Linear Algebra Fundamentals

Contents

Fundamentals

The fundamental, root-of-it-all building block for linear algebra is the vector, illustrated at figure 1.

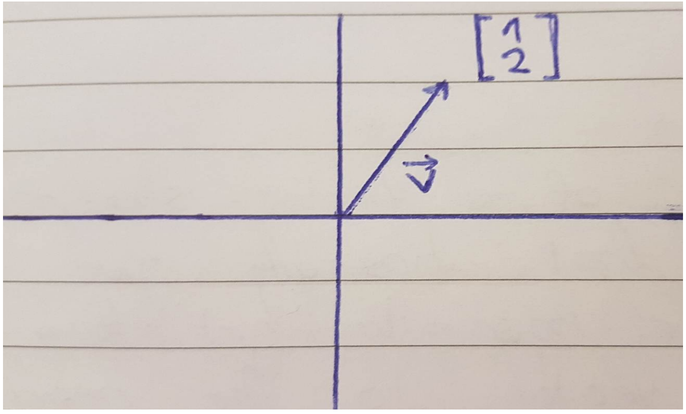

Figure 1: A basic vector

What defines a given vector is its **length and the direction it is pointing in**, but as long as these two facts are the same, you can move it all around it is still **the same vector**.

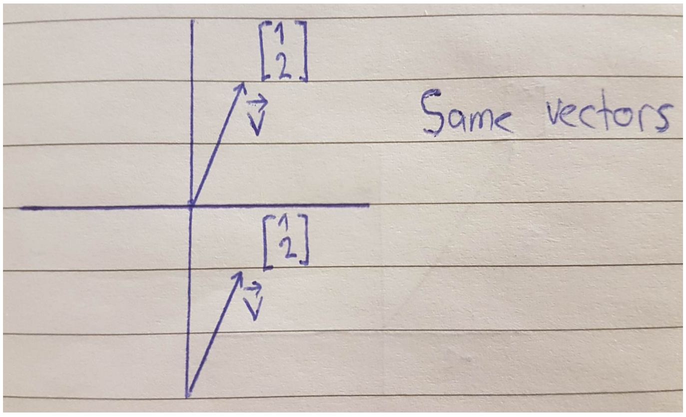

Figure 2: Identical vectors in different positions

In linear algebra, it is **almost always the case that your vector will be rooted at the origin**. The coordinates of a vector are a pair of numbers that give instructions for how to get from the tail of the vector to the head (also called, the tip), represeted inside square brackets, e.g. $\begin{bmatrix}X\\Y\end{bmatrix}$

The first number tells you now far to walk along the X-axis and the second number tells you how far to walk parallel to the Y-axis.

To distinguish vectors from points, the convention is to write the pair of numbers vertically with square brackets around them.

- $\begin{bmatrix}-2\\3\end{bmatrix}$ as a Vector
- $(-2, 3)$ as a Point

Every pair of numbers gives you **one and only one** vector.

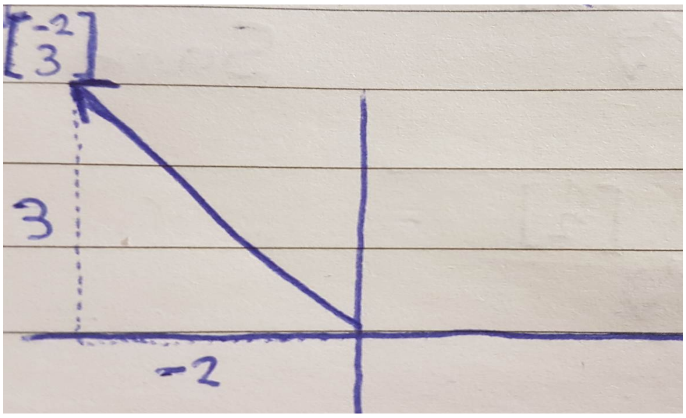

Figure 3: Example vector

Difference between Vector and a Point

The difference is precisely that between location and displacement.

- Points are **locations** in space
- Vectors are **displacement** in space

An analogy with time works well.

