CS 107, Probability, Spring 2019
Lecture 20

Michael Poghosyan

AUA

11 March 2019

Content

- CDF and its Properties
- Discrete r.v.

LZ

Welcome back from the Spring Break $\ddot\smile$

Properties of the CDF, reminder

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These four properties characterize completely CDFs!

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Probabilities through CDF

- $\mathbb{P}(X = a) = F(a) - F(a-)$;
- $\mathbb{P}(a < X \le b) = F(b) - F(a)$;
- $\mathbb{P}(a \le X \le b) = F(b) - F(a-)$;
- $\mathbb{P}(a \le X < b) = F(b-) - F(a-)$;
- $\mathbb{P}(a < X < b) = F(b-) - F(a)$;

Here we can have also $a = -\infty$ or/and $b = +\infty$

Reading Probabilities from the graph of CDF

Problem: Is the following a CDF of some r.v. X? If yes, calculate $\mathbb{P}(X = 2)$, $\mathbb{P}(2 < X \leq 3)$, $\mathbb{P}(X \leq 5)$, $\mathbb{P}(X > 2)$:

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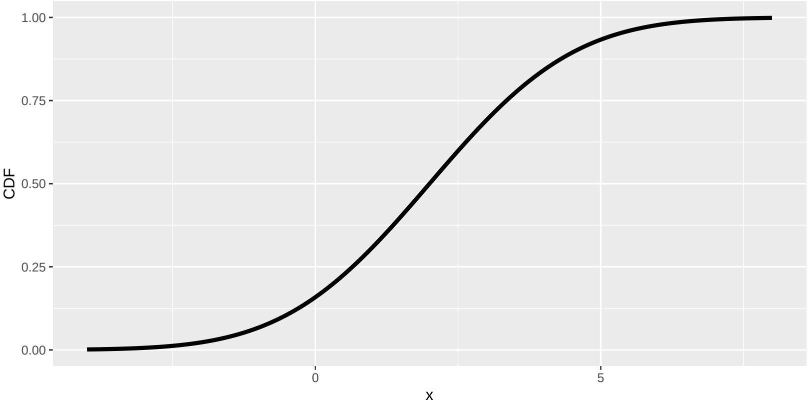

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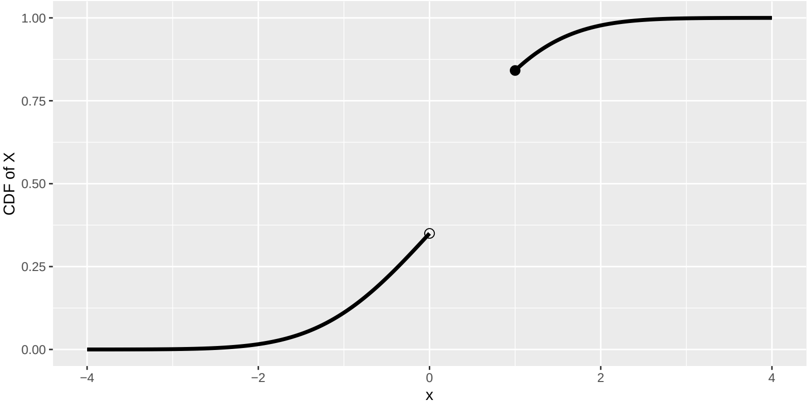

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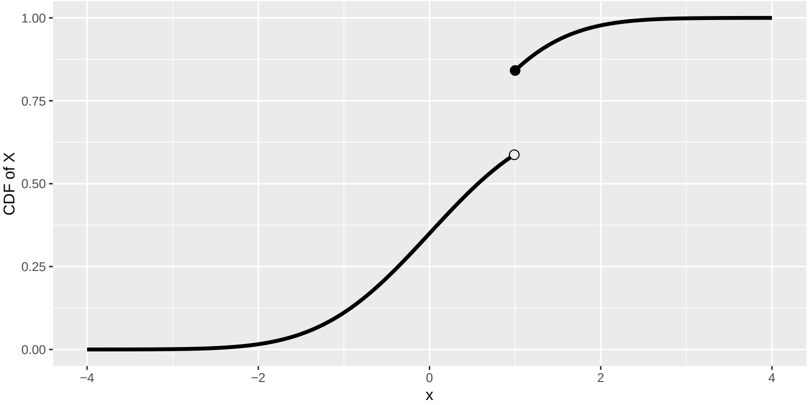

