Spectral Analysis and POD of Turbulent Pipe

M. Raba

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Two POD approaches

- **Two branches of code**: snapshot pod and classic pod. Follow Smits2017
  - Both have switches for direct mutliply for correlation.
  - Snapshot
    - branch Snap.A. fft-th->xcorr(t,t')->fft(x)
    - branch Snap.B. fft-th->fft(x)->correlate via direct mult->average in time
    - pod via following eq 2.4 and then 2.5.
- Classic
  - branch A. fft-th->xcorr(x,x')->fft(x)
  - branch b. fft-th->direct mult corr->fft(x)
  - pod follows equation 2.1
- Snapshot is only more expedient if using full velocity components spectral density matrix. Otherwise that has greatly more steps for getting the pod modes.
  - the problem is not $3r \times 3r$ transformed to $t\times$.
- **Snapshot procedure**: form correlation $\mathbf{R}(k; m; t, t') = \int_r \mathbf{u}(k; m; r, t)\mathbf{u}^*(k; m; r, t')\, r\, \mathrm{d}r$; then solve 2.4 for $\alpha^{(n)}$, then solve 2.4 for $\Phi_T^{(n)}$.
- **Classic POD**: form correlation $\mathbf{S}(k; m; r, r') = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{u}(k; m; r, t)\mathbf{u}^*(k; m; r', t)\, \mathrm{d}t$; Just do SVD/eig solve;

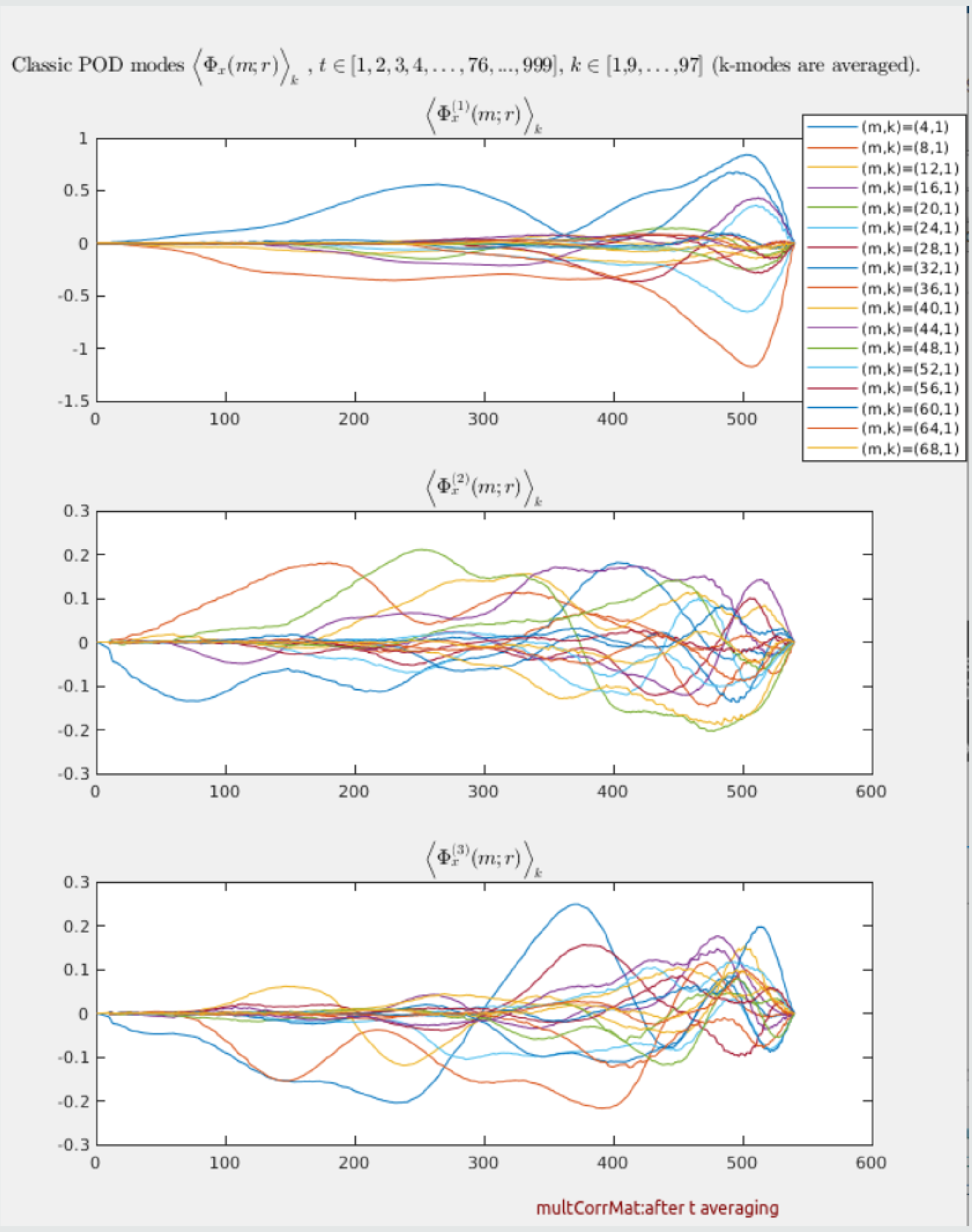
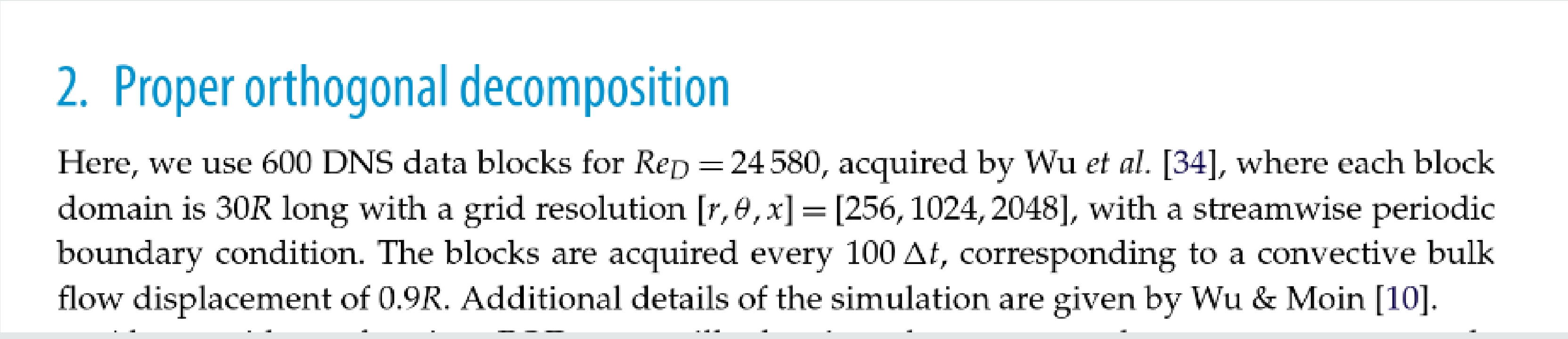


Figure 1: Classic POD Modes closely following Smits2017 procedure (direct mult used); medium size run (full time step and every 9th crosssection); link to .Mat file

Worry about Convergence of streamwise FFT

- Zigunov's advice:
  - we need long time series to get converged statistics (say, for example in the case of Fourier Analysis of V, we need as many entries in V as possible). This is particularly the case in experiments, where noise can contaminate the spectrum and spectral averaging can "clean up" our data by a good amount.
  - In general, Fourier analysis needs long data sets to capture periodicity with a good amount of certainty.
  - Upshot: need to show fft(x) is actually converged by varying density of crosssections. If not, more cs necessary !

