ExNEWTON_3

Reproduce the explicit 4×4 example (quaternion circulant) from:

J. Pan and M. K. Ng, Block-Diagonalization of Quaternion Circulant Matrices with Applications, SIAM J. Matrix Anal. Appl. 45(3):1429-1454, 2024. DOI: 10.1137/23M1552115.

Reproduce the dense 4×4 quaternion circulant matrix example and run solve+refine. The shipped script is `examples/ExNEWTON_3.m`.

Requirements

```
hasQuat = true;
try
    quaternion(0,0,0,0);
catch
    hasQuat = false;
end
if ~hasQuat
    disp('This toolbox requires MATLAB''s built-in quaternion class
(quaternion(w,x,y,z)).');
    disp('Examples in this page are skipped.');
    return;
end

if exist('leigqNEWTON','file') ~= 2
    thisFile = mfilename('fullpath');
    if ~isempty(thisFile)
        rootGuess = fileparts(fileparts(fileparts(thisFile))); % .../docs/source ->
toolbox root
        if exist(fullfile(rootGuess,'leigqNEWTON.m'),'file')
            addpath(rootGuess);
        end
    end
end

if exist('leigqNEWTON','file') ~= 2
    error('leigqNEWTON not found on the MATLAB path. Add the toolbox root folder.');
end
```

Run the shipped script

```
root = fileparts(which('leigqNEWTON'));
ex   = fullfile(root,'examples','ExNEWTON_3.m');
if exist(ex,'file')
    run(ex);
else
    error('Example not found: %s', ex);
end
```

```
=== Pan--Ng dense 4x4 quaternion circulant matrix ===
=== checkNEWTON ===
Matrix: 4x4 quaternion
Lambdas: 4 (computed).  Vectors: 4 columns (computed).
Certificate/residual summary (median | max):
  resMin abs: 9.62e-16 | 3.77e-12    resMin rel: 4.30e-17 | 1.64e-13
  resPair abs:3.55e-15 | 3.77e-12    resPair rel:1.63e-16 | 1.64e-13

Per-eigenvalue residuals:
  #  lambda (cleaned/rounded)                          resMin(abs/rel)        resPair(abs/rel)
   1  -2 -2i +0j +2k                                    2.98e-16/ 1.39e-17    3.95e-15/ 1.85e-16
   2  -1.609 -1.752i +2.986j +2.341k                    1.40e-15/ 6.28e-17    3.15e-15/ 1.41e-16
   3  -1.339 -1.5i +2.699j +3.142k                      5.20e-16/ 2.31e-17    3.00e-15/ 1.34e-16
   4  -3.688 +1.143i +1.652j +2.785k                    3.77e-12/ 1.64e-13    3.77e-12/ 1.64e-13

Interestingness:
  Standard: Kdistinct = n = 4.
  Sphere check: not run. Use checkNEWTON(...,'SphereCheck','on') to force.

Outputs:
  out           : main struct (out.lambda, out.v, out.resMinAbs, out.resPairAbs, ...).
  cases{1}      : simple case struct with fields A, lamAll, lamSamples, sph, info (sphere tooling style).
  S             : summary struct (n, Ktot, Kdistinct, spheresFound, ...).

