CSE 365: Computer Vision

Image Enhancement in the Frequency Domain

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Summer 2020

Image Enhancement in the Frequency Domain

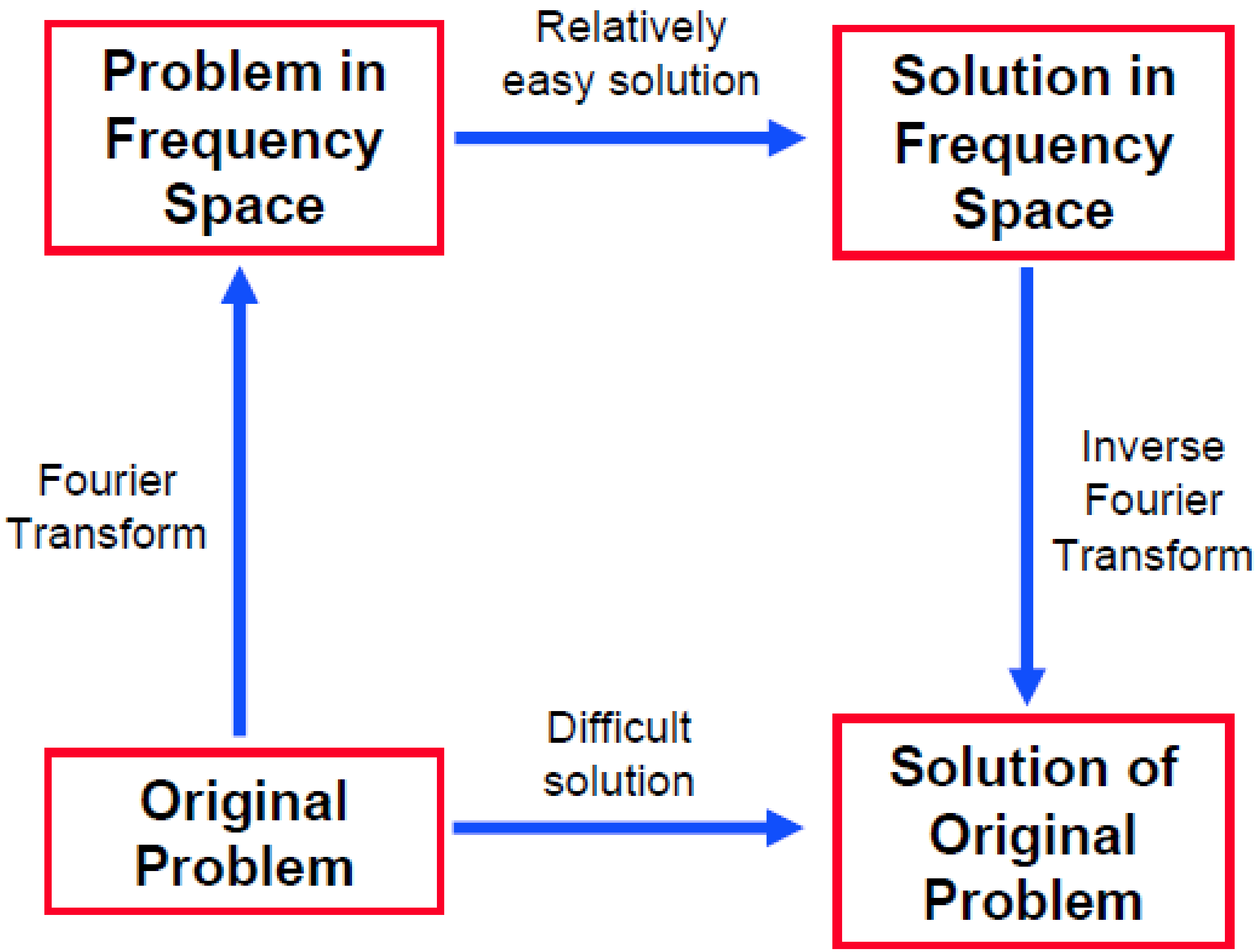

Image Enhancement in the Frequency Domain

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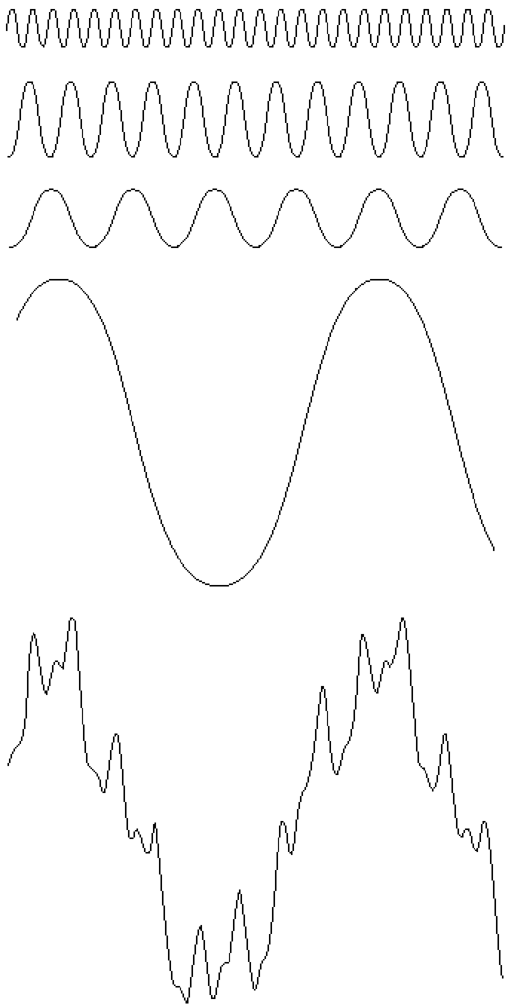


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

The 1D Continuous Fourier Transform

The continuous Fourier transform

$$F(\omega) = \int_x f(x)\ e^{-i2\pi\omega x}\, dx$$

The inverse continuous Fourier transform

$$f(x) = \int_\omega F(\omega)\ e^{i2\pi\omega x}\, d\omega$$

The 1D Discrete Fourier Transform (DFT)

The 1D discrete Fourier transform

$$F(k) = \sum_{x=0}^{N-1} f(x) e^{\frac{-2\pi i k x}{N}}$$

k = 0, 1, 2, ..., N-1

The inverse 1D discrete Fourier transform

$$f(x) = \frac{1}{N} \sum_{k=0}^{N-1} F(k) e^{\frac{2\pi i k x}{N}}$$

x = 0, 1, 2, ..., N-1

The 1D Discrete Fourier Transform (DFT)

$f(x) = [2 \ \ 3 \ \ 4 \ \ 4]$

$$F(0) = \sum_{x=0}^{3} f(x)\, e^{\frac{-2\pi i 0x}{4}} = \sum_{x=0}^{3} f(x)\, 1$$

$$= (f(0) + f(1) + f(2) + f(3)) = (2+3+4+4) = 13$$

$$F(1) = \sum_{x=0}^{3} f(x)\, e^{\frac{-2\pi i x}{4}} = [2e^{0}+3e^{-i\pi/2}+4e^{-\pi i}+4e^{-i3\pi/2}] = [-2+i]$$

$$F(2) = \sum_{x=0}^{3} f(x)\, e^{\frac{-4\pi i x}{4}} = [2e^{0}+3e^{-i\pi}+4e^{-2\pi i}+4e^{-3\pi i}] = [-1-0i] = -1$$

$$F(3) = \sum_{x=0}^{3} f(x)\, e^{\frac{-6\pi i x}{4}} = [2e^{0}+3e^{-i3\pi/2}+4e^{-3\pi i}+4e^{-i9\pi/2}] = [-2-i]$$

DFT of $[2 \ \ 3 \ \ 4 \ \ 4]$ is $[\, 13 \ \ (-2+i) \ \ -1 \ \ (-2-i) \,]$

The 1D Discrete Fourier Transform (DFT)

The Delta Function:

- Let $f(x) = \delta(x)$

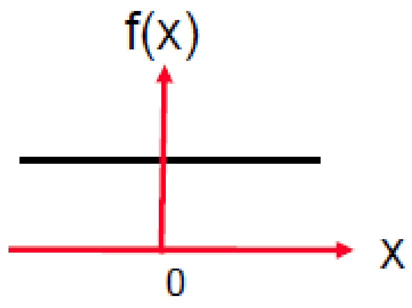
$$F(\omega) = \int_{-\infty}^{\infty} \delta(x) \cdot e^{-i2\pi\omega x}\, dx = 1$$

The 1D Discrete Fourier Transform (DFT)

The Constant Function:

- Let $f(x) = 1$

$$F(\omega) = \int_{-\infty}^{\infty} e^{-i2\pi\omega x}\, dx = \delta(\omega)$$



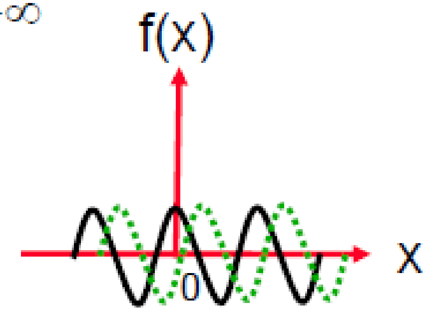
The 1D Discrete Fourier Transform (DFT)

A Basis Function:

- Let $f(x) = e^{i2\pi\omega_0 x}$

$$F(\omega) = \int_{-\infty}^{\infty} e^{i2\pi\omega_0 x} e^{-i2\pi\omega x}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-i2\pi(\omega-\omega_0)x}\, dx = \delta(\omega - \omega_0)$$



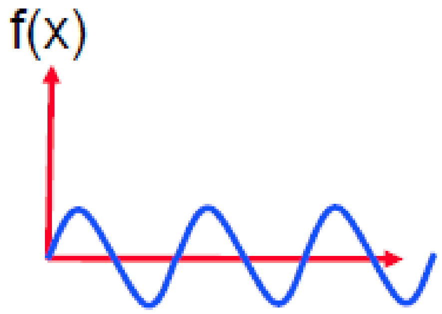
The 1D Discrete Fourier Transform (DFT)

The Cosine wave:

- Let $f(x) = \cos(2\pi\omega_0 x)$

$$F(\omega) = \int_{-\infty}^{\infty} \frac{1}{2}\left(e^{i2\pi\omega_0 x} + e^{-i2\pi\omega_0 x}\right) \cdot e^{-i2\pi\omega x}\, dx =$$

$$= \frac{1}{2}\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$$



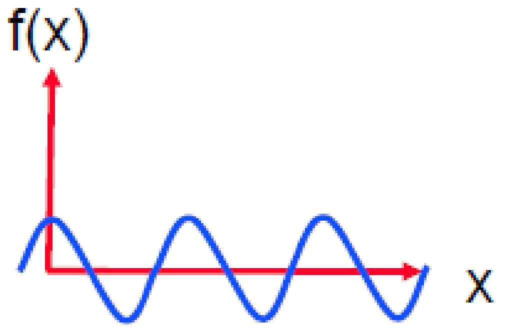
The 1D Discrete Fourier Transform (DFT)

The Sine wave:

- Let $f(x) = \sin(2\pi\omega_0 x)$

$$F(\omega) = \int_{-\infty}^{\infty} \frac{i}{2}\left(e^{-i2\pi\omega_0 x} - e^{i2\pi\omega_0 x}\right) \cdot e^{-i2\pi\omega x}\, dx =$$

$$= \frac{i}{2}\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$$

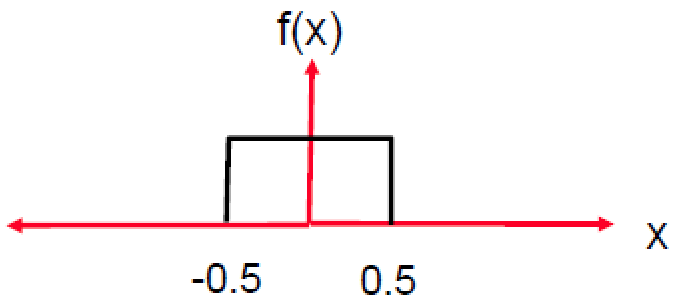


The 1D Discrete Fourier Transform (DFT)

The Window Function (rect):

- Let $\text{rect}_{1/2}(x) = \begin{cases} 1 & \text{if } |x| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$

$$F(\omega) = \int_{-0.5}^{0.5} e^{-i2\pi\omega x}\, dx = \frac{\sin(\pi\omega)}{\pi\omega} = \text{sinc}(\pi\omega)$$



The 2D Discrete Fourier Transform (DFT)

The 2D discrete Fourier transform

$$F(u,v) = \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} f(x,y) e^{-2\pi i (ux/N + vy/M)}$$

$u = 0, 1, 2, \ldots, N\text{-}1$

$v = 0, 1, 2, \ldots, M\text{-}1$

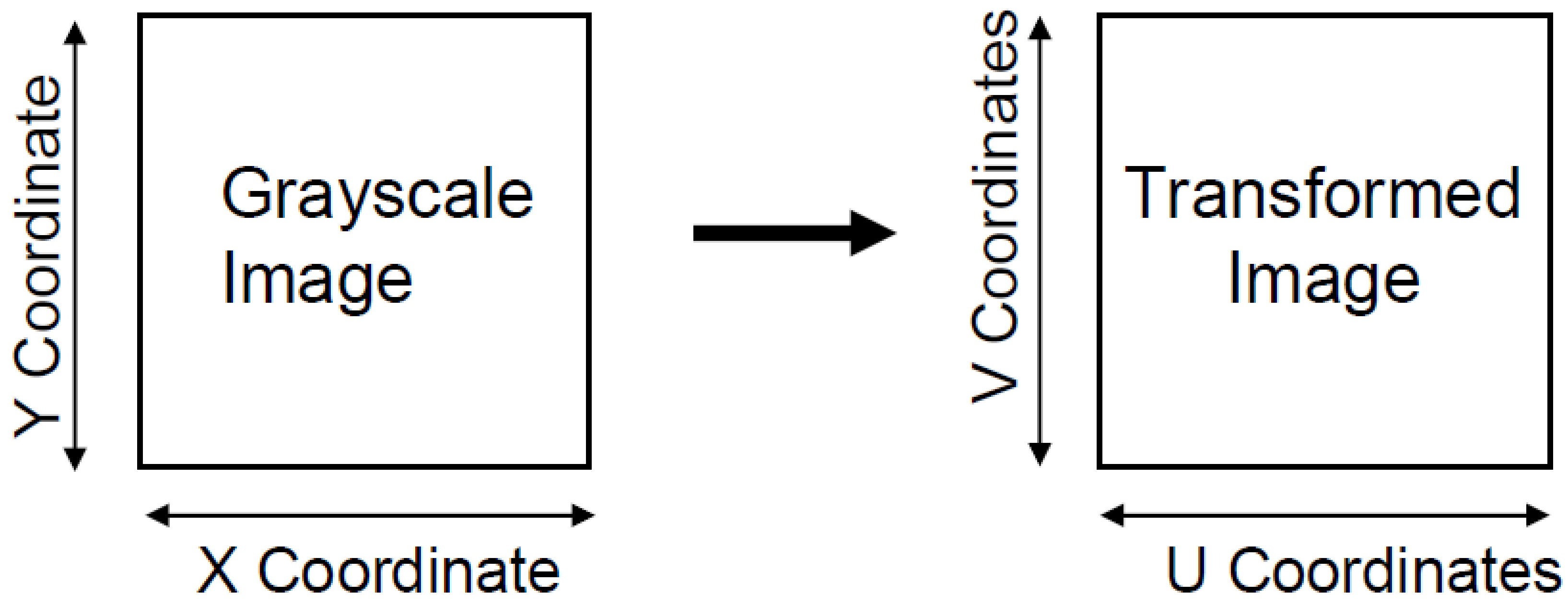
The 2D inverse discrete Fourier transform

