Relativity and Quantum Physics Summary

Mathematical Notes

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Contents

1 Special Relativity

1.1 Postulates

1. **Principle of Relativity**: The laws of physics are the same in all inertial reference frames.
2. **Constancy of Light Speed**: The speed of light in vacuum is constant ($c = 2.998 \times 10^8$ m/s) in all inertial frames.

1.2 Lorentz Transformations

For frames moving with relative velocity v along x-axis:

$$x' = \gamma(x - vt) \tag{1}$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{2}$$
$$y' = y \tag{3}$$
$$z' = z \tag{4}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = \frac{v}{c}$.

1.3 Time Dilation and Length Contraction

Theorem 1.1 (Time Dilation)**.**

$$\Delta t = \gamma \Delta t_0$$

where Δt_0 is the proper time (time in the rest frame).

Theorem 1.2 (Length Contraction)**.**

$$L = \frac{L_0}{\gamma}$$

where L_0 is the proper length (length in the rest frame).

1.4 Relativistic Velocity Addition

For velocities u and v along the same direction:

$$u' = \frac{u + v}{1 + \frac{uv}{c^2}}$$

1.5 Relativistic Energy and Momentum

- **Relativistic momentum**: $\vec{p} = \gamma m \vec{v}$
- **Total energy**: $E = \gamma mc^2$
- **Rest energy**: $E_0 = mc^2$
- **Kinetic energy**: $K = (\gamma - 1)mc^2$
- **Energy-momentum relation**: $E^2 = (pc)^2 + (mc^2)^2$

1.6 Four-Vectors

Definition 1.1. A **four-vector** is a quantity that transforms under Lorentz transformations like (ct, x, y, z).

- **Position four-vector**: $x^\mu = (ct, \vec{r})$
- **Energy-momentum four-vector**: $p^\mu = (E/c, \vec{p})$
- **Four-velocity**: $u^\mu = \gamma(c, \vec{v})$
- **Four-force**: $F^\mu = \gamma(\vec{F} \cdot \vec{v}/c, \vec{F})$

2 General Relativity

2.1 Equivalence Principle

Definition 2.1. The **weak equivalence principle** states that the gravitational and inertial masses are equivalent.

Definition 2.2. The **strong equivalence principle** states that the effects of gravity are locally indistinguishable from acceleration.

2.2 Spacetime Curvature

Definition 2.3. Spacetime is a four-dimensional manifold with metric tensor $g_{\mu\nu}$.

The line element is:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

2.3 Einstein Field Equations

Theorem 2.1 (Einstein Field Equations)**.**

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

where:

- $G_{\mu\nu}$ is the Einstein tensor
- Λ is the cosmological constant
- $T_{\mu\nu}$ is the stress-energy tensor

2.4 Schwarzschild Metric

For a spherically symmetric mass M:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

2.5 Black Holes

Definition 2.4. The **Schwarzschild radius** is $r_s = \frac{2GM}{c^2}$.

Definition 2.5. An **event horizon** is a boundary beyond which events cannot affect an outside observer.

3 Quantum Mechanics Fundamentals

3.1 Wave-Particle Duality

Definition 3.1. de Broglie wavelength: $\lambda = \frac{h}{p}$ where h is Planck's constant.

Definition 3.2. Photoelectric effect: $E_k = h\nu - \phi$ where ϕ is the work function.

3.2 Uncertainty Principle

Theorem 3.1 (Heisenberg Uncertainty Principle)**.**

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

where $\hbar = \frac{h}{2\pi}$.

3.3 Schrödinger Equation

Theorem 3.2 (Time-Dependent Schrödinger Equation)**.**

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$$

where $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r}, t)$ is the Hamiltonian operator.

Theorem 3.3 (Time-Independent Schrödinger Equation)**.**

$$\hat{H}\psi = E\psi$$

for stationary states with energy E.

3.4 Quantum States and Operators

Definition 3.3. A **quantum state** is represented by a wave function $\psi(\vec{r}, t)$ normalized as $\int |\psi|^2 d^3r = 1$.

Definition 3.4. An **observable** is represented by a Hermitian operator $\hat{A}$ with eigenvalues a_n and eigenstates $|n\rangle$.

3.5 Measurement and Collapse

Definition 3.5. Upon measurement of observable $\hat{A}$, the state collapses to an eigenstate $|n\rangle$ with probability $|\langle n|\psi\rangle|^2$.

4 Quantum Systems

4.1 Particle in a Box

For a particle in a one-dimensional box of length L:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

4.2 Harmonic Oscillator

Definition 4.1. The **quantum harmonic oscillator** has energy levels:

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

where $n = 0, 1, 2, \ldots$

The wave functions involve Hermite polynomials:

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}} x\right) e^{-\frac{m\omega x^2}{2\hbar}}$$

4.3 Hydrogen Atom

Theorem 4.1 (Bohr Model)**.**

$$r_n = \frac{n^2\hbar^2}{me^2}\frac{4\pi\epsilon_0}{e^2}$$

$$E_n = -\frac{me^4}{2\hbar^2(4\pi\epsilon_0)^2}\frac{1}{n^2}$$

Theorem 4.2 (Quantum Mechanical Solution)**.** The wave function is $\psi_{nlm}(r, \theta, \phi) = R_{nl}(r)Y_l^m(\theta, \phi)$ where:

- n is the principal quantum number
- l is the orbital angular momentum quantum number
- m is the magnetic quantum number

5 Angular Momentum

5.1 Orbital Angular Momentum

Definition 5.1. The **orbital angular momentum operator** is $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$.

