CS 107, Probability, Spring 2019
Lecture 37

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AUA

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Content

- Independent Random Variables

Independence of R.V.s

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- If X and Y are Jointly Continuous, then $f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$ for any[a] $x, y \in \mathbb{R}$, where $f_{X,Y}$, f_X, f_Y are the Joint PDF of X, Y and the Marginal PDFs of X

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- Assume $X \sim Unif[0,3]$, $Y \sim Exp(2)$ and $X \perp\!\!\!\perp Y$. Find $\mathbb{P}(X^2 + Y^2 \leq 1)$.

Facts About Multivariate Uniform Distribution:

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Fact 2: Inversely, if the r.vector $(X_1, ..., X_n)$ has a Multivariate Uniform Distribution on a set, which is a Cartesian Product of n sets $A_1, ..., A_n \subset \mathbb{R}$, i.e., if

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then $X_1, ..., X_n$ are Independent and $X_k \sim Unif(A_k)$, $k = 1, ..., n$.

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where

$$\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}, \qquad \text{and} \qquad \Sigma = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix}.$$

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Fact 2: If we have a r.vector $(X_1, \dots, X_n)$ with $(X_1, \dots, X_n) \sim \mathcal{N}(\mu, \Sigma)$, where μ and Σ are as above, then $X_1, \dots, X_n$ are Independent, and

$$X_k \sim \mathcal{N}(\mu_k, \sigma_k^2), \qquad k = 1, \dots, n.$$

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