$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{N-1} \sum_{v=0}^{M-1} F(u,v) e^{2\pi i (ux/N + vy/M)}$$

$y = 0, 1, 2, \ldots, N\text{-}1$

$x = 0, 1, 2, \ldots, M\text{-}1$

The 2D Discrete Fourier Transform (DFT)



The 2D Discrete Fourier Transform (DFT)

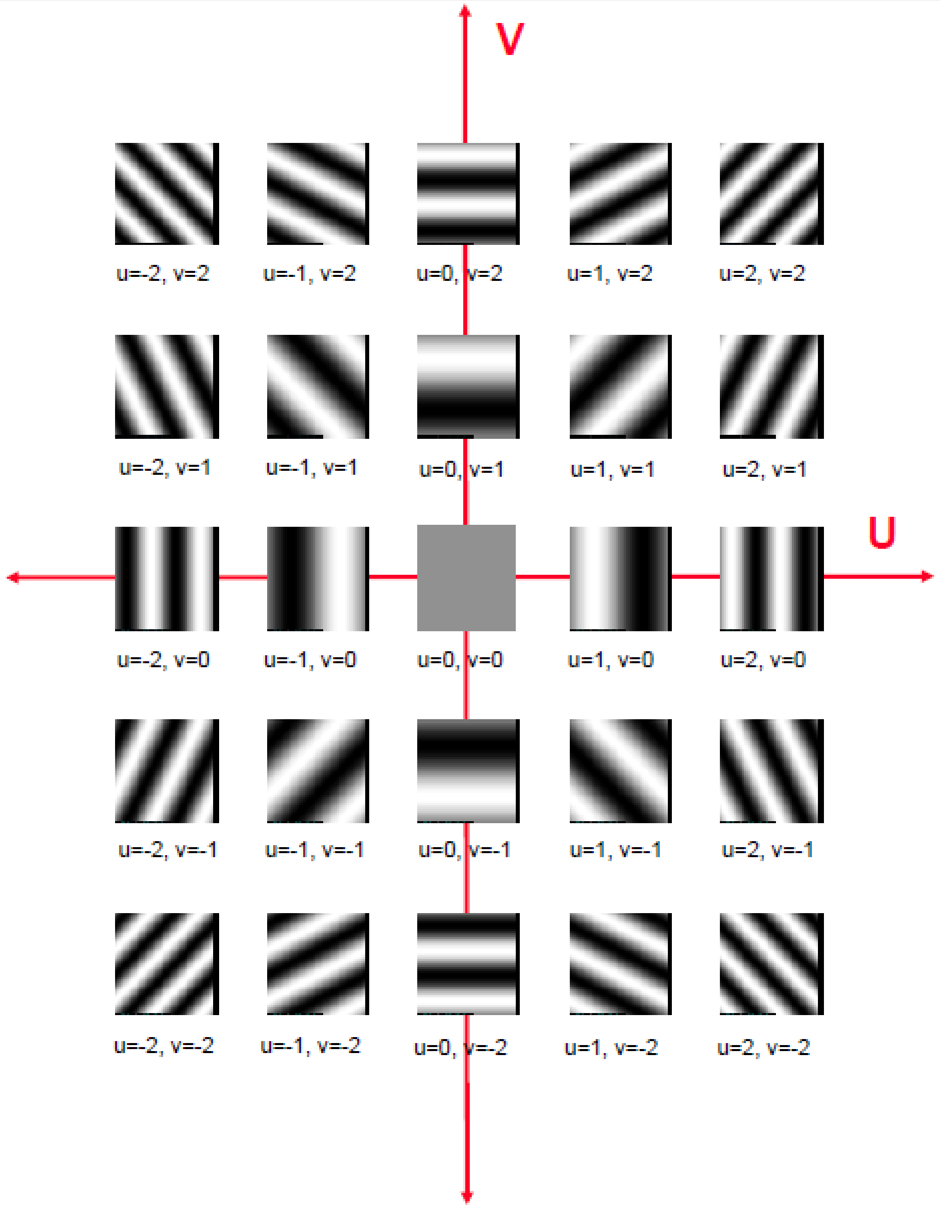


Image Enhancement in the Frequency Domain

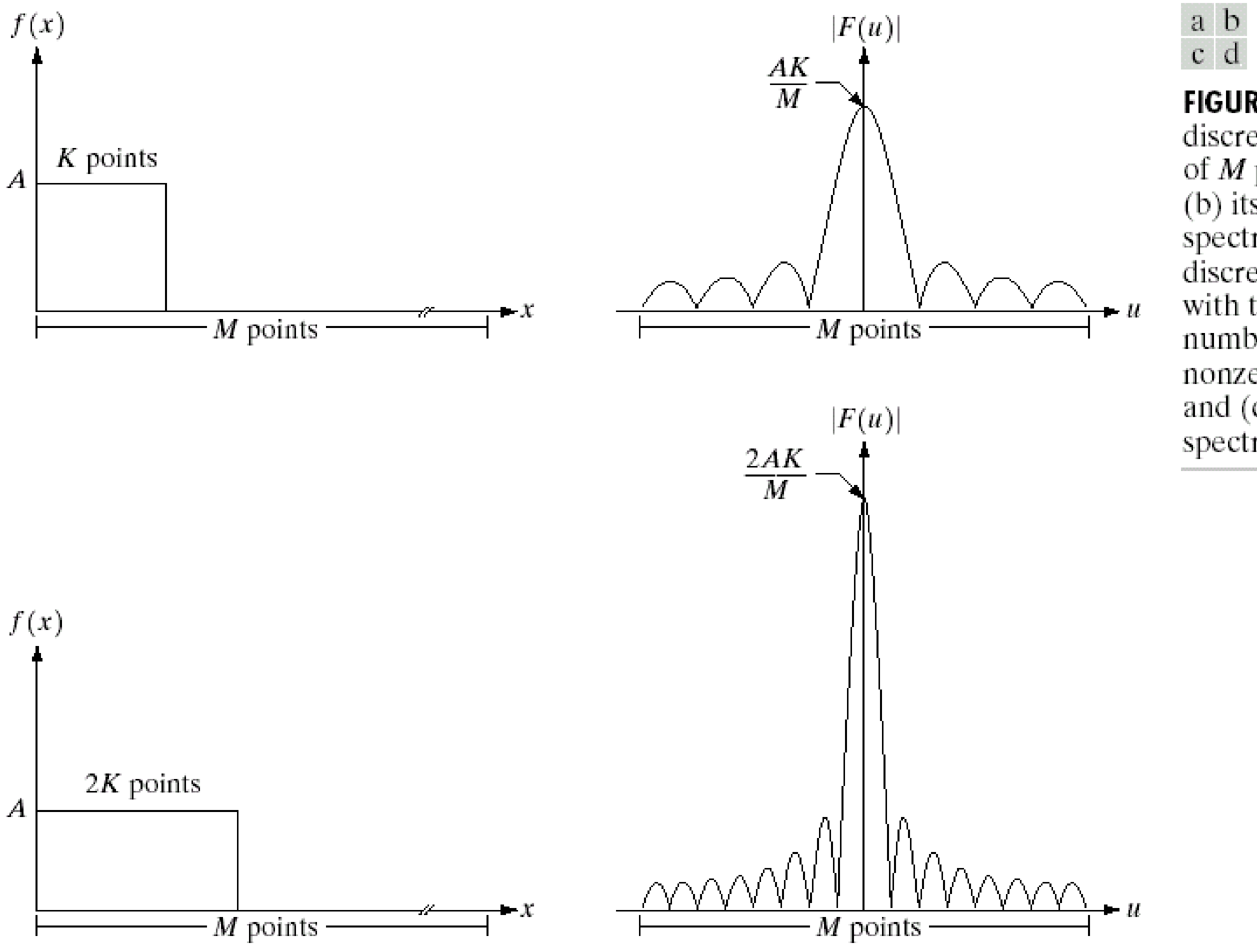
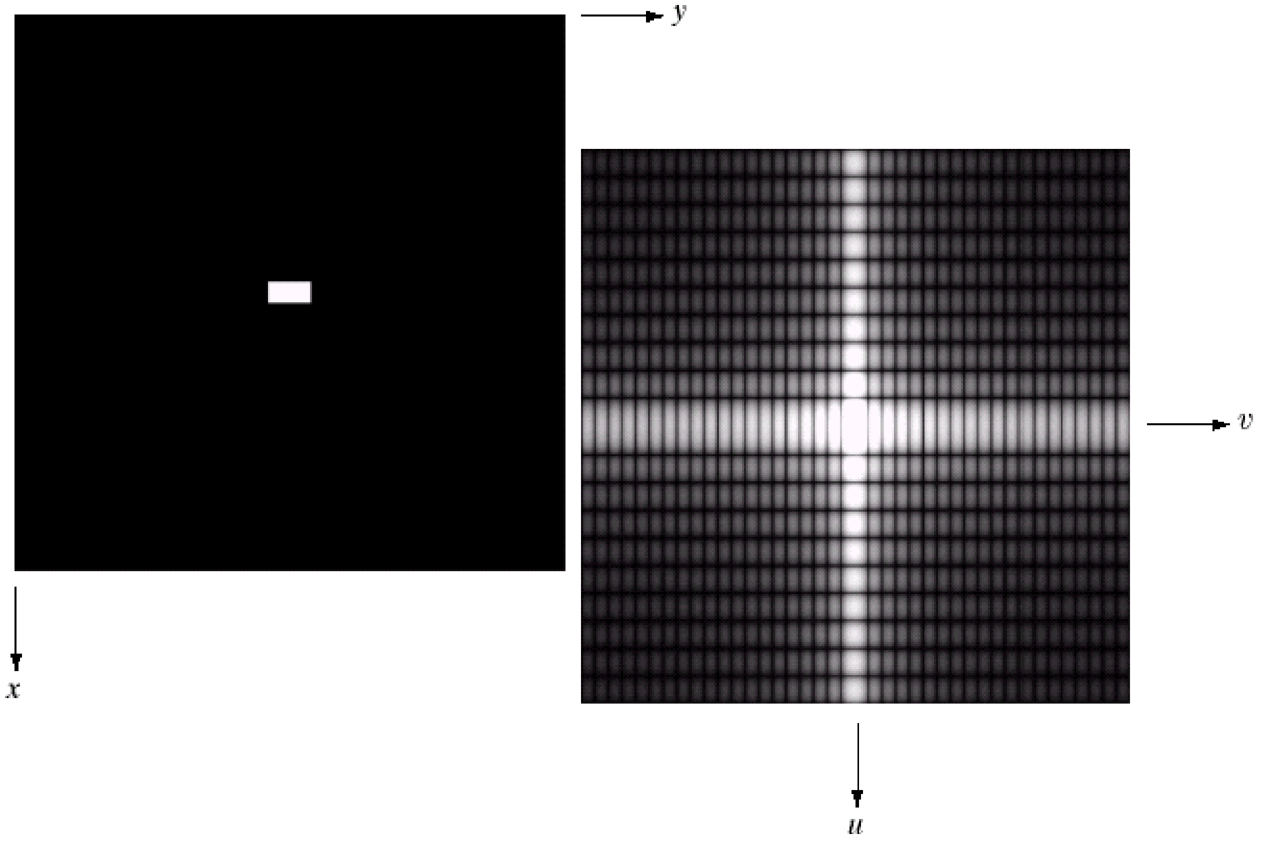


FIGURE 4.2 (a) A discrete function of M points, and (b) its Fourier spectrum. (c) A discrete function with twice the number of nonzero points, and (d) its Fourier spectrum.

