Complex Analysis Summary

Mathematical Notes

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Contents

1 Complex Numbers

1.1 Definition and Basic Properties

Definition 1.1. A **complex number** is an expression of the form $z = x + iy$ where $x, y \in \mathbb{R}$ and $i^2 = -1$. The set of all complex numbers is denoted $\mathbb{C}$.

1.2 Algebraic Operations

For $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$:

- **Addition**: $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$
- **Multiplication**: $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$
- **Conjugate**: $\overline{z} = x - iy$
- **Modulus**: $|z| = \sqrt{x^2 + y^2}$

1.3 Polar Form

Definition 1.2. For $z = x + iy \neq 0$, we can write $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ where:

- $r = |z| = \sqrt{x^2 + y^2}$ (modulus)
- $\theta = \arg z$ (argument)

1.4 De Moivre's Theorem

Theorem 1.1. For any integer n and complex number $z = r(\cos\theta + i\sin\theta)$:

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)) = r^n e^{in\theta}$$

1.5 Roots of Unity

The n-th roots of unity are:

$$\omega_k = e^{2\pi ik/n} = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$$

for $k = 0, 1, \ldots, n-1$.

2 Complex Functions

2.1 Definition

Definition 2.1. A **complex function** is a function $f : D \to \mathbb{C}$ where $D \subseteq \mathbb{C}$.

2.2 Representation

A complex function $f(z) = f(x + iy)$ can be written as:

$$f(z) = u(x, y) + iv(x, y)$$

where u and v are real-valued functions of two real variables.

2.3 Limits

Definition 2.2. $\lim_{z \to z_0} f(z) = L$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(z) - L| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

2.4 Continuity

Definition 2.3. A function f is **continuous** at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.

3 Analytic Functions

3.1 Complex Differentiability

Definition 3.1. A function f is **complex differentiable** at z_0 if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. This limit is called the **derivative** of f at z_0, denoted $f'(z_0)$.

3.2 Analyticity

Definition 3.2. A function f is **analytic** (or **holomorphic**) at z_0 if it is complex differentiable in some neighborhood of z_0. A function is **entire** if it is analytic on all of $\mathbb{C}$.

3.3 Cauchy-Riemann Equations

Theorem 3.1. Let $f(z) = u(x, y) + iv(x, y)$ be defined in a neighborhood of $z_0 = x_0 + iy_0$. Then f is complex differentiable at z_0 if and only if:

1. u and v are differentiable at (x_0, y_0)
2. The Cauchy-Riemann equations hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

3.4 Properties of Analytic Functions

Theorem 3.2. If f is analytic at z_0, then f is continuous at z_0.

Theorem 3.3. If f and g are analytic at z_0, then so are $f + g$, $f - g$, fg, and f/g (if $g(z_0) \neq 0$).

Theorem 3.4 (Chain Rule)**.** If f is analytic at z_0 and g is analytic at $f(z_0)$, then $g \circ f$ is analytic at z_0 and $(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$.

4 Harmonic Functions

4.1 Definition

Definition 4.1. A real-valued function $u(x, y)$ is **harmonic** if it satisfies Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

4.2 Relationship to Analytic Functions

Theorem 4.1. If $f(z) = u(x,y) + iv(x,y)$ is analytic, then both u and v are harmonic functions.

Definition 4.2. If $f = u + iv$ is analytic, then v is called a **harmonic conjugate** of u.

5 Power Series

5.1 Definition

Definition 5.1. A **power series** centered at z_0 is a series of the form:

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

5.2 Radius of Convergence

Theorem 5.1. For any power series $\sum a_n (z - z_0)^n$, there exists $R \in [0, \infty]$ such that:

- The series converges absolutely for $|z - z_0| < R$
- The series diverges for $|z - z_0| > R$

R is called the **radius of convergence**.

5.3 Analyticity of Power Series

Theorem 5.2. A power series defines an analytic function within its radius of convergence.

5.4 Taylor Series

Theorem 5.3. If f is analytic in a disk $|z - z_0| < R$, then f has a Taylor series expansion:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

valid for $|z - z_0| < R$.

