Data Representation

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Overview

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Motivation for study

- If you've programmed before, you may have had some surprises in the results of your computations; for example,
  - consider that `1500000000+1500000000`
  - evaluates to `-1294967296` on my machine
  - a result that mathematicians would consider wrong

Motivation for study

- If you've programmed before, you may have had some surprises in the results of your computations; for example,
  - consider that `1500000000+1500000000`
  - evaluates to `-1294967296` on my machine
  - a result that mathematicians would consider wrong
- As an additional example,
  - consider that `.15+.15`
  - evaluates to `0.29999999999999999` on my machine
  - a result that mathematicians would also consider wrong

Motivation for study

- The results that we've observed are a reflection of the way that numbers are stored in computers
- This can impact how your programs work
- So, it is worth taking a bit of time to discuss the representation of data in the computer

Overview

Binary numbers and modern computers

- All information stored on modern computers is represented with numbers
- Binary (base-2) numbers are used to represent these numbers in modern computers
- Electronics in the computer alternate between two states: on and off
  - Sequences of 'these' represent numbers, which represent data

Overview

What is a positional number system?

- Any positive integer $b > 1$ can be chosen as a base for a positional number system
  - decimal system ($b = 10$)
  - binary system ($b = 2$)
- Such a system uses b symbols for the integers (know as the digits)

$$0, 1, ..., b - 1$$

- Any integer N is represented by a sequence of base-b digits

$$N = a_n a_{n-1} ... a_1 a_0$$

- Such that b^k is the place value of a_k and

$$N = a_n \times b^n + a_{n-1} \times b^{n-1} + ... + a_1 \times b^1 + a_0 \times b^0$$

Overview

The binary number system

- Positional numeration system using base-2
- The two digits used are zero (0) and one (1); are called *bits*
- Binary numbers are a sequence of bits; can have an embedded binary point
- Binary integers are binary numbers without a fractional part
- The integral parts of a binary number are represented as a sequence of bits

$$N_I = a_n a_{n-1} ... a_1 a_0$$

- Such that in N_I 2^k is the place value of a_k and

$$N_I = a_n \times 2^n + a_{n-1} \times 2^{n-1} + ... + a_1 \times 2^1 + a_0 \times 2^0$$

- A similar notion holds for the fractional part; for instance, consider 1001.001_2

$$1001.001_2 = 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 + 0 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3}$$

Overview

Binary to decimal conversion

- A binary number N_2 can be converted to base-10 by writing N_2 in expanded notation and calculating by decimal arithmetic
- We can also use the following algorithm (this should look relatively familiar)

Integral part
  i. Double the leftmost digit
  ii. Add the result to next digit to the right
  iii. Double that sum
  iv. Add the result to the next digit
  v. Repeat until the last integral digit is added; the final sum is the decimal equivalent

Fractional part
  i. Multiply the rightmost fractional digit by $\frac{1}{2}$
  ii. Add the next digit to the left to that product
  iii. Multiply that sum by $\frac{1}{2}$
  iv. Add the next digit to that product
  v. Repeat until the leftmost digit (prior to decimal point) is added
  vi. Then multiply that sum by $\frac{1}{2}$

Overview

Decimal to binary conversion

- Given decimal number N, with integral part N_I and fractional part N_F, N can be converted to base-2 using the following algorithm:

  Integral part

  i. Subtract the largest possible power of base-2 from N_I
  ii. Subtract the largest possible power of base-2 from the result
  iii. Continue this process until a difference of zero is obtained
  iv. Place a bit value of 1 in the place values of those powers subtracted and a bit value of 0 elsewhere

  Fractional part

  i. Multiply N_F and the *fractional* portion of each succeeding product by 2 until a zero (or repeating) fractional part is observed
  ii. The resultant sequence of integral parts in-order gives the corresponding representation of N_F in base-2

Overview

Binary addition

- When adding two binary numbers, you need to know the following facts:
  - $0 + 0 = 0$
  - $1 + 0 = 1$
  - $0 + 1 = 1$
  - $1 + 1 = 0$, with carry 1
  - $1 + 1 + 1 = 1$, with carry 1
- Let's consider the following example,

$$\begin{array}{r} 1111 \\ +\,110 \\ \hline \end{array}$$

Binary addition example

Step 1.) $1 + 0 = 1$

$$\begin{array}{r} 1111 \\ +\,0110 \\ \hline 1 \end{array}$$

Binary addition example

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Step 2.) 1+1=0, with carry 1

$$\begin{array}{r} {\scriptstyle 1} \\ 1111 \\ +\,0110 \\ \hline 01 \end{array}$$

