Coherent Structures in Turbulence

J. L. Lumley

1. INTRODUCTION.

During the past decade there has been a very active interest in coherent structures. An excellent reference for this work is the paper of Cantwell (1981).

The general awareness of the existence of organized structures in turbulent motions can probably be traced to the work of Townsend (1956), who referred to a " double structure" of turbulent flows, consisting of more or less organized "big eddies", and a less well-organized, smaller scale, background turbulence. Townsend's work referred to so-called fully developed turbulence; that is, in general, flows that had attained selfsimilarity and were consequently many characteristic dimensions (body diameters, etc) from their origin. Townsend found that the energy in the big eddies was rather small, say of the order of one-fifth of the total turbulent energy.

The coherent structures that have attracted the interest of the turbulence community recently differ somewhat from Townsend's big eddies. For the most part, the flows that have been examined recently are younger, closer to their origins, and it has been found (usually by the use of flow visualization) that there are often present in these flows structures much more organized and more energetic than those present in fully developed turbulence. This is a matter of some controversy at the moment: one school (Brown & Roshko, 1971, 1974;

ISBN 0-12-493240-1

Roshko, 1981; Cantwell, 1981; Browand & Troutt, 1980) appears to feel that these structures are more characteristic of all turbulence than we previously thought; that we overlooked the presence of such structures in turbulence previously examined principally because we did not use visualization, and because we made measurements using ordinary statistical approaches, as opposed to the conditioned sampling widely used by this group. The claim (to which there is some truth) is, that the usual statistical approaches tend to smear out, and thus conceal, the organization truly present in the flow, which can be identified only by the use of conditioned sampling and visualization.

The opposing school, (for example: Bradshaw, 1966; Hussain, 1981) feels that there is substantial evidence that the degree of organization in these flows decreases as the flows age ;that one reason we were not previously aware of the existence of these well organized, energetic structures is that we never measured in the early part of turbulent flows, saying that the flows were not yet fully developed; that, in fact, the well-organized structures observed may be attributable in part to the care with which these flows have been set up - that is, if extreme care is taken to remove adventitious disturbances from the oncoming flow (as is now more usual than formerly), the instability of the initial flow will be of a single type, and the transition flow will be dominated by the nonlinear evolution of this instability. The initial development of the turbulence will probably be influenced for some downstream distance by these structures.

Whether these structures are the same as those that will be present when the flow is fully developed is difficult to say; certainly there is evidence that in some cases they are not (compare Bevilaqua & Lykoudis, 1971 and Payne & Lumley, 1967). There is some feeling that the structures present in the early flow are probably more characteristic of the initial instability, while those present in the fully developed turbulence are probably characteristic in some sense of the fully developed flow (represent, say, some type of instability of that flow), and are consequently not necessarily the same, unless the initial flow has been set up so that the dominant instability present there is of the same type as that present in the fully developed flow. There is evidence for this in the development of the flat plate wake of Chevray & Kovasznay (1969), which is formed without vortex shedding, and is quite different from the cylinder wake of Townsend (1947); the same is true of the spheroid wake of Chevray (1968) and the porous disk of Bevilaqua & Lykoudis (1971), which do not contain vortical structures, in contrast to the sphere wake of Bevilaqua & Lykoudis (1971). This is also consistent with the results of second order modeling (which does not include the effect of large coherent structures), which forms a progression with the experimental results cited: from the wake with structures, to that without, to the model, the latter two being quite close together (see Taulbee & Lumley, 1981). This suggests that the wake supposedly without structures actually has weak ones.

Roshko (1981) points out that the mixing layer (perhaps uniquely) will appear thin to disturbances of sufficiently large scale at any stage of its development, and consequently should be subject to the same type of (Kelvin-Helmholtz) instability at all ages; he would thus expect to find energetic, well-organized structures of the same type at all ages. The same might be said of other nearly-parallel shear flows; the wake, for example, should be even thinner than the shear layer. The instability here, however, would not be of the Kelvin-Helmholtz type, but would presumably be associated with the inflectionary profile; the inflection points are buried in the small-scale turbulence, however, and this instability is probably strongly damped by the momentum transport of this fine-grained turbulence. This might be a reason why energetic, well-organized structures would not be observed in the late, equilibrium (similarity) wake. The mixing layer may be unique in this respect, although one might expect a certain amount of disorganization to creep in with age - that is, the older mixing layer is already quite disturbed by turbulence of all scales, including non-linearly evolved older three-dimensional somewhat disorganized structures; although subject to a Kelvin-Helmholtz type of instability, this could be damped and modified by the momentum transport of the other eddies present, resulting in a somewhat weaker and less-well organized coherent structure than one might find in an early mixing layer which has just undergone transition. The measurements of Browand & Troutt (1980) appear to support this view; they find that the mixing layer begins to evolve in a self-similar way after some time, and that although there are structures present that extend across the flow, they are somewhat disorganized (skewed and branched). It is certainly true that ordinary statistical measures do not make clear the cross-stream extent of these structures.

In naturally occuring flows, and flows in machines, the possibility of encountering fully developed turbulence is relatively remote (with the exception of the boundary layer). Jet noise is produced by the part of the jet within a few diameters of the nozzle; mixing layers of interest in the design of slots and flaps are quite close to their origin. We may consequently expect that these flows will be characterized by much more energetic and well-organized structures than are present in the academically interesting fully-developed turbulence. Whether these structures will be as well-organized as those currently under investigation by the coherent-structures community is an open question; it seems likely that the flows occuring in machines and in nature will be much more disturbed than those currently under investigation, and hence that the structures will lose some of their organization.

Whether more or less organized, and more or less energetic, it is clear that some form of organized structure is present in turbulent shear flows of all types. While it is of academic interest to calculate these structures in fully developed turbulence, it is not usually of practical importance; in most cases, the organized structures contain a sufficiently small part of the total energy, and play a sufficiently secondary role in transport, that calculations made, say, by second order modeling, which ignores the existence of these structures, are satisfactory. In the flows of more practical importance, however, which are for the most part young, the organized structures play an undeniable role, and a calculation technique that takes their presence into account is necessary. We will discuss such a technique below.

