CS 107 Section A - Probability

Spring 2020, AUA

Homework No. 02

Due time/date: 09:35AM, 07 February, 2020

Note: Supplementary Problems will not be graded, but you are very advised to solve them and to discuss later with TA or Instructor.

Problem 1. Let $A, B \subset \Omega$ are events. We denote by $A \triangle B$ the Event that *exactly one of the Events A and B happened*, i.e. either A or B happened, but not both[1].

a. Express $A \triangle B$ in terms of A and B;
b. Express, with a proof (using the axioms and properties of a Probability Measure) the probability $\mathbb{P}(A \triangle B)$ in terms of $\mathbb{P}(A \cup B)$ and $\mathbb{P}(A \cap B)$.
c. Express, again with a proof, the probability $\mathbb{P}(A \triangle B)$ in terms of $\mathbb{P}(A), \mathbb{P}(B)$ and $\mathbb{P}(A \cap B)$.

Problem 2. Assume $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and

$$4\mathbb{P}(\{\omega_1\}) = 2\mathbb{P}(\{\omega_2\}) = \mathbb{P}(\{\omega_3\}).$$

Calculate

$$\mathbb{P}(\{\omega_1, \omega_3\}).$$

Problem 3. A biased die is rolled (biased means that the probabilities of getting 1, 2, ..., 6 are not the same, and even the probability of some outcomes can be 0). Our Sample Space is

$$\Omega = \{1, 2, 3, 4, 5, 6\}.$$

Assume we know that

$$\mathbb{P}(\{1\}) = 0.2 \qquad \text{and} \qquad \mathbb{P}(\{2\}) = 0.1.$$

a. Find the probability that the number on the top face of our die will be greater than or equal to 3;
b. Find the minimal possible value of $\mathbb{P}(\{1, 2, 3\})$.
c. Give an example (i.e., choose the probabilities of each side of our die) such that the minimum in Part **b.** is attained.

Problem 4. Let A, B, C be some events such that A and C are mutually exclusive, and

$$\mathbb{P}(A) = 0.3, \quad \mathbb{P}(B) = \mathbb{P}(C) = 0.4, \quad \mathbb{P}(A \cap B) = \mathbb{P}(B \cap C) = 0.1.$$

Our aim is to calculate the probability $\mathbb{P}(A \cup \overline{B} \cup \overline{C})$. You are free to follow the steps below or do in another way[2].

[1] In Mathematics, this is called the symmetric difference between the sets A and B

[2] In fact, there are much easier ways to calculate this probability, but the idea of these steps is to learn to work with the properties of the Probability Measure.

a. Write the formula to expand $\mathbb{P}(A \cup \overline{B} \cup \overline{C})$;

b. Use the relationship $A \cap \overline{B} = A \setminus (A \cap B)$ to calculate $\mathbb{P}(A \cap \overline{B})$;

c. Use the de Morgan's rule for $\overline{B} \cap \overline{C}$ to calculate the probability $\mathbb{P}(\overline{B} \cap \overline{C})$;

d. Calculate now $\mathbb{P}(A \cup \overline{B} \cup \overline{C})$;

e. Give an explanation for the result geometrically, by using the Venn diagrams;

f. (Supplementary) Find an easy way to calculate the required probability.

Problem 5. Assume $\mathbb{P}(A) = 0.6$ and $\mathbb{P}(B) = 0.7$.

a. Prove that
$$0.3 \leq \mathbb{P}(A \cap B) \leq 0.6.$$

b. (Supplementary) Prove that, in general,
$$\mathbb{P}(E) + \mathbb{P}(G) - 1 \leq \mathbb{P}(E \cap G) \leq \min\{\mathbb{P}(E), \mathbb{P}(G)\}.$$

c. (Supplementary) Find a similar exact bound for $\mathbb{P}(E \cup G)$ in terms of $\mathbb{P}(E)$ and $\mathbb{P}(G)$.

d. (Supplementary) Give examples showing that $\mathbb{P}(A \cap B) = 0.6$ is possible, and another example showing that $\mathbb{P}(A \cap B) = 0.3$ is possible in part **a.**.

Problem 6. Assume we are choosing randomly a natural number, so our Sample Space is
$$\Omega = \mathbb{N} = \{1, 2, 3, 4, ...\}$$
and assume the probability to choose the number $k \in \mathbb{N}$ is $\frac{c}{3^k}$, where c is some fixed constant, i.e.
$$\mathbb{P}(\{k\}) = \frac{c}{3^k}, \qquad k \in \mathbb{N}.$$

a. Find c;

b. Write the probabilities in the table form;

c. Calculate the probability of the following event: the chosen number is divisible by 4.

Problem 7. We roll two symmetrical dice. Find the probability of the following events:

a. The sum of the numbers shown will be 6;

b. The sum of the numbers shown will be 8;

c. The sum of the numbers shown is in the set $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$;

d. Both numbers are odd.

Problem 8. (from [R][3]) An elementary school is offering 3 language classes: one in Spanish, one in French, and one in German. The classes are open to any of the 100 students in the school. There are 28 students in the Spanish class, 26 in the French class, and 16 in the German class. There are 12 students who are in both Spanish and French, 4 who are in both Spanish and German, and 6 who are in both French and German. In addition, there are 2 students taking all 3 classes.

[3] Our textbook by Ross.

a. If a student is chosen randomly, what is the probability that he or she is not in any of the language classes?
b. If a student is chosen randomly, what is the probability that he or she is taking exactly one language class?
c. If 2 students are chosen randomly, what is the probability that at least 1 is taking a language class?