AUA CS 108, Statistics, Fall 2019

Lecture 12

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- How to check (visually) if a Dataset is coming from a Normal Distribution?

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- How to check (visually) if a Dataset is coming from a Normal Distribution?
- How to check (visually) if a Dataset is coming from a Pareto Distribution?

Q-Q Plots, Theoretical vs Theoretical Distribution

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To answer this question, we again take some levels of quantiles, say, for some n,

$$\alpha = \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}$$

and then draw the points (q_α^F, q_α^G), where q_α^F is the α-quantile of the Theoretical Distribution with the CDF F, and q_α^G is the α-quantile of the Theoretical Distribution with the CDF G.

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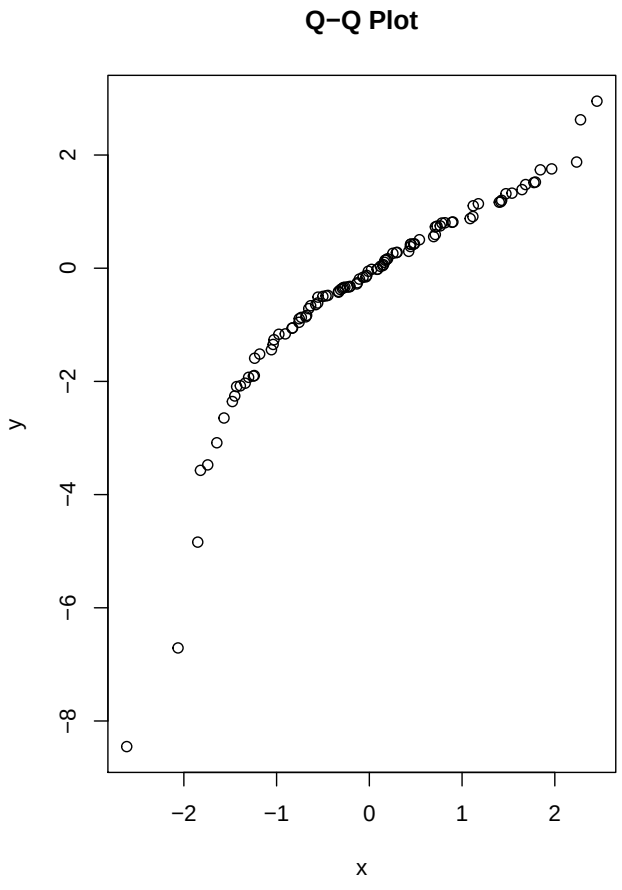
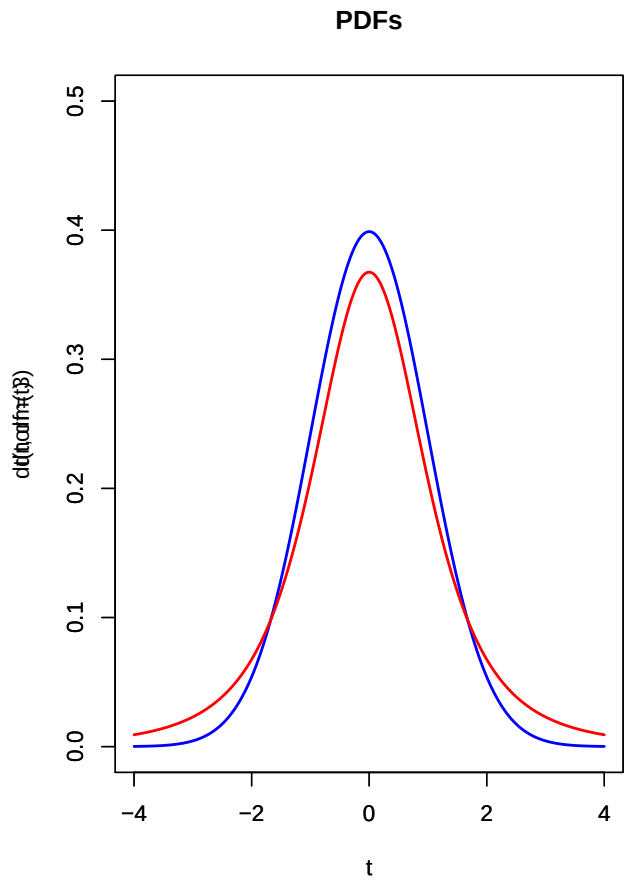
Idea: If G has fatter tails on both sides than F, then we will have graphically some cubic-function graph shape Quantiles.

