Advanced Calculus Summary

Mathematical Notes

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Contents

1 Multivariable Functions

1.1 Limits and Continuity

Definition 1.1. A function $f : \mathbb{R}^n \to \mathbb{R}$ has limit L at point $\mathbf{a}$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that:

$$0 < \|\mathbf{x} - \mathbf{a}\| < \delta \Rightarrow |f(\mathbf{x}) - L| < \epsilon$$

1.2 Partial Derivatives

Definition 1.2. The **partial derivative** of $f(x, y)$ with respect to x is:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

1.3 Clairaut's Theorem

Theorem 1.1. If f has continuous second partial derivatives, then:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

1.4 Chain Rule for Multivariable Functions

Theorem 1.2. If $z = f(x, y)$ where $x = g(t)$ and $y = h(t)$, then:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

1.5 Implicit Differentiation

For $F(x, y, z) = 0$:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

2 Directional Derivatives and Gradients

2.1 Directional Derivative

Definition 2.1. The **directional derivative** of f at $\mathbf{a}$ in direction $\mathbf{u}$ is:

$$D_{\mathbf{u}} f(\mathbf{a}) = \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a})}{h}$$

2.2 Gradient

Definition 2.2. The **gradient** of $f(x, y, z)$ is:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

2.3 Relationship

Theorem 2.1.

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$$

where $\mathbf{u}$ is a unit vector.

2.4 Properties of Gradient

- ∇f points in the direction of maximum increase
- $|\nabla f|$ is the maximum rate of change
- ∇f is perpendicular to level curves/surfaces

3 Tangent Planes and Linear Approximation

3.1 Tangent Plane

For surface $z = f(x, y)$ at point $(a, b, f(a, b))$:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

3.2 Linear Approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

3.3 Total Differential

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

4 Maximum and Minimum Values

4.1 Critical Points

Definition 4.1. A **critical point** of $f(x, y)$ is a point (a, b) where either:

- $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or
- $f_x(a, b)$ or $f_y(a, b)$ doesn't exist

4.2 Second Derivative Test

Theorem 4.1. Let $D = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$ at critical point (a, b):

- If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a local minimum
- If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a local maximum
- If $D < 0$, then (a, b) is a saddle point
- If $D = 0$, the test is inconclusive

4.3 Lagrange Multipliers

Theorem 4.2. To find extrema of $f(x,y,z)$ subject to constraint $g(x,y,z)=k$:

$$\nabla f = \lambda \nabla g$$

for some scalar λ.

5 Multiple Integrals

5.1 Double Integrals

Definition 5.1.

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

5.2 Properties of Double Integrals

- $\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$
- $\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA$
- If $f(x,y) \geq g(x,y)$ on R, then $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$

5.3 Polar Coordinates

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

5.4 Triple Integrals

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

5.5 Cylindrical Coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$
$$dV = r\,dz\,dr\,d\theta$$

5.6 Spherical Coordinates

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$
$$dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

6 Vector Fields

6.1 Definition

Definition 6.1. A **vector field** on $\mathbb{R}^n$ is a function $\mathbf{F}$ that assigns to each point $\mathbf{x}$ a vector $\mathbf{F}(\mathbf{x})$.

6.2 Gradient Fields

Definition 6.2. A vector field $\mathbf{F}$ is **conservative** if $\mathbf{F} = \nabla f$ for some scalar function f.

6.3 Divergence

Definition 6.3.

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

6.4 Curl

Definition 6.4.

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

7 Line Integrals

7.1 Line Integral of Scalar Function

$$\int_C f(x, y, z)\, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\, dt$$

7.2 Line Integral of Vector Field

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

7.3 Fundamental Theorem for Line Integrals

Theorem 7.1. If $\mathbf{F} = \nabla f$ and C is any curve from A to B, then:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$$

7.4 Independence of Path

Theorem 7.2. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if and only if $\mathbf{F}$ is conservative.

8 Green's Theorem

8.1 Green's Theorem

Theorem 8.1.

$$\oint_C P\, dx + Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

where C is the positively oriented boundary of region D.

8.2 Area Using Green's Theorem

$$A = \frac{1}{2} \oint_C x\, dy - y\, dx$$

9 Surface Integrals

9.1 Parametric Surfaces

A surface S can be parametrized as:

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

9.2 Surface Area

$$A = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

9.3 Surface Integral of Scalar Function

$$\iint_S f(x,y,z) \, dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

9.4 Surface Integral of Vector Field

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

10 Divergence Theorem

10.1 Divergence Theorem (Gauss's Theorem)

Theorem 10.1.