1. Times, (Also called instants or datetimes) are locations in time
2. Durations are displacements in time

So, in time;

- 4:00pm, noon, midnight, 12:20am, etc... are all **times**
- +3 hours, -2.5 hours, +17 seconds, etc... are all **durations**

Notice how durations can be positive or negative; this gives them **direction** in addition to their pure scalar value. Now the best way to mentally distinguish times and durations is by the operations they support.

1. Given a time, you can add a duration to get a new time, e.g. 3:00pm + 2 hours = 5:00pm
2. You can subtract two times to get a duration, e.g. 7:00pm - 1:00pm = 6 hours
3. You can add durations to get another, e.g. 3 hours + 50 minutes = 3 hours 50 minutes

But, we cannot add two times, 3:15am + 4:00pm = ??

Space example for Vectors and Points

Let's carry the analogy over to now talk about space:

- (3,5), (-2.25, 7), (0, -1), etc... are all **points**
- $\begin{bmatrix} 4 \\ -5 \end{bmatrix}$ is a **vector**, meaning 4 units east then 5 south, assuming north is up

Now we have exactly the same analogous operations operations in space as we did with time:

1. You can add a point and a vector. Starting at (4, 5) and going $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ takes you to the point (3, 8)
2. You can subtract two points to get a displacement between them: (10, 10) - (3, 1) = $\begin{bmatrix} 7 \\ 9 \end{bmatrix}$, which is the displacement you would take from the second location to get to the first.
3. You can add two displacements to get a compound displacement: $\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -5 \\ 8 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$, that is going 1 step east then 3 north **and then** going 5 west then 8 north is the **same thing** as just going 4 west and then 11 north.

But, you **cannot add two points**. In more concrete terms:

New Zealand + $\begin{bmatrix} 200kmEast \\ 7000kmNorth \end{bmatrix}$ is another location (Point) somewhere on earth, but New Zealand + Australia makes no sense.

Summary

To summarize, a location is where (or when) you are, and displacement is how to get from one location to another. Displacements have both magnitude (how far to go) and a direction (which in time, a one-dimensional space, is simply positive or negative). In space, locations are points and displacements are vectors. In time, locations are (points in) time, a.k.a instants and displacements displacements are durations.

Vector Addition

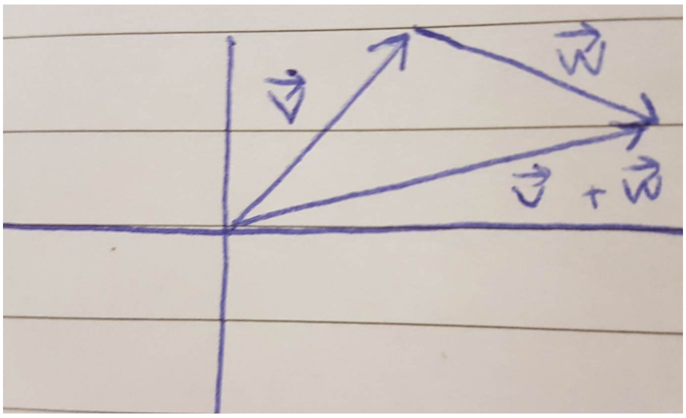

Figure 4: Example addition of vectors

If you take a step along the first vector, then take a step in the direction and distance described by the second vector. The overall effect is just the same as if you moved along the sum of those two vectors to start with.

You can think about this as an extension of how we think about adding numbers on a number line.

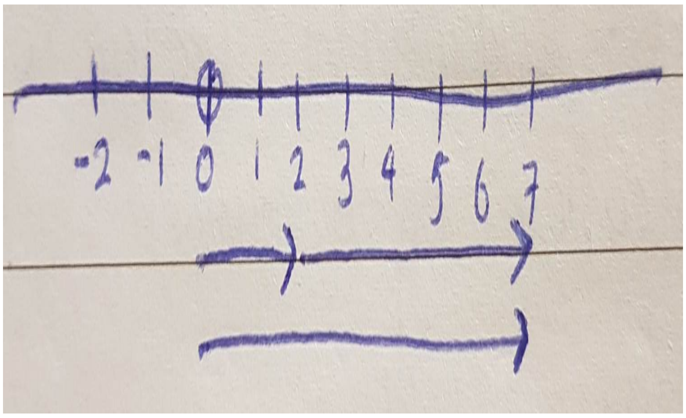

Figure 5: Numberline analogy of vector additions

$2 + 5 = 7$, that is, two steps to the right followed by another 5 steps to the right, the overall effect is the same as if you took 7 steps to the right.