We have previously proposed a technique for identifying these structures (Lumley, 1967, 1970; Payne, 1966; Bakewell, 1966) which uses conventional statistical approaches. I am a little uneasy about the use of conditioned sampling, since it is necessary to introduce the prejudices of the experimenter in order to supply the condition. To quote Loren Eisely (1979; p. 199), "Man, irrespective of whether he is a theologian or a scientist, has a strong tendency to see what he hopes to see". I will show below that one can find in statistical data irrelevant structures with high probability; I cannot call them non-existent, since they are there, but they are formed by chance juxtaposition of other, relevant, structures, and have no significance. They are rather like the birds in an Escher woodcut (Hofstadter, 1980), that are formed by the spaces between other birds. Of course, I cannot generalize; in the hands of a superb experimenter, probably even the most unsatisfactory technique will not lead one astray. In addition, the experimentalist cannot be a *tabula rasa*, as Liepmann points out (private communication): if the experiment is to be a success, the experimentalist must introduce his prejudices in one way or another. There are, finally, experimental situations that are clearer physically than others, where there is less likeihood of error. Other things being equal, however, I prefer a less prejudicial technique for identifying the presence of coherent structures; specifically, I prefer the technique I will describe below.

The use of conventional statistical approaches in general has recently been very much criticised. Laufer, for example, has said (during the present meeting), "[The use of conventional statistical approaches] never got us anywhere". I find this an immoderate view, and feel that its implication, that these approaches should therefore be abandoned, is unwarrented. It is probably true that conventional statistical approaches did not help much to elucidate the organized structure of shear flows, but that is not the only problem in turbulence; I believe they have been very helpful in other respects. That they have not been helpful in this respect I believe is attributable not to an inherent shortcoming in these approaches, but to their misuse. Relatively few measurements of moments higher than second have been made, and these have not usually been tied to the physics. We shall show below that much of the information which is felt to be missing from the conventional statistical descriptions may be found in the third and fourth moments. Cantwell (1981) says, "To an investigator of the 1920s or 1930s, turbulence was essentially a stochastic phenomenon...", and later, "The last twenty years ...have seen a growing realization that ...most turbulent shear flows are dominated by ...motions that are not random". Cantwell to the contrary notwithstanding, most turbulence investigators still consider turbulence to be essentially stochastic, or random, in nature; this does not mean that there are not organized structures, but that these structures occur at times and places, and with strengths and shapes, that jitter, the extent of the jitter depending on the flow situation. It is this jitter that requires the use of what is called boot-strapping by the coherent structures people - the search for a delay that will maximize the correlation. This technique works as well, of course, to improve the correlation of an irrelevant signal as of a relevant one, and has a history in statistics of being extremely dangerous (in the sense that, if a finite piece of the variable is in question, a lag can be found at which the correlation is as good as desired, regardless of the relation or lack of it).

Cantwell (1981) says further that "...there does not exist a unique relationship between the correlation tensor and the unsteady flow that produces it". While this is formally true, it is misleading. As we shall show below, the correlation tensor gives a decomposition of the unsteady flow that produces it which is unique except for phase, which must be determined from third moments. That is, the structure of the organized motions (except for phase) is determined by the correlation tensor - other aspects of the occurrence of the motions must be determined from higher order statistics.

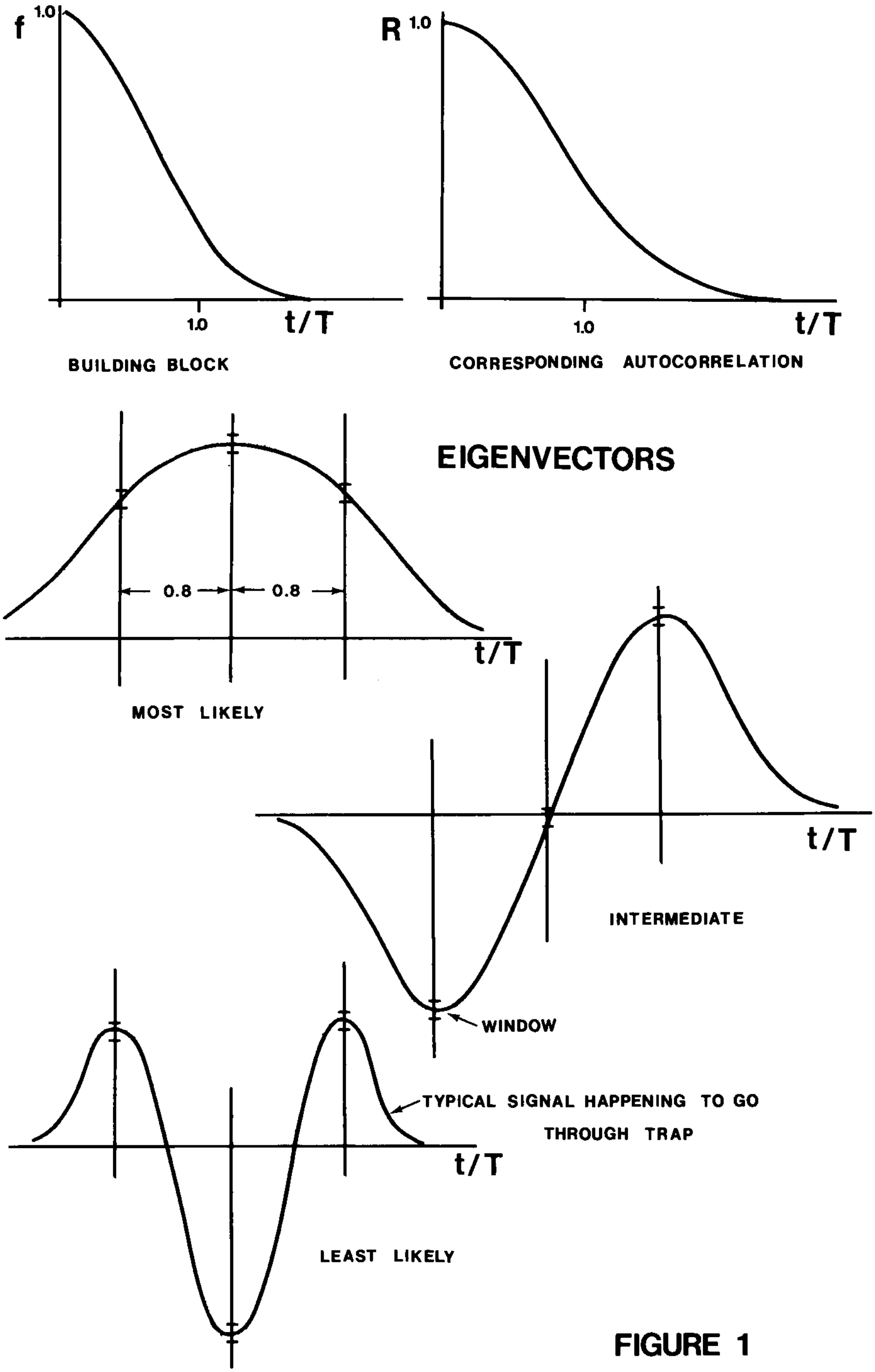
f
1.0
1.0
t/T
BUILDING BLOCK
R
1.0
1.0
t/T
CORRESPONDING AUTOCORRELATION
EIGENVECTORS
0.8
0.8
t/T
MOST LIKELY
t/T
INTERMEDIATE
WINDOW
TYPICAL SIGNAL HAPPENING TO GO THROUGH TRAP
t/T
LEAST LIKELY