Smits 2017 used Such a large streamwise resolution:

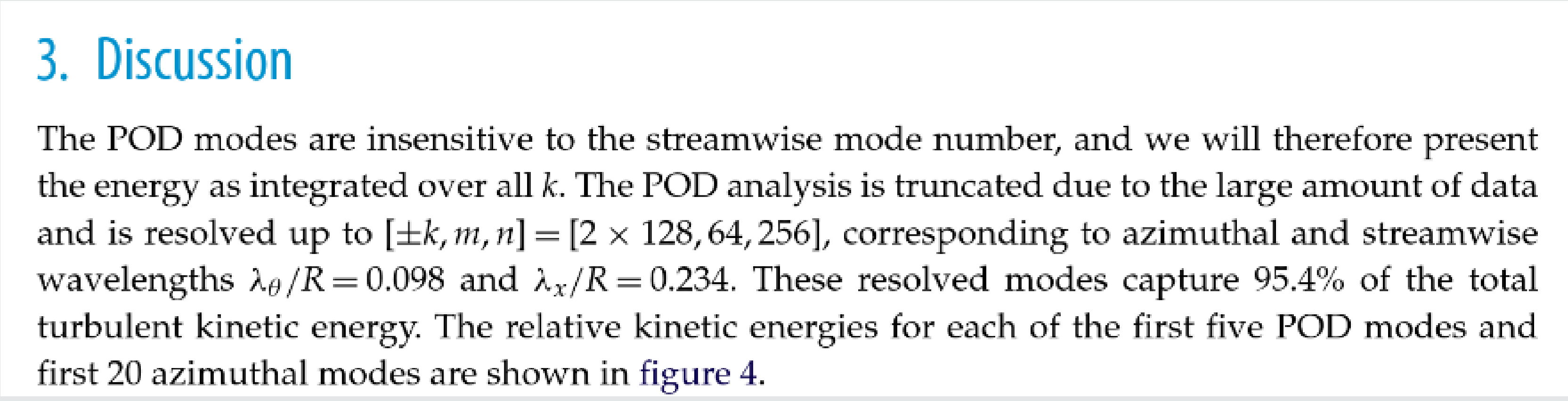


2. Proper orthogonal decomposition

Here, we use 600 DNS data blocks for $Re_D = 24\,580$, acquired by Wu *et al.* [34], where each block domain is $30R$ long with a grid resolution $[r, \theta, x] = [256, 1024, 2048]$, with a streamwise periodic boundary condition. The blocks are acquired every $100\,\Delta t$, corresponding to a convective bulk flow displacement of $0.9R$. Additional details of the simulation are given by Wu & Moin [10].

Figure 2: **Smits2017 data resolution**. This shows **2048 crosssections** were used. That was perhaps necessary to achieve convergence in the fft streamwise. In order to show that is indeed not necessary, a convergence study must be done

Then they saved a fraction of that to file.



3. Discussion

The POD modes are insensitive to the streamwise mode number, and we will therefore present the energy as integrated over all k. The POD analysis is truncated due to the large amount of data and is resolved up to $[\pm k, m, n] = [2 \times 128, 64, 256]$, corresponding to azimuthal and streamwise wavelengths $\lambda_\theta/R = 0.098$ and $\lambda_x/R = 0.234$. These resolved modes capture 95.4% of the total turbulent kinetic energy. The relative kinetic energies for each of the first five POD modes and first 20 azimuthal modes are shown in figure 4.

Figure 3: **Smits2017 data resolution**. This shows that 128 streamwise modes were then saved to file and averaged. But 2048 crosssections were initially processed.

Not necessary?

- Then show that with actual convergence study.
- Also compare with more papers.

Goal Project resolution (viz: finish Msc)

- If this data set is too large, then cannot replicate Smits' result.
- a valid procedure has been developed, other phyiscs can be gleaned by smaller data sets
- Downside of this project: lot of data, perhaps intractable for a short term project
  - Lot of unneceessary uncertainty introduced by predecessor's results (lack of version control, no record of code's output, code needs to be documented to a higher standard: no intermediate results and incomplete procedure).

Timeline A of Project conclusion (+)

- **Expected Time to get more crosssecion data** I dont have a good estimate for that. If one expects Smits2017 results then one needs $2048 \times ntimesteps$.
  - Tecplot did not seem to extract that data in a parallel way, even when parallel was called.
- **Finish December** Would be able to get paper but requires a bit more time

Timeline B of Project conclusion (-)

- work with current data
- do spectral analysis on limited data, but won't be able to get results for JFM paper
- This is not desirable
- **Finish at end of Summer**. No paper but just get done. Scope of project was overestimated.

Changes for Date: *<2022-06-06 Mon>*

Done A: Snapshot POD

- Correlation matrix is computed two ways (switch implemented for this)
  - A) either using xcorr (form assuming stationary and ergodic hypothesis, see below pages for that explanation)
  - B) direct calculation.
- Graph of classic POD run — looks rough

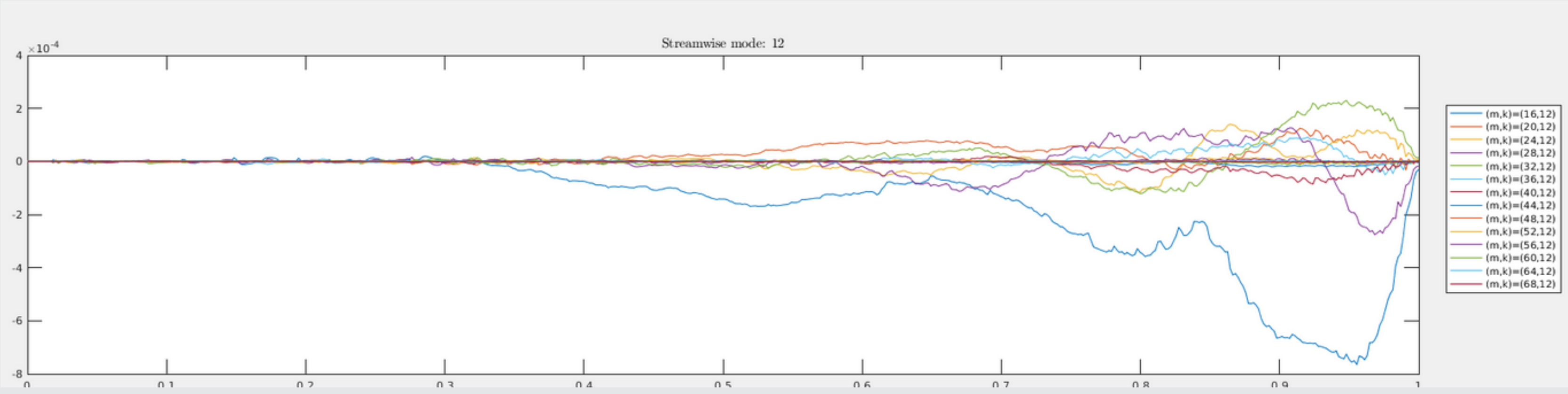


Figure 4: **snapshot pod run** update this.

