AUA CS108, Statistics, Fall 2020

Lecture 41

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02 Dec 2020

Contents

Test for the Mean of the Normal, σ is known: Z-Test

Model: Our Data comes from $\mathcal{N}(\mu, \sigma^2)$, σ is known; Our (unknown) Parameter is μ

Hypothesis: We are given some μ_0, and we want to Test:

- Case 1: $\mathcal{H}_0: \ \mu = \mu_0 \qquad vs \qquad \mathcal{H}_1: \ \mu \neq \mu_0$
- Case 2: $\mathcal{H}_0: \ \mu = \mu_0 \qquad vs \qquad \mathcal{H}_1: \ \mu > \mu_0$
- Case 3: $\mathcal{H}_0: \ \mu = \mu_0 \qquad vs \qquad \mathcal{H}_1: \ \mu < \mu_0$

Significance Level: $\alpha \in (0, 1)$;

Random Sample: We take $X_1, X_2, ..., X_n \sim \mathcal{N}(\mu, \sigma^2)$;

Test for the Mean of the Normal, σ is known: Z-Test

Test Statistics: We take

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}.$$

Under $\mathcal{H}_0$,

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1).$$

Test for the Mean of the Normal, σ is known: Z-Test

Rejection Region: Now we choose the **RR**. The idea is:

If $\mathcal{H}_0$ is True, then Z is close to 0

We consider our 3 cases:

Case 1: for Testing $\mathcal{H}_0: \ \mu = \mu_0 \quad \text{vs} \quad \mathcal{H}_1: \ \mu \neq \mu_0$

$$RR = \{|Z| > c\}.$$

Case 2: for Testing $\mathcal{H}_0: \ \mu = \mu_0 \quad \text{vs} \quad \mathcal{H}_1: \ \mu > \mu_0$

$$RR = \{Z > c\}.$$

Case 3: for Testing $\mathcal{H}_0: \ \mu = \mu_0 \quad \text{vs} \quad \mathcal{H}_1: \ \mu < \mu_0$

$$RR = \{Z < c\}.$$

Here the Critical Value c is yet to be determined.

Choosing the Critical Value

Now, let us choose c.

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We choose c from the requirement to have a Test with Significance Level α:

$$\alpha =$$

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This means, geometrically, that

$$c = z_{1-\alpha/2},$$

the $1 - \frac{\alpha}{2}$-level quantile of the Standard Normal Distribution.

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So, finally, we have the Test for the Case 1: given μ_0, σ, Observations and Significance Level α, calculate $Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$.

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So, finally, we have the Test for the Case 1: given μ_0, σ, Observations and Significance Level α, calculate $Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$.

- If $|Z| > z_{1-\alpha/2}$, **Reject** $\mathcal{H}_0$;
- If $|Z| \leq z_{1-\alpha/2}$, **Do Not Reject** $\mathcal{H}_0$.

Z-Test, Complete Version

Model: $X_1, X_2, ..., X_n \stackrel{IID}{\sim} \mathcal{N}(\mu, \sigma^2)$, σ **is known**, the Parameter (our unknown) is μ;

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$\mathcal{H}_1$ is	RR is
$\mu \neq \mu_0$	$\lvert Z \rvert > z_{1-\frac{\alpha}{2}}$
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$\mu > \mu_0$	$t > t_{n-1,1-\alpha}$
$\mu < \mu_0$	$t < t_{n-1,\alpha}$

t-test Example

Example: I have generated in **R** a Sample of Size 20 from $\mathcal{N}(3.12, 2^2)$ and made some rounding:

```
set.seed(20112019)
n <-20; sigma <- 2
obs <-  rnorm(n, mean = 3.12, sd = sigma)
obs <- round(obs, digits = 2); obs
```

```
##  [1]  1.80  5.60  1.10  3.20  4.91  5.15  1.76  2.47  0.
## [13]  3.98  4.79  1.98  4.50  3.52  4.13 -0.08  3.87
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Now, let us forget about the fact that the actual value of μ is 3.12 and that $\sigma = 2$, and do some Testing, just assuming that our Observation is coming from a Normal Distribution. Say, let us Test, at the 5% Significance Level, the following Hypothesis:

$$\mathcal{H}_0: \ \mu = 4 \qquad \textit{vs} \qquad \mathcal{H}_1: \ \mu \neq 4.$$

Example, Cont'd

First, we calculate *t*-statistic:

```
mu0 <- 4;
x.bar <- mean(obs); s <- sd(obs);
t <- (x.bar - mu0)/(s/sqrt(n)); t
```

```
## [1] -1.795358
```

Example, Cont'd

First, we calculate t-statistic:

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mu0 <- 4;
x.bar <- mean(obs); s <- sd(obs);
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```
## [1] -1.795358
```

Now, we calculate the critical value, the quantile $t_{n-1,1-\alpha/2}$:

```
a <- 0.05
c <- qt(1-a/2, df = n-1); c
```

```
## [1] 2.093024
```

Example, Cont'd

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Finally, we check if t is in RR, i.e., if $|t| > t_{n-1,1-\alpha/2}$:

```r
abs(t) > c
```

```
## [1] FALSE
```

Example, Cont'd

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So the decision is:

Example, Cont'd

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Finally, we check if t is in RR, i.e., if $|t| > t_{n-1,1-\alpha/2}$:

```
abs(t) > c
```

```
## [1] FALSE
```

So the decision is: **Fail to Reject** $\mathcal{H}_0$ at 5% level.

