CS 107, Probability, Spring 2020
Lecture 04

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Content

- Properties of the Probability Measure

LZ

Theorem: You can decompose a solid ball into finitely many pieces (5 pieces will suffice) and reassemble them using only rigid motions (rotations + translations) to get two solid balls of the same size.

In fact, you can decompose an orange into finitely many pieces, and then reassemble that pieces to obtain the Earth. Well, the material will be the orange one, of course, but the size will be the Earth one.

This is the famous Banach-Tarski Paradox. See `https://www.youtube.com/watch?v=s86-Z-CbaHA`

Last Lecture ReCap

Last time we have defined what the Probability (Measure) is: we have

- An Experiment with a Sample Space Ω;
- $\mathcal{F}$ is the set of all Events (in our case, $\mathcal{F}$ is the power set of Ω);
- and a function $\mathbb{P}$, defined for any Event, with the Properties:

P1: For any $A \in \mathcal{F}$, $\mathbb{P}(A) \geq 0$;

P2: $\mathbb{P}(\Omega) = 1$;

P3: For any sequence $A_n \in \mathcal{F}$, $n = 1, 2, \ldots$ with $A_i \cap A_j = \varnothing$ for $i \neq j$,

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

$\mathbb{P}(A)$ is called the Probability of the Event A.

Properties of the Probability Measure

1. $\mathbb{P}(\varnothing) = 0$;
2. if $A, B \in \mathcal{F}$ are mutually exclusive events, i.e., if $A \cap B = \varnothing$, then
$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B);$$
3. for any event $A \in \mathcal{F}$,
$$\mathbb{P}(\overline{A}) = 1 - \mathbb{P}(A);$$
Here $\overline{A} = A^c = \Omega \setminus A$.

Properties of the Probability Measure

4. If $A_1, A_2, ..., A_n \in \mathcal{F}$ are pairwise disjoint (mutually exclusive), i.e., if $A_i \cap A_j = \varnothing$ for $i \neq j$, then

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i);$$

5. for any events $A, B \in \mathcal{F}$ (not necessarily disjoint),

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B);$$

Intermezzo

Recall the problem from the last lecture:

Assume our mobile phone Weather App says that there is a 50% chance of snow for this Saturday, and also 50% chance of snow this Sunday.
Is it true that it will snow for sure (i.e., with probability 1) this weekend?

What is your answer now?

Properties of the Probability Measure

6. If $A_1, A_2, ..., A_n \in \mathcal{F}$ are events, not necessarily disjoint, then

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + ... + \mathbb{P}(A_n) -$$

$$-\mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_3) - ... - \mathbb{P}(A_{n-1} \cap A_n) +$$

$$+\mathbb{P}(A_1 \cap A_2 \cap A_3) + \mathbb{P}(A_1 \cap A_2 \cap A_4) + ... + \mathbb{P}(A_{n-2} \cap A_{n-1} \cap A_n) - ...$$

$$... + (-1)^{n-1}\mathbb{P}(A_1 \cap A_2 \cap ... \cap A_n).$$

This is the general version of the previous property, and is called the inclusion-exclusion principle.

Example:

$$\mathbb{P}(A \cup B \cup C) =$$

Properties of the Probability Measure

7. (Boole's Inequality) For any sequence A_n of Events, $n = 1, 2, 3, \ldots$,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i);$$

8. for any events[1] $A, B \in \mathcal{F}$

$$\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A \cap B);$$

9. if $A, B \in \mathcal{F}$ are two events such that $A \subset B$, then

$$\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A);$$

[1] Recall that

$$B \setminus A = \{x : x \in B \text{ and } x \notin A\}.$$

Properties of the Probability Measure

10. if A and B are two events such that $A \subset B$, then

$$\mathbb{P}(A) \leq \mathbb{P}(B);$$

11. for any event $A \in \mathcal{F}$,

$$0 \leq \mathbb{P}(A) \leq 1;$$

Example

Assume that in some university, for the CS department, the probability that a sophomore student knows Java programming is 50%, and the probability that he/she knows C++ programming is 60%. We consider the following Events:

$$\textit{Java} = \text{Student knows Java programming}$$

$$\text{C++} = \text{Student knows C++ programming.}$$

Then

$$\mathbb{P}(\textit{Java}) = 0.5 \qquad \text{and} \qquad \mathbb{P}(\text{C++}) = 0.6.$$

- What is the Experiment here?
- Is it possible that no student with Java knowledge knows C++ programming? I.e., can *Java* and C++ be mutually exclusive?

Example, cont'd

Recall that

$$\mathbb{P}(\mathit{Java}) = 0.5 \qquad \text{and} \qquad \mathbb{P}(C++) = 0.6.$$

- What is the probability that a student knows Java or C++ programming?
- If we assume that the probability of knowing both Java and C++ programming is 25% for the student, what is the probability that the student knows Java or C++ programming?
- And what is the probability that a student doesn't know neither Java nor C++?
- What is the probability that the student knows exactly one of these languages?