CS 107, Probability, Spring 2019
Lecture 26

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AUA

25 March 2019

Content

- End of the story about Important Discrete R.V.s
- Examples of Important Continuous R.V.s

LZ

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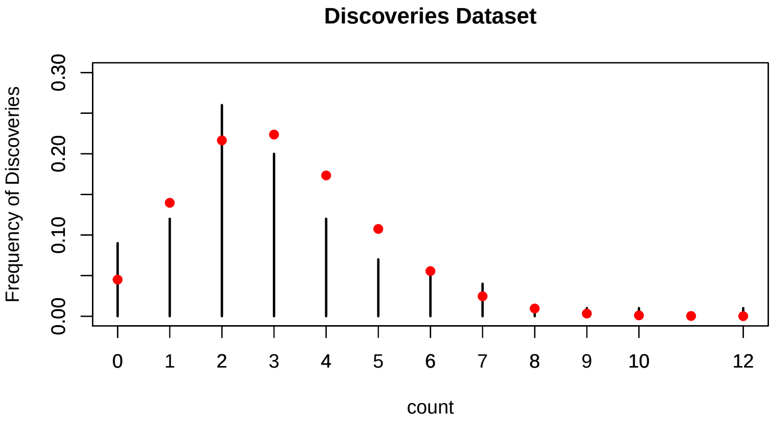
- 80% of comments are created by 20% of users $\ddot\smile$
- My phone is at 20% 80% of the time $\ddot\smile$
- Studied 20% of the material, get a 80% on the finals. Happened to no student ever. (MP: Exactly!)

Poisson Distribution: R Examples

R Code

```
help("discoveries")
disc <- discoveries
plot(disc)
hist(disc, breaks = seq(0,13,1)) #histogram
#Fitting the data by the Poisson
lambda = mean(disc)
table(disc)
plot(table(disc)/length(disc), xlim = c(0,max(disc)),
   ylim = c(0,0.3), main = "Discoveries Dataset",
   ylab = "Frequency of Discoveries", xlab = "count")
n = 0:max(disc)
m = dpois(n, lambda)
par(new = T) #To keep the previous graph
plot(n,m, xlim = c(0,max(disc)), ylim = c(0,0.3),
   pch = 19, col="red", main = "", xlab = "", ylab = "")
```

Discoveries Dataset Model Result

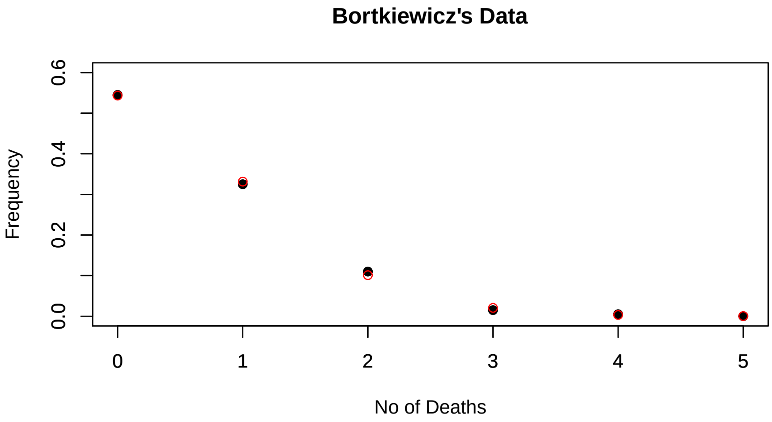


Poisson Distribution: R Examples

R Code

```
#Bortkiewicz's Data
#Deaths by horse-kick in Prussian Army cavalry corps
#(Bortkiewicz 1898). N is the number of units
#corresponding to each number of deaths.
hk<-data.frame(D = c(0,1,2,3,4,5),
   N = c(109,65,22,3,1,0))
plot(hk)
lambda <- sum(hk$D*hk$N)/sum(hk$N)
n = 0:5
m = dpois(n, lambda)
plot(n,m, xlim = c(0,5), ylim = c(0,0.6),
    col="red", type = "l", lwd = 3)
par(new = T)
plot(hk$D, hk$N/sum(hk$N) , xlim = c(0,5),
    ylim = c(0,0.6), pch=19)
```

Bortkiewicz Dataset Model Result



Approximation of Binomial Distribution through Poisson

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$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \approx e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

