*Version
November 6, 2011

$$(V)' \supseteq (H_0^1)' = H^{-1}.$$

$$\mathbf{u}\cdot\nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

(E)M i c h a e l I s a i a h R a b a
University of Kentucky
Fall 2011 Notes 773 Navier Stokes Eqns Sem. POT 845

Contents

Denn nichts ist für den
Menschen als Menschen etwas
wert, was er nicht mit
Leidenschaft tun kann.

Chapter 1

Linear & Nonlinear Static NSE

Friday, August 26

- Overview of NSE

1.1 Physical properties remarks

- These equations establish that changes in momentum (acceleration) of fluid particles are simply the product of changes in pressure and dissipative viscous forces (similar to friction) acting inside the fluid. These viscous forces originate in molecular interactions and dictate how sticky (viscous) a fluid is. Thus, the Navier-Stokes equations are a dynamical statement of the balance of forces acting at any given region of the fluid.

1.2 Derivation

$$\vec{F} = \vec{F}_{\text{int}} = \nabla \cdot \sigma$$

where $\sigma = \text{normal} + \underbrace{\text{sheer strength}}_{\text{viscosity - 0 in ideal fluid}}$ and

$$
\begin{aligned}
(\sigma_{ij}) &= \underbrace{\frac{-1}{3}}_{\text{normalize}} \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} + T \\
&= \underbrace{\frac{1}{3} \cdot (-pI_3)}_{\text{gravity}} + \underbrace{\tau}_{\text{tensor; material dependent}}
\end{aligned}
$$

where we assume the tensor σ is symmetric.

For a Newtonian fluid,

$$
\underbrace{\tau}_{\text{stress tensor}} \propto \underbrace{\frac{\partial u}{\partial y} + \left(\frac{\partial u}{\partial y}\right)^T}_{\text{velocity gradient - strain rate}}
$$

where τ is proportional to viscosity, which varies with temperature.

For $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ and p constant,

$$
\begin{aligned}
\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla \cdot \left(pI_3 + \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)\right) \boxed{???} \\
\underbrace{\nabla \cdot u}_{\text{incompressibility}} &= 0
\end{aligned}
$$

Calculating,

$$
\begin{aligned}
&\nabla \cdot (\nabla u + (\nabla u)^T) \quad \text{or} \quad \frac{\partial}{\partial x_j}\left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\right) && \text{take transpose} \\
&= \Delta u^i + \frac{\partial}{\partial x_i}\left(\frac{\partial u^j}{\partial x_j}\right) && \text{div } = 0 \because \text{ fluid incompressible} \\
&\Rightarrow \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mu \Delta \mathbf{u}. && \text{(NSE)}
\end{aligned}
$$

This is formally a coupled system in four unknowns $(u^1, u^2, u^3), p$ and 2 eqns, and is therefore determinable. To uncouple, write (?)

Difficulty: p is presumably a function of 4 variables, but my eqn doesn't tell me; this makes p difficult to handle.For fluid have boundary conds., but for p $\nexists$ initial condition

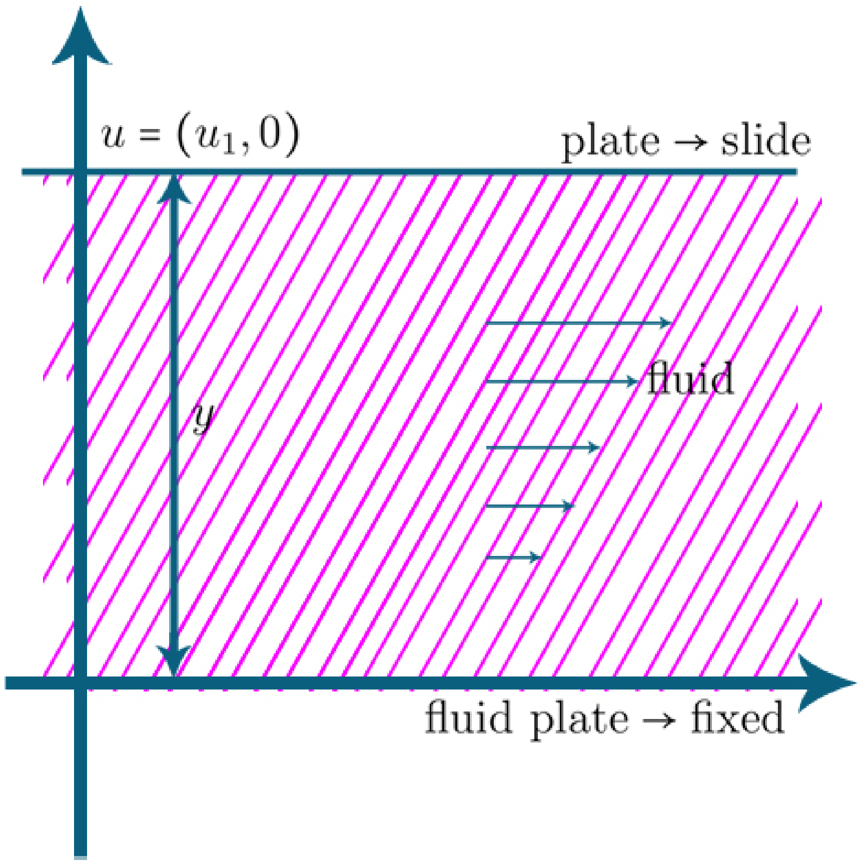


Figure 1.1: Measuring shear flow

$$\underbrace{\boxed{\Delta p = -\text{div}\ \big((\mathbf{u}\cdot\nabla)\mathbf{u}\big)}}_{\text{elliptic eqn}}$$

1.3 Shear flow

- An element of solid has a preferred shape, to which it relaxes when the external forces on it are withdrawn (whereas a fluid does not).
- Looking at a rectangular element $ABCD$, under the action of a shear force $\vec{F}$, the element assumes the shape $ABC'D'$. If the solid is per-

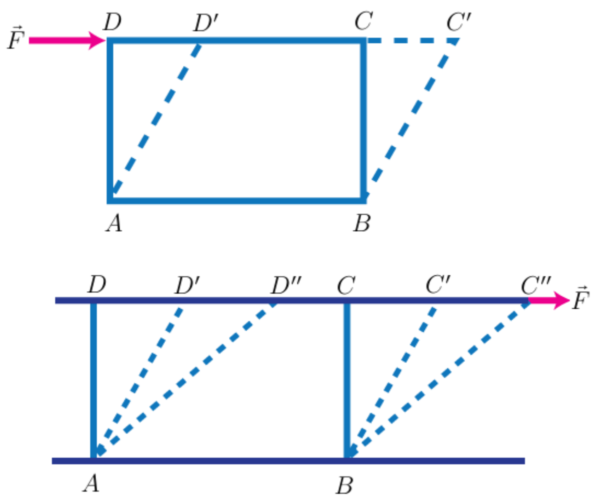


Figure 1.2: Deformation of solid (top) and fluid elements (bottom)

fectly elastic it goes back to its preferred shape $ABCD$ when F is withdrawn.

- In contrast a fluid deforms **continuously** under the action of a shear force (however small). Thus the element of the fluid $ABCD$ confined between parallel plates deforms to shapes such as $ABC'D'$ and $ABC''D''$ as long as the force F is maintained on the upper place.

How to Measure Viscosity

First measure the normal force that is exerted on the plate. From this we can $\Rightarrow$ determine pressure p:

$$\vec{F} = \underbrace{\mu}_{\text{This constant is to be determined}} \cdot A \cdot \underbrace{\frac{u}{y}}_{\textbf{strain rate: }\text{sort of like } \frac{du}{dy}}$$

measure area A, the strain rate $\frac{u}{y}$ to determine experimentally the pressure constant μ

1.4 Steady-state Stokes - Overview

$u\,|_{t=0} = u_0, \quad \nabla \cdot u_0 = 0.$

In NSE (or just Stokes) the following boundary conditions are commonplace:

- $u\mid_{\partial\Omega} = 0$ no slip boundary - fluid will have zero velocity relative to the boundary.

- $$\left.\begin{array}{l} \mathbf{u}\cdot\mu = 0 \\ \text{or} \quad \underbrace{(\nabla\times\mathbf{u})\times\mathbf{v} = 0}_{\text{vorticity - here no tang. component}} \end{array}\right\} \text{Navier Slip boundary condition}$$

A. Steady-State Stokes

$$\begin{aligned} -\mu\Delta\mathbf{u} + \nabla p &= 0 \\ \nabla\cdot\mathbf{u} &= 0 \\ \mathbf{u}\mid_{\partial\Omega} &= 0 \end{aligned}$$

drop t.

∇p **is harmonic**,so ∇p can be wild on boundary $\Leftarrow$ we have this from holomorphic functions

B. Evolution Stokes

$$\begin{aligned} \mathbf{u}_t - \mu\Delta\mathbf{u} + \nabla p &= 0 \\ \nabla\cdot\mathbf{u} &= 0 \\ \mathbf{u}\mid_{t=0} = u_0, \quad u\mid_{\partial\Omega} &= 0 \end{aligned}$$

C. Steady NSE

$$\begin{aligned} \mathbf{u}\cdot\nabla\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p &= 0 \\ \nabla\cdot\mathbf{u} &= 0 \\ \mathbf{u}\mid_{\partial\Omega} &= 0 \end{aligned}$$

1.5 Steady-Stokes - Variational form

Compare this to "Lax-Milgram-type" arguments for elliptic eqns. In fact, NS is "almost" elliptic:

$$\underbrace{-\mu\Delta u}_{2^{nd}\text{order function with divergent form}} + \overbrace{\nabla p}^{\text{but have this termso we must find a different solution space.}} = 0$$

For the first term

$$\left\langle \underbrace{-\mu\Delta \mathbf{u}}_{\mu\int_\Omega\langle\nabla\mathbf{u},\nabla\phi\rangle \quad \text{where } \phi\in C_c^\infty} + \nabla p, \phi \right\rangle = 0 \quad \Rightarrow \int_\Omega \langle -\mu\Delta\mathbf{u} + \nabla p, \phi\rangle = 0,$$

For the second term,

$$\langle \nabla p, \phi\rangle \overset{\text{Leibnitz rule}}{=} \underbrace{\text{div } (p\phi)}_{\text{trace}=0} - p\cdot(\nabla\cdot\phi) \quad \text{so}$$

$$\mu\int_\Omega \nabla\mathbf{u}, \nabla\phi - \int_\Omega \langle p, \nabla\cdot\phi\rangle = 0$$

where we have assumed that the test function ϕ is divergence-free:

$$V = \left\{\phi\in C_c^\infty(\Omega) \mid \text{div } \phi = 0\right\}.$$

Reverse direction:

$$\mu\int_\Omega \langle\nabla\mathbf{u}, \nabla\phi\rangle = 0 \quad \forall\phi\in V$$

or

$$\min\left\{\mu\int_\Omega |\nabla\mathbf{u}|^2 \mid u\in V\right\}.$$

We calculate first variation to get Euler-Langrange eqn (above)

The Problem with the Space $\mathcal{V}$

The space $\mathcal{V} = \left\{\phi\in C_c^\infty(\Omega) : \text{div } \phi = 0\right\})$ is not a complete Vector Space ! ! ! $\Rightarrow$ We need to complete this $\Rightarrow$ So how will we do this? $\Rightarrow$ depends on the space we want to close, i.e. H_0^1, L^2.

Monday, August 29

- Function spaces

Once we have the variational formulation of

$$\begin{aligned} \mu\Delta\mathbf{u} + \nabla p &= 0 \quad \text{in } \Omega \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

we have to set up the corresponding function space. The usual Sobolev space is deficient because it requires information about $\nabla\mathbf{u}$, whereas here only $\nabla\cdot\mathbf{u}$ is known. In the following, we consider only vector fields that are divergence-free.

1.6 $W^{k,p}, H^k, H^{1/2}$

The Inner Product $(\cdot,\cdot)$ and the Pairing $\langle\cdot,\cdot\rangle$

$(u,v) \Leftarrow$ we can use the L^2 inner product for two functions. $\langle u,v\rangle$ use for those arguments that aren't functions, for example $(-\Delta\mathbf{u},v)$.

$W^{k,p}$. Define $W^{k,p}(\Omega) = \left\{f \in L^p(\Omega) : D^\alpha df \in L^p(\Omega) \quad \forall\, |\alpha| \le k, k \ge 0, 1 \le p \le \infty\right\}$

$$\|f\|_{W^{k,p}} = \sum_{|\alpha|\le k} \left(\|D^\alpha f\|_{L^p}^p\right)^{\frac{1}{p}}$$

where $\left(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}}\right)$ is a Banach space.

Density theorem

We denote the space $C^\infty(\overline{\Omega})$ as the space of smooth functions up to the boundary. For $p = 2, 1 \le p < \infty$, this space is dense in $W^{k,p}(\Omega)$.

Note that $\overline{C_c^\infty}_{W^{k,p}} = W_0^{k,p}$ and $u = D^\alpha\mathbf{u} = 0 \forall\, |\alpha| \le k-1 \quad \text{on } \partial\Omega$. ?

H and V

For $\left\{u \in C_c^\infty(\Omega), \nabla\cdot\mathbf{u} = 0\right\} \subset H_0^1 \subset L^2$, define $H = \{\mathbf{u} \in C_c^\infty : \nabla\cdot\mathbf{u} = 0\}$. This space is not closed. After mollifying, define

- $V := \overline{\{\mathbf{u} \in C_c^\infty(\Omega) : \nabla \cdot \mathbf{u} = 0\}} = H_0^\perp$ Here $u \in H_0^1$ and $\nabla \cdot \mathbf{u} = 0$ preserve their definitions.
- $H := \overline{\{\mathbf{u} \in C_c^\infty : \nabla \mathbf{u} = 0 \text{ and } p \in H^1(\Omega)\}} = L^2$

Here we have $u_k =\in C_c^\infty$ and $\overset{u}{\underset{u_k}{\uparrow L^2}} \in C_c^\infty$ and $\nabla \cdot \mathbf{u}_k = 0$.

What is $\nabla \cdot \mathbf{u} = 0$ here? In this space, $\nabla \cdot \mathbf{u}$ is $= 0$, remember though we are taking distributional derivatives.

Since $\nabla \cdot \mathbf{u} = 0$, can we talk about $\mathbf{u}\,|_{\partial\Omega}$. If $\mathbf{u}$ is a L^2 function whose distributional derivative is zero, then do we have enough information? No: If only $\mathbf{u} \in L^2$, then the boundary can't be defined.

We have "partial information" with $\mathbf{u} \cdot \nu$. This expression is well defined when

$$\boxed{\mathbf{u} \cdot \nu \in H^{-1/2}(\partial\Omega)}$$

1.7 The Auxiliary space $E(\Omega)$

Define the auxiliary space $E(\Omega) := \left\{ \mathbf{u} \in L^2(\Omega) : \underbrace{\nabla \cdot \mathbf{u}}_{\text{distributional divergence}} \in L^2(\Omega) \right\}$.

Then

$$H^1 \subset E(\Omega) \subset L^2.$$

Define

$$(u, v)_{E(\Omega)} = \int_\Omega \mathbf{u} \cdot \nu + (\text{div } \mathbf{u})(\text{div } \mathbf{v})$$

well defined → each term is $\in L^2$.

and

$$\|u\|_{E(\Omega)} = \sqrt{(\mathbf{u}, \mathbf{u})_{E(\Omega)}}.$$

this function is onto but still not 1-1:
ker $\gamma_0 = H_0^1(\Omega)$ = whole space

Note that $(E(\Omega), \|\cdot\|)_{E(\Omega)}$ is Hilbert (check).

Density theorem for $E(\Omega)$. Assume $\partial\Omega \in C^{0,1}$. Then

$$C_c^\infty(\overline{\Omega}) \quad \text{is dense in } E(\Omega).$$

Proof. (...) (see s.31)

Trace theorem for $\boldsymbol{E(\Omega)}$. A trace theorem allows us to define $\mathbf{u}\cdot\nu\mid_\gamma$ for $\mathbf{u}\in E(\Omega)$.

To do this, we ask where the trace operator γ lives. The definition

$$\gamma_0 : H^1(\Omega) \subset\subset L^2(\partial\Omega)$$

has the deficiency that not every L^2 function can be the trace of a H^1 function: this mapping is <u>not</u> onto. Replace this definition with

$$\gamma_0 : H^1(\Omega) \subset\subset H^{\frac{1}{2}}(\partial\Omega)$$

$D^{\frac{1}{2}}$. ← can't define fractional derivatives pointwise. use fourier transform
this is for flat case. not if domain is nonflat. If have Lipschitz graph

$$\partial\Omega = \left\{ (x,\ \underbrace{\psi(x)}_{\text{Lipschitz}}\ ) \right\} : x \in \mathbb{R}^{n-1}$$

I push forward. ϕ is H^1 so this is equivalent

Define $H^s(\mathbb{R}^n) : |f|^2 + |\xi|^2 \left|\hat{f}\right| \in L^2 = \left\{ f : \mathbb{R}^n \to \mathbb{R} \quad \left(1 + |\xi|^2\right)^{\frac{s}{2}} \left|\hat{f}\right| \in L^2 \right\}$.

Consider

$$\widetilde{f}(x) \in H^{\frac{1}{2}}(\mathbb{R}^{n-1}) = f(x, \phi(x)) = \mathbb{R}^{n-1} \to \mathbb{R}$$

where $f \in H^{\frac{1}{2}}(\partial\Omega)$.

locally Lipschitz → Use Partition of unity and define globally.

1.8 Extension map

Define $\ell_\Omega : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$.

Find

$$f \to F$$

by solving

$$\Delta F = 0$$
$$F\mid_{\partial\Omega} = f.$$

The solution of this Laplace equation gives F, the definition of the extension map F. Note that f and F are reciprocals of each other:

$$\|F\|_{H^1} \le c\|f\|_{H^{\frac{1}{2}}},$$
$$\gamma \circ \ell_n = \ell_\Omega \circ \gamma_0 = \mathrm{id}$$

1.9 Trace and Stokes formula

For $\Omega \subset C^{1,1}$ there exists a linear continuous operator $\gamma_\nu : E(\Omega) \subset\subset H^{-\frac{1}{2}}(\partial\Omega)$ and

require the (stronger) $C^{1,1}$ regularity here

$$H^{-\frac{1}{2}}(\partial\Omega) = \left(H^{\frac{1}{2}}(\partial\Omega)\right)$$

dual space.

Furthermore the following hold:

1. $\gamma_\nu u = \mathbf{u} \cdot \nu \quad \forall \mathbf{u} \in C^1(\overline{\Omega})$.
2. Stokes formula holds:

   $\gamma \cdot \mathbf{u}$ restrict normal component

   $$\left(\text{div } (\omega \mathbf{u}) + \mathbf{u} \cdot \nabla \mathbf{u}\right) = \omega \left(\mathbf{u} \cdot \nu\right).$$

   $\gamma_0 \omega$ restrict

   this allows us to integrate by parts in the space $E(\Omega)$

3. $\ker(\gamma_\nu) = E_0(\Omega)$ and

   $$E_0(\Omega) = \overline{C_c^\infty(\overline{\Omega})_{E(\Omega)}}$$

   For Sobolev spaces we had that *the whole component on the boundary was zero*; here, we have no information about the tangent component, just the normal component.

For the function $\mathbf{u}$ the normal component is well defined, but grad $\mathbf{u}$ is not.

$Dx_1, Dx_2, \dots$ are not defined. which I need for the matrix representing the tangential components.

Friday, September 2

1.10 Preliminaries for Stokes Equation Existence

Hodge Decomposition

$$L^2(\Omega) = H \oplus H_1 \oplus H_2$$

where $H = \left\{ u \in L^2(\Omega) : \text{div } u = 0, \underbrace{\gamma_\nu}_{\text{"normal trace"}} = 0 \right\}$

$$H_1 = \left\{ u \in L^2(\Omega) : \mathbf{u} = \text{grad } p, p \in H^1(\Omega), \Delta p = 0 \right\}$$
$$H_2 = \left\{ u \in L^2(\Omega) : u = \text{grad } p, p \in H_0^1(\Omega) \right\}$$

no information on boundary data for H_1, but we do have $\Delta p = 0$.

Check:

1. $H_1 \cap H_2 = H \cap H^{-1} = H^{-1} H2 = \{0\}$.
2. $H_1, H_2 \subset H^\perp$ where $H^\perp = \left\{ u \in L^2(\Omega), u = \Delta p, p \in H^1(\Omega) \right\}$
3. $H_1 \perp H_2$

   Check (3) :

Stokes theorem can be relaxed, only the gradient need be in H_0^1 with the space $E(\Omega)$.

$$u = \nabla p, \Delta p = 0, \tag{1.1}$$
$$v = \nabla q, q \in H_0^1. \tag{1.2}$$

Stokes theorem applies:

$$(u, v) = (\overbrace{\nabla p}^{\text{well defined, } \because \nabla p \in E(\Omega)}, \underbrace{\nabla q)}_{\text{well defined, } \because q \in H_0^1} + (\text{div } \nabla p, q)$$
$$= (\underbrace{\gamma_0(\nabla p)}_{0 \because \text{Eq 1.2}}, \gamma_0 q)$$
$$= 0$$

4. Let $u \in L^2(\Omega)$ and $p \in H_0^1$ solves

$$\begin{aligned} \Delta p &= \text{div } u \quad \text{in } \Omega \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

By Lax-Milgram, there exists a solution.

$$\Rightarrow \text{div } (\nabla p - u) = 0 \Rightarrow u = (\mathbf{u}\nabla p) + \nabla p$$

Claim: The term $\nabla p - u$ is divergence-free but perhaps not trace free ($\gamma_\nu \neq 0$). To see this, write

$$\mathbf{u} - \nabla p \in \overset{\text{div -free}}{E}(\Omega) \Rightarrow \begin{cases} \text{normal trace} \\ \text{can be defined} \end{cases} \Rightarrow$$

$$\underbrace{\gamma_\nu(\mathbf{u}\nabla p)}_{\text{well-defined}} \in H^{-1/2}(\partial\Omega)$$

Let $q \in H^1$ solve

$$\Delta q = 0 \tag{1.3}$$

$$\frac{\partial q}{\partial \nu} = \gamma_\nu(u - \nabla p) \tag{1.4}$$

where γ_ν must satisfy $\langle \gamma_\nu(u - \nabla p), 1 \rangle = 0 \Rightarrow \quad \int_\Omega \underbrace{\text{div } (\mathbf{u}\nabla p)}_{=0} \Rightarrow$ (the above equation is solvable up to a constant. We write $\mathbf{u} \in H^1/\mathbb{R}$ to denote this ("modulo the constant").