FIGURE 1

2. AN OBJECT LESSON.

Let us examine the possibility of "identifying" an irrelevant structure in a stochastic signal. We may consider a signal which is composed of building blocks in the following manner:

$$u(t) = \int f(t-\tau)g(\tau)d\tau \qquad (2.1)$$

where f(t) is a deterministic function and g(t) is a stochastic function, independent in non-overlapping intervals and with a Gaussian probability density for amplitude. For our function f(t) we may select a Gaussian bell-shaped curve (figure 1), so that a realization of the random function u(t) might look like figure 2:

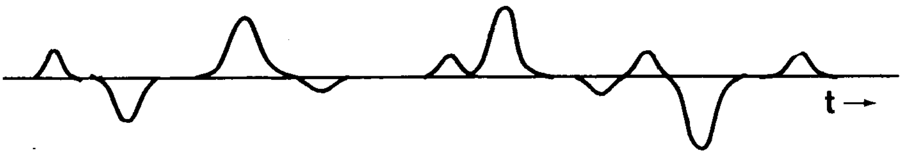

Figure 2. A realization of a function composed of randomly located bell-curves with Gaussian amplitudes.

To approach this problem, we may now consider the probability that a realization of this function passes through a series of windows spaced equally along the axis. Call the times at which the windows are located t_k, $k = 1,\ldots,n$, and the values of the function at those times $u(t_k) = u_k$. What we wish is the probability that u_1 has approximately one value, u_2 has another, and so on, or

$$\Pr\{c_k \leq u_k < c_k+\Delta c_k,\ k = 1,\ldots,n\}$$
$$\sim \exp[-R_{ij}^{-1}c_i c_j/2] = P\{\underline{c}\} \qquad (2.2)$$

where the form may be written down directly since we know that the amplitude behavior of u(t) is Gaussian. The quantity appearing in the exponent of equation (2.2) is the autocorrelation matrix:

$$R_{ij} = \overline{u_i u_j} \tag{2.3}$$

We may now ask what trap, or collection of windows, is most probable. We would expect that this would be something like the building blocks of which we constructed the function. If we extremize the probability for a fixed mean-squarevalue of the trap $c_i c_i = c^2$ we obtain a classical eigenvalue equation for the autocorrelation matrix:

$$R_{ij} X_j^{(k)} = \lambda^{(k)} X_i^{(k)}, \qquad X_i^{(k)} X_i^{(\ell)} = \delta_{k\ell} \tag{2.4}$$

For our particular case, with traps consisting of three windows spaced 0.8 integral scales apart, we have computed the eigenvectors, which are shown in figure 1. It is seen that the most likely is indeed similar to the building blocks of u(t). The second and third eigenvectors can be seen to correspond to the accidental occurrence of either two or three of the fundamental building blocks in juxtaposition, with alternating signs. Any trap can be constructed of these building blocks. In particular, a trap that is proportional to an eigenvector has an especially simple probability:

$$P\{c\underline{X}^{(k)}\} = \exp[-c^2/2\lambda^{(k)}] \tag{2.5}$$

Finally, we may integrate over the amplitude, to obtain the probability that u(t) passes through a trap of given shape regardless of amplitude:

$$\int P\{c\underline{X}^{(k)}\}dc \sim [\lambda^{(k)}]^{1/2} \tag{2.6}$$

The eigenvalues of our three-window trap are given by

$$\lambda^{(1)} = 1.92,\ \lambda^{(2)} = 0.870,\ \lambda^{(3)} = 0.214 \tag{2.7}$$

Hence, the ratio of the probability of the most likely to that of the next most likely is only 1.49, and to the least likely is only 3.00. If the experimenter set his trap to catch a function like the third eigenvector, which is the least likely event he could look for, he will find one for every three of the (most likely) building blocks.

As the number of windows in the trap is increased, the calculation, of course, becomes more complex. The result, however (the square root of the ratio of the first to the second eigenvalues), does not seem to be particularly sensitive to the number of windows in the trap, so long as the same range of the variable is covered, nor does it seem to be particularly sensitive to the exact form of the autocorrelation function (i.e.- to the exact form of the building block).

The moral I would like to draw from this is that it is dangerous to go hunting for a rare bird in a random field, since the field is filled with not-so-rare birds which only appear to exist.

3. A NON-PREJUDICIAL APPROACH

Let us consider a vector function of a single variable, $u_i(x_2)$, and let us suppose that $\underline{u}$ is inhomogeneous, and that the energy is integrable:

$$\int \overline{u_i u_i} dx_2 < \infty \tag{3.1}$$

We will deal with the homogeneous situation, and with combined situations, later. This material and that in sections 4. and 7. can be found in Lumley (1967, 1970). Now, when we think we see an organized structure in the random function $\underline{u}$, one way of testing this hypothesis is to take the correlation of the random function with the proposed candidate structure; find out, that is, how nearly parallel the proposed structure and the random function are in function space:

$$a = \int \phi_i^* u_i dx_2 / [\int \phi_i \phi_i^* dx_2]^{1/2} \tag{3.2}$$

where ( )* indicates the complex conjugate, which we include for generality. The quantity in (3.2) has been normalized by the amplitude of the candidate structure so that the projection of the candidate on the random function will not be affected by the amplitude of the candidate, only by its shape. If we adopt a probabilistic approach, we are interested not in the value of (3.2) in a single realization, but in some statistical measure of (3.2) over the ensemble of realizations. The simplest statistical measure is the mean square of the absolute value. Now, instead of guessing at various candidate structures and testing each one, we can ask if there is a structure that will maximize the mean square magnitude of the projection. This is a well-defined extremization problem:

$$\int R_{ij}(x_2,x_2')\phi_j^{(n)}(x_2')dx_2' = \lambda^{(n)}\phi_i^{(n)}(x_2) \qquad (3.3)$$

where the kernel is the autocorrelation matrix

$$R_{ij}(x_2,x_2') = \overline{u_i(x_2)u_j(x_2')} \qquad (3.4)$$

There is not just one, but a denumerable infinity of solutions which are orthogonal, and can be normalized:

