AUA CS 108, Statistics, Fall 2019

Lecture 19

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- Why we consider Random Samples?

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- Why we consider Random Samples?
- What is the problem we consider in the Parametric Statistics?

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Our Problem here is, using the observation $x_1, x_2, ..., x_n$, to estimate μ and σ^2.

Point Estimates

Motivating Example ⌣̈

Motivating Example $\ddot\smile$

Example: I have generated the following Data from a Normal Distribution:

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Moral: Statistics is like a Detective Story: you need to find the Unknown (murderer?) using some small amount of Observations, Data you have $\ddot\smile$

Statistics, Estimator and Estimate

Let us recall what is our Problem: assume we have a Dataset $x_1, ..., x_n$. We assume that this is a realization of a Random Sample $X_1, ..., X_n$, coming from one of the Distributions from some Parametric Family:

$$X_1, X_2, ..., X_n \sim \mathcal{F}_\theta, \qquad \theta \in \Theta.$$

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Definition: Any (measurable) function of the Random Sample $X_1, X_2, ..., X_n$ is called a **Statistics**. So Statistics is a r.v. of the form

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This is our third meaning of the term *Statistics*.

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$$X_1, \qquad \frac{X_1 + X_n}{3}, \qquad \sin(X_1 \cdot X_2 + X_3 + ... + X_n).$$

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Example: The Sampling Distribution of the Statistics

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is almost Normal, for large n, by the CLT.

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- g doesn't depend on the unknown θ;

then the Statistics $g(X_1, X_2, ..., X_n)$ is called an **Estimator** for θ, and it is usually denoted by

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The value of the Estimator at our observations, $g(x_1, x_2, ..., x_n)$, is called an **Estimate** for θ, and it is again (unfortunately) denoted by $\hat{\theta} = \hat{\theta}_n$.

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$$\{Exp(\lambda) : \ \lambda > 0\}$$

model, using the Random Sample

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And the following is not an estimator:

- $\hat{\lambda} = \dfrac{\lambda}{X_1 + X_n}$, since it depends on λ - the unkown parameter value.

Estimators and Estimates

Note: We require our Estimator to be independent of the Parameter θ, since we want to be able to calculate the value of the Estimator on the observation, i.e., we want to be able to calculate the Estimate, we want to make our Estimator *observable*. Since θ is unknown, then we cannot use it in the Estimator - we will not be able to calculate the Estimate.

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- **Estimate** is a number, it is the result of plugging the observation into the Estimator.

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where $b = 0$ and $g = 1$: this is to be able to use one of our standard Distributions. Next, from a Dataset we pass, for a generalization, to a Random Sample

$$X_1, X_2, X_3, X_4, X_5, X_6, X_7,$$

where X_k is the gender of the k-th child *before the observation was made* ($X_k = 1$ if the child will be a girl, and 0 otherwise).

Example, cont'd

Then we will have

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To estimate p, let us take the following **Estimator**:

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This is a r.v. . The **Estimate** for p, using our Observation, will be

$$\hat{p} = \frac{0 + 1 + 1 + 0 + 0 + 1 + 0}{7} = \frac{3}{7}.$$

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$$X_1, X_2, ..., X_n \sim Bernoulli(p),$$

and we want to estimate the Parameter p.

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Or, maybe

$$\hat{p} = \frac{X_{(1)} + X_{(n)}}{2} \qquad or \qquad \hat{p} = Median(X_1, ..., X_n)?$$

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And what about estimating σ^2? Can you suggest Estimators? Say, which one to choose:

$$\widehat{\sigma^2} = \left(\frac{\sum_{k=1}^n |X_k - \overline{X}_n|}{n}\right)^2 \qquad or \qquad \widehat{\sigma^2} = \frac{\sum_{k=1}^n (X_k - \overline{X}_n)^2}{n} \qquad or$$

$$\widehat{\sigma^2} = \frac{\sum_{k=1}^n (X_k - \overline{X}_n)^2}{n-1} \qquad or \qquad \widehat{\sigma^2} = \text{other Estimator?}$$

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If $X \sim \mathcal{F}_\theta$, then we will write

$$\mathbb{E}_\theta(X)$$

the Expected value of X: since $X \sim \mathcal{F}_\theta$, then the Expected value depends on θ.

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the Expected value of X: since $X \sim \mathcal{F}_\theta$, then the Expected value depends on θ. Say, if $X \sim \mathcal{F}_\theta$ is continuous, then

$$\mathbb{E}_\theta(X) = \int_{-\infty}^{+\infty} x \cdot f(x|\theta) dx.$$

Notation

Before starting, few Notations. Let us consider the Model

$$\{\mathcal{F}_\theta : \theta \in \Theta\}.$$

Assume $F_\theta(x) = F(x|\theta)$ is the CDF of the Distribution $\mathcal{F}_\theta$ and $f_\theta(x) = f(x|\theta)$ is the PD(M)F of $\mathcal{F}_\theta$.

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And we will use $Var_\theta(X)$ for the Variance of X.

Properties of Estimators

Risk, Mean Squared Error of the Estimator

Assume we have a Random Sample

$$X_1, X_2, ..., X_n \sim \mathcal{F}_\theta, \qquad \theta \in \Theta,$$

and we use the Estimator $\hat{\theta}$ to estimate θ.

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Definition: The **Mean Squared Error** or the **Quadratic Risk** of the estimator $\hat{\theta}_n$ of θ is

$$MSE(\hat{\theta}_n, \theta) = Risk(\hat{\theta}_n, \theta) = \mathbb{E}_\theta[(\hat{\theta}_n - \theta)^2].$$

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Definition: We say that the estimator $\hat{\theta}_n^1$ of θ is **preferable** to $\hat{\theta}_n^2$, another estimator of θ, if

$$MSE(\hat{\theta}_n^1, \theta) \leq MSE(\hat{\theta}_n^2, \theta), \qquad \forall \theta \in \Theta,$$

and there exists a θ s.t. $MSE(\hat{\theta}_n^1, \theta) < MSE(\hat{\theta}_n^2, \theta)$.