Binary addition example

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Step 3.) 1+1+1=1, with carry 1

$$\begin{array}{r} {\scriptstyle 1\,1} \\ 1111 \\ +\,0110 \\ \hline 101 \end{array}$$

Binary addition example

Step 1.) $1 + 0 = 1$

$$\begin{array}{r} 1111 \\ +\,0110 \\ \hline 1 \end{array}$$

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Step 3.) 1+1+1=1, with carry 1

$$\begin{array}{r} {\scriptstyle 1\,1} \\ 1111 \\ +\,0110 \\ \hline 101 \end{array}$$

Step 4.) 1+1=0, with carry 1

$$\begin{array}{r} {\scriptstyle 1\,1} \\ 1111 \\ +\,0110 \\ \hline 10101 \end{array}$$

Overview

Data representation in the computer

- Most modern computers deal with memory as chunks of bits of sizes that are powers of 2
- A byte of memory is usually defined as 8-bits
- Data is represented by a fixed number bytes, with the amount of bytes dependent on the kind of data

Overview

Integral numbers

- We only have a limited number of bytes to represent these data in the computer, with the amount dependent on the kind of data
- For the purposes of our discussion, let's say that we only have a half-byte (4-bits) to represent the integral numbers
- The positional system makes it easy to represent positive numbers using straight binary encoding:
1	0	1	1
  - Which could represent

$$1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 11$$

- This is great for positive numbers, but *how can we represent negative numbers along side positive numbers?*

Sign magnitude notation

- One idea is to reserve the most significant bit (that furtherest to the left) as a sign bit.
  - The idea being that if this bit is set (1), we have a negative number
  - Otherwise, we have a positive number
  - We could then encode -3 as

1	0	1	1

    would be represent

$$-1 \times 1 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = -3$$

  - Positive 3 would then be encoded as

0	0	1	1

    would be represent

$$-1 \times 0 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 3$$

Sign magnitude notation

- This method proves problematic with basic arithmetic:

$$\begin{array}{r} 0011 \\ +\,1011 \\ \hline 1110 \end{array}$$

- That is,

$$3 + -3 = -6$$

- A result that mathematicians would consider wrong

Sign magnitude notation

- Another 'problem' with this technique is that we have two values encoding zero:
  - | 1 | 0 | 0 | 0 | would represent

    $$-1 \times 1 + 0 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 = -0$$

  - | 0 | 0 | 0 | 0 | would represent

    $$-1 \times 0 + 0 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 = 0$$

A second idea: additive inverses

- Based on the limitations of sign magnitude notation, we'd really like an encoding scheme such that

$$x + (-x) = 0 \mid x \in \mathbb{Z}$$

  - That is, store negative numbers as the additive inverses of their positive counterparts

- How we encode a negative number is answered by figuring out what value we would add to its positive counterpart to get zero:

$$\begin{array}{r} 0011 \\ +\,???? \\ \hline 0000 \end{array}$$

  - There is no value that we can use in place of the question marks to make this work...

A second idea: additive inverses

- Recall, that our values can be represented by a limited number of bytes depending on the kind of value
- Anything beyond that is lost
- Therefore, we do not need the total to be zero, but need four digits of zero
- What we're really looking for is:

$$\begin{array}{r} 0011 \\ +\ ???? \\ \hline 10000 \end{array}$$

  - This problem is solvable because the most significant bit will be lost...

A second idea: additive inverses

- We would therefore encode -3 as 1101 because

$$\begin{array}{r} 0011 \\ +\,1101 \\ \hline 10000 \end{array}$$

- This way of encoding signed numbers is known as **two's complement**
  - To get the two's compliment of any positive integral binary number, flip all the bits and add one
  - Note: *the most significant bit here is not exactly a sign bit, though it can be used to determine whether the integral binary value encoded is positive or negative*

Two's complement

- The common way to represent integral numbers in memory is using binary form for positive numbers and by its twos complement when negative
- What range of values can we represent using straight binary coding and twos complement using half-byte of memory?
  - Largest (positive) integer:

0	1	1	1

    $= 2^{4-1} - 1 = 7$

  - Smallest (most negative) integer:

1	0	0	0

    $= -2^{4-1} = -8$

Overview

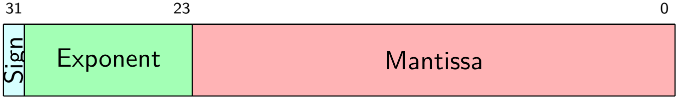
Normalized exponential form

- Every decimal number N_D can be written as a power of ten in *exponential form*
- This form is not unique

$$N_D = 111 = 0.111 \times 10^3 = 1.11 \times 10^2 = 11.1 \times 10^1 = 11100 \times 10^{-2}$$

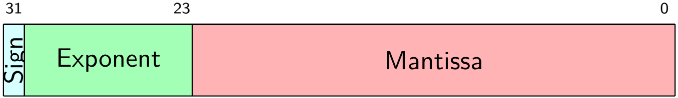
- However, we can write any non-zero N_D uniquely as a number M multiplied by a power of ten e by ensuring that the decimal point appears directly in front of the first non-zero digit in N_D
  - This is known as *normalized exponential form*
  - The number M is called the *mantissa* of N_D
  - The exponent e is the *exponent* of N_D
  - Note the difference between the definition of *normalized exponential form* and *scientific notation*
- Binary numbers, like decimal numbers, can be written in *normalized exponential form* using powers of 2 opposed to 10

Floating-point numbers



- Using normalized exponential form of the value to be encoded, we can store a floating-point number at a memory location partitioned into three fields

Floating-point numbers



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  - The first bit denotes the sign s of the number

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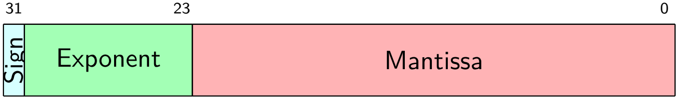
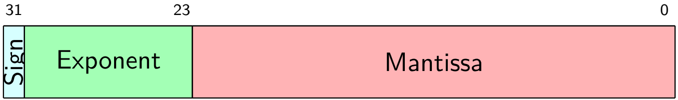


- Using normalized exponential form of the value to be encoded, we can store a floating-point number at a memory location partitioned into three fields
  - The first bit denotes the sign s of the number
  - The second field, the exponent e of the number
    - Few computers store e in binary form when positive or zero; twos complement when negative
    - Others use a bias, such that the exponent is calculated as $2^{(e-\text{bias})}$ where the *bias*is $2^{n-1} - 1$ such that n is the number of bits of the exponent field

Floating-point numbers



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  - The third field is the mantissa m of the number, and contains the fractional part of the number in normalized binary form

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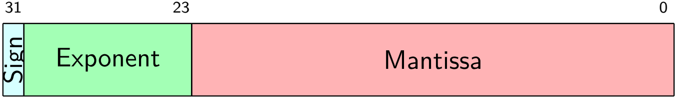

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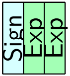
  - The third field is the mantissa m of the number, and contains the fractional part of the number in normalized binary form
- The floating-point value encoded can thus calculated as

$$(-1)^s * 2^{(e-\text{bias})} * (1 + m)$$

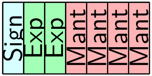
Floating-point numbers

- Let's look at a "toy" floating-point number system, an exponent range of -1 to 1, and four bits for the mantissa
- The left-most bit of this "toy" floating-point system will be the sign bit:

- We then need to be able to represent an exponent range of -1 to 1, a range of three values, meaning we need at two bits for the exponent:



- We then need to provide four bits for the mantissa:



Floating-point numbers

- Using our "toy" floating-point number system, we could represent 3.5 (with the exponent encoded with a bias and mantissa assuming a leading 1) as follows:

0	1	0	1	1	0	0

- We can then calculate the encoded floating-point value as

$$(-1)^s * 2^{(e-\text{bias})} * (1+m)$$

$$(-1)^0 * 2^{(2-((2^{2-1})-1))} * (1+0.75)$$

$$(-1)^0 * 2^{2-1} * 1.75$$

$$(-1)^0 * 3.5$$

$$3.5$$

Floating-point numbers

- Special encodings
  - There are positive and negative infinity values, where the exponent is all 1-bits and the mantissa is all 0-bits
  - There are special not a number (or NaN) values, where the exponent is all 1-bits and the mantissa is not all 0-bits
  - There are also a positive and a negative zero values, differing in the sign bit, where all other bits are 0