Apply Lax-Milgram to solve (1.3,4) .

Note: When applying Lax-Milgram with Neumann BC you have to be careful (?)

Thus

$$\boxed{\mathbf{u} = \underbrace{\mathbf{u} - \nabla p - \nabla q}_{\in H} + \overbrace{\nabla q}^{\in H^1} + \underbrace{\nabla p}_{\in H_2}}$$

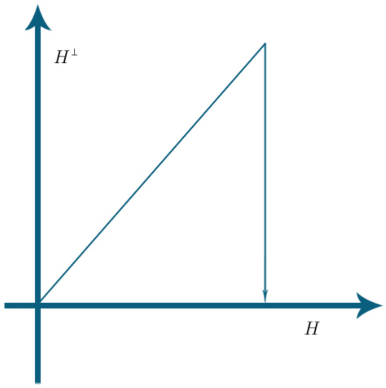


Figure 1.3: The $HH^\perp$ plane

1.11 Leray projection operator

Define $\mathbb{P}$ the bounded linear operator

$$\mathbb{P} : L^2(\Omega) \mapsto H$$

remember $L^2(\Omega) = H \oplus H^2$

which has the apriori estimate $\|Du\|_{L^2} \lesssim \|u\|_{L^2}$ where $u \mapsto \mathbf{u} - \nabla p - \nabla q$.

The Projection operator is used to interpret Stokes:

$$\begin{aligned} -\mu\Delta u + \nabla p &= f \quad \text{in } \Omega \\ \text{div } u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Apply op on both sides of the equation:

$$\mu\mathbb{P}\Delta u + \underbrace{\mathbb{P}(\nabla p)}_{\in H^2} = \overbrace{\mathbb{P}\underbrace{f}_{\in L^2}}^{\in L^2}$$

$\mathbb{P}\Delta u$ is Stokes operator, denoted by $\mathbb{A}$.

Later: Mild solutions $\rightsquigarrow$ Kâto, Giga, Fujita, (cf. Du Hamel form, where we use $\mathbb{P}$ to kill pressure p).

Avantage: $\mathbb{P}$ eliminates the pressure term p.
Disadvantage:

or

$$-\mu\mathbb{A}u = \widetilde{f}$$

1.12 Existence for Stokes Equation

$\Omega \subset \mathbb{R}^n$ is bounded Lipschitz

Use Leray projection operator to make p drop out:

$$\langle -\mu\Delta u + \nabla p, v\rangle = \langle f, v\rangle \quad \forall v \in V \quad \Rightarrow$$

$$-\mu(\nabla \mathbf{u} \nabla \mathbf{v} \overset{\text{int. by parts}}{=} (f, v) \quad \forall \mathbf{u} \in V)$$

For the space

$$V = \left\{u \in H_0^1(\Omega), \text{div } \mathbf{u} = 0\right\} \subset H_0^1$$

we can write the inner product

$$(\mathbf{u}, \mathbf{v})_V = \int_\Omega \nabla \mathbf{u} \cdot \mathbf{v}$$

and

$$\|u\|_V^2 = (\mathbf{u}, \mathbf{u})_V.$$

We look for $\mathbf{u} \in V$ that satisfies

$$-\mu(\mathbf{u}, \mathbf{v})_V = (\mathbf{f}, \mathbf{v}) \quad \forall v \in V$$

where $\mathbf{f} \in V'$ and $f \in H^{-1}$.

Because $\mathbf{u} \in V$, this $\mathbf{u}$ satisfies (A) incompressibility condition (B) trace=0

This implies

$$\underbrace{(-\mu\Delta \mathbf{u} + f, \mathbf{v})}_{\in \mathcal{D}'(\Omega)} = 0 \quad \forall \mathbf{v} \in V.$$

$$\underbrace{-\mu\Delta + f}_{\in H^{-1}} = \Delta \overbrace{p}^{\in L^2} \quad \text{for some } p \in \mathcal{D}'(\Omega)$$

Can recover p; see Notes last class.

Summarize:

Fact $u \in V$ solves $-\mu(\mathbf{u}, \mathbf{v})_V = (f, v)$ for $f \in V'$ if and only if

$$\begin{aligned} -\mu\Delta u + \nabla p = f &\quad \text{in } \Omega \\ \text{div } \mathbf{u} = 0 &\quad \text{in } \Omega \\ u = 0 &\quad \text{on } \partial\Omega \end{aligned}$$

for some $p \in L^2$

$\in V'$ is read "in the dual V'"

1.13 Energy approach

Define the Energy functional $E(u) = \mu\|u\|_V^2 - 2(\mathbf{f}, \mathbf{u})$.

The variational problem $u \in V$ solves $-\mu(\mathbf{u}, \mathbf{v})_V = (f, v) \quad \text{for } v \in V$. is equivalent to

1. u minimizes $E(u)$ over V.

2. $a(\mathbf{u}, \mathbf{v}) = \mu(\mathbf{u}, \mathbf{v})_V \quad \forall \mathbf{u}, \mathbf{v} \in V$.

cf. Laplace Energy functional: $\Delta u = f \Leftrightarrow \int_\Omega \frac{1}{2}|\Delta u|^2 - f$, where we minimize the functional over H_0^1; Here we minimize over V.

bilinear form is (A) bounded (B) Coersive so we can apply Lax-Milgram

Wednesday, September 7

- Existence
- Regularity

Theorem 1 *For*

$$
\begin{aligned}
-\mu\Delta\mathbf{u} + \nabla p = \mathbf{f} &\quad \textit{in } V \subset H_0^1 \\
\textit{div }\mathbf{u} = \mathbf{f} &\quad \textit{in } V' \supset H_0^1 \\
\mathbf{u} = 0. &
\end{aligned}
$$

you can enlarge the data here

✓ *so its in a larger space*

There exists a unique $\mathbf{u} \in V$ *and unique* $p \in L^2(\Omega)/\mathbb{R}$ *solving homogeneous system with estimate*

$$\|\mathbf{u}\|_V + \|p\|_{L^2/\mathbb{R}} \leq \|\mathbf{u}\|_{H^{-1}}.$$

Proof. $a(\mathbf{u},\mathbf{v}) = \mu(\nabla\mathbf{u},\nabla\mathbf{v})_{L^2} \quad \forall \mathbf{u},\mathbf{v} \in V.$
The following hold for a:

$$
\begin{aligned}
\left|a(\mathbf{u},\mathbf{v})\right| &\lesssim |\mathbf{u}|_V\,|\mathbf{v}|_V \\
a(\mathbf{u},\mathbf{v}) &\geq \alpha\,|\mathbf{u}|_V^2
\end{aligned}
$$

boundedness

coersivity

Lax-Milgram gives that there exists a unique $\mathbf{v} \in V$ which satisfies

$$a(\mathbf{u},\mathbf{v}) = \langle \mathbf{f},\mathbf{v}\rangle \quad \forall \mathbf{v} \in V$$

It follows from Notes from last class that

$$\Rightarrow -\mu\Delta u + \nabla p = \mathbf{f} \quad \text{for some } p \in \mathcal{D}'$$

Note that

$$p \in L^2 \overset{\text{Poincaire}}{\Leftarrow} \nabla p + \mathbf{f} + \mu\Delta\mathbf{u} \in H^{-1}$$

Initially p is just a distribution. $\Rightarrow$ using info $\Rightarrow p \in L^2$.

1.14 Energy Method proof.

The energy functional E is given by $E(u) = \mu|\mathbf{u}|_L^2 - (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in V$.

Claim $\mathbf{u} \in V$ solves (*) if and only if $E(u) = \min_{\mathbf{v} \in V} E(v)$.

We know the minimization is always obtained.

Pf. $\mu(\Delta \mathbf{u}, \Delta \mathbf{v})_{L^2} = (\mathbf{f}, \mathbf{v}) \qquad \forall \mathbf{v} \in V$. Let

$$\mathbf{v} = \mathbf{u} - \mathbf{w} \quad \text{where } \mathbf{w} \in V.$$

Then

$$\mu(\nabla \mathbf{u}, \nabla \mathbf{u})_{L^2} = (\mathbf{f}, \mathbf{u}) - (\mathbf{f}, \mathbf{w}) + \mu(\nabla \mathbf{u}, \nabla \mathbf{w})_{L^2}$$

Using that

$$\mu(\nabla \mathbf{u}, \nabla \mathbf{u})_{L^2} \overset{\text{CS}}{\leq} \frac{\mu}{2}\|\nabla \mathbf{u}\|_{L^2}^2 + \frac{\mu}{2}\|\nabla \mathbf{w}\|_{L^2}^2$$

we have

$$\begin{aligned} \Rightarrow \mu(\nabla \mathbf{u}, \nabla \mathbf{u})_{L^2} &= (\mathbf{f}, \mathbf{u}) - (\mathbf{f}\mathbf{w}) + \mu(\nabla \mathbf{u}, \nabla \mathbf{w})_{L^2} \\ &\leq (\mathbf{f}, \mathbf{u}) - (\mathbf{f}, \mathbf{w}) + \frac{\mu}{2}\|\nabla \mathbf{u}\|_{L^2}^2 + \frac{\mu}{2}\|\nabla \mathbf{w}\|_{L^2}^2 \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{1}{2}E(u) &= \frac{\mu}{2}\|\nabla \mathbf{u}\|_{L^2}^2 - (\mathbf{f}\mathbf{u}) \\ &\leq \frac{\mu}{2}\|\nabla \mathbf{w}\|_{L^2}^2 - (\mathbf{f}, \mathbf{w}) = \frac{1}{2}E(v) \end{aligned}$$

$\Leftarrow$

$$0 = \frac{d}{dt}E(\mathbf{u} + tv) = 2\mu(\nabla \mathbf{u}, \nabla \mathbf{v})_{L^2} - 2(f, \mathbf{v}). \quad \square \tag{1.5}$$

In solving the Laplace equation using Energy method, $\mathbf{u} \in H_0^1$. Here $u \in V$.

1.15 Existence for Inhomogeous Stokes

$f \perp$ constant

$L^2(\Omega)/\mathbb{R}$ means $(f,1)_{L^2} \overset{\blacktriangle}{=} \int_\Omega f \cdot 1 = \int_\Omega f = 0$

Theorem 2 *For*

$$(IS) \quad \begin{cases} \mu\Delta\mathbf{u} - \nabla p &= \mathbf{f} \quad in\ \Omega \\ \operatorname{div}\mathbf{u} &= g \quad in\ \Omega \\ \mathbf{u} &= \boldsymbol{\phi} \quad on\ \partial\Omega. \end{cases}$$

and $\Omega \subset \mathbb{R}^n$ *bounded,* $\partial\Omega \in C^2$*, let* $f \in H^{-1}, g` \in L^2, \phi \in H^{1/2}(\partial\Omega)$ *and*

$$\int_\Omega g = \int \nabla \cdot \mathbf{u} \overset{\text{divergence theorem}}{=} \int_{\partial\Omega} \phi \cdot \nu$$

$!\exists\mathbf{u} \in H_1$ *and* $!\exists\ p \in L^2/\mathbb{R}$ *solving 1.7 and*

$$\|\mathbf{u}\|_{H^1} + \|p\|_{L^2}/\mathbb{R} \lesssim \left(\|\mathbf{f}\|_{H^{-1}} + \|g\|_{L^2} + \|\mathbf{f}\|_{H^{\frac{1}{2}}}\right)$$

The approach for the inhomogeneous is cook up new function that is divergence-free and has zero trace. Note that $\mathbf{f}$ changes.

Remember: A function $f \in H_0^1 \Rightarrow$ div $\mathbf{f} \in L^2$

A Property of the divergence operator

The divergence operator $\nabla\cdot : H_0^1(\Omega) \overset{\text{onto}}{\to} L^2(\Omega) \cap \left\{\int_\Omega f = 0\right\}$

Define the restriction operator $\gamma_0 : H^1(\Omega) \hookrightarrow H^{1/2}(\partial\Omega) \Rightarrow \exists u_0 \in H^1$ such that $\gamma_0(u_0) = \phi$.

first extend it

Because of the Compatibility Condition,

$$\int_\Omega (\underbrace{\operatorname{div}\ \mathbf{u}_0}_{\int_{\partial\Omega}\phi\cdot\nu} - \overbrace{g}^{\int_\Omega g}) = g$$

$$\int_{\partial\Omega} \phi \cdot \nu - \int_\Omega g \overset{\text{Compatibility Condition}}{=} 0$$

$\Rightarrow \exists u_1 \in H_0^1$ such that div $\mathbf{u}_1 = g -$ div $\mathbf{u}_0$

Somehow, $\mathbf{u}_0, \mathbf{u}_1$ kills g.

Let $\mathbf{v} = \mathbf{u} - \mathbf{u}_0 - \mathbf{u}_1$. Then

$$\begin{aligned} \operatorname{div}\ \mathbf{v} &= \operatorname{div}\ \mathbf{u} - \operatorname{div}\ \mathbf{u}_0 - \operatorname{div}\ \mathbf{u}_1 \\ &= g - \operatorname{div}\ \mathbf{u}_0 - (g - \operatorname{div}\ \mathbf{u}_0) \\ &= 0. \end{aligned}$$

Trace

$$\begin{aligned}\gamma_0(V) &= \gamma_0(\mathbf{u}) - \gamma_0(\mathbf{u}_0) - \gamma_0(\mathbf{u}_1)\\ &= \phi - \phi = 0.\end{aligned}$$

The Eq (IS) in $\mathbf{v}$ becomes

$$\begin{aligned}-\mu\Delta\mathbf{v} = \nabla p &= -\mu(\nabla\mathbf{u} - \nabla\mathbf{u}_0 - \nabla\mathbf{u}_1) + \nabla p\\ &= \mathbf{f} + \mu(\underbrace{\Delta u_0}_{\in H^1} + \underbrace{\Delta\mathbf{u}_1}_{\in H^1}) = \underbrace{\widetilde{f}}_{\in H^{-1}}\end{aligned}$$

Since $\Delta\mathbf{u}_0 \in H^1$ and $\mathbf{u}_1 \in H^1$, then $\Delta\mathbf{u} \in H^{-1}$.

Then

$$\begin{aligned}\nabla\cdot\mathbf{v} &= 0 \quad \text{in } \Omega\\ \mathbf{v} &= 0 \quad \text{on } \partial\Omega.\end{aligned}$$

Now its time to prove that

$$H_0^1(\Omega) \overset{\text{onto}}{\hookrightarrow} L^2(\Omega) \cap \left\{\int f = 0\right\} \tag{1.6}$$

Not 1-1: add divergence-free function and we still have $= f$ (no uniqueness).

The term ∇p: The gradient operator $\nabla : L^2(\Omega) \to H^{-1}$ After taking out constants, this space is 1-1.

slash out constants so we can have $\nabla f + c = \nabla f$.

Adjoint:

$$\langle\nabla^*, g\rangle = \langle f, \nabla g\rangle \tag{1.7}$$

This equality 1.7 holds because $\langle -\text{div }\mathbf{f}, \mathbf{g}\rangle \overset{\text{integrate by parts}}{=} \langle -\text{div }\mathbf{f}, g\rangle$

1.16 Regularity

If $\Omega \in \mathbb{R}^n$ is bounded and $\partial\Omega \in C^{m+2}$ and $\mathbf{u} \in \mathbf{W}^{2,\alpha}, p \in W^{1,\alpha}, 1 < \alpha < +\infty$.

For $\mathbf{u} \in \mathbf{W}^{2,\alpha}$, $\Delta\mathbf{u}$ is pointwise defined and ∇p is well defined If additionally $f \in \mathbf{W}^{m,\alpha}, g \in \mathbf{W}^{m+1}$, and $\phi(\Omega) \in W^{m+2-\frac{1}{\alpha}}(\partial\Omega)$,

strong solution instead of weak solution.

$$\mathbf{u} \in \mathbf{W}^{m+2,\alpha}(\Omega), p \in W^{m+1,\alpha}$$

and

$$\begin{aligned}&\|\mathbf{u}\|_{\mathbf{W}^{m+2,\alpha}} + \|p\|_{\mathbf{W}^{m+1,\alpha}}\\ &\lesssim \left(\|\mathbf{f}\|_{\mathbf{W}^{m,\alpha}} + \|\mathbf{g}\|_{\mathbf{W}^{m+1}} + \|\phi\|_{\mathbf{W}^{m+2-\frac{1}{\alpha}}(\partial\Omega)} + \|\mathbf{u}\|_{L^2}\right)\end{aligned}$$

Question: Why do we have $\mathbf{u} \in \mathbf{W}^{2,\alpha}$? Ans: If $\mathbf{f} \in \mathbf{W}^{\alpha,1}$ $\boxed{??}$

Theorem 3 *If* $\mathbf{f} \in L^2, g \in H^1, \phi \in H^{3/2}$, *then*

$$\mathbf{u} \in \mathbf{W}^{2,2} \quad p \in W^{1,2}.$$

Why? Suppose $g = \phi = 0$ for the moment and $\mu = 0 \Rightarrow$

$$\begin{aligned}\Delta\mathbf{u} + \nabla p &= \mathbf{f}\\ \nabla\cdot\mathbf{u} &= 0\\ \mathbf{u} &= 0\end{aligned} \tag{1.8}$$

For this eq 1.16, which has only two terms on the lhs,

$$f \in L^2 \Rightarrow \nabla\cdot\mathbf{u} \in L^2 \Rightarrow \text{ then automatically } \Rightarrow p \in W^{1,\alpha}.$$

We must show, using finite quotients, that

$$(\nabla\mathbf{u}, \nabla\mathbf{v}) \overset{(*)}{=} (\mathbf{f}, \mathbf{v})$$

use $\mathbf{u} \mapsto \left(\frac{\mathbf{u}(x+he)-\mathbf{u}(x)}{h}\right) \in H^1$ uniformly with respect to h. $\Rightarrow D^2\mathbf{u} \in L^2$.

Approach: Use the finite difference quotient as test function, but have to cut out.

Friday, September 9

- regularity for $n = 2$.

Theorem 4 (regularity for $n = 2$)
For $\boldsymbol{f} \in \mathbf{W}^{m,\alpha}, \boldsymbol{g} \in \boldsymbol{W}^{m+1,\alpha}, \boldsymbol{\phi} \in \mathbf{W}^{m+2-\frac{1}{\alpha},\alpha}(\partial\Omega), \partial\Omega \in \boldsymbol{C}^{m+2}$ *and* $\int_\Omega \boldsymbol{g} = \int_{\partial\Omega} \boldsymbol{\phi} \cdot \boldsymbol{\nu}$ *then*

Remember that (B) and (C) below are satisfied if and only if the Compatibility Condition is satisfied.

$$
\begin{aligned}
-\mu\Delta\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega \\
\operatorname{div}\mathbf{u} &= g \quad \text{in } \Omega \quad (B) \\
\mathbf{u} &= \phi \quad \text{on } \partial\Omega (C)
\end{aligned}
$$

has unique solution (modulo constant) such that $\mathbf{u} \in \mathbf{W}^{m+2,\alpha}, \boldsymbol{p} \in \mathbf{W}^{m+1,\frac{1}{\alpha}}$ *and*

$$
\begin{aligned}
&\|\boldsymbol{u}\|_{\mathbf{W}^{m+2,\alpha}} + \|\boldsymbol{p}\|_{\boldsymbol{W}^{m+1,\alpha}} \\
&\leq \left(\|\mathbf{f}\|_{\boldsymbol{W}^{m,\alpha}} + \|\boldsymbol{g}\|_{\mathbf{W}^{m+\frac{1}{\alpha}}} + \|\phi\|_{\mathbf{W}^{m+2-\frac{1}{\alpha},\alpha}} + \|\mathbf{u}\|_{L^2}\right).
\end{aligned}
$$

✓

Approach: We find auxillary equationto kill (B),(C). In particular: find gradient potential $\Rightarrow$ kill pressure $p \Rightarrow$ get formally biharmonic eq - coupled 2nd order (like 2nd order Elliptic equation) for which Sobolev $W^{k,p}$ theory holds.

Claim $\exists \mathbf{v} \in \mathbf{W}^{m+2,\alpha}$ such that

$$
\begin{aligned}
\operatorname{div}\mathbf{v} &= g \quad \text{in } \Omega \\
\mathbf{v} &= \boldsymbol{\phi} \quad \text{on } \partial\Omega
\end{aligned}
$$

(Prove later)

If Claim 1 holds, then $\exists \mathbf{w} = \mathbf{u} - \mathbf{v} \Rightarrow$ where w satisfies

$$
(*') \quad \begin{cases}
\nu\Delta\mathbf{w} + \nabla p &= \mathbf{f} + \nu\nabla\mathbf{v} = \widetilde{\mathbf{f}} \\
\operatorname{div}\mathbf{w} &= 0 \\
\mathbf{w} &= 0
\end{cases}
$$

$\widetilde{\mathrm{f}}$ has same regularity as f.

Case 1: Ω is 1-connected

$\because \mathrm{div}\,\mathbf{w} = 0 \Rightarrow$ can find can find extr in 1-form that is closed $\Rightarrow dW = 0$.

