# Real Analysis Summary

Mathematical Notes

October 19, 2025

## Contents

# 1 The Real Numbers

## 1.1 Field Axioms

The real numbers $\mathbb{R}$ form a field with operations $+$ and $\cdot$ satisfying:

- **Associativity**: $(a+b)+c = a+(b+c)$, $(ab)c = a(bc)$
- **Commutativity**: $a+b = b+a$, $ab = ba$
- **Identity**: $a+0 = a$, $a \cdot 1 = a$
- **Inverses**: $a+(-a) = 0$, $a \cdot a^{-1} = 1$ for $a \neq 0$
- **Distributivity**: $a(b+c) = ab+ac$

## 1.2 Order Axioms

There exists a relation $<$ on $\mathbb{R}$ such that:

- **Trichotomy**: For any $a, b \in \mathbb{R}$, exactly one of $a < b$, $a = b$, or $b < a$ holds
- **Transitivity**: If $a < b$ and $b < c$, then $a < c$
- **Addition**: If $a < b$, then $a + c < b + c$
- **Multiplication**: If $a < b$ and $c > 0$, then $ac < bc$

## 1.3 Completeness Axiom

**Definition 1.1.** A set $S \subseteq \mathbb{R}$ is **bounded above** if there exists $M \in \mathbb{R}$ such that $s \leq M$ for all $s \in S$. Such an $M$ is called an **upper bound**.

**Definition 1.2.** The **supremum** (least upper bound) of $S$, denoted $\sup S$, is the smallest upper bound of $S$.

**Theorem 1.1** (Completeness Axiom)**.** Every non-empty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

## 1.4 Archimedean Property

**Theorem 1.2.** For any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.

## 1.5 Density of Rationals

**Theorem 1.3.** Between any two real numbers, there exists a rational number.

# 2 Sequences

## 2.1 Definition and Convergence

**Definition 2.1.** A **sequence** is a function $a : \mathbb{N} \to \mathbb{R}$, denoted $(a_n)$.

**Definition 2.2.** A sequence $(a_n)$ **converges** to $L \in \mathbb{R}$ if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N$. We write $\lim_{n\to\infty} a_n = L$.

## 2.2 Properties of Convergent Sequences

**Theorem 2.1.** If $(a_n)$ converges to $L$, then $L$ is unique.

**Theorem 2.2.** If $(a_n)$ converges, then $(a_n)$ is bounded.

**Theorem 2.3.** If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then:

- $\lim_{n\to\infty}(a_n + b_n) = L + M$
- $\lim_{n\to\infty}(a_n b_n) = LM$
- $\lim_{n\to\infty}(a_n/b_n) = L/M$ (if $M \neq 0$)

## 2.3 Monotone Sequences

**Definition 2.3.** A sequence $(a_n)$ is **monotone increasing** if $a_{n+1} \geq a_n$ for all $n$.

**Theorem 2.4** (Monotone Convergence Theorem)**.** A monotone sequence converges if and only if it is bounded.

## 2.4 Subsequences

**Definition 2.4.** A **subsequence** of $(a_n)$ is a sequence $(a_{n_k})$ where $(n_k)$ is a strictly increasing sequence of natural numbers.

**Theorem 2.5** (Bolzano-Weierstrass)**.** Every bounded sequence has a convergent subsequence.

## 2.5 Cauchy Sequences

**Definition 2.5.** A sequence $(a_n)$ is **Cauchy** if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $n, m \geq N$.

**Theorem 2.6.** A sequence converges if and only if it is Cauchy.

# 3 Limits of Functions

## 3.1 Definition

**Definition 3.1.** Let $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$, and let $c$ be a limit point of $A$. We say $\lim_{x\to c} f(x) = L$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$ and $x \in A$.

## 3.2 Sequential Characterization

**Theorem 3.1.** $\lim_{x\to c} f(x) = L$ if and only if for every sequence $(x_n)$ in $A \setminus \{c\}$ that converges to $c$, we have $\lim_{n\to\infty} f(x_n) = L$.

## 3.3 One-Sided Limits

**Definition 3.2.** $\lim_{x\to c^+} f(x) = L$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < x - c < \delta$.

### 3.4 Limit Laws

**Theorem 3.2.** If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then:

- $\lim_{x \to c}[f(x) + g(x)] = L + M$
- $\lim_{x \to c}[f(x)g(x)] = LM$
- $\lim_{x \to c}[f(x)/g(x)] = L/M$ (if $M \neq 0$)

## 4 Continuity

### 4.1 Definition

**Definition 4.1.** A function $f : A \to \mathbb{R}$ is **continuous** at $c \in A$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$ and $x \in A$.

