Category Theory Summary

Mathematical Notes

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Contents

1 Basic Definitions

1.1 Categories

Definition 1.1. A **category** $\mathcal{C}$ consists of:

- A collection of **objects** $\mathrm{Ob}(\mathcal{C})$
- For each pair of objects A, B, a set $\mathrm{Hom}(A, B)$ of **morphisms** (or arrows)
- For each object A, an **identity morphism** $1_A : A \to A$
- A **composition** operation $\circ : \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$

satisfying:

- **Associativity**: $(f \circ g) \circ h = f \circ (g \circ h)$
- **Identity**: $f \circ 1_A = f = 1_B \circ f$ for $f : A \to B$

1.2 Morphisms

Definition 1.2. A morphism $f : A \to B$ is:

- **Monic** (monomorphism) if $f \circ g = f \circ h$ implies $g = h$
- **Epic** (epimorphism) if $g \circ f = h \circ f$ implies $g = h$
- **Iso** (isomorphism) if there exists $g : B \to A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$

1.3 Examples of Categories

Example 1.1.

- **Set**: Objects are sets, morphisms are functions
- **Grp**: Objects are groups, morphisms are group homomorphisms
- **Top**: Objects are topological spaces, morphisms are continuous maps
- **Vect**$_k$: Objects are vector spaces over field k, morphisms are linear maps
- **Pos**: Objects are partially ordered sets, morphisms are order-preserving maps

2 Functors

2.1 Definition

Definition 2.1. A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of:

- A function $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$
- For each $f : A \to B$ in $\mathcal{C}$, a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$

satisfying:

- $F(1_A) = 1_{F(A)}$
- $F(f \circ g) = F(f) \circ F(g)$

2.2 Types of Functors

Definition 2.2. A functor $F : \mathcal{C} \to \mathcal{D}$ is:

- **Covariant** if $F(f : A \to B) = F(f) : F(A) \to F(B)$
- **Contravariant** if $F(f : A \to B) = F(f) : F(B) \to F(A)$
- **Full** if $\text{Hom}(A, B) \to \text{Hom}(F(A), F(B))$ is surjective
- **Faithful** if $\text{Hom}(A, B) \to \text{Hom}(F(A), F(B))$ is injective

2.3 Examples of Functors

Example 2.1.
- **Forgetful functor**: $U : \text{Grp} \to \text{Set}$ sends groups to their underlying sets
- **Free functor**: $F : \text{Set} \to \text{Grp}$ sends sets to free groups
- **Hom functor**: $\text{Hom}(A, -) : \mathcal{C} \to \text{Set}$ sends B to $\text{Hom}(A, B)$
- **Power set functor**: $P : \text{Set} \to \text{Set}$ sends sets to their power sets

3 Natural Transformations

3.1 Definition

Definition 3.1. A **natural transformation** $\eta : F \Rightarrow G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$ consists of:

- For each object A in $\mathcal{C}$, a morphism $\eta_A : F(A) \to G(A)$

such that for any morphism $f : A \to B$, the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow{\scriptstyle \eta_A} & & \downarrow{\scriptstyle \eta_B} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

3.2 Natural Isomorphism

Definition 3.2. A natural transformation $\eta : F \Rightarrow G$ is a **natural isomorphism** if each η_A is an isomorphism.

4 Limits and Colimits

4.1 Cones and Cocones

Definition 4.1. Given a diagram $D : \mathcal{J} \to \mathcal{C}$, a **cone** over D consists of:

- An object C in $\mathcal{C}$
- For each object j in $\mathcal{J}$, a morphism $c_j : C \to D(j)$

such that for any morphism $f : j \to j'$ in $\mathcal{J}$, we have $D(f) \circ c_j = c_{j'}$.

Definition 4.2. A **limit** of a diagram $D : \mathcal{J} \to \mathcal{C}$ is a cone (L, λ) that is universal: for any other cone (C, c), there exists a unique morphism $u : C \to L$ such that $\lambda_j \circ u = c_j$ for all j.

4.2 Colimits

Definition 4.3. A **colimit** of a diagram $D : \mathcal{J} \to \mathcal{C}$ is a cocone (C, c) that is universal: for any other cocone (L, λ), there exists a unique morphism $u : C \to L$ such that $u \circ c_j = \lambda_j$ for all j.

4.3 Specific Limits and Colimits

Definition 4.4.

