Differential Equations Summary

Mathematical Notes

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Contents

1 Ordinary Differential Equations

1.1 First-Order ODEs

Definition 1.1. A first-order ODE has the form:

$$\frac{dy}{dx} = f(x, y)$$

1.1.1 Separable Equations

Definition 1.2. A separable equation has the form:

$$\frac{dy}{dx} = g(x)h(y)$$

Solution: $\int \frac{dy}{h(y)} = \int g(x)\,dx + C$

1.1.2 Linear Equations

Definition 1.3. A linear first-order ODE has the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Solution: $y = e^{-\int P(x)\,dx}\left[\int Q(x)e^{\int P(x)\,dx}\,dx + C\right]$

1.1.3 Exact Equations

Definition 1.4. An equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

1.2 Second-Order Linear ODEs

Definition 1.5. A second-order linear ODE has the form:

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = f(x)$$

1.2.1 Homogeneous Case

For $f(x) = 0$, the general solution is:

$$y = c_1y_1(x) + c_2y_2(x)$$

where y_1 and y_2 are linearly independent solutions.

1.2.2 Characteristic Equation

For constant coefficients $ay'' + by' + cy = 0$:

$$ar^2 + br + c = 0$$

- Two real roots: $y = c_1e^{r_1x} + c_2e^{r_2x}$
- One real root: $y = (c_1 + c_2x)e^{rx}$
- Complex roots: $y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$

1.3 Systems of ODEs

Definition 1.6. A system of first-order ODEs:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x})$$

where $\mathbf{x} = (x_1, \ldots, x_n)^T$.

1.3.1 Linear Systems

For $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$:

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$

where e^{At} is the matrix exponential.

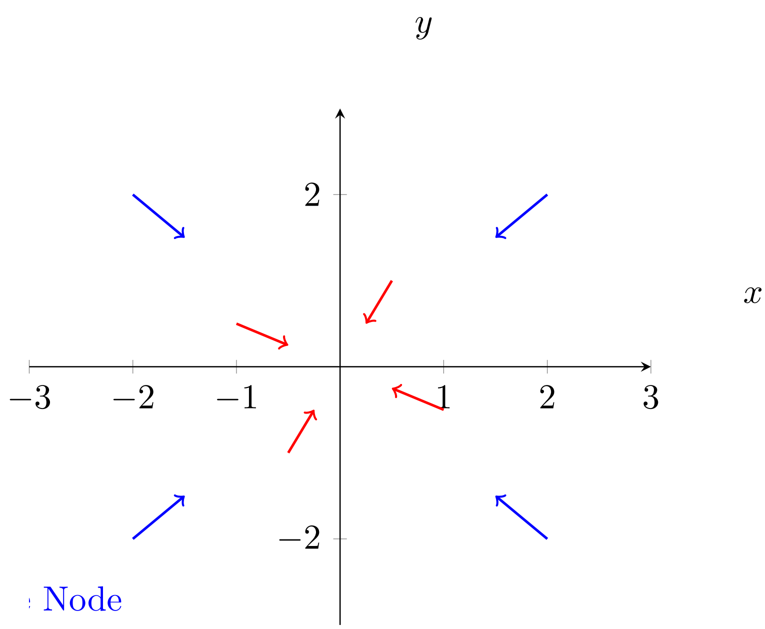

Figure 1: Phase portraits for linear systems

2 Existence and Uniqueness

2.1 Picard-Lindelöf Theorem

Theorem 2.1. If $f(t, y)$ is continuous and Lipschitz in y on a rectangle R, then the IVP:

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

has a unique solution on some interval containing t_0.

2.2 Lipschitz Condition

Definition 2.1. A function $f(t, y)$ satisfies a Lipschitz condition if:

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

for some constant $L > 0$.

3 Stability Theory

3.1 Equilibrium Points

Definition 3.1. An equilibrium point of $\frac{dx}{dt} = f(x)$ is a point x^* such that $f(x^*) = 0$.