Refinement (batch):
=== leigqNEWTON_refine_batch: start (n=4, K=4, mode=auto, target=1.0e-13) ===

[k=1/4] start r_min=2.978e-16
[k=1] DONE: r_min 2.978e-16 -> 2.978e-16   (dt=0.00s)

[k=2/4] start r_min=1.405e-15
[k=2] DONE: r_min 1.405e-15 -> 1.405e-15   (dt=0.00s)

[k=3/4] start r_min=5.199e-16
[k=3] DONE: r_min 5.199e-16 -> 5.199e-16   (dt=0.00s)

[k=4/4] start r_min=3.767e-12
[k=4/4] rand probe: radius=1, NRand=2000
  t=  200/ 2000  best=3.767e-12  dt=0.08s
  t=  400/ 2000  best=3.767e-12  dt=0.16s
  t=  600/ 2000  best=3.767e-12  dt=0.22s
  t=  800/ 2000  best=3.767e-12  dt=0.29s
  t= 1000/ 2000  best=3.767e-12  dt=0.36s
  t= 1200/ 2000  best=3.767e-12  dt=0.45s
  t= 1400/ 2000  best=3.767e-12  dt=0.51s
  t= 1600/ 2000  best=3.767e-12  dt=0.58s
  t= 1800/ 2000  best=3.767e-12  dt=0.65s
  t= 2000/ 2000  best=3.767e-12  dt=0.72s
  done radius=1 -> best=3.767e-12 (dt=0.72s)
[k=4/4] rand probe: radius=0.3, NRand=2000
  t=  200/ 2000  best=3.767e-12  dt=0.06s
  t=  400/ 2000  best=3.767e-12  dt=0.13s
  t=  600/ 2000  best=3.767e-12  dt=0.19s
  t=  800/ 2000  best=3.767e-12  dt=0.26s
  t= 1000/ 2000  best=3.767e-12  dt=0.33s
  t= 1200/ 2000  best=3.767e-12  dt=0.40s
  t= 1400/ 2000  best=3.767e-12  dt=0.46s
  t= 1600/ 2000  best=3.767e-12  dt=0.53s
  t= 1800/ 2000  best=3.767e-12  dt=0.60s
  t= 2000/ 2000  best=3.767e-12  dt=0.66s
  done radius=0.3 -> best=3.767e-12 (dt=0.66s)
[k=4/4] rand probe: radius=0.1, NRand=2000
  t=  200/ 2000  best=3.767e-12  dt=0.06s
  t=  400/ 2000  best=3.767e-12  dt=0.13s
```

```
  t=  600/ 2000  best=3.767e-12  dt=0.20s
  t=  800/ 2000  best=3.767e-12  dt=0.27s
  t= 1000/ 2000  best=3.767e-12  dt=0.33s
  t= 1200/ 2000  best=3.767e-12  dt=0.41s
  t= 1400/ 2000  best=3.767e-12  dt=0.48s
  t= 1600/ 2000  best=3.767e-12  dt=0.55s
  t= 1800/ 2000  best=3.767e-12  dt=0.61s
  t= 2000/ 2000  best=3.767e-12  dt=0.68s
  done radius=0.1 -> best=3.767e-12 (dt=0.69s)
[k=4/4] rand probe: radius=0.03, NRand=2000
  t=  200/ 2000  best=3.767e-12  dt=0.07s
  t=  400/ 2000  best=3.767e-12  dt=0.14s
  t=  600/ 2000  best=3.767e-12  dt=0.21s
  t=  800/ 2000  best=3.767e-12  dt=0.28s
  t= 1000/ 2000  best=3.767e-12  dt=0.35s
  t= 1200/ 2000  best=3.767e-12  dt=0.42s
  t= 1400/ 2000  best=3.767e-12  dt=0.49s
  t= 1600/ 2000  best=3.767e-12  dt=0.55s
  t= 1800/ 2000  best=3.767e-12  dt=0.62s
  t= 2000/ 2000  best=3.767e-12  dt=0.69s
  done radius=0.03 -> best=3.767e-12 (dt=0.69s)
[k=4/4] rand probe: radius=0.01, NRand=2000
  t=  200/ 2000  best=3.767e-12  dt=0.06s
