CS 107, Probability, Spring 2020
Lecture 14

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Content

- Independence of Events
- Repeated, Independent Trials: Binomial Distribution

The Secretary/Marriage Problem

Assume you have *n* candidates for the place of your second half, your husband/wife. You will meet them one by one, and you do not know anything about them. After meeting a candidate, you can tell if he/she is better than the previous ones, i.e., you can rank all candidates you have met so far. You will not meet a person twice, you cannot go back to previous candidates - after the meeting with a candidate you are deciding either to marry or not that candidate.

Question: When to stop?

Answer: Reject the first $\frac{1}{e} \approx 0.368$ percent of the candidates, and then choose the first candidate who will be better than every applicant interviewed so far. If nobody will satisfy this, just choose the last one, and live a happy life $\ddot\smile$

Last Lecture ReCap

Last time we were talking about Independence of Events:

- Two Events $A, B \subset \Omega$ are called Independent, if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B);$$

- If $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$, then Independence of A and B is equivalent to $\mathbb{P}(A|B) = \mathbb{P}(A)$ or to $\mathbb{P}(B|A) = \mathbb{P}(B)$;
- If $A \cap B = \varnothing$, then A and B are Dependent.

Some Properties of Independent Events

Assume A, B are some events.

- If A and B are independent, then A and B cannot be disjoint (unless $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$);
- If A and B are independent, then so are A and $\overline{B}$, and also $\overline{A}$ and B, also $\overline{A}$ and $\overline{B}$;
- If A is independent of B and A is independent of C, and also $B \cap C = \varnothing$, then A is independent of $B \cup C$.

Independence of more that two events

Now assume we have several events in the Experiment: A_k, $k = 1, 2, ..., n$. We define two notions:

Pairwise Independence

We will say that the events $A_1, A_2, ..., A_n$ are **Pairwise Independent**, if every pair A_i and A_j are Independent, for any $i \neq j$, i.e., if

$$\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i) \cdot \mathbb{P}(A_j), \qquad i \neq j.$$

Independence of more that two events, cont'd

Mutual Independence

We say that $A_1, \dots, A_n$ are **Mutually Independent** or just **Independent**, if for any subgroup of events $A_{i_1}, A_{i_2}, \dots, A_{i_k}$,

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = \mathbb{P}(A_{i_1}) \cdot \mathbb{P}(A_{i_2}) \cdot \dots \cdot \mathbb{P}(A_{i_k}),$$

that is, if

$$\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i) \cdot \mathbb{P}(A_j), \qquad i \neq j,$$

$$\mathbb{P}(A_i \cap A_j \cap A_k) = \mathbb{P}(A_i) \cdot \mathbb{P}(A_j) \cdot \mathbb{P}(A_k), \qquad i \neq j, i \neq k, j \neq k,$$

...

$$\mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_n) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \dots \cdot \mathbb{P}(A_n).$$

Independence of more that two events, cont'd

Say, three Events A, B and C are Independent, iff

$$\mathbb{P}(A\cap B)=\mathbb{P}(A)\cdot\mathbb{P}(B),\ \ \mathbb{P}(A\cap C)=\mathbb{P}(A)\cdot\mathbb{P}(C),\ \ \mathbb{P}(B\cap C)=\mathbb{P}(B)\cdot\mathbb{P}(C),$$

and

$$\mathbb{P}(A\cap B\cap C)=\mathbb{P}(A)\cdot\mathbb{P}(B)\cdot\mathbb{P}(C).$$

Please note that if events are Mutually Independent, then they are also Pairwise Independent, but the inverse implication is not true in general!

Exercise: Give an Example.

Remark: From this point on, when talking about the Independence of several Events, we will mean the **Mutual Independence**.

Example:

Problem: (Network Reliability Problem) Assume we have some computer network joining two nodes through n intermediate nodes. The probability that each intermediate node is working is p. What is the probability[1] that the connection between the initial and terminal nodes is working, given that

- Intermediate nodes are connected in a series
- Intermediate nodes are connected in a parallel way

Solution: OTB

[1] We intuitively assume that the working/non-working states for nodes are independent

Remark

Remark: Assume we have three Events A, B and C in an Experiment. The question is to calculate the Probability

$$\mathbb{P}(A \cap B \cap C).$$

Now,

- If A, B and C are Independent, then

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C).$$

- In the general case, by the Chain Rule for Probabilities,

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(B|A) \cdot \mathbb{P}(C|A \cap B).$$