- **Product**: Limit of a discrete diagram
- **Coproduct**: Colimit of a discrete diagram
- **Equalizer**: Limit of a parallel pair
- **Coequalizer**: Colimit of a parallel pair
- **Pullback**: Limit of a cospan
- **Pushout**: Colimit of a span

5 Adjoint Functors

5.1 Definition

Definition 5.1. Functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are **adjoint** (written $F \dashv G$) if there exists a natural isomorphism:

$$\mathrm{Hom}_{\mathcal{D}}(F(A), B) \cong \mathrm{Hom}_{\mathcal{C}}(A, G(B))$$

5.2 Unit and Counit

Definition 5.2. For adjoint functors $F \dashv G$:

- The **unit** is $\eta : 1_{\mathcal{C}} \Rightarrow G \circ F$
- The **counit** is $\epsilon : F \circ G \Rightarrow 1_{\mathcal{D}}$

satisfying the triangle identities:

- $\epsilon_{F(A)} \circ F(\eta_A) = 1_{F(A)}$
- $G(\epsilon_B) \circ \eta_{G(B)} = 1_{G(B)}$

5.3 Examples of Adjoints

Example 5.1.

- **Free-Forgetful**: $F \dashv U : \mathrm{Grp} \to \mathrm{Set}$
- **Tensor-Hom**: $- \otimes A \dashv \mathrm{Hom}(A, -)$ in vector spaces
- **Product-Exponential**: $A \times - \dashv (-)^A$ in cartesian closed categories

6 Monads

6.1 Definition

Definition 6.1. A **monad** on a category $\mathcal{C}$ is a triple (T, η, μ) where:

- $T : \mathcal{C} \to \mathcal{C}$ is a functor
- $\eta : 1_{\mathcal{C}} \Rightarrow T$ (unit)
- $\mu : T^2 \Rightarrow T$ (multiplication)

satisfying:

- $\mu \circ T\mu = \mu \circ \mu T$ (associativity)
- $\mu \circ T\eta = \mu \circ \eta T = 1_T$ (unit laws)

6.2 Monad Algebras

Definition 6.2. A **T-algebra** for a monad (T, η, μ) is a pair (A, α) where:

- A is an object in $\mathcal{C}$
- $\alpha : T(A) \to A$ is a morphism

satisfying:

- $\alpha \circ \eta_A = 1_A$
- $\alpha \circ T(\alpha) = \alpha \circ \mu_A$

6.3 Examples of Monads

Example 6.1.
- **List monad**: $T(A) = \text{List}(A)$
- **Maybe monad**: $T(A) = A \cup \{\perp\}$
- **State monad**: $T(A) = S \to (A \times S)$
- **Continuation monad**: $T(A) = (A \to R) \to R$

7 Yoneda Lemma

7.1 Presheaves

Definition 7.1. A **presheaf** on a category $\mathcal{C}$ is a functor $\mathcal{C}^{\text{op}} \to \text{Set}$.

7.2 Yoneda Embedding

Definition 7.2. The **Yoneda embedding** is the functor $Y : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \text{Set}]$ defined by:

$$Y(A) = \text{Hom}(-, A)$$

7.3 Yoneda Lemma

Theorem 7.1 (Yoneda Lemma)**.** For any presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ and object A in $\mathcal{C}$:

$$[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}](\mathrm{Hom}(-, A), F) \cong F(A)$$

7.4 Corollary

Corollary 7.1. The Yoneda embedding is full and faithful.

8 Topoi

8.1 Definition

Definition 8.1. An **elementary topos** is a category $\mathcal{E}$ with:

- Finite limits
- A subobject classifier Ω
- Power objects

8.2 Subobject Classifier

Definition 8.2. A **subobject classifier** is an object Ω with a morphism $\mathrm{true} : 1 \to \Omega$ such that for any monomorphism $m : A \to B$, there exists a unique morphism $\chi_m : B \to \Omega$ making the following diagram a pullback:

$$\begin{tikzcd}
A \arrow[r, "m"] \arrow[d] & B \arrow[d, "\chi_m"] \\
1 \arrow[r, "\mathrm{true}"] & \Omega
\end{tikzcd}$$

8.3 Examples of Topoi

Example 8.1.
- **Set**: The category of sets
- **Sh**(X): Sheaves on a topological space X
- **Sh**(C, J): Sheaves on a site (C, J)

9 Higher Category Theory

9.1 2-Categories

Definition 9.1. A **2-category** is a category enriched over Cat, consisting of:

- Objects
- 1-morphisms between objects
- 2-morphisms between 1-morphisms

with horizontal and vertical composition satisfying interchange laws.

9.2 ∞-Categories

Definition 9.2. An ∞**-category** (or $(\infty, 1)$-category) is a simplicial set satisfying the weak Kan condition, where morphisms can be composed up to higher homotopies.