"1 homotopy cohomology is exact $\Rightarrow$" translated $\Rightarrow \exists \boldsymbol{\phi}$ such that

$$\nabla^{\perp}\boldsymbol{\phi} = \left(\frac{\partial\boldsymbol{\phi}}{\partial x_2}, -\frac{\partial\boldsymbol{\phi}}{\partial x_1}\right) \quad \Leftrightarrow \quad \text{Condition (B)}$$

We get rid of the divergence-free condition and plug in:

$$\text{Take } \phi = 1 \quad \Rightarrow \quad -\Delta D_2\boldsymbol{\phi} + D_1 p = \underbrace{\widetilde{\mathbf{f}}^1}_{\text{first component of the vector } \widetilde{\mathbf{f}}}$$

Take derivatives & subtract to eliminate $\widetilde{\mathbf{f}}$ $\Rightarrow$:

mixed derivatives cancel

$$\begin{aligned}
-D_2(\Delta D_2\boldsymbol{\phi}) + D_{12}^2 p &= D_2\widetilde{\mathbf{f}}^1 \\
D_1(\Delta D_1\boldsymbol{\phi}) + D_{21}^2 p &= D_1\widetilde{\mathbf{f}}^2 \\
\Rightarrow -\Delta(D_2^2 + D_1^2)\phi &= D_2\widetilde{\mathbf{f}}^1 - D_1\widetilde{\mathbf{f}}^2 \\
\Rightarrow -\Delta^2\boldsymbol{\phi} &= D_2\widetilde{\mathbf{f}}' - D_1\widetilde{\mathbf{f}}^2 \quad \in W^{m-1,\alpha}
\end{aligned} \tag{1.9}$$

Note that 1.9, an equation in $\boldsymbol{\phi}$ does not involve p. Further, it is a first order differential operator $\Rightarrow$ lost 1-order regularity.

$$dV = 0 \Rightarrow \begin{cases} \text{tangent derivative} \\ \text{normal derivative} \end{cases} = 0$$

$$\frac{\partial\boldsymbol{\phi}}{\partial\nu} = 0 \qquad \text{on } \partial\Omega$$

For the second order operator need a second condition; this problem as it stands is not closed $\Rightarrow$ tangent gradient $= 0$

Note that the potential $\boldsymbol{\phi}$ is constant, but because $\boldsymbol{\phi}$ is not unique, pick $\boldsymbol{\phi} = 0$ on $\partial\Omega \Rightarrow$

$$\begin{aligned}
\Delta\mathbf{u} &= \mathbf{f} \in \mathbf{W}^{m,2} \\
\mathbf{u} &= 0 \\
&\Rightarrow \mathbf{u} \in \mathbf{W}^{m+2,2}
\end{aligned}$$

Using finite quotients and integrate by parts, get 2 more derivative estimate. This is possible $\because \alpha = 2$ For $\mathbf{u} \in \mathbf{W}^{m+2,\alpha}$,

Calderon-Zygmund theory of harmonic analysis does this for higher order α.

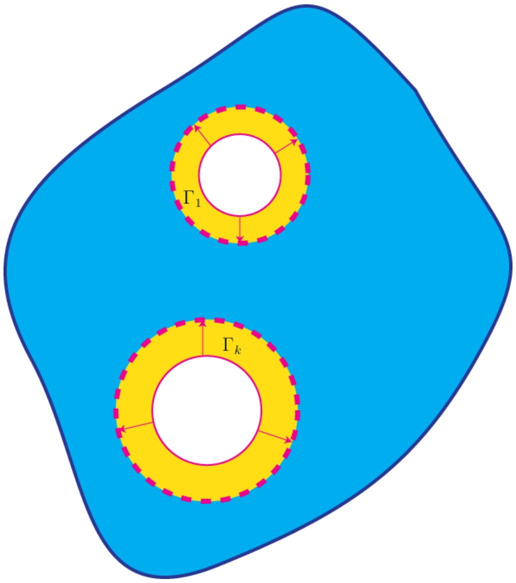


Figure 1.4: Difficulty: cannot shrink closed paths to 0. The 1-cohomology of the domain is like copies $H^1(\Omega) = \mathbb{Z}^k \Rightarrow$ have k freedoms $\Rightarrow$ cannot not specify constants c_i

$$\Delta^2\mathbf{u} = \mathbf{f} \quad \mathbf{u} \in \mathbf{W}^{m+2,\alpha}$$
$$\frac{\partial \mathbf{u}}{\partial \nu} = 0$$
$$\Rightarrow \boldsymbol{\phi}\mathbf{W}^{m-1+4,\alpha} \Rightarrow \mathbf{w} \in \mathbf{W}^{m+2,\alpha} \quad \square\,.$$

the needed second condition

From $\phi' \quad \Rightarrow \quad \underbrace{\nabla p}_{p \in W^{m+1,\alpha}} = \widetilde{\mathbf{f}} + \Delta \in W^{m,\alpha}$.

Case 2: Ω is k-connected

- Suppose 2-connected for simplicity.
- We cut k (here: 2) times to make 1-connected

Ω is $(k+1)$-connected, $k \geq 1$. Let $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_k$. $\exists \phi : \Omega \to \mathbb{R}$ such that

$$\mathbf{w} = (D_2\phi, -D_1\phi) = \begin{cases} \phi = 0 \text{ on } \phi_0, \phi = c \text{ on } \Gamma_i, & 1 \leq i \leq k \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

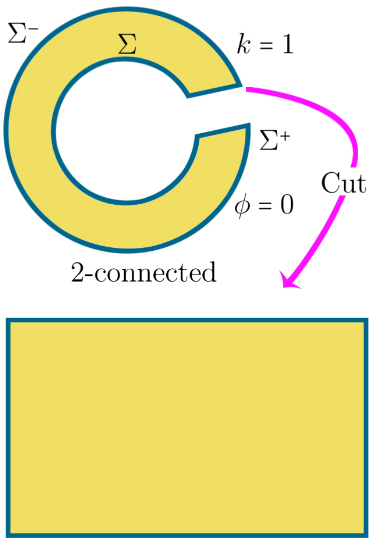


Figure 1.5: ϕ is defined except on cut. For ϕ to be well-defined there too $\Rightarrow$ extend ϕ $\Rightarrow$ make well-defined trace

For the **Multi-Connected case**, equation same as in single connected case. The biharmonic equation is uniquely solable; but the dirichlet + neumann boundary conditions are difficult $\boxed{?}$ Rest of argument is exactly same.

- For $x \in \Sigma \quad \phi^+ = \phi^+(x_0) + \int_\Sigma d\omega$.
- $\because$ lhs,rhs have same limit $\Rightarrow$ we can smoothly extend

Proofs of Case 1

Omitted. (see p7,8 handcopy)

Wednesday, September 14

- For the Navier-Stokes Equations we introduction the nonlinear advection term $\mathbf{u}\cdot\nabla\mathbf{u}$, i.e. the advection operator is
$$\mathbf{u}\cdot\nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$
- Picture advection as transport of salt dumped in a river. If the river is originally fresh water and is flowing quickly, the predominant form of transport of the salt in the water will be advective, as the water flow itself would transport the salt. If the river were not flowing, the salt would simply disperse outwards from its source in a diffusive manner, which is not advection.
- Very few exact solutions of NSE are known in closed form. In general, exact solutions are possible only when the nonlinear terms vanish identically.

1.17 Navier-Stokes Existence

- NS Existence
- Triple product $B[\cdot,\cdot,\cdot]$
- Brouwer fixpoint theorem

$$(NS)\quad \begin{cases} \mu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \nabla p &= \mathbf{f} \quad \text{in } \Omega \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in } \Omega \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Variation form: $((\mathbf{u},\mathbf{v})) + B[\mathbf{u},\mathbf{u},\mathbf{v}] = (f,\mathbf{v}) \quad \forall \mathbf{v}\in\mathcal{V}$

$((\mathbf{u},\mathbf{v}))$ like L^2 inner product but with gradients

$B[\mathbf{u},\mathbf{v}\mathbf{w}] = \int_\Omega \underbrace{\mathbf{u}}_{\in L^{\frac{2n}{n-2}}} \cdot \overbrace{\nabla\mathbf{v}}^{L^2} \cdot \underbrace{\mathbf{w}}_{L^n} \quad \mathbf{u},\mathbf{v}\in V, \mathbf{w} \underbrace{\in\mathcal{V}}_{\therefore\text{bounded}}$ We to to extend to the bigger space
$$\widetilde{V} = \overline{\mathcal{V}}.$$

Note that $\overline{\mathcal{V}} \notin H_0^1 \Rightarrow$ take closure instead of the space

$$H_0^1 \cap L^n$$

which has norm

$$\|\mathbf{u}\|_{H_0^1 \cap L^n} = \|\mathbf{u}\|_{H_0^1} + \|\mathbf{u}\|_{L^n}$$

for $n = 2$, $H_0^1 \cap L^n = V$.

Lemma For $n \geq 2, B : V \times V \times \widetilde{V} \to \mathbb{R}$ is bounded:

$$\left|B[\mathbf{u}, \mathbf{vw}]\right| \underbrace{\leq}_{\text{hö}} \|\mathbf{u}\|_{L^{2^*}} \|\nabla \mathbf{v}\|_{L^2} \|\mathbf{w}\|_{\boxed{\widetilde{V}}}$$

replaces $\square$ with L^n

$$\overbrace{\leq}^{\text{Sobolev embedding}} \quad c(n)\|\mathbf{u}\|_{H_0^1}\|\mathbf{v}\|_{H_0^1}\|\mathbf{u}\|_{\widetilde{V}}$$

Remember: For $2 \leq n \leq 4$ check that

$$\widetilde{V} = V$$

but in general $H_0^1 \cap L^n \subset V \quad \because \frac{2n}{n-2} \geq n \Leftrightarrow 2 \geq n-2 \Leftrightarrow n \leq 4$

In particular, $B : V \times V \times V \to R$ is well defined for $2 \leq n \leq 4$; for $m \geq 5, \widetilde{V} \not\subset V \quad \because B$ is not well-defined.

We need to understand trilinear form better; It can induce bilinear form $B[\mathbf{u}, \mathbf{v}] : V \times V \to \boxed{\mathbb{R}}$

replace $\square$ with $\widetilde{V}'$

Lemma

1. $B[u, \mathbf{v}, \mathbf{v}] = 0 \quad \forall \mathbf{v} \in \widetilde{V}$ and $\mathbf{w} \in \widetilde{V}$.
2. $B[\mathbf{u}, \mathbf{v}, \mathbf{w}] = -B[\mathbf{u}, \mathbf{w}, \mathbf{v}] \quad \forall \mathbf{u} \in V, \mathbf{v}, \mathbf{w} \in \widetilde{V}$

skew symmetry in last two vars.

Proof (1)

$$B[\mathbf{u}, \mathbf{v}, \mathbf{v}] \overset{\text{df}}{=} \int_\Omega \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{v} = \int_\Omega u^i \partial_i v^j v^j$$

$$= \int_\Omega u^i \partial_i \left(\frac{|v|^2}{2}\right) \overset{\text{ibp}}{=} -\int_\Omega \partial_i u^j \left(\frac{|v|^2}{2}\right) = 0 \quad \square$$

(2) $B[\mathbf{u}, \mathbf{v}+\mathbf{w}] = 0$ by (1).

$$\text{Lhs} = \overbrace{B[\mathbf{u}, \mathbf{v}, \mathbf{v}]}^{0} + B[\mathbf{u}, \mathbf{v}, \mathbf{w}] = B[\mathbf{u}, \mathbf{wv}] + \underbrace{B[\mathbf{u}, w, w]}_{0}$$

$$= 0.$$

1.18 Existence - Galerkin

Goal: Solve $u \in V$ for

$$\nu((\mathbf{u}, \mathbf{v})) + B[\mathbf{u}, \mathbf{u}, \mathbf{v}] = (\mathbf{fv}) \quad \forall \mathbf{v} \in \widetilde{V}.$$

Let $\{w_k\}_{k=1}^{\infty}$ be an orthonormal basis (ONB) of $\boxed{\mathbf{L}^2(\Omega)}$ replace $\square$ with V.

The space L^2 is infinite-dimensional vector space ⇒ cut off ⇒ finite like $\mathbb{R}^n$ ⇒ have family of solutions ⇒ if we can prove apriori estimate, we can take limit ⇒ function can solve the limiting equation ⇒ done.

Let $X^m = \text{span } \{w_1, \ldots, w_m\}$ $\quad m \geq 1$ and let $\mathbf{u}_m = \sum_{i=1}^{m} a_i^m w_i$ coeff tb determined

$$\nu((\mathbf{u}_m, w_j)) + B[\mathbf{u}_m, \mathbf{u}_m, w_j] = (\mathbf{f}, \mathbf{w}_j) \quad 1 \leq j \leq m$$

choose w_j to be basis

N O T E: ONB $\in L^2$, but $\notin H_0^1$

$$\nu a_j^m + \boxed{A_{il}^j a_i^m a_l^{m'}} = (\mathbf{f}, w_j)$$

We ask if we can solve the term $\boxed{A_{il}^j a_i^m a_l^{m'}}$, being nonlinear. Formally, this term is *not* like solving $A\mathbf{x} = B$, but rather like solving $A\mathbf{x} + B\mathbf{x}\mathbf{x}^T = C$; does solution exist ⇒ maybe ⇒ ∵ equal number of equations ⇒ solution maybe not unique though ∵ nonlinear terms. Therefore we need to turn to the following

Theorem 5 (Brouwer)
If $\boldsymbol{f} : \mathbb{B}_{\boldsymbol{R}} \to \mathbb{B}_{\boldsymbol{R}}$ is continuous ⇒ $\exists \boldsymbol{x_0} \in \mathbb{B}_{\mathbb{R}}$ such that

$$\boldsymbol{f}(\boldsymbol{x_0}) = \boldsymbol{x_0},$$

where $\mathbb{B}_{\boldsymbol{R}}$ are closed balls. ✓

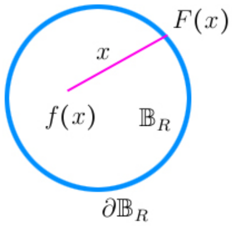


Figure 1.6: The closed ball $\mathbb{B}_R$

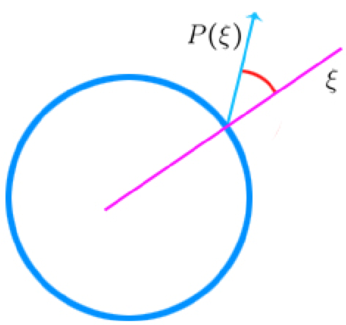


Figure 1.7: ξ and $P(\xi)$

Proof (contradiction)

If $f(x) \notin x \quad x \in \mathbb{B}_n \Rightarrow F(x) = f(x)x \cap \partial\mathbb{B}_R$;

$F : \mathbb{B}_R \to \partial\mathbb{B}_R$. $F(x) = x$ if $x \in \partial B \Rightarrow F$ is a continuous map and w_i is fixed on the boundary $\Rightarrow F\,|_{\partial\mathbb{B}_R} = \mathrm{Id} \Rightarrow$ impossible by degree theorems $\Leftarrow$ $x = 0$ on in $\mathbb{B}_R$, but boundary is of degree 1.

infinite-d versions of Browder thm very useful for existence of nonlinear pde.

Lemma (Variation of the Brouwer Fixpoint Theorem) Let X be a finite-dimensional Hilbert space equipt with inner product $[\cdot,\cdot]$ and norm $[\cdot]$; $P : X \to X$ is continuous (but not necessarily linear). Suppose $[P(\xi)] \quad \forall\, |\xi| = k > 0 \quad \Rightarrow \exists \xi \in X$ with $[\xi]\xi k$ such that $P(\xi) = 0$.

Proof If $p(\xi) \neq 0 \quad \forall \xi \in \mathbb{B}_R = \left\{\xi \in X : [\xi] \leq k\right\}$, then define

$$Q(\xi) = -k\frac{p(\xi)}{[p(\xi)]} \quad \xi \in \mathbb{B}_R$$

and

$$[Q(\xi)] = k.$$

Now $Q : \mathbb{B}_R \to \partial\mathbb{B}_R$ continuously and $\Rightarrow$ Brouwer $\Rightarrow \exists \xi_0 \in \mathbb{B}_R$ such that $Q(\xi_0) = \xi_0$ where automatically $Q(x_0)$ is on the boundary and $[Q(\xi_0), \xi_0] = k$.

$$= [-k\frac{p(\xi_0)}{[p(\xi_0),\xi_0]}] = k\frac{[p(\xi_0),\xi_0]}{[p(\xi_0)]}$$

which is impossible "because ≤ 90degree"

Define $P_m(\mathbf{u}) : X^m \to X^m$ by

$$[P_m(\mathbf{u}),\mathbf{v}] = \nu((\mathbf{u},\mathbf{v})) + B[\mathbf{u},\mathbf{u},\mathbf{v}] - (\mathbf{f},\mathbf{v}).$$

Need to verify: $B[(\xi),\xi] > 0.$

$$\nu((\mathbf{u},\mathbf{u})) - (\mathbf{f},\mathbf{u}) = \nu\|\nu\|\ \mathbf{u}_V^2 - \|\mathbf{f}\|_V\|\mathbf{u}\|_V = \|\mathbf{u}\|_{\mathbf{v}}\left(\nu\|\mathbf{u}\| - \|\mathbf{f}\|_{V'}\right) > 0$$

if $\|\mathbf{u}\|_V = k$ where $m > \frac{\|\mathbf{f}\|_{V'}}{V} \Rightarrow \exists\mathbf{u}_m \in X^m$ with $\|u_m\|_V < k$ such that $p_m(\mathbf{u}_m) =\Leftrightarrow$ equivalent to $\nu a_j^m + A_{il}^j a_i^m a_l^m = (\mathbf{f},w_j)$. In fact $\|u_m\|_V \le \frac{\|f\|_{V'}}{V}$.

$$\text{Since}[p_m(\mathbf{u}_m),\mathbf{u}_m]\nu\|\mathbf{u}_m\|_V^2 - (\mathbf{f}\mathbf{u}_m) \Rightarrow \nu\|\mathbf{u}_m\|_V^2 = (\mathbf{f}\mathbf{u}_m) \le \|f\|_{V'}\|\mathbf{u}_m\|_V$$

$$\Rightarrow \boxed{\nu\|\mathbf{u}_m\|_V \le \|f\|_{V'}}$$

which gives the sought after apriori estimate with uniform bound (so that we can now take limits).

Challange

Write down Lax-Milgram proof using a Galerkin method approach

Friday, September 16

- Variational Form for Inhomogeneous Navier-Stokes
- Uniqueness for Inhomogeneous Navier-Stokes

1.19 Variational Form for Inhomogeneous Navier-Stokes

We show that the variational form for inhomogeous Navier-Stokes is well posed.

Define the eigenfunction $\{X^m\}$ of Stokes operator having zero Dirichlet boundary conditions $X^m = \text{span}\,\{w_i\}_{i=1}^m$.

Find $u \in V$ such that

$$\nu((\mathbf{u},\mathbf{v})) + \underbrace{B[\mathbf{u},\mathbf{u},\mathbf{v}]}_{\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{v} \Leftarrow \text{well defined for } \mathbf{v}\in\widetilde{V}} = (\mathbf{f},\mathbf{v}) \quad \forall \mathbf{v}\in\widetilde{V}, \mathbf{f}\in V'$$

(From last time) For $u^m \in X^m$

$$\nu((u^m,\mathbf{v})) + \underbrace{B[\mathbf{u}^m,\mathbf{u}^m,\mathbf{v}]}_{\text{skew-symm in last two vars}} = (\mathbf{f},\mathbf{v}) \quad \forall \mathbf{v}\in X^m \tag{1.10}$$

where $X^m \subseteq \widetilde{V}$

- Use Brouwer fpt in finite space
- Write $\mathbf{v} = \mathbf{u}^m$. Then 1.10 gives

$$\nu((\mathbf{u}^m,\mathbf{u}^m)) = (\mathbf{f},\mathbf{u}^m) \quad \Rightarrow$$
$$\boxed{\|\mathbf{u}^m\|_V \lesssim \tfrac{1}{\nu}\|f\|_{V'}}$$

This is called an "apriori" estimate ∵ lhs holds for all sequences $\mathbf{u}^m$, & rhs is independent of the number m.

- Send $m \to \infty$.
- By weak compactness properties of V, bounded sequence in $V \Rightarrow$ take subsequence so sequence converges weakly:

$u^m \rightharpoonup \nabla\mathbf{u}$ in L^2 $\Rightarrow$ Rellich compactness $\Rightarrow \mathbf{u}^m \to \mathbf{u}$ in L^2 $\Rightarrow \mathbf{u}^m \rightharpoonup \mathbf{u}$ in V.

- $\forall \mathbf{v} \in X^m$,

$$\begin{aligned} \nu((\mathbf{u}^m, \mathbf{v})) &\to \nu((\mathbf{u}, \mathbf{v})), \\ (\mathbf{f}, \mathbf{v}) &\to (\mathbf{f}, \mathbf{v}) \quad \text{vacuously} \\ B[\mathbf{u}^m, \mathbf{u}^m, \mathbf{v}] &\to B[\mathbf{u}, \mathbf{u}, \mathbf{v}] \end{aligned} \tag{1.11}$$

The last of these 1.11 is not obvious. Write

$$B[\mathbf{u}^m, \mathbf{u}^m, \mathbf{v}] = \int_\Omega \mathbf{u}^m \cdot \nabla \mathbf{u}^m \cdot \mathbf{v} \overset{\text{skew-sym}}{=} -B[\mathbf{u}^m, \mathbf{v}, \mathbf{u}^m] = -\int_\Omega \underbrace{\mathbf{u}^m \overset{\text{tensor}}{\otimes} \mathbf{u}^m}_{\in L^1} \cdot \underbrace{\nabla \mathbf{v}}_{L^\infty}$$

$$\Rightarrow \text{so that } \Rightarrow = -\int_\Omega \mathbf{u} \otimes \mathbf{u} \cdot \nabla \mathbf{v} = -B[\mathbf{u}, \mathbf{v}, \mathbf{u}] = B[\mathbf{u}, \mathbf{v}, \mathbf{u}] = B[\mathbf{u}, \mathbf{u}, \mathbf{v}]$$

$$\Rightarrow \nu((\mathbf{u}, \mathbf{u})) + B[\mathbf{u}, \mathbf{u}, \mathbf{v}] = (\mathbf{f}, \mathbf{v}) \quad \boxed{\mathbf{v} \in X^m}$$

not for V' yet

Note: we don't want solution & test function to vary at the same time: fix solution and send test function $\to \infty$; fix test function & send solution $\to \infty$.

we have, then, that

For all $v \in \widetilde{V}$ there exists $\mathbf{v}^m \in X^m$ so that

$$\mathbf{v}^m \to \mathbf{v} \in \widetilde{V}.$$

and so

$$\boxed{\nu((\mathbf{u}, \mathbf{v})) + B[\mathbf{u}, \mathbf{u}, \mathbf{v}] = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \widetilde{V}, \mathbf{f} \in \widetilde{V}}.$$

1.20 Uniqueness for Inhomogeneous Navier-Stokes

Theorem 6 *For $n \le 4$ and $\nu \gg 1$ in the sense that $\nu^2 \ge c(n) \cdot \|\mathbf{f}\|_{V'}$ $\quad !\exists \mathbf{u} \in V$ for*

$$\nu \Delta U + \nabla p = \mathbf{f} - \mathbf{u} \cdot \nabla \mathbf{u}.$$

Brouwer FPT is great for non-linear ∵ you can have many fix-points; LM is only for 1 fixpoint.