$$\int \phi_i^{(p)}\phi_i^{*(q)}dx_2 = \delta_{pq} \qquad (3.5)$$

which is to say that the structures of various orders have nothing in common with each other. The random function $\underline{u}$ can be represented in terms of the eigenfunctions:

$$u_i(x_2) = \sum a_n\phi_i^{(n)}(x_2), \quad a_n = \int u_i\phi_i^{*(n)}dx_2 \qquad (3.6)$$

where the random coefficients of different orders are uncorrelated, and their mean square value is given by the eigenvalues:

$$\overline{a_n a_m^*} = 0,\ n \neq m; = \lambda^{(n)},\ n = m \qquad (3.7)$$

This is probably the most significant part of the representation theorem: the random function may truly be reconstructed from these structures with random coefficients. Note that the representation converges optimally fast - the first coefficient is (in mean square) as large as possible; of the remainder that could not be incorporated in the first term (because it was orthogonal to it) the coefficient of the next term is as large as possible, and so forth. There are a number of other properties that need not concern us here (which may be found in Lumley 1970): there is a representation for the autocorrelation matrix in terms of the eigenfunctions, the eigenvalues are positive (from their definition), and their sum converges, there are straightforward ways of calculating the eigenvalues and eigenfunctions, and even of estimating how many terms in the series are necessary for a representation (essentially the ratio of the length scale characteristic of the inhomogeneity to that of the energy containing eddies, a ratio which is seldom more than three).

This is a non-prejudicial way of extracting organized structures from an inhomogeneous random function, on an energy-weighted basis; that is, the structures are those that contribute most to the energy. This representation is known in the literature of probability theory as the proper orthogonal decomposition theorem.

4. HOMOGENEOUS DIRECTIONS

The representation of section 3, while useful, does not directly address the situation we discussed in section 2, in which the random function was stationary or homogeneous. If we apply the decomposition of section 3 to such a situation, we find that the eigenfunctions are no longer discrete, but there is now a continuous spectrum of them, and they are in fact the Fourier modes. The proper orthogonal decomposition theorem reduces to the harmonic orthogonal decomposition theorem. While a Fourier representation is, of course, useful for many purposes, it suffers from the disadvantage that the eigenfunctions are not confined principally to one region of space or time; they are not eddies, in the sense in which one usually thinks of them, since they do not correspond to physical entities that one can see in flow visualization.

Fortunately, another decomposition exists which is appropriate for homogeneous directions. This is an outgrowth of the shot-effect expansion (see Rice, 1944; Lumley, 1970). Any homogeneous function may be written as

$$u_i(x) = \int f_i(x-x')g(x')dx' \tag{4.1}$$

where $f_i(x-x')$ is a deterministic function and $g(x')$ is a stochastic function. One has a certain amount of freedom in choosing how much of the behavior of $\underline{u}$ to put in $\underline{f}$, and how much to put in g. The usual choice, and the one that seems most convenient, is to make g white; that is, uncorrelated in non-overlapping intervals. Specifically, if we pick

$$\overline{g(x)g(x')} = \delta(x'-x) \tag{4.2}$$

then we have Campbell's theorem:

$$R_{ij}(\xi) = \int f_i(x)f_j(x+\xi)dx \tag{4.3}$$

If we take Fourier transforms, we can write for the spectrum

$$\Phi_{ij}(k) = \hat{f}_i\hat{f}_j^* \tag{4.4}$$

where we are indicating the Fourier transform of $\underline{f}$ by a circumflex. It is clear that (4.4) determines $\underline{f}$ or its Fourier transform to within a phase angle:

$$\hat{f}_\alpha = \Phi_{\alpha\alpha}^{1/2}\exp[i\theta(k)] \tag{4.5}$$

This is not to say that the phase angle is undeterminable, simply that it is undeterminable from second order statistics. In the next section we will show how the phase angle may be determined. Once this has been done, we will have exactly what we want - a representation of the random function as a series of coherent structures occurring at stochastic locations with stochastic strengths. Just as we cannot determine the phase from the second order statistics, we also cannot say anything about the way in which the structures repeat - with overlapping, or without overlapping, and with what periodicity, etc., on the basis of second order statistics. Again, this is not to say that this information is undeterminable, simply that it cannot be determined from second order statistics. In section 5 we will consider the retrieval of phase information, and in section 6 the retrieval of information on overlap and spacing.

5. RETRIEVING PHASE INFORMATION

Let us consider a signal composed of building blocks as in (2.1). We will make a specific choice of f(t) as

$$f(t) = (d/dt)[2\pi]^{-1/2}\sigma^{-1}\exp[-t^2/2\sigma^2] \tag{5.1}$$

so that the building blocks are odd functions; these might correspond, for example, to measurement of transverse velocity due to a random sprinkling of vortices. We will take the function g(t) to be independent in nonoverlapping intervals (so that these building blocks are truly distributed chaotically), and to have the following probability structure:

$$g(t)dt = 1 \text{ with prob } \mu dt, = 0 \text{ with prob } 1-\mu dt \tag{5.2}$$

The vortices are all of the same sign and strength, now; although the weighting function does not have zero mean, the building-block function does, so that the function u(t) does also. Now, we obtain from Campbell's theorem

$$S = \hat{f}\hat{f}^* = \omega^2\exp[-\sigma^2\omega^2] \tag{5.3}$$

where S is the spectrum. Hence, we have for the transform of f

$$\hat{f} = S^{1/2}\exp[i\theta], \quad S^{1/2} = |\omega|\exp[-\sigma^2\omega^2/2] \tag{5.4}$$

Actually, although we cannot determine the phase angle from the spectrum, we know from the definition of f in (5.1) that it is -90° for positive frequencies and +90° for negative frequencies. That the proper phase angle makes a considerable difference can be seen from an examination of figure 3; if f is reconstituted with the proper amplitude but with zero phase angle, the resulting function is even, rather than odd, and would give quite a misleading impression.

