Linear Algebra Summary

Mathematical Notes

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Contents

1 Vector Spaces

1.1 Definition

A vector space V over a field F is a set with two operations:

- Vector addition: $+ : V \times V \to V$
- Scalar multiplication: $\cdot : F \times V \to V$

1.2 Axioms

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $a, b \in F$:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity)
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (associativity)
3. $\exists \mathbf{0} \in V : \mathbf{v} + \mathbf{0} = \mathbf{v}$ (zero vector)
4. $\forall \mathbf{v} \exists (-\mathbf{v}) : \mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ (additive inverse)
5. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ (distributivity)
6. $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$ (distributivity)
7. $(ab)\mathbf{v} = a(b\mathbf{v})$ (associativity)
8. $1\mathbf{v} = \mathbf{v}$ (identity)

1.3 Subspaces

A subset $W \subseteq V$ is a subspace if:

- $\mathbf{0} \in W$
- $\mathbf{u}, \mathbf{v} \in W \Rightarrow \mathbf{u} + \mathbf{v} \in W$
- $\mathbf{v} \in W, a \in F \Rightarrow a\mathbf{v} \in W$

2 Linear Independence and Basis

2.1 Linear Independence

Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent if:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0} \Rightarrow c_1 = c_2 = \cdots = c_n = 0$$

2.2 Span

The span of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n : c_i \in F\}$$

2.3 Basis

A basis for vector space V is a linearly independent set that spans V.

2.4 Dimension

The dimension of V is the number of vectors in any basis for V.

3 Linear Transformations

3.1 Definition

A function $T : V \to W$ is linear if:

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
- $T(a\mathbf{v}) = aT(\mathbf{v})$

3.2 Kernel and Image

- $\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$
- $\mathrm{Im}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\}$

3.3 Rank-Nullity Theorem

$$\dim(\ker(T)) + \dim(\mathrm{Im}(T)) = \dim(V)$$

4 Matrices

4.1 Matrix Operations

For matrices A, B and scalar c:

- $(A + B)_{ij} = A_{ij} + B_{ij}$
- $(cA)_{ij} = cA_{ij}$
- $(AB)_{ij} = \sum_k A_{ik}B_{kj}$

4.2 Special Matrices

- Identity: $I_{ij} = \delta_{ij}$
- Transpose: $(A^T)_{ij} = A_{ji}$
- Symmetric: $A = A^T$
- Skew-symmetric: $A = -A^T$
- Orthogonal: $A^T A = I$

4.3 Matrix Inverses

A^{-1} exists if and only if $\det(A) \neq 0$.

5 Determinants

5.1 Definition

For $n \times n$ matrix A:

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)}$$

5.2 Properties

- $\det(AB) = \det(A)\det(B)$
- $\det(A^T) = \det(A)$
- $\det(cA) = c^n \det(A)$
- Swapping rows changes sign
- Adding multiple of row to another doesn't change determinant

6 Eigenvalues and Eigenvectors

6.1 Definition

For matrix A, λ is an eigenvalue with eigenvector $\mathbf{v}$ if:

$$A\mathbf{v} = \lambda\mathbf{v}$$

6.2 Characteristic Polynomial

$$p(\lambda) = \det(A - \lambda I) = 0$$

6.3 Properties

- Sum of eigenvalues = trace of A
- Product of eigenvalues = determinant of A
- Eigenvectors corresponding to distinct eigenvalues are linearly independent

7 Diagonalization

7.1 Definition

Matrix A is diagonalizable if there exists invertible P such that:

$$P^{-1}AP = D$$

where D is diagonal.

7.2 Conditions

A is diagonalizable if and only if:

- A has n linearly independent eigenvectors, or
- Geometric multiplicity = algebraic multiplicity for each eigenvalue

8 Inner Product Spaces

8.1 Definition

An inner product on V is a function $\langle \cdot, \cdot \rangle : V \times V \to F$ satisfying:

- $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$
- $\langle a\mathbf{u} + b\mathbf{v}, \mathbf{w} \rangle = a\langle \mathbf{u}, \mathbf{w} \rangle + b\langle \mathbf{v}, \mathbf{w} \rangle$
- $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ with equality iff $\mathbf{v} = \mathbf{0}$

8.2 Norm

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

8.3 Orthogonality

Vectors $\mathbf{u}, \mathbf{v}$ are orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

9 Gram-Schmidt Process

Given linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$:

$$\mathbf{u}_1 = \mathbf{v}_1 \tag{1}$$

$$\mathbf{u}_k = \mathbf{v}_k - \sum_{i=1}^{k-1} \frac{\langle \mathbf{v}_k, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i \tag{2}$$

Then $\{\mathbf{u}_1/\|\mathbf{u}_1\|, \mathbf{u}_2/\|\mathbf{u}_2\|, \ldots, \mathbf{u}_n/\|\mathbf{u}_n\|\}$ is orthonormal.

10 Singular Value Decomposition

For any $m \times n$ matrix A:

$$A = U\Sigma V^T$$

where:

- U is $m \times m$ orthogonal
- V is $n \times n$ orthogonal
- Σ is $m \times n$ diagonal with non-negative entries

11 Key Theorems

11.1 Fundamental Theorem of Linear Algebra

For $m \times n$ matrix A:

$$\mathbb{R}^n = \ker(A) \oplus \mathrm{Im}(A^T)$$
$$\mathbb{R}^m = \ker(A^T) \oplus \mathrm{Im}(A)$$

11.2 Cayley-Hamilton Theorem

Every square matrix satisfies its own characteristic equation:

$$p(A) = 0$$

11.3 Spectral Theorem

For symmetric matrix A, there exists orthogonal Q such that:

$$Q^T A Q = D$$

where D is diagonal.