Mathematical Inconsistencies, Paradoxes, and Contradictions

Mathematical Notes

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Contents

1 Foundational Paradoxes

1.1 Russell's Paradox

Paradox 1.1 (Russell's Paradox)**.** Let $R = \{x : x \notin x\}$ be the set of all sets that do not contain themselves. Then:

- If $R \in R$, then by definition $R \notin R$ (contradiction)
- If $R \notin R$, then by definition $R \in R$ (contradiction)

Theorem 1.1. The naive set theory (unrestricted comprehension) is inconsistent.

1.2 Cantor's Paradox

Paradox 1.2 (Cantor's Paradox)**.** Let U be the universal set containing all sets. Then $P(U)$ (the power set of U) has cardinality $2^{|U|}$. But since U contains all sets, $P(U) \subseteq U$, which implies $2^{|U|} \leq |U|$, contradicting Cantor's theorem that $|A| < |P(A)|$ for any set A.

1.3 Burali-Forti Paradox

Paradox 1.3 (Burali-Forti Paradox)**.** Let Ω be the set of all ordinal numbers. Then Ω is well-ordered and transitive, so Ω is an ordinal. But then $\Omega \in \Omega$, which contradicts the fact that no ordinal can be an element of itself.

2 Logical Paradoxes

2.1 The Liar Paradox

Paradox 2.1 (The Liar Paradox)**.** Consider the statement: "This statement is false."

- If the statement is true, then it is false (contradiction)
- If the statement is false, then it is true (contradiction)

2.2 Epimenides Paradox

Paradox 2.2 (Epimenides Paradox)**.** A Cretan says: "All Cretans are liars."

- If the Cretan is telling the truth, then all Cretans are liars, including himself
- If the Cretan is lying, then not all Cretans are liars, so he might be telling the truth

2.3 Berry Paradox

Paradox 2.3 (Berry Paradox)**.** Consider the smallest positive integer not definable in fewer than twelve words. This integer is defined in eleven words, creating a contradiction.

2.4 Grelling-Nelson Paradox

Paradox 2.4 (Grelling-Nelson Paradox)**.** A word is **heterological** if it does not describe itself. For example, "long" is heterological because "long" is not a long word. But is "heterological" heterological?

- If "heterological" is heterological, then it describes itself, so it's not heterological
- If "heterological" is not heterological, then it doesn't describe itself, so it is heterological

3 Set-Theoretic Paradoxes

3.1 Skolem's Paradox

Paradox 3.1 (Skolem's Paradox)**.** Skolem's paradox states that if Zermelo-Fraenkel set theory (ZFC) is consistent, then it has a countable model, even though ZFC proves the existence of uncountable sets.

3.2 The Banach-Tarski Paradox

Paradox 3.2 (Banach-Tarski Paradox)**.** Given a solid ball in 3-dimensional space, there exists a decomposition of the ball into a finite number of disjoint subsets, which can then be put back together in a different way to yield two identical copies of the original ball.

Theorem 3.1 (Banach-Tarski Theorem)**.** If A and B are bounded subsets of $\mathbb{R}^3$ with non-empty interior, then A and B are equidecomposable.

3.3 Hausdorff Paradox

Paradox 3.3 (Hausdorff Paradox)**.** There exists a countable set D such that $S^2 \setminus D$ is equidecomposable with two copies of itself.

4 Probability Paradoxes

4.1 Bertrand's Paradox

Paradox 4.1 (Bertrand's Paradox)**.** What is the probability that a random chord of a circle is longer than the side of an inscribed equilateral triangle? Different methods give different answers:

- Method 1: Random endpoints on circumference $\rightarrow P = \frac{1}{3}$
- Method 2: Random midpoint in circle $\rightarrow P = \frac{1}{4}$
- Method 3: Random perpendicular distance $\rightarrow P = \frac{1}{2}$

4.2 Monty Hall Problem

Paradox 4.2 (Monty Hall Problem)**.** In a game show, you choose one of three doors. Behind one door is a car, behind the others are goats. The host opens a door revealing a goat and offers you the chance to switch. Should you switch?

