Calculus Summary

Mathematical Notes

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1 Limits

1.1 Definition

The limit of $f(x)$ as x approaches a is L if:

$$\lim_{x \to a} f(x) = L$$

For every $\epsilon > 0$, there exists $\delta > 0$ such that:

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$$

1.2 Limit Laws

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then:

- $\lim_{x \to a}[f(x) + g(x)] = L + M$
- $\lim_{x \to a}[f(x) - g(x)] = L - M$
- $\lim_{x \to a}[cf(x)] = cL$
- $\lim_{x \to a}[f(x)g(x)] = LM$
- $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ (if $M \neq 0$)

1.3 One-Sided Limits

- $\lim_{x \to a^-} f(x)$: limit from the left
- $\lim_{x \to a^+} f(x)$: limit from the right
- $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

1.4 Infinite Limits

- $\lim_{x \to a} f(x) = \infty$: $f(x)$ grows without bound
- $\lim_{x \to \infty} f(x) = L$: horizontal asymptote at $y = L$

2 Continuity

2.1 Definition

A function f is continuous at a if:

$$\lim_{x \to a} f(x) = f(a)$$

2.2 Continuity on an Interval

f is continuous on $[a, b]$ if it's continuous at every point in (a, b) and:

- $\lim_{x \to a^+} f(x) = f(a)$ (continuous from the right at a)
- $\lim_{x \to b^-} f(x) = f(b)$ (continuous from the left at b)

2.3 Intermediate Value Theorem

If f is continuous on $[a, b]$ and N is between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = N$.

3 Derivatives

3.1 Definition

The derivative of f at a is:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

3.2 Notation

- $f'(x)$ or $\frac{df}{dx}$: first derivative
- $f''(x)$ or $\frac{d^2 f}{dx^2}$: second derivative
- $f^{(n)}(x)$ or $\frac{d^n f}{dx^n}$: n-th derivative

3.3 Basic Differentiation Rules

- $\frac{d}{dx}[c] = 0$ (constant rule)
- $\frac{d}{dx}[x^n] = nx^{n-1}$ (power rule)
- $\frac{d}{dx}[cf(x)] = cf'(x)$ (constant multiple rule)
- $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$ (sum rule)
- $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$ (difference rule)

3.4 Product and Quotient Rules

- Product rule: $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
- Quotient rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

3.5 Chain Rule

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

3.6 Implicit Differentiation

To find $\frac{dy}{dx}$ when y is implicitly defined by $F(x, y) = 0$:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

3.7 Derivatives of Common Functions

$$\frac{d}{dx}[\sin x] = \cos x \tag{1}$$
$$\frac{d}{dx}[\cos x] = -\sin x \tag{2}$$
$$\frac{d}{dx}[\tan x] = \sec^2 x \tag{3}$$
$$\frac{d}{dx}[\ln x] = \frac{1}{x} \tag{4}$$
$$\frac{d}{dx}[e^x] = e^x \tag{5}$$
$$\frac{d}{dx}[a^x] = a^x \ln a \tag{6}$$
$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}} \tag{7}$$
$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2} \tag{8}$$

4 Applications of Derivatives

4.1 Tangent Lines

The equation of the tangent line to $y = f(x)$ at $(a, f(a))$ is:

$$y - f(a) = f'(a)(x - a)$$

4.2 Related Rates

When two or more quantities are related by an equation, their rates of change are also related.

4.3 Linear Approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

for x near a.

4.4 Newton's Method

To approximate a root of $f(x) = 0$:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

5 Extreme Values

5.1 Critical Points

A critical point of f is a number c where $f'(c) = 0$ or $f'(c)$ doesn't exist.

5.2 First Derivative Test

If c is a critical point:

- If f' changes from positive to negative at c, then $f(c)$ is a local maximum
- If f' changes from negative to positive at c, then $f(c)$ is a local minimum

5.3 Second Derivative Test

If $f'(c) = 0$ and $f''(c)$ exists:

- If $f''(c) > 0$, then $f(c)$ is a local minimum
- If $f''(c) < 0$, then $f(c)$ is a local maximum
- If $f''(c) = 0$, the test is inconclusive

5.4 Extreme Value Theorem

If f is continuous on $[a, b]$, then f attains both an absolute maximum and absolute minimum on $[a, b]$.