Some Experiments

```
par(mfrow = c(1,2))
x <- rnorm(100, mean=0, sd=1); y <- rt(100, df = 3)
qqplot(x,y, main = "Q-Q Plot")
t <- seq(-4,4,0.01)
plot(t, dnorm(t), type = "l", xlim = c(-4,4), ylim = c(0, 0.5), col ="blue", lwd = 2, main = "PDFs")
par(new = TRUE)
plot(t, dt(t, df = 3), type = "l", xlim = c(-4,4), ylim = c(0, 0.5), col ="red", lwd = 2)
```

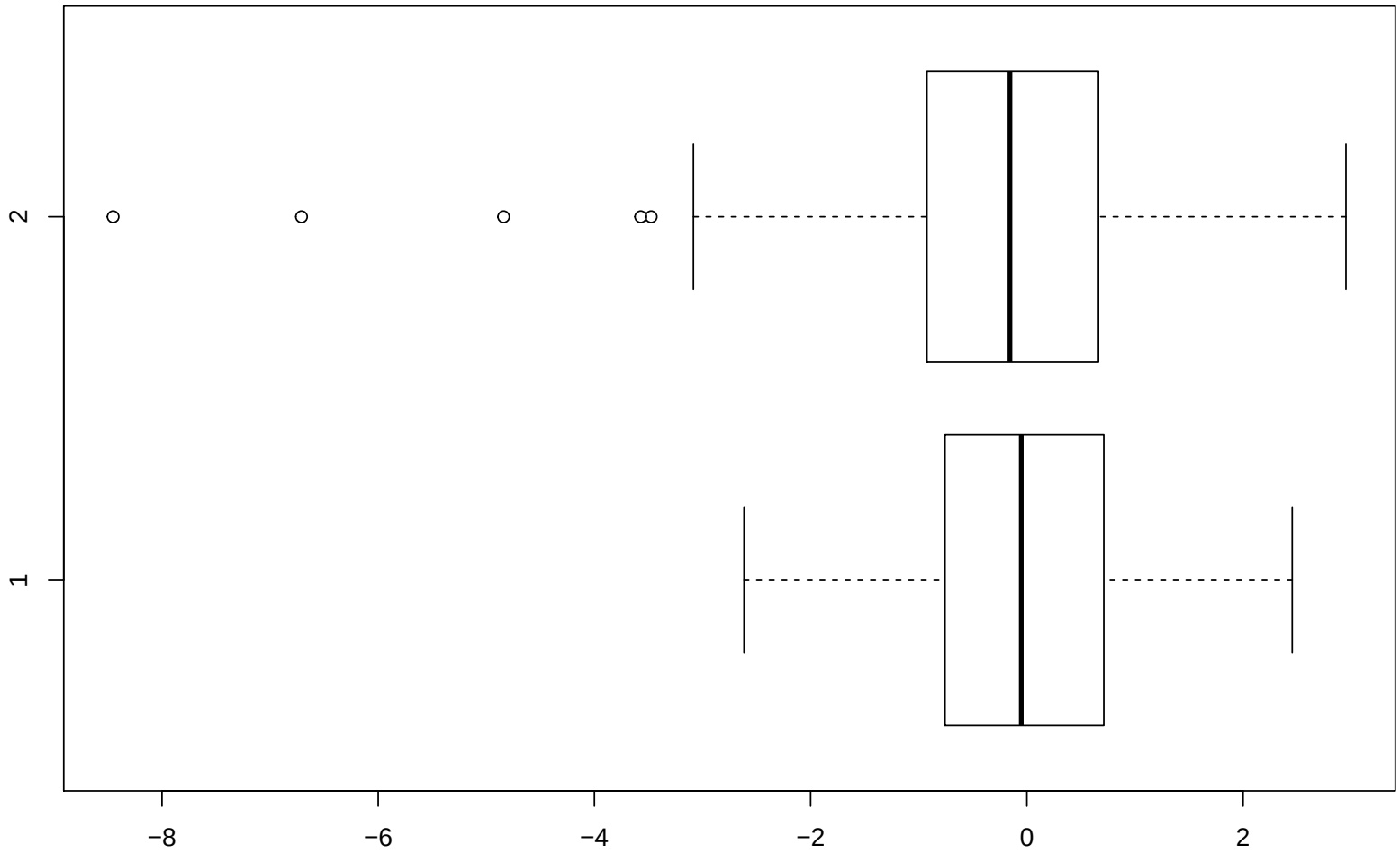




Some Experiments

The above Datasets, using BoxPlots:

```
boxplot(x,y, horizontal = T)
```



Some Experiments

```
par(mfrow = c(1,2))
x <- rnorm(100, mean=2, sd=1); y <- rchisq(200, df = 3)
qqplot(x,y, main = "Q-Q Plot")
t <- seq(0,5,0.01)
plot(t, dnorm(t), type = "l", xlim = c(0,5), ylim = c(0, 0.5), col ="blue", lwd = 2,  main = "PDFs")