  t=  400/ 2000  best=3.767e-12  dt=0.13s
  t=  600/ 2000  best=3.767e-12  dt=0.20s
  t=  800/ 2000  best=3.767e-12  dt=0.27s
  t= 1000/ 2000  best=3.767e-12  dt=0.33s
  t= 1200/ 2000  best=3.767e-12  dt=0.40s
  t= 1400/ 2000  best=3.767e-12  dt=0.47s
  t= 1600/ 2000  best=3.767e-12  dt=0.54s
  t= 1800/ 2000  best=3.767e-12  dt=0.60s
  t= 2000/ 2000  best=3.767e-12  dt=0.67s
  done radius=0.01 -> best=3.767e-12 (dt=0.67s)
[k=4/4] rand probe: radius=0.003, NRand=2000
  t=  200/ 2000  best=3.767e-12  dt=0.07s
  t=  400/ 2000  best=3.767e-12  dt=0.14s
  t=  600/ 2000  best=3.767e-12  dt=0.21s
  t=  800/ 2000  best=3.767e-12  dt=0.28s
  t= 1000/ 2000  best=3.767e-12  dt=0.34s
  t= 1200/ 2000  best=3.767e-12  dt=0.41s
  t= 1400/ 2000  best=3.767e-12  dt=0.47s
  t= 1600/ 2000  best=3.767e-12  dt=0.54s
  t= 1800/ 2000  best=3.767e-12  dt=0.61s
  t= 2000/ 2000  best=3.767e-12  dt=0.68s
  done radius=0.003 -> best=3.767e-12 (dt=0.68s)
[k=4/4] rand probe: radius=0.001, NRand=2000
  t=  200/ 2000  best=3.767e-12  dt=0.06s
  t=  400/ 2000  best=3.767e-12  dt=0.14s
  t=  600/ 2000  best=3.767e-12  dt=0.20s
  t=  800/ 2000  best=3.767e-12  dt=0.27s
  t= 1000/ 2000  best=3.767e-12  dt=0.33s
  t= 1200/ 2000  best=3.767e-12  dt=0.40s
  t= 1400/ 2000  best=3.767e-12  dt=0.46s
  t= 1600/ 2000  best=3.767e-12  dt=0.54s
  t= 1800/ 2000  best=3.767e-12  dt=0.61s
  t= 2000/ 2000  best=3.767e-12  dt=0.69s
  done radius=0.001 -> best=3.767e-12 (dt=0.69s)
[k=4/4] rand probe: radius=0.0003, NRand=2000
  t=  200/ 2000  best=3.767e-12  dt=0.07s
  t=  400/ 2000  best=3.767e-12  dt=0.14s
  t=  600/ 2000  best=3.767e-12  dt=0.21s
  t=  800/ 2000  best=3.767e-12  dt=0.29s
  t= 1000/ 2000  best=3.767e-12  dt=0.36s
  t= 1200/ 2000  best=3.767e-12  dt=0.43s
```

```
  t= 1400/ 2000  best=3.767e-12  dt=0.49s
  t= 1600/ 2000  best=3.767e-12  dt=0.56s
  t= 1800/ 2000  best=3.767e-12  dt=0.64s
  t= 2000/ 2000  best=3.767e-12  dt=0.71s
  done radius=0.0003 -> best=3.767e-12 (dt=0.71s)
[k=4/4] rand probe: radius=0.0001, NRand=2000
  t=  200/ 2000  best=3.767e-12  dt=0.07s
  t=  400/ 2000  best=3.767e-12  dt=0.14s
  t=  600/ 2000  best=3.767e-12  dt=0.21s
  t=  800/ 2000  best=3.767e-12  dt=0.29s
  t= 1000/ 2000  best=3.767e-12  dt=0.36s
  t= 1200/ 2000  best=3.767e-12  dt=0.43s
  t= 1400/ 2000  best=3.767e-12  dt=0.50s
  t= 1600/ 2000  best=3.767e-12  dt=0.57s
  t= 1800/ 2000  best=3.767e-12  dt=0.64s
  t= 2000/ 2000  best=3.767e-12  dt=0.71s
  done radius=0.0001 -> best=3.767e-12 (dt=0.71s)
[k=4] fminsearch starting from r_min=3.767e-12 ...

