AUA CS 108, Statistics, Fall 2019

Lecture 20

Michael Poghosyan
YSU, AUA
michael@ysu.am, mpoghosyan@aua.am

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- Give some Estimators for $Geom(p)$ parameter p.

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- We find a good **Estimator** $\hat{\theta}$;
- We plug the Dataset values into the Estimator to get the **Estimate** for θ

Properties of Estimators

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Story about Good and Bad Estimators $\ddot\smile$

Risk, Mean Squared Error of the Estimator

From the last lecture:

Definition: The **Mean Squared Error** or the **Quadratic Risk** of the estimator $\hat{\theta}_n$ of θ is

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Definition: We say that the estimator $\hat{\theta}_n^1$ of θ is **preferable** to $\hat{\theta}_n^2$, another estimator of θ, if

$$MSE(\hat{\theta}_n^1, \theta) \leq MSE(\hat{\theta}_n^2, \theta), \qquad \forall \theta \in \Theta,$$

and there exists a θ s.t. $MSE(\hat{\theta}_n^1, \theta) < MSE(\hat{\theta}_n^2, \theta)$.

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Solution: OTB: Use the fact that for $X \sim Pois(\lambda)$,

$$\mathbb{E}(X) = \lambda = Var(X).$$

Best MSE Estimators

Now, having the idea of the Error of estimation for an Estimator, we can try to find the best Estimator in the sense of MSE, i.e., we can try to solve

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So we move forward considering other Properties of Estimators making them useful in the estimation process.

Bias, Biased and Unbiased Estimators

Definition: The **Bias** of Estimator $\hat{\theta}$ of θ is

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Note: We require the above hold **for any parameter value** since, if, say, it is correct for *some* values of θ, then it can happen that the true value of our unknown θ is exactly that value, for whihc we do not have the equality.

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- The idea of Unbiased Estimator is the following: if we will calculate Estimates many-many times using our Unbiased Estimator, and then average the obtained Estimates, we will obtain the (almost exact, exact when many-many$\to \infty$) value of our Parameter.

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- So we can say about an Unbiased Estimator as: **In average, it is Exact**
- Bias can be interpreted, in some sense, as the *accuracy* of the Estimator.

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Now, we choose several Estimators to estimate μ:

$$\hat{\mu}_1 = X_1, \qquad \hat{\mu}_2 = \frac{X_1 + X_3}{2}, \qquad \hat{\mu}_3 = \frac{X_1 + X_4}{10},$$

$$\hat{\mu}_4 = \overline{X}_n = \frac{X_1 + X_2 + ... + X_n}{n}.$$

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Problem: Check if each Estimator is Biased or Unbiased.

Solution: OTB