Poisson Distribution: R Examples

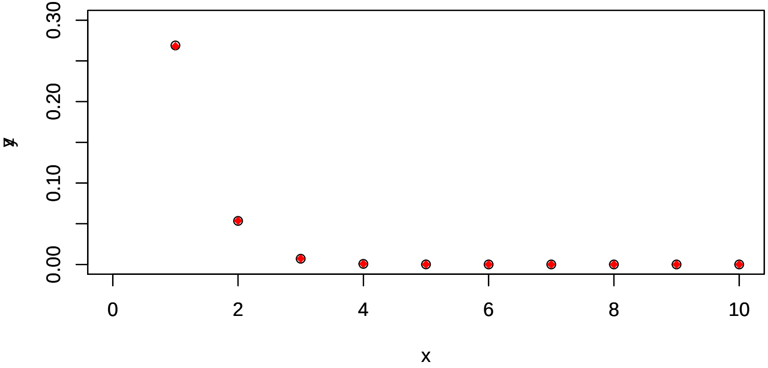
R Code

```
# Approximation of Binomial by Poisson
n <- 100
p <- 0.004
lambda <- n*p

x <- 0:20
y <- dbinom(x, size = n, prob = p)
plot(x,y, ylim = c(0,0.4))
z <- dpois(x, lambda = lambda)
par(new=T)
plot(x,z, pch = 18, col = "red", ylim = c(0,0.4))

#Plotting the difference
plot(x, abs(y-z))
```

Poisson approximation of Binomial Distribution



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$$\mathbb{P}(X = 3) = \binom{20000}{3} \cdot p^3 \cdot (1 - p)^{20000-3} \approx e^{-\lambda} \cdot \frac{\lambda^3}{3!} = e^{-2} \cdot \frac{2^3}{3!} \approx 0.18.$$

Note about the Poisson Distribution

Assume the average number of downloads in a day for some software (or some mobile app) is λ. So if we want to model the number of daily downloads X of that software (X is a r.v.), we can model that as $X \sim Poisson(\lambda)$.

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Now, we want to model the number Y of downloads for 2 days. Since the average number of downloads in 2 days for the same software will be 2λ, then we can use $Y \sim Poisson(2\lambda)$.

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Important Continuous Distributions

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$$\mathbb{P}(X \in [a, b]) = F(b) - F(a) = \int_a^b f(x)dx.$$

Uniform Distribution

Uniform Distribution on $[a, b]$

We say that the r.v. X has a uniform distribution in $[a, b]$, and we will write $X \sim Unif[a, b]$ if its PDF is given by

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The Graph of the PDF: On the board.

The CDF of the Uniform Distribution

Fact: The CDF of $X \sim Unif[a, b]$ is given by:

$$F(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & x \in [a, b] \\ 1, & x > b \end{cases}$$

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Graph on the Board

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Uniform Distribution Examples

- Recall the experiments of picking a random number from some interval - we were talking, actually, about the Uniform Distribution. So if we are picking from $[a, b]$, and if we will denote the randomly chosen number by X, then $X \sim Unif[a, b]$ (if, of course, we are choosing *uniformly* $\ddot\smile$).

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- If the Baghramyan Metro Station trains arrive in 10 min intervals, and I am visiting that station at some random time instant, then my waiting time X (in minutes) can be modeled by $X \sim Unif[0, 10]$.

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- Will the age distribution (exact age, not rounded to years) in AUA be Uniform ?

Uniform Distributions: Example

Example: Assume $X \sim \mathit{Unif}[1,4]$. Calculate $\mathbb{P}(X = 2)$, $\mathbb{P}(X \leq 2)$, $\mathbb{P}(2 \leq X \leq 3.5)$.

Uniform Distribution: R Examples

R Code

```
a <- 0
b <- 2
x <- seq(from = a-1, to = b+1, by = 0.01)
y <- dunif(x, min = a, max = b)
par(mfrow = c(1,2))
#plot the PDF
plot(x,y, type = "l", lwd = 3,
      main = "Uniform Distribution PDF")

z <- punif(x, min = a, max = b)
#plot the CDF
plot(x,z, type = "l", lwd = 3,
      main = "Uniform Distribution CDF")
```

Uniform Distribution: R Examples

R Code

```
#generating 10 uniformly distributed numbers in [0,1]
smpl <- runif(10) #equivalent to runif(10, min=0, max=1)
smpl

#generating 10000 uniformly distributed numbers in [0,1]
smpl <- runif(10000)
hist(smpl) # the histogram
```

Exponential Distribution

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We say that the r.v. X has an Exponential Distribution with the parameter $\lambda > 0$ (rate), and we write $X \sim Exp(\lambda)$, if its PDF is given by

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Exercise: Check that this function is a PDF of some r.v.
Exercise: Prove that the CDF of $X \sim Exp(\lambda)$ is given by:

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0 \end{cases}$$