par(new = TRUE)
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```
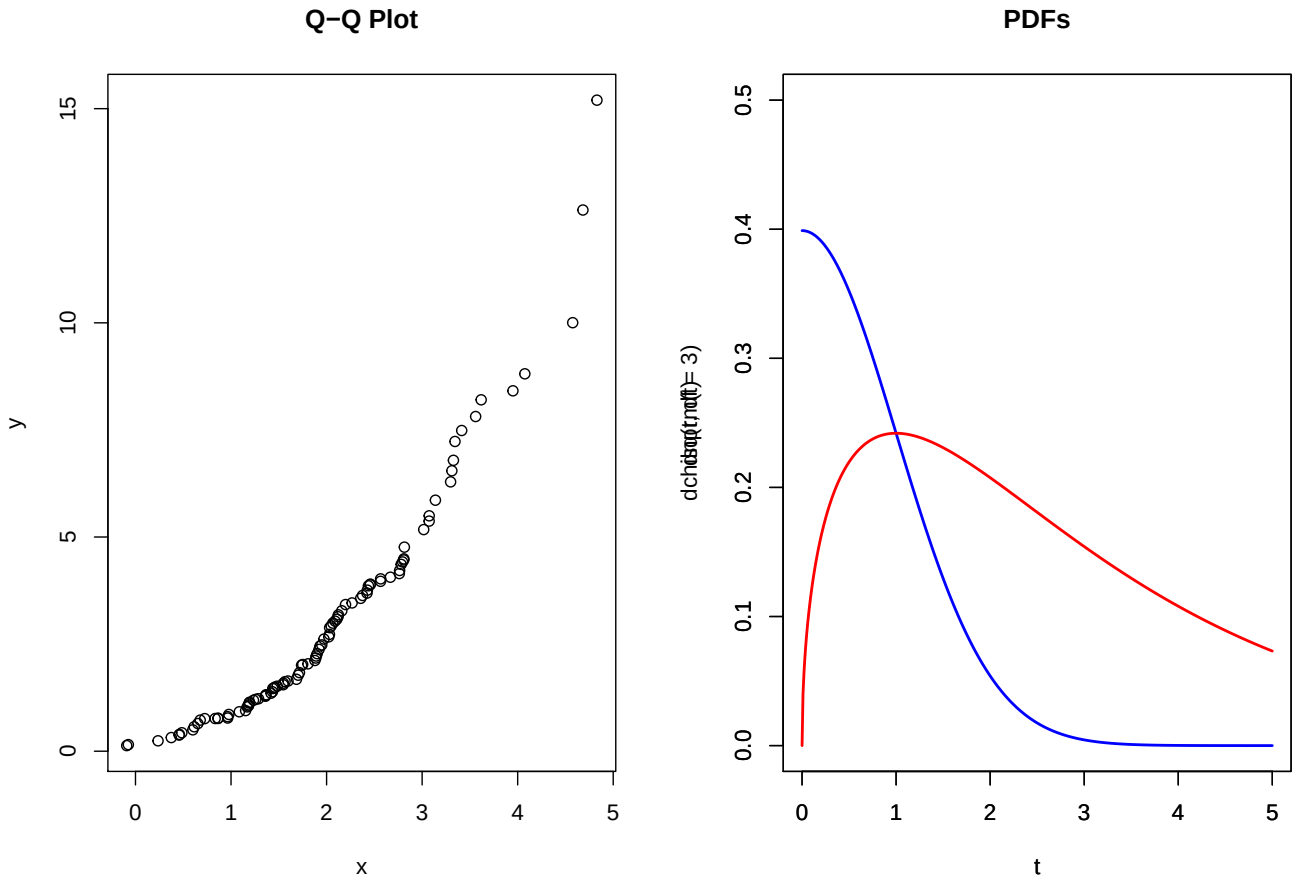
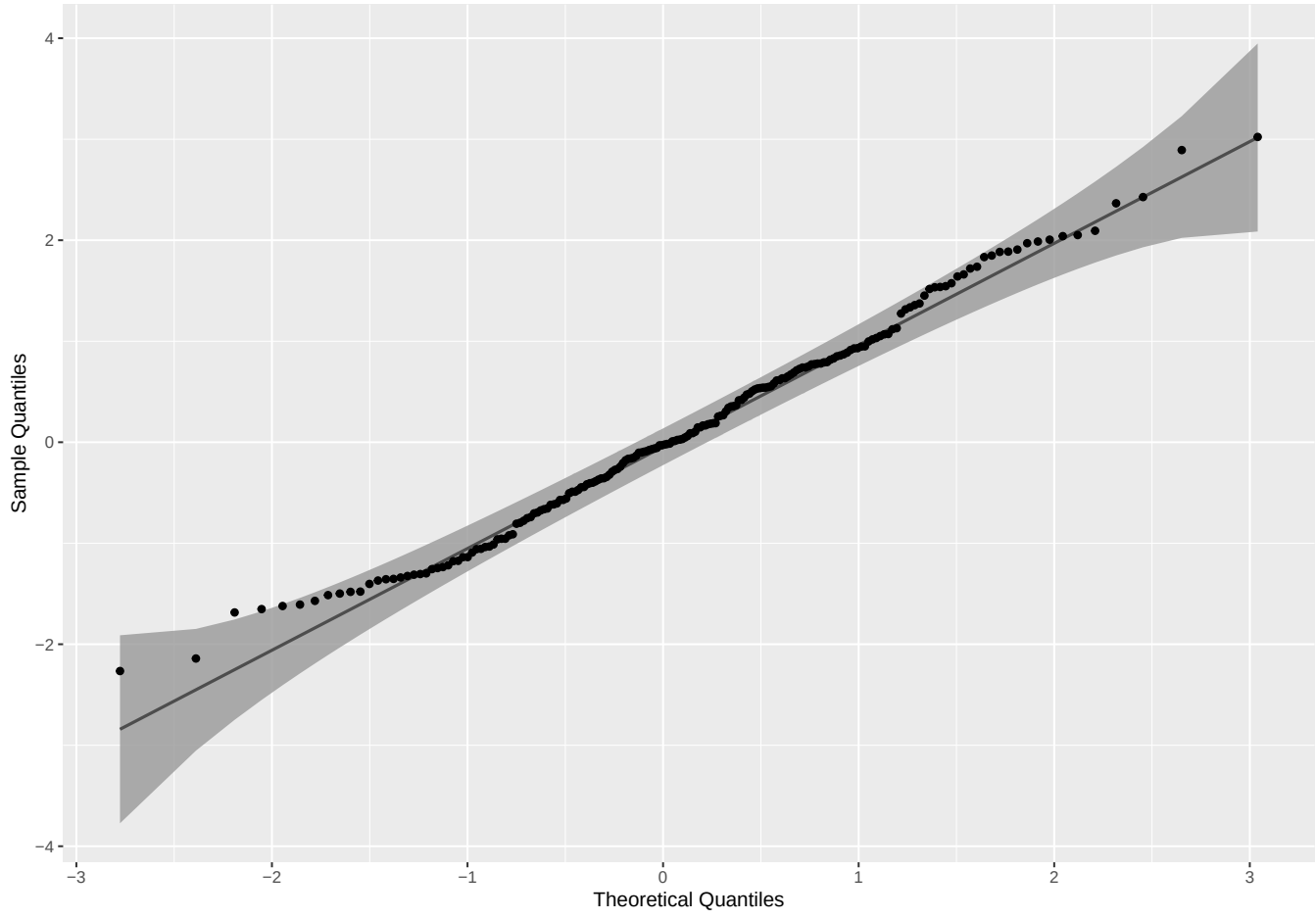



Addition, Q-Q Plot

here you can find some interpretaitons of different shapes of Q-Q Plots: StackExchange Page.

Addition, Q-Q Plot with a Confidence Band

