Numerical Analysis Summary

Mathematical Notes

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Contents

1 Error Analysis

1.1 Sources of Error

Definition 1.1.

- **Modeling Error**: Error in mathematical model
- **Data Error**: Error in input data
- **Truncation Error**: Error from finite approximations
- **Round-off Error**: Error from finite precision arithmetic

1.2 Error Types

Definition 1.2. For approximation $\tilde{x}$ of exact value x:

- **Absolute Error**: $|x - \tilde{x}|$
- **Relative Error**: $\frac{|x-\tilde{x}|}{|x|}$ (if $x \neq 0$)
- **Forward Error**: $|f(x) - f(\tilde{x})|$
- **Backward Error**: $|\tilde{x} - x|$ where $f(\tilde{x}) = f(x)$

1.3 Conditioning

Definition 1.3. A problem is **well-conditioned** if small changes in input produce small changes in output. The **condition number** measures sensitivity:

$$\kappa = \lim_{\delta \to 0} \sup_{|\Delta x| \leq \delta} \frac{|\Delta f|}{|\Delta x|} \cdot \frac{|x|}{|f(x)|}$$

2 Root Finding

2.1 Bisection Method

Theorem 2.1. If f is continuous on $[a, b]$ and $f(a)f(b) < 0$, then the bisection method converges to a root with error bound:

$$|x_n - x^*| \leq \frac{b-a}{2^{n+1}}$$

2.2 Newton's Method

Definition 2.1. Newton's method for finding roots of $f(x) = 0$:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Theorem 2.2. If $f'(x^*) \neq 0$ and f'' is continuous near x^*, then Newton's method converges quadratically:

$$|x_{n+1} - x^*| \leq C|x_n - x^*|^2$$

2.3 Secant Method

Definition 2.2. The secant method uses two previous points:

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

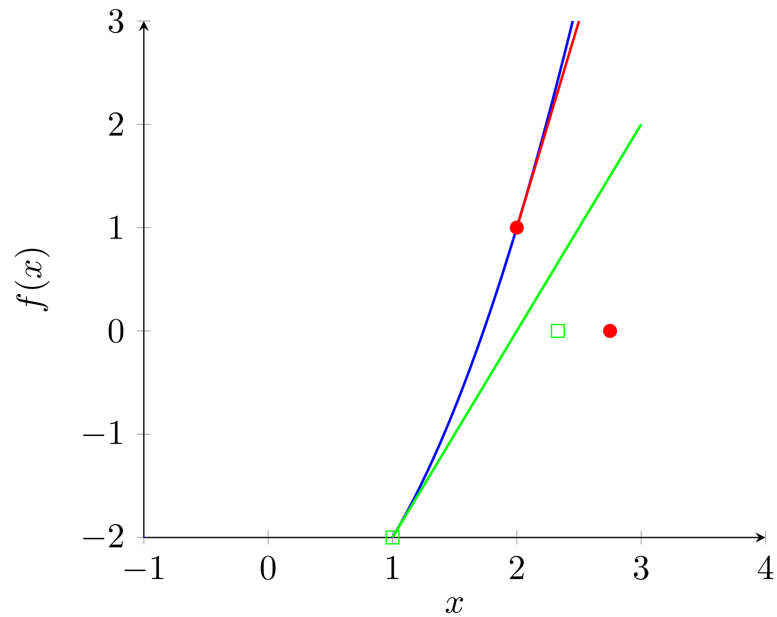

Figure 1: Root finding methods

3 Interpolation

3.1 Lagrange Interpolation

Definition 3.1. Given points (x_i, y_i), the Lagrange interpolating polynomial is:

$$P_n(x) = \sum_{i=0}^{n} y_i L_i(x)$$

where $L_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$

3.2 Newton's Divided Differences

Definition 3.2. The Newton form of the interpolating polynomial:

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, \ldots, x_n](x - x_0)\cdots(x - x_{n-1})$$

where $f[x_i, \ldots, x_j] = \frac{f[x_{i+1}, \ldots, x_j] - f[x_i, \ldots, x_{j-1}]}{x_j - x_i}$

3.3 Error in Interpolation

Theorem 3.1. If $f \in C^{n+1}[a,b]$, then for $x \in [a,b]$:

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

for some $\xi \in [a,b]$.

4 Numerical Integration

4.1 Newton-Cotes Formulas

Definition 4.1. The n-point Newton-Cotes formula:

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} w_i f(x_i)$$

where $x_i = a + ih$ and $h = \frac{b-a}{n}$.

4.1.1 Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx \frac{h}{2}[f(a) + 2f(a+h) + \cdots + 2f(b-h) + f(b)]$$

4.1.2 Simpson's Rule

$$\int_a^b f(x)\,dx \approx \frac{h}{3}[f(a) + 4f(a+h) + 2f(a+2h) + \cdots + 4f(b-h) + f(b)]$$

4.2 Gaussian Quadrature

Definition 4.2. Gaussian quadrature uses optimal nodes and weights:

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

where x_i are roots of Legendre polynomials.

