CS 107, Probability, Spring 2020
Lecture 20

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Content

- The Cumulative Distribution Function
- Discrete Random Variables

Last Lecture ReCap

- RVs are functions from Ω, the Sample Space of an Experiment, to $\mathbb{R}$, i.e., some numerical quantities associated with the outcome of the Experiment;
- The general problem we want to solve concerning Random Variables is to be able to calculate the Probabilities $\mathbb{P}(X \in A)$, where X is the RV, and $A \subset \mathbb{R}$.
- Example: Let X be the number of goals in the upcoming (concrete) football game. What is the Probability that X will be less than 3, i.e. what is $\mathbb{P}(X < 3)$?
- It turns out that, to calculate Probabilities $\mathbb{P}(X \in A)$, it is enough to be able to calculate the Probabilities $\mathbb{P}(X \leq x)$ for any x, and we define the CDF of X to be

$$F(x) = F_X(x) = \mathbb{P}(X \leq x), \qquad x \in \mathbb{R}.$$

Properties of the CDF

Assume $F(x)$ is the CDF of the r.v. X:

$$F(x) = F_X(x) = \mathbb{P}(X \leq x), \qquad x \in \mathbb{R},$$

Then:

Properties of CDF

- $0 \leq F(x) \leq 1$, for any $x \in \mathbb{R}$;
- $F(-\infty) = 0$ and $F(+\infty) = 1$;
- F is an increasing function, i.e., if $x_1 \leq x_2$, then $F(x_1) \leq F(x_2)$;
- F is right-continuous at every point, i.e. $F(x_0+) = F(x_0)$ at any $x_0 \in \mathbb{R}$

Characterization of the CDF

Now we want to answer the inverse question: which functions F can serve as CDFs for some r.v.? It turns out that the above properties **completely characterize CDFs**:

Characterization of CDFs

Assume $F : \mathbb{R} \to \mathbb{R}$ is a function satisfying

- $0 \leq F(x) \leq 1$, for any $x \in \mathbb{R}$;
- $F(-\infty) = 0$ and $F(+\infty) = 1$;
- F is an increasing function, i.e., if $x_1 \leq x_2$, then $F(x_1) \leq F(x_2)$;
- F is right-continuous at every point, i.e. $F(x_0+) = F(x_0)$ at any $x_0 \in \mathbb{R}$

Then there exists an Experiment with a Sample Space Ω, Probability Measure $\mathbb{P}$ and a r.v. $X : \Omega \to \mathbb{R}$ such that $F(x)$ is the CDF of X: $F(x) = F_X(x), \qquad x \in \mathbb{R}$.

Graphical Example of a CDF

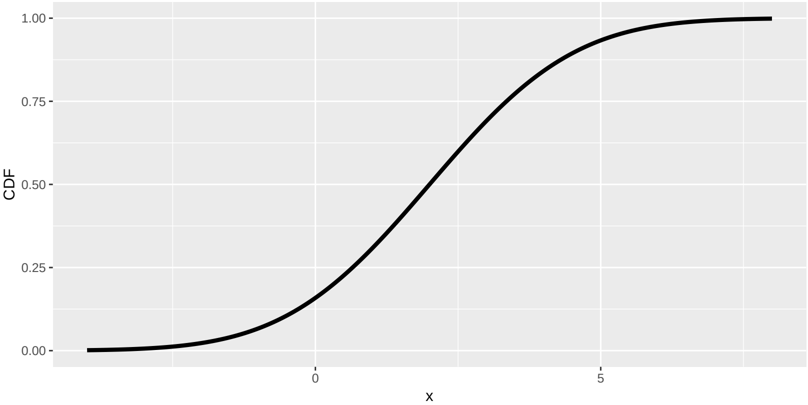

Figure: CDF of some r.v. X

Examples

Example: Which of the following functions define a CDF of some Random Variable:

- $F(x) = x^2$, $x \in \mathbb{R}$;
- $F(x) = \sqrt{x}$, $x \geq 0$;
- $F(x) = \dfrac{1}{1+x^2}$, $x \in \mathbb{R}$;
- $F(x) = 0$, if $x \leq 0$ and $F(x) = 1$, if $x > 0$;
- $F(x) = 0$, if $x < 0$ and $F(x) = 1$, if $x \geq 0$;
- $F(x) = 0$, if $x \leq 0$ and $F(x) = 1 - e^{-x}$, if $x > 0$;
- $\sigma(x) = sigmoid(x) = \dfrac{1}{1+e^{-x}}$, $x \in \mathbb{R}$.

Solution: OTB

Examples

Example: Assume we are tossing a fair coin 2 times, and let X be the number of Heads in that Experiment. Find the CDF of X analytically and draw the graph of that CDF.

Solution: OTB

Using CDFs to calculate Probabilities

Now recall that our general Problem was to be able to calculate the Probabilities of the type $\mathbb{P}(X \in A)$, where X is a rv. and $A \subset \mathbb{R}$. The following Proposition is helping us to calculate these Probabilities. Assume X is a r.v., and $F(x)$ is its CDF. Then:

Probabilities through CDF

- $\mathbb{P}(X = a) = F(a) - F(a-)$;
- $\mathbb{P}(a < X \leq b) = F(b) - F(a)$;
- $\mathbb{P}(a \leq X \leq b) = F(b) - F(a-)$;
- $\mathbb{P}(a \leq X < b) = F(b-) - F(a-)$;
- $\mathbb{P}(a < X < b) = F(b-) - F(a)$;

Here it is possible that $a = -\infty$ or/and $b = +\infty$

Reading Probabilities from the graph of CDF

Problem: Is the following a CDF of some r.v. X? If yes, calculate $\mathbb{P}(X = 2)$, $\mathbb{P}(2.5 < X \le 5)$, $\mathbb{P}(X \le 5)$, $\mathbb{P}(X > 0)$:

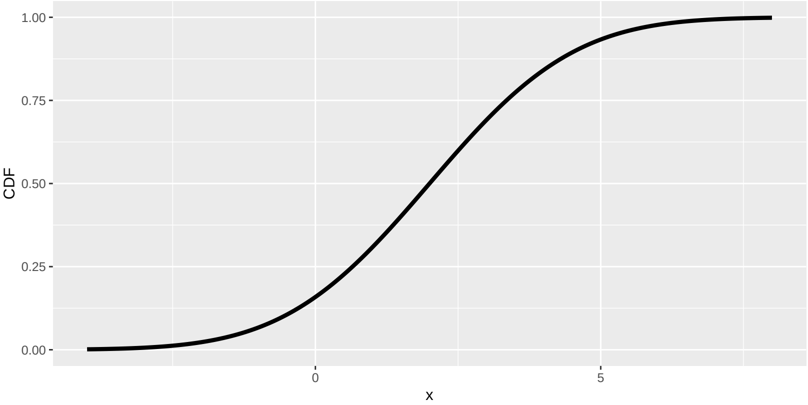

Figure: CDF of some r.v. X