AUA CS108, Statistics, Fall 2020

Lecture 33

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Contents

- Method of Moments (MoM, MME)
- Maximum Likelihood Method (MLE)

The Method of Moments, m-D case

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Example

Example: Find the MoM Estimator for (μ, σ^2) in the $\mathcal{N}(\mu, \sigma^2)$ Model.

Solution: OTB

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Note: Sometimes we are interested not in the Estimate of the Parameter itself, but in the Estimate for some function of that Parameter. Say, we want to Estimate $h(\theta)$, where θ is our unknown Parameter. Then one of the approaches is the Plug-in Principle: find an Estimator $\hat{\theta}$ for θ, say, using the MoM, and then plug that in h, to obtain $h(\hat{\theta})$ as an Estimator for $h(\theta)$.

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The Maximum Likelihood Method

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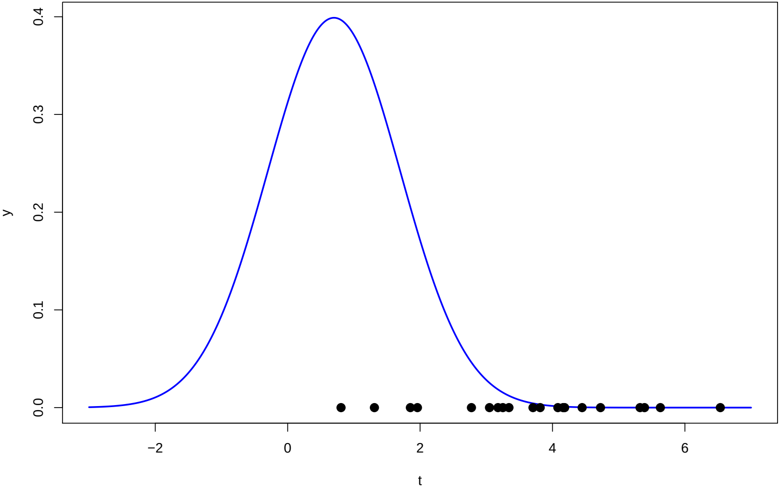
Idea of Maximum Likelihood Estimation: We choose that value of our parameter, under which **our Observation is the most Probable**.

Idea of the MLE

Guess which Distribution makes our Data most Probable:



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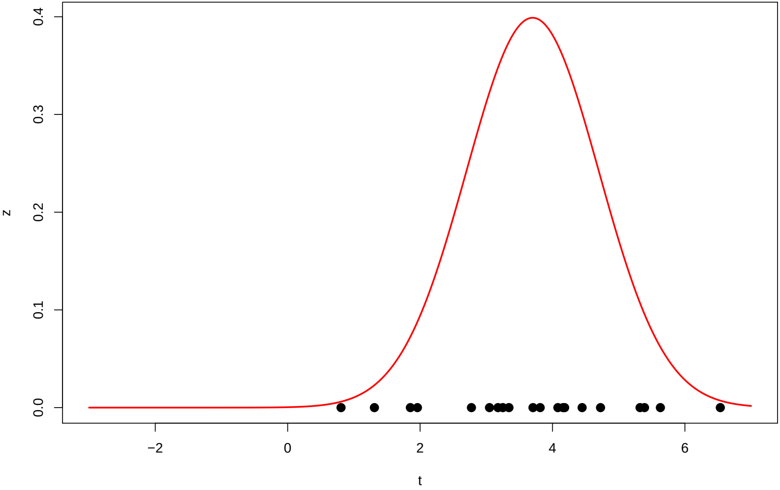