- I'm thinkign that this can be better resovled if the temporal correlation is completely uncorrelated (this is the opposite of classic pod, where we want the correlation in radial)
  - papers say something ot this effect (add).

Done B: Classic POD, Large run done.

- change fft(θ) to fourier(θ) (function *fourier*2.*m*)
- Large run is all timesteps, 1/2 of the crosssection
  - runtime: ~13 hours $\because$ code is parallized and does fourier(θ) first.

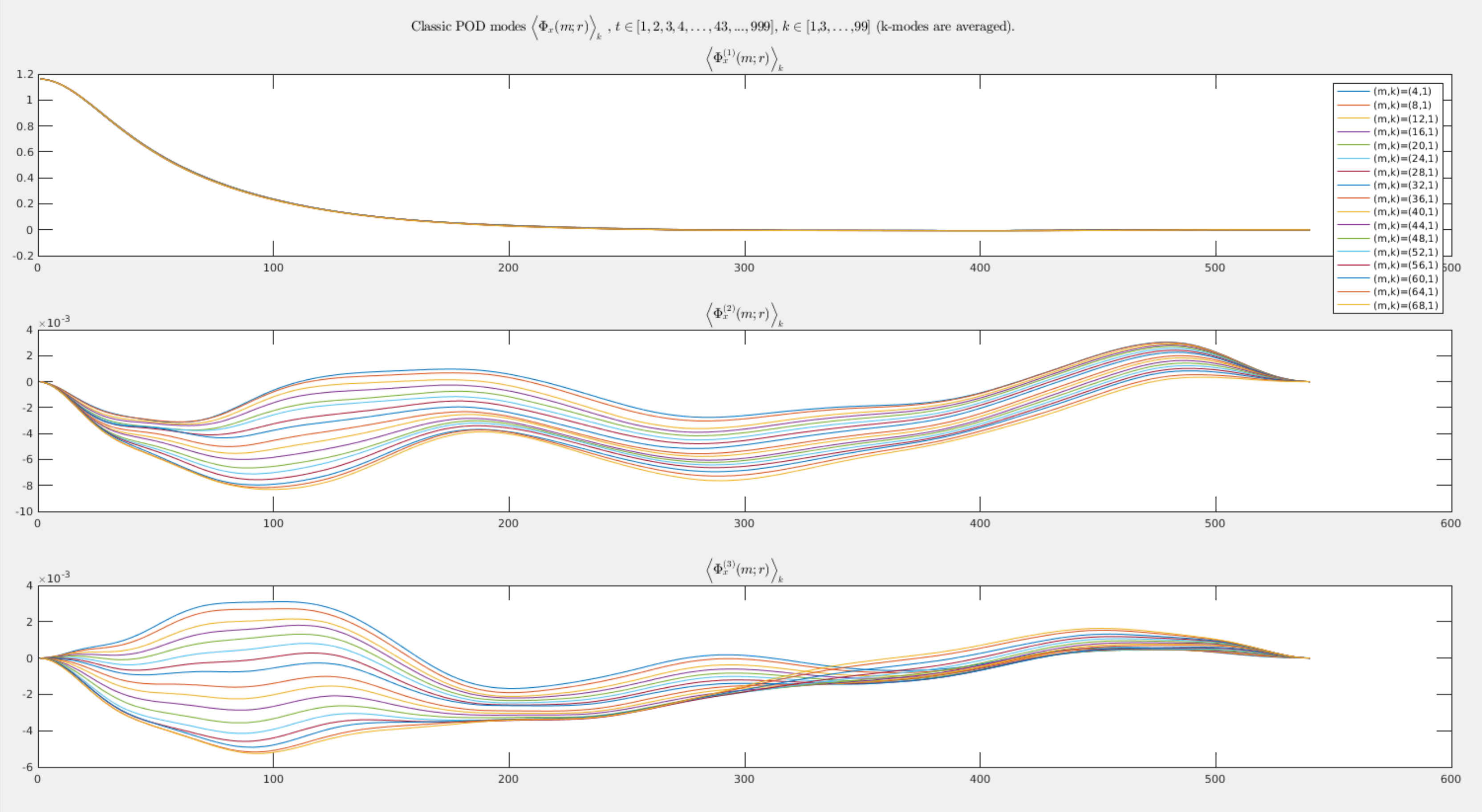


Figure 5: *classic pod run1 sized run (not a lot of crosssections used):

Need to do:

- Classic POD, with **full components**: u, v, w correlation matrix ...currently, just using streamwise component.
  - Needs a switch for direct correlation matrix calc. Cannot use xcorr for this as can be done for time-correlation (because not homogeneous/stationary in the radial direction.)
    - nb need to extract full interpolated components to disk ~ 5 TB.
- **Answer Major question:** Should we have correlation in time at all?
  - Tentative answer: **No. velocity fluctuations should have no time correlation at all, and any spikes in the time correlation would effect the POD graph (make it look like aliasing occurs)**.
    - **Idea:** by increasing timestep, that can decrease correlation , and therefore make the spectral analysis result less jittery.

