CS 107, Probability, Spring 2020
Lecture 05

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- Classical Probability Models: Finite Sample Spaces
- Classical Probability Models: Countably Infinite Sample Spaces
- Classical Probability Models: Finite Sample Spaces with Equally Likely

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Let us do a scientific experiment: Take a deep breath. Keep it for at least for few seconds. Do you remember Julius Caesar and his last words?

Figure: Gaius Julius Caesar

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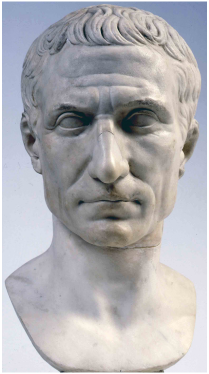

Figure: Gaius Julius Caesar

Let us do a scientific experiment: Take a deep breath. Keep it for at least for few seconds. Do you remember Julius Caesar and his last words?

You too, Brutus.

Vnimanie, the question: what are the chances that you just inhaled a molecule which the great Caesar exhaled when saying his last words?

The answer is: the probability is larger than 99%! $\ddot\smile$

Last Lecture ReCap

Last time we talked about/proved some properties of a Probability:

- $\mathbb{P}(\varnothing) = 0$;
- $\mathbb{P}(\overline{A}) = 1 - \mathbb{P}(A)$;
- $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$; in particular, if $A \cap B = \varnothing$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$;
- $\mathbb{P}(A \cup B \cup C \cup D) =$ (OTB)
- $\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A \cap B)$; in particular, if $A \subset B$, then $\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A)$;
- if $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$;
- $0 \leq \mathbb{P}(A) \leq 1$.

Example

Recall the last lecture Example: Assume that in some university, for the CS department, the probability that a sophomore student knows Java programming is 50%, and the probability that he/she knows C++ programming is 60%. So

$$\mathbb{P}(Java) = 0.5 \qquad \text{and} \qquad \mathbb{P}(C++) = 0.6.$$

- In our initial problem, what is the highest possible probability for a student to know Java and C++ programming?
- In our initial problem, what is the smallest possible probability for a student to know Java and C++ programming?

Classical Probability Models: Discrete Sample Spaces

Discrete = Sample Space is Finite or Countably Infinite

Here we give the general construction of the Probability Models for the Discrete Sample Spaces. The continuous case is much more complicated.
To give Probability Models, Probability Spaces, we need to give:

- The Sample Space Ω;
- The set of Events $\mathcal{F}$;
- The Probability Measure $\mathbb{P}$.

Classical Probability Models: Finite Sample Spaces

Assume the Sample Space Ω is finite:

- Our Sample Space is $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$.
- Every subset of Ω is an Event, i.e., $\mathcal{F} = 2^\Omega$, the power set of Ω.
- We take any real numbers $p_1, p_2, ..., p_n$ with

$$p_1 \geq 0, ..., p_n \geq 0, \quad p_1 + p_2 + ... + p_n = 1,$$

  and define

$$\mathbb{P}(\{\omega_1\}) = p_1, \quad \mathbb{P}(\{\omega_2\}) = p_2, \quad ..., \quad \mathbb{P}(\{\omega_n\}) = p_n.$$

Classical Probability Models: Finite Sample Spaces

We write this in a more convenient table form:

Outcome	ω_1	ω_2	...	ω_n
$\mathbb{P}(\{\omega_k\})$	p_1	p_2	...	p_n

We are not done yet! We define, for any event A,

$$\mathbb{P}(A) = \sum_{\omega_i \in A} p_i,$$

and also add $\mathbb{P}(\varnothing) = 0$.

I FSYO!

Examples

UnBiased Coin:

Outcome	H	T
$\mathbb{P}(\{\omega\})$	0.5	0.5

Biased Coin:

Outcome	H	T
$\mathbb{P}(\{\omega\})$	0.3	0.7

UnBiased Die:

Outcome	1	2	3	4	5	6
$\mathbb{P}(\{\omega\})$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Classical Probability Models: Countably Infinite Sample Spaces

We continue considering Discrete Models. We assume that we have a countably infinite Sample Space:

$$\Omega = \{\omega_1, \omega_2, ..., \omega_n, \omega_{n+1} ...\},$$

We take $\mathcal{F} = 2^{\Omega}$. Then we take arbitrary real numbers p_k, $k \in \mathbb{N}$ with

$$p_k \geq 0, \quad k \in \mathbb{N} \qquad \text{and} \qquad \sum_{k=1}^{\infty} p_k = 1,$$

and define

$$\mathbb{P}(\{\omega_k\}) = p_k, \qquad k \in \mathbb{N}.$$

Classical Probability Models: Countably Infinite Sample Spaces

In the table form, this looks like

Outcome	ω_1	ω_2	ω_3	...	ω_n	...
Probability	p_1	p_2	p_3	...	p_n	...

Now, we define for any nonempty event $A \in \mathcal{F}$ (i.e., for any nonempty subset $A \subset \Omega$),

$$\mathbb{P}(A) = \sum_{\omega_k \in A} p_k,$$

and also $\mathbb{P}(\varnothing) = 0$.
Now we have the complete Model.

Examples

Example: Say, we have an Experiment with non-negative Integer outcomes, with Probabilities:

Outcome	0	1	2	3	4	...
$\mathbb{P}(\{\omega\})$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6} - \frac{1}{10}$	$\frac{1}{10} - \frac{1}{20}$	$\frac{1}{20} - \frac{1}{30}$	...

Example: Say, we have an Experiment with outcomes ω_k, $k = 1, 2, 3, ...$, with Probabilities:

Outcome	ω_1	ω_2	ω_3	ω_4	ω_5	...
$\mathbb{P}(\{\omega\})$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	...

Moral

The Moral is:

For the Discrete case (i.e., when the Sample Space is either finite or countably infinite), it is enough to define the Probability Measure only for the outcomes: $\mathbb{P}(\{\omega_k\})$.

The Probability of any other Event can be obtained from these outcome probabilities.

Example:

Equally Likely Outcomes

Now assume we have a Discrete Model with finitely many outcomes:

$$\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$$

One of the important Probability Models is the Equally Likely Outcomes (or the uniform discrete distribution) case:

$$\mathbb{P}(\{\omega_1\}) = \mathbb{P}(\{\omega_2\}) = ... = \mathbb{P}(\{\omega_n\}) = \frac{1}{n}.$$

In the table form:

Outcome	ω_1	ω_2	ω_3	...	ω_n
Probability	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	...	$\frac{1}{n}$

Equally Likely Outcomes, cont'd

Now, consider an event $A \subset \Omega$. Then

$$\mathbb{P}(A) = \sum_{\omega_k \in A} p_k = \sum_{\omega_k \in A} \frac{1}{n} =$$

$$= \frac{1}{n} \cdot (\text{number of elements in } A) = \frac{\text{number of elements in } A}{\text{total number of elements in } \Omega}.$$

Or, in other words,

$$\mathbb{P}(A) = \frac{\text{number of elements favorable for the event } A}{\text{total number of possible outcomes}} = \frac{\#A}{\#\Omega}.$$

Some Questions to Answer

- What is the Equally Probable Outcomes (or the Discrete Uniform) Model?
- Give an example of an Equally Probable Outcomes model.
- Is it possible to define the Equally Probable Outcomes Model for the Countably Infinite Sample Space case?
- How to calculate the probability of an event in the Equally Probable Outcomes Case?