Example, Cont'd

Now, the same, but with an **R** built-in function `t.test`:

```r
t.test(obs, mu = mu0, conf.level = 0.95)
```

```
##
##  One Sample t-test
##
## data:  obs
## t = -1.7954, df = 19, p-value = 0.08852
## alternative hypothesis: true mean is not equal to 4
## 95 percent confidence interval:
##  2.524009 4.112991
## sample estimates:
## mean of x
##    3.3185
```

Example, Cont'd

Now, for the same Data, let us Test, at the 10% Significance Level, the following Hypothesis:

$$\mathcal{H}_0 : \ \mu = 3 \qquad \textit{vs} \qquad \mathcal{H}_1 : \ \mu > 3.$$

Example, Cont'd

Now, for the same Data, let us Test, at the 10% Significance Level, the following Hypothesis:

$$\mathcal{H}_0: \ \mu = 3 \qquad vs \qquad \mathcal{H}_1: \ \mu > 3.$$

```
t.test(obs, mu=3,alternative="greater", conf.level=0.9)
```

```
##
##  One Sample t-test
##
## data:  obs
## t = 0.83906, df = 19, p-value = 0.2059
## alternative hypothesis: true mean is greater than 3
## 90 percent confidence interval:
##  2.814508      Inf
## sample estimates:
## mean of x
##    3.3185
```

Note

Note: In **R** `t.test` command, the default values for parameters are:

- `mu = 0`
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Important Note: There are different ways to test a Hypothesis: one is by Test Statistics, another one is by CIs, and the next, easiest one is by *p*-Values.

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Important Note: There are different ways to test a Hypothesis: one is by Test Statistics, another one is by CIs, and the next, easiest one is by *p*-Values. All Statistical Software are calculating *p*-Values when doing testing. And the Decision based on the *p*-Value is:

- If *p*-Value$< \alpha$, then we Reject $\mathcal{H}_0$
- If *p*-Value$\geq \alpha$, then we Fail to Reject $\mathcal{H}_0$

t-Test Example

Example: Recall the Etruscans-Italians Problem: Scientists have a data about 84 skull sizes (widths) of adult Etruscans, and the problem was to see if Etruscans were Italians.

Also Scientists believe that the skull size is not changing much through time, and modern adult Italians skull size is in average 132.4mm.

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We make a Hypothesis:

$$\mathcal{H}_0 : \text{Etruscans were Italians} \quad vs \quad \mathcal{H}_1 : \text{They were not}$$

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In mathematical terms,

$$\mathcal{H}_0 : \mu = 132.4 \qquad \textit{vs} \qquad \mathcal{H}_1 : \mu \neq 132.4$$

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In mathematical terms,

$$\mathcal{H}_0 : \mu = 132.4 \qquad vs \qquad \mathcal{H}_1 : \mu \neq 132.4$$

Now let's test, at 95%, this Hypo in **R**:

t-Test Example

```
library(Rlab)
data <- etruscan
x <- data$width[data$group == "ancient"]

t.test(x, mu = 132.4)
```

```
##
##  One Sample t-test
##
## data:  x
## t = 17.46, df = 83, p-value < 2.2e-16
## alternative hypothesis: true mean is not equal to 132.4
## 95 percent confidence interval:
##  142.4781 145.0695
## sample estimates:
## mean of x
##  143.7738
```

Test for the Normal Variance, μ is known

Model: $X_1, X_2, ..., X_n \stackrel{IID}{\sim} \mathcal{N}(\mu, \sigma^2)$, μ **is known**, the Parameter (our unknown) is σ^2;

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$\mathcal{H}_1$ is	RR is
$\sigma^2 \neq \sigma_0^2$	$\chi^2 \notin \left[\chi^2_{n,\frac{\alpha}{2}}, \chi^2_{n,1-\frac{\alpha}{2}}\right]$
$\sigma^2 > \sigma_0^2$	$\chi^2 > \chi^2_{n,1-\alpha}$
$\sigma^2 < \sigma_0^2$	$\chi^2 < \chi^2_{n,\alpha}$

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Null Hypothesis: $\mathcal{H}_0 : \ \sigma^2 = \sigma_0^2$

Significance Level: $\alpha \in (0, 1)$;

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$\sigma^2 \neq \sigma_0^2$	$\chi^2 \notin \left[\chi^2_{n-1,\frac{\alpha}{2}}, \chi^2_{n-1,1-\frac{\alpha}{2}}\right]$
$\sigma^2 > \sigma_0^2$	$\chi^2 > \chi^2_{n-1,1-\alpha}$
$\sigma^2 < \sigma_0^2$	$\chi^2 < \chi^2_{n-1,\alpha}$

Large Sample Hypothesis Testing

aka

Asymptotic Testing

Asymptotic Test for the Mean of General Distribution

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Test Statistics: $W = \dfrac{\hat{\theta}^{MLE} - \theta_0}{1/\sqrt{n \cdot \mathcal{I}(\theta_0)}}$ or $W = \dfrac{\hat{\theta}^{MLE} - \theta_0}{1/\sqrt{n \cdot \mathcal{I}\left(\hat{\theta}^{MLE}\right)}}$

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