Reminder of Goal: reconstruct the following

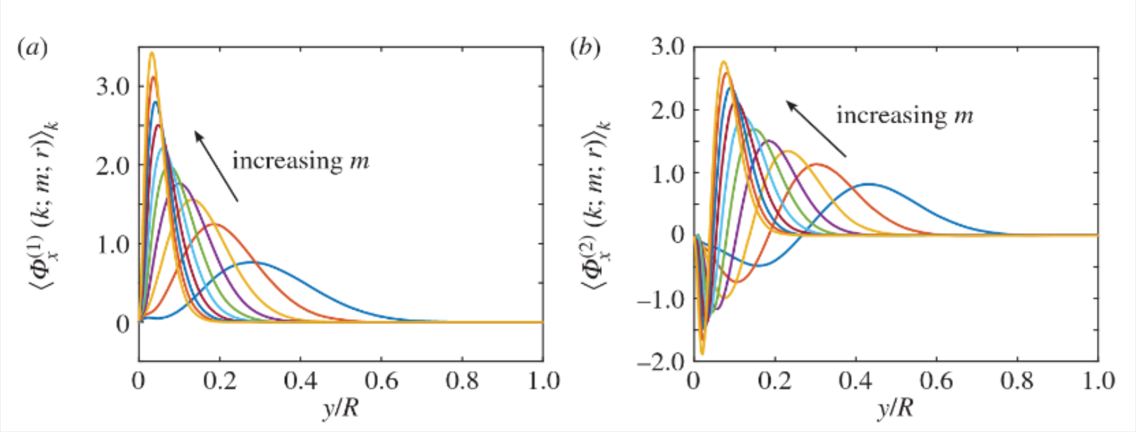


Figure 6: Smits 2017

Classic POD ⇝

- Radial Correlation — for antipating Classic POD Result (show graph:)

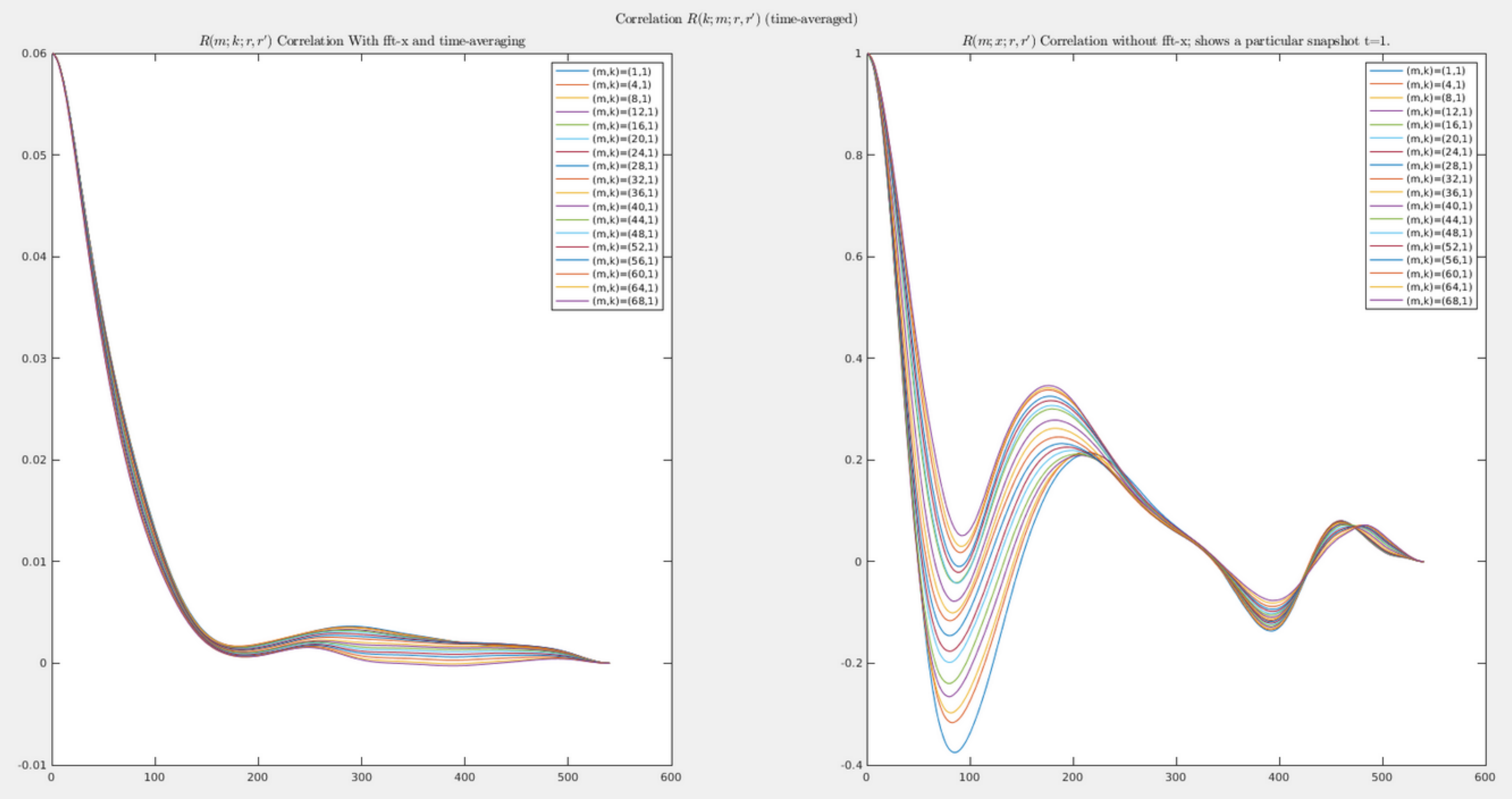


Figure 7: Radial Correlation. **left** After applying fft in x-dir. **Right:** no fft-x applied yet (just fft-θ and correlating).