```
require(qqplotr)
x <- data.frame(variable = rnorm(200))
ggplot(data = x, mapping = aes(sample = variable)) + stat_qq_band() +
  stat_qq_line() + stat_qq_point() + labs(x = "Theoretical Quantiles", y = "Sample Quantiles")
```



Numerical Summaries for Bivariate Data

Sample Covariance and the Correlation Coefficient

Assume now we have a bivariate Dataset

$$(x_1, y_1), ..., (x_n, y_n),$$

or just two 1D Datasets of the same size:

$$x: \ x_1, ..., x_n \qquad \textit{and} \qquad y: \ y_1, ..., y_n.$$

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Our aim is to see if some linear relationship, association exists between x and y. Of course, the best way is to visualize our Dataset by a ScatterPlot.

Now we want to answer, numerically, how strong/week is the linear relationship between our variables x and y.

Sample Covariance

The **Sample Covariance** of Variables (1D Datasets) x and y is

$$cov(x, y) = s_{xy} = \frac{\sum_{k=1}^{n} (x_k - \overline{x}) \cdot (y_k - \overline{y})}{n}$$

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Note: Recall that for a r.v. X, $Cov(X, X) = Var(X)$.

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Note: Recall that for a r.v. X, $Cov(X, X) = Var(X)$. Here, for Datasets, we have two definitions for the Sample Variance $var(x)$. And we give two definitions of the Sample Covariance, so the property $cov(x, x) = var(x)$ will hold in both cases.

Sample Covariance

Definition: We say that the Variables (Datasets) x and y are **uncorrelated**, if $cov(x, y) = 0$.

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Remark: In Probability, we have 2 notions: *Independence* and *Corelation*. Here, in the case of Datasets, we do not have the notion of *Independence*

Example

Here is the **R** code to calculate the Covariance between the `Speed` and `Dist` variables in the `cars` Dataset:

```
cov(cars$speed, cars$dist)
```

```
## [1] 109.9469
```

Sample Correlation Coefficient

Another measure of the linear relationship between the Variables x and y of Bivariate Dataset is the *Pearson's Correlation Coefficient*:

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Definition: The **Sample Correlation Coefficient** of x and y is

$$cor(x,y) = \rho_{xy} = \frac{cov(x,y)}{\sqrt{Var(x) \cdot Var(y)}} = \frac{cov(x,y)}{sd(x) \cdot sd(y)} = \frac{s_{xy}}{s_x \cdot s_y},$$

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If $s_x = 0$ or $s_y = 0$, then we take $cor(x,y) = 0$ by definition.

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If $s_x = 0$ or $s_y = 0$, then we take $cor(x,y) = 0$ by definition.

Note: Please note that we need to calculate the Standard Deviations and Covariance by using the same denominator: either everywhere take n, or take everywhere $n-1$.

Sample Correlation Coefficient

In both cases, when one calculates Standard Deviations and Covariance by using n simultaneously or $n-1$ simultaneously in the denominator, we will obtain

$$cor(x,y) = \rho_{xy} = \frac{\sum_{k=1}^{n}(x_k - \overline{x}) \cdot (y_k - \overline{y})}{\sqrt{\sum_{k=1}^{n}(x_k - \overline{x})^2 \cdot \sum_{k=1}^{n}(y_k - \overline{y})^2}}$$

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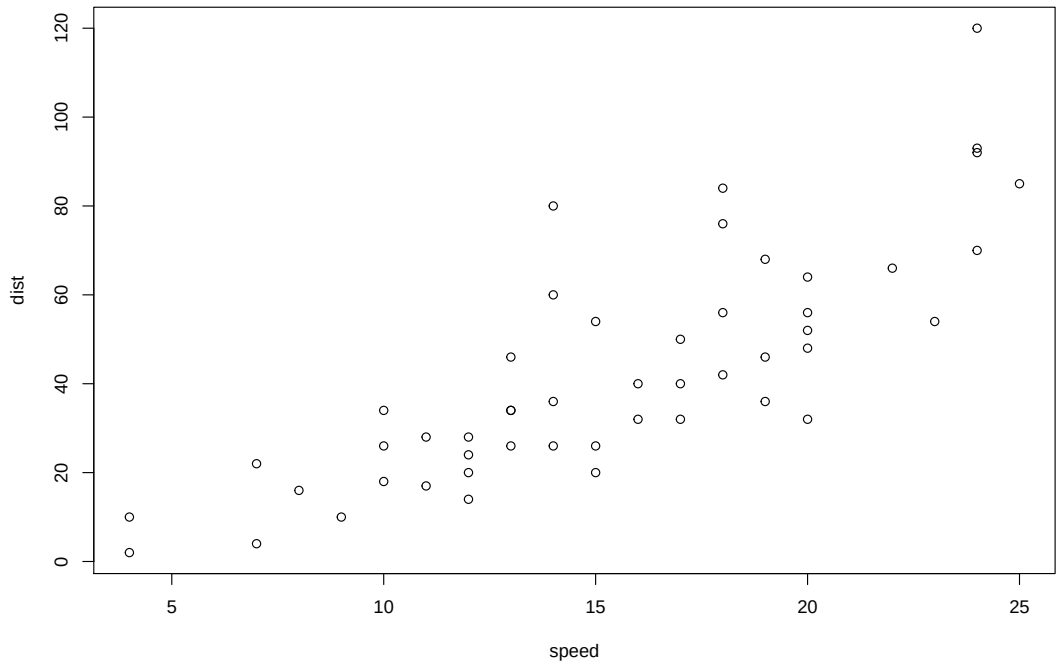
Another formula to calc the correlation coefficient is

$$cor(x,y) = \rho_{xy} = \frac{\sum_{k=1}^{n} x_k y_k - n \cdot \overline{x} \cdot \overline{y}}{\sqrt{\sum_{k=1}^{n} x_k^2 - n \cdot (\overline{x})^2} \cdot \sqrt{\sum_{k=1}^{n} y_k^2 - n \cdot (\overline{y})^2}}.$$

Examples:

Now, the **R** code to calculate the Covariance between the `Speed` and `Dist` variables in the `cars` Dataset:

```
plot(dist~speed, data = cars)
```



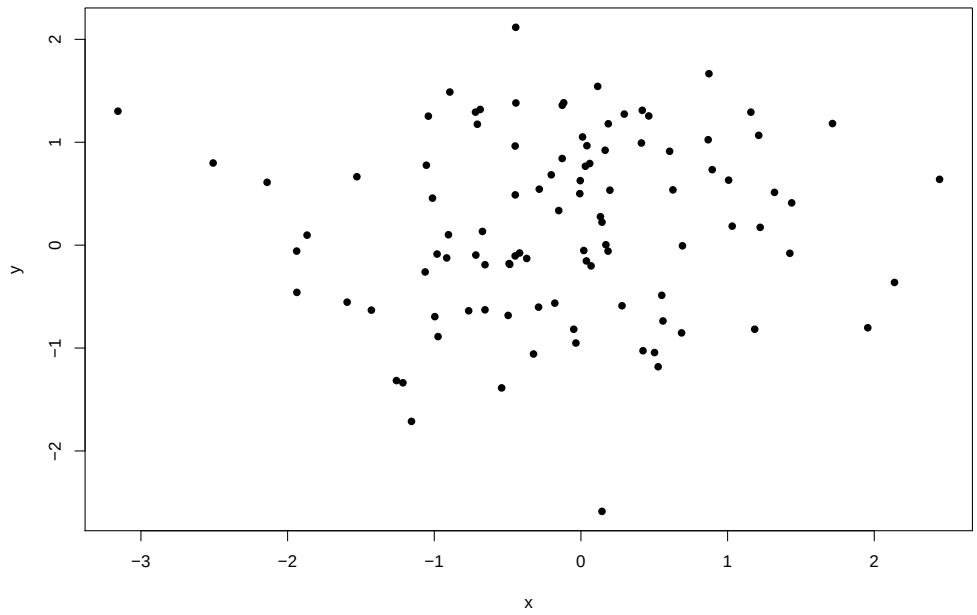