This last formula, given Independence of A, B, C, is the same as above, since, if we have Independence,

$$\mathbb{P}(B|A) = \mathbb{P}(B) \qquad \text{and} \qquad \mathbb{P}(C|A \cap B) = \mathbb{P}(C).$$

Gambler's Fallacy

Problem: (Gambler's Fallacy) Assume we are tossing a fair die 20 times. The first 19 tosses were Heads. Are the chances that in the next toss we will have Heads higher than Tails?
Solution: OTB

Repeated Independent Trials Model

Consider the following Experiment: We are rolling 3 fair dice. This experiment can be considered as:

- One experiment, repeated just once, with possible outcomes: $(1,1,1)$, $(1,1,2)$,..., $(6,6,6)$;
- A 3-times repetition of a simple experiment of rolling a fair die

Here we will consider Experiments that consist of repetitions of a single (simple) Experiment. Each single (simple) Experiment in this case is called a Trial.
And if the Trials are independent (i.e., the knowledge of the result of some Trial is not changing the Probabilities of results in the other Trial), then we say that we have Repeated Independent Trials model.

Repeated Independent Trials Model

Here we consider two problems: The first one is the Binomial Model (of repeated trials):

- We have a Simple Experiment, called a Trial;
- We have an Event in this Experiment, say, A;
- If A will happen/appear, we call that a **Success**;
- We repeat our Simple Experiment n times (n=Number of Trials);
- We are interested in: how many times A will appear during that n Trials?

Examples:

Some Examples:

- We are rolling a die 4 times. And we are interested in how many 6s we will have in that 4 rolls. Here the simple Experiment, the Trial is rolling a die. And we have 4 Trials in our Experiment. And our Trials are independent. Our Event A is rolling 6;
- We have a box (or an urn) full of white, red, blue balls. 20 times we draw a ball at random, fix its color, without returning the ball to the box. Here the Simple Experiment, our Trial is drawing just one ball out of the box. We have 20 Trials. The Trials are not Independent, so we will not consider this type of problems in this lecture.

Examples:

Some Examples:

- Again, from the above box, 20 times we draw a ball at random, fix its color, and return that ball into the box. Here the Simple Experiment, our Trial is again drawing just one ball out of the box. We have 20 Trials. And the Trials are Independent. We are interested how many white or red balls will be drawn during that 20 Trials. So the Event A is we are drawing either a white or a red ball.
- We are rolling 2 dice 50 times (say, during the Nardi game). And we are interested in how many Iqi-Birs we have during the rolls. Here the Simple Experiment, Trial is rolling 2 dice. The number of Trials is 50. Trials are Independent. Our Event A is rolling Iqi-Bir.

Examples:

Some Examples:

- We are rolling 4 dice. And we are interested in the probability of having the sum 12. Well, this is not of the type of problems we will consider here, because it is not about some Event in a Trial, and about how many times that Event appeared during 4 repetitions.
- Can you give some real-world type problems?

Repeated Indep Trials: Binomial Probabilities

Now Assume:

- We have a Simple Experiment (Trial);
- We have an Event A in this simple Experiment; When A happens, we call it a **Success**, otherwise - **Failure**. So we consider Binary case: either we can have a "Success" or a "Failure". Say, Male/Female, Even/Odd, 0/1, on/off, pass/fail, win/loose, defective/non-defective, Klassik/Non-Klassik, heavier than 65Kg/lighter than or equal to 65 Kg etc.
- We assume the probability of having A in one Trial is p, so $\mathbb{P}(A) = p$;
- We repeat our Trial n times;
- We are interested in the probability that we will have exactly k successes in these n Trials, i.e., k times we will have A, and $n - k$ times we will have $\overline{A}$.

Repeated Indep Trials: Binomial Probabilities

Binomial Probabilities

For any $k = 0, 1, ..., n$,

$$\mathbb{P}(\text{Exactly } k \text{ successes in } n \text{ trials}) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}.$$

or, if we will denote the number of Successes in that n Trials by X,

$$\mathbb{P}(X = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}.$$

Intuition: OTB

Repeated Indep Trials: Binomial Probabilities

We can represent this in the form of a Table: Let X be the number of Successes in n Trials. Then the *Distribution* of X is given by:

X	0	1	2	...	n
$\mathbb{P}(X=k)$	$\binom{n}{0}p^0(1-p)^n$	$\binom{n}{1}p^1(1-p)^{n-1}$	$\binom{n}{2}p^2(1-p)^{n-2}$	...	$\binom{n}{n}p^n(1-p)^0$