- Compare to graphs of correlation data from eg Eggels et al

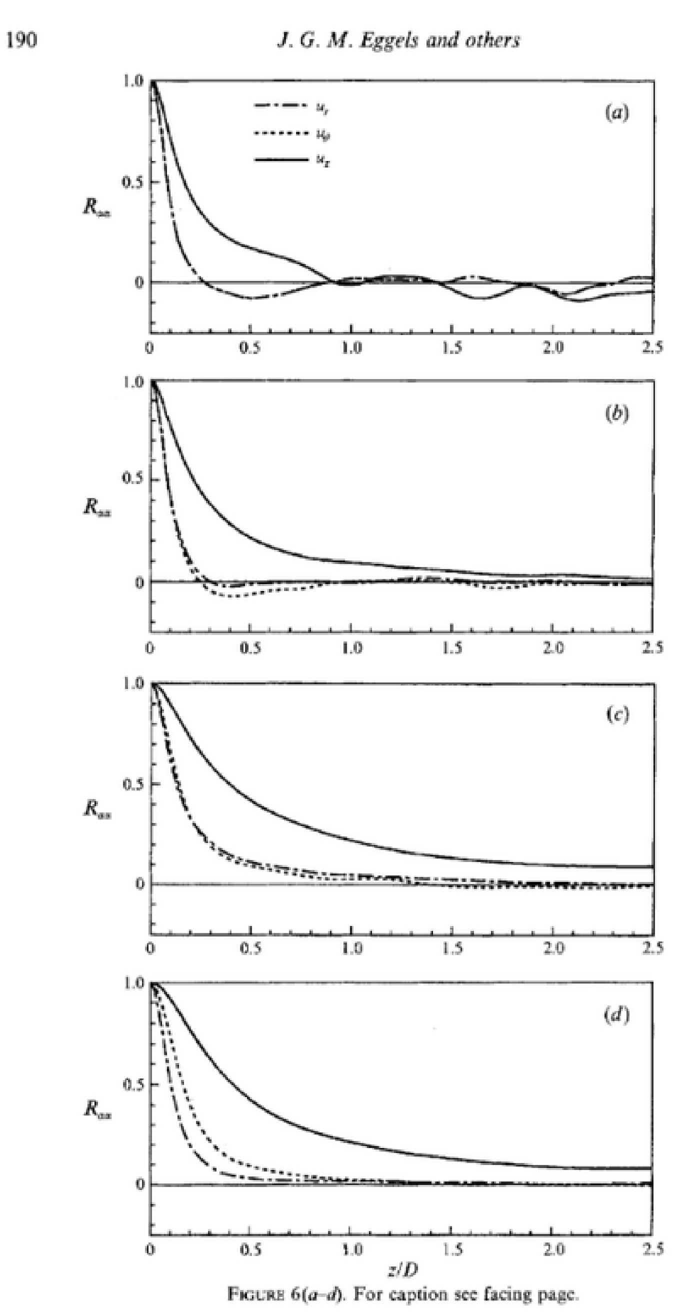


Figure 8: Eggels et al Correlation, FigURe 6. Two-point correlation coefficients of the thrce fluctuating velocity components computed from the DNS(E) data as functions of the streamwise separation distance z/D : $(a) r/D = 0.008$, $y^+ = 177.2$; $(b) r/D = 0.247, y^+ = 90.9$; $(c) r/D = 0.451, y^+ = 17.8$; $(d) r/D = 0.487, y^+ = 4.7$

Classic correlation matrix Formation

- Form symmetric positive definite matrix.

$$S(r,r';m;k) = \begin{bmatrix} u(0)u^H(0) & u(0)u^H(1) & u(0)u^H(2) & \dots & u(0)u^H(R) \\ u(1)u^H(0) & u(1)u^H(1) & u(1)u^H(2) & \dots & u(1)u^H(R) \\ \vdots & & & & \\ u(m)u^H(0) & u(m)u^H(1) & u(m)u^H(2) & \dots & u(m)u^H(R) \end{bmatrix} \tag{1}$$

- The matrix needs to be formed with u, v, w components, which are stacked as , eg for 2d, Wu Kuisheng (2018) do:

$$U = \begin{bmatrix} u_1(x_1) & u_2(x_1) & \cdots & u_N(x_1) \\ u_1(x_2) & u_2(x_2) & \cdots & u_N(x_2) \\ . & . & . & . \\ u_1(x_M) & u_2(x_M) & \cdots & u_N(x_M) \\ v_1(x_1) & v_2(x_1) & \cdots & v_N(x_1) \\ v_1(x_2) & v_2(x_2) & \cdots & v_N(x_2) \\ . & . & . & . \\ . & . & \cdots & . \\ v_1(x_M) & v_2(x_M) & \cdots & v_N(x_M) \\ . & . & . & . \\ w_1(x_1) & w_2(x_1) & \cdots & w_N(x_1) \\ w_1(x_2) & w_2(x_2) & \cdots & w_N(x_2) \end{bmatrix} \tag{2}$$

- Then correlation matrix $\mathbf{C}$ to describe the temporal correlation of flow field is $\mathbf{C} = \frac{1}{N}\mathbf{U}^H\mathbf{U}$.
- In particular, in contast to snapshot POD, we need to form the correlation matrix with all 3 flow field components;
- **Compare with snapshot pod**. That is why snapshot pod is more efficient. we just need to find correlation matrix in time, so dont have component for u, v, w, just would have t. Then the streamwise POD mode $\alpha^{(n)}$ is found as a projection onto w, the streamwise component.

Snapshot POD

- Forming the temporal correlation tensor, for statistically stationary data, we can write the correlation matrix as,

$\rightsquigarrow$

$$S(r, r'; m; k) = \begin{bmatrix} u(0)u^H(0) & u(0)u^H(1) & u(0)u^H(2) & \dots & u(0)u^H(m) \\ u(1)u^H(0) & u(1)u^H(1) & u(1)u^H(2) & \dots & u(1)u^H(m) \\ \vdots & & & & \\ u(m)u^H(0) & u(m)u^H(1) & u(m)u^H(2) & \dots & u(m)u^H(m) \end{bmatrix} \tag{3}$$

- Since assumed homogeneity and statistically stationary and ergodic signal,

$$S(t, t'; m; k) = \begin{bmatrix} S(0) & S(1) & S(2) & \dots & S(m) \\ S(1) & S(0) & S(1) & \dots & S(m-1) \\ S(2) & S(1) & S(0) & \dots & S(m-2) \\ \vdots & & & & \\ S(m) & S(m-1) & S(m-2) & \dots & S(0) \end{bmatrix} \tag{4}$$

- Snapshot correlation Result:

