CS 107 Section B - Probability

Spring 2019, AUA

Homework No. 10

Due time/date: 10:35AM, 19 April, 2019

Note: Supplementary Problems will not be graded, but you are very advised to solve them and to discuss later with TA or Instructor.

Note: Please provide your answers in the form of a decimal number, by calculating and simplifying fractions, with the accuracy of 2 digits after the period.

Problem 1. Assume X and Y are Jointly Distributed r.v.s. Is it true, in general, that

$$\mathbb{P}(X \leq a, Y \leq b) = 1 - \mathbb{P}(X > a, Y > b) \ ?$$

Justify your answer.

Problem 2. Assume we have a box containing 5 white, 4 green and 7 black balls. We pick at random 3 balls. Let X be the number of white balls taken, and Y be the number of black balls taken.

a. Find the Joint PMF of X and Y, if the balls are taken with replacement, i.e., we return each time the taken ball into the box;
b. Calculate $\mathbb{P}(X \leq 2, Y \geq 2)$ and $\mathbb{P}(X - Y \leq 2)$;
c. (Supplementary) Find the Joint PMF of X and Y, and the above probabilities, if the balls are taken without replacement, i.e., we are not returning the taken ball into the box.

Problem 3. Fig. 1 show the part of the graph of some function $F(x, y)$. Is it a Joint CDF for some r.v.'s X and Y? Explain your reasoning.

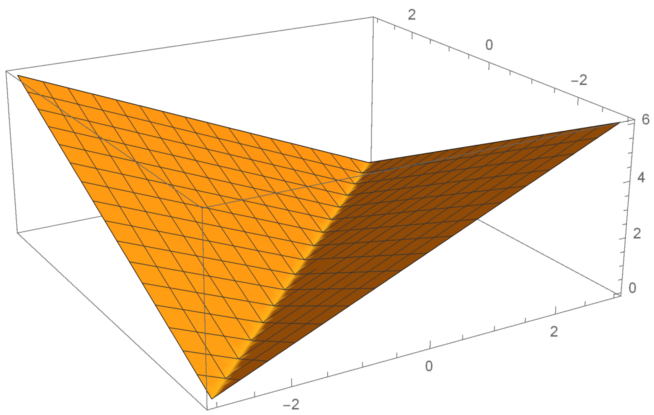

Figure 1: The graph of $F(x, y)$

Problem 4. Is the function $f(x, y)$ a Joint PDF for some r.v.'s X and Y? Explain your reasoning.

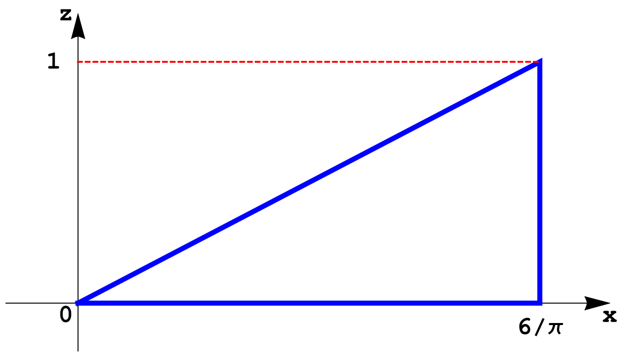

Figure 2: Triangle

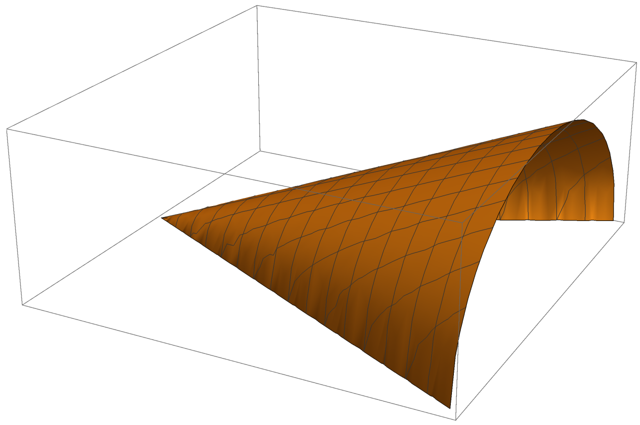

Figure 3: The graph of $f(x, y)$

a. $f(x,y) = \begin{cases} 2, & (x,y) \in [0,1] \times [0,0.5]; \\ 1, & (x,y) \in [1,2] \times [0.5,1]; \\ 0, & otherwise. \end{cases}$

b. $f(x,y) = \begin{cases} 4, & (x,y) \in [0,0.5] \times [0,0.5]; \\ 1, & (x,y) \in \{3\} \times [0.5,1]; \\ 0, & otherwise. \end{cases}$

c. $f(x,y) = \begin{cases} x \cdot y, & \text{if } 0 \le x \le 1,\ 0 \le y \le x; \\ 0, & otherwise. \end{cases}$

