Signal Processing Summary

Mathematical Notes

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Contents

1 Discrete-Time Signals

1.1 Basic Definitions

Definition 1.1. A **discrete-time signal** is a sequence $x[n]$ where $n \in \mathbb{Z}$ is the discrete time index.

Definition 1.2. A signal is **causal** if $x[n] = 0$ for $n < 0$.

Definition 1.3. A signal is **finite-length** if $x[n] = 0$ for $n < 0$ and $n \geq N$.

1.2 Elementary Signals

- **Unit impulse**: $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$
- **Unit step**: $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$
- **Complex exponential**: $e^{j\omega_0 n}$
- **Sinusoid**: $\cos(\omega_0 n + \phi)$

1.3 Signal Properties

Definition 1.4. A signal is **periodic** with period N if $x[n] = x[n + N]$ for all n.

Definition 1.5. The **energy** of a signal is $E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$.

Definition 1.6. The **power** of a periodic signal with period N is $P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$.

2 Linear Time-Invariant Systems

2.1 System Properties

Definition 2.1. A system is **linear** if $T[ax_1[n] + bx_2[n]] = aT[x_1[n]] + bT[x_2[n]]$.

Definition 2.2. A system is **time-invariant** if $T[x[n - k]] = y[n - k]$ for any k.

2.2 Impulse Response

Definition 2.3. The **impulse response** of an LTI system is $h[n] = T[\delta[n]]$.

Theorem 2.1 (Convolution Sum)**.** For an LTI system with impulse response $h[n]$, the output is:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n - k]$$

2.3 System Properties from Impulse Response

- **Causal**: $h[n] = 0$ for $n < 0$
- **Stable**: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$
- **FIR**: Finite impulse response (finite length)
- **IIR**: Infinite impulse response (infinite length)

3 Discrete Fourier Transform

3.1 Definition

Definition 3.1. The **Discrete Fourier Transform** (DFT) of a length-N sequence $x[n]$ is:

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1$$

Definition 3.2. The **Inverse DFT** is:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N}, \quad n = 0, 1, \ldots, N-1$$

3.2 DFT Properties

- **Linearity**: $\text{DFT}[ax[n] + by[n]] = aX[k] + bY[k]$
- **Circular shift**: $\text{DFT}[x[(n-m)_N]] = X[k]e^{-j2\pi km/N}$
- **Parseval's theorem**: $\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$
- **Circular convolution**: $\text{DFT}[x[n] \circledast y[n]] = X[k]Y[k]$

3.3 Fast Fourier Transform

Theorem 3.1 (FFT Algorithm)**.** The DFT can be computed in $O(N \log N)$ operations using the FFT algorithm.

4 Z-Transform

4.1 Definition

Definition 4.1. The **Z-transform** of a sequence $x[n]$ is:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

4.2 Region of Convergence

Definition 4.2. The **Region of Convergence** (ROC) is the set of z values for which the Z-transform converges.

4.3 Z-Transform Properties

- **Linearity**: $\mathcal{Z}[ax[n] + by[n]] = aX(z) + bY(z)$
- **Time shift**: $\mathcal{Z}[x[n-k]] = z^{-k}X(z)$
- **Convolution**: $\mathcal{Z}[x[n] * y[n]] = X(z)Y(z)$
- **Multiplication by** n: $\mathcal{Z}[nx[n]] = -z\frac{dX(z)}{dz}$

4.4 Inverse Z-Transform

- **Partial fraction expansion**
- **Power series expansion**
- **Contour integration**: $x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$

5 Digital Filters

5.1 FIR Filters

Definition 5.1. A **Finite Impulse Response** filter has impulse response:

$$h[n] = \begin{cases} b_n & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

The system function is:

$$H(z) = \sum_{n=0}^{M} b_n z^{-n}$$

5.2 IIR Filters

Definition 5.2. An **Infinite Impulse Response** filter has system function:

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

The difference equation is:

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]$$

5.3 Filter Design

- **Window method** for FIR filters
- **Impulse invariance** for IIR filters
- **Bilinear transform** for IIR filters
- **Least squares** design

6 Frequency Response

6.1 Definition

Definition 6.1. The **frequency response** of an LTI system is:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

6.2 Magnitude and Phase

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\angle H(e^{j\omega})}$$

6.3 Filter Types

- **Lowpass**: Passes low frequencies, attenuates high frequencies
- **Highpass**: Passes high frequencies, attenuates low frequencies
- **Bandpass**: Passes frequencies in a specific band
- **Bandstop**: Attenuates frequencies in a specific band

7 Sampling Theory

7.1 Nyquist-Shannon Sampling Theorem

Theorem 7.1 (Sampling Theorem)**.** A bandlimited signal with maximum frequency f_m can be perfectly reconstructed from its samples if the sampling rate $f_s \geq 2f_m$.

7.2 Aliasing

Definition 7.1. Aliasing occurs when the sampling rate is insufficient, causing high-frequency components to appear as low-frequency components.

7.3 Anti-aliasing Filter

An anti-aliasing filter is a lowpass filter applied before sampling to prevent aliasing.