### 4.2 Equivalent Characterizations

**Theorem 4.1.** The following are equivalent for $f : A \to \mathbb{R}$ at $c \in A$:

1. $f$ is continuous at $c$
2. $\lim_{x \to c} f(x) = f(c)$
3. For every sequence $(x_n)$ in $A$ converging to $c$, $\lim_{n \to \infty} f(x_n) = f(c)$

### 4.3 Properties of Continuous Functions

**Theorem 4.2.** If $f$ and $g$ are continuous at $c$, then so are $f + g$, $f - g$, $fg$, and $f/g$ (if $g(c) \neq 0$).

**Theorem 4.3** (Composition)**.** If $f$ is continuous at $c$ and $g$ is continuous at $f(c)$, then $g \circ f$ is continuous at $c$.

### 4.4 Continuity on Intervals

**Definition 4.2.** A function $f$ is **continuous on an interval** $I$ if it is continuous at every point in $I$.

**Theorem 4.4** (Intermediate Value Theorem)**.** If $f$ is continuous on $[a, b]$ and $f(a) < k < f(b)$ (or $f(b) < k < f(a)$), then there exists $c \in (a, b)$ such that $f(c) = k$.

**Theorem 4.5** (Extreme Value Theorem)**.** If $f$ is continuous on $[a, b]$, then $f$ attains its maximum and minimum values on $[a, b]$.

## 5 Uniform Continuity

### 5.1 Definition

**Definition 5.1.** A function $f : A \to \mathbb{R}$ is **uniformly continuous** on $A$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$ and $x, y \in A$.

### 5.2 Properties

**Theorem 5.1.** If $f$ is uniformly continuous on $A$, then $f$ is continuous on $A$.

**Theorem 5.2** (Uniform Continuity Theorem)**.** If $f$ is continuous on a closed and bounded interval $[a, b]$, then $f$ is uniformly continuous on $[a, b]$.

## 6 Differentiation

### 6.1 Definition

**Definition 6.1.** A function $f : A \to \mathbb{R}$ is **differentiable** at $c \in A$ if the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. This limit is called the **derivative** of $f$ at $c$, denoted $f'(c)$.

### 6.2 Properties

**Theorem 6.1.** If $f$ is differentiable at $c$, then $f$ is continuous at $c$.

**Theorem 6.2** (Product Rule)**.** If $f$ and $g$ are differentiable at $c$, then $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.

**Theorem 6.3** (Chain Rule)**.** If $f$ is differentiable at $c$ and $g$ is differentiable at $f(c)$, then $(g \circ f)'(c) = g'(f(c))f'(c)$.

### 6.3 Mean Value Theorems

**Theorem 6.4** (Rolle's Theorem)**.** If $f$ is continuous on $[a, b]$, differentiable on $(a, b)$, and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

**Theorem 6.5** (Mean Value Theorem)**.** If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

**Theorem 6.6** (Cauchy's Mean Value Theorem)**.** If $f$ and $g$ are continuous on $[a, b]$ and differentiable on $(a, b)$, and $g'(x) \neq 0$ for all $x \in (a, b)$, then there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

## 7 Integration

### 7.1 Riemann Sums

**Definition 7.1.** Let $f : [a, b] \to \mathbb{R}$ be bounded. A **partition** of $[a, b]$ is a finite set $P = \{x_0, x_1, \ldots, x_n\}$ where $a = x_0 < x_1 < \cdots < x_n = b$.

**Definition 7.2.** The **upper Riemann sum** is $U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i$ where $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$.

The **lower Riemann sum** is $L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i$ where $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$.

### 7.2 Riemann Integrability

**Definition 7.3.** A function $f : [a, b] \to \mathbb{R}$ is **Riemann integrable** if

$$\sup_P L(f, P) = \inf_P U(f, P)$$

This common value is called the **Riemann integral** of $f$ over $[a, b]$, denoted $\int_a^b f(x)\,dx$.

### 7.3 Properties of Integrable Functions

**Theorem 7.1.** If $f$ is continuous on $[a, b]$, then $f$ is Riemann integrable on $[a, b]$.

**Theorem 7.2.** If $f$ is monotone on $[a, b]$, then $f$ is Riemann integrable on $[a, b]$.

**Theorem 7.3.** If $f$ is bounded and has only finitely many discontinuities on $[a, b]$, then $f$ is Riemann integrable on $[a, b]$.

### 7.4 Fundamental Theorem of Calculus

**Theorem 7.4** (FTC Part 1)**.** If $f$ is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, then $F$ is differentiable on $[a, b]$ and $F'(x) = f(x)$.

**Theorem 7.5** (FTC Part 2)**.** If $f$ is continuous on $[a, b]$ and $F$ is any antiderivative of $f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

## 8 Series

### 8.1 Definition

**Definition 8.1.** A **series** is an expression of the form $\sum_{n=1}^{\infty} a_n$ where $(a_n)$ is a sequence.

**Definition 8.2.** The **partial sums** of the series are $S_n = \sum_{k=1}^{n} a_k$. The series **converges** to $S$ if $\lim_{n\to\infty} S_n = S$.

