Recursive functions

Michael Nowak

Texas A&M University

Overview

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$$n! = \begin{cases} 1 & \text{if } n < 2, \\ n \times (n-1)! & \textit{otherwise} \end{cases}$$
  - Using this definition, we are defining factorial in terms of factorial

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    - In this case, we define $n!$ as $(n-1)!$

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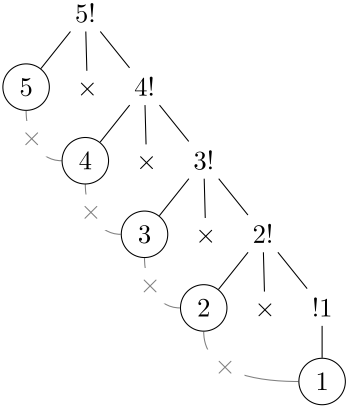
- Using our recursive definition of factorial, we would solve 5! as:



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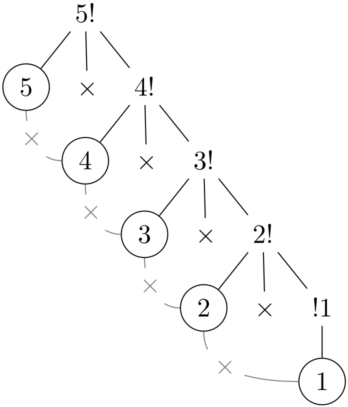

- That is, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

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```
int fact(int val)
{
    int res = 1;
    while(val > 1) {
        res *= val;
        val -= 1;
    }
    return res;
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```
int fact(int n)
{
    if(n < 2)
        return 1;
    else
        return n*fact(n-1);
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Writing a recursive function

- When writing a recursive function, we must always write:
  - One or more base cases that prompt our function to return without further recursion
  - One or more recursive cases that moves us closer towards meeting the base case(s)

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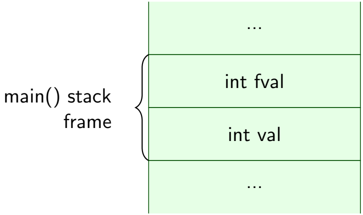
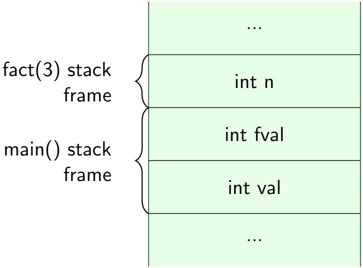
Recursive functions and the call stack: factorial

- Let's consider the state of the `call stack` as our program uses our recursive function `fact` to solve 5!
- Our program starts from `main()`, so a `stack frame` (`activation record`) for `main()` is pushed to the stack
  - Assume that main has two local variables, `int val` and `int fval` storing the value to calculate the factorial of and the return value of `fact(val)` respectively



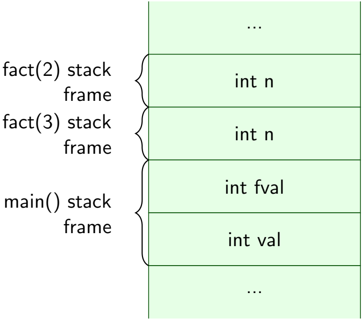
Recursive functions and the call stack: factorial

- Let's assume that our recursive function `fact` is called from main with the argument `3`
- This prompts a `stack frame` for `fact(3)` to be pushed to the `stack`
- `fact(3)` stores its argument `3` in the `local variable` `n` in its `stack frame`
- Execution has been transfered from `main()` to `fact(3)`



Recursive functions and the call stack: factorial

- As $n \geq 2$, we will execute the recursive case of the `fact` function, `return n*fact(2)`; `fact(2)` must be evaluated before the expression in the return statement can be evaluated
- A `stack frame` for fact(2) is thus pushed to the stack and execution is transfered to fact(2)
- `fact(2)` stores its argument `2` in the `local variable` `n` in its `stack frame`