10 Applications

10.1 Algebraic Topology

- Fundamental groupoid
- Homology and cohomology as functors
- Spectral sequences
- Fibrations and cofibrations

10.2 Algebraic Geometry

- Schemes as functors
- Sheaves and presheaves
- Étale cohomology
- Derived categories

10.3 Logic and Computer Science

- Curry-Howard correspondence
- Type theory
- Domain theory
- Coalgebras

10.4 Physics

- Quantum mechanics
- String theory
- Topological quantum field theory
- Categorical quantum mechanics

11 Universal Properties

11.1 Initial and Terminal Objects

Definition 11.1.

- An object I is **initial** if for any object A, there exists a unique morphism $I \to A$
- An object T is **terminal** if for any object A, there exists a unique morphism $A \to T$

11.2 Universal Elements

Definition 11.2. A **universal element** of a functor $F : \mathcal{C}^{\text{op}} \to \text{Set}$ is a pair (A, x) where A is an object and $x \in F(A)$, such that for any other pair (B, y) with $y \in F(B)$, there exists a unique morphism $f : B \to A$ with $F(f)(x) = y$.

12 Enriched Categories

12.1 Definition

Definition 12.1. A category **enriched over** a monoidal category $\mathcal{V}$ is a category $\mathcal{C}$ where:

- $\text{Hom}(A, B)$ is an object in $\mathcal{V}$
- Composition is a morphism $\text{Hom}(B, C) \otimes \text{Hom}(A, B) \to \text{Hom}(A, C)$
- Identity is a morphism $I \to \text{Hom}(A, A)$

satisfying associativity and unit laws.

12.2 Examples

Example 12.1.

- **Ordinary categories**: Enriched over Set
- **Preorders**: Enriched over $\{0, 1\}$
- **Metric spaces**: Enriched over $([0, \infty], \geq)$
- **Abelian categories**: Enriched over Ab

13 Coends and Ends

13.1 Definition

Definition 13.1. For a functor $F : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D}$, the **coend** is the colimit of the diagram formed by $F(A, A)$ for all objects A, with morphisms induced by $F(f, 1_A)$ and $F(1_A, f)$.

Definition 13.2. For a functor $F : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D}$, the **end** is the limit of the diagram formed by $F(A, A)$ for all objects A, with morphisms induced by $F(f, 1_A)$ and $F(1_A, f)$.

13.2 Examples

Example 13.1.

- **Tensor product**: $\int^A F(A) \otimes G(A)$
- **Hom functor**: $\int_A \text{Hom}(F(A), G(A))$
- **Geometric realization**: $\int^n \Delta^n \times X_n$

14 Monoidal Categories

14.1 Definition

Definition 14.1. A **monoidal category** is a category $\mathcal{C}$ equipped with:

- A bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- A unit object I
- Natural isomorphisms for associativity and unit

satisfying coherence conditions.

14.2 Symmetric Monoidal Categories

Definition 14.2. A **symmetric monoidal category** is a monoidal category with a natural isomorphism $\sigma_{A,B} : A \otimes B \to B \otimes A$ satisfying $\sigma_{B,A} \circ \sigma_{A,B} = 1_{A\otimes B}$.

14.3 Examples

Example 14.1.
- **Set** with cartesian product
- **Vect** with tensor product
- **Ab** with tensor product
- **Cat** with cartesian product

15 Important Theorems

15.1 Adjoint Functor Theorem

Theorem 15.1 (Adjoint Functor Theorem)**.** A functor $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint if and only if:

- G preserves limits
- $\mathcal{C}$ is complete
- The solution set condition holds

15.2 Freyd's Theorem

Theorem 15.2 (Freyd's Theorem)**.** A small category with all small limits is a preorder.

15.3 Brown Representability

Theorem 15.3 (Brown Representability)**.** In the homotopy category of pointed CW complexes, a contravariant functor F is representable if and only if:

- F satisfies the wedge axiom
- F satisfies the Mayer-Vietoris axiom

16 Conclusion

Category theory provides a unifying framework for mathematics by abstracting common patterns across different fields. Key concepts include:

- Categories, functors, and natural transformations
- Limits, colimits, and universal properties
- Adjoint functors and monads
- Topoi and higher categories
- Enriched categories and monoidal categories

These concepts have found applications in:

- Algebraic topology and geometry
- Logic and computer science
- Physics and quantum mechanics
- Functional programming
- Database theory

Category theory continues to be a powerful tool for understanding mathematical structures and their relationships.