✓ for small Reynolds number

Proof $n \le 4 \quad \Rightarrow \widetilde{V} = V$.

$$\underbrace{\nu(\nabla \mathbf{u}, \nabla \mathbf{v})_{L^2}}_{\because \nu\|\nabla \mathbf{u}\|_{L^2} \le \|f\|_{V'}} + \overbrace{B[\mathbf{u}, \mathbf{u}, \mathbf{v}] = (\mathbf{f}, \mathbf{v})}^{0 \because \text{skew-symm}}$$

the solution is unique "in a ball" $\boxed{?}$

Suppose $\mathbf{u}_1, \mathbf{u}_2$ unique solutions. Let $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$. Then

$\nu(\nabla \mathbf{u}_i, \nabla \mathbf{v}_i) + B[\mathbf{u}_i, \mathbf{u}_i, \mathbf{v}] = (\mathbf{f}, \mathbf{v}) \Rightarrow$ replace $\mathbf{v}$ with $\mathbf{w} \Rightarrow \nu(\nabla \mathbf{w}, \mathbf{v}) + B[u_1, \mathbf{u}_1, \mathbf{v}]$

Note that $\int(\mathbf{u}_1 \nabla \mathbf{u}_1 - \mathbf{u}_2 \nabla \mathbf{u}_2) = \int\big((\mathbf{u}_1 - \mathbf{u}_2)\nabla \mathbf{u}_2 + \mathbf{u}_2 \nabla(\mathbf{u}_1 - \mathbf{u}_2)\big) \cdot \mathbf{v} = \int \mathbf{w} \nabla \mathbf{u}_1 \mathbf{v} - \mathbf{u}^2 \nabla \mathbf{w} \mathbf{v}$

$\Rightarrow \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + B[\mathbf{w}, \mathbf{u}, \mathbf{v}] + B[\mathbf{u}_2, \mathbf{w}, \mathbf{v}] = 0.$

Test with $\mathbf{v} = \mathbf{w} \Rightarrow \nu\,|\nabla \mathbf{w}|_2^2 + B[\mathbf{w}, \mathbf{u}, \mathbf{w}] + \underbrace{B[\mathbf{u}_2, \mathbf{w}, \mathbf{w}]}_{0} = 0$

$$\Rightarrow \nu\|\nabla \mathbf{w}\|_2^2 \le \int |\mathbf{w}|\,|\nabla \mathbf{u}|\,|\mathbf{w}| \underbrace{=}_{\text{switch}} \int \underbrace{|\mathbf{v}|}_{\in L^4}\,\underbrace{|\mathbf{u}|}_{\in L^4}\,\underbrace{|\nabla \mathbf{w}|}_{\in L^2} \le \|\mathbf{w}\|_5 \|\mathbf{u}\|_4 \|\nabla w\|_2 \underbrace{\le}_{\text{Sobolev}} \|\mathbf{u}\|_4 \cdot c(4) \cdot \|\nabla \mathbf{w}\|_2 .$$

Use $\|\mathbf{u}\|_4 \le \dfrac{c(n) \cdot \|\mathbf{f}\|_V'}{\nu} \quad \Rightarrow \underbrace{\left(\nu - \dfrac{c(4)\|\mathbf{f}\|_{V'}}{\nu}\right)}_{<0} \|\nabla \mathbf{w}\|_2^2 \le 0.$

1.21 Inhomogeneous NS Uniqueness

For $\Omega \subset \mathbb{R}^n$ bounded, $\partial\Omega \in C^2$, $\quad f \in H^{-1}$, find $u \in H^1$ such that

<u>not</u> $\mathbf{u} \in V \Leftarrow$ not 0 on Γ.

$$(*) \quad \begin{cases} -\nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p & = \mathbf{f} \quad \text{in } \Omega \\ \nabla \cdot \mathbf{u} & = 0 \quad \text{in } \Omega \\ \mathbf{u} & = \phi \quad \text{on } \Gamma. \end{cases}$$

we can find $\mathbf{u}$. To help proving uniquess, introduce $\xi \in H^2, D\xi \in L^n, \xi \in L^\infty$ such that

$$\mathbf{u} = \text{curl } \xi.$$

Curl

- For $n = 2$, $\xi = (\xi^1, \xi^2)$, curl $\xi = D_2\xi^1 - D_1\xi^2$.
- For $n = 3$, $\xi = (\xi^1, \xi^2, \xi^3)$,

$$\text{curl }\xi = (\nabla \times \xi) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \xi^1 & \xi^2 & \boxed{\xi^3} \end{pmatrix}$$

For $n = 2$ the term $\boxed{}$ vanishes
$\Rightarrow$ not a vector

- $n \geq 4$, $\xi = (\xi^1, \ldots, \xi^n)$,

$$\text{curl }\xi = (R_1\xi, \ldots, R_n\xi)$$

where

$$R_i\xi = \sum_{j,k=1}^{n} a_{ijk} D_j \xi^k$$

and

$$\sum_{j,k} a_{ij}k = 0 \quad \forall i.$$

We want $\phi \in \text{curl }\xi \in H^1 \cap L^n$.

Theorem 7 *$\exists$ at least one $u \in H^1$ and $p \in \mathcal{D}'(\Omega)$ solving 1.21, $\psi \in H^1 \cap L^n$ such that div $\psi = 0$ and $\psi = \phi$ on Γ* ✓

Proof. Let $\hat{\mathbf{u}} = \mathbf{u} - \psi$ for $\begin{cases} \nabla \cdot \hat{\mathbf{u}} = 0 \\ \hat{\mathbf{u}} = 0 \quad \text{on } \partial\Gamma \end{cases}$ In terms of $\hat{\mathbf{u}}$:

$$-\nu\Delta(\hat{\mathbf{u}} + \psi) + (\hat{\mathbf{u}} + \psi) \cdot \nabla \cdot (\hat{\mathbf{u}} + \psi) + \nabla p = \mathbf{f}$$

$$\Rightarrow \underbrace{-\nu\Delta\hat{\mathbf{u}} + \hat{\mathbf{u}} \cdot \nabla\hat{\mathbf{u}}}_{\text{difficult quadratic terms}} + \overbrace{\psi \cdot \nabla\hat{\mathbf{u}} + \hat{\mathbf{u}} \cdot \nabla\psi + \nabla p}^{\text{linear first order}} = \underbrace{\mathbf{f} + \nu\Delta\psi + \psi \cdot \nabla\psi}_{\widetilde{\mathbf{f}}}$$

solve this $\Rightarrow$ gives variational form

For all $v \in \widetilde{V}$,

$$\nu(\nabla\hat{\mathbf{u}}\nabla\mathbf{v}) + B[\hat{\mathbf{u}}, \hat{\mathbf{u}}, \mathbf{v}] + B[\mathbf{v}, \hat{\mathbf{u}}, \mathbf{v}] + B[\hat{\mathbf{u}}, \mathbf{w}, \mathbf{v}] = (\widetilde{\mathbf{f}}, \mathbf{v}).$$

$B[\cdot,\cdot,\cdot]$ are difficult but can control

Verify that $\widetilde{\mathbf{f}} \in H^{-1}$. Suppose $\phi \in H_0^1$. Then

$$\langle \psi \cdot \nabla\psi, \phi \rangle \overset{\text{well-defined}\checkmark}{=} \left| \int_\Omega \psi \nabla \psi \phi \right| \le \int_\Omega \underbrace{\psi}_{\in L^n} \overbrace{|\nabla\psi|}^{\in L^2} \cdot \underbrace{|\phi|}_{\in H_0^1} \le \|\psi\|_{L^n} \cdot \|\nabla\psi\|_{L^2} \cdot \|\phi\|_{H_0^1}$$

$\Rightarrow \psi \cdot \nabla\psi \in H^{-1}$.

Monday, September 19

- coersivity

For

$$\begin{aligned} \nu\Delta\mathbf{u} - \nabla p + \mathbf{u}\cdot\nabla\mathbf{u} &= \mathbf{f} \\ \mathrm{div}\mathbf{u} &= 0 \\ \mathbf{u} &= \phi \end{aligned}$$

Suppose $\psi \in H^2 \cap L^n$ satisfies $\mathrm{div}\psi = 0$, $\psi = 0$ on $\partial\Omega$, $\hat{\mathbf{u}} = \mathbf{u} - \psi + \nabla\hat{\mathbf{u}} + \nabla p = \mathbf{f}$ where $\widetilde{\mathbf{f}} = \mathbf{f} + \nu\Delta\psi - \psi\cdot\nabla\psi \in H^{-1}$.

The corresponding weak form is

$$\nu((\hat{\mathbf{u}},\mathbf{v})) + B[\hat{\mathbf{u}},\hat{\mathbf{u}},\mathbf{v}] + B[\hat{\mathbf{u}},\psi,\mathbf{v}] + B[\psi,\hat{\mathbf{u}},\mathbf{v}] = (\hat{\mathbf{f}},\mathbf{v}) \quad \forall \mathbf{v} \in V$$

We need the coersivity estimate for LM:

For $\alpha > 0$

$$\begin{aligned} &\nu((\hat{\mathbf{u}},\hat{\mathbf{u}})) + B[\hat{\mathbf{u}},\hat{\mathbf{u}},\hat{\mathbf{u}}] + B[\hat{\mathbf{u}},\psi,\hat{\mathbf{u}}] + B[\psi,\hat{\mathbf{u}},\hat{\mathbf{u}}] \geq \alpha\|\hat{\mathbf{u}}\|^2 \\ &\Leftrightarrow \nu((\mathbf{u},\hat{}\mathbf{u})) + B[\hat{\mathbf{u}},\psi,\hat{\mathbf{u}}] \geq \alpha\|\hat{\mathbf{u}}\|^2. \end{aligned}$$

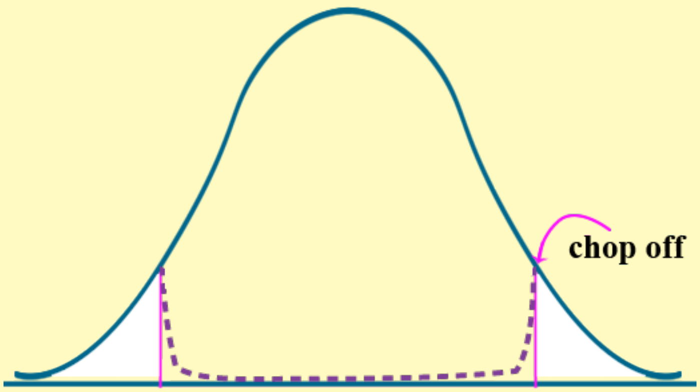
It suffices to show that

$$\left|B[\hat{\mathbf{u}},\psi,\hat{\mathbf{u}}]\right| \leq \frac{\nu}{2}\|\hat{\mathbf{u}}\|^2.$$

$$B[\hat{\mathbf{u}},\psi,\hat{\mathbf{u}}] = -B[\hat{\mathbf{u}},\hat{\mathbf{u}},\psi] = -\int_\Omega \hat{\mathbf{u}}\cdot D\hat{\mathbf{u}}\psi \leq \|\hat{\mathbf{u}}\|_{L^{\frac{2n}{n-2}}}\|D\hat{\mathbf{u}}\|_{L^2}\cdot\|\psi\|_{L^2} \overset{\text{Sobolev embedding}}{\leq} c(n)\|\hat{\mathbf{u}}\|^2\|\psi\|_{L^n}$$

1.22 Lemmas

How and why we cut off



Why: Chop off to make norm $\|\psi\|_{L^n}$ small in the above inequality

$$B[\hat{\mathbf{u}}, \psi, \hat{\mathbf{u}}] \le \cdots \le c(n)\|\hat{\mathbf{u}}\|^2 \cdot \boxed{\|\psi\|_{L^n}}$$

How: We introduce the function $\phi = \text{ curl } \xi \quad \because \quad$ cannot chop off indirectly: we can cut potential, but cant cut curl:

$$\phi = \underbrace{\text{curl } \overbrace{\xi}^{\text{not here}}}_{\text{cut here}}.$$

If we cut ξ, the divergence-free condition breaks down!

Additionally $\xi \in H^2 \cap W^{1,n} \cap \underbrace{L^\infty}_{bounded}$ and $\phi = \text{curl } \xi$; $\xi \in H^2 \Rightarrow \partial\phi \in H^{1/2}$.

$$\left\|\text{curl } (\theta\xi)\right\|_{L^n} + \|\theta\nabla\xi\|_{L^n},$$
$$\left\|\frac{c}{\delta}\right\|_{L^n(2\Omega\smallsetminus\Omega_\delta)} + \|\nabla\xi\|_{L^n(\Omega_{2\delta})}$$

Lemma 1 *Let $\rho(x) = dist\ (x, \Gamma)$ $\quad \forall\varepsilon > 0$ there exists $\theta_\varepsilon \in C^2(\bar{\Omega})$ such that*

$$\begin{cases} \theta_\varepsilon = 1 & \text{if } \rho(x) \le \delta_\varepsilon^2 \\ \theta_\varepsilon = 0 & \text{if } \rho(x) \ge \delta_\varepsilon, \quad \delta_\varepsilon = \exp\left(\frac{-1}{\varepsilon}\right) \\ |\nabla\theta_\varepsilon| \le \frac{\varepsilon}{\rho(x)} & \text{if } \rho(x) \le \delta_\varepsilon \end{cases}$$

✓

Proof Define $\Gamma_\delta = \left\{\in \Omega : \rho(x) \le \delta\right\}$. If $\Gamma \in C^2 \quad \Rightarrow \quad \exists \delta_0 = \delta_0(\Gamma) > 0$ such that $\rho \in C^2(\Gamma_\delta)$

$\left\{\begin{array}{l}\text{from} \\ \text{differential geometry}\end{array}\right.$

Let $\eta = \begin{cases} t & t \le \delta_\varepsilon^2 \\ \varepsilon \log\left(\frac{\delta_\varepsilon}{t}\right) & \delta_\varepsilon^2 \le t \le \delta_2 \\ 0 & t \ge \delta_\varepsilon \end{cases}$ Then $\theta_?(x) = \eta(\rho(x))$; $\theta_\varepsilon = \eta_\varepsilon(\rho(x))\nabla\rho(x)$ by chain rule.

Why is $\|\nabla\rho\| \le 1$? Since

$$\begin{cases} \rho(x_1) = \rho(x_2) \le |x_1 - x_2| \\ \mathrm{Lip}(\rho) \le 1 \\ \|\nabla\rho\| \le \mathrm{Lip}\ (\rho) \le 1 \end{cases}$$

we have

$$\begin{aligned} |\nabla\theta_\varepsilon| \le \left|\eta_\varepsilon(\rho(x))\right|\left|\rho(x)\right| &\le \left|\eta_\varepsilon'(\rho(x))\right| \\ &\le \frac{\varepsilon}{\rho(x)}. \end{aligned}$$

So (?)

$$\left\|\frac{\varepsilon}{\rho}\mathbf{f}\xi\right\|_{L^n} + \|\theta_\varepsilon\nabla\xi\|_{L^n} \le \left\|\mathrm{curl}\ (\theta_2\xi)\right\|_{L^n}.$$

We need to estimate $\varepsilon\left\|\frac{\xi}{\rho}\right\|_{L^n}$. Be careful: ρ is small near by $\Rightarrow$ use Poincaire inequality near the boundary ("Poincaire inequality with weight") $\Rightarrow$

Lemma 2 (Hardy inequality)

$$\boldsymbol{\int_\Omega \frac{\mathbf{v}^2}{\rho^2} \le c_1 \int_\Omega |\nabla\mathbf{v}|^2 \quad \forall \mathbf{v} \in H_0^1} \qquad \checkmark$$

This is like a Poincaire inequality with weight. Cf:

$$\int_B \mathbf{v}^2 \le c_1 \int_{B_r} (\nabla\mathbf{v}^2)$$

Proof. Near boundary, of $\frac{0}{0}$ type. It suffices to show $\exists\delta > 0$ such that

$$\int_{\Omega\cap\{\rho\le\delta\}} \frac{\mathbf{v}^2}{\rho^2} \le c\int_\Omega |\nabla\mathbf{v}|^2$$

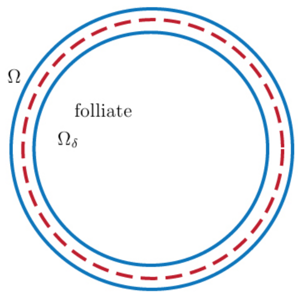


Figure 1.8: Folliate.

We want to estimate

$$\int_{\Omega\cap\{\rho\leq\delta\}} \frac{\mathbf{v}^2}{\rho}.$$

To do this we use Federer's co-area formula for the annular region

For

$$\int_{\Omega\cap\{\rho>\delta\}} \leq \frac{1}{\delta^2}\int_\Omega \mathbf{v}^2$$

use ordinary poinaire away from boundary

Federers Co-area formula (Class: 767)

$$\int_{\mathbb{R}^n} \mathbf{f}\,|\nabla g| = \int_0^\infty \left(\int_{\{g=t\}} \mathbf{f}\, dH^{n-1}\right) dt$$

where $f \in L^1, g \in \mathrm{Lip}$.

- Like a curvilinear Fubini
- For level surfaces
- For good Lipschitz surfaces, this is just Fubini; for curvilinear its Federer.

Then

$$\int_{\Omega\cap\{\rho\le\delta\}} \frac{\mathbf{v}^2}{\rho} = \int_0^\delta \frac{1}{t^2}\left(\int_{\{\rho=t\}} \mathbf{v}^2\, d\sigma\right) dt = \int_0^\delta \left(\int_{\{\rho=t\}} \mathbf{v}^2\, d\sigma\right) d\left(-\frac{1}{t}\right)$$

$$\overset{\text{ibp}}{=} -\frac{1}{t}\int_{\rho=t} \mathbf{v}^2 \,|_0^\delta + \int_0^\delta \frac{1}{t}\frac{d}{dt}\left(\int_{\rho=t} \mathbf{v}^2\right) dt = -\frac{1}{\delta^2}\int_{\rho=\delta} \mathbf{v}^2 + \int_0^\delta \frac{1}{t}\left(\int_{\rho=t} |\mathbf{v}|\,|\nabla\mathbf{v}|\, d\sigma\right) dt$$

$\rho = t$ is parallel surface of boundary

$$\le 2\int_0^\delta \frac{1}{t}\left(\int_{\rho=t} |\mathbf{v}|^2\right)\left(|\nabla\mathbf{v}|^2\right) dt \qquad \text{throw neg terms away \& hö}$$

$$\overset{\text{hö - 2}}{=} 2\left(\int_0^\delta \frac{1}{t^2}\int_{\rho=t} \mathbf{v}^2\, d\sigma dt\right)^{\frac{1}{2}} \left(\int_0^\delta \int_{\rho=t} |\nabla\mathbf{v}|^2\, d\sigma dt\right)^{\frac{1}{2}}$$

$$\le \int_{\Omega_\delta} \frac{\mathbf{v}^2}{\rho^2} \le 2\left(\int_{\Omega_\delta} \frac{\mathbf{v}^2}{\rho^2}\right)^{\frac{1}{2}} \left(\int_{\Omega_\delta} |\nabla\mathbf{v}|^2\right)^{\frac{1}{2}},$$

$\Omega_\delta = \Omega \cap \{\rho \le \delta\}$.

Wednesday, September 21

- regularity estimate

(Last time) $\int_\Omega \frac{\mathbf{v}^2}{\rho^2} \le c \int_\Omega |\nabla \mathbf{v}|^2 \quad \forall \mathbf{v} \in H_0^1$.

