CS 107, Probability, Spring 2019
Lecture 28

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AUA

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Content

- Normal Distribution
- Functions of Random Variables (aka Transformations of Random Variables)

LZ

This could be in one of our Midterms $\ddot\smile$

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Your answer is not correct! :P

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Recall also that if $X \sim \mathcal{N}(0, 1)$, then we say that X is Standard Normal r.v. (or has a Standard Normal Distribution).

Some Examples

Example: Assume that the heights of women in Armenia are Normally distributed with the Mean 158.1cm and Standard Deviation 5.7cm[1].

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- Calculate the Probability that the (randomly chosen) woman height will be smaller than 158.1cm.
- Calculate the Probability that the (randomly chosen) woman height will be larger than 170cm.

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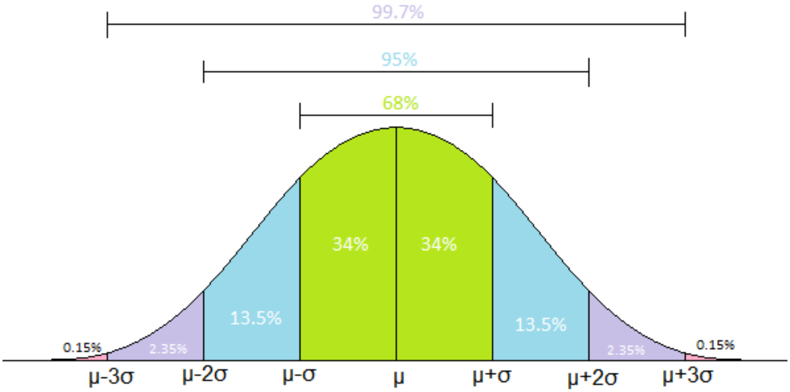

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- If $X \sim \mathcal{N}(0, 1)$, then
$$Y = \mu + \sigma \cdot X \sim \mathcal{N}(\mu, \sigma^2)$$

Functions of Random Variables: Making new R.V.s from the old ones

Functions of Random Variables

Given a r.v. X, one can form new r.v.s by applying functions on X.

[2] (18+) Measurable!

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Now, if X is a r.v., and $g : \mathbb{R} \to \mathbb{R}$ is some function[2], then the r.v.

$$Y = g(X)$$

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Example: For example, $Y = X^3$, $Y = \ln(X)$, $Y = \frac{X}{1+X}$, $Z = \sin(X)$,... are all (if defined, of course) new r.v.s obtained from the r.v. X.

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- Assume $X \sim Unif[-2, 2]$. What can be said about $Y = 3X + 1$?

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- Assume $X \sim Unif[-2, 2]$. What can be said about $Y = 3X + 1$? What about $Z = X^2$?

Simulation of Transformed RVs: R Examples

R Code

```
x <- runif(50000, min = -2, max = 2)
hist(x)
hist(x, freq = F, col = "cyan")
abline(h = 0.25, col = "red", lwd = 2)

y <- 3*x +1
hist(y, freq = F, col = "lightblue") #shows uniform

z <- x^2
hist(z, freq = F, col = "lightcyan") #shows non-uniform!
```