Differential Geometry Summary

Mathematical Notes

October 19, 2025

Contents

1 Manifolds

1.1 Definition and Basic Properties

Definition 1.1. A **topological manifold** of dimension n is a topological space M such that:

1. M is Hausdorff
2. M is second countable
3. Every point $p \in M$ has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$

Definition 1.2. A **smooth manifold** is a topological manifold equipped with a smooth atlas, i.e., a collection of charts (U_α, ϕ_α) such that the transition maps $\phi_\beta \circ \phi_\alpha^{-1}$ are smooth.

1.2 Examples of Manifolds

Example 1.1.
- $\mathbb{R}^n$ is an n-dimensional manifold
- $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ is an n-dimensional sphere
- The torus $T^2 = S^1 \times S^1$ is a 2-dimensional manifold
- The projective plane $\mathbb{RP}^2$ is a 2-dimensional manifold

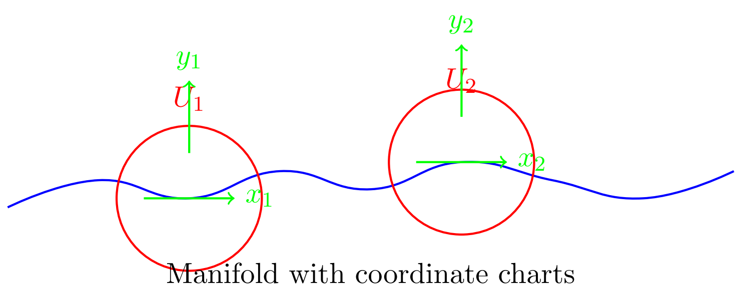


Figure 1: A manifold with overlapping coordinate charts

2 Tangent Spaces and Vectors

2.1 Tangent Vectors

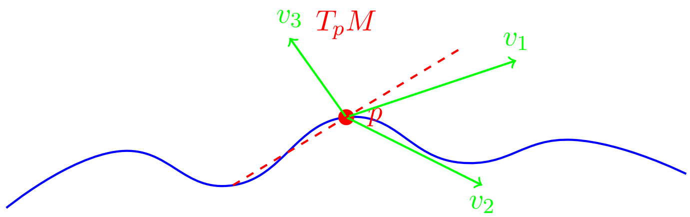
Definition 2.1. A **tangent vector** at point $p \in M$ is an equivalence class of smooth curves $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$, where two curves are equivalent if they have the same derivative in any coordinate chart.

Definition 2.2. The **tangent space** T_pM at point p is the vector space of all tangent vectors at p.

2.2 Tangent Bundle

Definition 2.3. The **tangent bundle** TM is the disjoint union of all tangent spaces:

$$TM = \bigcup_{p \in M} T_pM$$



Tangent space at point p

Figure 2: Tangent vectors and tangent space

3 Vector Fields

3.1 Definition

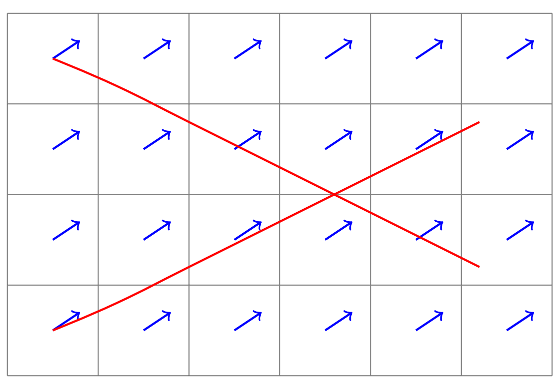
Definition 3.1. A **vector field** on manifold M is a smooth assignment of a tangent vector to each point of M.

3.2 Lie Bracket

Definition 3.2. The **Lie bracket** of vector fields X and Y is the vector field $[X, Y]$ defined by:

$$[X, Y]f = X(Yf) - Y(Xf)$$

for any smooth function f.



Vector field with integral curves

Figure 3: Vector field and its integral curves

4 Differential Forms

4.1 Definition

Definition 4.1. A **differential k-form** on manifold M is a smooth assignment of an alternating k-linear form on each tangent space.

4.2 Exterior Derivative

Definition 4.2. The **exterior derivative** d is a linear operator on differential forms satisfying:

1. $d^2 = 0$
2. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$
3. For functions f, df is the usual differential

4.3 Stokes' Theorem

Theorem 4.1. Let M be an oriented n-dimensional manifold with boundary, and let ω be an $(n-1)$-form. Then:

$$\int_M d\omega = \int_{\partial M} \omega$$

5 Riemannian Geometry

5.1 Riemannian Metric

Definition 5.1. A **Riemannian metric** on manifold M is a smooth assignment of an inner product on each tangent space.

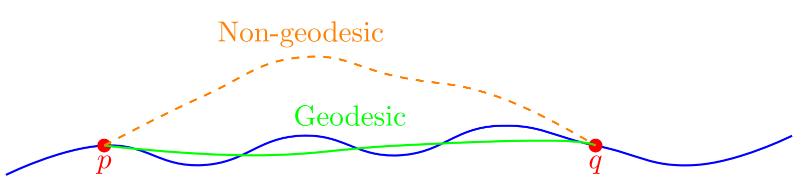
Definition 5.2. A **Riemannian manifold** is a smooth manifold equipped with a Riemannian metric.