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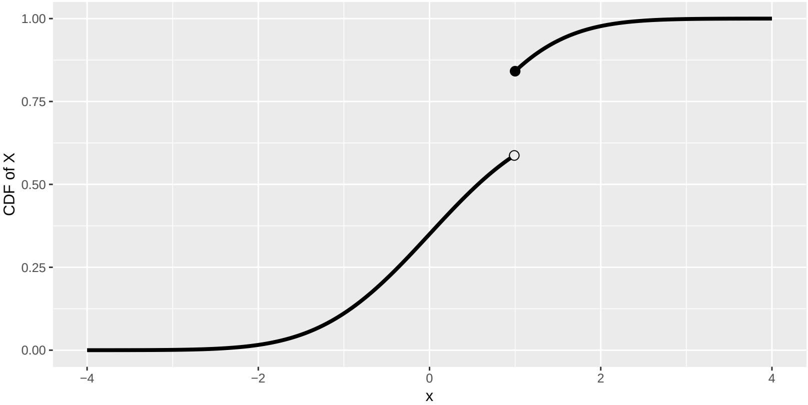

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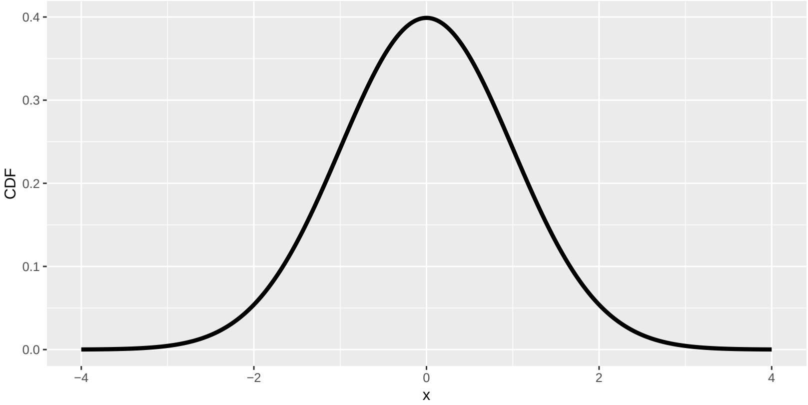

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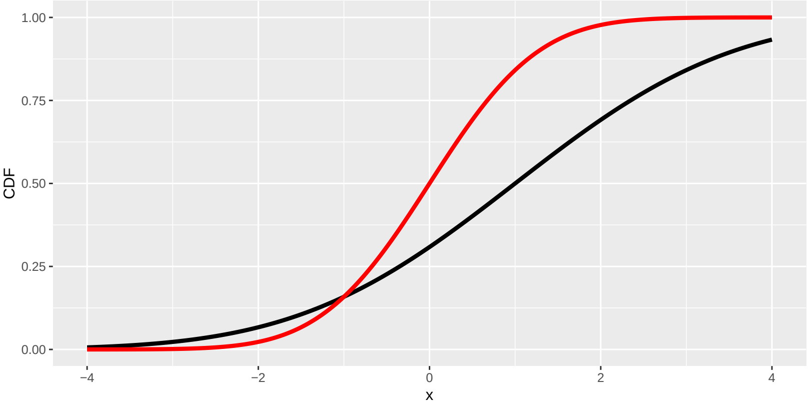

Figure: CDFs of r.v. X and Y

Discrete R.V.s

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So if X is Discrete, then the Range of X can be written as

$$Range(X) = \{x_1, x_2, x_3, ...\},$$

where the set on the RHS[1] can be also finite.

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- If Ω is not Discrete, then X CAN BE Discrete: Example: for the Darts game, let X show the points you will get

Characterization of Discrete R.V.s

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Question: What is the difference between Discrete Probability Models and Discrete r.v.s?

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For example,

$$\mathbb{P}(a \le X \le b) = \sum_{a \le x_k \le b} p_k.$$

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