- If not comfortable with coarea formula do: $\Omega = B_1, \quad \rho(x) = 1 - |x| \Rightarrow$

$$\int_{B_1} \frac{\mathbf{v}^2}{\left(1 - |x|^2\right)} \lesssim \int_{B_1} |\nabla \mathbf{v}|^2$$

- can reduce to polar coors

$$\text{lhs} = \int_0^1 \frac{1}{(1-r)^2} \left(\int_{\partial B_r} \mathbf{v}^2 \, d\sigma \right) dr$$

- do next steps as in Notes last class

Lemma 3 *$\forall \nu > 0 \exists \psi \in h^1 \cap L^n$ such that $\left|B(\mathbf{u}, \psi, \mathbf{u})\right| \le \nu \|\mathbf{u}\|^2$* ✓ *need for* $\nu((\mathbf{u},\mathbf{u})) + B[\mathbf{u},\psi,\mathbf{u}] \ge \alpha \|\mathbf{u}\|^2$; *choose* $\alpha = \frac{\nu}{2}$

Proof.

$$\left|B[\mathbf{u}, \psi, \mathbf{u}\right| = \left|-B[\mathbf{u}, \mathbf{u}, \psi]\right| \le \left| \int_\Omega \boxed{\mathbf{u}} \cdot \nabla \mathbf{u} \cdot \boxed{\psi} \right|$$
$$\le \|\mathbf{u}\|^2 \|\mathbf{u} \otimes w\|_{L^2} = \|\mathbf{u}\| \cdot \left\|u^i \psi^j\right\|_{L^2}.$$

$\boxed{} \in L^2$

boundary data ψ has **curl extension**; cut off interior part: $\psi = \text{curl } (\theta_\varepsilon \xi)$ $\xi \in H^2 \cap W^{1,n} \cap L^\infty$ and

$$|\nabla \theta_\varepsilon| \le \frac{\varepsilon}{\rho}$$

Use to cut off

$$\|\mathbf{u}\psi\|_{L^2} \le \left\| |\mathbf{u} \nabla \theta_\varepsilon| \, |\xi| + |\mathbf{u}| \, |\theta_\varepsilon| \, |\nabla \xi| \right\|_{L^2}$$

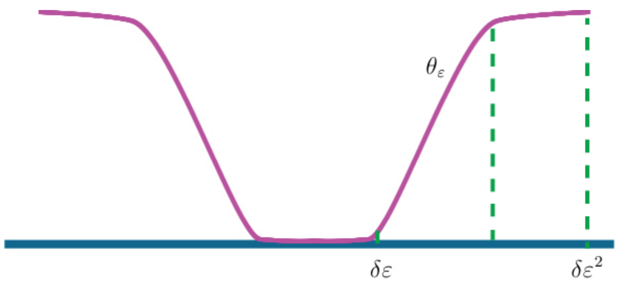


Figure 1.9: the bump function θ_ε

set $\Omega_\delta = \{x \in \Omega : \rho(x) \le \delta\}$. This is the tubular neighborhood $\nabla\theta$. Has support in annular region; ξ has support near boundary.

$$\lesssim \left\| |\mathbf{u}| \left| \boxed{\nabla\theta_\varepsilon} \right| \right\|_{L^2(\Omega_{\delta_\varepsilon})} + \left\| |\mathbf{u}| \, |\nabla\xi| \right\|_{L^2(\Omega_{\delta_\varepsilon})}$$

$\Omega_{\delta_\varepsilon}$: we restrict on

$\delta_\varepsilon = \exp\left(-\frac{1}{\varepsilon}\right)$;

$\nabla\theta_\varepsilon \in L^2$.

max est

$$\le \varepsilon \left\| \frac{|\mathbf{u}|}{\rho} \right\|_{L^2(\Omega_{\delta_\varepsilon})} + \|\mathbf{u}\|_{L^{\frac{2n}{n-2}}} \|\nabla\xi\|_{L^n(\Omega_{\delta_\varepsilon})}$$

Use $\frac{1}{2} = \frac{1}{n} + \frac{1}{\frac{2n}{n-2}}$ $\Rightarrow \le$ poincaire $\Rightarrow$ Sobolev embedding $\Rightarrow$

$$\|\mathbf{u}\| \cdot \|\nabla\xi\|_{L^n(\Omega_{\delta_\varepsilon})} \lesssim \left(\varepsilon + \|\nabla\xi\|_{L^n(\Omega_{\delta_\varepsilon})}\right)\|\mathbf{u}\|$$

Now we can make the coefficient as small as possible by **absolute continuity**:

$$\boxed{\lim_{\varepsilon\to 0}\|\nabla\xi\|_{L^n(\Omega_{\delta_\varepsilon})} = 0.}$$

1.23 Regularity for NSE for $n = 2, 3, 4, 5, 6, 7$

- $n = 2$
- $n = 3$
- $n = 4$ critical - papers
- $n = 5$ supercritical - this dimension is the heart of the matter
- $n = 6$ solved
- $n = 7$ open problem

n=2,3 Any weak solution $\mathbf{u} \in V$ to

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f} - \mathbf{u} \cdot \nabla \mathbf{u}$$

is smooth up to $\partial\Omega$ provided $\partial\Omega \in C^\infty$ and $f \in C^\infty(\bar{\Omega})$.

$n = 2$

We begin with minimum regularity: $u \in H_0^1$. Assume $f = 0$.

$$\mathbf{u} \cdot \nabla \mathbf{u} = \nabla \cdot (\mathbf{u} \otimes \mathbf{u})$$

$u^j \frac{\partial u}{\partial x_j} = \frac{\partial}{\partial x_j}\left(u^j u\right) - \underbrace{\frac{\partial}{\partial x_j} u^j u}_{0 \because \mathrm{div}\,\mathbf{u}=0.}$

sob emb (crit dim)

$$\mathbf{u} \in H_0^1(\Omega) \Rightarrow \mathbf{u} \in L^q \text{for some } q \in \boxed{< +\infty}$$

"as long as finite"

$$\Rightarrow |\mathbf{u} \otimes \mathbf{u}| \le |\mathbf{u}|^2 \in L^q \quad \forall q < +\infty \quad \Rightarrow \nabla(L^q) \in W^{1,q'} \quad \left(q' = \frac{q}{q-1}\right)$$

Last time, if $-\Delta \mathbf{u} + \nabla p \in W^{-1,q} \Rightarrow$ add 2nd order regularity $\Rightarrow \mathbf{u} \in \mathbf{W}^{1,q}, \quad p \in L^q \quad \forall q$.

Now go back:

$$\underbrace{\mathbf{u}}_{\in L^p} \cdot \underbrace{\mathbf{u}}_{\in L^p} = \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) \Rightarrow \text{product } \in L^q \quad \Rightarrow \mathbf{u} \in \mathbf{W}^{2,q} \quad p \in W^{1,q}.$$

Go back:

$$\nabla(\mathbf{u} \cdot \nabla \mathbf{u}) = \underbrace{\mathbf{u} \cdot \nabla^2 \mathbf{u}}_{\in L^q} + \nabla \mathbf{u} \otimes \nabla \mathbf{u} \quad \Rightarrow \nabla \cdot \nabla \in W^{1,q} \quad \Rightarrow \text{Apply Stokes operator}$$

$$\Rightarrow \mathbf{u} \in W^{3,q} \quad p \in W^{2,q} \quad \Rightarrow \mathbf{u} \in W^{k,q} \quad p \in W^{k-1,q} \quad \forall k, q$$

Apply Morrey embedding $\Rightarrow u \in C^\infty(\bar{\Omega}), p \in C^\infty(\bar{\Omega})$. $\square$

$n = 3$.

$\nabla \mathbf{u} \in L^2 \underbrace{\Rightarrow}_{W^{1,2} embedding} \mathbf{u} \in L^6.$

in L^6 for $n=3$

$\underbrace{\mathbf{u}}_{6} \cdot \underbrace{\nabla \mathbf{u}}_{2} \in L^{3/2} \quad \Rightarrow -\Delta \mathbf{u} + \nabla p \in L^{3/2} \quad \Rightarrow \mathbf{u} \in W^{2,3/2}, p \in W^{1,3/2}$ Not $\in L^2$ lol

$\Rightarrow$ in 3d, product = dim $\Rightarrow \mathbf{u} \in L^q \Rightarrow \mathbf{u} \cdot \nabla \mathbf{u} = \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) \in W^{-1,q}$

$\Rightarrow \mathbf{u} \in W^{1,q}, p \in L^q \quad \Rightarrow$ keep going $\Rightarrow \quad \square$

n=4

Previous argument gets stuck:

$$W^{1,2}(\mathbb{R}^4) \hookrightarrow L^4 \Rightarrow \mathbf{u} \cdot \nabla \mathbf{u} = \nabla \cdot \underbrace{\mathbf{u} \otimes \mathbf{u}}_{L^2} \in W^{-1,2} \Rightarrow \quad -\Delta \mathbf{u} + \nabla p \in W^{-1,2} \Rightarrow \mathbf{u} \in W^{1,2}, p \in L^2 \Rightarrow$$

$\mathbf{u} \in W^{2,\frac{4}{3}} p \in W^{1,\frac{4}{3}}$.

This is **good** because we have one more derivative. Its **bad** because we now have lower integrability (!):

$$\nabla \mathbf{u} \in W^{1,\frac{4}{3}}(\mathbb{R}^4) \hookrightarrow \boxed{L^2}$$

$\square$ stop!

Our efforts are wasted because nothing becomes better.

Idea for $n = 4$

Consider instead Burger's equation (ignore pressure p):

$$- \nabla \mathbf{u} = \nabla \cdot \overbrace{\mathbf{u} \otimes \mathbf{u}}^{L^2}$$

$$\nabla^2 \mathbf{u} = \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) \Rightarrow \text{cancel 1 derivative}$$

$$\Rightarrow \nabla \mathbf{u} = (\mathbf{u} \otimes \mathbf{u}) \Rightarrow |\nabla \mathbf{u}| \approx |\mathbf{u}|^2 \Rightarrow \left\| |\mathbf{u}| \right\| \approx \left\| |\mathbf{u}|^2 \right\|_4^2 = \|\mathbf{u}\|_4^2$$

Sobolev emb

$$\|\mathbf{u}\|_4 \leq c \|\mathbf{u}\|_4^2 \tag{1.12}$$

Note that *the integrability does not improve, but the power in 1.12 does*

Theorem 8 *Any solution* $\mathbf{u} \in L^4(\mathbb{R}^4)$ *of* $-\Delta \mathbf{u} = \nabla \cdot (\mathbf{u} \otimes \mathbf{u})$ *is* ***zero*** $\Rightarrow$

$$\|\mathbf{u}\|_{L^4(\mathbb{R}^4)} \leq c \|\mathbf{u}\|^2_{L^4(\mathbb{R}^4)} .$$

Then

$$1 \leq c \|\mathbf{u}\|_{L^4(\mathbb{R}^4)} \Rightarrow \|\mathbf{u}\|_{L^4(\mathbb{R}^4)} \geq \frac{1}{c}$$

is not true. ✓

Monday, October 3

Lemma 4 *For all $m \in \mathbb{R}$ if $f \in H^{m,2}(B)$ & $\mathbf{u}|_{\partial B} \in H^{m+2-\frac{1}{2},2}(\partial B)$ then*

$$\begin{cases} -\Delta \mathbf{v} + \nabla p & = \mathbf{f} \\ \nabla \cdot \mathbf{v} & = 0 \\ \mathbf{v} = \mathbf{u} & \text{on } \partial B \end{cases}$$

has a unique solution $\mathbf{v} \in H^{m+2,2}(B)$ and

$$\|\mathbf{v}\|_{H^{m+2}} \leq c\left(\|\mathbf{f}\|_{H^m} + \|\mathbf{u}\|_{H^{m+2-\frac{1}{2},2}}(\partial B)\right)$$

[From last time - needs to be proved].

For $m = -1, m \in \mathbb{Z}$ Lemma has been known via Solonikhov's theorem.

For $m = -2$, by interpolation, it sufficies to prove the lemma for $m \in \mathbb{Z}_-$.

- If two integers true, then between two integers also true

use standard interpolation.

$$\Delta \mathbf{u} = \mathbf{f}$$
$$\left\|\nabla^2 \mathbf{u}\right\|_{L^{p_1}} \leq \|\mathbf{f}\|_{L^{p_1}}$$
$$\left\|\nabla^2 \mathbf{u}\right\|_{L^{p_2}} \leq \|\mathbf{f}\|_{L^{p_2}}$$

Then for all $q \in [p_1, p_2]$,

$$\left\|\nabla^2 \mathbf{u}\right\|_{L^q} \leq c\|\mathbf{f}\|_{L^q}$$

- If $k > 0, H^{-k} = \left(H_0^k\right)^*$

Hölder expontent negative, so use duality to characterize.

$$\Sigma^m = \begin{cases} \left\{\mathbf{v} \in H^{m,2}(B) : \operatorname{div} \mathbf{v} = 0\right\} & m \geq 0 \\ \left\{\mathbf{v} \in H^m(B) : \operatorname{div} \mathbf{v} = 0\right\} & m < 0. \end{cases}$$

Sobolev space

and

$$\Gamma^m = \begin{cases} \left\{\mathbf{u} \in H^{m-\frac{1}{2},2}(\partial B), \int_{\partial B} \mathbf{u} \cdot \nu \, d\sigma = 0\right\} & m \geq \frac{1}{2} \\ \left\{\mathbf{u} \in H^{m-\frac{1}{2}}(\partial B) : \langle \mathbf{u}, \nu \rangle_{(S)} = 0\right\} & m < \frac{1}{2} \end{cases}$$

where $\left(H^{-m+\frac{1}{2}}(S)\right)' = H^{m-\frac{1}{2}}(S) \ldots$ so define pair

For any $\phi \in \Gamma^m$, $\exists!$ $\mathbf{v} \in \Sigma^m$ such that

$$
\begin{aligned}
-\Delta \mathbf{v} + \nabla p &= 0 \\
\nabla \cdot \mathbf{v} &= 0 \\
\mathbf{v}\,|_{\partial B} &= \phi.
\end{aligned}
$$

$(\mathrm{div}\mathbf{v})(\phi) = -\mathbf{v}(\nabla\phi)$.

nonpositive integer case was last class bitch!

To prove above, $m = -k, k \in \mathbb{Z}$. We want to estimate

$$\|\mathbf{v}\|_{H^{-k}(B)} \le C\|\phi\|_{r^{-k}} .$$

Decompose:

$$\mathbf{v} = \mathbf{v}_1 + \nabla s$$

where s satisfies

$$(*) \quad \begin{cases} \nabla s & = 0 \\ \frac{\partial s}{\partial \nu} & = \phi \cdot \nu \quad \text{on } \partial B. \end{cases}$$

In order for this to be solvable, the average on the boundary must be 0. We know that by Lax-Milgram (?)

We know

$$\|\nabla s\|_{H^{m,2}(B)} \le C\|\phi\|_{\Gamma^m} .$$

$$
\begin{aligned}
\mathbf{v}_1 &= \mathbf{u} - \nabla s \\
\Delta \mathbf{v}_1 + \nabla p &= 0 \quad \text{in } B_1 \\
\nabla \cdot \mathbf{v}_1 &= 0 \quad \text{in } B_1
\end{aligned}
$$

From the boundary condition of (*), $\mathbf{v}_1 = \mathbf{v} - \nabla s$ on ∂B and

$$\langle \mathbf{v}_1, \nu \rangle_{(S)} = 0$$

"is compatible"

It suffices to estimate

$$\|\mathbf{v}_1\|_{H^{-k}(B)} \le C\|\phi\|_{H^{-k}(\Omega B)}$$

That is,

$$\|\mathbf{v}_1\|_{H^{-k}(B)} = \sup_{f^* \in H^k(B)} \langle \mathbf{v}_1, \mathbf{f}^* \rangle / \|\mathbf{f}^*\|_{H^k(B)}$$

Let $0 \neq \mathbf{f}^* \in H_0^k(B)$. Let $\mathbf{v}^* \in \Sigma^{k+2}$ solve

$$\begin{cases} -\Delta \mathbf{v}^* + \nabla q^* & = \mathbf{f}^* \\ \nabla \cdot \mathbf{v}^* & = 0 \\ \mathbf{v}^* & = 0 \end{cases}$$

Solenekov's estimate

$$\|\mathbf{v}^*\|_{H^{k+2}(B)} \leq c\|f^*\|_{H^k(B)}$$

and

$$\begin{aligned}
\langle \mathbf{v}_1, \mathbf{f}^* \rangle &= \langle \mathbf{v}_1 - \Delta \mathbf{v}^* + \nabla q^* \rangle \\
&= \langle \mathbf{v}_1 - \Delta \mathbf{v}^* \rangle + \langle \mathbf{v}_1, \nabla q^* \rangle \\
&\operatorname{div} \langle \mathbf{v}_1, q^* \rangle - \langle \operatorname{div} \mathbf{v}, q^* \rangle \\
&= \int_{\partial B} \left\langle \mathbf{v}_1, \frac{\partial \mathbf{v}^*}{\partial \nu} \right\rangle - \left\langle \frac{\partial \mathbf{v}_1}{\partial \nu}, \mathbf{v}_1^* \right\rangle \\
&= - \int_{\partial B} \boxed{\left\langle \mathbf{v}_1, \tfrac{\partial \mathbf{v}^*}{\partial \nu} \right\rangle} + \left\langle \mathbf{v}_1, \underbrace{\nabla q^*}_{0} \right\rangle - \left\langle \nabla p, \underbrace{\mathbf{v}^*}_{0 \because \operatorname{div}\langle p, \mathbf{v}^* \rangle - \langle p \operatorname{div} \mathbf{v}^* \rangle} \right\rangle
\end{aligned}$$

ibp

where $\boxed{}$ is 0 because $\operatorname{div}(\langle \mathbf{v}_1, q^* \rangle - \int_{\partial B} \langle \mathbf{v}_1, q^* \rangle)\vec{n} = 0$. Then

$$\begin{aligned}
\|\mathbf{v}_1\|_{H^{-k}} &= \sup_{\substack{\mathbf{f}^* \in H_0^k \\ \mathbf{f}^* \neq 0}} \left| \int_{\partial B} \right| \left\langle \mathbf{v}_1, \frac{\partial \mathbf{v}^*}{\partial \nu} \right\rangle \\
\|\mathbf{f}^*\|_{H_0^k} &= 1.
\end{aligned}$$

Continuing,

$$\leq \left\| \frac{\partial \mathbf{v}^*}{\partial \nu} \right\|_{H^{m+\frac{1}{2}}(\partial B)} \|\mathbf{v}_1\|_{H^{-k-\frac{1}{2}}(\partial B)} \leq \|\mathbf{v}^*\|_{H^{k+2}(B)} \left[\|\mathbf{v}\|_{H^{-k-\frac{1}{2}}} + \|\nabla s\|_{H^{-k-\frac{1}{2}}(\partial B)} \right]$$

$$\leq \|\mathbf{f}^*\|_{H_0^k(B)} \left[\|\mathbf{v}\|_{H^m(\partial B)} + \|\phi\|_{H^{m-\frac{1}{2}}(\partial B)} \right]$$

Using

$$\begin{aligned}
\nabla s &= ? \\
\frac{\partial s}{\partial \nu} &= \phi \cdot \nu
\end{aligned}$$

and

$$\|\nabla s\| \leq \|\phi\|_{H^{m-\frac{1}{2}}(\partial B)}$$

- I can talk about trace if I loose $\frac{1}{2}$ derivatives.

$$\leq c\|\mathbf{f}^*\|_{H_0^k(B)}\|\phi\|_{H^{m-\frac{1}{2}}(\partial B)} \leq c\|\phi\|_{H^{m-\frac{1}{2}}(\partial B)}$$

- $\boxed{\text{Stokes operator is self-dual, so operator same operator}}$.
- Conclusion: for every $m \in \mathbb{R}$,

$$S : \Gamma^m \to Z^m \text{is an isomorphism } \phi \to V.$$

$$\begin{aligned} -\Delta\mathbf{v} + \nabla p &= 0 \\ \nabla \cdot \mathbf{v} &= 0 \\ \mathbf{v} &= \phi \end{aligned}$$

$$\|\mathbf{u}\|_{\Sigma^n} \leq c\|\phi\|_{\Gamma^m(\partial B)}\,.$$

- interpolate $\Rightarrow$ true for all whole numbers.
- duality $\Rightarrow$ interpolate $\Rightarrow$ negative numbers.
- Using conclusion,

$$\begin{aligned} -\Delta\mathbf{v} + \nabla p &= 0 \\ \nabla \cdot \mathbf{v} &= 0 \\ \mathbf{v}\,|_{\partial B} &= \mathbf{u}\,|_{\partial B} \in H^1(\partial B). \end{aligned}$$

- Global integrability/ differentiablity $\Rightarrow$

$$\boxed{\mathbf{v} \in H^{\frac{3}{2}}(B) \cap W^{1,\frac{8}{3}}(B.)}$$

Claim$v \in C^{\omega}$. Because $\Delta p = 0,, \Rightarrow p \in C^{\omega}$ and $ddp \in C^{\omega}$ so $\Delta \mathbf{v} \in C^{\omega}$ and $v \in C^{\omega}$.

Need to prove for $0 < \rho < \frac{1}{2}$,

$$\|\mathbf{v}\|_{L^2(B)}^2 \le c\rho^4 \|\mathbf{v}\|_{L^2(B)}^2$$
$$\|\nabla \mathbf{v}\|_{L^2(B_\rho)}^2 \le c\rho^4 \|\mathbf{v}\|_{L^2(B)}^2 .$$

This estimate is not obvious, because of ∇p term: Let $\overline{v} = \mathbf{v} - \fint_{\partial B} \mathbf{u}$ Then

average on boundary still satisfies boundary conditions.

$$-\|\nabla \mathbf{v}\|_{L^{\frac{8}{3}}(B)} \overset{\text{embedding}}{=} \|\mathbf{v} - \overline{\mathbf{u}}\|_{H^{\frac{3}{2}}(B)} \le \|\mathbf{u} - \overline{\mathbf{u}}\|_{H^1(\partial B)} \overset{\text{Poincaire}}{\le} \|\nabla \mathbf{u}\|_{L^2(\partial B)}$$

For q, we need estimate. Assume $\fint p = 0$.

$$\|p\|_{L^2(B)} \le \|\nabla \mathbf{v}\|_{L^2(B)} \le C\|\nabla \mathbf{u}\|_{L^2(\partial B)} .$$

- p is harmonic we can control L^2 norm, control all high order derivatives of harmonic fnction inside

$$\Rightarrow \|\nabla q\|_{H^{m,2}(\widetilde{B})} \le c\|\nabla \mathbf{u}\|_{L^2(\partial B)} .$$

By embedding

$$\underbrace{\left\|\nabla^{k+1} p\right\|_{L^\infty(\widetilde{B})}}_{\le \|\nabla^{k+2}\mathbf{v}\|_{L^\infty(\widetilde{B})}} \le c\|\nabla \mathbf{u}\|_{L^2(B)}$$

This concludes Streuwe's paper reduced to the $n = 4$ case.