We can recover the phase information from the triple correlation. By an extension of Campbell's theorem we can write (with our assumptions on the behavior of g(t)):

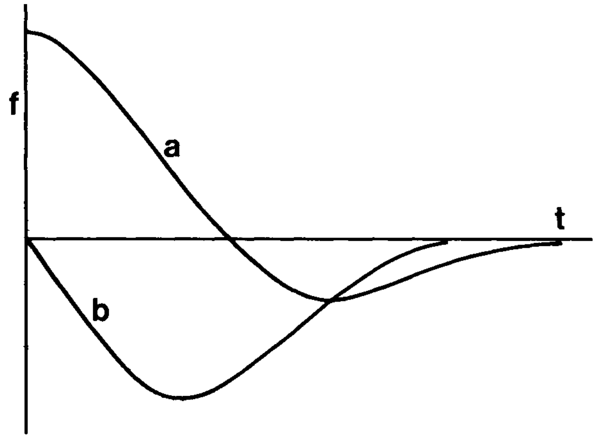

Figure 3. The function f reconstituted with the correct amplitude, but in the case (a)with zero phase angle. (b) has the correct phase.

$$\overline{u(t)u(t+\tau_1)u(t+\tau_2)} = \mu\int f(x)f(x+\tau_1)f(x+\tau_2)dx$$
$$= R_2(\tau_1,\tau_2) \qquad (5.5)$$

Taking the double Fourier transform of this, we obtain the bi-spectrum described and discussed by Rosenblatt and his co-workers (Brillinger & Rosenblatt, 1967a,b; Rosenblatt, 1966; Rosenblatt & Van Ness, 1965):

$$S_2(\omega_1,\omega_2) = \iint \exp[i\omega_1\tau_1+i\omega_2\tau_2]R_2(\tau_1,\tau_2)d\tau_1 d\tau_2 \qquad (5.6)$$

$$= \mu\hat{f}(\omega_1)\hat{f}(\omega_2)\hat{f}^*(\omega_1+\omega_2) \qquad (5.7)$$

$$= \mu S^{1/2}(\omega_1)S^{1/2}(\omega_2)S^{1/2}(\omega_1+\omega_2) \qquad (5.8)$$
$$\times\exp[i\{\theta(\omega_1)+\theta(\omega_2)-\theta(\omega_1+\omega_2)\}]$$

Notice that if the phase angle is proportional to the frequency, the exponent vanishes. However, an exponent proportional to frequency corresponds to a simple time delay; the shape of the building block is not affected, but the time at which the center occurs is shifted. In the general case, we can move the three frequencies as close together as possible, and solve (5.8) to obtain the phase angle of f in terms of that of the bi-spectrum. We can make the upper and lower frequencies define essentially the edges of an eddy (see Tennekes & Lumley, 1972) if we make

$$\omega_1 = \alpha\omega,\ \omega_2 = \omega,\ \omega_1+\omega_2 = 1/\alpha,\ 1/\alpha = 1+\alpha,\quad \alpha = 0.62 \tag{5.9}$$

The upper and lower frequencies are then equally divided logarithmically from the center frequency. If we define

$$\theta(\omega) = a+b\omega+c\omega^2+d\omega^3+\ldots \tag{5.10}$$
$$\psi = \theta(\alpha\omega) + \theta(\omega) - \theta(\omega/\alpha)$$

then we find

$$\psi = a-2c\alpha\omega^2-d(3\alpha^2+3\alpha)\omega^3-\ldots \tag{5.11}$$

so that all the coefficients in the expansion of the phase angle can be determined except for the linear term (which is irrelevant, as discussed above). It is necessary, of course, to carry out the analysis separately for positive and negative frequencies. In our case, c and d are zero, and a takes on the value -90° for positive frequencies and +90° for negative, so that the phase angle is determined.

We have admittedly simplified things a bit by our assumptions regarding the structure of u(t) in this example. In more complex situations, the right hand side of (5.7) would be multiplied by the bi-spectrum of g(t). Since in a general representation, only the second order properties of g have been selected up to this point, we can specify g(t) more precisely by making further choices now, consistent with the measurements.

6. OVERLAP AND RECURRENCE

Hussain (1981) refers to the suggested decomposition (2.1), but says that it is, unfortunately, not useful because it implies that the eddies overlap. In fact, this is not true. The only assumption that is made is that the function $g(t)$ is uncorrelated in non-overlapping intervals. This does not imply, however, that it is independent in non-overlapping intervals. It is well-known (see, for example, Tennekes & Lumley 1972) that lack of correlation and independence are not the same thing. Let us consider the probability density for $g(t)$:

$$B(u,v;\tau)dudv = \Pr\{u<g(t)dt<u+du, v<g(t+\tau)dt<v+dv\} \qquad (6.1)$$

The autocorrelation of $g(t)$ is given by

$$\overline{g(t)g(t+\tau)}dt^2 = \int uvBdudv \qquad (6.2)$$

If the probability density B is symmetric for non-zero lags, then $g(t)$ will be uncorrelated. It is fairly easy to display a B which suppresses the occurrence of a second structure for a time T, but is statistically independent thereafter. In the following example, by making the starred standard deviation as small as we like relative to the unstarred, we may suppress as much as we like the occurrence of two structures closer than T:

$$B(u,v;\tau) = \{\exp[-u^2/2\sigma^2 - v^2/2\sigma_*^2] + \exp[-u^2/2\sigma_*^2 - v^2/2\sigma^2]\}/4\pi\sigma\sigma_*, 0<t<T \qquad (6.3)$$

$$= \{\exp[-u^2/2\sigma^2]/\sigma + \exp[-u^2/2\sigma_*^2]/\sigma_*\} \times \{\exp[-v^2/2\sigma^2]/\sigma + \exp[-v^2/2\sigma_*^2]/\sigma_*\}/8\pi, \quad t>T \qquad (6.4)$$

This is sketched in figure 4 for a moderate ratio of the standard deviations.

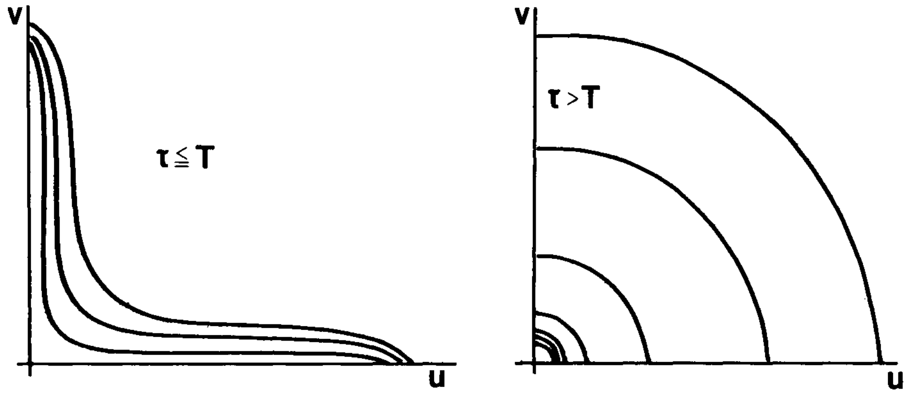

Figure 4. Qualitative sketch of iso-probability contours for the joint density of equations (6.3) and (6.4), for a moderate value of the ratio of standard deviations.