Theorem 4.1. Switching gives you a $\frac{2}{3}$ probability of winning, while staying gives only $\frac{1}{3}$.

4.3 Simpson's Paradox

Paradox 4.3 (Simpson's Paradox)**.** A trend appears in different groups of data but disappears or reverses when these groups are combined.

Example 4.1. Consider two treatments A and B:

- Treatment A: 20/40 patients recover (50%)
- Treatment B: 10/20 patients recover (50%)
- Overall: Treatment A appears better
- But when stratified by severity: Treatment B is better in both groups

5 Geometric Paradoxes

5.1 The Missing Square Puzzle

Paradox 5.1 (Missing Square Puzzle)**.** Rearrange four pieces of a right triangle to form the same triangle, but with a "missing" square unit.

5.2 The Coastline Paradox

Paradox 5.2 (Coastline Paradox)**.** The length of a coastline depends on the scale of measurement. As the measurement scale becomes smaller, the measured length increases without bound.

5.3 Zeno's Paradoxes

Paradox 5.3 (Zeno's Paradox of Motion)**.** To reach a destination, you must first reach the halfway point. To reach the halfway point, you must reach the quarter-way point, and so on. Since there are infinitely many points to reach, motion is impossible.

Paradox 5.4 (Achilles and the Tortoise)**.** Achilles runs after a tortoise. When Achilles reaches the tortoise's starting position, the tortoise has moved ahead. When Achilles reaches that position, the tortoise has moved ahead again, and so on. Achilles never catches the tortoise.

6 Infinite Series Paradoxes

6.1 Grandi's Series

Paradox 6.1 (Grandi's Series)**.** The series $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ can be evaluated in different ways:

- Grouping: $(1 - 1) + (1 - 1) + \cdots = 0 + 0 + \cdots = 0$
- Alternative grouping: $1 + (-1 + 1) + (-1 + 1) + \cdots = 1 + 0 + 0 + \cdots = 1$
- Cesàro sum: $\frac{1}{2}$

6.2 Riemann Rearrangement Theorem

Theorem 6.1 (Riemann Rearrangement Theorem)**.** If a series $\sum a_n$ converges conditionally, then for any real number L, there exists a rearrangement of the series that converges to L.

6.3 The Harmonic Series

Paradox 6.2 (Harmonic Series Divergence)**.** The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges to $\ln(2)$.

7 Computational Paradoxes

7.1 The Halting Problem

Theorem 7.1 (Halting Problem)**.** There is no algorithm that can determine whether an arbitrary computer program will halt or run forever.

Proof. Assume there exists a function $H(P, I)$ that returns true if program P halts on input I, false otherwise. Define:

$$D(P) = \begin{cases} \text{halt} & \text{if } H(P, P) = \text{false} \\ \text{loop forever} & \text{if } H(P, P) = \text{true} \end{cases}$$

Then $H(D, D)$ cannot be determined without contradiction. □

7.2 Rice's Theorem

Theorem 7.2 (Rice's Theorem)**.** For any non-trivial property of partial functions, it is undecidable whether a given Turing machine computes a partial function with that property.

7.3 The Busy Beaver Problem

Definition 7.1. The **busy beaver function** $\Sigma(n)$ is the maximum number of 1s that can be written by an n-state Turing machine before halting.

Theorem 7.3. The busy beaver function is non-computable and grows faster than any computable function.

8 Mathematical Inconsistencies

8.1 Inconsistent Axiom Systems

Definition 8.1. An axiom system is **inconsistent** if it can prove both a statement and its negation.

8.2 Naive Set Theory

Theorem 8.1. Naive set theory with unrestricted comprehension is inconsistent.

8.3 Inconsistent Mathematics

Definition 8.2. Paraconsistent logic allows for the study of inconsistent theories without everything becoming provable (explosion principle).

9 Philosophical Paradoxes

9.1 The Sorites Paradox

Paradox 9.1 (Sorites Paradox (Heap Paradox)).