- $\hat{L}^2|l, m\rangle = l(l+1)\hbar^2|l, m\rangle$
- $\hat{L}_z|l, m\rangle = m\hbar|l, m\rangle$
- $l = 0, 1, 2, \ldots$ and $m = -l, -l+1, \ldots, l-1, l$

5.2 Spin

Definition 5.2. **Spin** is an intrinsic angular momentum property of particles.

For spin-1/2 particles:

$$\hat{S}^2|s, m_s\rangle = s(s+1)\hbar^2|s, m_s\rangle = \frac{3}{4}\hbar^2|s, m_s\rangle$$

$$\hat{S}_z|s, m_s\rangle = m_s\hbar|s, m_s\rangle$$

where $m_s = \pm\frac{1}{2}$.

5.3 Pauli Matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

6 Identical Particles

6.1 Symmetrization Postulate

Definition 6.1. For identical particles, the wave function must be either:

- **Symmetric** (bosons): $\psi(1, 2) = \psi(2, 1)$
- **Antisymmetric** (fermions): $\psi(1, 2) = -\psi(2, 1)$

6.2 Pauli Exclusion Principle

Theorem 6.1. No two identical fermions can occupy the same quantum state.

6.3 Exchange Symmetry

- **Bosons** (integer spin): Follow Bose-Einstein statistics
- **Fermions** (half-integer spin): Follow Fermi-Dirac statistics

7 Quantum Field Theory

7.1 Field Quantization

Definition 7.1. A **quantum field** is an operator-valued function $\hat{\phi}(x)$ that creates and annihilates particles.

7.2 Creation and Annihilation Operators

- $\hat{a}^\dagger$ creates a particle
- $\hat{a}$ annihilates a particle
- $[\hat{a}, \hat{a}^\dagger] = 1$ (bosons)
- $\{\hat{a}, \hat{a}^\dagger\} = 1$ (fermions)

7.3 Feynman Diagrams

Definition 7.2. Feynman diagrams are graphical representations of particle interactions in quantum field theory.

8 Relativistic Quantum Mechanics

8.1 Klein-Gordon Equation

For spin-0 particles:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\psi = 0$$

8.2 Dirac Equation

For spin-1/2 particles:

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

where γ^μ are the Dirac matrices.

8.3 Antimatter

Definition 8.1. Antiparticles have the same mass but opposite charge and quantum numbers as their corresponding particles.

9 Quantum Entanglement

9.1 Bell States

The maximally entangled two-qubit states:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{5}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \tag{6}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{7}$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{8}$$

9.2 Bell's Theorem

Theorem 9.1. No local hidden variable theory can reproduce all the predictions of quantum mechanics.

9.3 EPR Paradox

The Einstein-Podolsky-Rosen paradox questions the completeness of quantum mechanics and introduces the concept of "spooky action at a distance."

10 Applications

10.1 Quantum Computing

- Quantum gates and circuits
- Quantum algorithms (Shor's, Grover's)
- Quantum error correction

10.2 Quantum Cryptography

- Quantum key distribution (BB84 protocol)
- Quantum teleportation
- Quantum secure communication

10.3 Quantum Sensing

- Quantum metrology
- Gravitational wave detection
- Quantum magnetometry

10.4 Cosmology

- Big Bang theory
- Cosmic microwave background
- Dark matter and dark energy
- Inflation theory

11 Important Constants

- Speed of light: $c = 2.998 \times 10^8$ m/s
- Planck constant: $h = 6.626 \times 10^{-34}$ J·s
- Reduced Planck constant: $\hbar = 1.055 \times 10^{-34}$ J·s
- Gravitational constant: $G = 6.674 \times 10^{-11}$ N·m²/kg²
- Electron charge: $e = 1.602 \times 10^{-19}$ C
- Electron mass: $m_e = 9.109 \times 10^{-31}$ kg
- Proton mass: $m_p = 1.673 \times 10^{-27}$ kg
- Fine structure constant: $\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} \approx \frac{1}{137}$
- Bohr radius: $a_0 = \frac{4\pi\epsilon_0 \hbar^2}{me^2} = 5.292 \times 10^{-11}$ m
- Rydberg constant: $R_\infty = \frac{me^4}{8\epsilon_0^2 h^3 c} = 1.097 \times 10^7$ m^{-1}

12 Key Theorems

12.1 No-Cloning Theorem

Theorem 12.1. It is impossible to create an identical copy of an arbitrary unknown quantum state.

12.2 Unitarity

Theorem 12.2. Quantum evolution is unitary: $\hat{U}^\dagger \hat{U} = \hat{I}$.

12.3 Conservation Laws

- Energy conservation
- Momentum conservation
- Angular momentum conservation
- Charge conservation
- Baryon and lepton number conservation

13 Modern Developments

13.1 Quantum Field Theory

- Standard Model of particle physics
- Quantum electrodynamics (QED)
- Quantum chromodynamics (QCD)
- Electroweak theory

13.2 Quantum Gravity

- String theory
- Loop quantum gravity
- Causal dynamical triangulation
- Asymptotic safety

13.3 Quantum Information

- Quantum error correction
- Quantum communication
- Quantum simulation
- Quantum machine learning