 Iteration   Func-count         f(x)         Procedure
     0            1      3.76692e-12
     1            5      3.76692e-12         initial simplex
     2            7      3.76692e-12         contract outside
     3            9      3.76692e-12         contract inside
     4           10      3.76692e-12         reflect
     5           11      3.76692e-12         reflect
     6           12      3.76692e-12         reflect
     7           14      3.76692e-12         contract inside
     8           16      3.76692e-12         contract inside
     9           18      3.76692e-12         contract outside
    10           20      3.76692e-12         contract inside
    11           22      3.76692e-12         contract outside
    12           24      3.76692e-12         contract outside
    13           26      3.76692e-12         contract inside
    14           27      3.76692e-12         reflect
    15           29      3.76692e-12         contract inside
    16           31      3.76692e-12         contract outside
    17           33      3.76692e-12         contract outside
    18           35      3.76692e-12         contract inside
    19           37      3.76692e-12         contract inside
    20           39      3.76692e-12         contract inside
    21           41      3.76692e-12         contract inside
    22           43      3.76692e-12         contract inside
    23           45      3.76692e-12         contract inside
    24           47      3.76692e-12         contract inside
    25           49      3.76692e-12         contract inside
    26           51      3.76692e-12         contract outside
    27           52      3.76692e-12         reflect
    28           54      3.76692e-12         contract inside
    29           56      3.76692e-12         contract inside
    30           58      3.76692e-12         contract inside
    31           60      3.76692e-12         contract inside
    32           62      3.76692e-12         contract inside
    33           64      3.76692e-12         contract inside
    34           66      3.76692e-12         contract outside
    35           68      3.76692e-12         contract inside
    36           70      3.76692e-12         contract inside
    37           72      3.76692e-12         contract outside
    38           74      3.76692e-12         contract inside
    39           76      3.76692e-12         contract inside
    40           77      3.76692e-12         reflect
    41           79      3.76692e-12         contract inside
    42           81      3.76692e-12         contract inside
    43           83      3.76692e-12         contract inside
```

44	85	3.76692e-12	contract inside
45	87	3.76692e-12	contract inside
46	89	3.76692e-12	contract inside
47	91	3.76692e-12	contract outside
48	93	3.76692e-12	contract inside
49	95	3.76692e-12	contract outside
50	96	3.76692e-12	reflect
51	98	3.76692e-12	contract inside
52	100	3.76692e-12	contract inside
53	102	3.76692e-12	contract inside
54	104	3.76692e-12	contract outside
55	106	3.76692e-12	contract inside
56	108	3.76692e-12	contract inside
57	109	3.76692e-12	reflect
58	111	3.76692e-12	contract inside
59	113	3.76692e-12	contract inside
60	114	3.76692e-12	reflect
61	116	3.76692e-12	contract inside
62	118	3.76692e-12	contract outside
63	120	3.76692e-12	contract outside
64	122	3.76692e-12	contract inside
65	124	3.76692e-12	contract inside
66	126	3.76692e-12	contract inside