3.2 Linear Stability Analysis

Definition 3.2. For a linear system $\frac{dx}{dt} = Ax$, the stability is determined by the eigenvalues of A:

- All eigenvalues have negative real parts: asymptotically stable
- Any eigenvalue has positive real part: unstable
- Zero real parts: need further analysis

3.3 Lyapunov Stability

Definition 3.3. An equilibrium point x^* is:

- **Stable** if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|x(0) - x^*| < \delta$ implies $|x(t) - x^*| < \epsilon$ for all $t \geq 0$
- **Asymptotically stable** if it's stable and $\lim_{t\to\infty} x(t) = x^*$

3.4 Lyapunov's Method

Theorem 3.1. If there exists a Lyapunov function $V(x)$ such that:

- $V(x^*) = 0$ and $V(x) > 0$ for $x \neq x^*$
- $\dot{V}(x) \leq 0$ for all x

then x^* is stable. If $\dot{V}(x) < 0$ for $x \neq x^*$, then x^* is asymptotically stable.

4 Partial Differential Equations

4.1 Classification

Definition 4.1. A second-order linear PDE in two variables:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

is classified by the discriminant $\Delta = B^2 - 4AC$:

- $\Delta > 0$: Hyperbolic
- $\Delta = 0$: Parabolic
- $\Delta < 0$: Elliptic

4.2 Wave Equation

Definition 4.2. The wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

General solution: $u(x,t) = f(x - ct) + g(x + ct)$

4.3 Heat Equation

Definition 4.3. The heat equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

Solution by separation of variables: $u(x,t) = X(x)T(t)$

4.4 Laplace's Equation

Definition 4.4. Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Solutions are harmonic functions.

5 Method of Characteristics

5.1 First-Order PDEs

Definition 5.1. For the PDE $a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)$, the characteristic equations are:

$$\frac{dx}{ds} = a, \quad \frac{dy}{ds} = b, \quad \frac{du}{ds} = c$$

6 Green's Functions

6.1 Definition

Definition 6.1. A Green's function $G(x,\xi)$ for the operator L satisfies:

$$LG(x,\xi) = \delta(x - \xi)$$

where δ is the Dirac delta function.

6.2 Solution Representation

Theorem 6.1. If $Lu = f$ with homogeneous boundary conditions, then:

$$u(x) = \int G(x,\xi) f(\xi)\, d\xi$$

7 Fourier Methods

7.1 Fourier Series

Definition 7.1. For a periodic function $f(x)$ with period $2L$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx$$

7.2 Fourier Transform

Definition 7.2. The Fourier transform of $f(x)$ is:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} \, dx$$

Inverse transform:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} \, d\xi$$

8 Applications

8.1 Physics

Differential equations model:

- Classical mechanics (Newton's laws)
- Electromagnetism (Maxwell's equations)
- Quantum mechanics (Schrödinger equation)
- Fluid dynamics (Navier-Stokes equations)

8.2 Biology

Applications include:

- Population dynamics
- Epidemiology
- Chemical kinetics
- Neural networks

8.3 Engineering

Used in:

- Control systems
- Signal processing
- Heat transfer
- Structural analysis

9 Important Theorems

9.1 Existence and Uniqueness for Systems

Theorem 9.1. If $\mathbf{f}(t, \mathbf{x})$ is continuous and satisfies a Lipschitz condition in $\mathbf{x}$ on a domain D, then the IVP $\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x})$, $\mathbf{x}(t_0) = \mathbf{x}_0$ has a unique solution.

9.2 Sturm-Liouville Theory

Theorem 9.2. For the Sturm-Liouville problem:

$$-\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = \lambda w(x)y$$

with appropriate boundary conditions, the eigenvalues are real and the eigenfunctions are orthogonal.

9.3 Maximum Principle

Theorem 9.3. For Laplace's equation in a bounded domain, the maximum and minimum values occur on the boundary.

10 Numerical Methods

10.1 Finite Differences

Definition 10.1. Finite difference approximations:

- Forward: $f'(x) \approx \frac{f(x+h)-f(x)}{h}$
- Backward: $f'(x) \approx \frac{f(x)-f(x-h)}{h}$
- Central: $f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$

10.2 Finite Elements

Definition 10.2. The finite element method approximates the solution by piecewise polynomial functions on a mesh.