5.2 Length and Distance

Definition 5.3. The **length** of a curve $\gamma : [a, b] \to M$ is:

$$L(\gamma) = \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))}\, dt$$

Definition 5.4. The **distance** between two points is the infimum of lengths of curves connecting them.



Geodesic vs. non-geodesic path

Figure 4: Geodesic as shortest path on a curved surface

6 Curvature

6.1 Levi-Civita Connection

Definition 6.1. The **Levi-Civita connection** is the unique torsion-free, metric-compatible connection on a Riemannian manifold.

6.2 Riemann Curvature Tensor

Definition 6.2. The **Riemann curvature tensor** is defined by:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

6.3 Scalar and Ricci Curvature

Definition 6.3. The **Ricci curvature tensor** is:

$$\mathrm{Ric}(X,Y) = \mathrm{tr}(Z \mapsto R(Z,X)Y)$$

Definition 6.4. The **scalar curvature** is the trace of the Ricci tensor.

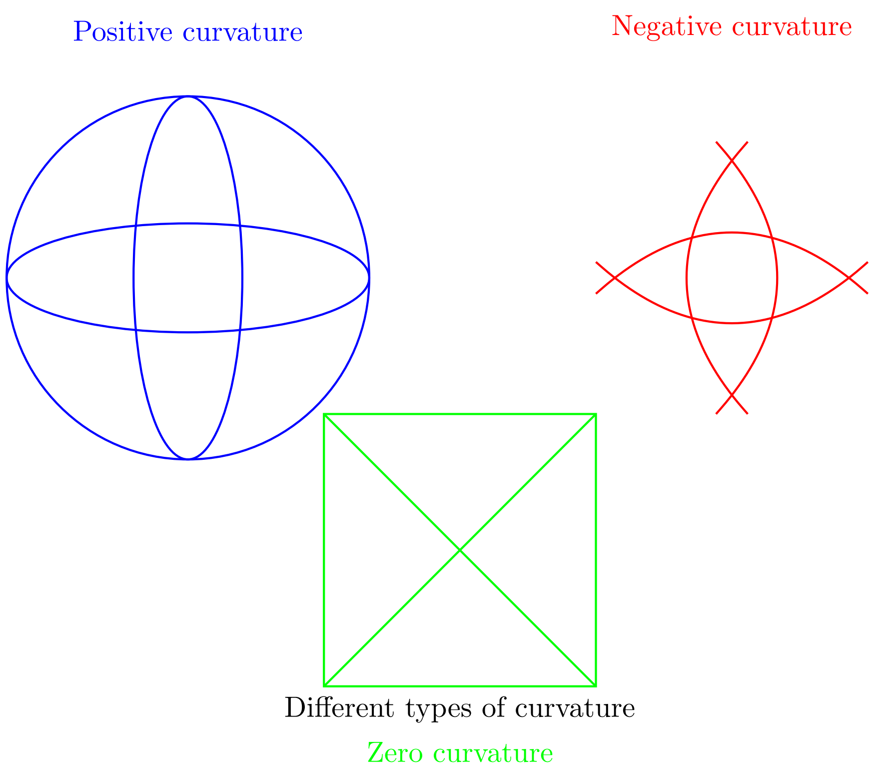


Figure 5: Positive, negative, and zero curvature surfaces

7 Geodesics

7.1 Definition

Definition 7.1. A **geodesic** is a curve whose tangent vector is parallel transported along itself.

7.2 Geodesic Equation

Theorem 7.1. A curve $\gamma(t)$ is a geodesic if and only if it satisfies the geodesic equation:

$$\frac{d^2\gamma^k}{dt^2} + \Gamma^k_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0$$

where Γ^k_{ij} are the Christoffel symbols.

8 Submanifolds

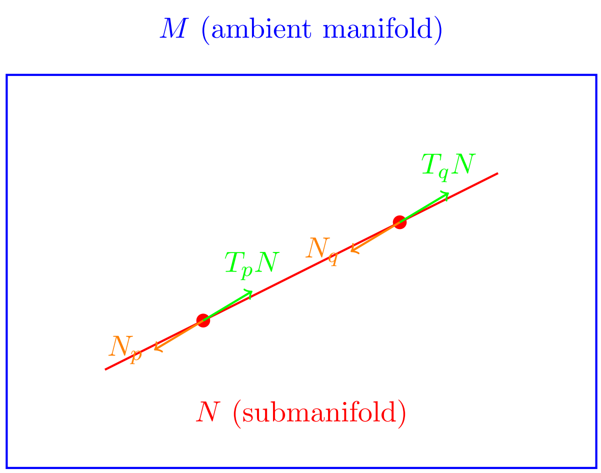
8.1 Definition

Definition 8.1. A **submanifold** of manifold M is a subset $N \subseteq M$ that is itself a manifold with the induced topology and smooth structure.

