Cross Product

The cross product is defined only for three-dimensional vectors. If a and b are two three-dimensional vectors, then their cross product, written as $a \times b$ and pronounced "a cross b", **is another three-dimensional vector**. We define this cross product $a \times b$ by the following three requirements.

1. $a \times b$ is a vector, that is **perpendicular** to both a and b
2. The magnitude (or length) of the vector $a \times b$, written as $\|a \times b\|$, is the area of the parallelogram spanned by a and b
3. The direction of $a \times b$ is determined by the right-hand rule. This means that if we curl the fingers of the right hand from a to b, then the thumb points in the direction of $a \times b$

The below figure illustrates how using trigonometry, we can calculate that the area of the parallelogram spanned by a and b is $\|a\|\|b\|\sin(\theta)$ where θ is the angle between a and b. The figure shows the parallelogram as having a base of length $\|b\|$ and perpendicular height of $\|a\|\sin(\theta)$.

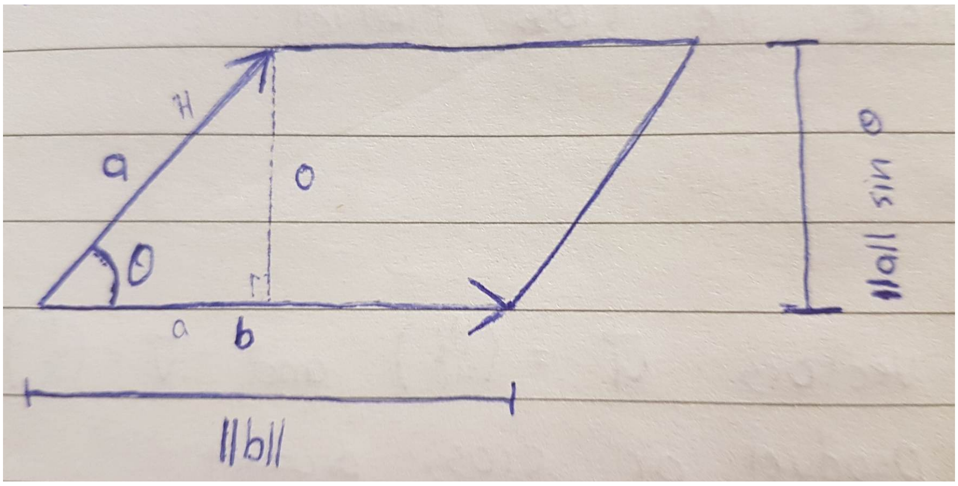

Figure 1: Cross Product

This formula shows that the magnitude of the cross product is largest when a and b are perpendicular. On the other hand, if a and b are parallel or if either vector is the zero vector, then the cross product is the zero vector.

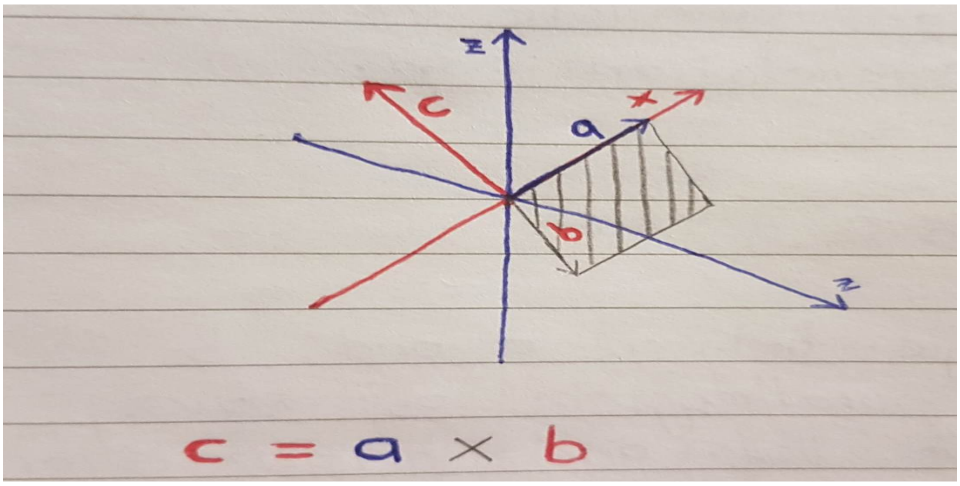

Figure 2: Cross Product

How to calculate the Cross Product

Given two vectors $\vec{u} = \begin{bmatrix} \mathrm{u}_1 \\ \mathrm{u}_2 \\ \mathrm{u}_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \\ \mathrm{v}_3 \end{bmatrix}$, the vector product or cross product $\vec{u} \times \vec{v}$ can be calculated using the following;

$$\vec{u} \times \vec{v} = (u_2v_3 - v_2v_3)\vec{\hat{i}} \text{ - } (u_1v_3 - v_1u_3)\vec{\hat{j}} + (u_1v_2 - v_1u_2)\vec{\hat{k}}$$

or

$$\vec{u} \times \vec{v} = (u_2v_3 - v_2v_3)\vec{\hat{i}} + (v_1u_3 - u_1v_3)\vec{\hat{j}} + (u_1v_2 - v_1u_2)\vec{\hat{k}}$$

Example

$\vec{u} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}$

$\vec{u} \times \vec{v} = (-3 \times 6 - 0 \times 2)\hat{i} + (4 \times 2 - 1 \times 6)\hat{j} + (1 \times 0 - 4 \times -3)\hat{k}$ $\vec{u} \times \vec{v} = -\vec{18}\hat{i} + 2\vec{\hat{j}}\ \vec{12}\hat{k}$