Exploring the Moduli Space of Vacua for SU(N) Yang-Mills Theories with N=4 Supersymmetry

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Abstract

The moduli space of vacua is the term used to describe the space of all the possible ground states of a Yang-Mills quantum field theory. This work explores the moduli space of vacua for an SU(N) Yang-Mills theory with N=4 supersymmetry. This poster will describe this space of vacua and explore the properties of the theories which this space describes. In particular, the purpose of this work is to attempt to understand what happens when we take limits in this moduli space in a specific way, corresponding to taking isogenies between the abelian varieties associated to each point in the space.

Introduction

Yang-Mills Theories

Every compact Lie group then has an associated Yang-Mills theory which is constructed by taking a basis of the corresponding Lie algebra and associating each basis vector with an operator whose role is to create a massless spin-1 particle. A Lagrangian for the theory is then constructed using the Lie algebra which gives equations of motion for the theory. In general, given a Lie group G such that dim(G) = D and rank(G) = N, the associated Yang-Mills theory has D massless spin-1 particles, of which N are neutral (i.e. they don't interact with each other and so are analogous to photons) and the remaining D-N have charges with respect to each of the N massless spin-1 particles.

The Moduli Space of Vacua

As stated in the abstract, the moduli space of vacua is the term used to describe the space of all the vacuum solutions to a given theory. In the case of an SU(N) Yang-Mills theory with N=4 supersymmetry, this space is continuous and manifold-like. In particular, the space turns out to be isomorphic to $\mathbb{C}^{3(N-1)}/\Gamma$ where Γ is the Weyl group of SU(N) which is just the group of permutations on N objects. In this space of vacua, associated to each point is an abelian variety, which can be thought of as a complex torus. The objective of our work is to understand how the complex structure of these tori change as we approach singularities in this space of vacua. A visual representation of the space of vacua is shown in figure 1.

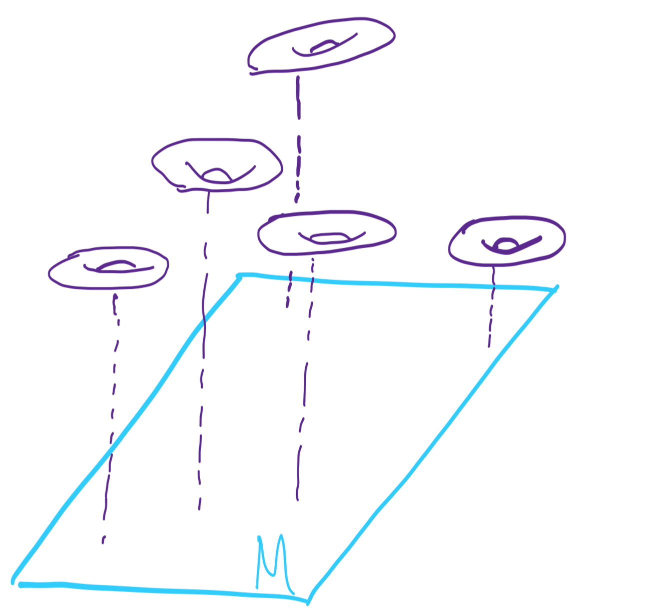

Figure 1: Shown here is a cartoon version of what the moduli space of vacua, *M,* looks like. Associated to each point in the space of vacua, there is an abelian variety (a torus) which contains information about the theory we are studying.

Abelian Varieties

Associated to every point in the moduli space of vacua, there is an abelian variety, which can be thought of as a torus. Abelian varieties are constructed by taking N-dimensional complex space and modding it out by a lattice of rank-2N. So, an abelian variety is an object $A = \mathbb{C}^N/\Lambda$ where Λ is a rank-2N lattice. This is shown pictorially in figure 2.