8 Multirate Signal Processing

8.1 Downsampling

Definition 8.1. Downsampling by factor M: $y[n] = x[Mn]$

In frequency domain:

$$Y(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} X(e^{j(\omega-2\pi k)/M})$$

8.2 Upsampling

Definition 8.2. Upsampling by factor L: $y[n] = \begin{cases} x[n/L] & n = 0, \pm L, \pm 2L, \ldots \\ 0 & \text{otherwise} \end{cases}$

In frequency domain:

$$Y(e^{j\omega}) = X(e^{jL\omega})$$

8.3 Rational Sampling Rate Conversion

To change sampling rate by factor L/M:

1. Upsample by L
2. Filter to prevent aliasing
3. Downsample by M

9 Adaptive Filters

9.1 Least Mean Squares (LMS)

Definition 9.1. The **LMS algorithm** updates filter coefficients as:

$$\mathbf{w}[n+1] = \mathbf{w}[n] + \mu e[n]\mathbf{x}[n]$$

where μ is the step size and $e[n]$ is the error signal.

9.2 Recursive Least Squares (RLS)

Definition 9.2. The **RLS algorithm** minimizes the weighted least squares cost function:

$$J[n] = \sum_{i=0}^{n} \lambda^{n-i}|d[i] - \mathbf{w}^T[n]\mathbf{x}[i]|^2$$

where λ is the forgetting factor.

10 Statistical Signal Processing

10.1 Power Spectral Density

Definition 10.1. The **power spectral density** of a wide-sense stationary process is:

$$S_{xx}(e^{j\omega}) = \sum_{m=-\infty}^{\infty} R_{xx}[m]e^{-j\omega m}$$

where $R_{xx}[m] = E[x[n]x^*[n-m]]$ is the autocorrelation function.

10.2 Wiener Filter

Definition 10.2. The **Wiener filter** minimizes the mean-square error between the desired signal and filter output.

For FIR Wiener filter:

$$\mathbf{w}_{\text{opt}} = \mathbf{R}_{xx}^{-1}\mathbf{r}_{xd}$$

where $\mathbf{R}_{xx}$ is the autocorrelation matrix and $\mathbf{r}_{xd}$ is the cross-correlation vector.

11 Image Processing

11.1 2D Discrete Fourier Transform

Definition 11.1. The **2D DFT** of an $M \times N$ image $f[m,n]$ is:

$$F[u,v] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi(um/M+vn/N)}$$

11.2 Image Filtering

- **Spatial domain**: Convolution with filter kernel
- **Frequency domain**: Multiplication with frequency response
- **Edge detection**: Sobel, Prewitt, Laplacian operators
- **Smoothing**: Gaussian, median filtering

12 Compression

12.1 Lossless Compression

- **Huffman coding**: Variable-length coding based on symbol probabilities
- **Lempel-Ziv**: Dictionary-based compression
- **Predictive coding**: Encode prediction errors

12.2 Lossy Compression

- **Transform coding**: DCT, wavelet transforms
- **Quantization**: Reduce precision of coefficients
- **JPEG**: DCT-based image compression
- **MPEG**: Motion-compensated video compression

13 Applications

13.1 Communication Systems

- Modulation and demodulation
- Channel equalization
- Error correction coding
- Synchronization

13.2 Audio Processing

- Speech recognition
- Audio compression (MP3, AAC)
- Noise reduction
- Echo cancellation

13.3 Biomedical Signal Processing

- ECG analysis
- EEG signal processing
- Medical image analysis
- Heart rate variability

13.4 Radar and Sonar

- Target detection
- Range and velocity estimation
- Beamforming
- Clutter suppression

14 Important Algorithms

14.1 Fast Convolution

- **Overlap-add method**
- **Overlap-save method**
- Use FFT for efficient computation

14.2 Filter Banks

- **Analysis filter bank**: Decompose signal into subbands
- **Synthesis filter bank**: Reconstruct signal from subbands
- **Perfect reconstruction**: Input equals output

14.3 Wavelets

- **Continuous wavelet transform**
- **Discrete wavelet transform**
- **Wavelet packet decomposition**
- Applications in compression and denoising

15 Implementation Considerations

15.1 Finite Wordlength Effects

- **Quantization noise**
- **Overflow and underflow**
- **Limit cycles**
- **Coefficient sensitivity**

15.2 Computational Complexity

- **FIR filters**: $O(N)$ per output sample
- **IIR filters**: $O(N)$ per output sample
- **FFT**: $O(N \log N)$ for length-N transform
- **Convolution**: $O(NM)$ for length-N and length-M sequences

16 Key Theorems

16.1 Parseval's Theorem

Theorem 16.1.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

16.2 Convolution Theorem

Theorem 16.2.

$$\mathcal{F}[x[n] * y[n]] = X(e^{j\omega})Y(e^{j\omega})$$

16.3 Modulation Theorem

Theorem 16.3.

$$\mathcal{F}[x[n]e^{j\omega_0 n}] = X(e^{j(\omega-\omega_0)})$$

17 Important Transforms

17.1 Discrete Cosine Transform

Definition 17.1. The **DCT** is defined as:

$$X[k] = \sum_{n=0}^{N-1} x[n] \cos\left(\frac{\pi(2n+1)k}{2N}\right)$$

17.2 Walsh-Hadamard Transform

Definition 17.2. The **WHT** uses Walsh functions as basis functions and is computationally efficient.

17.3 Karhunen-Loève Transform

Definition 17.3. The **KLT** is the optimal transform for decorrelating signals with known statistics.