Floating-point numbers

- It is also important to note that floating-point numbers are only approximations
0	0	0	0	0	0	1

    $$(-1)^0 * 2^{0-1} * (1 + .0625) = 0.53125$$
0	0	0	0	0	1	0

    $$(-1)^0 * 2^{0-1} * (1 + .125) = 0.5625$$
  - $0.56250 - 0.53125 = 0.03125$

Floating-point numbers

- Note: the gap between adjacent floating point numbers increases with each power of two:
0	0	1	0	0	0	0

    $$(-1)^0 * 2^{1-1} * 1.00 = 1.000$$
0	0	1	0	0	0	1

    $$(-1)^0 * 2^{1-1} * (1 + .0625) = 1.0625$$
  - $1.0625 - 1.0 = 0.0625$

Floating-point numbers

- Floating-point numbers fundamentally imprecise because they represent fractional values with a finite number of bits
- Therefore, it makes sense that
  - `.15 + .15`
  - evaluates to `0.29999999999999999` on my machine
  - because `.15` cannot be represented exactly and instead is stored as `.14999999999999999`
- because the digits simply cut off when you get to the end of the mantissa
- so floating-point numbers are not exactly represented in the computer; a a rounding scheme is employed to deal with this
- an implication of this is that you cannot be certain that two floating point numbers are equal if they were calculated using arithmetic

Floating-point values

- Try calculating the value of 0.6 in binary by hand... would you feel comfortable if your bank used floating-point values to store monetary values?

Conversion errors

- A terminating decimal fraction could be converted into a nonterminating binary fraction
  - An example is decimal value 0.6, which is $0.\overline{1001}$ in binary
- This would indeed result in an approximation of the true value in the computer
- Such conversion errors are normally small; however, they can be propagated through calculations

Propagation of errors

- Assume computer truncates all numerical values to four decimal digits
- A floating-point number $A = \frac{2}{3}$ would be stored as 0.6666; the relative error would then be

$$r = \frac{\frac{2}{3} - 0.6666}{\frac{2}{3}} = 0.0001$$

- Notice what happens when A is added to itself six times:
  - $0.6666 + 0.6666 = 1.333$
  - $1.333 + 0.6666 = 1.999$
  - $1.999 + 0.6666 = 2.665$
  - $2.665 + 0.6666 = 3.331$
  - $3.331 + 0.6666 = 3.997$
- In this case, the relative error is propagated across the additions

$$r = \frac{4 - 3.997}{4} = 0.00075$$

Overview

Character data

- Another way to represent data in the computer is using binary-coded decimal (BCB) codes
- Initially, 6-bit BCB codes were used

B	A	8	4	2	1

  - The light green bits, labeled B and A, encoded zone bits
  - The light blue bits, labeled 8, 4, 2, and 1, encoded numeric bits
  - This scheme allowed 2^6 64 characters to be encoded (10 digits, 26 letters, and 28 special characters)
- This 6-bit code is usually represented in the computer as 7-bits, with the additional bit is known as a check bit

P	B	A	8	4	2	1

- Frequently, we require more than the 28 special characters provided under any 6-bit BCB code
  - 8-bit codes, comprised of four zone bits and four 8-4-2-1 bits, were therefore developed
  - Extended Binary-Coded Decimal Interchange Code (EBCDIC) by IBM
  - American Standard Code for Information Interchange (ASCII-8)

Overview

References

- Lewis, M. C. (2015). *Introduction to the art of programming using Scala*. CRC Press.
- Lipschutz, S. (1987). *Schaum's Outline of Essential Computer Mathematics*. McGraw-Hill.
- Stroustrup, B. (2014). *Programming: principles and practice using C++* (2nd ed.). Pearson Education.

Overview

Overview

Base-b to decimal conversion

- A base-b number N_b can be converted to base-10 by writing N_b in expanded notation and calculating by decimal arithmetic
- We can also perform this conversion through the following algorithm:

  Integral part
  i. Multiply the leftmost digit by base b
  ii. Add the next digit to the right to that product
  iii. Multiply that sum by the base b
  iv. Add the next digit to that product
  v. Repeat until the right-most digit (that before the decimal point) is added

  Fractional part
  i. Multiply the rightmost fractional digit by $\frac{1}{b}$
  ii. Add the next digit to the left to that product
  iii. Multiply that sum by $\frac{1}{b}$
  iv. Add the next digit to that product
  v. Repeat until the leftmost digit (prior to decimal point) is added
  vi. Then multiply that sum by $\frac{1}{b}$

Overview

Decimal to base-b conversion

- Given decimal number N, with integral part N_I and fractional part N_F, N can be converted to base-b using the following algorithm:

  Integral part
  1. Divide N_I and each succeeding quotient by b until a zero quotient is obtained
  2. The sequence of remainders in reverse order is the corresponding representation of N_I in base-b