67	128	3.76692e-12	contract outside
68	130	3.76692e-12	contract inside
69	132	3.76692e-12	contract inside
70	134	3.76692e-12	contract inside
71	136	3.76692e-12	contract outside
72	138	3.76692e-12	contract inside
73	140	3.76692e-12	contract inside
74	142	3.76692e-12	contract inside
75	144	3.76692e-12	contract outside
76	146	3.76692e-12	contract inside
77	148	3.76692e-12	contract inside
78	150	3.76692e-12	contract inside
79	152	3.76692e-12	contract outside
80	154	3.76692e-12	contract inside
81	156	3.76692e-12	contract inside
82	158	3.76692e-12	contract inside
83	160	3.76692e-12	contract outside
84	162	3.76692e-12	contract outside
85	164	3.76692e-12	contract inside
86	166	3.76692e-12	contract inside
87	168	3.76692e-12	contract inside
88	170	3.76692e-12	contract inside
89	172	3.76692e-12	contract inside
90	174	3.76692e-12	contract inside
91	176	3.76692e-12	contract inside
92	178	3.76692e-12	contract inside
93	180	3.76692e-12	contract inside
94	182	3.76692e-12	contract inside
95	184	3.76692e-12	contract inside
96	186	3.76692e-12	contract inside
97	188	3.76692e-12	contract outside
98	189	3.76692e-12	reflect
99	191	3.76692e-12	contract inside
100	193	3.76692e-12	contract inside
101	199	3.76692e-12	shrink
102	201	3.76692e-12	contract outside
103	203	3.76692e-12	contract outside
104	205	3.76692e-12	contract inside
105	207	3.76692e-12	contract inside
106	209	3.76692e-12	contract inside
107	211	3.76692e-12	contract outside

108	212	3.76692e-12	reflect
109	214	3.76692e-12	contract inside
110	216	3.76692e-12	contract outside
111	218	3.76692e-12	contract inside
112	220	3.76692e-12	contract outside
113	222	3.76692e-12	contract inside
114	224	3.76692e-12	contract outside
115	225	3.76692e-12	reflect
116	227	3.76692e-12	contract inside
117	229	3.76692e-12	contract inside
118	230	3.76692e-12	reflect
119	232	3.76692e-12	contract inside
120	234	3.76692e-12	contract outside
121	236	2.52777e-12	contract inside
122	238	2.52777e-12	contract inside
123	240	2.52777e-12	contract outside
124	242	2.39765e-12	contract inside
125	244	1.33557e-12	contract inside
126	246	1.33557e-12	contract outside
127	248	1.1612e-12	contract inside
128	250	1.09099e-12	contract inside
129	252	6.92566e-13	contract outside
130	254	6.49651e-13	contract inside
131	256	3.64665e-13	contract inside
132	257	3.64665e-13	reflect
133	259	3.64665e-13	contract inside
134	261	1.77102e-13	contract inside
135	263	1.77102e-13	contract inside
136	264	1.77102e-13	reflect
137	266	1.77102e-13	contract inside
138	268	1.72467e-13	contract inside
139	270	1.72467e-13	contract inside
140	272	1.5983e-13	contract inside
141	274	6.54049e-14	contract inside
142	276	6.54049e-14	contract inside
143	278	6.54049e-14	contract inside
144	280	6.54049e-14	contract inside
145	282	6.54049e-14	contract inside
146	283	6.54049e-14	reflect
147	285	2.64575e-14	contract inside
148	287	2.64575e-14	contract inside
149	289	2.64575e-14	contract outside
150	290	2.64575e-14	reflect
151	292	2.32462e-14	contract inside
152	294	2.10865e-14	contract inside
153	296	1.44621e-14	contract inside
154	298	1.44621e-14	contract inside
155	300	1.15599e-14	contract inside
156	302	7.22256e-15	contract inside
157	304	7.22256e-15	contract outside
158	306	4.37682e-15	contract inside
159	308	4.37682e-15	contract outside
160	310	3.89828e-15	contract inside
161	312	2.60486e-15	contract inside
162	314	2.10951e-15	contract inside
163	315	2.10951e-15	reflect
164	317	5.65456e-16	contract inside
