CS 107, Probability, Spring 2019
Lecture 08

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AUA

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Content

- Classical Probability Models: Finite Sample Spaces with Equally Likely Outcomes = Combinatorial Problems, Cont'd
- Classical Probability Models: Geometric Probabilities

The Monty Hall Problem

https://www.youtube.com/watch?v=mhlc7peGlGg

Example: Birthday Problem

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Assumptions: There are 365 days in a year, and the probability of being born on each day is the same.

Example: Birthday Problem, another version

Problem: We have 36 participants in our group of Probability class, including the instructor. What is the probability that at least one of our students will share the instructor's (MP's) birthday?

Example: Julius Caesar Problem

Do you remember our Caesar?

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Figure: Gaius Julius Caesar

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Problem: What are the chances that I have just inhaled a molecule which the great Caesar exhaled when saying his last words?
Assume we have N molecules of air in the world, and Caesar exhaled n of them, and they are uniformly spread over the world. And assume I have just inhaled m molecules.
Physics says that $N \sim 10^{44}$, $n, m \sim 2.2 \cdot 10^{22}$

The rest on the board!

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Correct? No!

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Correct? Aha!

Geometric Probabilities

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- a cube in $\mathbb{R}^3$; lifetimes for 3 different parts of the above marshutka;
- an n-dimensional cube in $\mathbb{R}^n$, arrival times of our Probability Students to our class today (? on Wed?)

We want to give a basic Probabilistic model for our Experiment.

Geometric Probabilities, Cont'd

So assume that our Experiment's Sample Space is $\Omega \subset \mathbb{R}^n$.

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What we mean by that:

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Geometric Probabilities, Cont'd

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Assumption

We assume that we have **uniform probability distribution**.

What we mean by that:

- **Incorrect interpretation:** the probability of choosing each point in Ω is the same. Well, in this Geometric Model (non-countably infinite Sample Space!) we will have that the probability of choosing any particular point is 0.
- **Correct Interpretation:** the probability of choosing any equal-measure (length/area/volume) subsets (Events) is the same.

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- Our Experiment's Sample Space is $\Omega \subset \mathbb{R}^n$;
- We assume that Ω is measurable[1], i.e., it has a finite length (if $\Omega \subset \mathbb{R}$) or it has a finite area (if $\Omega \subset \mathbb{R}^2$) or it has a finite volume (if $\Omega \subset \mathbb{R}^n$, $n \geq 3$).

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- If $A \subset \Omega$ is an Event (if A has a finite measure), the we define

$$\mathbb{P}(A) = \frac{measure(A)}{measure(\Omega)}.$$

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  $\mathbb{P}(A) = \frac{length(A)}{length(\Omega)} = \frac{0}{12} = 0.$

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- How to define a Probability on the Finite Sample Space?
- How to calculate the Probability of an Event in the Equiprobable outcomes (Finite Sample Space) case?
- What is the definition of the Geometric Probability Model?