  Fractional part
  1. Multiply N_F and the *fractional* portion of each succeeding product by b until a zero (or repeating) fractional part is observed
  2. The resultant sequence of integral parts in-order gives the corresponding representation of N_F in base-b

Overview

Binary multiplication

- Can reduce multiplication of binary numbers to multiplication by bits and addition
- Simpler than decimal multiplication; multiplying a number by the multiplicand 1 or 0 yields either the multiplier or zero
- Let's consider the following example:

$$\begin{array}{r} 10011 \\ \times\ 110 \\ \hline \end{array}$$

Binary Multiplication Example

Step 1.) Multiplication: 10011×0

$$\begin{array}{r} 10011 \\ \times\ 110 \\ \hline 00000 \end{array}$$

Binary Multiplication Example

Step 1.) Multiplication: 10011×0

$$\begin{array}{r} 10011 \\ \times\ 110 \\ \hline 00000 \end{array}$$

Step 2.) Multiplication: 10011×1

$$\begin{array}{r} 10011 \\ \times\ 110 \\ \hline 00000 \\ 10011\_ \end{array}$$

Binary Multiplication Example

Step 1.) Multiplication: 10011×0

$$\begin{array}{r} 10011 \\ \times\, 110 \\ \hline 00000 \end{array}$$

Step 2.) Multiplication: 10011×1

$$\begin{array}{r} 10011 \\ \times\, 110 \\ \hline 00000 \\ 10011\_ \end{array}$$

Step 3.) Summation

$$\begin{array}{r} 10011 \\ \times\, 110 \\ \hline 00000 \\ +\, 10011\_ \\ \hline 100110 \end{array}$$

Binary Multiplication Example

Step 1.) Multiplication: 10011×0

$$\begin{array}{r} 10011 \\ \times\, 110 \\ \hline 00000 \end{array}$$

Step 2.) Multiplication: 10011×1

$$\begin{array}{r} 10011 \\ \times\, 110 \\ \hline 00000 \\ 10011\_ \end{array}$$

Step 3.) Summation

$$\begin{array}{r} 10011 \\ \times\, 110 \\ \hline 00000 \\ +\, 10011\_ \\ \hline 100110 \end{array}$$

Step 4.) Multiplication: 10011×1

$$\begin{array}{r} 10011 \\ \times\, 110 \\ \hline 00000 \\ +\, 10011\_ \\ \hline 100110 \\ 10011\_\_ \end{array}$$

Binary Multiplication Example

Step 1.) Multiplication: 10011×0

$$\begin{array}{r} 10011 \\ \times\ 110 \\ \hline 00000 \end{array}$$

Step 2.) Multiplication: 10011×1

$$\begin{array}{r} 10011 \\ \times\ 110 \\ \hline 00000 \\ 10011\underline{\ \ } \end{array}$$

Step 3.) Summation

$$\begin{array}{r} 10011 \\ \times\ 110 \\ \hline 00000 \\ +\ 10011\underline{\ \ } \\ \hline 100110 \end{array}$$

Step 4.) Multiplication: 10011×1

$$\begin{array}{r} 10011 \\ \times\ 110 \\ \hline 00000 \\ +\ 10011\underline{\ \ } \\ \hline 100110 \\ 10011\underline{\ \ \ \ } \end{array}$$

Step 5.) Summation

$$\begin{array}{r} 10011 \\ \times\ 110 \\ \hline 00000 \\ +\ 10011\underline{\ \ } \\ \hline {\scriptstyle 1\,1}\ \ \ \ \ \ \\ 100110 \\ +\ 10011\underline{\ \ \ \ } \\ \hline 1110010 \end{array}$$

Overview

Binary subtraction

- Binary subtraction can be conducted similarly to the decimal subtraction:
  - If the subtrahend digit is greater than the minuend digit, appropriate from the next column to the left
  - Subtract the lower value from the upper value
    - $0 - 0 = 0$
    - $1 - 0 = 1$
    - $1 - 1 = 0$
    - $0 - 1 = 1$, with a borrow of 1 from the next column
- Let's consider the following example:

$$\begin{array}{r} 10110 \\ -\ 01101 \\ \hline \end{array}$$

Binary subtraction example

Step 1.) $1(\text{borrowed}) - 1 = 1$

$$
\begin{array}{r}
{\scriptstyle 0\,1} \\
101\cancel{1}\cancel{0} \\
-\ 01101 \\
\hline
1
\end{array}
$$