```
cor(cars$speed, cars$dist)
```

```
## [1] 0.8068949
```

Examples:

Some simulations:

```
x <- rnorm(100); y <- rnorm(100);
plot(x,y, pch=16)
```



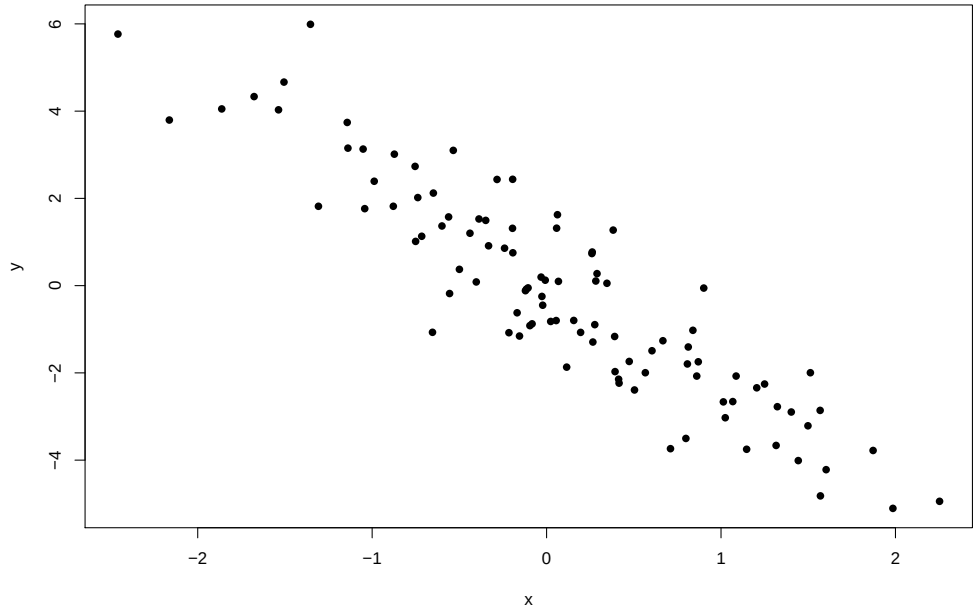

```
c(cor(x,y), cov(x,y))
```

```
## [1] 0.06988425 0.06020739
```

Examples:

Some simulations:

```
x <- rnorm(100); y <- -2.4*x + rnorm(100);
plot(x,y, pch=16)
```




```
c(cor(x,y), cov(x,y))
```

```
## [1] -0.9118475 -2.0734118
```

Examples:

Let us now use the `state.x77` Dataset from **R**:

```
head(state.x77)
```

```
##            Population Income Illiteracy Life Exp Murder HS Gr
## Alabama          3615   3624        2.1    69.05   15.1    41
## Alaska            365   6315        1.5    69.31   11.3    66
## Arizona          2212   4530        1.8    70.55    7.8    58
## Arkansas         2110   3378        1.9    70.66   10.1    39
## California      21198   5114        1.1    71.71   10.3    62
## Colorado         2541   4884        0.7    72.06    6.8    63
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```
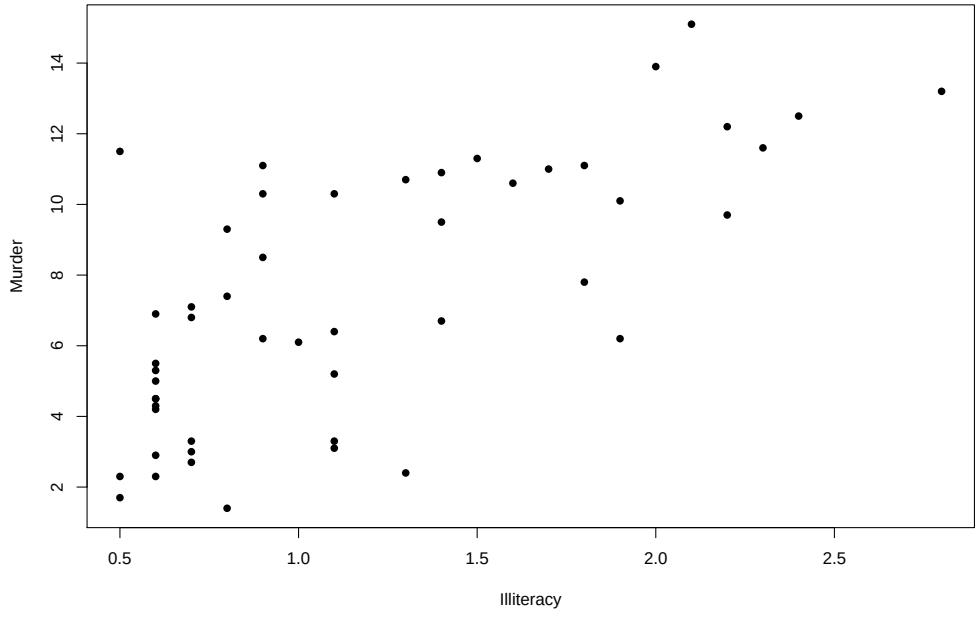

