CS 107, Probability, Spring 2019
Lecture 44

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AUA

- 2 lectures

Content

- Intro to Markov Chains

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Definition of the Markov Chain

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Assume $X_0, X_1, X_2, \ldots$ is a sequence of r.v.s (Discrete Stochastic Process), which take values from $S = \{1, 2, \ldots, N\}$. We say that X_n, $n = 0, 1, 2, \ldots$ is a (Finite State Discrete Time) **Markov Chain**, if

$$\mathbb{P}(X_{t+1} = j | X_t = i, X_{t-1} = k, \ldots, X_0 = m) = \mathbb{P}(X_{t+1} = j | X_t = i)$$

for any time t, for any state $j, i, k, \ldots, m$.

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for any time t, for any state $j, i, k, \dots, m$.

Interpretation: Given today's State, tomorrow's State is independent of the past States. Or, in other words, today's information is enough to completely determine the probabilities of Tomorrow's states.

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We will consider only Time-Homogeneous Markov Chains (MC). We will say that our MC is Time-Homogeneous, if

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The $N \times N$ Matrix $P = [p_{ij}]$ is called the Transition Probability Matrix. Row sums of P are equal to 1.

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Of course, in this case we need to have $\sum_{j=1}^{N} \pi_i = 1$.

Markov Chain Transition Probability Graph

Sometimes people describe the Markov Chain by drawing the Transition Probability Graph: OTB

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Give the Transition Probability Graph OTB!

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- And Google itself is based on Markov Chains! The heart and idea of the Google is PageRank Algorithm developed by Larry Page and Sergei Brin, who later founded Google. See https://en.wikipedia.org/wiki/PageRank or again just Google "PageRank Markov Chain"

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- Random Walks are examples of Markov Chains
- Ideas of Markov Chains and Processes are important in Language Models (n-gram Models), Machine Learning (HMM, Hidden Markov Models), Reinforcement Learning (Markov Decision Processes), in Simulations (MCMC, Markov Chain Monte Carlo) etc.

Some Problems Concerning Markov Chains

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It is easy to see, using the Markov property, that this can be calculated as:

$$\mathbb{P}(X_0 = i_0, X_1 = i_1, ..., X_n = i_n) =$$

$$\mathbb{P}(X_0 = i_0) \cdot \mathbb{P}(X_1 = i_1 | X_0 = i_0) \cdot \mathbb{P}(X_2 = i_2 | X_1 = i_1) \cdot ... \cdot \mathbb{P}(X_n = i_n | X_{n-1} = i_{n-1})$$

$$= \mathbb{P}(X_0 = i_0) \cdot p_{i_0 i_1} \cdot p_{i_1 i_2} \cdot ... \cdot p_{i_{n-1} i_n}.$$

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Having the ability to calculate the Probabilities of Paths

$$\mathbb{P}(X_0 = i_0, X_1 = i_1, ..., X_n = i_n),$$

we can calculate, say, what is the most probable path of movement, say, for the next 3 time instants or what is the most probable state at time $t = 3$.