Binary subtraction example

Step 1.) $1(\text{borrowed}) - 1 = 1$

$$\begin{array}{r} 0\,1 \\ 101\cancel{10} \\ -\,01101 \\ \hline 1 \end{array}$$

Step 2.) $0 - 0 = 0$

$$\begin{array}{r} 0\,1 \\ 101\cancel{10} \\ -\,01101 \\ \hline 01 \end{array}$$

Binary subtraction example

Step 1.) $1(\text{borrowed}) - 1 = 1$

$$\begin{array}{r} 101\overset{0}{\cancel{1}}\overset{1}{\cancel{0}} \\ -\ 01101 \\ \hline 1 \end{array}$$

Step 2.) $0 - 0 = 0$

$$\begin{array}{r} 101\overset{0}{\cancel{1}}\overset{1}{\cancel{0}} \\ -\ 01101 \\ \hline 01 \end{array}$$

Step 3.) $1 - 1 = 0$

$$\begin{array}{r} 101\overset{0}{\cancel{1}}\overset{1}{\cancel{0}} \\ -\ 01101 \\ \hline 001 \end{array}$$

Binary subtraction example

Step 1.) $1(\text{borrowed}) - 1 = 1$

$$\begin{array}{r} {\scriptstyle 0\,1} \\ 101\cancel{10} \\ -\;01101 \\ \hline 1 \end{array}$$

Step 2.) $0 - 0 = 0$

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Step 3.) $1 - 1 = 0$

$$\begin{array}{r} {\scriptstyle 0\,1} \\ 101\cancel{10} \\ -\;01101 \\ \hline 001 \end{array}$$

Step 4.) $1(\text{borrowed}) - 1 = 1$

$$\begin{array}{r} {\scriptstyle 0\,1\quad 0\,1} \\ \cancel{10}1\cancel{10} \\ -\;01101 \\ \hline 1001 \end{array}$$

Binary subtraction example

Step 1.) $1(\text{borrowed}) - 1 = 1$

$$\begin{array}{r} 101\overset{0}{\cancel{1}}\overset{1}{\cancel{0}} \\ -\ 01101 \\ \hline 1 \end{array}$$

Step 2.) $0 - 0 = 0$

$$\begin{array}{r} 101\overset{0}{\cancel{1}}\overset{1}{\cancel{0}} \\ -\ 01101 \\ \hline 01 \end{array}$$

Step 3.) $1 - 1 = 0$

$$\begin{array}{r} 101\overset{0}{\cancel{1}}\overset{1}{\cancel{0}} \\ -\ 01101 \\ \hline 001 \end{array}$$

Step 4.) $1(\text{borrowed}) - 1 = 1$

$$\begin{array}{r} \overset{0}{\cancel{1}}\overset{1}{\cancel{0}}1\overset{0}{\cancel{1}}\overset{1}{\cancel{0}} \\ -\ 01101 \\ \hline 1001 \end{array}$$

Step 5.) $0 - 0 = 0$

$$\begin{array}{r} \overset{0}{\cancel{1}}\overset{1}{\cancel{0}}1\overset{0}{\cancel{1}}\overset{1}{\cancel{0}} \\ -\ 01101 \\ \hline 01001 \end{array}$$

Binary subtraction example 2

$$\begin{array}{r} 100 \\ -\,011 \\ \hline \end{array}$$

Binary subtraction example 2

Step 1.)

$$\begin{array}{r} {\scriptstyle 0\,1\,1} \\ \cancel{100} \\ -\,011 \\ \hline 1 \end{array}$$

$$\begin{array}{r} 100 \\ -\,011 \\ \hline \end{array}$$

Binary subtraction example 2

$$
\begin{array}{r}
100 \\
-\ 011 \\
\hline
\end{array}
$$

Step 1.)

$$
\begin{array}{r}
{\scriptstyle 0\,1\,1} \\
\cancel{100} \\
-\ 011 \\
\hline
1
\end{array}
$$

Step 2.)

$$
\begin{array}{r}
{\scriptstyle 0\,1\,1} \\
\cancel{100} \\
-\ 011 \\
\hline
01
\end{array}
$$

Binary subtraction example 2

$$\begin{array}{r} 100 \\ -\ 011 \\ \hline \end{array}$$

Step 1.)

$$\begin{array}{r} {\scriptstyle 0\,1\,1} \\ \cancel{100} \\ -\ 011 \\ \hline 1 \end{array}$$

Step 2.)

$$\begin{array}{r} {\scriptstyle 0\,1\,1} \\ \cancel{100} \\ -\ 011 \\ \hline 01 \end{array}$$