Integrating over one of the variables, we obtain the expression for the single density:

$$\int B(u,v;\tau)dv = \{\exp[-u^2/2\sigma^2]/\sigma + \exp[-u^2/2\sigma_*^2]/\sigma_*\}/2(2\pi)^{1/2} \quad (6.5)$$

and it is evident that the function has an intermittency factor of 0.5. That is, the function g(t) is essentially on 50% of the time and off 50% of the time. In figure 5, we show a rough sketch of this density. Of course, this choice of densities by no means exhausts the possibilities; densities could be easily constructed in which the loss of suppression was gradual, or in which the density did not become statistically independent after the time T, but became increasingly stimulative, so that as time went on, a second occurrence would become more and more certain. These are just two among many interesting variations. So far as data processing is concerned, once the phase of f(t) is known, relation (2.1) can be Fourier transformed and the individual values of the transform of u(t) divided by the transform of f(t) to give the transform of g(t), which can then be Fourier transformed back to real space; now, from the real g, the probability density can be obtained.

In the case of our simple example, the information on recurrence is most easily obtained from higher moments of g(t). Specifically, the fourth moment is the lowest one to contain this information. If we form the autocorrelation of the square

$$\rho_2 = \overline{[g^2(t)-\overline{g^2(t)}][g^2(t+\tau)-\overline{g^2(t+\tau)}]}/\overline{[g^2(t)-\overline{g^2(t)}]^2} \quad (6.6)$$

we obtain (for a vanishingly small ratio of the standard deviations) the sketch in figure 6:

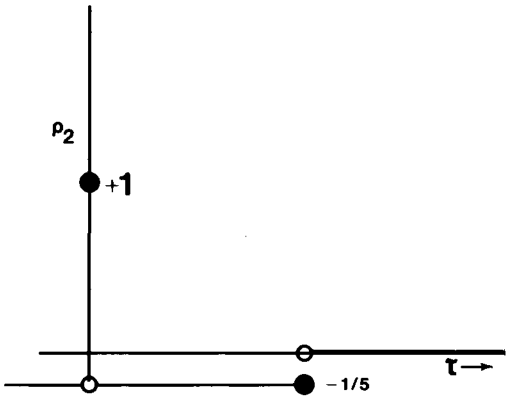

Figure 5, (Left). A sketch of the single density corresponding to equation (6.5). Figure 6, (Right). A sketch of equation (6.6) for the model of eq. (6.3-4).

The fact that this correlation drops to -1/5 instead of -1 is only a reflection of the fact that it is a fourth order correlation; the value of -1/5 indicates perfect anti-correlation, or complete suppression, if you will. If, instead of becoming statistically independent, the process became stimulative (as opposed to suppressive), we would expect the correlation to rise to plus one. The moral is, that a model such as (2.1) makes no suppositions regarding overlap and recurrence; this information is contained in the higher statistics.

7. NEARLY PARALLEL SHEAR FLOWS

In jets, wakes, shear layers, boundary layers, and so forth, we have a situation that is approximately homogeneous in the x_1 and x_3 directions, and approximately stationary, while being of integrable energy in the x_2 direction. In such a flow, we must combine the representation of section 2. with that of section 3. We can write for the fluctuating velocity field

$$u_i = \sum_n \int \exp[i(k_1x_1+k_3x_3+\omega t)]\phi_i^{(n)} a_n dk_1 dk_3 d\omega \tag{7.1}$$

where phi is the deterministic eigenfunction and a_n is the random coefficient. The coefficients of different orders are uncorrelated:

$$\overline{a_n a_m^*} = \lambda^{(n)} \delta_{nm} \tag{7.2}$$

and the mean square value is the eigenvalue:

$$\int \Phi_{ij}(x_2,x_2')\phi_j^{(n)}(x_2')dx_2' = \lambda^{(n)} \phi_i^{(n)}(x_2) \tag{7.3}$$

where the kernel is the cross-spectral density (we are suppressing the wavenumbers and frequencies):

$$\overline{\hat{u}_i(k_1,\ldots)\hat{u}_j^*(k_1',\ldots)} = \delta(k_1-k_1')\ldots\Phi_{ij} \tag{7.4}$$

$$u_i = \int \exp[i(k_1x_1+\ldots)\hat{u}_i dk_1 dk_3 d\omega$$

Again, the eigenfunctions are orthogonal, and can be normalized:

$$\int \phi_i^{(p)} \phi_i^{(q)*} dx_2 = \delta_{pq} \tag{7.5}$$

We may identify the organized structure by

$$\begin{aligned}\int \exp[i(k_1x_1+\ldots)\phi_i^{(1)} a_1 dk_1 \ldots \\ = \int f_i(x_1-\xi_1,\ldots)g(\xi_1,\ldots)d\xi_1\ldots\end{aligned} \tag{7.6}$$

where the weighting function is uncorrelated in non-overlapping intervals:

$$\overline{g(\xi_1,\ldots)g(\xi_1',\ldots)} = \delta(\xi_1-\xi_1')\ldots \tag{7.7}$$

Finally, the deterministic function which is sprinkled randomly is given by:

$$\begin{aligned} f_i &= \int \exp[i(k_1x_1+\ldots)][\lambda^{(1)}/(2\pi)^3]^{1/2}\phi_i^{(1)}dk_1\ldots \\ &= \int \psi_i^{(1)}dk_1\ldots \text{ say} \end{aligned} \tag{7.8}$$

From a physical point of view, these shear flows present the most interesting case. If we are to develop equations to predict the form of these organized structures, it is for the function of (7.8) that we must do so.