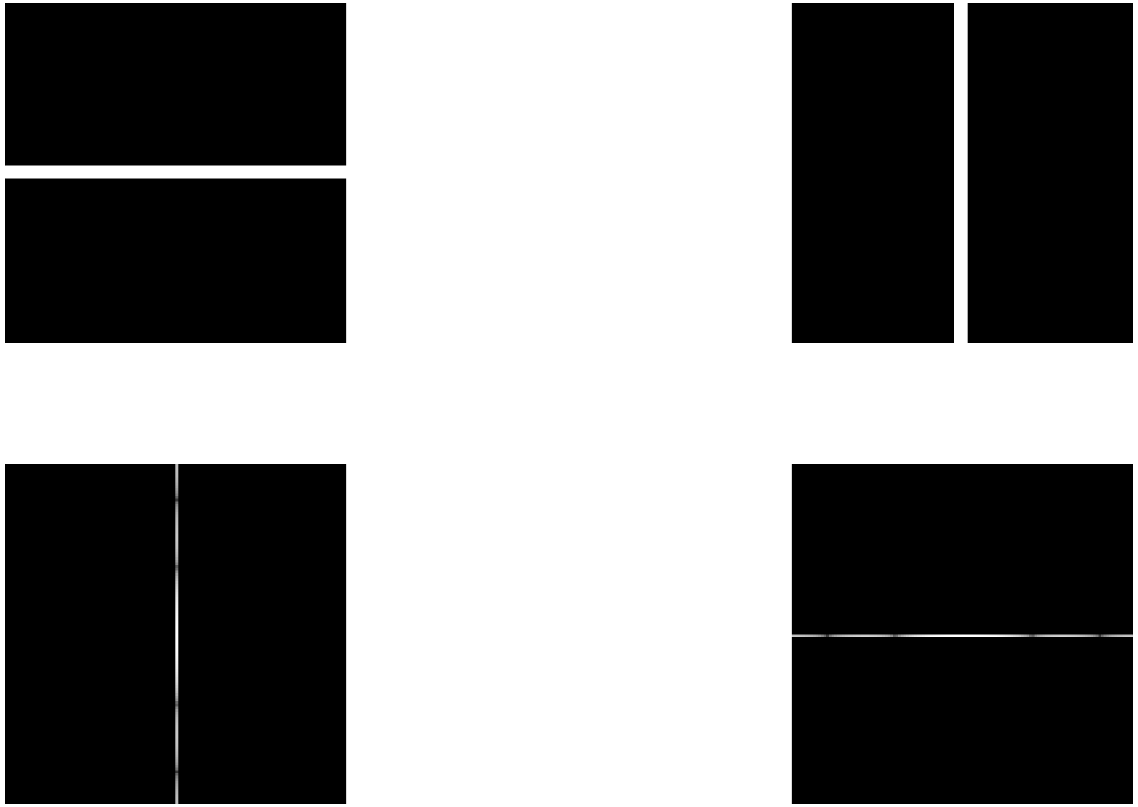
Image Enhancement in the Frequency Domain

a b

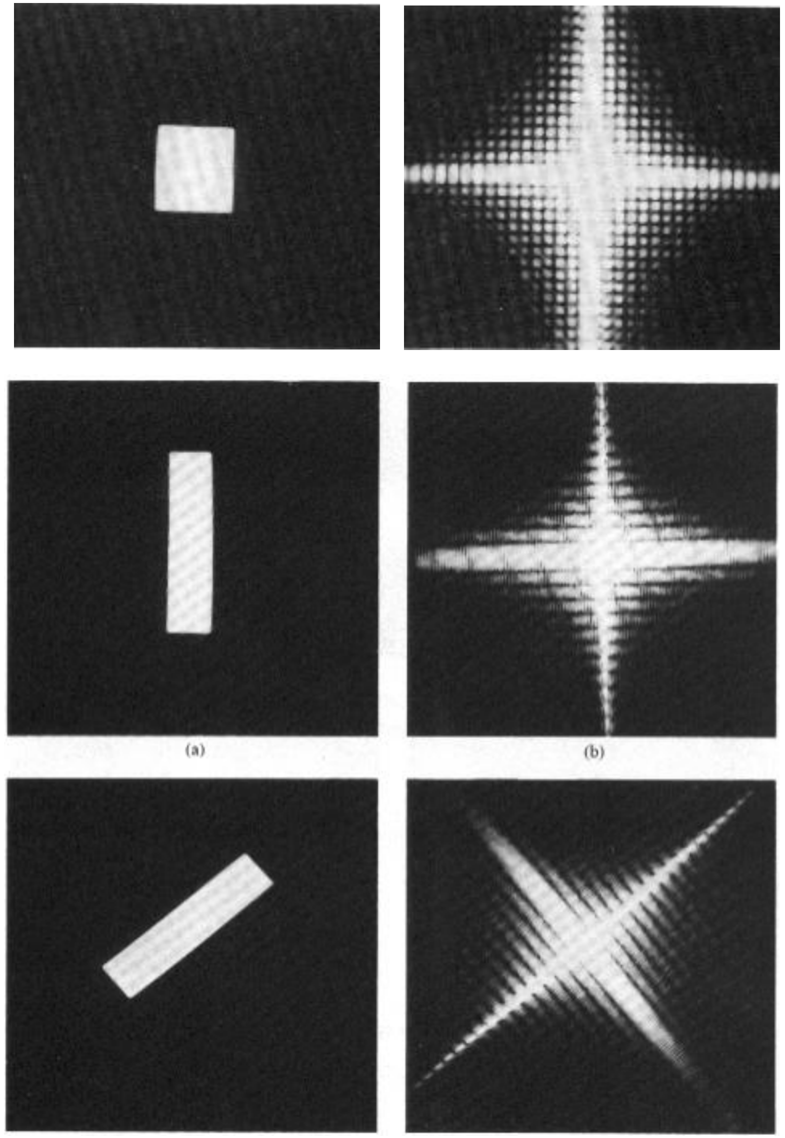
FIGURE 4.3 (a) Image of a 20×40 white rectangle on a black background of size 512×512 pixels. (b) Centered Fourier spectrum shown after application of the log transformation given in Eq. (3.2-2). Compare with Fig. 4.2.



Fourier Transform - Examples



Fourier Transform - Examples



Fourier Transform - Examples

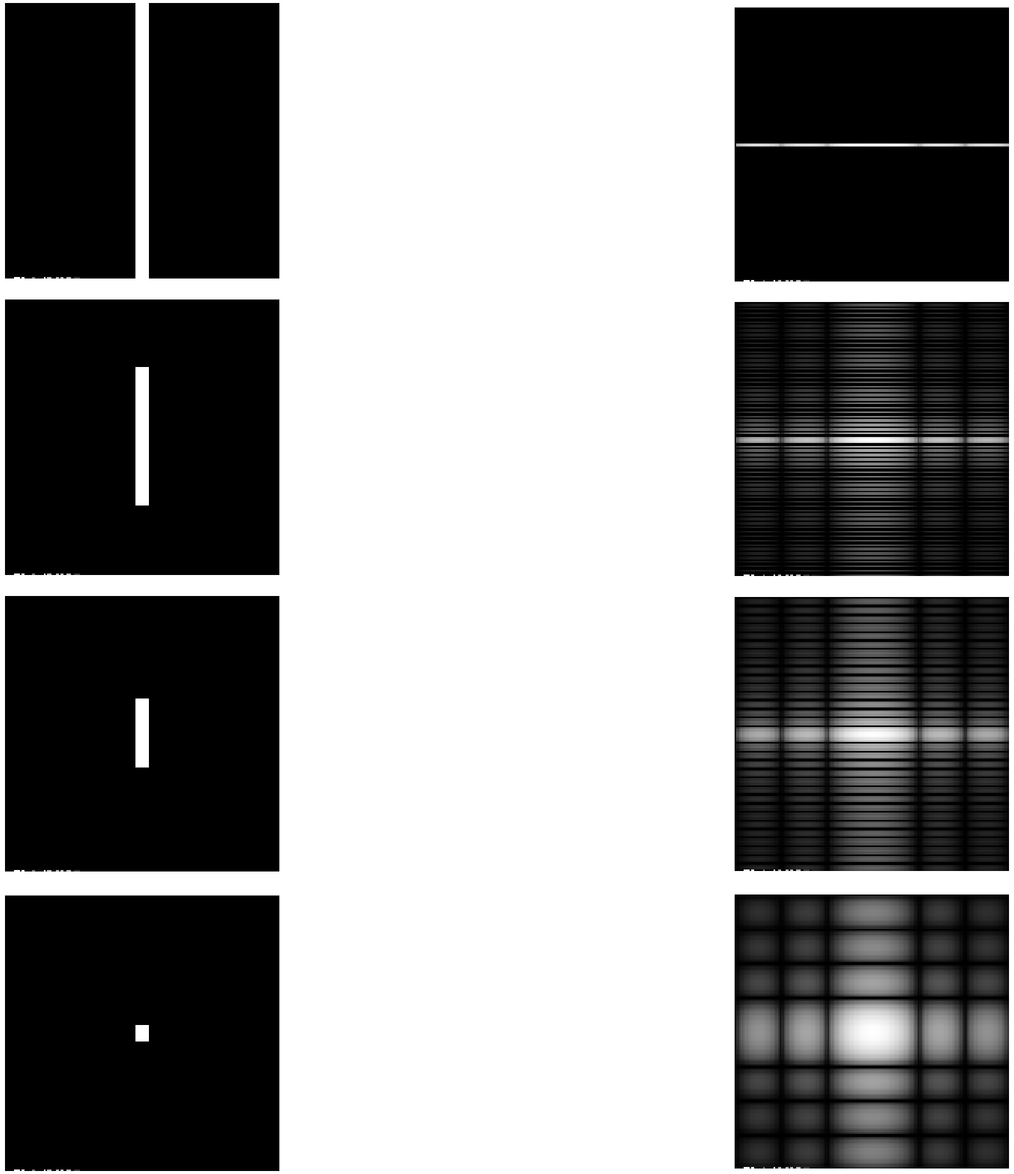
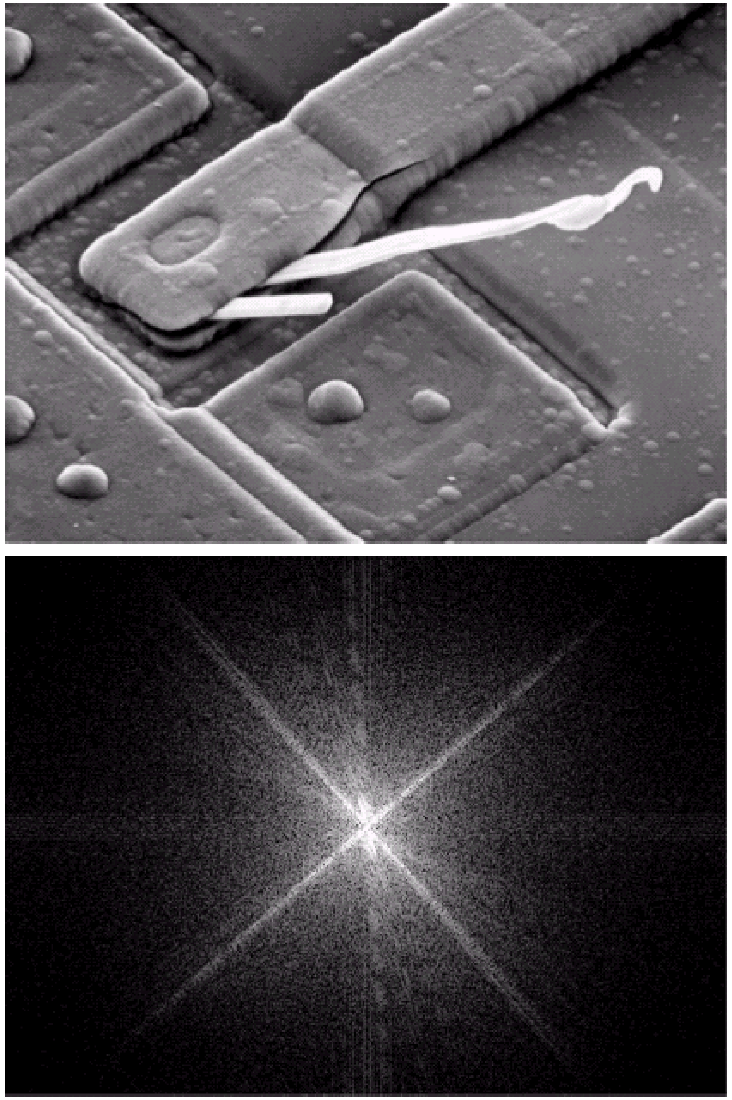


Image Enhancement in the Frequency Domain



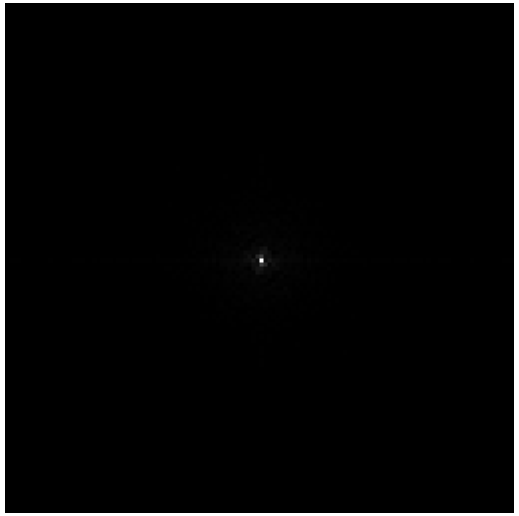
a
b

FIGURE 4.4
(a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

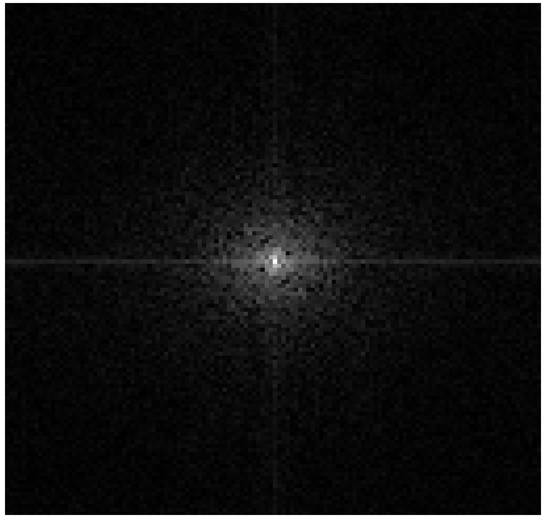
The Fourier Image

Image f

Fourier spectrum |F(u,v)|



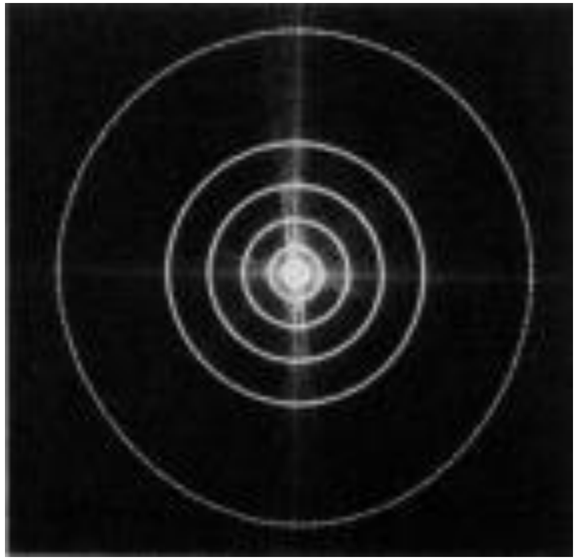
Fourier spectrum log(1 + |F(u,v)|)



Frequency Bands

Image

Fourier Spectrum



Percentage of image power enclosed in circles (small to large) :