Step 3.)

$$\begin{array}{r} {\scriptstyle 0\,1\,1} \\ \cancel{100} \\ -\ 011 \\ \hline 001 \end{array}$$

Overview

Binary division

- Binary division can be reduced to multiplying the divisor by individual digits of the dividend followed by subtraction
- Let's consider the following example:

$$11\overline{)10111}$$

Binary division example

Step 1.)

$$
\begin{array}{r@{\,}l l}
 & 1 & \\
11 & \overline{)10111} & \\
 & \underline{11} & \text{(Mult.)} \\
 & 101 & \text{(Subt.)}
\end{array}
$$

Binary division example

Step 1.)

$$
\begin{array}{r@{}l}
 & 1 \\
11 & \big| \overline{10111} \\
 & \underline{11} \quad \text{(Mult.)} \\
 & 101 \quad \text{(Subt.)}
\end{array}
$$

Step 2.)

$$
\begin{array}{r@{}l}
 & 11 \\
11 & \big| \overline{10111} \\
 & \underline{11} \quad \text{(Mult.)} \\
 & 101 \quad \text{(Subt.)} \\
 & \underline{11} \quad \text{(Mult.)}
\end{array}
$$

Binary division example

Step 1.)

$$\begin{array}{rll} & 1 & \\ 11 \,| & \overline{10111} & \\ & \underline{11} & \text{(Mult.)} \\ & 101 & \text{(Subt.)} \end{array}$$

Step 2.)

$$\begin{array}{rll} & 11 & \\ 11 \,| & \overline{10111} & \\ & \underline{11} & \text{(Mult.)} \\ & 101 & \text{(Subt.)} \\ & \underline{11} & \text{(Mult.)} \end{array}$$

Step 3.)

$$\begin{array}{rll} & 11 & \\ 11 \,| & \overline{10111} & \\ & \underline{11} & \text{(Mult.)} \\ & 101 & \text{(Subt.)} \\ & \underline{11} & \text{(Mult.)} \\ & 101 & \text{(Subt.)} \end{array}$$

Binary division example

Step 1.)

$$\begin{array}{rl} 1 & \\ 11\overline{)10111} & \\ \underline{11} & \text{(Mult.)} \\ 101 & \text{(Subt.)} \end{array}$$

Step 2.)

$$\begin{array}{rl} 11 & \\ 11\overline{)10111} & \\ \underline{11} & \text{(Mult.)} \\ 101 & \text{(Subt.)} \\ \underline{11} & \text{(Mult.)} \end{array}$$

Step 3.)

$$\begin{array}{rl} 11 & \\ 11\overline{)10111} & \\ \underline{11} & \text{(Mult.)} \\ 101 & \text{(Subt.)} \\ \underline{11} & \text{(Mult.)} \\ 101 & \text{(Subt.)} \end{array}$$

Step 4.)

$$\begin{array}{rl} 111 & \\ 11\overline{)10111} & \\ \underline{11} & \text{(Mult.)} \\ 101 & \text{(Subt.)} \\ \underline{11} & \text{(Mult.)} \\ 101 & \text{(Subt.)} \\ \underline{11} & \text{(Mult.)} \end{array}$$

Binary division example

Step 1.)

$$\begin{array}{rl}
1 & \\
11|\overline{10111} & \\
\underline{11} & \text{(Mult.)}\\
101 & \text{(Subt.)}
\end{array}$$

Step 2.)

$$\begin{array}{rl}
11 & \\
11|\overline{10111} & \\
\underline{11} & \text{(Mult.)}\\
101 & \text{(Subt.)}\\
\underline{11} & \text{(Mult.)}
\end{array}$$

Step 3.)

$$\begin{array}{rl}
11 & \\
11|\overline{10111} & \\
\underline{11} & \text{(Mult.)}\\
101 & \text{(Subt.)}\\
\underline{11} & \text{(Mult.)}\\
101 & \text{(Subt.)}
\end{array}$$

Step 4.)

$$\begin{array}{rl}
111 & \\
11|\overline{10111} & \\
\underline{11} & \text{(Mult.)}\\
101 & \text{(Subt.)}\\
\underline{11} & \text{(Mult.)}\\
101 & \text{(Subt.)}\\
\underline{11} & \text{(Mult.)}
\end{array}$$

Step 5.)

