Plasma Physics Summary

Mathematical Notes

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Contents

1 Introduction to Plasma Physics

1.1 What is Plasma?

Definition 1.1. A **plasma** is a state of matter consisting of a collection of charged particles (ions and electrons) that is electrically neutral on average and exhibits collective behavior.

1.2 Plasma Parameters

- **Plasma frequency**: $\omega_p = \sqrt{\frac{n_e e^2}{m_e \epsilon_0}}$
- **Debye length**: $\lambda_D = \sqrt{\frac{\epsilon_0 k_B T_e}{n_e e^2}}$
- **Plasma parameter**: $g = \frac{1}{n_e \lambda_D^3}$
- **Collision frequency**: $\nu_{ei} = \frac{n_i e^4 \ln \Lambda}{4\pi \epsilon_0^2 m_e^{1/2} (k_B T_e)^{3/2}}$

1.3 Plasma Conditions

For a gas to be considered a plasma:

1. $\lambda_D \ll L$ (Debye length much smaller than system size)
2. $N_D = n_e \lambda_D^3 \gg 1$ (many particles in Debye sphere)
3. $\omega_p \tau \gg 1$ (plasma frequency much larger than collision time)

2 Magnetohydrodynamics (MHD)

2.1 MHD Equations

The MHD equations describe the macroscopic behavior of plasmas:

Theorem 2.1 (MHD Continuity Equation)**.**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

Theorem 2.2 (MHD Momentum Equation)**.**

$$\rho \frac{D\vec{v}}{Dt} = -\nabla p + \vec{J} \times \vec{B} + \rho \vec{g}$$

Theorem 2.3 (MHD Induction Equation)**.**

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \eta \nabla^2 \vec{B}$$

where $\eta = \frac{1}{\mu_0 \sigma}$ is the magnetic diffusivity.

Theorem 2.4 (MHD Energy Equation)**.**

$$\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = 0$$

where γ is the adiabatic index.

2.2 MHD Approximations

- **Ideal MHD**: $\sigma \to \infty$ (perfect conductor)
- **Resistive MHD**: Finite conductivity included
- **Hall MHD**: Electron inertia effects included

2.3 Magnetic Reynolds Number

Definition 2.1. The **magnetic Reynolds number** is:

$$R_m = \frac{vL}{\eta} = \mu_0 \sigma v L$$

where v is characteristic velocity and L is characteristic length.

2.4 Frozen-in Theorem

Theorem 2.5 (Alfvén's Frozen-in Theorem)**.** In ideal MHD, magnetic field lines are "frozen" into the plasma flow:

$$\frac{d\Phi}{dt} = 0$$

where Φ is magnetic flux through a surface moving with the plasma.

3 Plasma Waves and Instabilities

3.1 Electromagnetic Waves in Plasma

Theorem 3.1 (Dispersion Relation for Electromagnetic Waves)**.**

$$\omega^2 = \omega_p^2 + c^2 k^2$$

3.2 Electrostatic Waves

- **Electron plasma waves**: $\omega^2 = \omega_p^2 + \frac{3k_B T_e}{m_e} k^2$
- **Ion acoustic waves**: $\omega^2 = \frac{k_B T_e}{m_i} k^2$
- **Upper hybrid waves**: $\omega^2 = \omega_p^2 + \omega_c^2$
- **Lower hybrid waves**: $\omega^2 = \frac{\omega_{ci}\omega_{ce}}{1+\frac{\omega_p^2}{\omega_c^2}}$

3.3 Magnetohydrodynamic Waves

- **Alfvén waves**: $v_A = \frac{B}{\sqrt{\mu_0 \rho}}$
- **Magnetosonic waves**: $v_{ms} = \sqrt{v_A^2 + c_s^2}$
- **Slow magnetosonic waves**: $v_s = \frac{v_A c_s}{\sqrt{v_A^2 + c_s^2}}$

3.4 Plasma Instabilities

Definition 3.1. A **plasma instability** occurs when small perturbations grow exponentially in time.

3.4.1 Rayleigh-Taylor Instability

Theorem 3.2 (Rayleigh-Taylor Growth Rate)**.**

$$\gamma = \sqrt{gk\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}}$$

where $\rho_2 > \rho_1$ and g is the acceleration.