Problem 5. The graph of the Joint PDF $f(x, y)$ of some r.v.s X and Y can be obtained in the following way: we first draw a right triangle on the plane XOZ, see Fig. 2, and then rotate that triangle around OX to obtain a half-cone, see Fig. 3. Everywhere else $f(x, y)$ is zero. Calculate the probability $\mathbb{P}(X > 1)$.

Problem 6. It can be proved that the following function

$$F(x,y) = \begin{cases} x^2 \cdot y^2, & \text{if } 0 \le x \le 1,\ 0 \le y \le 1; \\ x^2, & \text{if } 0 \le x \le 1,\ y > 1; \\ y^2, & \text{if } x > 1,\ 0 \le y \le 1; \\ 1, & \text{if } x > 1,\ y > 1; \\ 0, & otherwise. \end{cases}$$

is a Joint CDF for some r.v.'s X and Y (the graph is given in Fig. 4). Calculate

a. The (Marginal) CDF of X, F_X and of Y, F_Y;

b. $\mathbb{P}(X \in [2,3], Y \in [0,1])$;

c. $\mathbb{P}(X < 0.3)$;

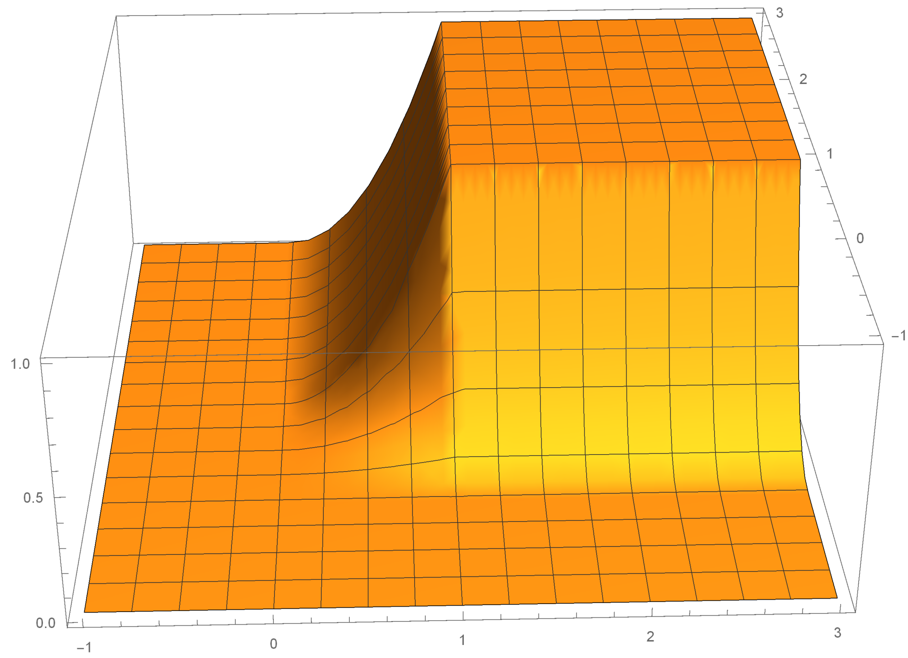

Figure 4: The Joint CDF of X and Y, $F(x,y)$

d. $\mathbb{P}(0.5 \le X \le 0.7, 0.1 \le Y \le 0.9)$.

Problem 7. It can be proved that the following function

$$F(x,y) = \begin{cases} (1-e^{-x}) \cdot (1-e^{-y}), & \text{if } x \ge 0,\ y \ge 0; \\ 0, & \textit{otherwise}. \end{cases}$$

is a Joint CDF for some r.v.'s X and Y. Calculate

a. the Joint PDF of X and Y, $f(x,y)$;

b. $\mathbb{P}((X,Y) \in [0,2] \times [0,4])$;

Note: Give methods to calculate both by using the Joint PDF and without using that Joint PDF.

c. $\mathbb{P}(X < -3, Y > 4)$;

d. $\mathbb{P}(X + Y \le 5)$;

e. $\mathbb{P}(X \le 3)$.