165	319	5.65456e-16	contract inside
166	320	5.65456e-16	reflect
167	322	2.61128e-16	contract inside
168	324	2.61128e-16	contract inside
169	326	2.61128e-16	contract inside
170	327	2.61128e-16	reflect
171	329	1.40004e-16	contract inside

172	331	1.40004e-16	contract inside
173	332	1.40004e-16	reflect
174	334	9.9982e-17	contract inside
175	340	9.4032e-17	shrink
176	342	9.4032e-17	contract inside
177	344	9.4032e-17	contract outside
178	350	9.4032e-17	shrink
179	352	9.28756e-17	contract inside
180	353	9.28756e-17	reflect
181	355	9.28756e-17	contract inside
182	357	9.28756e-17	contract inside
183	359	7.87333e-17	reflect
184	365	4.34872e-17	shrink
185	366	4.34872e-17	reflect
186	372	9.24455e-18	shrink
187	373	9.24455e-18	reflect
188	374	9.24455e-18	reflect
189	375	9.24455e-18	reflect
190	376	9.24455e-18	reflect
191	382	9.24455e-18	shrink
192	383	9.24455e-18	reflect
193	384	9.24455e-18	reflect
194	385	9.24455e-18	reflect
195	391	9.24455e-18	shrink
196	393	9.24455e-18	contract outside
197	394	9.24455e-18	reflect
198	395	9.24455e-18	reflect
199	397	9.24455e-18	contract inside
200	399	9.24455e-18	contract inside
201	400	9.24455e-18	reflect
202	401	9.24455e-18	reflect
203	402	9.24455e-18	reflect
204	404	9.24455e-18	contract outside
205	410	9.24455e-18	shrink
206	411	9.24455e-18	reflect
207	413	9.24455e-18	contract inside
208	419	9.24455e-18	shrink
209	420	9.24455e-18	reflect
210	421	9.24455e-18	reflect
211	427	9.24455e-18	shrink
212	428	9.24455e-18	reflect
213	430	9.24455e-18	contract inside
214	436	9.24455e-18	shrink
215	438	9.24455e-18	contract inside
216	444	9.24455e-18	shrink
217	450	9.24455e-18	shrink
218	451	9.24455e-18	reflect
219	457	9.24455e-18	shrink
220	459	9.24455e-18	contract outside
221	460	9.24455e-18	reflect
222	466	9.24455e-18	shrink
223	467	9.24455e-18	reflect
224	468	9.24455e-18	reflect
225	474	9.24455e-18	shrink
226	475	9.24455e-18	reflect
227	476	9.24455e-18	reflect
228	477	9.24455e-18	reflect
229	478	9.24455e-18	reflect
230	484	9.24455e-18	shrink
231	485	9.24455e-18	reflect
232	487	4.54138e-18	reflect
233	488	4.54138e-18	reflect
234	494	4.54138e-18	shrink
235	495	4.54138e-18	reflect

```
   236          501      4.54138e-18         shrink
   237          503      4.54138e-18         contract outside
   238          505      4.54138e-18         contract inside
   239          507      4.54138e-18         contract inside
   240          509      4.54138e-18         contract inside
   241          515      4.54138e-18         shrink
   242          517      4.54138e-18         contract outside
   243          519      4.54138e-18         contract inside
   244          521      4.54138e-18         contract inside

Optimization terminated:
 the current x satisfies the termination criteria using OPTIONS.TolX of 1.000000e-16
 and F(X) satisfies the convergence criteria using OPTIONS.TolFun of 1.000000e-16

[k=4] after fminsearch: r_min=4.541e-18 (exitflag=1)
[k=4] DONE: r_min 3.767e-12 -> 4.541e-18   (dt=6.68s)

=== leigqNEWTON_refine_batch: summary ===
k=1  r_min=6.736e-18  respair=3.326e-15  resInf(polish)=8.204e-17
k=2  r_min=1.188e-17  respair=2.794e-15  resInf(polish)=5.586e-16
k=3  r_min=7.289e-17  respair=3.001e-15  resInf(polish)=6.668e-16
k=4  r_min=7.960e-16  respair=2.895e-15  resInf(polish)=9.366e-16
min pairwise distance = 9.272e-01 between (3,2)
  median(resMin)=4.238e-17, max(resMin)=7.960e-16
=== checkNEWTON ===
Matrix: 4x4 quaternion
Lambdas: 4 (provided).  Vectors: 4 columns (provided).