8. DYNAMICAL EQUATIONS

We may approach the dynamical equations directly. If we begin with the 1-component of the equation for the fluctuating velocity:

$$\dot{u}_1+u_{1,1}U+U'u_2+u_{1,j}u_j = p_{,1}/\rho+\nu u_{1,jj} \tag{8.1}$$

Multiplying by a_k we obtain the Fourier transform:

$$\overline{a_ku_i} = \int \exp[ik_1x_1+..]\phi_i^{(k)}\lambda^{(k)}dk_1\ldots \tag{8.2}$$

Hence, we must multiply equation (8.1) by a_k, average, take the inverse Fourier transform, and divide by the eigenvalue. This leads to the following:

$$\begin{aligned} &ik_1(U+\omega/k_1)\phi_1^{(n)}+U'\phi_2^{(n)} \\ &+\textstyle\sum_{p,q}\int D_j\phi_1^{(p)}(k_1',\ldots)\phi_j^{(q)}(k_1-k_1',\ldots)\overline{a_ka_pa_q}/\lambda^{(k)}dk_1'\ldots+\ldots \end{aligned} \tag{8.3}$$

where we have written only the terms on the left hand side, since these are the ones that will cause difficulties. We are now faced with the classical closure problem of turbulence, though in a slightly unfamiliar form: we must somehow terminate this coupled hierarchy of equations. One possibility that comes to mind is a sort of Heisenberg approach (Monin & Yaglom, 1975), in which the higher order eigenfunctions are supposed to act like a viscosity on the lower order ones. This has greater likelihood of success here than it does in the homogeneous case (where it nevertheless gives results that are not qualitatively bad) because here there is between eigenfunctions a sort of spectral gap - successive eigenfunctions have relatively little in common with each other, being orthogonal.

There is, however, a simpler approach which may produce adequate results. We may begin from the equation for the Reynolds stress tensor, and introduce there one of the closures that is used in second order modeling, obtaining from this closed equation for the Reynolds stress an equation for the eigenfunction (from the integral equation for the eigenfunction: by multiplying by the eigenfunction and integrating). The Reynolds stress is given by

$$R_{ij} = \overline{u_i(\underline{x}',t')u_j(\underline{x},t)} \tag{8.4}$$

and the equation for the Reynolds stress (indicating only the terms for which closure is necessary; the pressure-strain correlation and the viscous terms do not require modeling in this approach):

$$\dot{R}_{ij}+\ldots+(\overline{u_i'u_ju_k})_{,k}+\ldots \tag{8.5}$$

Now, the most elementary sort of mixing length approach would replace the turbulent transport term by

$$\overline{u_i'u_ju_k} = -(R_{ij,\ell}\overline{u_ku_\ell}T)_{,k} \tag{8.6}$$

This is a slight extension of the closures used in second order modeling, since those closures do not ordinarily pertain to the Reynolds stress at two different points and times. If the implications of (8.6) are examined carefully, it is found that it is not disastrously unreasonable, except for the behavior of the time correlation, which does not decay under the assumption (8.6). We may introduce a further simplification if we note that in a first approximation the transport coefficient in (8.6) is constant over a cross-section of all the nearly parallel shear flows. Using assumption (8.6), and proceeding as described above, we may obtain an equation for the function defined by equation (7.8):

$$\dot{\psi}_i^{(1)}+\psi_{i,j}^{(1)}U_j+U_{i,j}\psi_j^{(1)} = -p_{,i}^{(1)}/\rho+(\psi_{i,\ell}^{(1)}\overline{u_k u_\ell}T)_{,k}\,, \qquad \psi_{i,i}^{(1)} = 0 \tag{8.7}$$

The viscous terms have been neglected, since they are small relative to the transport terms arising from the triple correlations. Note that this equation has the form of the linearized Navier-Stokes equation. It is not truly linear, since the transport coefficient in fact is given by

$$\overline{u_i u_j} = \sum_n \int \lambda^{(n)} \phi_i^{(n)} \phi_j^{(n)*} dk_1 \ldots \tag{8.8}$$

The equation is consequently cubic. Note that the exact equation (8.3) is quadratic; our closure scheme has changed a quadratic equation to a cubic one, and that is bound to have interesting consequences. The time scale which appears in (8.6) and in (8.7) is proportional to the ratio of turbulent energy to dissipation rate.

The attractive feature of equation (8.7) is that it may be solved by a sort of self-consistent field approximation. We take, as a first approximation, an isotropic turbulent field; the transport coefficient is then an isotropic eddy viscosity, which we take constant as a first approximation. Equation (8.7) then leads directly to the classical Orr-Sommerfeld equation; that is, we must solve the linear stability problem for the flow in question. We can accept only neutral disturbances, since the frequency and wavenumber must be real; the frequency and wavenumber arise not by an arbitrary Fourier expansion, but from the representation theorem which embodies the fact that the turbulence is stationary and homogeneous. We do not know the value of the eddy viscosity, but we know that the Reynolds number based on it must be at least above the critical value in order to have a solution with real frequency and wavenumber. If the Reynolds number is above the critical value, we will have a spread of such frequencies and wavenumbers. The disturbance will presumably grow slowly (so as not to violate the stationarity assumption); as it grows, the eddy viscosity will grow, and the value of the Reynolds number will be reduced. The most energetic possible disturbance will then be that which has reduced the Reynolds number to its minimum critical value, at which point only a single disturbance (a single wavenumber and frequency) will remain. This is the marginal stability idea that has been put forward speculatively by many writers (Malkus, 1956; Malkus & Veronis, 1958; Lessen, 1978): that the eddy viscosity should have such a value that the turbulent flow should be marginally stable to small disturbances. However, we have had to make no assumptions to arrive here, other than the closure assumption.

To implement the self-consistent field approximation, we should now construct a transport coefficient from our newly obtained critical disturbance, and solve the resulting equation again to obtain a better approximation to the critical disturbance. However, there is probably no point in this until a number of questions have been satisfactority resolved. For example: at the critical state, we are reduced to a single disturbance. This is not a true representation of reality - real flows are disturbed by a spread of wavenumbers and frequencies. How can this be arranged? We need a better closure of the Reynolds stress equation, one that better represents the time behavior of the correlation, since this surely will have an impact on the behavior of the eigenfunctions. We have not said anything about how to specify a length scale (essentially, we have a velocity scale-to specify a time scale, we need a length scale). It is not clear how to weight the eigenfunctions of various orders in reconstructing the transport coefficient - if only the first eigenfunction were present, our technique will permit us to determine the first eigenvalue, but if more than one is present, it is not clear how to determine the higher ones. Finally, it is not clear how to deal with flows, like the mixing layer, that have no minimum critical Reynolds number. Presumably many of these questions would be answered by a better closure, which would behave more sensibly. What we have here is a very crude model which nevertheless shows us a crude mechanism.

9. APPLICATIONS TO MEASUREMENTS

The decomposition of section 7. is being applied to the early part of a circular jet by W. K. George, Jr. and his colleagues at SUNY Buffalo. They hope to shed light on the nature of the coherent structures that are responsible for noise generation in this flow. This work is being carried out with the support of the U. S. Air Force Office of Scientific Research.