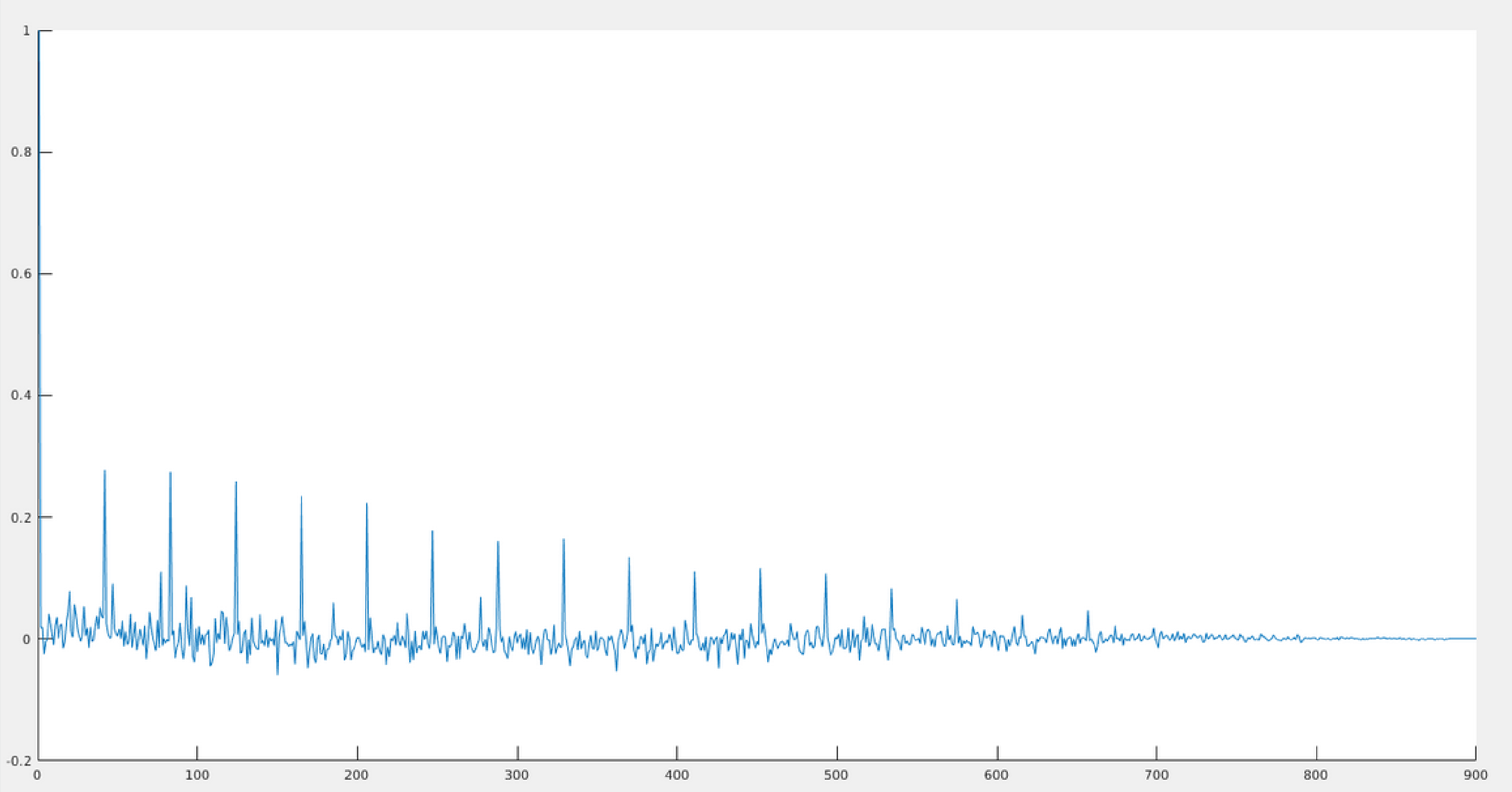


Figure 9: **Temporal Correlation.** Result of fft-θ then forming the correlation matrix. The graph is a row of that matrix, ie $\{S(0), S(1), S(2), \dots, S(m)\}$, where $S(t_i)$ is the correlation at various lags t_i.

- either too periodic (correlated) or not smooth enough (no correlation)
  - according to (source) $\alpha^{(n)}$ should be totally uncorrealted.

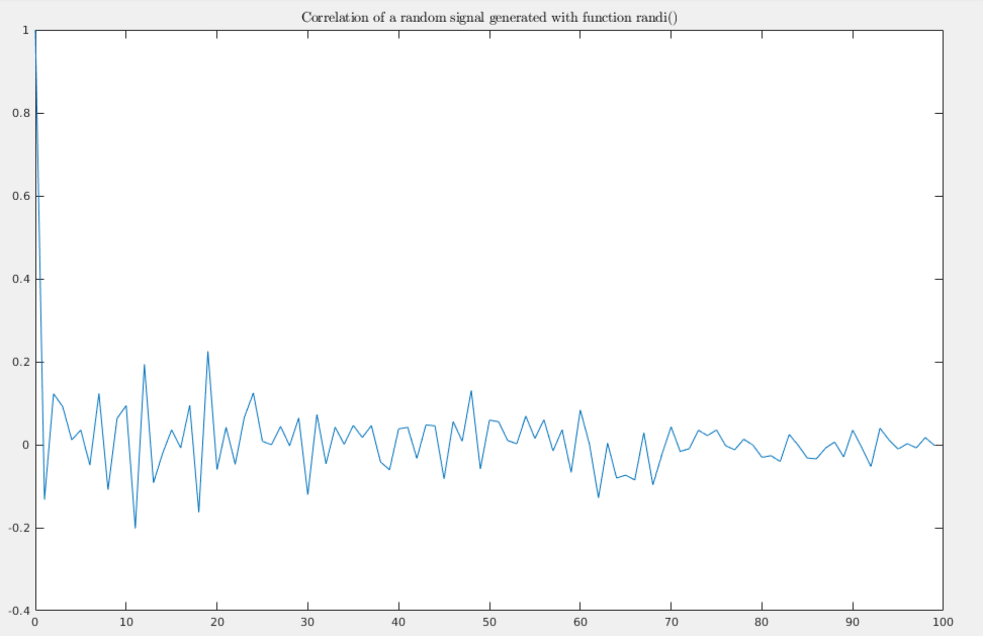


Figure 10: **Correlation of random signal.** In order to smooth out our signal, the temporal correlation ought to look completlely uncorrelated (just like a random signal).

- The following was changed: change fft(θ) to fourier(θ) (function *fourier2.m*)