Sunday, October 28

- 3d regularity For $n = 3, f \in L^\infty(0,T;H), f' \in L^1(0,T;H), u_0 \in H^2 \cap V$. If $\nu \gg 1$ or u_0 sufficiently small then $\exists!\ u \in \mathbb{R}^3$ such that

$$u' \in L^2(0,T;V) \cap L^\infty(0,T;H) \Rightarrow u \in L^\infty(0,T;H^2(\Omega))$$

and

$$f \in L^2(0,T;V') \quad u \in L^2 \Rightarrow u \in L^\infty(0,T;H) \cap L^\infty(0,T;V).$$

Proof (tricky)

$$u_t - \nu\Delta u + \underbrace{u \cdot \nabla u}_{\text{trouble term}} + \nabla p = f$$

$$\nabla \cdot u = 0$$

- assume smooth, carry out argument

$$u' \cdot \left(u_t' - \nu\Delta u' + \nabla p' + (u \cdot \nabla u)'\right) = f'$$

$$\frac{d}{dt}\int_\Omega \frac{|u'|^2}{2} + 2\nu\int_\Omega |\nabla u'| + 2B[u',u,u'] = 2\langle f', u'\rangle \le 2\|f'\|_{L^2}\|u'\|_{L^2},$$

$$B[\underbrace{u'}_{4}, \underbrace{u}_{2}, \underbrace{u'}_{4}] \le \|\nabla u'\|_{L^2}^2\|\nabla u\|_{L^2} \le C\|\nabla u'\|_{L^2}^2\|\nabla u\|_{L^2}$$

where we have used that, in 3d, $\|u'\|_4^2 \le \|u\|_2^{\frac{1}{4}}\left(\|\nabla u'\|_2^{\frac{3}{4}}\right)^2 \overset{\text{poincaire}}{\le} \|\nabla u'\|_2^2$

$$\underbrace{\frac{d}{dt}\|u'\|_2^2}_{\in L^\infty} + \underbrace{2\left(\nu - c\|\nabla u\|_2\right)}_{\text{if } \ge 0 \text{ then ok.}}\boxed{\|\nabla u'\|^2} \qquad u' \in L^2$$

$$\le 2\|f'\|_{L^2}\|u'\|_{L^2}$$

If the term is not ≥ 0, $\Rightarrow$ we impose condition on the initial data: $\nu - c\|\nabla u_0\|_2 >$ positive. When we turn on evolution the term $\nu - c\|\nabla u\|_2$

grows, so it could become negative. We estimate:

$$
\begin{aligned}
u\left(u' - \nu\Delta u + \nabla p + u\cdot\nabla u\right) &= f\cdot u,\\
\nu\|\nabla u\|_{L^2}^2 &= (f,u) - 2(u,u')\\
&\le \frac{1}{\nu}\|f\|_{V'} + \frac{\nu}{2}\|\nabla u\|_{L^2}^2 + (u,u')\\
&\Rightarrow \nu\|\nabla u\|_{L^2}^2 \le \frac{1}{\nu}\|f\|_{V'}^2 - 2(u,u').
\end{aligned}
$$

Call

$$
\begin{aligned}
d_1 &= \left\|f(0)\right\|_{L^2} + \nu c_0\|u_0\|_{H^2} + c_0\|u_0\|_{H^2}^2,\\
d_2 &= \|f\|_{L^\infty},\\
d_3 &= \frac{d^2}{\nu} + 2\left(\|u_0\|_{L^2}^2 + \frac{Td_2}{\nu}\right)^{\frac{1}{2}}\cdot d_1\\
d_4 &= \frac{d_2}{\nu} + \left(1+d_1^2\right)\left(\|u_0\|_{L^2}^2 + \frac{Td_2}{\nu}\right)\exp\int_0^T\left\|f'(s)\right\|_{L^2} \le \int\frac{\nu^3}{c^2}
\end{aligned}
$$

- can make true for either large coefficient or small data.
- Clearly $d_3 \le d_4 \Rightarrow \nu\left\|\nabla u(0)\right\|_{L^2}^2 \le d_3 \le d_4 < \frac{\nu^3}{c^2}$.
- We can show that, even at max time T_*.
- Choose T_* near zero such that $0 < T_* < T$ and such that $\nu - c\left\|\nabla u(t)\right\|_{L^2} > 0$.
- We show that T_* can be extended all the way to T.

$$
\begin{aligned}
\frac{d}{dt}\|u'\|_{L^2}^2 &\le 2\|f'\|_{L^2}\|u'\|_{L^2} \overset{\text{add}}{\Rightarrow}\\
\frac{d}{dt}\left(1+\|u'\|_{L^2}^2\right) &\le \left(1+\|u'\|_{L^2}^2\right)\|f'\|_{L^2}\\
\left(1+\|u'\|_{L^2}^2\right)\exp - \int_0^t\left\|f'(s)\right\|_{L^2}ds &\le 0
\end{aligned}
$$

as long as positive we have gronwell inequality, which tells initial control

Then

$$1 + \left\| u'(t) \right\|_{L^2}^2 \leq \left(1 + d_1^2\right) \exp \int_0^T \left\| f'(s) \right\|_{L^2} ds$$

and

$$\nu \|\nabla u\| \geq d_4$$
$$\Rightarrow \nu - c\|\nabla u\|$$
$$geq 0.$$

Contradiction $\Rightarrow$ cannot stop.

- Need to prove that next time $L^\infty \cap L^2$.

Chapter 2

Evolution NSE

Wednesday, October 5

We study Evolution NSE on

- bounded domain
- entire space

- $((\mathbf{u}, \mathbf{v})) = \int_\Omega Du \cdot Dv \quad \mathbf{u}, \mathbf{v} \in V,$
- $\|\mathbf{u}\| = \sqrt{((\mathbf{u}, \mathbf{u}))}$
- $(H, |\cdot|)$ and $(V, \|\cdot\|)$ are Hilbert.
- $V \subseteq H$

"inclusion map"

- $\hookrightarrow H$ compact linear
- H' bounded linear functional $H' \subset V'$
- $V \hookrightarrow H \boxed{\doteq} H' \subset \boxed{V'}$

sort of like divergence-free: $= H^{-1} \equiv (H_0^1)'$; Equivalent, because its actually

$$(V)' \supseteq (H_0^1)' = H^{-1}.$$

2.1 Stokes operator

The space $u \in V$ can be viewed as an element in the dual as follows: $\mathbf{v}$ is a bounded linear functional on V, so by RRT $\exists$ another element,

"identify" $\Rightarrow$ use Riesz Reprentation Theorem

$Au \in V$ such that

$$((\mathbf{u},\mathbf{v})) = \langle A\mathbf{u},\mathbf{v}\rangle_{V',V}.$$

- For $\mathbf{u} \mapsto A\mathbf{u}, V \to V'$. Then Stokes operator is "clearly" a bounded linear functional.

- For $\mathbf{u} \in H_0^1$, A is $-\Delta$ if drop the divergence-free condtion. Then

$$A\langle \mathbf{u},\mathbf{v}\rangle = \int A\mathbf{u}\cdot\mathbf{v}$$

This is (somehow) like

$$\int_\Omega D\mathbf{u}\cdot D\mathbf{v} \quad \mathbf{u},\mathbf{v} \in V.$$

- Stokes operator for $\frac{\partial \mathbf{u}}{\partial t} - \underbrace{\nu\Delta\mathbf{u} + \nabla p}_{\text{Stokes}} = \mathbf{f}$.

- When Ω unbounded, $\boxed{\text{need}}$ to adjust inner product . Then

$\boxed{\because}$ Poincaire not hold anymore

$$[[\mathbf{u},\mathbf{v}]] = \int_\Omega (\mathbf{u}\mathbf{v} + D\mathbf{u}\cdot D\mathbf{v})$$

like inner product in H^1 except $\mathbf{u},\mathbf{v} \in V$ instead of H^1.

- can define

$$[[\mathbf{u},\mathbf{v}]] = \langle A\mathbf{u},\mathbf{v}\rangle_{V',V}$$

"but A not as good as linear case":

$$A = -\Delta + I.$$

2.2 Function spaces

- For $1 \le \alpha \le +\infty$,

$$L^\alpha(a,b;\boxed{X}) = \left\{\mathbf{f} : (a,b) \to X \mid \|\mathbf{f}\|_{L^\alpha(a,b;X)<\infty}\right\}$$

dont forget - the spatial variation X is Banach, i.e. it has the inner product $\|\cdot\|_X$.

where $\|\mathbf{f}\|_{L^\alpha(a,b;X)} = \begin{cases} \int_a^b \|\mathbf{f}(t)_X^\alpha\| \, dt & 1 \le \alpha < +\infty \\ \text{ess sup}_{a\le t<b} \|\mathbf{f}(t)\|_X & \alpha = +\infty \end{cases}$

- $C(a,b;X) = \left\{ \mathbf{f} : \boxed{(a,b)} \to X \mid \mathbf{f} \text{ is continuous} \right\}$

  A. Case $[a,b]$: can define norm:

  $$\|\mathbf{f}\|_{C(a,b;X)} = \sup_{a \le t \le b} \left\| \mathbf{f}(t) \right\|_X .$$

  - cannot discount measure zero set (!)

  -Compare with norm having ess sup $\Leftarrow$ not defined everywhere, so we can discount.

  Lemma. Let X Banach space with dual X' and $u, g \in L^1(a,b;X)$. Following are equiv:

  1. $\mathbf{u}$ a.e. integrable equal to primative function of g

  $$\mathbf{u}(t) = \xi + \int_0^t g(s)\, ds$$

  2. For every test function $\phi = \mathcal{D}((a,b))$,

  $$\int_a^b \mathbf{u}(t)\phi'(t)\, dt = -\int_a^b \mathbf{g}(t)\phi(t)\, dt$$

  read $\mathcal{D}$ as "smooth with compact support".

  3. $\frac{d}{dt}\langle \mathbf{u}, \nu\rangle = \langle \mathbf{g}, \nu\rangle$ in the sense of distribution.

  -If $(i)-(iii)$ hold, $\mathbf{u}$ is a.e. equal to a continuous function from $[a,b] \to X$ **after mollifying**.

  Proof

- (i) $\to$ (ii) Easy to see
- (iii) $\to$ (ii). For all $\phi \in \mathcal{D}((a,b))$, true in sense of distributions:

  $$\int_a^b \langle \mathbf{u}, \nu\rangle\, \phi'(t)\, dt = -\int_a^b \langle \mathbf{g}, \nu\rangle\, \phi(t)\, dt$$

  $$\int \Big\langle \mathbf{u}(t), \mathbf{u}'(t) + g(t)\phi(t)\nu \Big\rangle\, dt = 0 \quad \text{or}$$

  get rid of t var.

  $$\underbrace{\left\langle \int_a^b \Big(\mathbf{u}(t)\phi'(t) + g(t)\phi(t)\Big)\, dt, \nu \right\rangle}_{\text{must }=0} \qquad \Rightarrow \int_a^b b(\mathbf{u}(t)\phi'(t) + g(t)\phi(t))\, dt = 0$$

  which proves (iii) $\to$(ii).

- (ii) → (i). Suppose $g \equiv 0$.
- need to prove that $\mathbf{u}(t) = \xi + \underbrace{\int g(s)}_{\text{gone}} = \text{constant}$. Then

$$u' = 0.$$

- Let $\phi_0 \in \mathcal{D}((a,b))$ be such that the mass $\int_a^b \phi_0(t)\,dt = 1$. Then

$$\int_a^b (\underbrace{\phi - \lambda\phi_0}_{\text{compact support/smooth}})(t)\,dt = 0.$$

$\Rightarrow \quad \exists \psi \in \mathcal{D}((a,b))$ such that $\psi'(\phi - \lambda\phi_0)$

$$\psi(t) = \int_a^t (\phi(t) - \lambda\phi_0(\tau))\,d\tau$$

- Use (ii).

$$\int_a^b \mathbf{u}(t)\psi'(t)\,dt = 0.\text{For } g = 0 \text{ and } \phi = \psi \quad \Rightarrow$$

$$\int_a^b \mathbf{u}(t)(\phi(t)\,dt) = \underbrace{\left(\int_a^b \phi(t)\,dt\right)}_{\int_a^b \xi\phi(t)\,dt} \cdot \underbrace{\left(\int_a^b \mathbf{u}(t)\phi_0(t)\,dx\right)}_{\xi_0 = \text{ constant}} \quad \Rightarrow$$

$$\int_a^b \underbrace{(\mathbf{u}(\tau) - \xi)}_{=0}\phi(t)\,dt = 0 \quad \Rightarrow \quad \mathbf{u}(t) - \xi = 0 \text{ a.e.}$$

- If $g \neq 0$, look at $\mathbf{v}_0(t) = \int_a^t g(\tau)\,d\tau$. Then

$$\int_a^b (\mathbf{u} - \mathbf{v}_0)\phi'(t)\,dt = 0 \quad \Rightarrow \quad \mathbf{u} - \mathbf{v}_0 = \xi$$

so $\mathbf{v}_0 = \int g(t)$.

- If $f \in L^1(a,b;X)$ satisfies

$$\int_a^b \mathbf{f}(t)\phi(t)\,dt = 0 \quad \forall \phi \in \mathcal{C}(a,b;\boxed{\mathbb{R}})$$

then

$$f = 0 \quad \text{a.e.}$$

Since, actually the spatial variation is the Banach space X, he replaces $\mathbb{R}$ with X.

Friday, October 7

For $u \in L^2(0,t;V)$ $\quad u'(\underbrace{0,T}_{\text{like } L^2};\overbrace{V'}^{H^{-1}})$, then

$$\mathbf{u} \in \mathcal{C}([0,T],H)$$

like $L^2 \to L^2$ but preserves continuity.

$$\frac{d}{dt}|\mathbf{u}|^2 = \langle \mathbf{u}(t), \mathbf{u}'(t)\rangle_{V,V'} \boxed{\in} L^1(0,T)$$
$$\Rightarrow |\mathbf{u}|^2(t) \in W^{1,1}(0,T) \hookrightarrow \mathcal{C}([0,T])$$

$\square$ on real number function (not X).

Stokes system

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{u} = 0 = \mathbf{u}_0 \end{cases} \Rightarrow \text{has two cond:} \begin{cases} \nabla\cdot\mathbf{u}_0 = 0 \\ \mathbf{u}_0\,|_{\partial\Omega} = 0 \end{cases}$$

- match initial data. cannot talk a priori about **trace**. But with regularity above we **can**.
- Question: for $f \in L^2(0,T;V')$, $\mathbf{u}_0 \in \boxed{H}$ are given, we seek $u \in L^2(0,T;V)$ such that

compatibility condition

$$\frac{d}{dt}(\mathbf{u},\mathbf{v}) + \nu((\mathbf{u},\mathbf{v})) = \langle \mathbf{f},\mathbf{v}\rangle \quad \forall v \in V$$
$$\mathbf{u}(0) = \mathbf{u}_0.$$

2.3 Galerkin

- solve using limiting process
- **adjust nonlinearity** to look more linear.
- Let $\{\mathbf{w}_i\}_{i=1}^{\infty}$ be complete (but not orthogonal) base of V & orthonormal base in H.

$$(\mathbf{w}_{ij})_{L^2} = \delta_{ij} \quad \text{for } i \neq j$$
$$((\mathbf{w}_i,\mathbf{w}_j)) = 0 \quad \text{otherwise.}$$

- $V_m = \text{span}\{\mathbf{w}_i\}_{i=1}^{\infty} \quad m \to \infty$.
- Find $\sum_{i=1}^{m} g_{im}(t)w(x)$ such that

$$\frac{d}{dt}(\mathbf{u}_m, \mathbf{w}_j) + \nu((\mathbf{u}_m, \mathbf{w}_j)) = \left\langle \boxed{\mathbf{f}}, \mathbf{w}_j \right\rangle$$

not f_m $\because$ tail disappears.

The boundary $\mathbf{u}(0) = \mathbf{u}_0$:

$$u_m(0) = \mathbf{u}_0^m \overset{\text{“project”}}{=} \sum_{i=1}^{m} (\mathbf{u}_0, \mathbf{w}_i)\mathbf{w}_i \quad \Rightarrow$$

$$(\mathbf{u}_m, \mathbf{w}_j) = \sum g_{im}(t) \underbrace{(\mathbf{w}_i, \mathbf{w}_j)}_{\text{orthon.}}$$

$$\underbrace{((\mathbf{u}_m, \mathbf{w}_j))}_{H_0^1 \text{inner product}} = \sum_{j=1}^{m} g_{im}(t)((\mathbf{w}_i, \mathbf{w}_j)) \qquad = \sum_{j=1}^{m} \beta_{ij} g_{im}(t) \quad \text{where } \beta_{ij} = ((\mathbf{w}_i, \mathbf{w}_j))$$

$$F_j(t) = \langle \mathbf{f}, \mathbf{w}_j \rangle \quad g_{im}(0) = (\mathbf{u}_0, \mathbf{w}_i)$$

and

$$g'_{im}(t) = \sum_{j=1}^{m} \beta_{ji} g_{jm}(t) = F_i(t) for \quad l \le i \le m$$
$$g_{im}(0) = (\mathbf{u}_0, \mathbf{w}_i).$$

$$\frac{d}{dt}\begin{pmatrix} g_{1m} \\ \vdots \\ g_{mm} \end{pmatrix} + \begin{pmatrix} \beta_{11} & \cdots & \beta_{m1} \\ \vdots & & \vdots \\ \beta_{1m} & \cdots & \beta_{mm} \end{pmatrix} \begin{pmatrix} g_{1m} \\ \vdots \\ g_{mm} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_m \end{pmatrix} \quad \Rightarrow$$

$$\begin{cases} X' + AX & = B(t) \\ \quad X(0) & = X_0 \end{cases}$$

Note: We do not need to use Brouwer fixpoint here, because we have time.

time is actually nice right?

- take limit: Clearly $\{\mathbf{u}_m\} \subset L^2(0, T; V_m)$.

- how to take limit? Use compactness. Compactness introduces some form of control (trapped in a ball) - use apriori estimate.

$$\frac{d}{dt}|\mathbf{u}_m|^2 = 2\left\langle \frac{d}{dt}\mathbf{u}_m, \mathbf{u}_m \right\rangle \quad \Rightarrow$$

$\frac{d}{dt}(\mathbf{u}_m, \mathbf{u}_j) = (\mathbf{u}_m, \frac{d}{dt}\mathbf{u}_j)$

$$\frac{d}{dt}\left(\frac{1}{2}|\mathbf{u}_m|^2\right) = \left(\frac{d}{dt}\mathbf{u}_m, \mathbf{u}_m\right) = \left\langle \frac{d}{dt}\mathbf{u}_m, \mathbf{u}_m \right\rangle = \left\langle \frac{d\mathbf{u}_m}{dt}, \mathbf{w}_j \right\rangle$$

Using that $2ab \le \varepsilon a^2 + \frac{1}{\varepsilon}b^2$,

$$\frac{d}{dt}\left(\frac{1}{2}|\mathbf{u}_m|^2 + \nu((\mathbf{u}_m, \mathbf{u}_m))\right) = \langle \mathbf{f}, \mathbf{u}_m \rangle$$

$$\frac{d}{dt}|\mathbf{u}_m|^2 + 2\nu\|\mathbf{u}_m\|^2 \overset{\text{schwarz}}{\le} \frac{1}{\nu}\|\mathbf{f}\|_{V'}^2 + \nu\|\mathbf{u}_m\|_V^2$$

$$|\mathbf{u}_m|^2(t) + \nu\int_0^t \|\mathbf{u}_m(\tau)\|^2\,d\tau \le \frac{1}{\nu}\int_0^t \|\mathbf{f}\|_{V'}^2\,d\tau \quad \forall m$$

$$\sup_{0\le t\le T}|\mathbf{u}_m|_H^2(t) + \nu\int_0^T \|\mathbf{u}_m(\tau)\|\,d\tau \le |\mathbf{u}_0^m|^2 + \frac{1}{\nu}\int_0^t \|\mathbf{f}\|_{V'}^2\,d\tau$$

Lax-Hopf condition

so (or where)

$$\mathbf{u}_m \in L_t^\infty L_x^2 \cap L_t^2 H_x^1$$

Then

$$\|\mathbf{u}_m\|_{L^\infty(0,T;H)}^2 + \nu\|\mathbf{u}_m\|_{L^2(0,T;V)}^2 \le |\mathbf{u}_0|_H^2 + \frac{1}{\nu}\int_0^t \|\mathbf{f}(t)\|_V^2\,dt$$

- pass limit.

cf Heat equation.

Monday, October 10

2.4 Pass to limits

- in finite dimension, weak convergence & strong convergence same thing
- take subsequence , we may Without loss of generality $\exists \mathbf{v} \in L^\infty(0,T;H)$ and $w \in L^2(0,T;\mathbf{V})$ such that $\mathbf{u}_m^* \rightharpoonup \mathbf{v}$ in $\boxed{L^\infty}(\boxed{0,T};\boxed{\text{H}})$ & $\mathbf{u}_m \rightharpoonup \mathbf{w}$ in $L^2(0,T;\mathbf{V})$

= $(L^1)^*$;
is Banach sp but not H-sp;
is H-sp

- $B_1 \subset L^\infty$ is weak star compact [1]. This means:
  $\langle u_m, \phi \rangle \to \langle v_1, \phi \rangle$ i.e. $\langle u_m, \psi \rangle \to \langle w, \phi \rangle \ \forall \psi \in L^2(0,T;v) \quad \Rightarrow$
- $|u_m|, \|u_m\|_{L^\infty} \le 1 \Rightarrow \exists u \in L^\infty$ such that $u_m \to u \quad \forall \quad \phi \in L^1, \quad \langle u_m, \phi \rangle \to \langle u, \phi \rangle$ or
$$\int u_m \phi \to \int u \phi \quad \forall \phi \in L^1(0,T;H)$$
- Claim: $v = w$.

  Note for $T < +\infty$, $L^2(0,T;V) \subseteq L^1(0,T;H)$.

  If I pick an element in the smaller test space, then $L^2 = L^1$ - they are the same. Remember that we can cook up a sequence having 2 different limits in different spaces, for example
$$\sin\left(\frac{x}{n}\right) \rightharpoonup 0 \quad \text{in } L^2$$
  but
$$\sin\left(\frac{x}{n}\right) \to \delta$$
  where δ is the Dirac mass in the Radon measure space $\mathcal{M}([0,1])$.

[1] The weak star topolgy is by definition, the weakest one that makes all functionals
$$x^* \to \langle x, x^* \rangle$$
continuous

So $\Rightarrow \langle v, \psi \rangle = \langle w, \psi \rangle \quad \forall \psi \in L^2(0,T;V) \Rightarrow$

$$v = w$$

.

- pass to limit: $\boxed{\left\langle \frac{d}{dt} u_m, w_i \right\rangle} + \nu((\underbrace{\mathbf{u}_m}_{\in L^2 \to \int \nabla u \cdot \nabla \mathbf{w}_i} , \overbrace{\mathbf{w}_i}^{\text{fixed}})) = \underbrace{\langle \mathbf{f}, \mathbf{w}_i \rangle}_{\to \langle f \rangle, \mathbf{w}_i}.$

  The term $\boxed{}$ is converges:

$$\frac{d}{dt} \langle w_m, w_i \rangle \to \langle w, w_i \rangle .$$

Distributions of convergent sequences converge

If function convergent $\Rightarrow$ Distribution also convergent. Moreover, the distribution converges in the distributional derivative sense

$$\frac{d}{dt} f_i \phi \to \frac{d}{dt} f \phi$$

Shift test function to see it converges:

$$\int \frac{d}{dt} f \phi = - \int f \frac{d}{dt} \phi.$$

Then the sequence u_0^m is convergent,

$$u_0^m = \sum_{i=0}^{m} g_{im}(0) w_i \to u_0 \quad \text{in } H.$$

- Then

$$\frac{d}{dt}(\mathbf{u}, \mathbf{w}_i) + \nu((\mathbf{u}, \mathbf{w}_i)) = \langle \mathbf{f}, \mathbf{w}_i \rangle \quad \Rightarrow$$

$$\frac{d}{dt}(\mathbf{u}, \mathbf{v}) + \nu((\mathbf{u}, \mathbf{w}_i)) = \langle \mathbf{f}, \mathbf{w}_i \rangle \text{ for all } v \in V.$$

v satisfies the equation in the distributional sense and the boundary value

$$\nabla \cdot \mathbf{u} = 0$$

is kosher because of the yellow boxed explanation above.