90%, 95%, 98%, 99%, 99.5%, 99.9%

Image Enhancement in the Frequency Domain

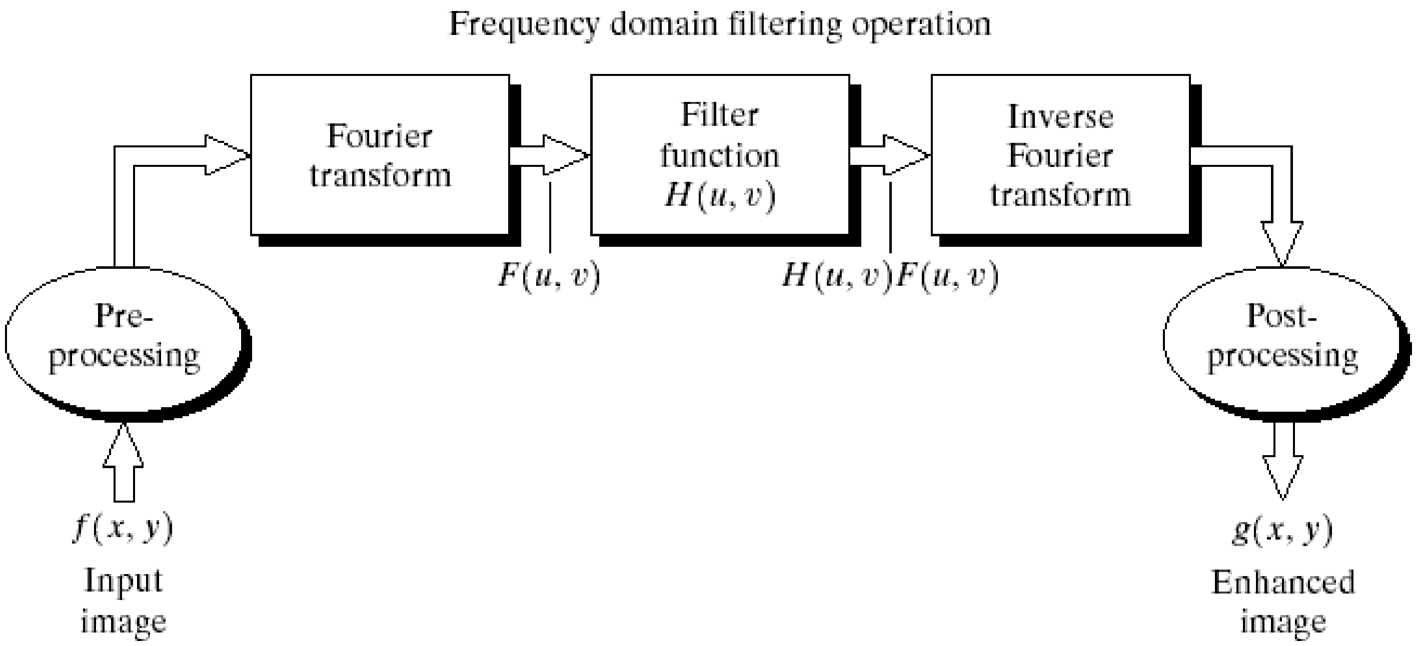


FIGURE 4.5 Basic steps for filtering in the frequency domain.

Image Enhancement in the Frequency Domain

FIGURE 4.6 Result of filtering the image in Fig. 4.4(a) with a notch filter that set to 0 the $F(0, 0)$ term in the Fourier transform.

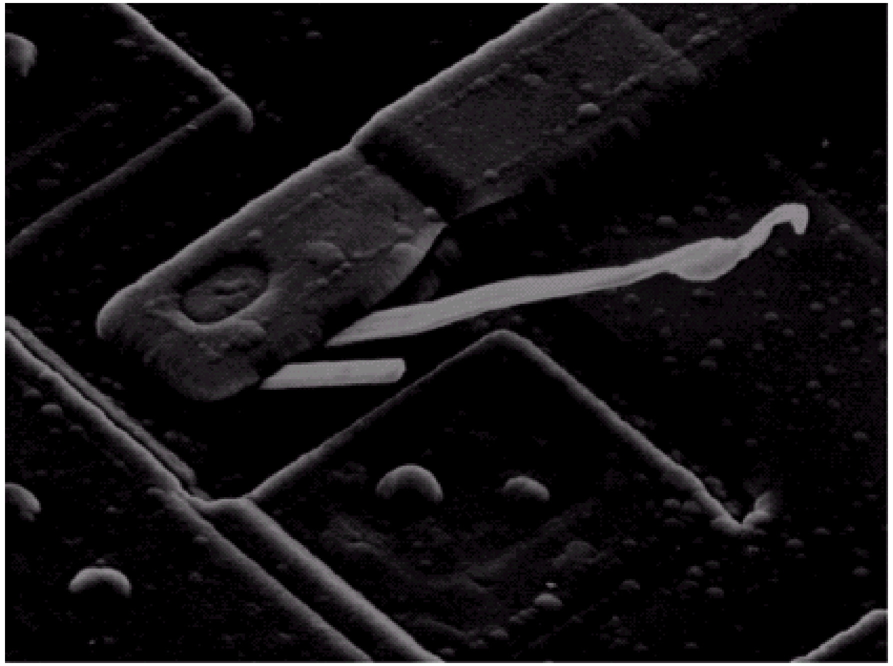


Image Enhancement in the Frequency Domain

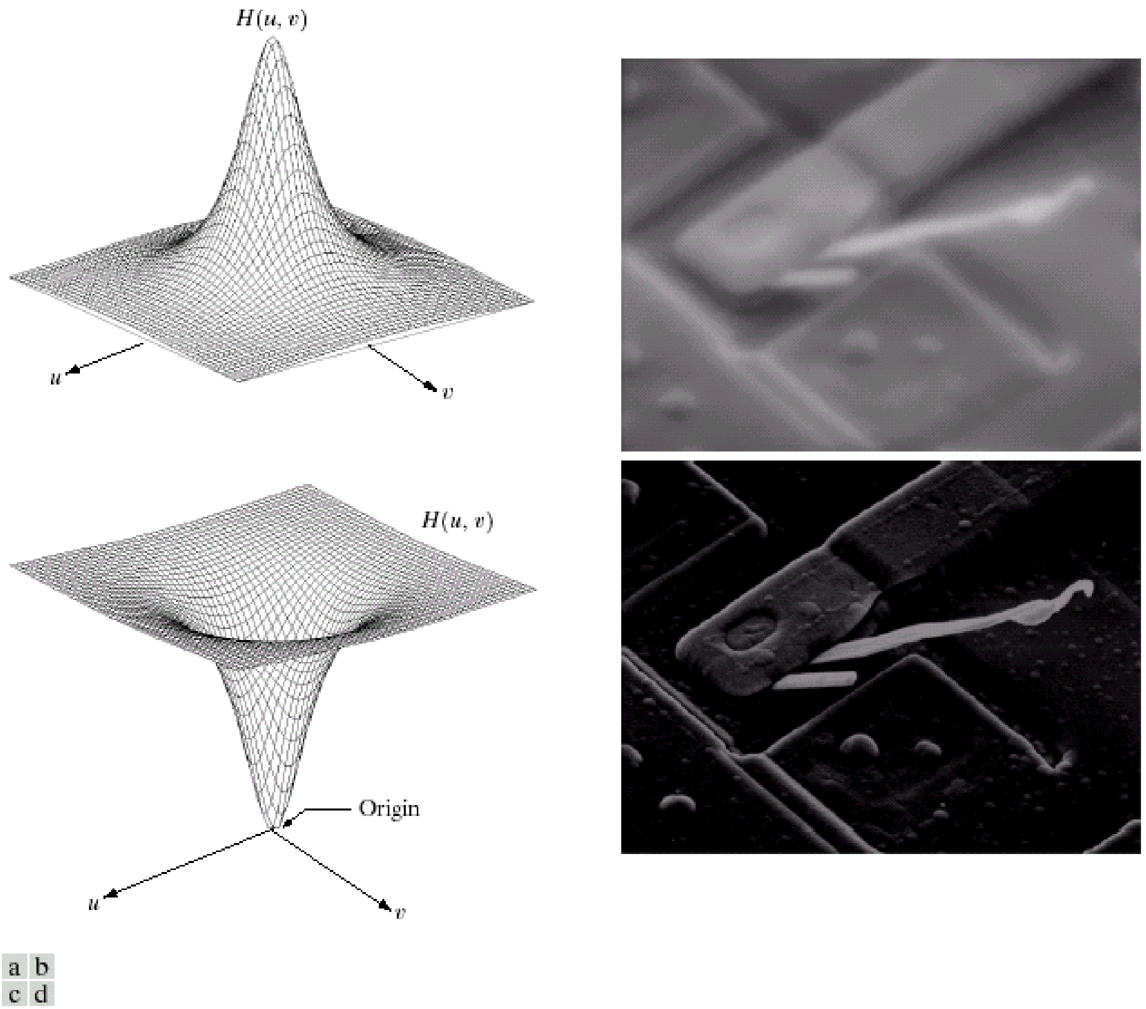


FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

Image Enhancement in the Frequency Domain

FIGURE 4.8
Result of highpass filtering the image in Fig. 4.4(a) with the filter in Fig. 4.7(c), modified by adding a constant of one-half the filter height to the filter function. Compare with Fig. 4.4(a).

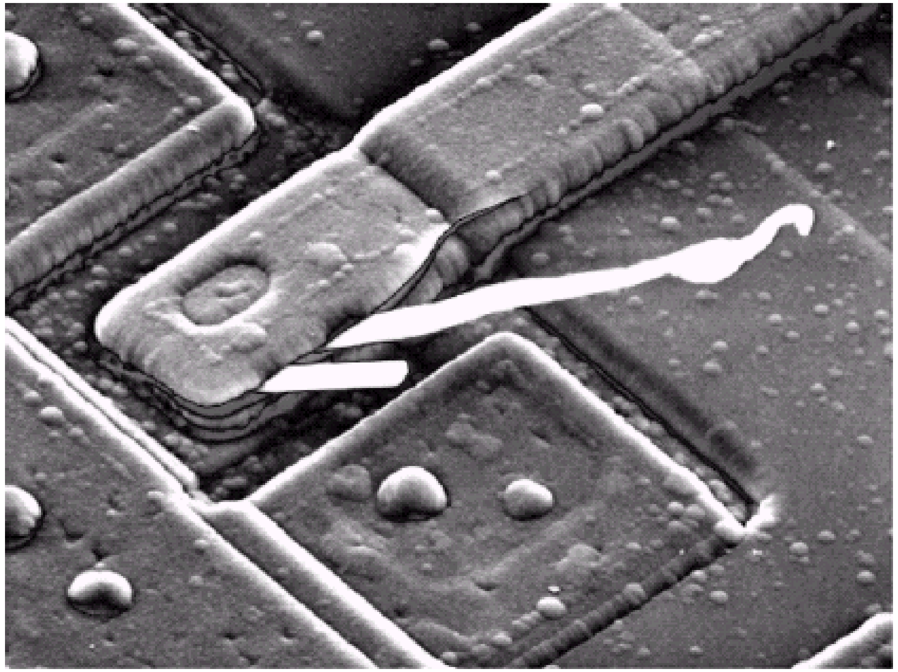
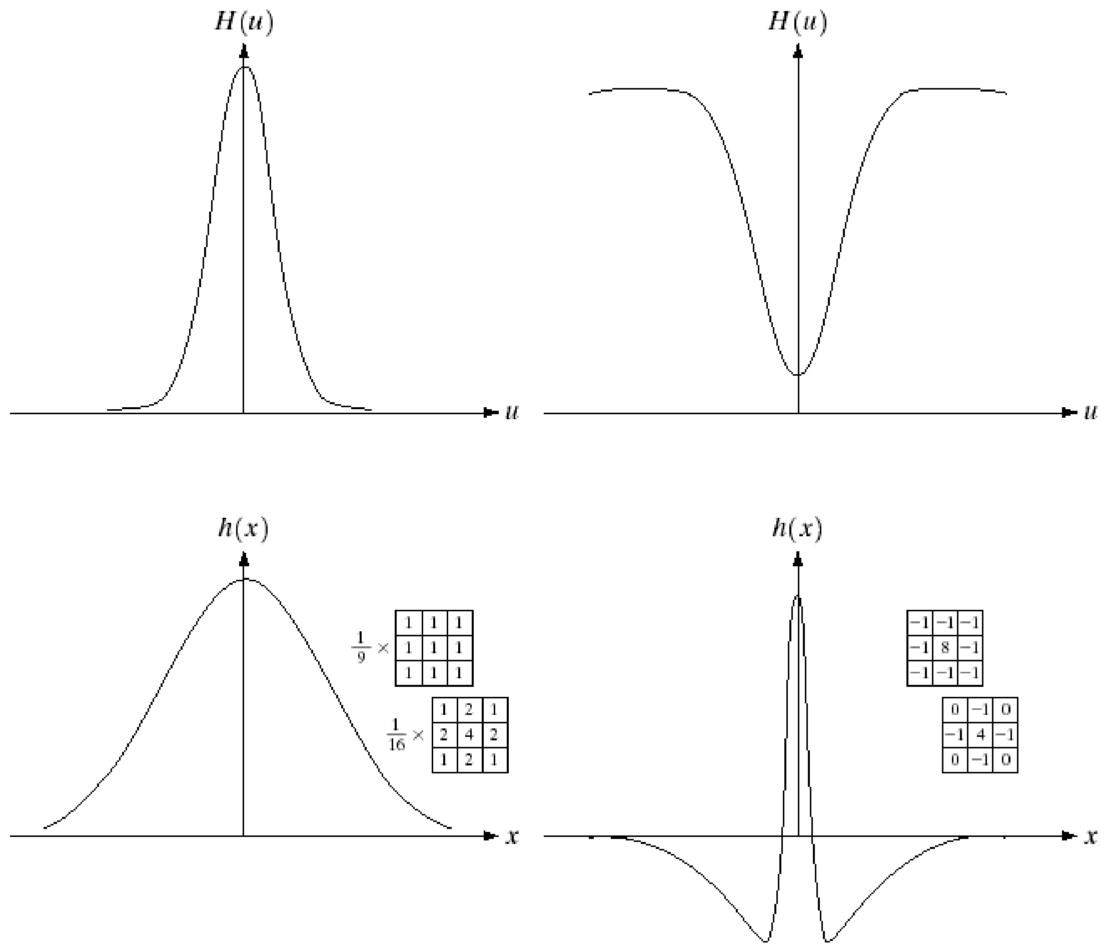


Image Enhancement in the Frequency Domain



a b
c d

FIGURE 4.9
(a) Gaussian frequency domain lowpass filter.
(b) Gaussian frequency domain highpass filter.
(c) Corresponding lowpass spatial filter.
(d) Corresponding highpass spatial filter. The masks shown are used in Chapter 3 for lowpass and highpass filtering.