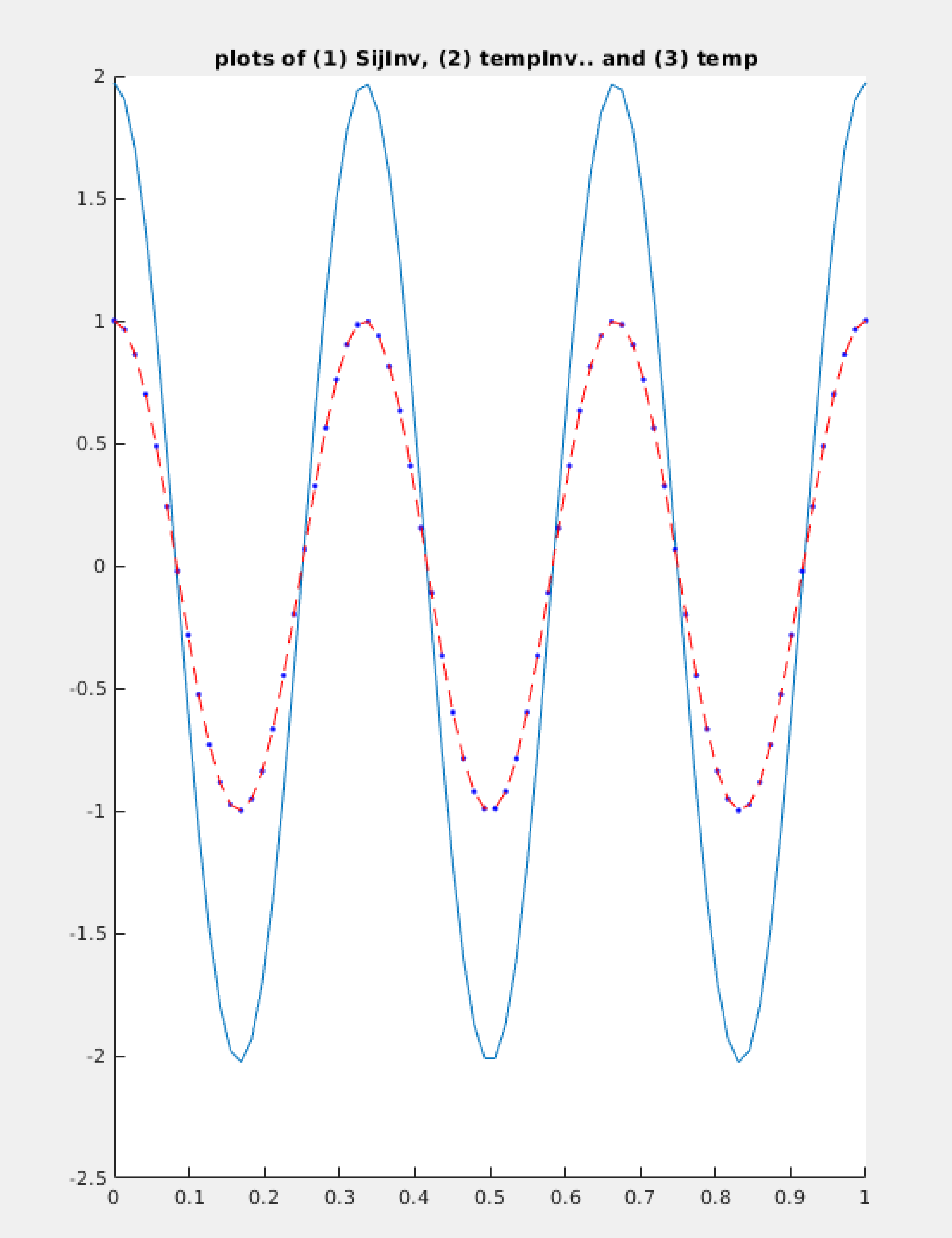


Figure 11: Shown: (1) original curve (blue dots) with reconstructed fourier approximation (red line); solid blue is real part of fourier coefficient.

Definition (Full Procedure for Master Branch *<2022-05-27 Fri>*)

- Spectral procedure follows Smits 2017.
- Pod procedure follows Smits 2017.

- Part 1. Spectral Analysis
  - Step A. take fft azimuthally
    - use half of θ data to avoid aliasing
    - Note: in my opinion $\sum_{m=0}^{M}(fft(theta))\,(cos(\theta) + i * sin(\theta))$ rather than just the fft ought to be used. This is done in (cite).
    - #TODO: include this in next update
  - Step B. find correlation in t' described in Smits2017.below.eq.2.4.

$$\mathsf{R}\left(km; t, t'\right) = \frac{1}{T}\int_r \mathsf{u}(k; m; r, t)\mathsf{u}^*\left(k; m; r, t'\right) r\,\mathrm{d}r \equiv \left\langle \mathsf{u}(k; m; r, t)\mathsf{u}^*\left(k; m; r, t'\right) \right\rangle_r \tag{5}$$

- Create this option: use of function xcorr used/not used
  - currently: (when function m5.m on master branch *<2022-05-27 Fri>* is used) — the above equation for R done as a explicitly as $\int uu^*$.

- Note that in the standard textbook case, we are correlating spatially and forming the ensemble time average. In anticipating the snapshot POD, the opposite is done: temporal correlation is found, and then the weighted (with r) spatial average (via integration) is found.

- Step C. take fft in x of th above correlation to get k modes.

- Part 2. Snapshot POD
  - the crossspectra for the kernal of the pod is given by the r -averaged function

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^{\tau}\mathsf{R}\left(k; m; t, t'\right)\alpha^{(n)}\left(k; m; t'\right)\mathrm{d}t' = \lambda^{(n)}(k; m)\alpha^{(n)}(k; m; t) \tag{2}$$

  - Note that $\alpha^{(n)}$ act as the eigenfunctions in the above Second Type Fredholm integral equation. This is simply the formulation of the snapshot POD.
  - Step D. Find the (sorted) eigenvalues $\alpha^{(n)}$ found in (2) to solve for $\Phi^{(n)}$,

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^{\tau}\mathsf{u}_{\mathrm{T}}(k; m; r, t)\alpha^{(n)^*}(k; m; t)\mathrm{d}t = \Phi_{\mathrm{T}}^{(n)}(k; m; r)\lambda^{(n)}(k; m)$$

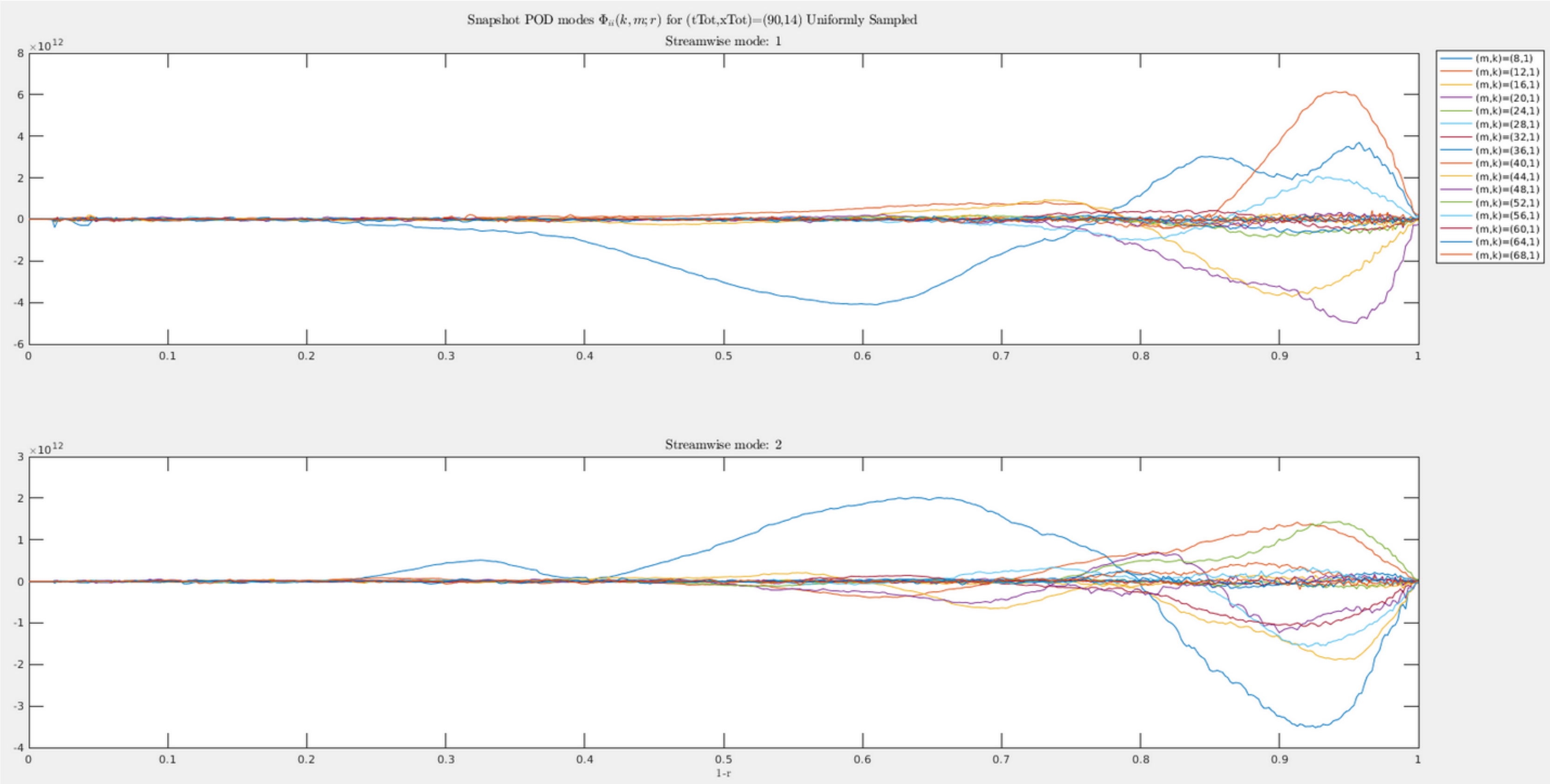


Figure 12: Shows snapshot POD for differen k modes; the timestep and crosssection data is uniformly spaced, with 90 timesteps and 14 crossections used. The small data sample is shown since this code branch must be parallized (in next update). **Note also**, that Smits2017 averages all k -mode graphs.