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \nabla \cdot \mathbf{F} \, dV$$

where S is the boundary of solid region E.

11 Stokes' Theorem

11.1 Stokes' Theorem

Theorem 11.1.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

where C is the boundary of surface S.

12 Sequences and Series of Functions

12.1 Pointwise Convergence

Definition 12.1. Sequence $\{f_n\}$ converges pointwise to f if:

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ for each } x \in D$$

12.2 Uniform Convergence

Definition 12.2. Sequence $\{f_n\}$ converges uniformly to f if:

$$\lim_{n\to\infty} \sup_{x\in D} |f_n(x) - f(x)| = 0$$

12.3 Weierstrass M-Test

Theorem 12.1. If $|f_n(x)| \leq M_n$ for all $x \in D$ and $\sum M_n$ converges, then $\sum f_n(x)$ converges uniformly.

13 Power Series

13.1 Radius of Convergence

Theorem 13.1. For power series $\sum_{n=0}^{\infty} a_n(x-c)^n$:

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$$

13.2 Operations on Power Series

- **Addition**: $\sum a_n x^n + \sum b_n x^n = \sum (a_n + b_n)x^n$
- **Multiplication**: $(\sum a_n x^n)(\sum b_n x^n) = \sum c_n x^n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$
- **Differentiation**: $\frac{d}{dx}[\sum a_n x^n] = \sum n a_n x^{n-1}$
- **Integration**: $\int [\sum a_n x^n]\, dx = \sum \frac{a_n}{n+1} x^{n+1} + C$

14 Fourier Series

14.1 Fourier Coefficients

For function f with period 2π:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\, dx$$

14.2 Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

14.3 Complex Form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$

14.4 Parseval's Theorem

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2)$$

15 Partial Differential Equations

15.1 Heat Equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

15.2 Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

15.3 Laplace's Equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

15.4 Method of Separation of Variables

Assume solution of form $u(x,t) = X(x)T(t)$ and substitute into PDE.

16 Complex Analysis

16.1 Complex Functions

Definition 16.1. A **complex function** is a function $f : \mathbb{C} \to \mathbb{C}$.

16.2 Cauchy-Riemann Equations

For $f(z) = u(x,y) + iv(x,y)$ to be differentiable:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

16.3 Cauchy's Integral Theorem

Theorem 16.1. If f is analytic in simply connected domain D and C is a closed curve in D, then:

$$\oint_C f(z)\,dz = 0$$

16.4 Cauchy's Integral Formula

Theorem 16.2. If f is analytic inside and on simple closed curve C, then for any point a inside C:

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

16.5 Residue Theorem

Theorem 16.3.

$$\oint_C f(z)\,dz = 2\pi i \sum \operatorname{Res}(f, a_k)$$

where the sum is over all isolated singularities inside C.

17 Important Theorems

17.1 Mean Value Theorem for Integrals

Theorem 17.1. If f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that:

$$\int_a^b f(x)\,dx = f(c)(b - a)$$

17.2 Fubini's Theorem

Theorem 17.2. If f is continuous on rectangle $R = [a, b] \times [c, d]$, then:

$$\iint_R f(x, y)\,dA = \int_a^b \int_c^d f(x, y)\,dy\,dx = \int_c^d \int_a^b f(x, y)\,dx\,dy$$

17.3 Change of Variables

Theorem 17.3.

$$\iint_R f(x, y)\,dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\,dv$$

17.4 Implicit Function Theorem

Theorem 17.4. If $F(x, y) = 0$ and $\frac{\partial F}{\partial y} \neq 0$ at (a, b), then there exists a function $y = f(x)$ such that $F(x, f(x)) = 0$ near (a, b).

18 Applications

18.1 Optimization Problems

- Find extrema of functions of several variables
- Constrained optimization using Lagrange multipliers
- Applications in economics, physics, engineering

18.2 Volume and Surface Area

- Triple integrals for volume calculations
- Surface integrals for surface area
- Applications in geometry and physics

18.3 Flux and Circulation

- Line integrals for work and circulation
- Surface integrals for flux
- Applications in fluid dynamics and electromagnetism

18.4 Heat and Wave Propagation

- Fourier series for periodic phenomena
- PDEs for modeling physical processes
- Applications in physics and engineering