AUA CS 108, Statistics, Fall 2019

Lecture 28

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Contents

Last Lecture ReCap

- How to estimate 3D unknown Parameter using MoM?

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- How to estimate 3D unknown Parameter using MoM?
- Give some reasons to use MoM.

The Maximum Likelihood Method

Idea of the Maximum Likelihood Method

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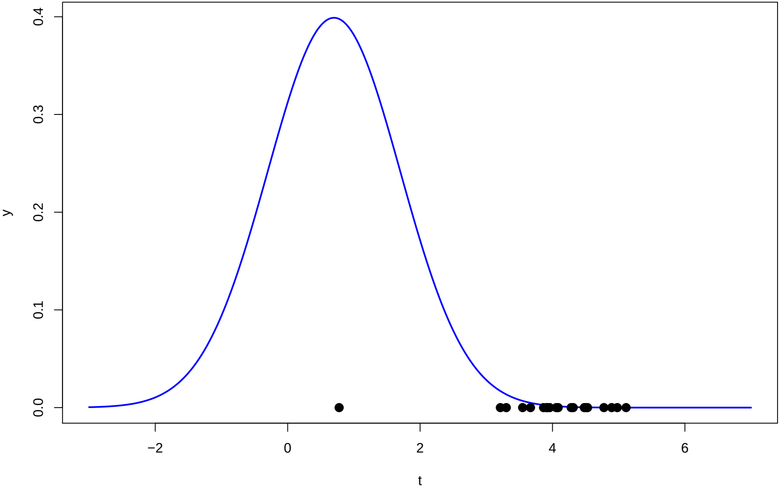
Idea of Maximum Likelihood Estimation: We choose that value of our parameter, under which **our Observation is the most Probable**.

Idea of the MLE

Guess which Distribution makes our Data most Probable:



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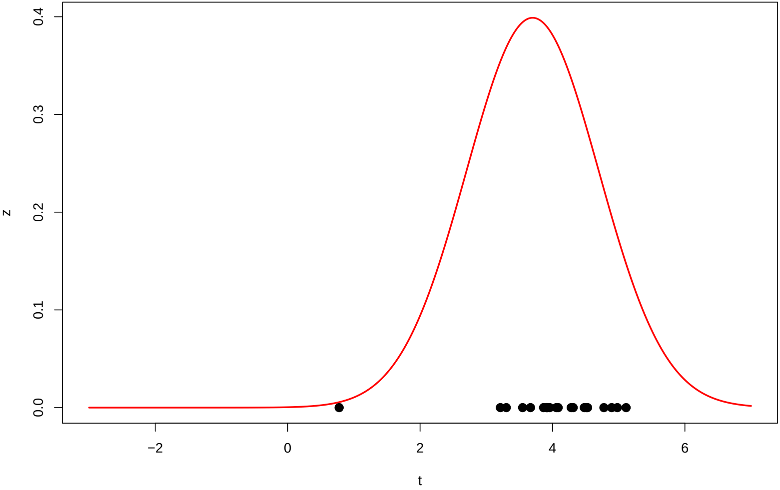

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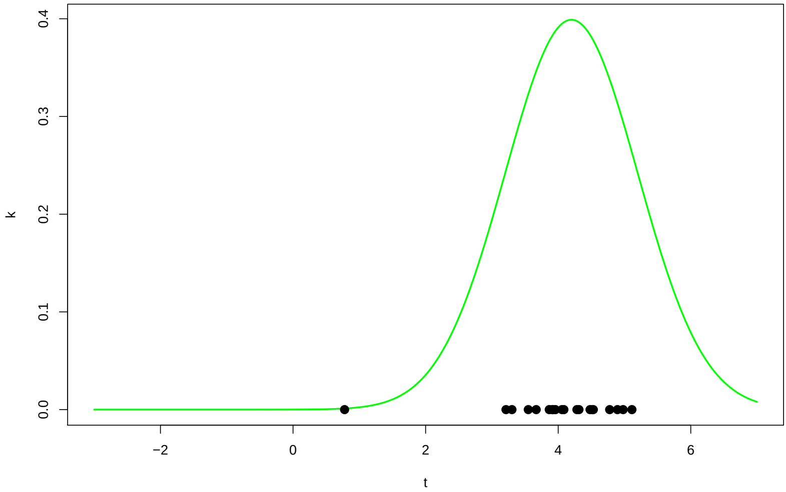

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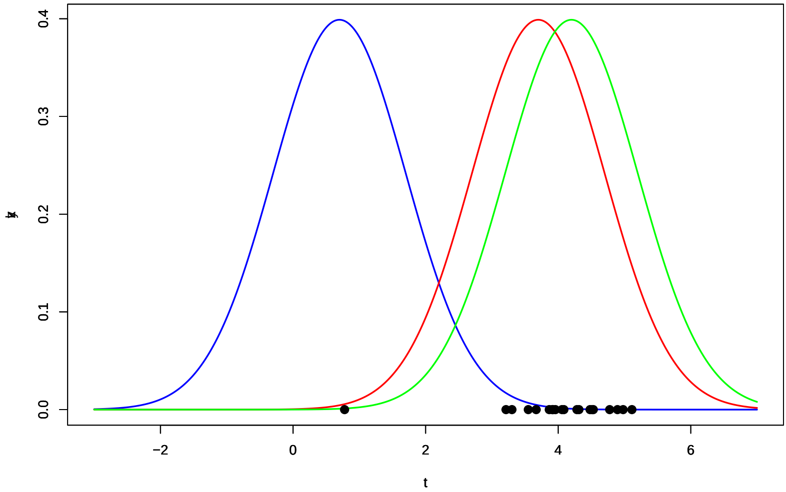

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And the Maximum Likelihood Method is saying: **choose that value of θ, under which it is most likely to get** $X_1, X_2, ..., X_n$.

Likelihood

Definition: The **Likelihood Function** for the above Model and Random Sample is the Joint PD(M)F of $X_1, ..., X_n$, **considered as a function of the parameter** θ, and **calculated at the Random Sample**, i.e., it is given by[1]

$$\mathcal{L}(\theta) = \mathcal{L}_n(X_1, ..., X_n|\theta) = f(X_1|\theta) \cdot f(X_2|\theta) \cdot ... \cdot f(X_n|\theta), \qquad \theta \in \Theta.$$

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Also we define the **Negative Log-Likelihood Function** to be

$$-\ell(\theta) = -\ln \mathcal{L}(\theta).$$

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And in the case if we have an Observation $x: \; x_1, x_2, ..., x_n$ from the above Model (from one of the Distributions of that Model), the **Maximum Likelihood Estimate** (again **MLE**) of the parameter θ is the value of $\hat{\theta}^{MLE}$ on our Observation.

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and then find the maximum point for this function, over $\theta \in \Theta$.

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i.e., the points of maximum of $\mathcal{L}(\theta)$ and $\ln \mathcal{L}(\theta)$ coincide. And, in the rest, we will find the Max points of the **Log-Likelihd** function.

Calc 1 + Calc 3 Refresher

Here it is desirable to have a slide about how to find the maximum points of a function $\ell(\theta)$ for $\theta \in \Theta$, considering:

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I know that you can fill this slide, so I am keeping it to you*.

* In fact, I realized that one slide will not be enough, and was lazy to prepare them ⌣̈

Examples

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Example: Find the MLE Estimator for θ in the *Unif*$[0, \theta]$ Model.

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