- Issues and Guiding Principles.
  - #TODO: Unfortunatelty, the maximum value is not occuring along the diagonal., as should occur with correlation coefficient matrices (!)
  - The matrix is positive semidefinite however (positive $\sqrt{\sigma_i} > 0 \forall i$.)
  - As alternative to uu^H calculation, suggest using
    1. xcorr() so: $xcorr(u, u^H)$ and form the symmetric matrix with zero lag along the diagonal. Make sure xcorr correctly conjugates the complex part.
  - Alternately use $corrcoef(u, u^H)$. The good point with this is the diagnonal entries are 1 automatically.
  - Value of r. $r \in [0, 0.5]$. That seems to be equallly spaced (but check that).
    - see file `file:///mnt/archLv/mike/podTimeCoeffCopy/tests/run/fftCode/snapWithXYonly.dat`
    - The value of $dr = \ldots$; presumable dr $\approx$ 0.5/540$.

- Example correlation coefficient matrix R.
  - The maximum values should occur along the diagonal since this is 0 lag occurs (but do not have that)
  - Here is the integrated correlation tensor with the $\int ruu^*\, dr$ minimalbeispiel,

$$\mathrm{R}\left(x_1, m_1; t, t'\right) = \begin{bmatrix} -1.9672 & -3.3689 & -3.6159 & -2.7419 & -2.5511 \\ -3.3689 & -5.7692 & -6.1922 & -4.6955 & -4.3688 \\ -3.6159 & -6.1922 & -6.6463 & -5.0398 & -4.6891 \\ -2.7419 & -4.6955 & -5.0398 & -3.8216 & -3.5557 \\ -2.5511 & -4.3688 & -4.6891 & -3.5557 & -3.3083 \end{bmatrix}, \text{ntimesteps} = 5 \tag{6}$$

which is indeed symmetric. This is matlabcorrMatSmits(1).dat.

- Intermediate Results
- Intermediate Spectral Results Graphs
  - Check if correlation matrix is formed properly. Sometimes (depending on how they were obtained), it turns out that the set of correlations doesn't form a proper correlation matrix. One way to check whether you do is to take the singular value decomposition and check all the singular values are non-negative.

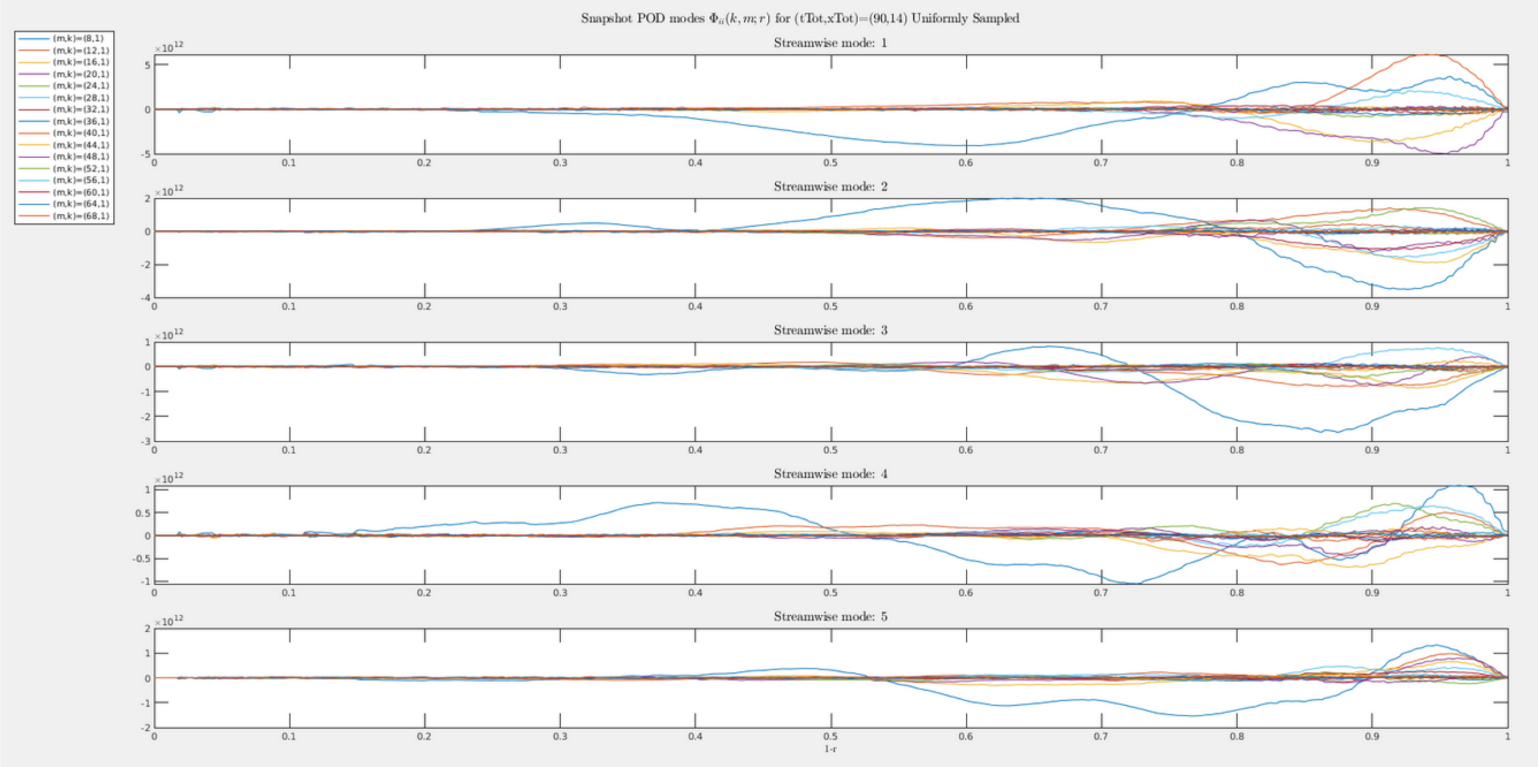


Figure 13: Shows snapshot POD for 5 differen k modes (5 shown, total is 14); These need to be averaged