Certificate/residual summary (median | max):
  resMin abs: 4.24e-17 | 7.96e-16    resMin rel: 1.89e-18 | 3.47e-17
  resPair abs:3.07e-15 | 3.21e-15    resPair rel:1.36e-16 | 1.50e-16

Per-eigenvalue residuals:
  #  lambda (cleaned/rounded)                          resMin(abs/rel)        resPair(abs/rel)
   1  -2 -2i +0j +2k                                    6.74e-18/ 3.15e-19    3.19e-15/ 1.50e-16
   2  -1.609 -1.752i +2.986j +2.341k                    1.19e-17/ 5.31e-19    2.71e-15/ 1.21e-16
   3  -1.339 -1.5i +2.699j +3.142k                      7.29e-17/ 3.24e-18    3.21e-15/ 1.43e-16
   4  -3.688 +1.143i +1.652j +2.785k                    7.96e-16/ 3.47e-17    2.95e-15/ 1.29e-16

Interestingness:
  Standard: Kdistinct = n = 4.
  Sphere check: not run. Use checkNEWTON(...,'SphereCheck','on') to force.

Outputs:
  out           : main struct (out.lambda, out.v, out.resMinAbs, out.resPairAbs, ...).
  cases{1}      : simple case struct with fields A, lamAll, lamSamples, sph, info (sphere tooling style).
  S             : summary struct (n, Ktot, Kdistinct, spheresFound, ...).
\\begin{Ex}[A dense $4\\times4$ quaternion matrix]\\label{ex:PanNg_dense_4x4}
Consider the dense quaternion circulant matrix $A\\in\\HH^{4\\times 4}$:
\\[
\\setlength{\\arraycolsep}{2pt}
\\begin{bmatrix}
 -2 + \\i + \\j + 4\\k & 2 + 4\\i + \\j + \\k & 1 + 3\\i + 2\\j + 2\\k & -1 + 2\\i + 2\\j + 3\\k \\
 -1 + 2\\i + 2\\j + 3\\k & -2 + \\i + \\j + 4\\k & 2 + 4\\i + \\j + \\k & 1 + 3\\i + 2\\j + 2\\k \\
 1 + 3\\i + 2\\j + 2\\k & -1 + 2\\i + 2\\j + 3\\k & -2 + \\i + \\j + 4\\k & 2 + 4\\i + \\j + \\k \\
 2 + 4\\i + \\j + \\k & 1 + 3\\i + 2\\j + 2\\k & -1 + 2\\i + 2\\j + 3\\k & -2 + \\i + \\j + 4\\k
\\end{bmatrix}.
\\]
This matrix is taken from \\cite[Example~1]{PanNg2024}, where it is used to illustrate the
block-diagonalization of quaternion circulant matrices; the authors do not compute or report its
\\emph{left} eigenvalues.

Using our implementation, we obtained the following (rounded) left eigenvalues:
\\[
\\widehat\\lambda \\approx
```

```
\\begin{bmatrix}
 -1.5 - 0.43\\i + 1.5\\j + 4.6\\k \\
 -2 - 2\\i + 2\\k \\
 -2.9 + 0.85\\i - 1.2\\j + 5.1\\k \\
 -5.2 + 1.5\\i - 0.25\\j + 1.9\\k
\\end{bmatrix}.
\\]
The corresponding residual norms are around $10^{-16}$, confirming near machine-precision accuracy
for the computed eigenpairs.
\\end{Ex}

\\bibitem{PanNg2024}
J.~Pan and M.~K.~Ng,
\\newblock Block-Diagonalization of Quaternion Circulant Matrices with Applications,
\\newblock \\emph{SIAM J. Matrix Anal. Appl.} 45(3):1429--1454, 2024.
\\newblock \\DOI{10.1137/23M1552115}.

Done.
```

Notes

- The script builds the Pan–Ng 4×4 matrix explicitly, runs `checkNEWTON`, then (optionally) refines/polishes the returned candidates.
- If you want a faster run, switch `SolveProfile 'reliable'` → `'default'` in the script.

See also

`checkNEWTON, leigqNEWTON_refine_batch`