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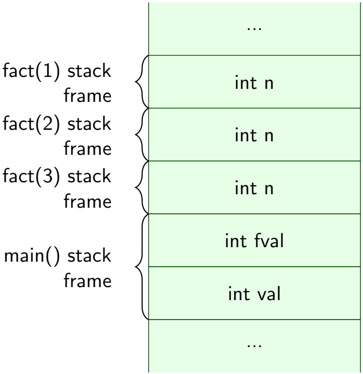
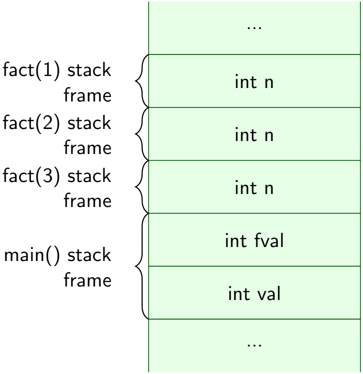


Recursive functions and the call stack: factorial

- As $n < 2$, we have finally arrived at the base case of the `fact` function, `return 1;` this statement is evaluated immediately
- `fact(1)` returns the value `1` to its caller, `fact(2)`



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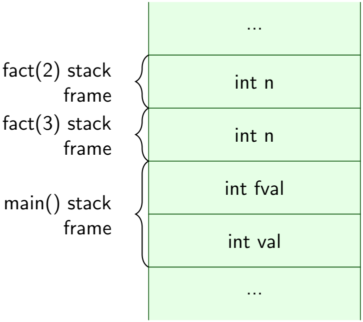
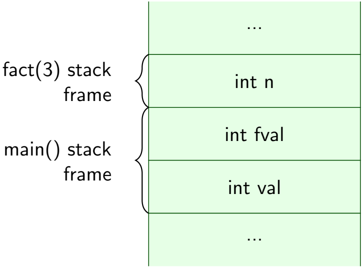
- When `fact(1)` returns the value `1`, its `stack frame` is popped from the `stack`
- Execution picks back up where things left off in `fact(2)` at the `return n*fact(1)` statement
- The return value of `fact(1)` is used in place of `fact(1)` call and `fact(2)` returns the product of `2*1` (`2`) to its caller, `fact(3)`



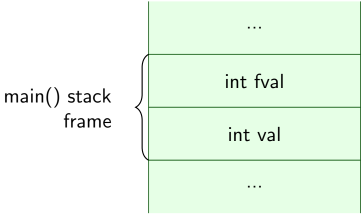
Recursive functions and the call stack: factorial

- When `fact(2)` returns the value 2, its `stack frame` is popped from the `stack`
- Execution picks back up where things left off in `fact(3)` at the `return n*fact(2)` statement
- The return value of `fact(2)` is used in place of `fact(2)` call and `fact(3)` returns the product of `3*2` (6) to its caller, `main()`



Recursive functions and the call stack: factorial

- When `fact(3)` returns the value 6, its `stack frame` is popped from the `stack` and our calculation of 3! using our recursive function is complete



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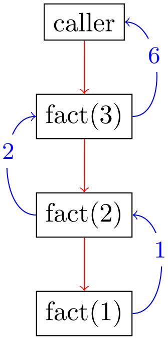
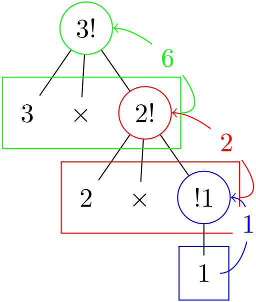


Recursive functions and the call stack: factorial

- Perhaps the following diagram will help detail better what's going on and where
  - The blue, red, and green circles represent the function call to `fact(1)`, `fact(2)`, and `fact(3)` respectively
  - The blue, red, and green rectangles represent the expressions evaluated in `fact(1)`, `fact(2)`, and `fact(3)` respectively
  - The blue, red, and green curved lines detail the return value of the expressions evaluated in `fact(1)`, `fact(2)`, and `fact(3)` respectively



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- We will assume function calls are processed from left to right; in C++ the order of such evaluation is up to the implementation (undefined)

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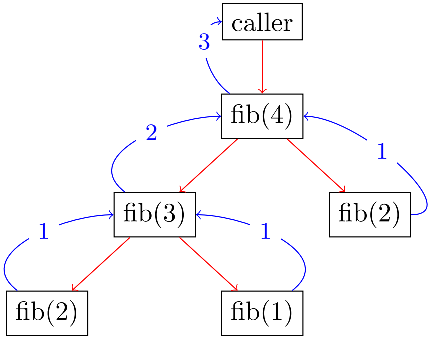

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- In general, anything solved recursively has an iterative solution
  - Sometimes the iterative version is more efficient, other times it is not
  - In some problems, a recursive solution maybe shorter to write and/or more elegant in nature; this may not be the case for other problems

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References

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