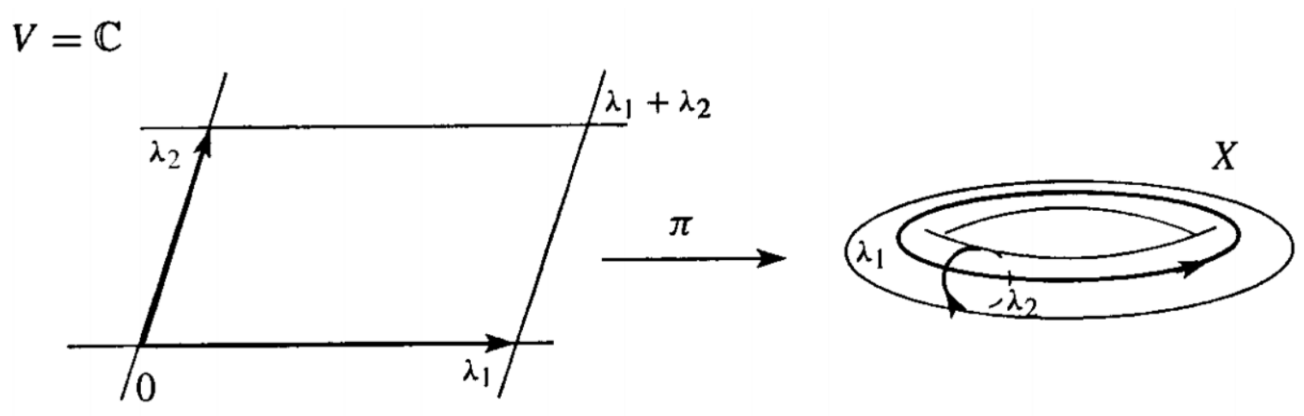

Figure 2: This shows how modding out the complex plane by a rank-2 lattice gives the classical torus shape. Extending this to higher dimensions is straightforward. This figure has been taken from reference 1.

Isogenies

An isogeny is a map $h: X \to X'$ from one abelian variety to another where Λ, the lattice which constructs X, is a sublattice of Λ', the lattice which constructs X'. The map h must be surjective with a finite kernel and must be "structure preserving". Structure preserving here has a different meaning depending on the specific context of the problem. When working in the context of linear algebra, a structure preserving map is just a linear transformation, but when working in the context of group theory as we are here, a structure preserving map is a group homomorphism. An isogeny is said to have index-n if $|\ker(h)| = n$. An example of lattices which would induce an isogeny between their respective abelian varieties is shown in figure 3. This particular isogeny would have index 4 because the map h sends the set of points $\{0, \frac{1}{2}, \frac{i}{2}, \frac{1}{2} + \frac{i}{2}\}$ to zero. Thus, $|\ker(h)| = 4$.

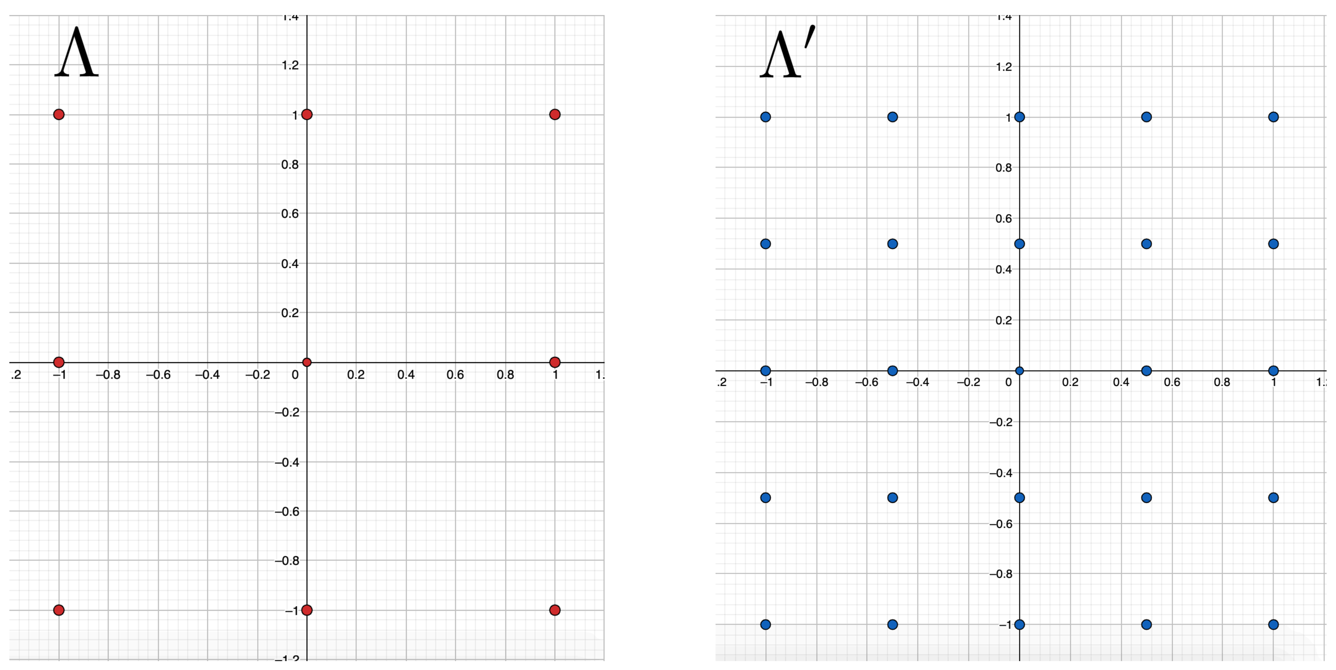

Figure 3: Shown here are two lattices Λ and Λ'. The important thing to notice here is that Λ is a sublattice of Λ', that is, every point in Λ is also a point in Λ'. This means that an isogeny is induced between the abelian varieties associated with these lattices.