- initial-time condition holds:

$$\mathbf{u}\,|_t = 0 = \mathbf{u}_0$$

Show: has well-defined limit in L^2-sp. $\mathbf{u} \in C([0,1], H) \quad \Rightarrow \quad \mathbf{u}$ in H-norm as function of t has well-defined limit.

pf Using

$$\frac{d}{dt} = \langle \mathbf{u} \rangle^2 = 2\,\langle \mathbf{u}', \mathbf{u} \rangle_{V',V}$$

read "in V', V duel"

in the sense of distribution in $(0,T)$.

- Once prove this then prove that L^2 norm in X-var is continuous in t-var.

- Notice that

$$2\,\langle \mathbf{u}', \mathbf{u} \rangle_{V',V} \in L^1(0,T)$$

by Hölder.

$$\begin{aligned}
|u| \in H, \quad |\mathbf{u}|^2\, L^1(0,T), \quad \frac{d}{dt}\left(|\mathbf{u}|^2\right) \in L^1(0,T) \quad &\Rightarrow \\
|\mathbf{u}|^2\,(t) \in W^{1,1}(0,T) \quad \underbrace{\subset}_{\text{embedding}} \quad C^{\frac{1}{2}}(0,T) \quad &\Rightarrow \\
|\mathbf{u}|^2\,(t) - |\mathbf{u}|^2\,(t_0) \overbrace{=}^{\text{ftc}} 2 \boxed{\int_{t_0}^{t}} \langle \mathbf{u}', \mathbf{u} \rangle\,(s)\, dS \quad &\Rightarrow \\
|\mathbf{u}|^2_{L^2}\,(t) \to |\mathbf{u}|^2 \quad \text{as } t \to t_0 \quad &\Rightarrow \\
\left|u(t) - u(t_0)\right|^2 \to 0 &
\end{aligned}$$

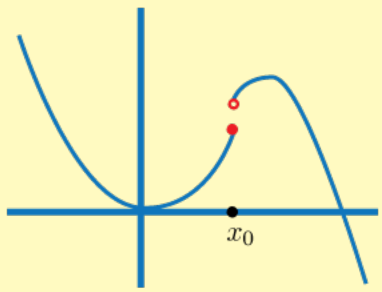
$\boxed{}$ $\to 0$ as $t \to t_0$ by absolute continuity, because the L^2-norm is continuous...

The last convergence holds because $\left|\mathbf{u} - \mathbf{u}(t_0)\right| < \varepsilon \quad \Leftrightarrow \quad \left|\mathbf{u}(t)\right|^2 + \left|\mathbf{u}(t_0)\right|^2 - 2\left\langle u(t), \mathbf{u}(t_0)\right\rangle \to 0 \quad \Leftrightarrow \quad 2\left|u(t_0)\right|^2 - 2\left\langle u(t_0), \mathbf{u}(t_0)\right\rangle = 0.$

- Leray-Hopf type energy inequality:

$$\sup_{0 \le t \le T} \left|\mathbf{u}(t)\right|^2 + \nu \underbrace{\int_0^T \|\mathbf{u}_m\|^2\, dt}_{\text{lower semi-continuity}} \le |\mathbf{u}_0|^2 + \frac{1}{\nu}\int_0^T \left\|\mathbf{f}(t)\right\|^2_{V'}\, dt$$

Lower semicontinuity



- weaker than continuity
- A function f is upper (lower) semi-continuous at a point x_0 if, roughly speaking, the function values for arguments near x0 are either close to $f(x_0)$ or less than $f(x_0)$.
- A lower semi-continuous function. The solid red dot indicates $f(x_0)$
- In particular, f is lower semi-continuous at x_0 if for every $\varepsilon > 0$ there exists a neighborhood U of x_0 such that $f(x) \geq f(x_0) - \varepsilon$ for all $x \in U$. Equivalently, this can be expressed as

$$\liminf_{x \to x_0} f(x) \geq f(x_0)$$

- A function is lower semi-continuous iff $\{x \in X : f(x) > \alpha\}$ is an **open set** for every $\alpha \in \mathbb{R}$.
- A function is continuous at x_0 iff it is upper **and** lower semi-continuous there.

Using Stokes operator

$$u' = \nu A\mathbf{u} + \mathbf{f}$$

$\mathbf{f} \in H^{-1}$, $\quad Au \in L^2(0, T, V')$, $\mathbf{f} \in L^2(0, T; V')$ so

$$\mathbf{u} \in L^\infty(0, T; H) \cap L^2(0, T; V)$$

Stability implies uniqueness. Like Heat equation: instead of laplace operator its Stokes operator.

- Prove: $\frac{d}{dt}|\mathbf{u}|^2 = 2\langle \mathbf{u}', \mathbf{u}\rangle_{V',V}$

$$
\begin{gathered}
C^\infty(0,T) \boxed{\subset} L^2 \\
C^\infty(0,T;V) \subset L^2(0,T;V)\text{dense} \\
C^\infty(0,T;V') \subset L^2(0,T;V')\text{dense.}
\end{gathered}
$$

dense in L^2

In fact, mollification in t-var $\mathbf{u}_m = \eta_\varepsilon * \mathbf{u}$ $\quad \exists\{\mathbf{v}_m\} \subset C^\infty$ such that

$$\mathbf{v}_m \to \mathbf{u} \quad \text{in } L^2(0,T;V)$$

and

$$\mathbf{v}_m \to \mathbf{u}' \quad \text{in } L^2(0,T;V')$$

use 1 sequence to do both mollification

Use Hölder

$$\boxed{2\langle \mathbf{u}_m', \mathbf{v}_m\rangle \to \langle \mathbf{u}', \mathbf{u}\rangle \quad \text{in } L^1(0,T)}$$

using that $\frac{d}{dt}|\mathbf{v}_m|^2 \to \frac{d}{dt}|\mathbf{u}|^2$.

2.5 Uniqueness

2.6 Stability Properties

Wednesday, October 12

Wednesday, October 14

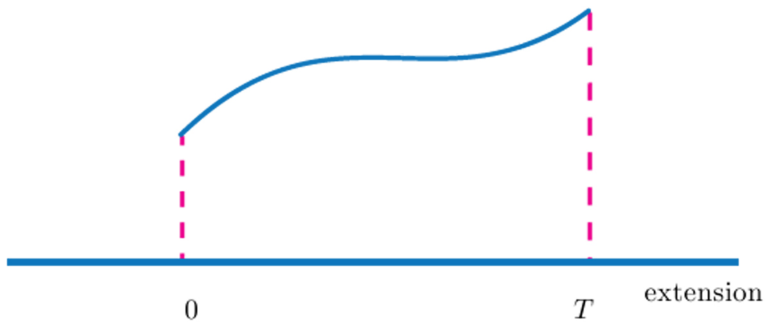


Figure 2.1: jump and extension

Wednesday, October 19

Suppose $\mathbf{u} \in L^2(0,T;X_0)$ $\quad \mathbf{u}' \in L^1(0,T;X_1) \hookrightarrow L^2$. Then there exists $\alpha < \frac{1}{2}$ such that

$$(*) \quad D_t^{\alpha}\mathbf{u} \in L^2(0,T;X_1)$$

- use Fourier transform

$$\widetilde{\mathbf{u}} = \begin{cases} u & t \in [0,T] \\ 0 & \text{otherwise} \end{cases}$$

Then $\overbrace{\widetilde{u}'}^{\widetilde{\hat{u}}} = \underbrace{u'\chi_{[0,T]}}_{g} \boxed{+u(0)\delta_0 - u(T)\delta_T}$ extension generates

- Fourier transform can be applied to distribution.

$$2\pi i\tau\widetilde{\hat{\mathbf{u}}}(\tau) = \hat{g} + \mathbf{u}(0)\cdot 1 - u(\tau)e^{-2\pi i T\cdot\tau}$$
$$\delta_0(T) = \int e^{-2\pi i\tau}\delta_0(t)\,dt$$

Then $\hat{g} \in L^2$.

Condition (*) is equivalent to, in terms of $\widetilde{\hat{u}}$,

$$\boxed{\Leftrightarrow \int_{\mathbb{R}} |\tau|^{2\alpha} \left|\widetilde{\hat{\mathbf{u}}}(\tau)\right|^2_{X_1} \, d\tau < +\infty}$$

for every L^1 function, $\hat{f}(\tau) = \int e^{-2\pi i t T}$ is bounded, and $L^1 \to L^1 \cap L^\infty \subset$ $\boxed{H^2}$.

by Hölder: $\int f^2 \leq |f|_\infty \int |f| \quad \|f\|^2_{L^2} \leq \|f\|_{L^1} \|f\|_{L^\infty}$

$$|\tau|^2 \cdot \boxed{\left\|\widetilde{u}(\tau)\right\|^2_{X_1}} \leq \underbrace{\left\|\hat{g}(t)\right\|^2_{X_1}}_{\text{bounded}} + \overbrace{\left\|\mathbf{u}(0)\right\|^2_{X_1} + \left\|\mathbf{u}(\tau)\right\|^2_{X_1}}^{\text{just numbers - bounded}} \leq C$$

cannot integrate; derivatives in L^2, but this term not L^2 $\Rightarrow$ so sacrifice derivatives $\Rightarrow$

$$|\tau|^{2\alpha} \leq \frac{2(1+\tau^2)}{1+\tau^{2(1-\alpha)}} \qquad \because \quad |\tau|^{2\alpha} + |\tau|^2 \leq 2(1+\tau^2) \quad |\tau|^{2\alpha} < 2 + \tau^2.$$

We need to estimate

$$\int_{\mathbb{R}} |\tau|^{2\alpha} \left\|\widetilde{\hat{\mathbf{u}}}(t)\right\|^2_{X_1} \, d\tau \lesssim \int_{\mathbb{R}} \frac{1+\tau^2}{1+(\tau^2)^{1-\alpha}} \left\|\widetilde{\mathbf{u}}(\tau)\right\|^2_{X_1}$$

$$\overset{\text{split}}{\leq} \int \underbrace{\frac{1}{1+\tau^{2(1-\alpha)}}}_{\leq 1} \underbrace{\left\|\widetilde{\hat{\mathbf{u}}}(\tau)\right\|^2_{X_1}}_{\text{parseval: } \int_{\mathbb{R}} \|\widetilde{\mathbf{u}}\|^2_{X_1}} \, d\tau + \int_{\mathbb{R}} \frac{\tau^2}{1+\tau^{2(1-\alpha)}} \left\|\widetilde{\hat{\mathbf{u}}}\right\|_{X_1} \, d\tau$$

$$\underbrace{\int_{\mathbb{R}} \frac{1\tau^2}{1+\tau^{2(1-\alpha)}}}_{\leq 1} \underbrace{\left(\|\hat{g}\|^2_{X_1} + C\right)}_{\in L^2} \leq \int_{\mathbb{R}} \left\|\hat{g}(\tau)\right\|^2_{X_1} + \int_{\mathbb{R}} \boxed{\frac{C}{1+\tau^{(1-\alpha)}}} \, d\tau$$

singularity at ∞
but $\int_0^\infty \frac{1}{\tau^{2(1-\alpha)}} \, d\tau < \infty$ if $\alpha < \frac{1}{2}$

- compactness follows. □

Theorem. For $n \leq 4$, $\mathbf{f} \in L^2([0,T]; V'), \mathbf{u}_0 \in H$, there exists at least one solution $\mathbf{u} \in L^2(0,T;V), \mathbf{u}' \in \boxed{L^2}(0,T;V')$ such that

Error. Actually, this is $\in L^1$; see below

$$\partial_t \mathbf{u} - \nu \Delta \mathbf{u} + \underbrace{\mathbf{u} \cdot \nabla \mathbf{u}}_{\text{nonlinear}} + \nabla p = \mathbf{f}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\mathbf{u}(0) = \mathbf{u}_0$$

- recall Trilinear form $B[\underbrace{\mathbf{u}}_{L^{10/3}}, \underbrace{\mathbf{v}}_{L^2}, \underbrace{\mathbf{w}}_{L^{10/3}}] = \int (\mathbf{u}\cdot\nabla\mathbf{v})\cdot\mathbf{w} \quad \forall \mathbf{u},\mathbf{v},\mathbf{w} \in V$ is well defined for $n \le 4$: after Hölder L^n not preserved.
- $B[\mathbf{u},\mathbf{v}] : V \to \mathbb{R}$.
- $B[\mathbf{u}] = B[\mathbf{u},\mathbf{u}]$
- NS can be reformulated into

$$\underbrace{\mathbf{u}_t}_{V'} - \nu \underbrace{A\mathbf{u}}_{H^{-1}} + \underbrace{B\mathbf{u}}_{V'} = \underbrace{\mathbf{f}}_{V'}$$

where it follows that $\mathbf{u}_t \in V'$ (after looking at where other terms are). Note that V' is integrable in t.

- Then the regularity of $\mathbf{u}$ is improved: $\mathbf{u} \in \underbrace{\mathcal{C}}_{\text{false, not continuous!!}} (0,T;H)$, by Albin-Lions above (AL: $V \subset\subset H \subset V'$)

  Error (from side note above). $\mathbf{u}' \in L^2$:

  $Au \in L^2(0,T;V'), \quad \mathbf{f} \in L^2(0,T;V), \quad \|B\mathbf{u}\|_{V'} \lesssim \|\mathbf{u}\|_V^2, \|B\mathbf{u}\|_{V'} = \left\|B[\mathbf{u},\mathbf{u}]\right\|_{V'} \le \sup_{\|\mathbf{u}\|_{V'}} \le C\|\mathbf{u}\|\|\mathbf{u}\|_V\|\mathbf{w}\|_V$,

  so $\|B\mathbf{u}\|_{V'}$ is L^1 integrable.

- not continuous. $\mathbf{u} \in C([0,T;H\text{weak}]$. weak continuous $\Rightarrow$ can still take weak limit. For all $\phi \in H$, $\langle \mathbf{u}(t), \phi\rangle \in C([0,T])$.
- nonlinear terms are trouble; take care of $B\mathbf{u}$, being nonlinear (because of square): $\int_0^T \|B\mathbf{u}\|_{V'} \lesssim \int_0^T \|\mathbf{u}\|_V^2$.

  <u>Proof</u> (Galerkin & Aubin-Lions Lem).

- $\{\mathbf{w}_i\}_{i=1}^\infty$, $V_m = \operatorname{span}\{\mathbf{w}_1, , \dots, \mathbf{w}_m\}$, $\quad \mathbf{u}_m = \sum_{i=1}^m g_{im}(t)\mathbf{w}_i(x)$.
- Solve $\frac{d}{dt}(\mathbf{u}_m, \mathbf{w}_j) + \nu \underbrace{((\mathbf{u}_m, \mathbf{w}_j))}_{\text{Dirichlet inner product}} + B[\mathbf{u}_m, \mathbf{u}_m, \mathbf{w}_j] = \langle \mathbf{f}(t), \mathbf{w}_j\rangle$.
- truncate initial data: $\mathbf{u}_m(0) = \operatorname{proj}_{V_m}(\mathbf{u}_0) = \mathbf{u}_{0m}$, where $u_{0m} = \sum_{i=1}^m (\mathbf{u}_0, \mathbf{w}_i)\mathbf{w}_i$

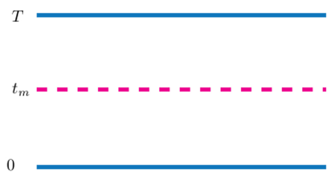


Figure 2.2: solution could blow up if $\in [0, T]$

- $\Rightarrow$ nonlin. ODE:

$$\boxed{\sum_{i=1}^{m} (\mathbf{w}_i, \mathbf{w}_j)} \dot{g_{im}}(t) + \nu \sum_{i=1}^{m} ((\mathbf{w}_i, \mathbf{w}_j)) g(t) + \sum_{i=1,\ell}^{m} B\mathbf{w}_i, \mathbf{w}_\ell, \mathbf{w}_j g_{im}(t) g_{\ell m}(t)$$

where the term is nondegenerate because it forms a base. $\Rightarrow$ non-singular. $(\mathbf{w}_i, \mathbf{w}_j)_{1 \le i, j \le m} \in GL(m)$.

- invert: $\dot{g}_{im} + \sum_{j=1}^{m} \alpha_{ij} g_{jm}(t) + \sum_{j,k=1}^{m} \alpha_{ijk} g_{jm} g_{km} = \sum_{j=1}^{m} \beta_{ij} \left\langle \mathbf{f}, \mathbf{w}_j \right\rangle$ where $\alpha_{ij}, \beta_{ij}, \alpha_{ijk} \in \mathbb{R}$.
- nonlinear, cannot solve globally (like $\dot{X} = X^2$.)
- can solve locally; Fixpoint, $\dot{X} = F(t, X)$.
- $\exists \mathbf{f}_m > 0$ & $g_{im}(0, \mathbf{f}_m) \to \mathbb{R}$
- If t_m is the maximum value then $\left| g_{im}(t) \right| \to \infty$ as $t_m \to \infty$; however,

$$\left(\frac{d}{dt} \mathbf{u}_m, \mathbf{u}_{mt} \right) \nu((\mathbf{u}_m, \mathbf{u}_m)) + \underbrace{B\mathbf{u}_m, \mathbf{u}_m, \mathbf{u}_m}_{=0} = \langle \mathbf{f}, \mathbf{u}_m \rangle .$$

- here have cancelation, so even though the equation is nonlinear, the estimate we have is still linear (!)

$$\frac{d}{dt} |\mathbf{u}_m|^2 + 2\nu \|\mathbf{u}_m\|^2 \le \|\mathbf{f}\|_{V'} \|\mathbf{u}_m\|_V$$

$\Rightarrow$ norm does not blow up, so we continue

$$\le \nu \|\mathbf{u}_m\|_V^2 + \frac{1}{\nu} \|\mathbf{f}\|_{V'}^2$$

$$\underbrace{\left|\mathbf{u}_m(t)\right|^2}_{\text{bounded}} + \nu \int_0^T \left\|\mathbf{u}_m(\tau)\right\|_V^2 \, dt \le \left|\mathbf{u}_m(0)\right|^2 + \frac{1}{\nu}\int_0^t \left\|\mathbf{f}(t)\right\|_{V'}^2 \, dt \overset{\text{orthon. proj}}{\le} \left|\mathbf{u}_0\right|^2 + \frac{1}{\nu}\int_0^t \left\|\mathbf{f}(\tau)\right\|_{V'}^2 \, d\tau$$

so $g_{im}(0) = (\mathbf{u}_0, \mathbf{w}_i)$.

Friday, October 21

- Galerkin - finish convergence
- uniqueness for $m = 2$.
- Leray-Hopf solution for $n = 3$
- regularity.

  Suppose $\mathbf{u} \in L^\infty(0,T;H) \cap L^2(0,T;V)$ solves

$$u_t - \Delta \mathbf{u} + \boxed{\mathbf{u} \cdot \nabla \mathbf{u}} + \nabla p = 0$$
$$\nabla \cdot \mathbf{u} = 0$$
$$\mathbf{u}\,|_{t=0} = \mathbf{u}_0.$$

zero

and suppose (mult by $\mathbf{u}$, integrate)

you can have turbulance and still have regularity...

$$\int_{\mathbb{R}^3} |\mathbf{u}(t)| + 2\int_0^T \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 \, dxdt \le \int_{\mathbb{R}^3} |\mathbf{u}_0|^2 .$$

- $\mathbf{u}_m$:

$$|u_m|^2 + \nu \int_0^T \int_{\mathbb{R}^3} |\nabla \mathbf{u}_m|^2 \le |\mathbf{u}_0| + \frac{1}{\nu}\int_0^t \int \|\mathbf{f}\|_{V'}^2$$

generalized energy inequalty.

- This says:

$$\|u_m\|_{L^\infty(0,T;H)} + \|\mathbf{u}_m\|_{(0,T;V)} \le C.$$

- pass to limit, possible for the subsequence.
-

$$\mathbf{u}_m \boxed{\rightharpoonup} \mathbf{u} \quad \text{in } L^\infty(0,T;H)$$

, where is weak star convergence because of the L^∞, and

$$\mathbf{u}_m \rightharpoonup u \quad \text{in } L^2(0,T;V)$$

- $\mathbf{u}_m' = \mathbf{f}_m'$ where $\mathbf{f}_m = \mathbf{f} - \underbrace{\nu A u_m}_{\in L^2(0,T;V')} - \underbrace{B\mathbf{u}_m}_{B[\mathbf{u}_m,\mathbf{u}_m]\in V^2}$ In fact, $B[\mathbf{u}_m,\mathbf{u}_m] \in V'$:

$$\|B\mathbf{u}_m\|_{V'} \le \sup_{\|\mathbf{w}_{V=1}\|} B[\mathbf{u}_m,\mathbf{u}_m,\mathbf{w}] = \sup_{\|\mathbf{w}\|_{V=1}} \int_{\mathbb{R}} \mathbf{u}\cdot\nabla\mathbf{u}_m\cdot\mathbf{w} \overset{\text{Sobolev}}{\le} \|\nabla\mathbf{u}_m\|_2^2,$$

so then $B\mathbf{u}_m \in L^1(0,t;V')$.
Moreover $\int_0^T \|B\mathbf{u}_m\|_{V'}\,dt \lesssim \int_0^T \|\nabla\mathbf{u}_m\|_2^2\,dt \le C$. Upshot:

$$\mathbf{f}_m \in L^1(0,T;V').$$

- have better control of derivatives,., so $\Rightarrow \mathbf{u}_m \to \mathbf{u}$ in $L^2(0,T;H)$.