Problem 8. Assume the PDF of the random vector (X,Y) is

$$f(x,y) = \begin{cases} K \cdot (x+y), & \text{if } 0 \le x \le 1,\ 0 \le y \le 1; \\ 0, & \textit{otherwise}. \end{cases}$$

for some constant K.

a. Find K;

b. Calculate the CDF of (X,Y), $F(x,y)$;

c. Calculate $\mathbb{P}((X,Y) \in D)$, where D is a trapezoid with vertices at $(0,0)$, $(0,1)$, $(1,1)$ and $(3,0)$;

d. Calculate $\mathbb{P}(X^2+Y^2 \leq 1)$.

Problem 9. Assume we are picking at random a point, uniformly, in $D \subset \mathbb{R}^2$, and let X and Y be the x- and y- coordinates of that point. We consider two cases, when

I. D is the triangle with vertices at $(-1,0)$, $(1,0)$ and $(0,1)$;

II. $D = \{(x,y) : 1 \leq x^2+y^2 \leq 4\}$.

For each case of D,

a. Find the Joint PDF of X and Y;

b. Find the (Marginal) PDFs of X and Y;

c. Calculate $\mathbb{P}(X \in [0,1], Y \in [0,1])$.

Problem 10. Assume

$$\mu = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}.$$

Let $(X,Y) \sim \mathcal{N}(\mu, \Sigma)$.

a. Construct the PDF of (X,Y);

b. Calculate, using **R** or any other software, the probabilities

$$\mathbb{P}(X^2+(Y-3)^2 < 3) \qquad \text{and} \qquad \mathbb{P}(X > 2);$$

c. Plot some level curves of the PDF of (X,Y).

d. (Supplementary) Plot, using some software, or your calculus knowledge, the graph of the Joint PDF of X and Y;

Problem 11. Express the Joint CDF (PDF) of U, V in terms of the Joint CDF (PDF) of X, Y, if

a. $U = 3X+2$, $V = 4Y-2$;

b. (Supplementary) $U = X+Y$, $V = X-Y$.

Problem 12. (Supplementary) Assume X and Y are discrete r.v.'s with values[1] $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$, respectively, and their Joint PMF is $\mathbb{P}(X = x_i, Y = y_j)$ for $i = 1,2,\ldots$, $j = 1,2,\ldots$. Express in terms of the Joint PMF:

a. Their Joint CDF $F(x,y)$;

b. $\mathbb{P}(a \leq X \leq b, c \leq Y \leq d)$;

c. The (Marginal) CDF of X, $F_X(x)$;

d. $\mathbb{P}(X = x, Y \leq y)$;

e. $\mathbb{P}(a \leq X \leq b)$.

[1] Finite or countably infinite, also not necessarily of the same size.

Problem 13. (Supplementary) Assume X and Y are Jointly Continuous with Joint CDF $F(x,y)$ and Joint PDF $f(x,y)$. Express (no proof is necessary):

a. $F(x,y)$ in terms of $f(x,y)$;

b. $f(x,y)$ in terms of $F(x,y)$;

c1. the (Marginal) CDF of X, $F_X(x)$ in terms of $F(x,y)$;

c2. the (Marginal) CDF of X, $F_X(x)$ in terms of $f(x,y)$;

d1. the (Marginal) PDF of Y, $f_Y(x)$ in terms of $F(x,y)$;

d2. the (Marginal) PDF of Y, $f_Y(x)$ in terms of $f(x,y)$;

e1. $\mathbb{P}(a \le X \le b, Y \ge c)$ in terms of $F(x,y)$;

e2. $\mathbb{P}(a \le X \le b, Y \ge c)$ in terms of $f(x,y)$;

f1. $\mathbb{P}(X \ge a, Y \le b)$ in terms of $F(x,y)$;

f2. $\mathbb{P}(X \ge a, Y \le b)$ in terms of $f(x,y)$;

g. $\mathbb{P}(X^4 + Y^4 \le 5)$ in terms of $f(x,y)$; write also the double integral in the iterated integrals form;

h. $\mathbb{P}(|X| + Y \le 5)$ in terms of $f(x,y)$; write also the double integral in the iterated integrals form;

i. $\mathbb{P}(X \in [0,2], Y \le \sin(X))$ in terms of $f(x,y)$; write also the double integral in the iterated integrals form;

j. the CDF of the 1D random variable $Z = X + Y$ in terms of $f(x,y)$;

k. the CDF of the 1D random variable $Z = \max\{X, Y\}$ in terms of $F(x,y)$ and $f(x,y)$;

l. the CDF of the 1D random variable $Z = \min\{X, Y\}$ in terms of $F(x,y)$ and $f(x,y)$.