Numberline analogy of vector addition

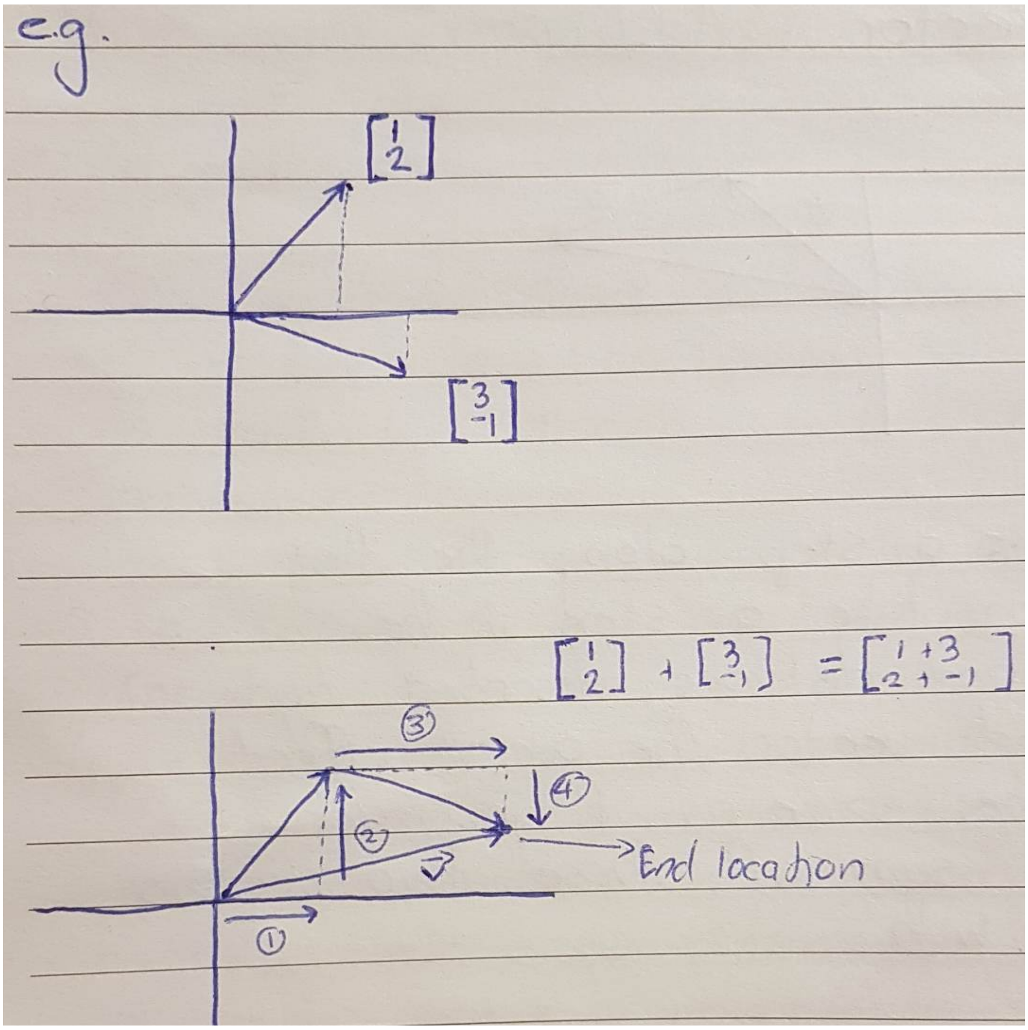

Figure 6: Example addition of vectors

Vector Subtraction

Vector subtraction is similar to that of vector addition with a slight difference in thought. **Remember**, we can shift vectors anywhere we want as long as it has not changed **direction** and **magnitude**.