8.2 Embedding and Immersion

Definition 8.2. A smooth map $f : N \to M$ is an **embedding** if it is a diffeomorphism onto its image.

Definition 8.3. A smooth map $f : N \to M$ is an **immersion** if its derivative is injective at every point.



Submanifold with tangent and normal spaces

Figure 6: Submanifold with tangent and normal vectors

9 Lie Groups

9.1 Definition

Definition 9.1. A **Lie group** is a smooth manifold that is also a group, where the group operations are smooth.

9.2 Lie Algebra

Definition 9.2. The **Lie algebra** of a Lie group G is the tangent space at the identity T_eG equipped with the Lie bracket.

9.3 Examples

Example 9.1.

- $GL(n, \mathbb{R})$ - general linear group
- $SO(n)$ - special orthogonal group
- $SU(n)$ - special unitary group
- S^1 - circle group

10 Fiber Bundles

10.1 Definition

Definition 10.1. A **fiber bundle** is a quadruple (E, B, F, π) where:

- E is the total space
- B is the base space
- F is the fiber
- $\pi : E \to B$ is the projection map

10.2 Vector Bundles

Definition 10.2. A **vector bundle** is a fiber bundle where each fiber is a vector space and the transition functions are linear.

11 Applications

11.1 Physics

Differential geometry is fundamental to:

- General relativity (spacetime as a Lorentzian manifold)
- Gauge theory (fiber bundles)
- Classical mechanics (symplectic geometry)
- Quantum mechanics (complex manifolds)

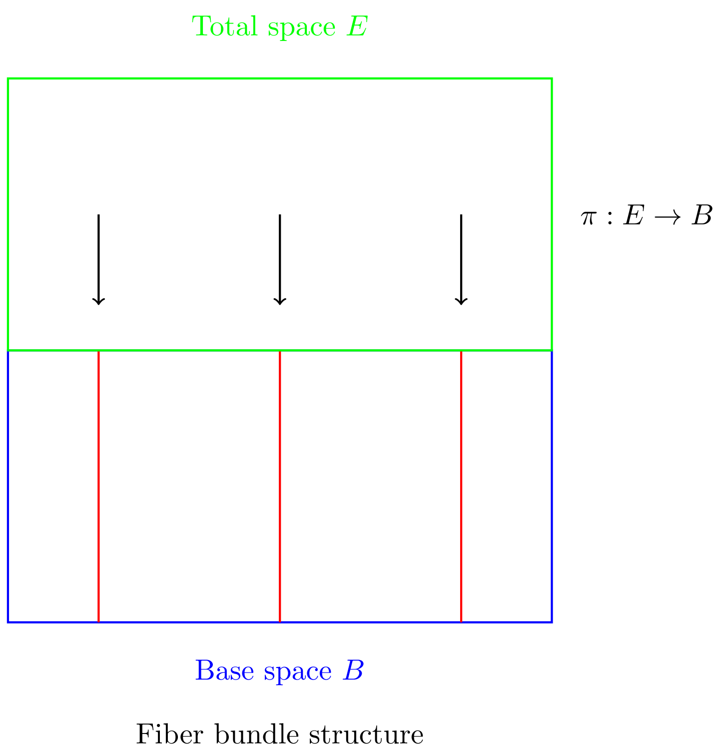


Figure 7: Fiber bundle with base space, total space, and fibers

11.2 Computer Graphics

Applications include:

- Surface modeling
- Mesh processing
- Shape analysis
- Animation

11.3 Robotics

Used in:

- Configuration spaces
- Motion planning
- Control theory
- Sensor fusion

12 Important Theorems

12.1 Gauss-Bonnet Theorem

Theorem 12.1. For a compact 2-dimensional Riemannian manifold M:

$$\int_M K\,dA = 2\pi\chi(M)$$

where K is the Gaussian curvature and $\chi(M)$ is the Euler characteristic.

12.2 Myers' Theorem

Theorem 12.2. If a complete Riemannian manifold has Ricci curvature bounded below by a positive constant, then it is compact.

12.3 Hopf-Rinow Theorem

Theorem 12.3. A Riemannian manifold is geodesically complete if and only if it is complete as a metric space.

12.4 Nash Embedding Theorem

Theorem 12.4. Every Riemannian manifold can be isometrically embedded in some Euclidean space.

13 Advanced Topics

13.1 Symplectic Geometry

Definition 13.1. A **symplectic manifold** is a smooth manifold equipped with a closed, non-degenerate 2-form.

13.2 Complex Geometry

Definition 13.2. A **complex manifold** is a manifold with an atlas of charts to $\mathbb{C}^n$ with holomorphic transition functions.

13.3 Kähler Manifolds

Definition 13.3. A **Kähler manifold** is a complex manifold with a Riemannian metric that is compatible with the complex structure.

13.4 Spin Geometry

Definition 13.4. A **spin manifold** is a Riemannian manifold that admits a spin structure.