construct the CDF of X.

Note: Use the definition of the CDF:

$$F(x) = \mathbb{P}(\omega : X(\omega) \leq x), \qquad x \in \mathbb{R}.$$

b. Find the PDF of X.