Laplacian Filter in the Frequency Domain

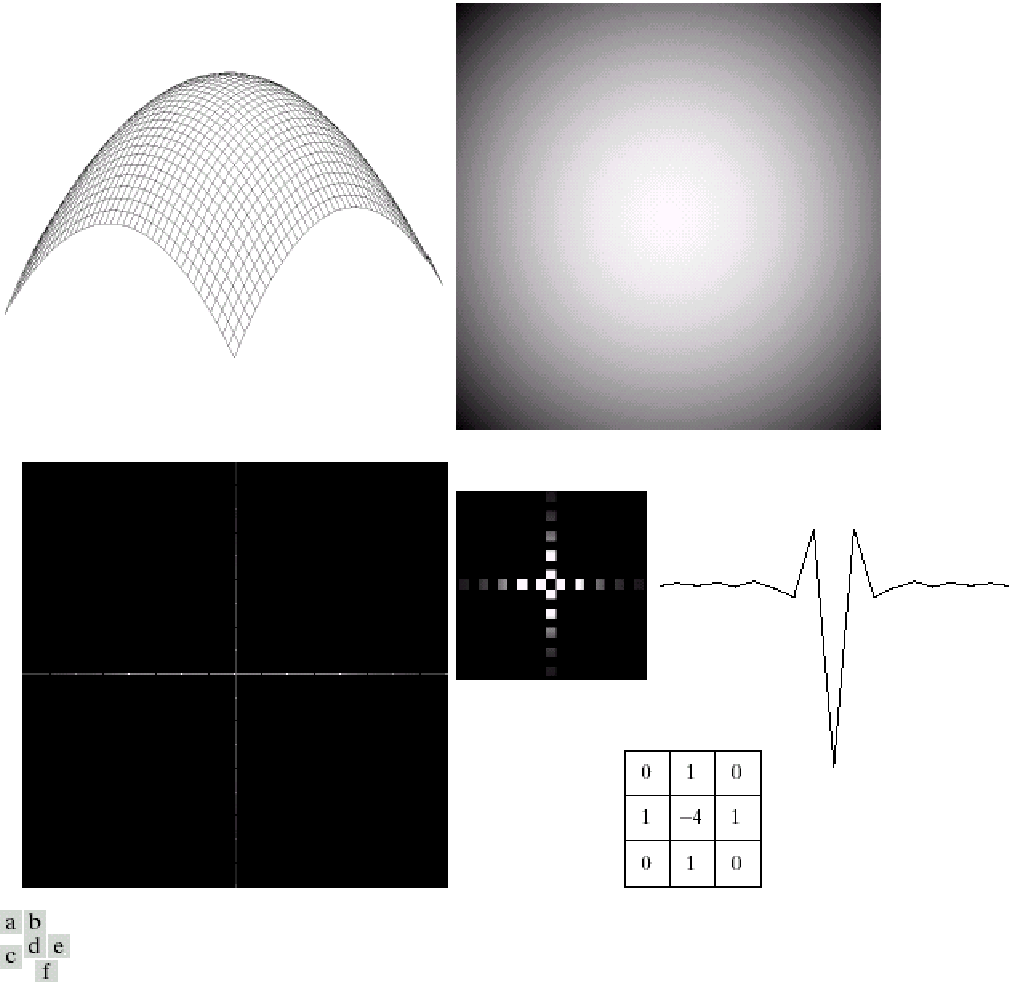


FIGURE 4.27 (a) 3-D plot of Laplacian in the frequency domain. (b) Image representation of (a). (c) Laplacian in the spatial domain obtained from the inverse DFT of (b). (d) Zoomed section of the origin of (c). (e) Gray-level profile through the center of (d). (f) Laplacian mask used in Section 3.7.

Problem 4.27

Suppose that you form a lowpass spatial filter that averages the four immediate neighbors of a point (x, y) but excludes the point itself.

(a) Find the equivalent filter H(u, v) in the frequency domain.

(b) Show that your result is a lowpass filter.

Problem 4.27 (Solution)

The spatial average (excluding the center term) is

$$g(x,y)=\frac{1}{4}\left[f(x,y+1)+f(x+1,y)+f(x-1,y)+f(x,y-1)\right].$$

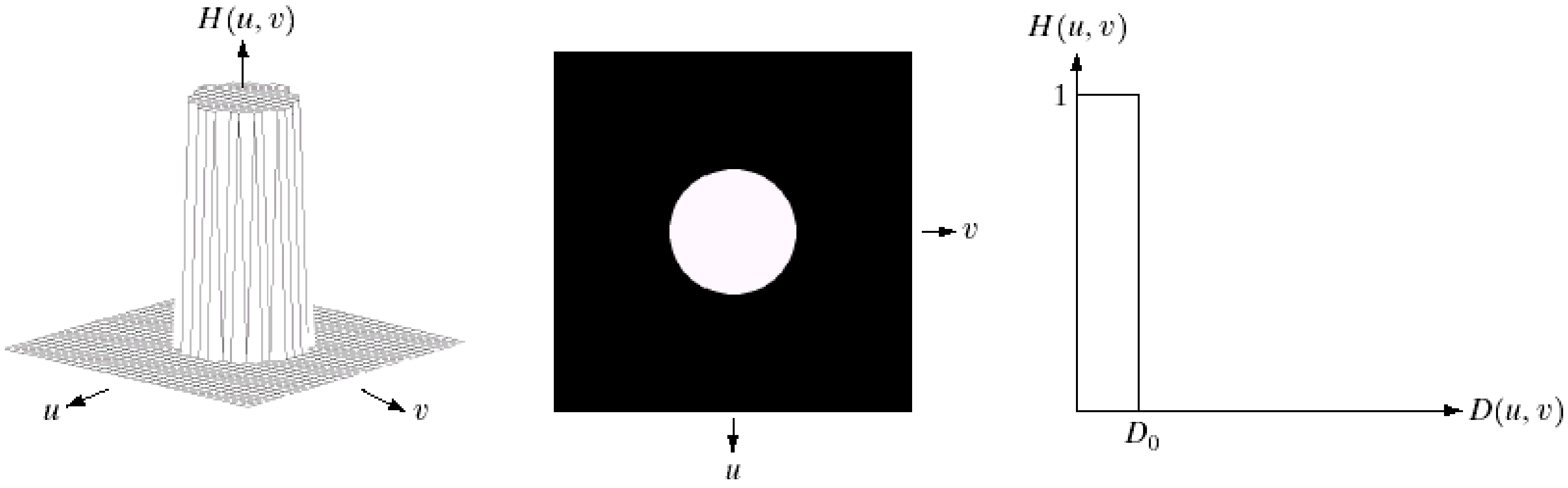
From property 3 in Table 4.3,

$$\begin{aligned} G(u,v) &= \frac{1}{4}\left[e^{j2\pi v/N}+e^{j2\pi u/M}+e^{-j2\pi u/M}+e^{-j2\pi v/N}\right]F(u,v) \\ &= H(u,v)F(u,v) \end{aligned}$$

where

$$H(u,v)=\frac{1}{2}\left[\cos(2\pi u/M)+\cos(2\pi v/N)\right]$$

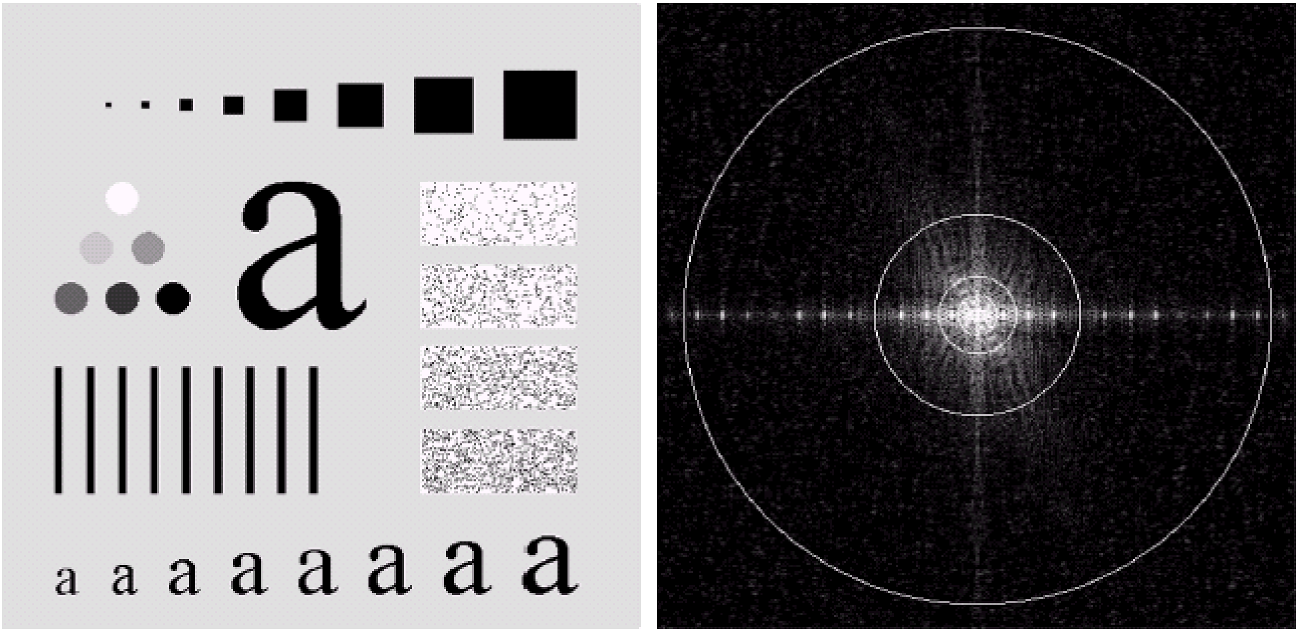
Ideal Lowpass Filters



a b c

FIGURE 4.10 (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

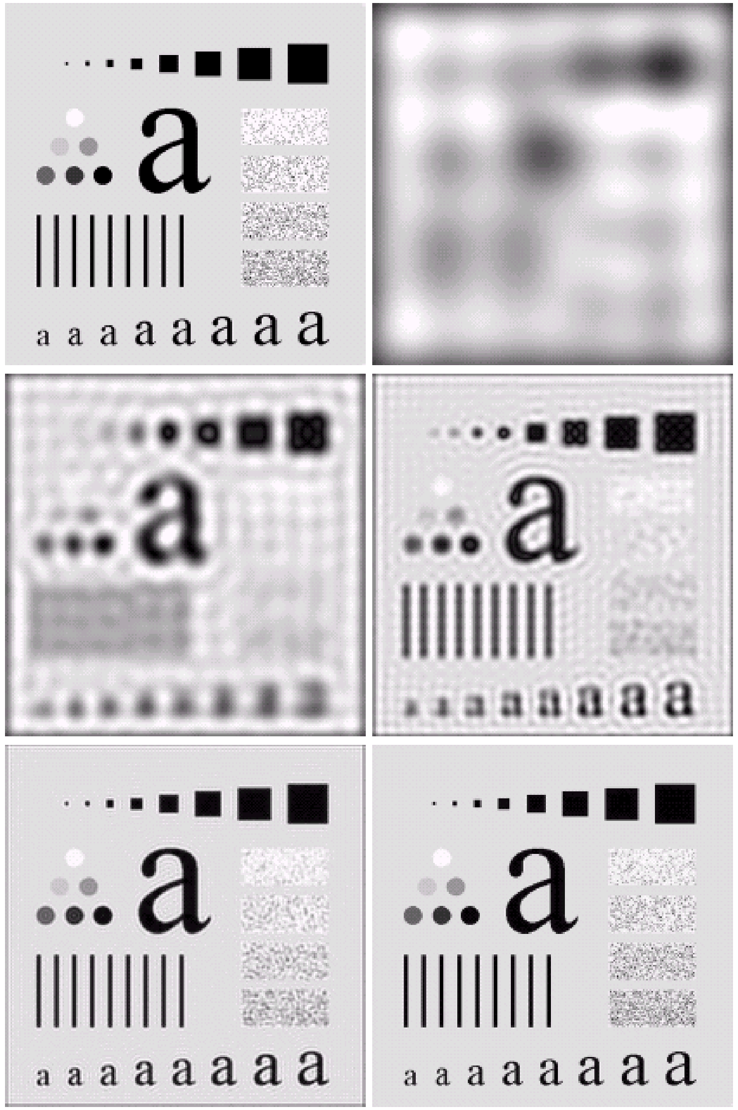
Ideal Lowpass Filters



a b

FIGURE 4.11 (a) An image of size 500 × 500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.