- ?? Of course need to apply Albin-Lions; need to control time derivatives, so

$$\|\mathbf{u}_m'\|_{L^1(0,T;V')} \le C$$

time derivative bounded (!)

where we used

$$\left|\int_{\mathbb{R}} \mathbf{u}_m\cdot\nabla\mathbf{u}_m\cdot\mathbf{w}\right| \le \|\mathbf{u}_m\|_4\|\nabla\mathbf{u}_m\|_2\|\mathbf{u}\|_4 \lesssim C\|\nabla\mathbf{u}_m\|_2^2 = C\|\mathbf{u}_m\|^2.$$

- so $\left\{\mathbf{u}_m \in L^2(0,T;V) \quad \& \quad \mathbf{u}_m' \in L^1(0,T;V')\right\} \subset\subset L^2(0,T;H)$

apply Albin-Lions, with $V \hookrightarrow\hookrightarrow H \hookrightarrow V'$

- indeed, is L^2 compact.

- Then $u_m \to \mathbf{u}$ in $L^2(0,T;H)$

strong convergence in H-space.

- $\boxed{\left(\frac{du_m}{dt},\mathbf{w}_j\right)} + \underbrace{\nu((\mathbf{u}_m,\mathbf{w}_j))}_{\to\nu((\mathbf{u}_m,\mathbf{w}_j))} + B[\mathbf{u}_m,\mathbf{u}_m,\mathbf{w}_j] = \underbrace{\langle\mathbf{f},\mathbf{w}_j\rangle}_{\text{constant}\Rightarrow\text{converge}}$

where ■ converge because $\mathbf{u}_m \rightharpoonup \mathbf{u} \Rightarrow \frac{\partial\mathbf{u}_m}{\partial t} \rightharpoonup \frac{\partial\mathbf{u}}{\partial t}$ in distribution & $\mathbf{w}$ is smooth $\Rightarrow \left(\frac{\partial\mathbf{u}}{\partial t},\mathbf{w}_j\right)$ in L^1 by lower semicontinuity.

- It suffices to show

$$B[\mathbf{u}_m,\mathbf{u}_m,\mathbf{w}_j] \to B[\mathbf{u},\mathbf{u},\mathbf{w}_j].$$

Note that $\int \mathbf{u}_m \boxed{\nabla\mathbf{u}_m} \mathbf{w}_j$ where ■ $\rightharpoonup \nabla\mathbf{u}$ in L^2.

- $\mathbf{u}_m \mathbf{w}_j \to \mathbf{u}\mathbf{w}_j$ (without $\mathbf{w}_j$ strong convergence.)
- $$\left\|\mathbf{u}_m \mathbf{w}_j - \mathbf{u}\mathbf{w}_j\right\|_2 = \left\|(\mathbf{u}_m - \mathbf{u})\mathbf{w}_j\right\|_2 \le \left\|\mathbf{w}_j\right\|_\infty \left\|\mathbf{u}_m - \mathbf{u}\right\|_2 \to 0$$

  which proves existence $\square$

  Need to prove:

  $$\mathbf{u} \in L^2(0,T;V), \quad \mathbf{u}' \in L_1(0,T;V') \hookrightarrow \mathcal{C}(0,T;H).$$

- initial data for approx solution $\Rightarrow$ pass to limit (will not repeat; see previous Notes).

  $$\left(\frac{d\mathbf{u}}{dt}, \mathbf{v}\right) + \nu((\mathbf{u}, \mathbf{v})) + B[\mathbf{u}, \mathbf{u}, \mathbf{v} = \langle \mathbf{f}, \mathbf{v}\rangle] \quad \forall v \in V$$

  $\Rightarrow$

  $$\frac{du}{dt} - \nu A + B\mathbf{u} = \mathbf{f}$$
  $$\mathbf{u}(0) = \mathbf{u}_0.$$

- The existence argument is due to Lions. Leray gives a different method; Schwartz's distribution was not yet invented. E. Hopf < Lions < Leray

  Leray's idea:

  $$\mathbf{u}_t + \boxed{\mathbf{u}} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}$$
  $$\nabla \cdot \mathbf{u} = 0$$

  Mollify the term ■ to u_ε, then equation is linear (Stokes equationwith first order term).

  $$\mathbf{v}_t + \mathbf{u}_\varepsilon \cdot \nabla \mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = \mathbf{f}$$
  $$\nabla \cdot \mathbf{v} = 0$$

  so that

  $$\mathbf{u} \to \mathbf{u}_\varepsilon \to \mathbf{v} = T(\mathbf{u}).$$

- find apriori est of T;
- build up function space in C^2
- T is contraction map on suitable function space
- $\mathbf{u} = T(\mathbf{u})$.
- $\mathbf{u}_t + \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}$
- ε varies and for each fixed ε have $\mathbf{v}$; send $\varepsilon \to 0$.

2.7 Uniqueness

Lemma 5 *For $n = 2$, $\Omega \subset \mathbb{R}^2$,*

$$\|\mathbf{u}\|_{L^4} \leq \sqrt{2}\|\mathbf{u}\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbf{u}\|_{L^2}^{\frac{1}{2}} \quad \mathbf{u} \in H_0^1$$

✓ Ladyzhenskaya inequality

in 2 dimension, H_0^1 is $H^1(\mathbb{R}^2) \hookrightarrow L^p \quad \forall p \in< +\infty$

Proof. Extend by Nirenberg higher dimension.

Lemma 6 (better estimate on trilinear form)

$$\left|\boldsymbol{B}[\mathbf{u}, \mathbf{v}, \mathbf{w}]\right| \leq \sqrt{2}\,|\mathbf{u}|^{\frac{1}{2}} \|\mathbf{u}\|^{\frac{1}{2}} \|\mathbf{v}\| \,|\mathbf{w}|^{\frac{1}{2}} \|\mathbf{w}\|^{\frac{1}{2}}$$

- If $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$, then

  $$B\mathbf{u} \in L^2(0,T;V'),$$

  where (compare previous) $B\mathbf{u}$ was $\in L^1$.

  $$\|B\mathbf{u}\|_{L^2(0,t;V')} \sqrt{2}\|\mathbf{u}\|_{L^\infty(0,T;H)} \|\mathbf{u}\|_{L^2(0,T;V)} \,.$$

- when can $\sqrt{2}$ be obtained (idk)

  for Sobolev inequality ppl interested in sharp constant $\Rightarrow$ geometric implications.

Theorem 9 (Uniqueness for $n = 2$)

Let $\mathbf{u}_1, \mathbf{u}_2$ be 2 solutions. Let $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$,

$$\partial_t - \Delta \mathbf{u} + \nabla p + \underbrace{\mathbf{u}_1 \nabla \mathbf{u}_1 - \mathbf{u}_2 \nabla \mathbf{u}_2}_{\text{subtract}} = 0$$

$$\mathbf{u} \cdot \left(\mathbf{u}_t - A\mathbf{u} + B[\mathbf{u}_1, \mathbf{u}_1] - B[\mathbf{u}_2, \mathbf{u}_2] \right) = 0$$

$$\left(\frac{|\mathbf{u}|^2}{2} + ((\mathbf{u}, \mathbf{u})) + B[\mathbf{u}_1, \mathbf{u}_1, \mathbf{u}] - B[\mathbf{u}_2, \boxed{\mathbf{u}_2}, \mathbf{u}] = 0 \right)$$

replace $\blacksquare$ by $\mathbf{u}_1$, equality preserved $\because B[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}] = B[\mathbf{u}_2, \mathbf{u}_2 - \mathbf{u}_1 + \mathbf{u}_1, \mathbf{u}] = \underbrace{-B[\mathbf{u}_2, \mathbf{u}, \mathbf{u}]}_{0} + B[\mathbf{u}_2, \mathbf{u}_1, \mathbf{u}]$

$$\frac{d}{dt} \left(\frac{|\mathbf{u}|^2}{2} + ((\mathbf{u})) + B[\mathbf{u}, \mathbf{u}_1, \mathbf{u}] \right) = 0$$

$$\left| B[\mathbf{u}, \mathbf{u}_1, \mathbf{u}] \right| \le C\, |\mathbf{u}|_4\, |\mathbf{u}|_4 \|\nabla \mathbf{u}_1\|_2 \le C\, |\mathbf{u}| \|\mathbf{u}\| \|\nabla \mathbf{u}_1\|_2$$

so

$$\frac{d}{dt} |\mathbf{u}|^2 + 2\|\mathbf{u}\|^2 \le 2\|\mathbf{u}\|^2 + C\, |\mathbf{u}|^2 \|\mathbf{u}_1\|^2$$

- Use Gronwell inequality, which says that

$$\left(\frac{d}{dt} |\mathbf{u}|^2 - c\|\mathbf{u}_1\|^2 |\mathbf{u}|^2 \right) \le 0$$

$$\frac{d}{dt} \left(e^{-c \int_0^t \|\mathbf{u}_1(t)^2\| \, dt |\mathbf{u}_0|^2} \right) \le 0$$

$$\text{integrate } \Rightarrow |\mathbf{u}|^2 (t) \le e^{c \int_0^t \|\mathbf{u}_1\|^2 \underbrace{|\mathbf{u}_0|^2}_{0}} = 0$$

so

$$\mathbf{u}_1 = \mathbf{u}_2 \quad \Rightarrow \quad \text{uniqueness.} \quad \square$$

- Lem2 proof direct corollary of Lem 1.

$$\begin{aligned} \left| B[\mathbf{u}, \mathbf{u}, \mathbf{v}] \right| &= \left| B[\mathbf{u}, \mathbf{v}, \mathbf{u}] \right| \le \|\mathbf{u}\|_2^{\frac{1}{2}} \|\nabla \mathbf{u}\|_2^{\frac{1}{2}} \|\nabla \mathbf{v}\|_2^2 \|\mathbf{u}\|_2^{\frac{1}{2}} \|\nabla \mathbf{u}\|_2^{\frac{1}{2}} \\ &\le \|\mathbf{u}\|_2 \|\nabla \mathbf{u}\|_2 \|\nabla \mathbf{w}\|_2 \\ &\Rightarrow \quad \int_0^T \|B[\mathbf{u}, \mathbf{u}]\|_{V'}^2 \le \int_0^T \|\mathbf{u}\|_2^2 \|\nabla \mathbf{u}\|_2^2 \\ &\le \|\mathbf{u}\|_{L^\infty(0,T;L^2)}^2 \int_0^T \|\nabla \mathbf{u}\|_2^2 \, dt. \end{aligned}$$

Monday, October 24

- proof of Ladyzhenskaya inequality
- existence theorem 3d version

Lemma 7 (n=2)

$$\|\mathbf{u}\|_{L^4} \le \sqrt{2}\underbrace{\|\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^2}^{\frac{1}{2}}}_{\text{even splitting}} \quad \forall \boldsymbol{u} \in \boldsymbol{H}_0^1.$$

✓

Lemma 8 (n=3)

$$\|\mathbf{u}\|_{L^4} \le \sqrt{2} \quad \underbrace{\|\mathbf{u}\|_{L^2}^{\frac{1}{4}}\|\nabla\mathbf{u}\|_{L^2}^{\frac{3}{4}}}_{\text{more weight on grad, which is harder to control}} \quad \forall \mathbf{u} \in \boldsymbol{H}_0^1$$

proof of lem 1 $W^{1,1}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}^2)$ $2 = \frac{2-1}{2-1\cdot 1}$

- prove for smooth compact support functions ⇒ pass limit (which we can do because dense).

$$\big\||\mathbf{u}|\big\|_{L^2} \le c\big\|\nabla(\mathbf{u}^2)\big\|_{L^1} = c\int 2\,|\mathbf{u}|\,|\nabla\mathbf{u}| \overset{\text{hölder}}{\le} c\left(\int|\mathbf{u}|^2\right)^{\frac{1}{2}}\left(\int|\nabla\mathbf{u}|^2\right)^{\frac{1}{2}} \quad \square$$

Proof of lem 2 $W^{1,2}(\mathbb{R}^3) \hookrightarrow L^6$ in 3d, is embedding in L^6.

- can interpolate: $L^2 < L^4 < L^6$
- how interpolate:

$$\frac{1}{4} = \frac{\theta}{2} + \frac{1-\theta}{6} \Rightarrow \theta = \frac{1}{4}.$$

$$\|\mathbf{u}\|_{L^4} \overset{\text{interpolation inequality}}{\le} \|\mathbf{u}\|_{L^2}^{\frac{1}{4}}\|\mathbf{u}\|_{L^6}^{\frac{3}{4}} \overset{\text{Sobolev inequality}}{\le} c\|\mathbf{u}\|_{L^2}^{\frac{1}{4}}\|\nabla\mathbf{u}\|_{L^2}^{\frac{3}{4}} \quad \square$$

$W^{2,2}(\mathbb{R}^3) \hookrightarrow L^6$

Show that $c = \sqrt{2}$ **in Lem 2 .** $\mathbf{u}(x_1, x_2) \boxed{=} -\int_{x_2} \frac{\partial}{\partial x_2}(\mathbf{u}^2)(x,\xi)\,d\xi$

smooth fc w compact spt; ending pt at ∞ is 0 because compact spt

expand int

$$= -2\int_{x_2}^{+\infty} \mathbf{u}(x,\xi)\left\|\frac{\partial}{\partial x_2}\mathbf{u}(x,\xi) \le d\xi\right\| \le \int_\Omega |\mathbf{u}(x,\xi)|\,|D_2\mathbf{u}(x,\xi)|\,d\xi$$

Similarly x_1-var

$$u^2(x_1,x_2) \le 2\int_\Omega |\mathbf{u}(\xi,x_2)|\,|D_1\mathbf{u}(\xi,x_2)|\,d\xi.$$

- $$\int \mathbf{u}^4(x,x_2)\,dx_1dx_2 \le 4\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{u}(x,\xi)D_2\mathbf{u}(x,\xi)\,d\xi dx_1 \le \int_{\mathbb{R}}\int_{\mathbb{R}} \|\mathbf{u}(x_2)\|\,|D_1\mathbf{u}(\xi,x_2)|\,dsdx_2$$
$$4\left(\int_{\mathbb{R}^2}\mathbf{u}^2(x_1,x_2)\right)\cdot\left(\int_{\mathbb{R}^2}|D_2\mathbf{u}|^2\right)^{\frac{1}{2}}$$
$$\le 2\int_{\mathbb{R}^2}\mathbf{u}^2\left(\int_{\mathbb{R}^2}|D_1\mathbf{u}|^2 + |D_2\mathbf{u}|^2\right) \quad \square$$

Theorem 10 *If $n = 3$ the solution we obtained via Galerkin method to the NSE satisfies*

$$u \in L^\infty(0,T;H) \cap L^2(0,T;V)$$

$$\mathbf{u} \in L^{\boxed{8/3}}(0,T;L^4(\Omega))$$

$\frac{2}{\frac{8}{3}} + \frac{3}{4} = \frac{3}{2} = 2 + \frac{3}{6}$

in 3d cannot do 2, which we used to prove uq above

$$u' \in L^{\boxed{4/3}}(0,T;V')(0,T;V')$$

✓

Proof For a.e. $t \in [0,T]$ we have $\mathbf{u}(t) \in H_0^1$. By the 3d interpolation inequality/ Sobolev inequality we have

$$\|\mathbf{u}(t)\|_{L^4} \lesssim \|\mathbf{u}(t)\|_{L^2}^{\frac{1}{4}}\|\nabla\mathbf{u}\|_{L^2}^{\frac{3}{4}} \Rightarrow (\|\mathbf{u}\|)_{L^4}^{8/3} \lesssim \left(\|\mathbf{u}\|_{L^2}^{\frac{1}{4}}|\nabla\mathbf{u}|_{L^2}^{3/4}\right)^{8/3} \lesssim \int_0^T \|\mathbf{u}\|_{L^2}^{\frac{2}{3}} \boxed{|\nabla\mathbf{u}|_{L^2}^2}\,dt$$

- usually cant split, but this is $L^1 s$

  $$\|\mathbf{u}\|_{L^\infty(0,T;H)}\|\mathbf{u}\|^2_{L^2(0,T;V)} \Rightarrow \|\mathbf{u}\|_{L^{8/3}(0,T;L^4)} \le \|\mathbf{u}\|_{L^\infty(0,T;H)}\|\mathbf{u}\|^{3/4}_{L^2(0,T;V)}$$

- can int in x-var something something in t-var

  proof $\mathbf{u}' = \underbrace{\nu\Delta\mathbf{u}}_{\in L^2(0,T;V')} + \boxed{B\mathbf{u}} + \underbrace{\mathbf{f}}_{L^2(0,T;V')}$ where does this term live?

  We claim $B\mathbf{u} \in L^{4/3}(0,T;V')$.

  If this is true, then we can write $u' \in L^2(0,T;V') + L^{4/3}(0,T;V') \supseteq L^{4/3}(0,T;V')$

- want to calculate duel norm

- est V' norm

  To show this, let $V \in V$

  For $1 < Bu, \nu > 1$,

  $$\left|B[u,\mathbf{u},\mathbf{v}]\right| \le \underbrace{|\mathbf{u}|_{L^4}}_{\int \boxed{\mathbf{u}}\,\boxed{\nabla\mathbf{u}\mathbf{v}}} \le \|\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}\|_{L^4}\left\|\boxed{\mathbf{v}}\right\|_{L^4}$$

  replace with $\nabla\mathbf{v}$; replace with $\mathbf{u}$; replace with $\nabla\mathbf{v}$

  $$\le \|\mathbf{u}\|^{\frac{1}{4}\cdot 2}_{L^2}\|\nabla\mathbf{u}\|^{3/4\cdot 2}_{L^2}\|\nabla\mathbf{u}\|_{L^2} \lesssim \|\mathbf{u}\|^{1/2}_{L^2}\|\nabla\mathbf{u}\|^{3/2}_{L^2}\|\mathbf{v}\|_V$$

  so

  $$\|Bu\|_{V'} \lesssim \|\mathbf{u}\|^{1/4}_{L^2}\|\nabla\mathbf{u}\|^{3/2}_{L^2} \in L^{4/3},$$

  $$\left\|\left|B(t)\right|_{V'}\right\|_{L^{4/3}(0,T)} \le \|\mathbf{u}\|_{L^\infty(0,T;L^2)}\|\nabla\mathbf{u}\|^{1/2}_{L^2(0,T;L^2)};$$

  This is the regularity we have; but this regularity we dont know fif unique;

- If try to extend 2d proof you will find we need more

- If $n = 3$ and $\mathbf{u}$ satisfies in addition the following

  $$u \in L^8(0,T;L^4(\Omega))$$

  then $\mathbf{u}$ is unique.

- There is a gap between $L^{3/8}$ (in 2d case above) almost everywherend L^8.

- Proof follows 2d proof.

  Proof Let $\mathbf{v}_1, \mathbf{v}_2$ be solution; $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$

$$\mathbf{u}\left(u_t - \nu\nabla\mathbf{u} + \nabla p\right) = \mathbf{u}\left(-\mathbf{u}_1 - \nabla\mathbf{u}_1 + \mathbf{u}_2\nabla\mathbf{u}_2\right) = \mathbf{u}\left(-\mathbf{u}_1\nabla\mathbf{u} - \mathbf{u}\nabla\mathbf{u}_2\right);$$ mult by u, int

$$\frac{d}{dt}\int_\Omega \frac{|\mathbf{u}|^2}{2} + \nu\int|\nabla\mathbf{u}|^2 = \underbrace{-B[u,\mathbf{u},\mathbf{u}]}_{0} - B[u,\nabla\mathbf{u}_2,\mathbf{u}] \overset{\text{switch}}{\leq} B[\underbrace{\mathbf{u}}_{L^4},\underbrace{\mathbf{u}}_{L^2},\underbrace{\mathbf{u}_2}_{L^4}]$$

$$\left\|\mathbf{u}(t)\right\|_{L^4}\left\|\nabla\mathbf{u}\right\|_{L2}\left\|\mathbf{u}_2\right\|_4^4 \overset{\text{int}}{\leq} \left\|\mathbf{u}\right\|_{L^2}^{\frac{1}{4}}\left\|\nabla\mathbf{u}\right\|_{L^2}^{\boxed{\frac{7}{4}}}\left\|\mathbf{u}_2\right\|_4^4 \qquad \tfrac{3}{4} + 1 = \tfrac{7}{4}$$

Using $ab \leq \varepsilon a^{\frac{8}{7}} + \frac{1}{\varepsilon}b^8$ where $\varepsilon = \nu$

$$\frac{\nu}{2}\left(\|u\|_{L^2}^{\frac{7}{4}}\right)^{\frac{8}{7}} + \left(\left\|u(t)\right\|_{L^2}^{\frac{1}{2}} + \left\|\mathbf{u}_2\right\|_{L^4}\right)^8 \leq \frac{\nu}{2}\int_\Omega|\nabla\mathbf{u}| + \left\|\mathbf{u}(t)\right\|_{L^2}^2\left\|\mathbf{u}_2\right\|_{L^4}^8$$

$$\Rightarrow \frac{d}{dt}\int_\Omega|\mathbf{u}|^2 + \underbrace{\nu\int_\Omega\left\|\nabla\mathbf{u}\right\|^2}_{\|\mathbf{u}_2\|_{L^4}}\underbrace{\left\|\mathbf{u}(t)\right\|_{L^2}^8}_{\int_\Omega 1|u|^2}$$

- Use Gronwell inequality

$$g' \leq c(t)g, \quad \frac{dg}{g} = c(t)\,dt, \quad \ln g - \ln g(0) \leq \int_0^t c(\tau)\,d\tau,$$

$$\frac{g(t)}{g(0)} \leq \exp\int_0^t c(\tau)\,d\tau, \quad g(\tau) \leq g(0)\exp\int_0^t c(\tau)\,d\tau \quad \Rightarrow$$

$$\int_\Omega\left|\mathbf{u}(t)\right|^2 \leq \left|\mathbf{u}(0)\right|^2\underbrace{\exp\int_0^t]\left\|\mathbf{u}\right\|_t(t)_4^8\,d\tau}_{\text{integrable}} \leq 0$$

so

$$\boxed{u \equiv 0 \quad \Rightarrow \quad \mathbf{u}_1 = \mathbf{u}_2}$$