It is not of the DataFrame format, so we change it to DataFrame:

```
state <- as.data.frame(state.x77)
```

Examples:

```
plot(Murder~Illiteracy, data = state, pch=16)
```




```
cor(state$Illiteracy, state$Murder)
```

```
## [1] 0.7029752
```

Examples:

Question: How to generate samples x, y with some given Correlation Coefficient?

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Answer: Exactly. I do not know it too ¨‿ Kidding, of course, I know.

Say, we want to have Datasets x, y of size n with $cor(x, y) = \rho \in (-1, 1)$.

One of the possible methods: take a Matrix

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

which is **Positive Definite**, take any 2D vector, say $\mu = [0, 0]^T$, and generate a Sample of size n from the Bivariate Normal Distribution $\mathcal{N}(\mu, \Sigma)$.

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One of the possible methods: take a Matrix

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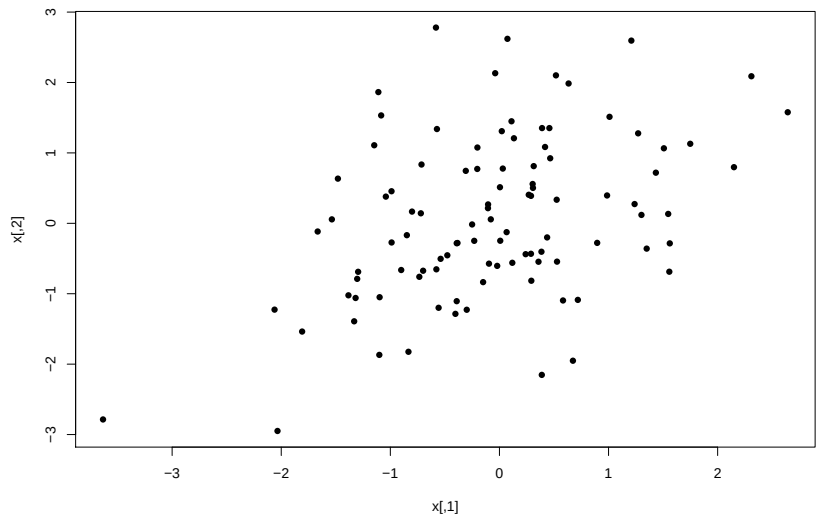
which is **Positive Definite**, take any 2D vector, say $\mu = [0, 0]^T$, and generate a Sample of size n from the Bivariate Normal Distribution $\mathcal{N}(\mu, \Sigma)$.

Then, the $cor(x, y)$ will be approximately ρ (and it will approach ρ as $n \to +\infty$).

Example

```
rho <- 0.35
covmatrix <- matrix(c(1,rho, rho, 1), nrow = 2)
mu <- c(0,0)
x <- mvtnorm::rmvnorm(100, mean = mu, sigma = covmatrix)
plot(x, pch = 16)
```




```
cor(x)
```

```
##             [,1]      [,2]
## [1,] 1.0000000 0.4215909
## [2,] 0.4215909 1.0000000
```