Ideal Lowpass Filters



Ideal Lowpass Filters

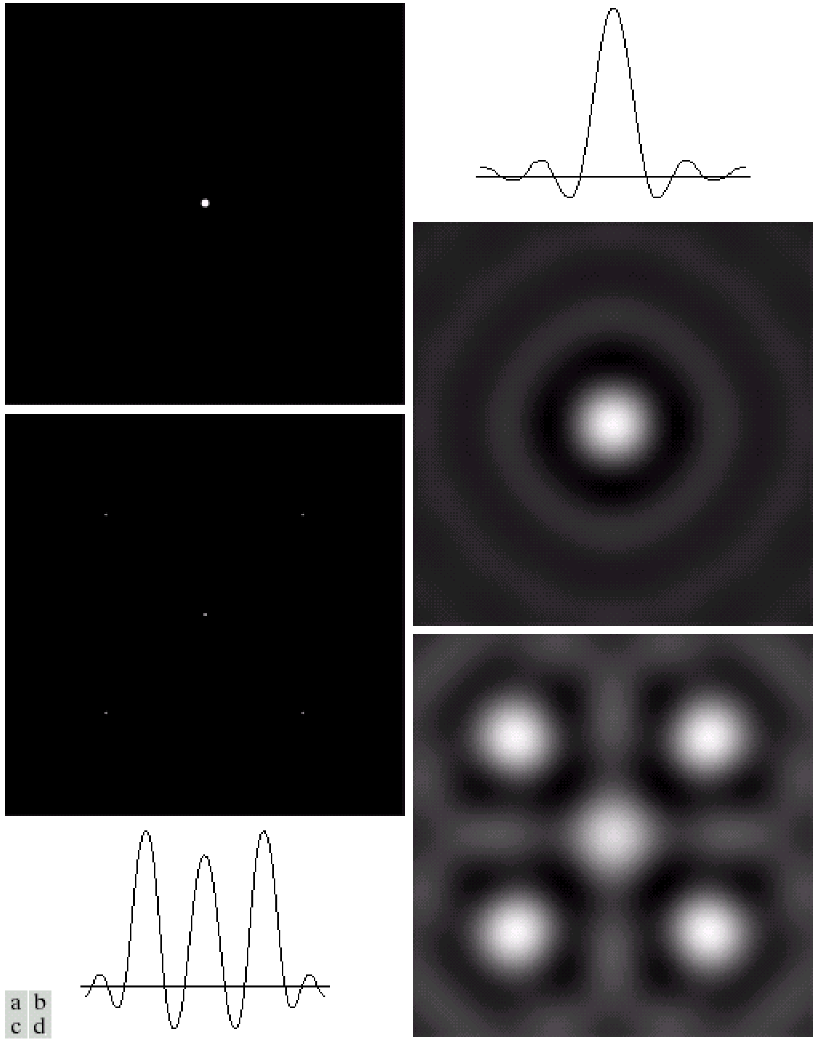
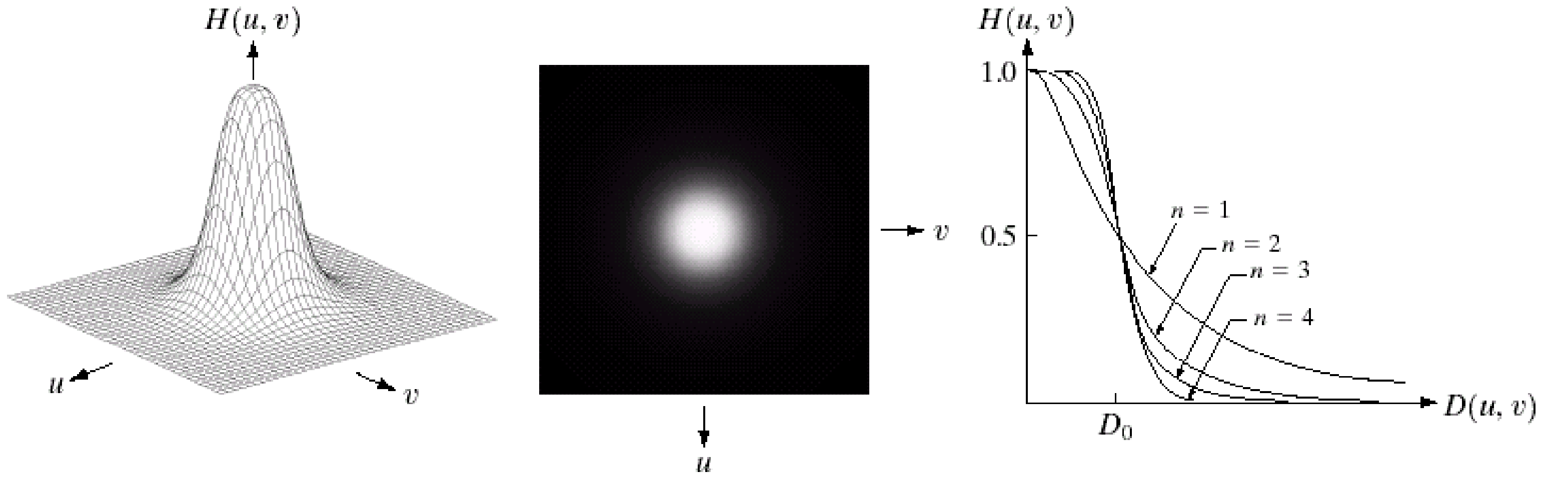


FIGURE 4.13 (a) A frequency-domain ILPF of radius 5. (b) Corresponding spatial filter (note the ringing). (c) Five impulses in the spatial domain, simulating the values of five pixels. (d) Convolution of (b) and (c) in the spatial domain.

Butterworth Lowpass Filters



a b c

FIGURE 4.14 (a) Perspective plot of a Butterworth lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

Butterworth Lowpass Filters

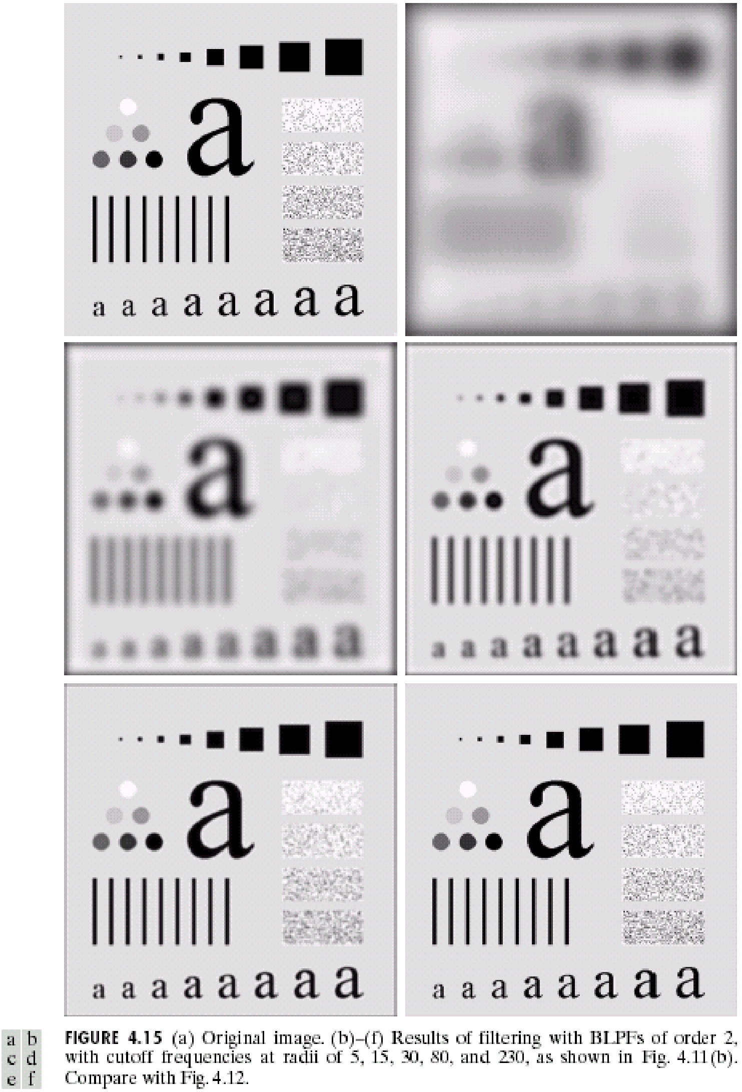


FIGURE 4.15 (a) Original image. (b)–(f) Results of filtering with BLPFs of order 2, with cutoff frequencies at radii of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Fig. 4.12.

Butterworth Lowpass Filters

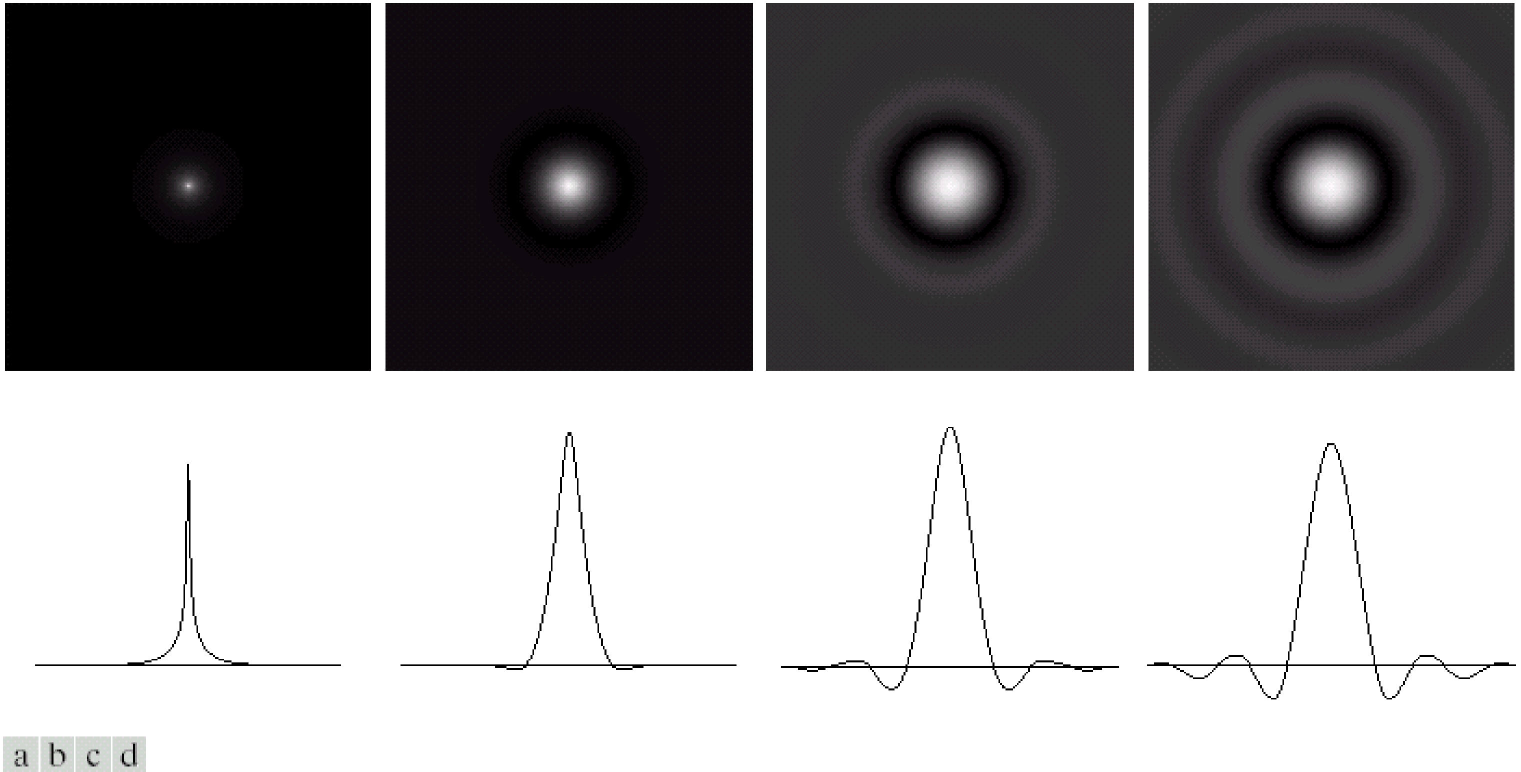
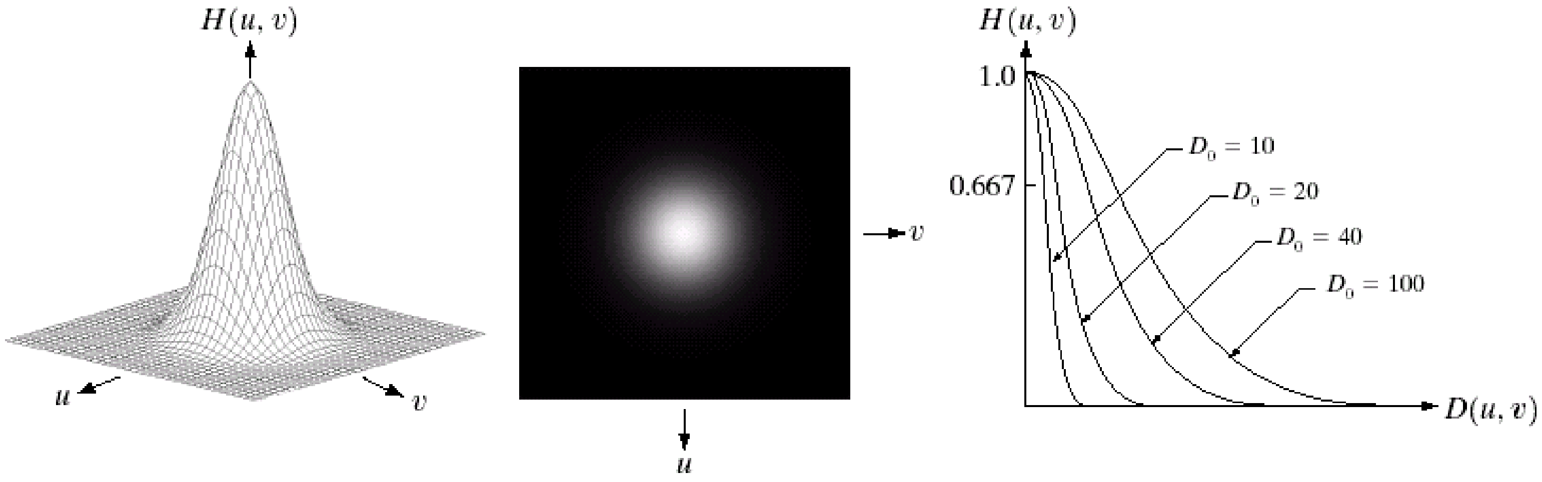


FIGURE 4.16 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding gray-level profiles through the center of the filters (all filters have a cutoff frequency of 5). Note that ringing increases as a function of filter order.

