CS 107, Probability, Spring 2019
Lecture 36

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AUA

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Content

- Transformation of Random Vectors
- Independent Random Variables

Transform of Jointly Continuous R.V.s

Now assume X and Y are Continuous r.v.s with Joint PDF $f_{X,Y}(x,y)$. Also, $g : \mathbb{R}^2 \to \mathbb{R}$ is a given function. Let $Z = g(X,Y)$. Again we are interested in the Distribution of Z.

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Note: Second, we can find the CDF of Z by:

$$F_Z(x) = \mathbb{P}(g(X, Y) \leq x) = \iint_{g(u,v) \leq x} f_{X,Y}(u, v)\, dudv.$$

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Here we will consider one very important case: we will restrict our attention to the sum of X and Y,

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- If X and Y are **Jointly Continuous** with the Joint PDF $f_{X,Y}(x, y)$, then $Z = X + Y$ will be a Continuous r.v. with the PDF

$$f_{X+Y}(x) = \int_{-\infty}^{\infty} f_{X,Y}(t, x - t)\, dt \qquad \forall x \in \mathbb{R}.$$

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Now, we define the Independence of 2 r.v. in a similar way.

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If X and Y are not Independent, then we say they are Dependent.

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- If X and Y are Jointly Continuous, then $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$ for any[a] $x, y \in \mathbb{R}$, where $f_{X,Y}$, f_X, f_Y are the Joint PDF of X, Y and the Marginal PDFs of X and Y, respectively.

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Independence of Discrete R.V.s

Assume X and Y are Discrete r.v.s with the PMFs

Values of X	x_1	x_2	...
$\mathbb{P}(X = x)$	p_1	p_2	...

Values of Y	y_1	y_2	...
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Then $X \perp\!\!\!\perp Y$ if and only if their Joint PMF has the form:

$Y \setminus X$	x_1	x_2	$\dots$	$\mathbb{P}(Y = y)$
y_1	$p_1 \cdot q_1$	$p_2 \cdot q_1$	$\dots$	q_1
y_2	$p_1 \cdot q_2$	$p_2 \cdot q_2$	$\dots$	q_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\mathbb{P}(X = x)$	p_1	p_2	$\dots$	

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Example: If X and Y are Independent, then so are X^2 and $\cos(2Y+1)$, i.e., $X^2 \perp\!\!\!\perp \cos(2Y+1)$.

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- Mutual Independence is that for any subsets $A_k \subset \mathbb{R}$, $\mathbb{P}(X_1 \in A_1, ..., X_n \in A_n) = \mathbb{P}(X_1 \in A_1) \cdot ... \cdot \mathbb{P}(X_n \in A_n)$

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- Mutual Independence implies Pairwise Independence, but the inverse implication is not correct, in general.
- Later, and in Statistics, we will use a lot the statement $X_1, ..., X_n$ *are Independent, Identically Distributed (IID) r.v.s*, meaning that X_k-s are Mutually Independent, and they are Identically Distributed.

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Example: Assume X and Y are Discrete with the following PMFs:

Values of X	-2	0	1
$\mathbb{P}(X = x)$	0.1	0.6	0.3

Values of Y	10	20
$\mathbb{P}(Y = y)$	0.2	0.8

Assume also that $X \perp\!\!\!\perp Y$.

- Find the Joint PMF of X and Y;
- Calculate $\mathbb{P}(X \cdot Y < 10)$.

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Type 2 Examples: Given Individual Distributions of X and Y, and Independence, we form the Joint Distribution and calculate Probabilities.

- Assume $X \sim Unif[0, 3]$, $Y \sim Exp(2)$ and $X \perp\!\!\!\perp Y$. Find $\mathbb{P}(X^2 + Y^2 \leq 1)$.