6 Concavity and Inflection Points

6.1 Concavity

- f is concave upward on (a, b) if $f''(x) > 0$ for all $x \in (a, b)$
- f is concave downward on (a, b) if $f''(x) < 0$ for all $x \in (a, b)$

6.2 Inflection Points

An inflection point is a point where the concavity changes, i.e., where $f''(x) = 0$ or $f''(x)$ doesn't exist.

7 L'Hôpital's Rule

If $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

8 Antiderivatives and Indefinite Integrals

8.1 Definition

An antiderivative of f is a function F such that $F'(x) = f(x)$.

8.2 Indefinite Integral

$$\int f(x)\,dx = F(x) + C$$

where C is the constant of integration.

8.3 Basic Integration Rules

$$\int k\,dx = kx + C \tag{9}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \tag{10}$$

$$\int \frac{1}{x}\,dx = \ln|x| + C \tag{11}$$

$$\int e^x\,dx = e^x + C \tag{12}$$

$$\int \sin x\,dx = -\cos x + C \tag{13}$$

$$\int \cos x\,dx = \sin x + C \tag{14}$$

$$\int \sec^2 x\,dx = \tan x + C \tag{15}$$

9 Definite Integrals

9.1 Definition

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and x_i^* is any point in the i-th subinterval.

9.2 Fundamental Theorem of Calculus

- Part 1: If $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$
- Part 2: $\int_a^b f(x)\,dx = F(b) - F(a)$ where F is any antiderivative of f

9.3 Properties of Definite Integrals

- $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
- $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
- $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

10 Integration Techniques

10.1 Substitution Rule

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

where $u = g(x)$.

10.2 Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

10.3 Partial Fractions

For rational functions, decompose into simpler fractions:

$$\frac{P(x)}{Q(x)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{Cx+D}{x^2+bx+c} + \cdots$$

10.4 Trigonometric Substitution

- For $\sqrt{a^2 - x^2}$: use $x = a\sin\theta$
- For $\sqrt{a^2 + x^2}$: use $x = a\tan\theta$
- For $\sqrt{x^2 - a^2}$: use $x = a\sec\theta$

11 Applications of Integration

11.1 Area Between Curves

$$A = \int_a^b |f(x) - g(x)|\,dx$$

11.2 Volume of Revolution

- Disk method: $V = \pi\int_a^b [f(x)]^2\,dx$
- Washer method: $V = \pi\int_a^b ([f(x)]^2 - [g(x)]^2)\,dx$
- Shell method: $V = 2\pi\int_a^b x f(x)\,dx$

11.3 Arc Length

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

11.4 Surface Area

$$S = 2\pi\int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx$$

12 Sequences and Series

12.1 Sequences

A sequence $\{a_n\}$ converges to L if:

$$\lim_{n\to\infty} a_n = L$$

12.2 Series

The series $\sum_{n=1}^{\infty} a_n$ converges if the sequence of partial sums $\{S_n\}$ converges, where:

$$S_n = \sum_{k=1}^{n} a_k$$

12.3 Geometric Series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{if } |r| < 1$$

12.4 Convergence Tests

- Divergence test: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges
- Integral test: If f is positive, continuous, and decreasing, then $\sum_{n=1}^{\infty} f(n)$ and $\int_1^{\infty} f(x)\,dx$ both converge or both diverge
- Comparison test: If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges
- Ratio test: If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum a_n$ converges
- Root test: If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\sum a_n$ converges

13 Power Series

13.1 Definition

A power series centered at a is:

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

13.2 Radius of Convergence

The radius of convergence R is such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$.

13.3 Taylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

13.4 Maclaurin Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

13.5 Common Maclaurin Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{16}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \tag{17}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \tag{18}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (|x| < 1) \tag{19}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \quad (|x| < 1) \tag{20}$$

14 Multivariable Calculus

14.1 Partial Derivatives

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

14.2 Gradient

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

14.3 Directional Derivative

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$$

where $\mathbf{u}$ is a unit vector.

14.4 Multiple Integrals

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

14.5 Change of Variables

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

15 Vector Calculus

15.1 Line Integrals

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt$$

15.2 Green's Theorem

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA$$

15.3 Divergence Theorem

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_E \nabla\cdot\mathbf{F}\,dV$$

15.4 Stokes' Theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$$