Charge Lattices and Polarization

As stated earlier, a general Yang-Mills theory has more than just one massless spin-1 particle (the analogue of a photon). Thus, each massive particle must have more than one charge. It must have an electric (and in general a magnetic) charge associated to each massless spin-1 particle. The dirac quantization condition tells us that given two massive particles in a Yang-Mills theory, there must exist an integral, antisymmetric, bilinear pairing (called the dirac pairing) between the two particle's charge vectors. This implies that the possible charges of any massive particles live in a rank-2(N-1) lattice. The specific form of the dirac pairing is what is called the polarization of the charge lattice. Each point on the moduli space of vacua has an associated lattice of possible charges. This lattice then has an embedding in complex N-space. Modding the complex N-space by the lattice gives an abelian variety, which is where the picture in figure 1 comes from.

Current Work

We want to understand what happens to the polarization of charge lattices in the space of vacua as we approach (or move around) its singular subspaces. In particular, we want to learn about the charge lattices of *SU(M)* (with M<N) sub-theories living in the singularities of the moduli space of the *SU(N)* theories. These charge lattices will have rank 2(M-1) and we want to understand how this rank-2(M-1) lattice will be embedded in the larger rank-2(N-1) lattice which comes from the total *SU(N)* theory. The physics of our situation tells us that this sublattice will induce an isogeny between the corresponding abelian varieties associated to the charge lattices. This will be the topic of our work as we move into the spring 2021 semester.

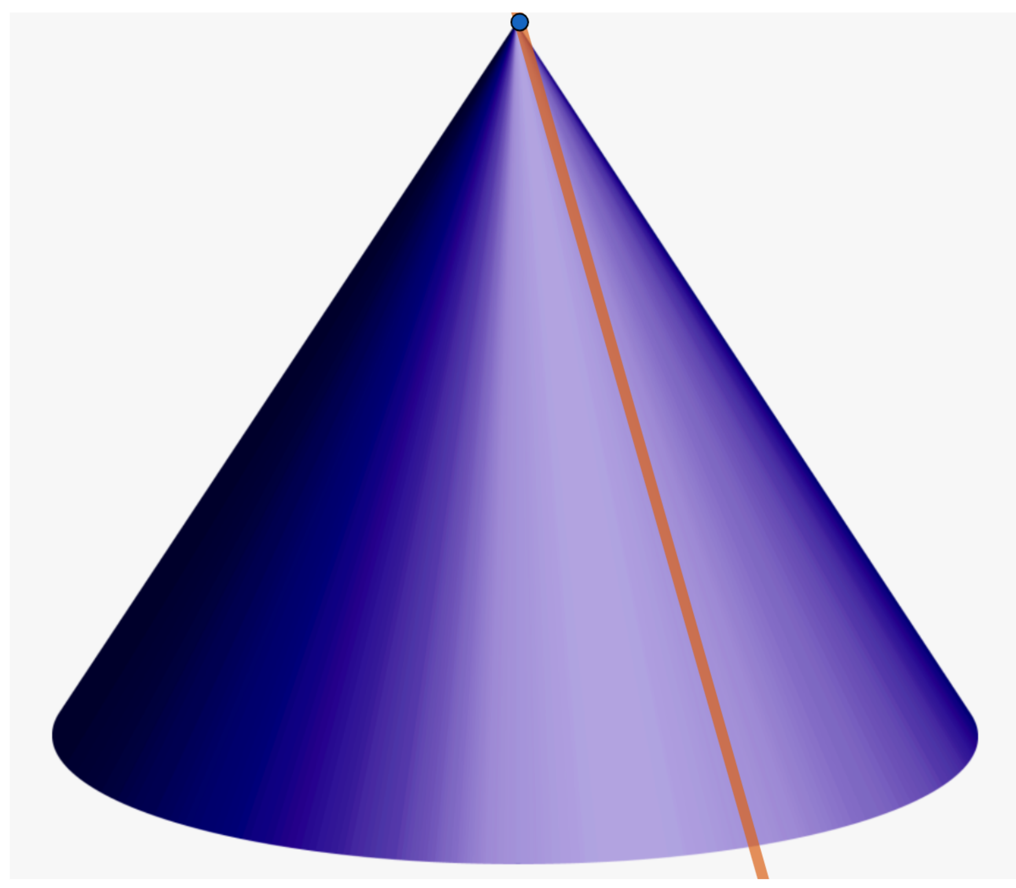

Figure 4: Shown here is a more accurate depiction of the moduli space of vacua for an SU(N) Yang-Mills theory with N=4 supersymmetry. The real space looks like the high dimensional analogue of a cone with nested singular subspaces (shown in this figure as an orange line).

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References

1. Christina Birkenhake and Herbert Lange, "Complex abelian varieties," *Springer, 2005*