4.3 Error Analysis

Theorem 4.1. For the trapezoidal rule with $f \in C^2[a,b]$:

$$\left| \int_a^b f(x)\,dx - T_n \right| \le \frac{(b-a)^3}{12n^2} \max_{x \in [a,b]} |f''(x)|$$

5 Numerical Differentiation

5.1 Finite Differences

Definition 5.1.

- **Forward Difference**: $f'(x) \approx \frac{f(x+h)-f(x)}{h}$
- **Backward Difference**: $f'(x) \approx \frac{f(x)-f(x-h)}{h}$
- **Central Difference**: $f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$

5.2 Error Analysis

Theorem 5.1. For central difference with $f \in C^3$:

$$f'(x) - \frac{f(x+h) - f(x-h)}{2h} = -\frac{h^2}{6} f'''(\xi)$$

for some $\xi \in [x-h, x+h]$.

6 Linear Systems

6.1 Gaussian Elimination

Definition 6.1. Gaussian elimination with partial pivoting solves $Ax = b$ by:

1. Forward elimination with row swaps
2. Back substitution

6.2 LU Factorization

Definition 6.2. If A can be factored as $A = LU$ where L is lower triangular and U is upper triangular, then $Ax = b$ becomes:

1. Solve $Ly = b$ for y
2. Solve $Ux = y$ for x

6.3 Iterative Methods

6.3.1 Jacobi Method

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j \neq i} a_{ij} x_j^{(k)} \right)$$

6.3.2 Gauss-Seidel Method

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j<i} a_{ij} x_j^{(k+1)} - \sum_{j>i} a_{ij} x_j^{(k)} \right)$$

6.4 Convergence

Theorem 6.1. The Jacobi and Gauss-Seidel methods converge if A is strictly diagonally dominant:

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \quad \forall i$$

7 Ordinary Differential Equations

7.1 Euler's Method

Definition 7.1. For $y' = f(t, y)$, $y(t_0) = y_0$:

$$y_{n+1} = y_n + hf(t_n, y_n)$$

where h is the step size.

7.2 Runge-Kutta Methods

Definition 7.2. The fourth-order Runge-Kutta method:

$$k_1 = hf(t_n, y_n) \tag{1}$$
$$k_2 = hf(t_n + h/2, y_n + k_1/2) \tag{2}$$
$$k_3 = hf(t_n + h/2, y_n + k_2/2) \tag{3}$$
$$k_4 = hf(t_n + h, y_n + k_3) \tag{4}$$
$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{5}$$

7.3 Error Analysis

Theorem 7.1. For Euler's method with $f \in C^1$:

$$|y(t_n) - y_n| \leq \frac{Mh}{2L}(e^{L(t_n - t_0)} - 1)$$

where $M = \max |f'|$ and L is the Lipschitz constant.

8 Approximation Theory

8.1 Best Approximation

Definition 8.1. The best approximation to f in norm $\|\cdot\|$ from subspace S is $p^* \in S$ such that:

$$\|f - p^*\| = \min_{p \in S} \|f - p\|$$

8.2 Chebyshev Approximation

Theorem 8.1. For $f \in C[a, b]$, there exists a unique best uniform approximation $p^* \in P_n$ such that:

$$\|f - p^*\|_\infty = \min_{p \in P_n} \|f - p\|_\infty$$

8.3 Least Squares Approximation

Definition 8.2. The least squares approximation minimizes:

$$\sum_{i=1}^{m}(f(x_i) - p(x_i))^2$$

for given data points $(x_i, f(x_i))$.

9 Fast Fourier Transform

9.1 Discrete Fourier Transform

Definition 9.1. The DFT of sequence $\{x_n\}$ is:

$$X_k = \sum_{n=0}^{N-1} x_n e^{-2\pi ikn/N}$$

9.2 FFT Algorithm

Theorem 9.1. The FFT computes the DFT in $O(N \log N)$ operations using the divide-and-conquer approach.

10 Eigenvalue Problems

10.1 Power Method

Definition 10.1. For dominant eigenvalue λ_1 of matrix A:

$$x^{(k+1)} = \frac{Ax^{(k)}}{\|Ax^{(k)}\|}$$

10.2 QR Algorithm

Definition 10.2. The QR algorithm for eigenvalues:

1. Factor $A_k = Q_k R_k$
2. Set $A_{k+1} = R_k Q_k$
3. Repeat until convergence

11 Applications

11.1 Scientific Computing

Numerical analysis is essential for:

- Solving differential equations
- Optimization problems
- Signal processing
- Computational fluid dynamics

11.2 Engineering

Applications include:

- Structural analysis
- Control systems
- Image processing
- Financial modeling

12 Important Theorems

12.1 Weierstrass Approximation Theorem

Theorem 12.1. For any $f \in C[a,b]$ and $\epsilon > 0$, there exists a polynomial p such that:

$$\|f - p\|_\infty < \epsilon$$

12.2 Intermediate Value Theorem

Theorem 12.2. If f is continuous on $[a,b]$ and $f(a)f(b) < 0$, then there exists $c \in (a,b)$ such that $f(c) = 0$.

12.3 Fixed Point Theorem

Theorem 12.3. If $g : [a,b] \to [a,b]$ is continuous and $|g'(x)| < 1$ for all $x \in [a,b]$, then g has a unique fixed point.