Gaussian Lowpass Filters



a b c

FIGURE 4.17 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

Image Enhancement in the Frequency Domain

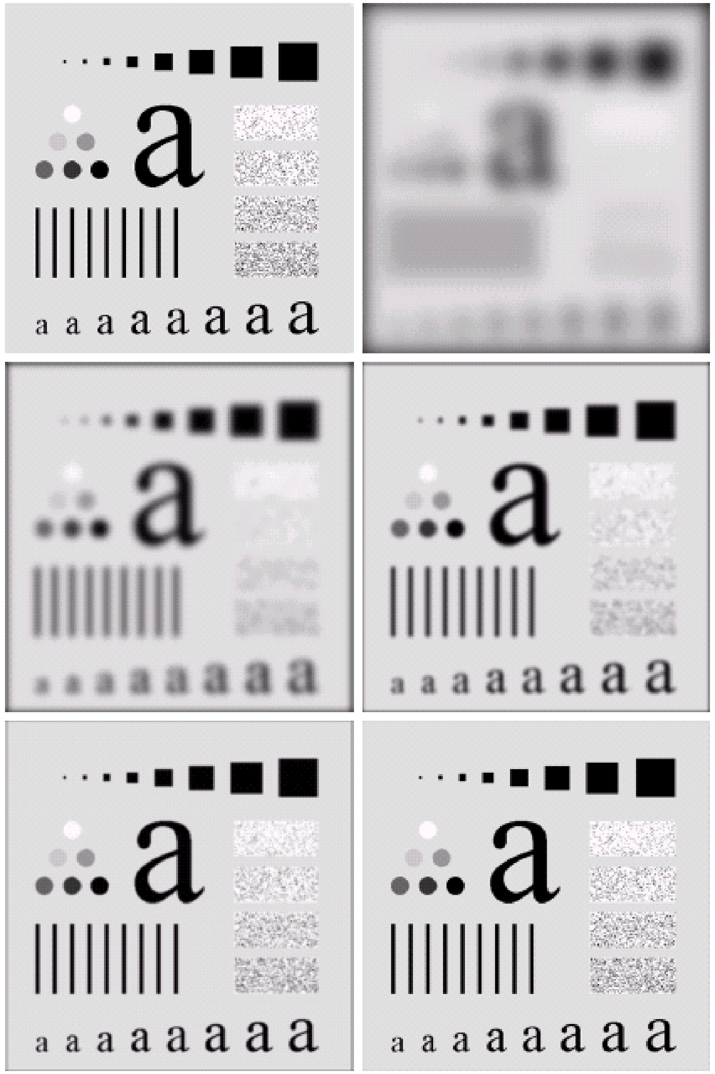
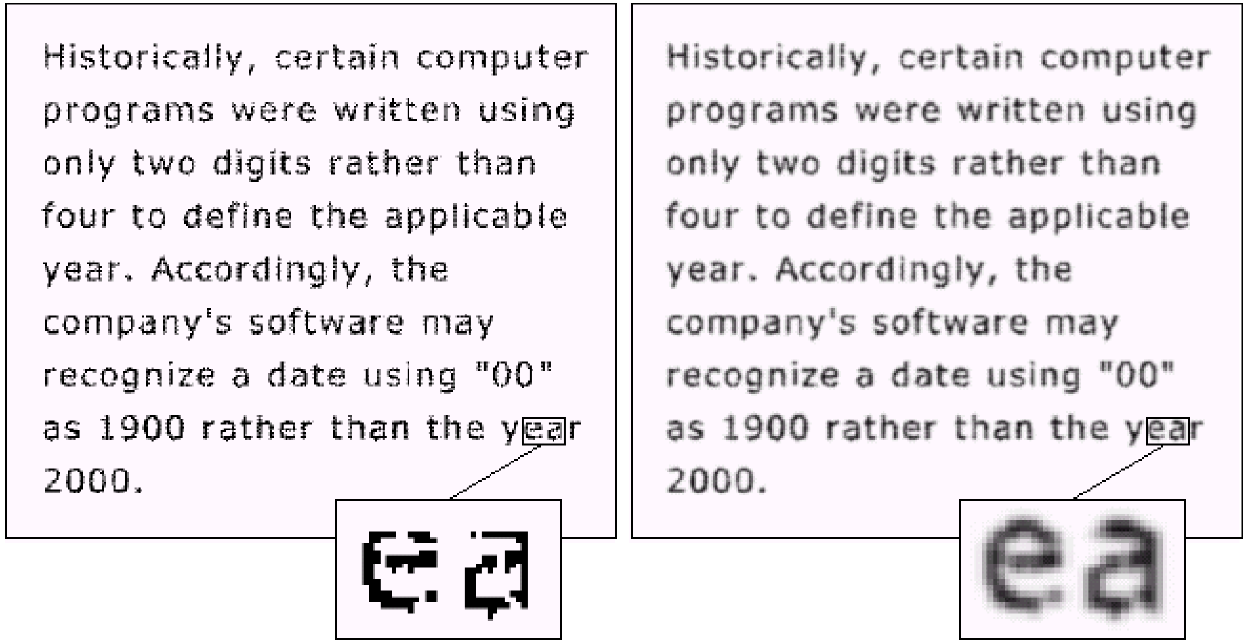


FIGURE 4.18 (a) Original image. (b)–(f) Results of filtering with Gaussian lowpass filters with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Figs. 4.12 and 4.15.

a b
c d
e f

Image Enhancement in the Frequency Domain



a b

FIGURE 4.19
(a) Sample text of poor resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).

Highpass Filters

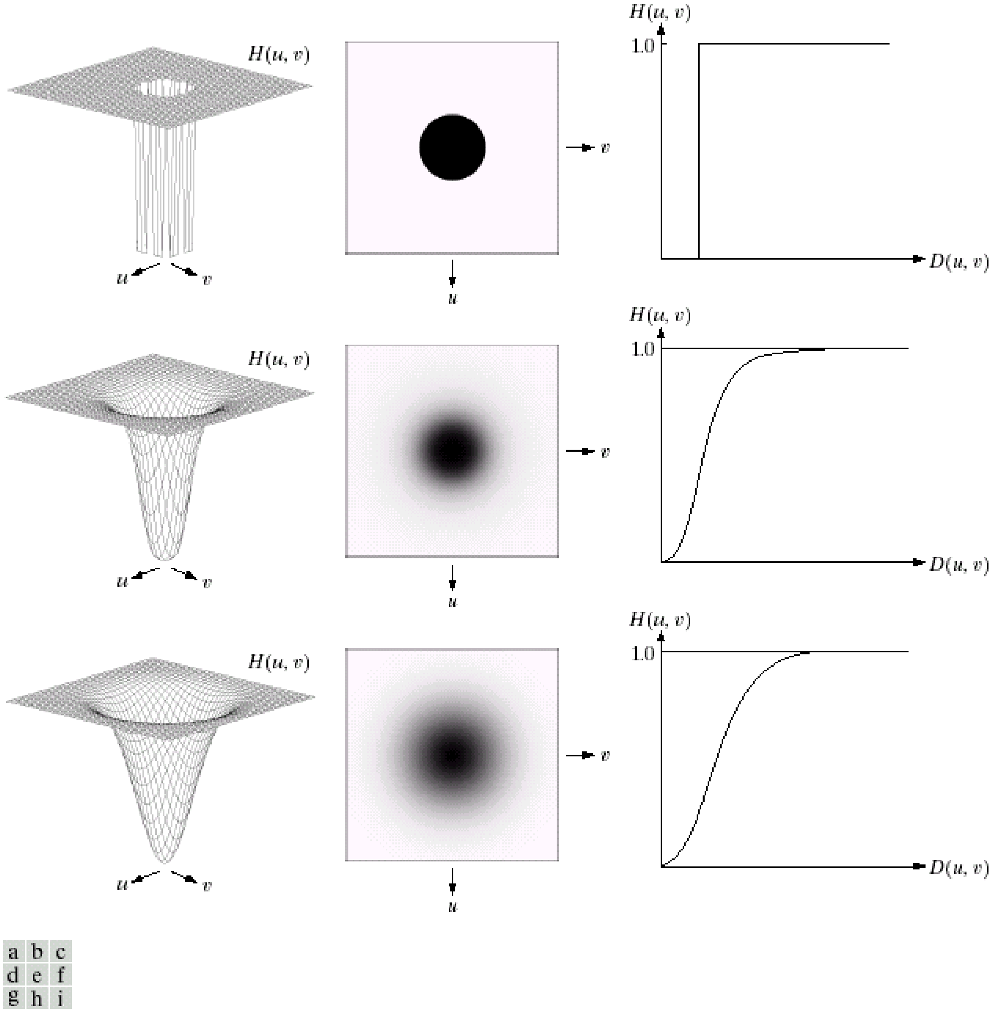


FIGURE 4.22 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

Highpass Filters

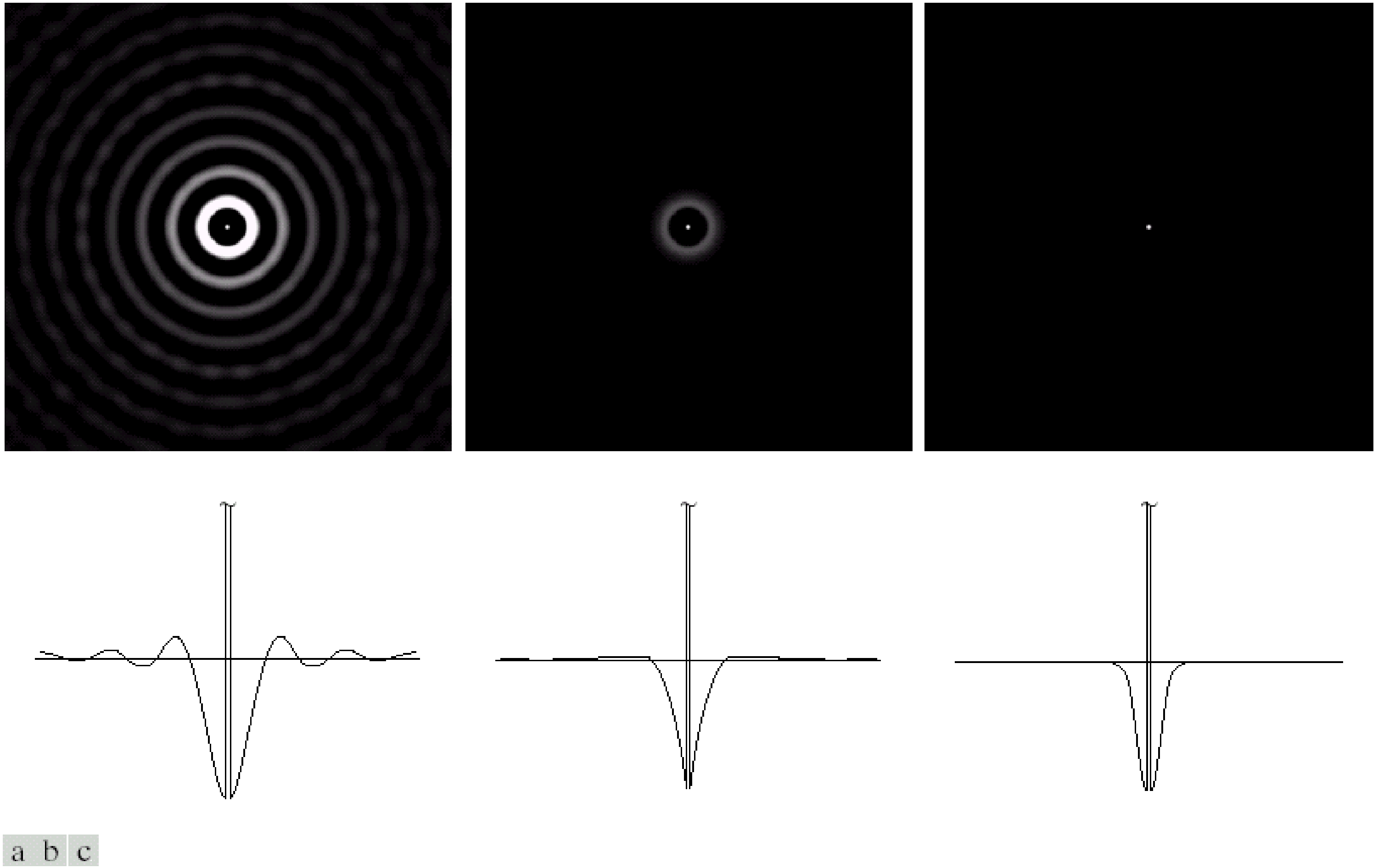
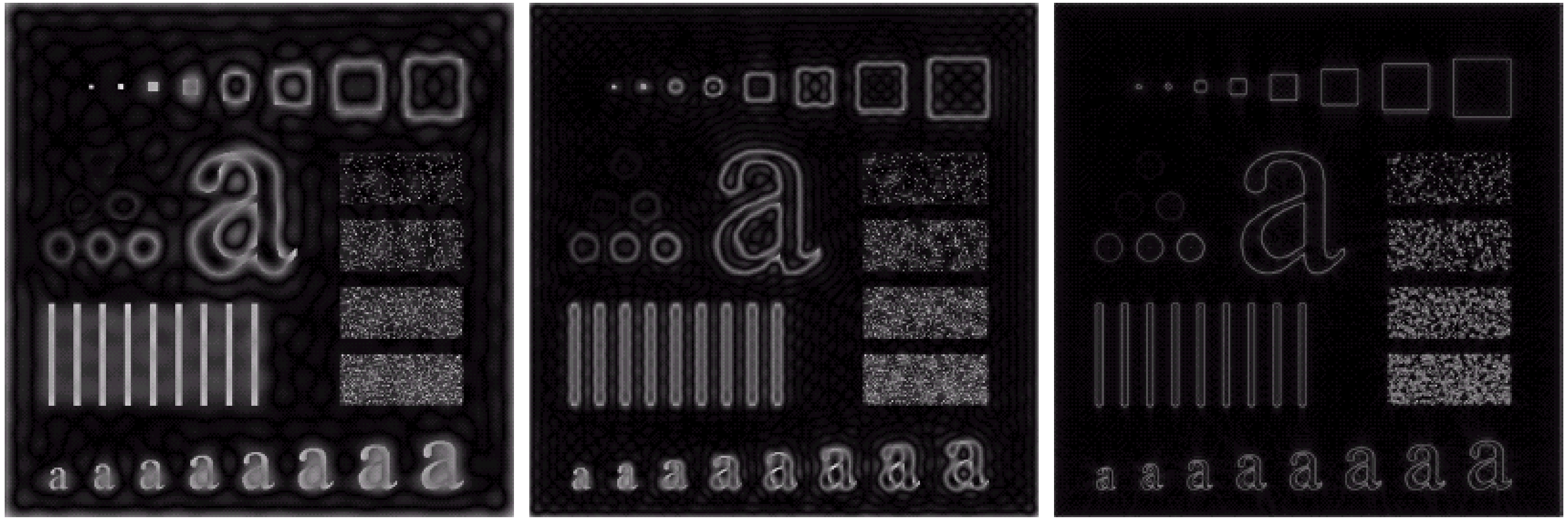


FIGURE 4.23 Spatial representations of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding gray-level profiles.

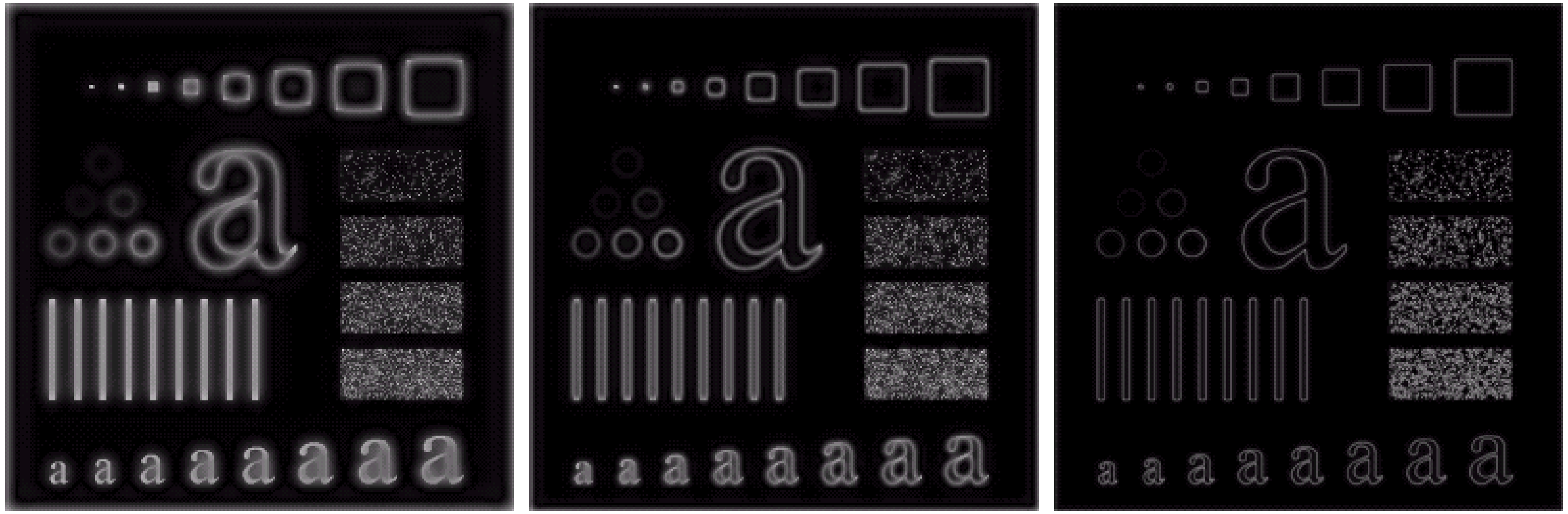
Ideal Highpass Filters



a b c

FIGURE 4.24 Results of ideal highpass filtering the image in Fig. 4.11(a) with $D_0 = 15$, 30, and 80, respectively. Problems with ringing are quite evident in (a) and (b).