6 Complex Integration

6.1 Contours

Definition 6.1. A **contour** (or **path**) is a continuous function $\gamma : [a, b] \to \mathbb{C}$.

Definition 6.2. A contour is **smooth** if $\gamma'(t)$ exists and is continuous on $[a, b]$.

Definition 6.3. A contour is **closed** if $\gamma(a) = \gamma(b)$.

6.2 Line Integrals

Definition 6.4. The **line integral** of f along contour γ is:

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt$$

6.3 Properties

Theorem 6.1. If f and g are continuous on γ and $c \in \mathbb{C}$, then:

- $\int_\gamma [f(z) + g(z)]\, dz = \int_\gamma f(z)\, dz + \int_\gamma g(z)\, dz$
- $\int_\gamma cf(z)\, dz = c \int_\gamma f(z)\, dz$
- $\int_{-\gamma} f(z)\, dz = -\int_\gamma f(z)\, dz$ (where $-\gamma$ is the reverse path)

7 Cauchy's Theorem

7.1 Simply Connected Domains

Definition 7.1. A domain D is **simply connected** if every closed contour in D can be continuously deformed to a point within D.

7.2 Cauchy's Theorem

Theorem 7.1 (Cauchy's Theorem)**.** If f is analytic in a simply connected domain D and γ is a closed contour in D, then:

$$\oint_\gamma f(z)\, dz = 0$$

7.3 Independence of Path

Theorem 7.2. If f is analytic in a simply connected domain D, then $\int_\gamma f(z)\, dz$ depends only on the endpoints of γ, not on the path itself.

8 Cauchy's Integral Formula

8.1 Cauchy's Integral Formula

Theorem 8.1. If f is analytic inside and on a simple closed contour γ, then for any point a inside γ:

$$f(a) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-a}\, dz$$

8.2 Derivatives of Analytic Functions

Theorem 8.2. If f is analytic inside and on a simple closed contour γ, then f has derivatives of all orders inside γ, and:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz$$

8.3 Morera's Theorem

Theorem 8.3. If f is continuous in a domain D and $\oint_\gamma f(z)\, dz = 0$ for every closed contour γ in D, then f is analytic in D.

9 Liouville's Theorem and Maximum Principle

9.1 Liouville's Theorem

Theorem 9.1. If f is entire and bounded, then f is constant.

9.2 Fundamental Theorem of Algebra

Theorem 9.2. Every non-constant polynomial with complex coefficients has at least one complex root.

9.3 Maximum Modulus Principle

Theorem 9.3. If f is analytic in a domain D and $|f|$ attains its maximum at a point in D, then f is constant in D.

9.4 Minimum Modulus Principle

Theorem 9.4. If f is analytic and non-zero in a domain D, then $|f|$ cannot attain its minimum at an interior point of D.

10 Singularities

10.1 Types of Singularities

Definition 10.1. A point z_0 is a **singularity** of f if f is not analytic at z_0 but is analytic in some punctured neighborhood of z_0.

Definition 10.2. A singularity z_0 is:

- **Removable** if $\lim_{z \to z_0} f(z)$ exists
- **Pole** if $\lim_{z \to z_0} |f(z)| = \infty$
- **Essential** if $\lim_{z \to z_0} f(z)$ does not exist and is not infinite

10.2 Laurent Series

Theorem 10.1. If f is analytic in an annulus $r < |z - z_0| < R$, then f has a Laurent series expansion:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

for any circle C in the annulus.

10.3 Residues

Definition 10.3. The **residue** of f at an isolated singularity z_0 is the coefficient a_{-1} in the Laurent series expansion of f around z_0.

10.4 Residue Theorem

Theorem 10.2. If f is analytic inside and on a simple closed contour γ except for isolated singularities $z_1, z_2, \ldots, z_n$ inside γ, then:

$$\oint_\gamma f(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k)$$

11 Conformal Mappings

11.1 Definition

Definition 11.1. A function f is **conformal** at z_0 if it preserves angles and orientation at z_0.