3.4.2 Kelvin-Helmholtz Instability

Theorem 3.3 (Kelvin-Helmholtz Growth Rate)**.**

$$\gamma = \frac{k\Delta v}{2}\sqrt{\frac{\rho_1\rho_2}{(\rho_1 + \rho_2)^2}}$$

where Δv is the velocity shear.

3.4.3 Two-Stream Instability

Theorem 3.4 (Two-Stream Instability Condition)**.**

$$\omega^2 = \omega_p^2\left(1 + \frac{v_0^2}{v_{th}^2}\right)$$

where v_0 is the relative velocity between streams and v_{th} is thermal velocity.

4 Fusion Physics

4.1 Thermonuclear Fusion

Definition 4.1. **Thermonuclear fusion** is the process of combining light atomic nuclei to form heavier nuclei, releasing energy.

4.2 Fusion Reactions

- **Deuterium-Tritium**: $D + T \rightarrow^4 He + n + 17.6$ MeV
- **Deuterium-Deuterium**: $D + D \rightarrow^3 He + n + 3.3$ MeV
- **Deuterium-Deuterium**: $D + D \rightarrow T + p + 4.0$ MeV
- **Proton-Proton**: $p + p \rightarrow D + e^+ + \nu_e + 0.42$ MeV

4.3 Fusion Conditions

Theorem 4.1 (Lawson Criterion)**.** For energy breakeven:

$$n\tau \geq \frac{12k_BT}{\langle\sigma v\rangle E_f}$$

where n is density, τ is confinement time, T is temperature, $\langle\sigma v\rangle$ is reaction rate, and E_f is fusion energy per reaction.

4.4 Tokamak Physics

Definition 4.2. A **tokamak** is a toroidal magnetic confinement device for controlled fusion.

4.4.1 Magnetic Confinement

- **Toroidal field**: $B_\phi = \frac{B_0R_0}{R}$
- **Poloidal field**: Generated by plasma current
- **Safety factor**: $q = \frac{rB_\phi}{RB_\theta}$

4.4.2 Plasma Equilibrium

Theorem 4.2 (Grad-Shafranov Equation)**.**

$$\Delta^*\psi = -\mu_0R^2\frac{dp}{d\psi} - F\frac{dF}{d\psi}$$

where ψ is the poloidal flux function and $F = RB_\phi$.

4.5 Fusion Power Density

Theorem 4.3 (Fusion Power Density)**.**

$$P_f = \frac{1}{4}n^2\langle\sigma v\rangle E_f$$

5 Space and Astrophysical Plasmas

5.1 Solar Wind

Definition 5.1. The **solar wind** is a stream of charged particles ejected from the Sun's corona.

5.1.1 Solar Wind Properties

- **Slow solar wind**: $v \approx 400$ km/s, $n \approx 10^7$ cm^{-3}
- **Fast solar wind**: $v \approx 800$ km/s, $n \approx 3 \times 10^6$ cm^{-3}
- **Temperature**: $T \approx 10^5 - 10^6$ K

5.2 Magnetosphere

Definition 5.2. The **magnetosphere** is the region around a planet dominated by its magnetic field.

5.2.1 Earth's Magnetosphere

- **Bow shock**: Standoff distance $\approx 10 - 15 R_E$
- **Magnetopause**: Boundary between solar wind and magnetosphere
- **Magnetotail**: Extended region downstream from Earth

5.3 Stellar Atmospheres

- **Photosphere**: Visible surface, $T \approx 5800$ K
- **Chromosphere**: Transition region, $T \approx 10^4$ K
- **Corona**: Outer atmosphere, $T \approx 10^6$ K

5.4 Interstellar Medium

- **Cold neutral medium**: $T \approx 100$ K, $n \approx 30$ cm^{-3}
- **Warm neutral medium**: $T \approx 8000$ K, $n \approx 0.3$ cm^{-3}
- **Warm ionized medium**: $T \approx 8000$ K, $n \approx 0.1$ cm^{-3}
- **Hot ionized medium**: $T \approx 10^6$ K, $n \approx 0.003$ cm^{-3}

6 Computational Plasma Physics

6.1 Particle-in-Cell (PIC) Method

Definition 6.1. The **Particle-in-Cell method** simulates plasmas by following individual particles in self-consistent electromagnetic fields.

