Definition (Full Procedure for Master Branch *<2022-05-27 Fri>*)

- Spectral procedure follows Smits 2017.
- Pod procedure follows Smits 2017.

- Part 1. Spectral Analysis
  - take fft azimuthally
    - use half of θ data to avoid aliasing
    - Note: in my opinion $\sum_{m=0}^{M}(fft(theta))(cos(\theta) + i * sin(\theta))$ rather than just the fft ought to be used.
    - #TODO: include this in next update
  - find correlation in t' described in Smits2017.below.eq.2.4.

$$\mathbf{R}(km; t, t') = \int_r \mathbf{u}(k; m; r, t)\mathbf{u}^*(k; m; r, t')\, r\, dr \qquad (1)$$

    - Note in particular that the function xcorr is not used (when function m5.m on master branch *<2022-05-27 Fri>* is used).
  - take fft in x of th above correlation to get k modes.
- Part 2. Snapshot POD
  - the crossspectra for the kernal of the pod is given by the r-averaged function

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \mathbf{R}\left(k; m; t, t'\right) \alpha^{(n)}\left(k; m; t'\right) dt' = \lambda^{(n)}(k; m)\alpha^{(n)}(k; m; t) \qquad (2)$$

  - Note that $\alpha^{(n)}$ act as the eigenfunctions in the above Second Type Fredholm integral equation. This is simply the formulation of the snapshot POD.
  - Find the (sorted) eigenvalues $\alpha^{(n)}$ found in (2) to solve for $\Phi^{(n)}$,

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \mathbf{u}_T(k; m; r, t)\alpha^{(n)*}(k; m; t)dt = \Phi_T^{(n)}(k; m; r)\lambda^{(n)}(k; m)$$

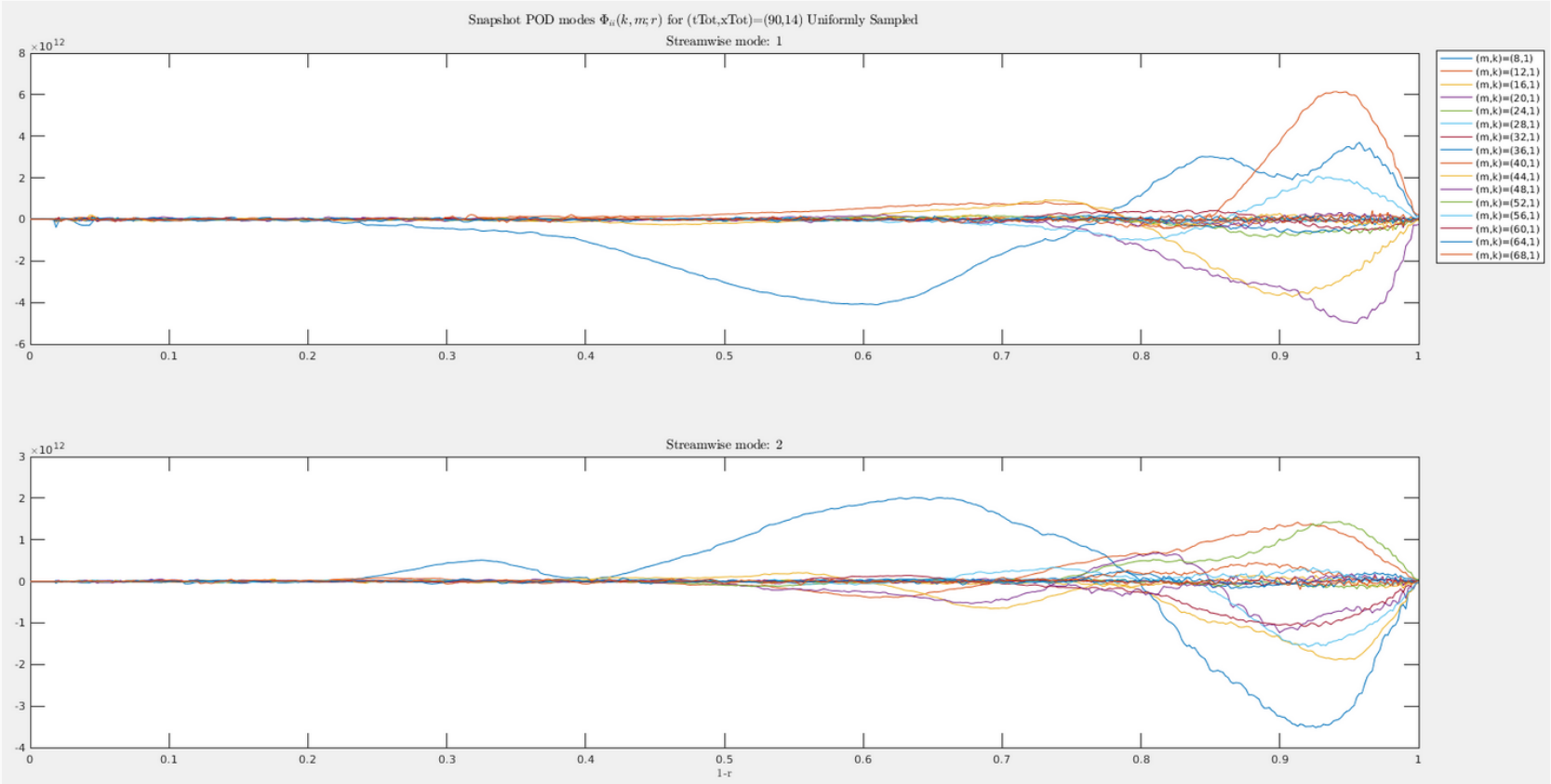Figure 1: Shows snapshot POD for differen k modes.

- Example correlation coefficient matrix R.

  - The maximum values should occur along the diagonal since this is 0 lag occurs (but do not have that)
  - This matrix is symmetric when not multiplied by the weight r, eg $\int uu^*\, dr$. For timestep =5, here is an example matrix without the r weight:

$$\begin{bmatrix} -3.3689 & -3.6159 & -2.7419 & -2.5511 \\ -5.7692 & -6.1922 & -4.6955 & -4.3688 \\ -6.1922 & -6.6463 & -5.0398 & -4.6891 \\ -4.6955 & -5.0398 & -3.8216 & -3.5557 \\ -4.3688 & -4.6891 & -3.5557 & -3.3083 \end{bmatrix} \qquad (1)$$

- #TODO: Unfortonuatelty, the maximum is not occuring along the diagonal.

- Here is the integrated correlation tensor with the $\int ruu^*\, dr$

$$\begin{bmatrix} -3.3689 & -3.6159 & -2.7419 & -2.5511 \\ -5.7692 & -6.1922 & -4.6955 & -4.3688 \\ -6.1922 & -6.6463 & -5.0398 & -4.6891 \\ -4.6955 & -5.0398 & -3.8216 & -3.5557 \\ -4.3688 & -4.6891 & -3.5557 & -3.3083 \end{bmatrix} \qquad (2)$$