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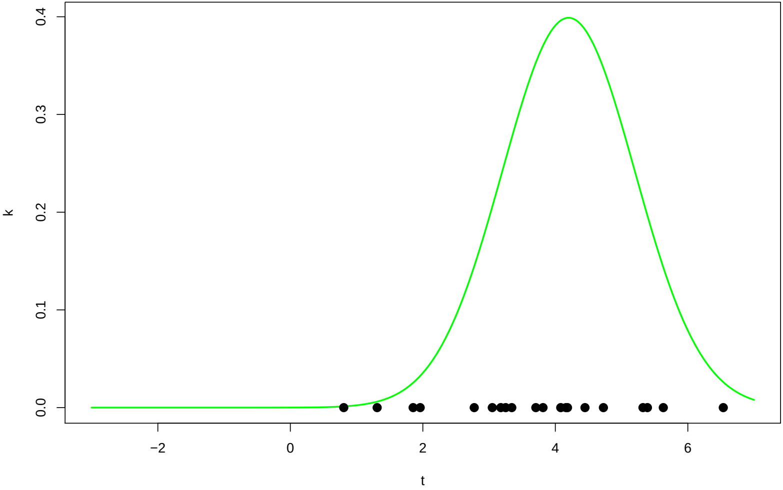

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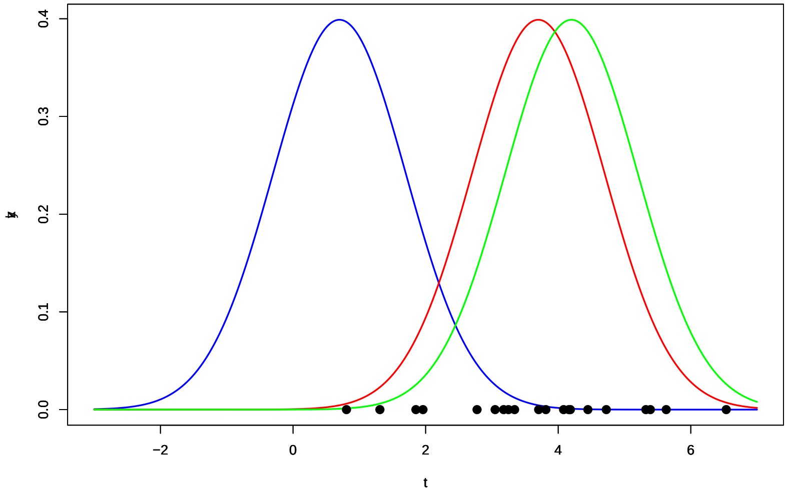

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Again, assume we have an Observation $x: \ x_1, ..., x_n$, from one of the Distributions of Parametric Family $\mathcal{F}_\theta$, with the PD(M)F $f(x|\theta)$, $\theta \in \Theta$.

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And the Maximum Likelihood Method is saying: **choose that value of θ, under which it is most likely to get** $X_1, X_2, ..., X_n$.

Likelihood

Definition: The **Likelihood Function** for the above Model and Random Sample is the Joint PD(M)F of $X_1, ..., X_n$, **considered as a function of the parameter** θ, and **calculated at the Random Sample**, i.e., it is given by[1]

$$\mathcal{L}(\theta) = \mathcal{L}_n(X_1, ..., X_n|\theta) = f(X_1|\theta) \cdot f(X_2|\theta) \cdot ... \cdot f(X_n|\theta), \qquad \theta \in \Theta.$$

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Also we define the **Negative Log-Likelihood Function** to be

$$-\ell(\theta) = -\ln \mathcal{L}(\theta).$$

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And in the case if we have an Observation $x: \ x_1, x_2, ..., x_n$ from the above Model (from one of the Distributions of that Model), the **Maximum Likelihood Estimate** (again **MLE**) of the parameter θ is the value of $\hat{\theta}^{MLE}$ on our Observation.

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Calc 1 + Calc 3 Refresher

Here it is important for you to refresh you knowledge from Calc1 + Calc3 about how to find the maximum points of a function $\ell(\theta)$ for $\theta \in \Theta$, considering:

- 1D Case
- *n*-D Case
- Sufficient Conditions.