$$\begin{array}{rl}
111 & \\
11|\overline{10111} & \\
\underline{11} & \text{(Mult.)}\\
101 & \text{(Subt.)}\\
\underline{11} & \text{(Mult.)}\\
101 & \text{(Subt.)}\\
\underline{11} & \text{(Mult.)}\\
R = 10 & \text{(Subt.)}
\end{array}$$

Overview

Decimal complements

- Given the decimal number N, the nines (radix-minus one) complement can be found by subtracting each digit of N from 9
  - Given $N = 1234$, the nines complement is:

$$\begin{array}{r} 9999 \\ -\,1234 \\ \hline 8765 \end{array}$$

- The tens (radix) complement of that number N can then be found by adding 1 its nines complement
  - Given the nines complement of $N = 1234$ (calculated above), the tens complement can be calculated by adding one:

$$\begin{array}{r} 8765 \\ +\,1 \\ \hline 8766 \end{array}$$

Decimal complements and subtraction

- Given integers X and Y such that $X < Y$ and each integer is composed of four digits, the difference $Z = Y - X$ can be rewritten as
  - $Z = Y - X + (9999 + 1 - 10000)$
  - $Z = Y + (9999 - X + 1) - 10000)$
  - $Z = Y + [(9999 - X) + 1] - 10000$
- We can thus calculate the difference Z
  - By adding the tens complement of X to Y
  - And subsequently subtracting 10000
    - Notice the implications of using a positional system here... subtracting 10000 is as simple as deleting the leading 1 from the sum in this particular instance

Decimal complements and subtraction example

- Let's consider $Z = Y - X$ when $Y = 1012$ and $X = 0381$ (case: $X < Y$)

- Using normal subtraction,

$$\begin{array}{r} {\scriptstyle 0\,9\,1} \\ \cancel{10}12 \\ -\ 381 \\ \hline 631 \end{array}$$

- Notice our need to borrow

- Using the tens complement of X, which is its nines complement 9618 plus 1 (9619)

$$\begin{array}{r} {\scriptstyle 1} \\ 1012 \\ +\ 9619 \\ \hline \cancel{1}0631 \end{array}$$

- Note: we added a trailing zero to X to make it the same 'length' as Y; when calculating its nines complement, $9 - 0$ is 9 at that place value

Decimal complements and subtraction example 2

- Let's now consider the $Z = Y - X$ when $Y = 1012$ and $X = 2017$ (case: $X > Y$)

- Using normal subtraction,

$$\begin{array}{r} 1012 \\ -\ 2017 \\ \hline -1005 \end{array}$$

- Using the tens complement of X, which is its nines complement 7982 plus 1 (7983)

$$\begin{array}{r} 1012 \\ +\ 7983 \\ \hline 8995 \end{array}$$

- As there is no overflow into the fifth-column, we need to take the negative of the tens complement of the sum to 'find' the desired difference *for our interpretation*
  - $$\begin{array}{r} 9999 \\ -\ 8995 \\ \hline 1004 \\ +\ 1 \\ \hline 1005 \end{array}$$
  - Negate $1005 \rightarrow -1005$

Binary complements

- For a binary number X, the ones complement (i.e., the base-2 radix-minus-one complement) can be obtained by subtracting each bit of X from 1; this is equivalent to inverting the bits
- Twos complement (i.e., the radix complement of the base-2 system) is then obtained by adding 1 to the ones complement of X

Binary complement and subtraction example

- We can evaluate the difference $Z = Y - X$, where $X = 1011$ and $Y = 1110$ by adding the twos complement of X to Y (case: $X < Y$)

$$\begin{array}{r} {\scriptstyle 1} \\ 1110 \\ +\,0101 \\ \hline \cancel{1}0011 \end{array}$$

- If we were working on a system that implemented 4-bit registers, the leading one would be lost automatically as the result would exceed the capacity of the register holding the it (Z)
  - When using complements to reduce subtraction to addition, overflow will always occur whenever X is less than Y

Binary complement and subtraction example 2

- Let's now evaluate the difference $Z = Y - X$, where $X = 1110$ and $Y = 1011$ by adding the twos complement of X to Y (case: $X > Y$)

$$\begin{array}{r} 1011 \\ +\,0010 \\ \hline 1101 \end{array}$$

- Again, assuming 4-bit registers, note in this case ($X > Y$) that there is no leading one that is automatically lost
  - We therefore need to take the negative of the twos complement of the sum to 'find' the desired difference *for our interpretation*
  -
$$\begin{array}{r} 1111 \\ -\,1101 \\ \hline 0010 \\ +\,1 \\ \hline 0011 \end{array}$$
  - Negate $0011 \to -0011$