AUA CS 108, Statistics, Fall 2019

Lecture 16

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30 Sep 2019

Contents

Last Lecture ReCap

- Give the definition of the convergence in the a.s./ Probability / QM / Distributions sense.

Example

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and X_n are defined on the same Probability Space. Which of the followings are true (use only the definitions):

- $X_n \xrightarrow{\mathbb{P}} 0$;
- $X_n \xrightarrow{qm} 0$;
- $X_n \xrightarrow{D} 0$?

Solution: OTB

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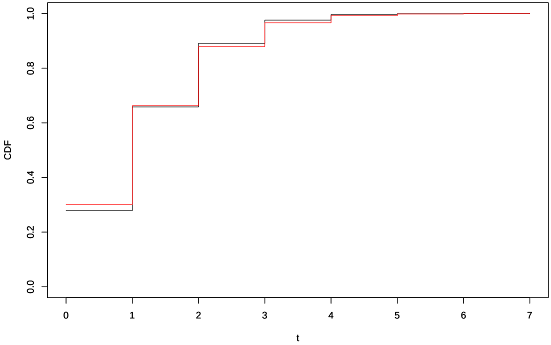
```
lambda <- 1.2; n <- 10; t <- seq(0,7, 0.1)
plot(t,pbinom(t, size = n, prob = lambda/n), type = "s", ylim = c(0,1), ylab = "CDF")
par(new = T)
plot(t, ppois(t, lambda = lambda), type = "s", col = "red", ylim = c(0,1), ylab = "CDF")
```



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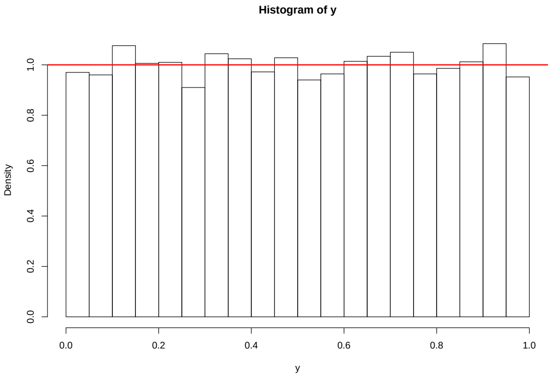
```r
n <- 10000 ## We use Y_n
m <- 10000 ## No. of generated numbers
y <- runif(m, min = 0, max = n)/n
hist(y, freq = F)
abline(h = 1, col = "red", lwd = 2)
```



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Theorem: Assume $X_n \stackrel{a.s.(P,qm,D)}{\longrightarrow} X$ and $Y_n \stackrel{a.s.(P,qm,D)}{\longrightarrow} Y$.

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Relationship between Convergence Types

Theorem: (Convergence Relationship Diagram)

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Note: Inverse implications are not always correct. But, say, the following holds: If $X_n \xrightarrow{D} X$ and $X \equiv \textit{constant}$, then $X_n \xrightarrow{P} X$ (X_n and X are defined on the same Probability space).

Limit Theorems

Sequence of IID r.v.

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- X_n-s are independent. Say, in particular,

$$Var(X_1 + X_2 + ... + X_n) = Var(X_1) + Var(X_2) + ... + Var(X_n) = n \cdot Var(X_1).$$

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