11.2 Characterization

Theorem 11.1. A function f is conformal at z_0 if and only if f is analytic at z_0 and $f'(z_0) \neq 0$.

11.3 Elementary Mappings

- **Translation**: $w = z + c$
- **Rotation**: $w = e^{i\theta} z$
- **Scaling**: $w = rz$ where $r > 0$
- **Inversion**: $w = 1/z$
- **Linear**: $w = az + b$ where $a \neq 0$
- **Power**: $w = z^n$
- **Exponential**: $w = e^z$
- **Logarithm**: $w = \log z$

12 Riemann Mapping Theorem

12.1 Statement

Theorem 12.1 (Riemann Mapping Theorem)**.** Any simply connected domain $D \subset \mathbb{C}$ (other than $\mathbb{C}$ itself) can be conformally mapped onto the unit disk $|z| < 1$.

13 Analytic Continuation

13.1 Definition

Definition 13.1. If f_1 is analytic in domain D_1 and f_2 is analytic in domain D_2 with $D_1 \cap D_2 \neq \emptyset$, and $f_1 = f_2$ on $D_1 \cap D_2$, then f_2 is an **analytic continuation** of f_1.

13.2 Uniqueness

Theorem 13.1. Analytic continuations are unique: if f_1 and f_2 are both analytic continuations of f to a domain D, then $f_1 = f_2$ in D.

14 Special Functions

14.1 Exponential Function

$$e^z = e^x(\cos y + i \sin y)$$

Properties:

- $e^{z_1+z_2} = e^{z_1}e^{z_2}$
- e^z is entire
- $e^z \neq 0$ for any z
- $e^{z+2\pi i} = e^z$ (periodic with period $2\pi i$)

14.2 Trigonometric Functions

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$
$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}$$

14.3 Hyperbolic Functions

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}$$
$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}$$

14.4 Logarithm

Definition 14.1. The **complex logarithm** is defined as:

$$\log z = \ln |z| + i \arg z$$

where $\arg z$ is any argument of z.

14.5 Power Functions

For $z \neq 0$ and $w \in \mathbb{C}$:

$$z^w = e^{w \log z}$$

15 Applications

15.1 Evaluation of Real Integrals

The residue theorem can be used to evaluate many real integrals:

- Integrals of rational functions
- Integrals involving trigonometric functions
- Improper integrals

15.2 Fluid Dynamics

Complex analysis is used to study:

- Potential flow
- Conformal mappings for flow around obstacles
- Stream functions and velocity potentials

15.3 Electrostatics

Applications include:

- Potential theory
- Conformal mappings for field problems
- Image methods

15.4 Signal Processing

Complex analysis is fundamental to:

- Fourier transforms
- Laplace transforms
- Z-transforms

16 Important Theorems

16.1 Argument Principle

Theorem 16.1. If f is meromorphic inside and on a simple closed contour γ and has no zeros or poles on γ, then:

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz = N - P$$

where N is the number of zeros and P is the number of poles of f inside γ (counting multiplicities).

16.2 Rouché's Theorem

Theorem 16.2. If f and g are analytic inside and on a simple closed contour γ and $|g(z)| < |f(z)|$ on γ, then f and $f + g$ have the same number of zeros inside γ.

16.3 Hurwitz's Theorem

Theorem 16.3. If (f_n) is a sequence of analytic functions that converges uniformly to f on compact sets, and each f_n has no zeros in a domain D, then either f is identically zero or f has no zeros in D.

16.4 Schwarz's Lemma

Theorem 16.4. If f is analytic in the unit disk, $|f(z)| \leq 1$ for $|z| < 1$, and $f(0) = 0$, then $|f(z)| \leq |z|$ for $|z| < 1$ and $|f'(0)| \leq 1$.

16.5 Picard's Theorems

Theorem 16.5 (Little Picard)**.** If f is entire and non-constant, then f takes every complex value except possibly one.

Theorem 16.6 (Great Picard)**.** If f has an essential singularity at z_0, then in any neighborhood of z_0, f takes every complex value except possibly one.