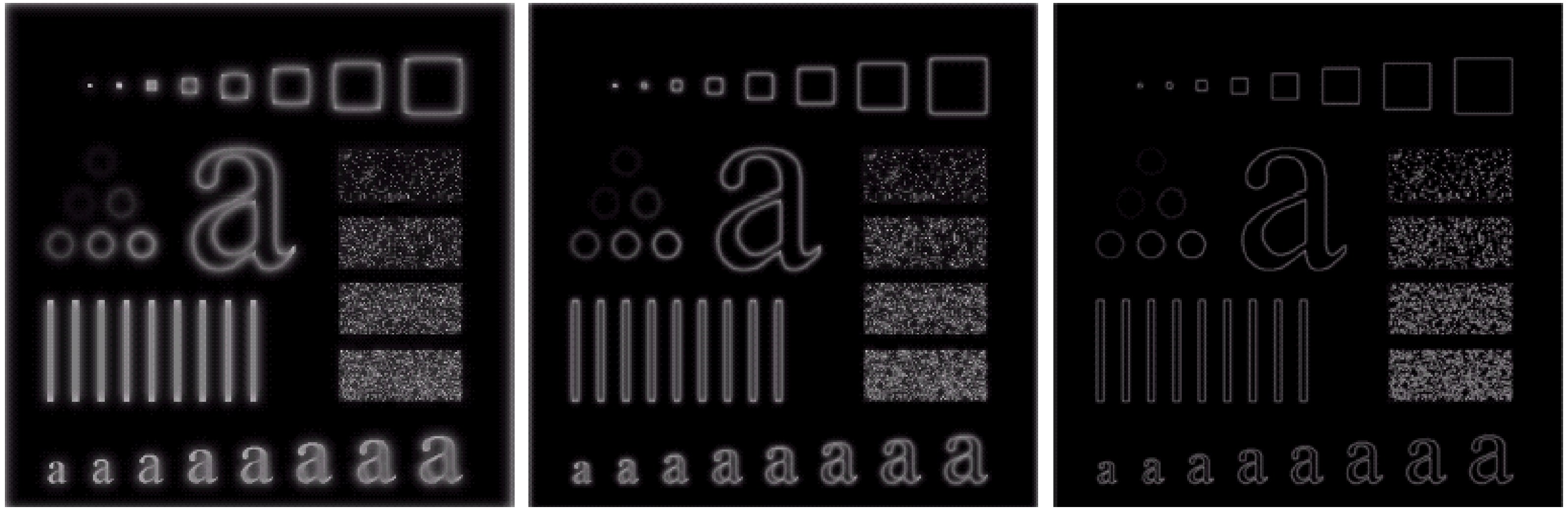
Butterworth Highpass Filters



a b c

FIGURE 4.25 Results of highpass filtering the image in Fig. 4.11(a) using a BHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. These results are much smoother than those obtained with an ILPF.

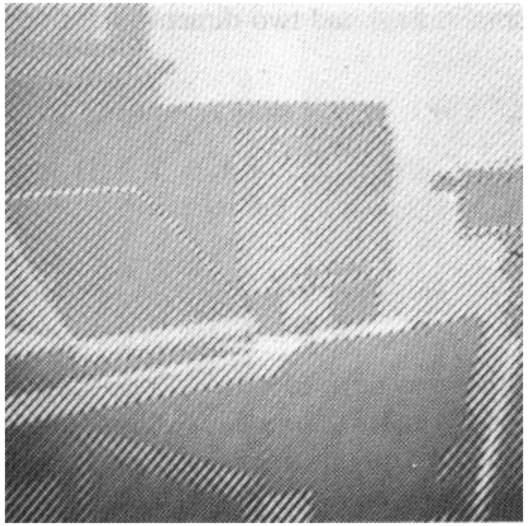
Gaussian Highpass Filters



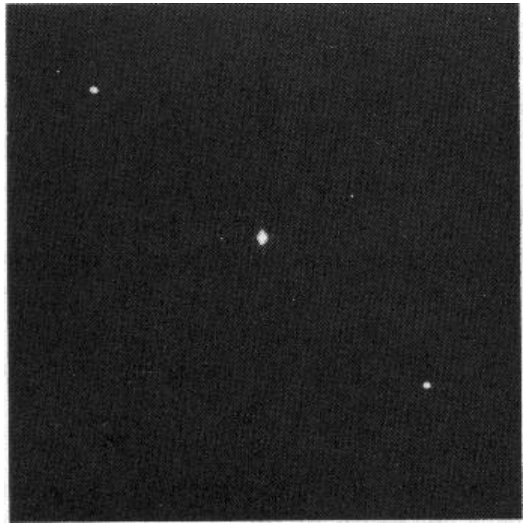
a b c

FIGURE 4.26 Results of highpass filtering the image of Fig. 4.11(a) using a GHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. Compare with Figs. 4.24 and 4.25.

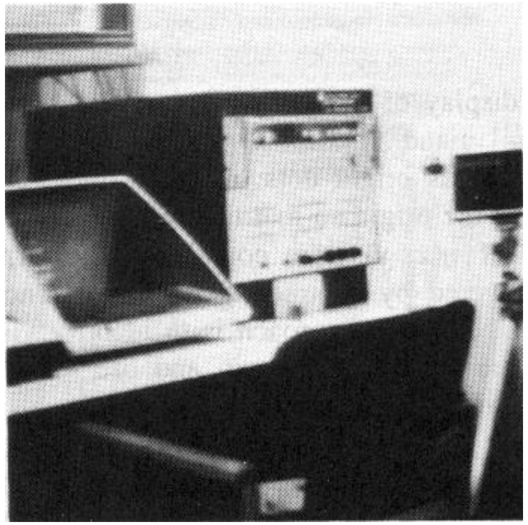
Noise Removal



Noisy image

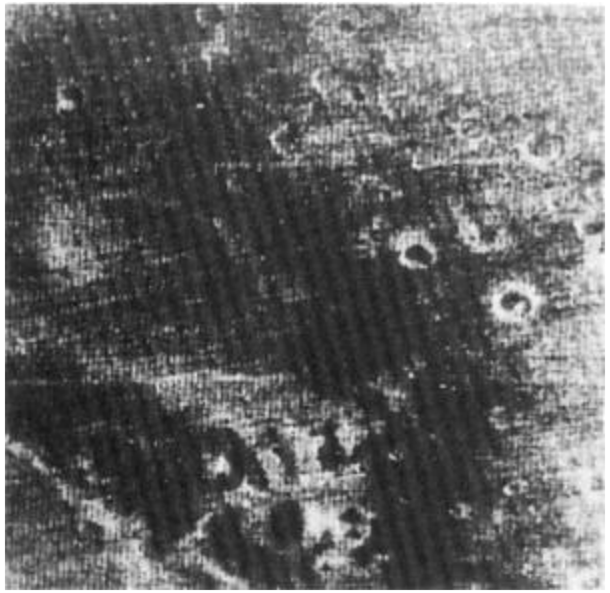


Fourier Spectrum

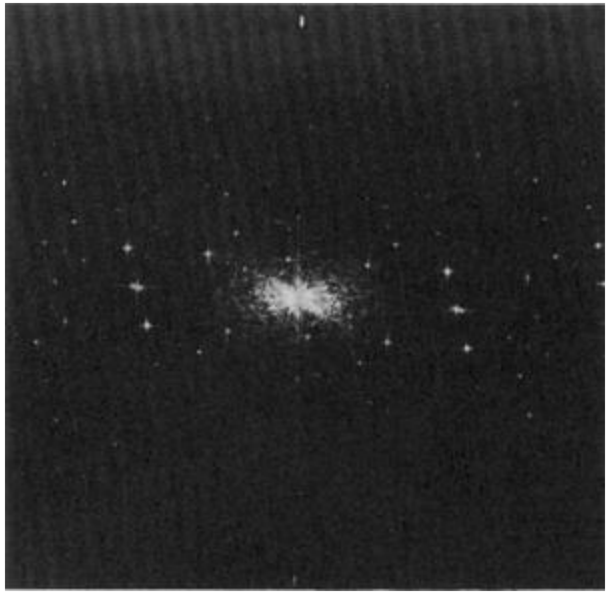


Noise-cleaned image

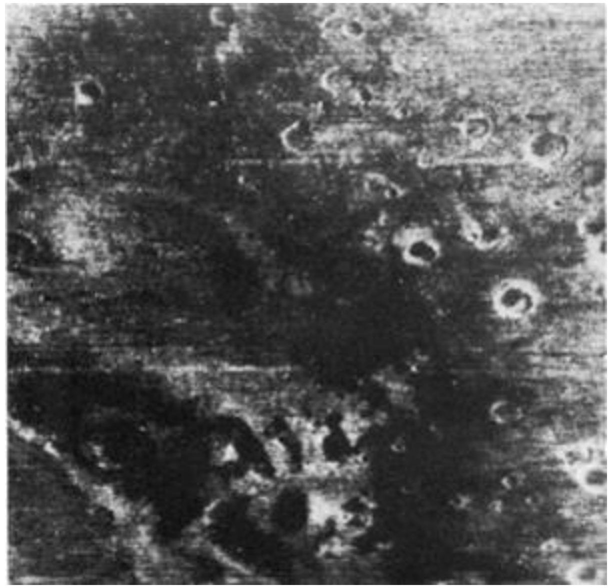
Noise Removal



Noisy image



Fourier Spectrum



Noise-cleaned image

Image Enhancement in the Frequency Domain

TABLE 4.1
Summary of some important properties of the 2-D Fourier transform.

Property	Expression(s)
Fourier transform	$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$
Inverse Fourier transform	$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$
Polar representation	$F(u,v) = \lvert F(u,v)\rvert e^{-j\phi(u,v)}$
Spectrum	$\lvert F(u,v)\rvert = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$, $R = \text{Real}(F)$ and $I = \text{Imag}(F)$
Phase angle	$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$
Power spectrum	$P(u,v) = \lvert F(u,v)\rvert^2$
Average value	$\bar{f}(x,y) = F(0,0) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$
Translation	$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$ $f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$ When $x_0 = u_0 = M/2$ and $y_0 = v_0 = N/2$, then $f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$ $f(x-M/2, y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$

Image Enhancement in the Frequency Domain

Conjugate symmetry	$F(u, v) = F^*(-u, -v)$ $\lvert F(u, v)\rvert = \lvert F(-u, -v)\rvert$	**TABLE 4.1** *(continued)*
Differentiation	$\dfrac{\partial^n f(x, y)}{\partial x^n} \Leftrightarrow (ju)^n F(u, v)$ $(-jx)^n f(x, y) \Leftrightarrow \dfrac{\partial^n F(u, v)}{\partial u^n}$	
Laplacian	$\nabla^2 f(x, y) \Leftrightarrow -(u^2 + v^2)F(u, v)$	
Distributivity	$\Im[f_1(x, y) + f_2(x, y)] = \Im[f_1(x, y)] + \Im[f_2(x, y)]$ $\Im[f_1(x, y) \cdot f_2(x, y)] \neq \Im[f_1(x, y)] \cdot \Im[f_2(x, y)]$	
Scaling	$af(x, y) \Leftrightarrow aF(u, v),\ f(ax, by) \Leftrightarrow \dfrac{1}{\lvert ab\rvert} F(u/a, v/b)$	
Rotation	$x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$ $f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$	
Periodicity	$F(u, v) = F(u + M, v) = F(u, v + N) = F(u + M, v + N)$ $f(x, y) = f(x + M, y) = f(x, y + N) = f(x + M, y + N)$	
Separability	See Eqs. (4.6-14) and (4.6-15). Separability implies that we can compute the 2-D transform of an image by first computing 1-D transforms along each row of the image, and then computing a 1-D transform along each column of this intermediate result. The reverse, columns and then rows, yields the same result.	

Image Enhancement in the Frequency Domain

Property	Expression(s)
Computation of the inverse Fourier transform using a forward transform algorithm	$\frac{1}{MN} f^*(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F^*(u, v) e^{-j2\pi(ux/M + vy/N)}$ This equation indicates that inputting the function $F^*(u, v)$ into an algorithm designed to compute the forward transform (right side of the preceding equation) yields $f^*(x, y)/MN$. Taking the complex conjugate and multiplying this result by MN gives the desired inverse.
Convolution†	$f(x, y) * h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) h(x - m, y - n)$
Correlation†	$f(x, y) \circ h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n) h(x + m, y + n)$
Convolution theorem†	$f(x, y) * h(x, y) \Leftrightarrow F(u, v) H(u, v)$; $f(x, y) h(x, y) \Leftrightarrow F(u, v) * H(u, v)$
Correlation theorem†	$f(x, y) \circ h(x, y) \Leftrightarrow F^*(u, v) H(u, v)$; $f^*(x, y) h(x, y) \Leftrightarrow F(u, v) \circ H(u, v)$

TABLE 4.1
(continued)

Image Enhancement in the Frequency Domain

TABLE 4.1
(continued)

Some useful FT pairs:

Impulse	$\delta(x, y) \Leftrightarrow 1$
Gaussian	$A\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2(x^2+y^2)} \Leftrightarrow Ae^{-(u^2+v^2)/2\sigma^2}$
Rectangle	$\mathrm{rect}[a, b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
Cosine	$\cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow \frac{1}{2}[\delta(u + u_0, v + v_0) + \delta(u - u_0, v - v_0)]$
Sine	$\sin(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow j\frac{1}{2}[\delta(u + u_0, v + v_0) - \delta(u - u_0, v - v_0)]$

† Assumes that functions have been extended by zero padding.