6.1.1 PIC Algorithm

1. Initialize particles and fields
2. Push particles using Lorentz force
3. Deposit charge and current densities
4. Solve Maxwell's equations for fields
5. Repeat for next time step

6.2 Magnetohydrodynamic Simulations

- **Finite difference methods**: Direct discretization of MHD equations
- **Finite volume methods**: Conservative formulation
- **Spectral methods**: Fourier decomposition
- **Adaptive mesh refinement**: Variable resolution

6.3 Kinetic Simulations

- **Vlasov equation**: $\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \frac{q}{m}(\vec{E} + \vec{v} \times \vec{B}) \cdot \nabla_v f = 0$
- **Gyrokinetic theory**: Averaged over gyromotion
- **Drift kinetic theory**: Includes drift motion

7 Plasma Diagnostics

7.1 Electromagnetic Diagnostics

- **Magnetic probes**: Measure magnetic field fluctuations
- **Electric probes**: Measure electric field and plasma potential
- **Interferometry**: Measure electron density
- **Thomson scattering**: Measure electron temperature and density

7.2 Particle Diagnostics

- **Langmuir probes**: Current-voltage characteristics
- **Retarding field analyzers**: Energy distribution functions
- **Neutral particle analyzers**: Ion temperature and density
- **Fast ion diagnostics**: Energetic particle measurements

7.3 Spectroscopic Diagnostics

- **Optical emission spectroscopy**: Line intensities and widths
- **X-ray spectroscopy**: Bremsstrahlung and line emission
- **Neutron diagnostics**: Fusion product measurements

8 Applications

8.1 Fusion Energy

- **Magnetic confinement fusion**: Tokamaks, stellarators
- **Inertial confinement fusion**: Laser and particle beam drivers
- **Magnetized target fusion**: Hybrid approaches

8.2 Space Physics

- **Solar-terrestrial interactions**: Space weather
- **Planetary magnetospheres**: Comparative planetology
- **Astrophysical plasmas**: Stellar and galactic physics

8.3 Industrial Applications

- **Plasma processing**: Semiconductor manufacturing
- **Plasma propulsion**: Electric spacecraft propulsion
- **Plasma medicine**: Medical applications
- **Materials processing**: Surface modification

9 Important Constants and Parameters

9.1 Fundamental Constants

- Electron charge: $e = 1.602 \times 10^{-19}$ C
- Electron mass: $m_e = 9.109 \times 10^{-31}$ kg
- Proton mass: $m_p = 1.673 \times 10^{-27}$ kg
- Permittivity of free space: $\epsilon_0 = 8.854 \times 10^{-12}$ F/m
- Permeability of free space: $\mu_0 = 4\pi \times 10^{-7}$ H/m
- Boltzmann constant: $k_B = 1.381 \times 10^{-23}$ J/K
- Speed of light: $c = 2.998 \times 10^{8}$ m/s

9.2 Plasma Parameters

- Classical electron radius: $r_e = \frac{e^2}{4\pi\epsilon_0 m_e c^2} = 2.818 \times 10^{-15}$ m
- Bohr radius: $a_0 = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} = 5.292 \times 10^{-11}$ m
- Fine structure constant: $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = 7.297 \times 10^{-3}$
- Electron cyclotron frequency: $\omega_{ce} = \frac{eB}{m_e}$
- Ion cyclotron frequency: $\omega_{ci} = \frac{eB}{m_i}$

9.3 Fusion Parameters

- DT fusion cross-section peak: $\sigma_{max} \approx 5 \times 10^{-28}$ m^2 at $T \approx 100$ keV
- Lawson criterion for DT: $n\tau \geq 10^{20}$ s/m^3 at $T \approx 10$ keV
- ITER parameters: $R = 6.2$ m, $a = 2.0$ m, $B = 5.3$ T, $I_p = 15$ MA