### 8.2 Convergence Tests

**Theorem 8.1** (Divergence Test)**.** If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

**Theorem 8.2** (Comparison Test)**.** If $0 \leq a_n \leq b_n$ for all $n$ and $\sum b_n$ converges, then $\sum a_n$ converges.

**Theorem 8.3** (Ratio Test)**.** If $\lim_{n\to\infty} |a_{n+1}/a_n| = L$, then:

- If $L < 1$, then $\sum a_n$ converges absolutely
- If $L > 1$, then $\sum a_n$ diverges
- If $L = 1$, the test is inconclusive

**Theorem 8.4** (Root Test)**.** If $\lim_{n\to\infty} |a_n|^{1/n} = L$, then:

- If $L < 1$, then $\sum a_n$ converges absolutely

- If $L > 1$, then $\sum a_n$ diverges
- If $L = 1$, the test is inconclusive

**Theorem 8.5** (Integral Test)**.** If $f$ is positive, continuous, and decreasing on $[1, \infty)$, then $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

### 8.3 Absolute and Conditional Convergence

**Definition 8.3.** A series $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges.

**Definition 8.4.** A series **converges conditionally** if it converges but does not converge absolutely.

**Theorem 8.6.** If a series converges absolutely, then it converges.

## 9 Power Series

### 9.1 Definition

**Definition 9.1.** A **power series** is a series of the form $\sum_{n=0}^{\infty} a_n(x-c)^n$.

### 9.2 Radius of Convergence

**Theorem 9.1.** For any power series $\sum a_n(x-c)^n$, there exists $R \in [0, \infty]$ such that:

- The series converges absolutely for $|x - c| < R$
- The series diverges for $|x - c| > R$

$R$ is called the **radius of convergence**.

### 9.3 Properties

**Theorem 9.2.** A power series converges uniformly on any closed interval contained in its interval of convergence.

**Theorem 9.3.** A power series can be differentiated and integrated term by term within its radius of convergence.

## 10 Uniform Convergence

### 10.1 Definition

**Definition 10.1.** A sequence of functions $(f_n)$ **converges uniformly** to $f$ on $A$ if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and all $x \in A$.

### 10.2 Properties

**Theorem 10.1** (Uniform Limit Theorem)**.** If $(f_n)$ is a sequence of continuous functions that converges uniformly to $f$ on $A$, then $f$ is continuous on $A$.

**Theorem 10.2.** If $(f_n)$ converges uniformly to $f$ on $[a, b]$ and each $f_n$ is Riemann integrable, then $f$ is Riemann integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

**Theorem 10.3.** If $(f_n)$ converges pointwise to $f$ on $[a, b]$, each $f_n$ is differentiable, and $(f'_n)$ converges uniformly on $[a, b]$, then $f$ is differentiable and $f' = \lim_{n\to\infty} f'_n$.

## 11 Compactness

### 11.1 Definition

**Definition 11.1.** A set $K \subseteq \mathbb{R}$ is **compact** if every open cover of $K$ has a finite subcover.

### 11.2 Heine-Borel Theorem

**Theorem 11.1.** A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

### 11.3 Properties

**Theorem 11.2.** If $f$ is continuous on a compact set $K$, then $f$ is uniformly continuous on $K$.

**Theorem 11.3.** If $f$ is continuous on a compact set $K$, then $f$ attains its maximum and minimum values on $K$.

## 12 Connectedness

### 12.1 Definition

**Definition 12.1.** A set $A \subseteq \mathbb{R}$ is **connected** if there do not exist disjoint open sets $U$ and $V$ such that $A \subseteq U \cup V$ and $A \cap U \neq \emptyset$, $A \cap V \neq \emptyset$.

### 12.2 Characterization

**Theorem 12.1.** A subset of $\mathbb{R}$ is connected if and only if it is an interval.

## 13 Important Theorems

### 13.1 Weierstrass Approximation Theorem

**Theorem 13.1.** Every continuous function on $[a, b]$ can be uniformly approximated by polynomials.

### 13.2 Stone-Weierstrass Theorem

**Theorem 13.2.** Let $A$ be an algebra of continuous functions on a compact set $K$ that separates points and contains the constant functions. Then $A$ is dense in $C(K)$.

### 13.3 Baire Category Theorem

**Theorem 13.3.** The intersection of countably many dense open sets in $\mathbb{R}$ is dense in $\mathbb{R}$.

### 13.4 Arzelà-Ascoli Theorem

**Theorem 13.4.** A sequence of functions in $C[a, b]$ has a uniformly convergent subsequence if and only if it is uniformly bounded and equicontinuous.

## 14 Applications

### 14.1 Existence Theorems

- Fixed point theorems (Brouwer, Banach)
- Existence of solutions to differential equations
- Existence of optima in constrained optimization

### 14.2 Approximation Theory

- Polynomial approximation
- Fourier series and transforms
- Numerical analysis foundations

### 14.3 Analysis of Functions

- Differentiability and smoothness
- Integration theory
- Measure theory foundations