This same decomposition will also be applied to measurements taken in the sub- and buffer layer of a turbulent boundary layer by Siegfried Herzog and the present author. Briefly, two components of velocity have been measured at 882 point pairs. The spacing increases by factors of two in each direction, the minimum spacing in the cross-stream directions being 0.7 (in dimensionless sublayer units), while that in the streamwise direction is 9.0. Three minutes of data were taken at each location; this should be sufficient to obtain statistics with better than 10% accuracy. These measurements differ from those of Bakewell (1966) in that two components have been measured (whereas Bakewell measured only one), and in the much increased streamwise separation.

We plan to obtain the third component of velocity by use of the continuity equation. That is, after the correlation curves have been interpolated by Fourier series, they can be differentiated, and the unmeasured correlations involving the third component obtained by solution of the differential equation deduced from the continuity condition.

We agree that the amount of data necessary to carry out this sort of decomposition is large. However, we feel (an unproven assertion) that the amount of effort is not greater than that required to obtain equivalent information of the same statistical accuracy by any other means. Many of the measurements obtained by conditioned sampling are not statistically stable, being based on relatively small statistical samples.

The measurements in the sublayer were begun under the sponsorship of the General Hydrodynamics Research Program of the David W. Taylor Naval Ship Research and Development Center, and were completed with the support of the Fluids Engineering Unit of the Applied Research Laboratory at the Pennsylvania State University, where they were carried out. The data Analysis is being carried out with the support of The NASA Ames Research Center (under Dr. G. T. Chapman), making use of the computational facilities of the NASA Langley Research Center, with the collaboration of Dr. T. Gatski.

REFERENCES

Bakewell, H. P. (1966) The viscous sublayer and adjacent wall region in turbulent pipe flow. Ph. D. Thesis. University Park: The Pennsylvania State University.

Bevilaqua, P. M. & Lykoudis, P. S. (1971) Mechanism of entrainment in turbulent wakes. AIAA Journal 9: 1657-1659.

Bradshaw, P. (1966) The effect of initial conditions on the development of a free shear layer. J. Fluid Mech. 26: 225-236.

Brillinger, D. R. & Rosenblatt, M. (1967a) Asymptotic theory of estimates of Kth order spectra. Spectral Analysis of Time Series. (ed. B. Harris). pp. 153-188. New York: Wiley.

Brillinger, D. R. & Rosenblatt, M. (1967b) Computation and interpretation of Kth order spectra. Spectral Analysis of Time Series. (ed. B. Harris). pp. 189-232. New York: Wiley.

Browand, F. K. & Troutt, T. R. (1980) A note on spanwise structure in the two-dimensional mixing layer. J. Fluid Mech. 917: 771-781.

Brown, G. & Roshko, A. (1971) Symposium on turbulent shear flows, p. 23.1. London: Imperial College.

Brown, G. & Roshko, A. (1974) On density effects and large structure in turbulent mixing layers. J. Fluid Mech. 64: 775-816.

Cantwell, B. J. (1981) Organized Motion in Turbulent Flow, in Ann. Rev. Fluid Mech. 13: 457-515. M. Van Dyke, J. V. Wehausen, J. L. Lumley, eds. Palo Alto: Annual Reviews, Inc.

Chevray, R. (1968) The turbulent wake of a body of revolution. J. Basic Engineering (Trans. ASME Series D) 90: 275-284.

Chevray, R. & Kovasznay, L. S. G. (1969) Turbulence measurements in the wake of a thin flat plate. AIAA J. 7: 1641-1643.

Eisely, L. (1979) Darwin and the Mysterious Mr. X: New Light on the Evolutionists. New York: E. P. Dutton.

Hofstadter, D. R. (1980) Goedel, Escher and Bach. New York: Vantage.

Hussain, A. K. M. F. (1981) Coherent structures and studies of perturbed and unperturbed jets, in Proceedings, An international conference on the role of coherent structures in modeling turbulence and mixing. Madrid: University Politechnica. (To appear in Lecture Notes in Physics).

Lessen, M. (1978) On the power laws for turbulent jets, wakes and shearing layers and their relationship to the principle of marginal stability. J. Fluid Mech. 88: 535-540.

Lumley, J. L. (1967) The structure of inhomogeneous turbulent flows. In Atmospheric Turbulence and Radio Wave Propagation, A. M. Yaglom and V. I. Tatarsky, eds., pp. 166-178. Moscow: NAUKA.

Lumley, J. L. (1970) Stochastic Tools in Turbulence. New York: Academic Press.

Malkus, W. V. R. (1956) Outline of a theory of turbulent shear flow. J. Fluid Mech. 1: 521-539.

Malkus, W. V. R. & Veronis, G. (1958) Finite amplitude cellular convection. J. Fluid Mech. 4: 225-260.

Monin, A. S. & Yaglom, A. M. (1975) Statistical Fluid Mechanics. (J. L. Lumley, ed.) Cambridge, MA: M. I. T. Press.

Payne, F. R. (1966) Large eddy structrure of the turbulent wake behind a circular cylinder. Ph. D. Thesis. University Park: The Pennsylvania State University.

Payne, F. R. & Lumley, J. L. (1967) Large eddy structure of the turbulent wake behind a circular cylinder. Physics of Fluids. 10: S194-S196.

Rice, S. O. (1944) Mathematical analysis of random noise. Bell System Tech. J. 23: 282-332.

Rosenblatt, M. (1966) Remarks on higher order spectra. Symposium on multivariate analysis. New York: Academic.

Rosenblatt, M. & Van Ness, X. (1965) Estimation of the bi-spectrum. Ann. Math. Stat. 36:1120-1136.

Roshko, A. (1981) The plane mixing layer; flow visualization results and three-dimensional effects. In Proceedings, An international conference on the role of coherent structures in modelling turbulence and mixing. Madrid: University Politecnica. (To appear in Lecture notes in Physics).

Taulbee, D. & Lumley, J. L. (1981) Prediction of the turbulent wake with a second order closure model. J. Fluid Mech. To be submitted.

Tennekes, H. & Lumley, J. L. (1972) A First Course in Turbulence. Cambridge, MA: M. I. T. Press.

Townsend, A. A. (1947) Measurements in the turbulent wake of a cylinder. Proc. Roy. Soc. A 190: 551-561.

This work supported in part by the U. S. NASA-Ames Research Center under Grant No. NSG-2382, and in part by the U. S. National Science Foundation, Engineering Division, under Grant No. 79-19817.

Sibley School of
Mechanical
and Aerospace
Engineering,
Cornell University
Ithaca, NY 14853