- One grain of sand is not a heap
- Adding one grain to a non-heap does not make it a heap
- Therefore, no number of grains makes a heap

9.2 The Ship of Theseus

Paradox 9.2 (Ship of Theseus). If all the parts of a ship are replaced over time, is it still the same ship?

9.3 The Raven Paradox

Paradox 9.3 (Raven Paradox (Hempel's Paradox)). The statement "All ravens are black" is logically equivalent to "All non-black things are non-ravens." But observing a white shoe seems to confirm the second statement, and thus the first, which seems counterintuitive.

10 Statistical Paradoxes

10.1 The Birthday Paradox

Paradox 10.1 (Birthday Paradox). In a group of 23 people, there is a 50% chance that two people share the same birthday.

10.2 The Prosecutor's Fallacy

Paradox 10.2 (Prosecutor's Fallacy). Confusing the probability of evidence given innocence with the probability of innocence given evidence.

10.3 The Base Rate Fallacy

Paradox 10.3 (Base Rate Fallacy). Ignoring the base rate when evaluating conditional probabilities.

11 Game-Theoretic Paradoxes

11.1 The Prisoner's Dilemma

Paradox 11.1 (Prisoner's Dilemma). Two prisoners must decide whether to cooperate or defect. The Nash equilibrium (both defect) is worse for both than mutual cooperation.

11.2 Newcomb's Problem

Paradox 11.2 (Newcomb's Problem). A predictor offers you two boxes: Box A (always contains $1000) and Box B (contains $1,000,000 if predicted you'd take only Box B, $0 otherwise). What should you choose?

11.3 The St. Petersburg Paradox

Paradox 11.3 (St. Petersburg Paradox)**.** A fair coin is flipped until heads appears. If heads appears on the n-th flip, you win $\$2^n$. The expected value is infinite, but most people wouldn't pay much to play.

12 Resolution Attempts

12.1 Axiomatic Set Theory

- Zermelo-Fraenkel (ZF) axioms
- Axiom of Choice
- Axiom of Regularity
- Axiom of Replacement

12.2 Type Theory

- Russell's type theory
- Simple type theory
- Dependent type theory

12.3 Category Theory

- Topos theory
- Internal logic
- Synthetic differential geometry

12.4 Constructive Mathematics

- Intuitionistic logic
- Brouwer's intuitionism
- Bishop's constructive analysis

13 Modern Approaches

13.1 Non-Standard Analysis

Definition 13.1. Non-standard analysis uses hyperreal numbers to rigorously handle infinitesimals and infinite numbers.

13.2 Paraconsistent Logic

Definition 13.2. Paraconsistent logic allows contradictions without everything being provable.

13.3 Modal Logic

Definition 13.3. **Modal logic** extends classical logic with operators for necessity and possibility.

14 Open Problems

14.1 Continuum Hypothesis

Theorem 14.1 (Gödel-Cohen)**.** The continuum hypothesis is independent of ZFC.

14.2 Consistency of Mathematics

Theorem 14.2 (Gödel's Second Incompleteness Theorem)**.** If a formal system is consistent, it cannot prove its own consistency.

14.3 The Axiom of Choice

Theorem 14.3 (Zermelo)**.** The axiom of choice is independent of ZF.

15 Implications for Mathematics

15.1 Foundational Crisis

- Crisis in set theory (early 20th century)
- Crisis in analysis (infinitesimals)
- Crisis in geometry (non-Euclidean)

15.2 Mathematical Pluralism

- Multiple valid mathematical frameworks
- Different axioms lead to different mathematics
- No single "true" mathematics

15.3 Computational Limitations

- Undecidable problems
- Incomputable functions
- Complexity barriers

16 Conclusion

Mathematical paradoxes and inconsistencies have played a crucial role in the development of mathematics, leading to:

- More rigorous foundations

- New mathematical structures
- Deeper understanding of logic
- Recognition of limitations
- Philosophical insights

These paradoxes remind us that mathematics is a human construction, subject to the limitations of our logical systems and requiring constant refinement and revision.