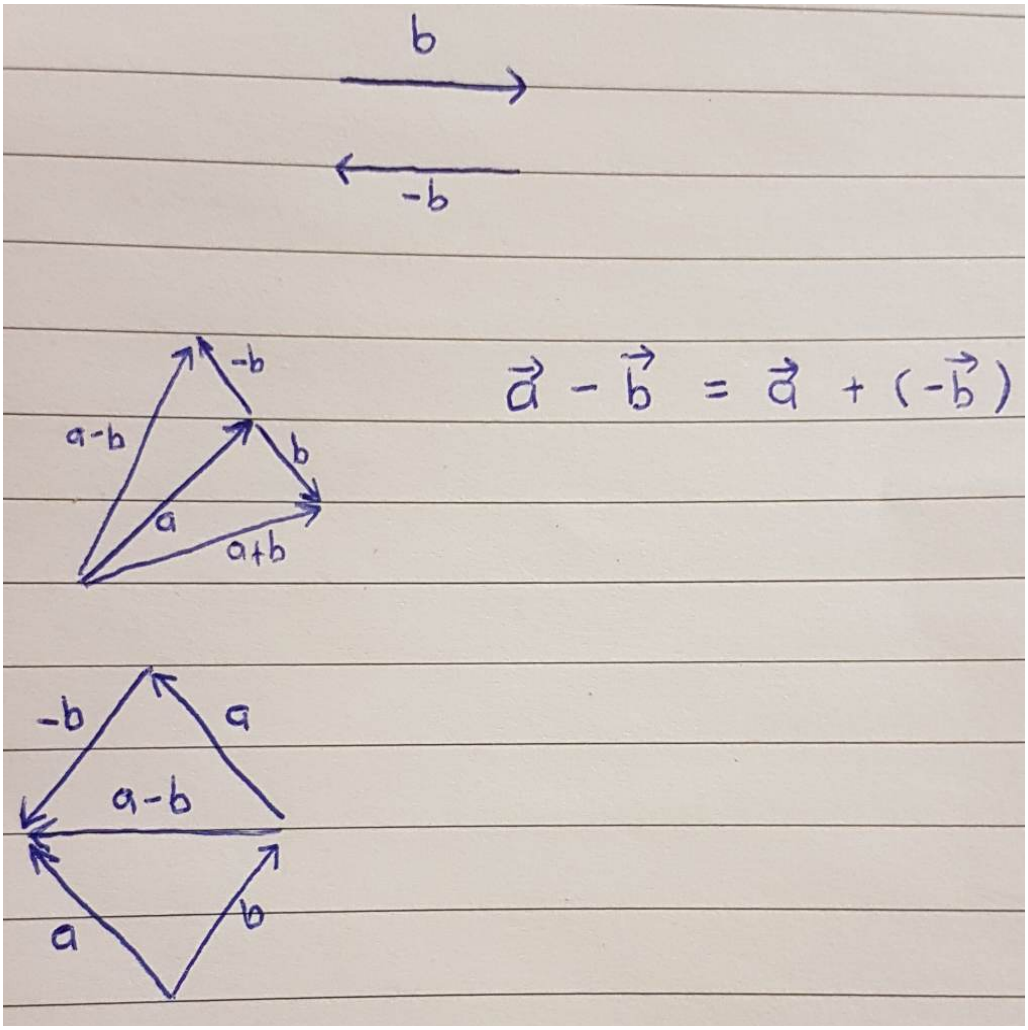

Figure 7: Example subtraction of vectors

Scaling

One of the final fundamental vector operations is multiplication by a number.

$$2\vec{v}$$

This means if you take the number 2 and multiply it by a given vector, it means you stretch out that vector so that it's 2 times as long as when you started.

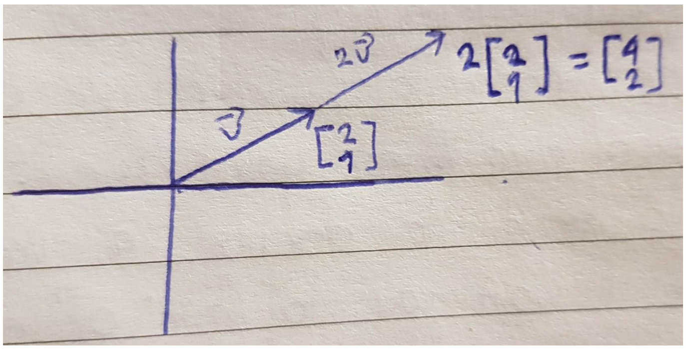

Figure 8: Example scaling