leigqNEWTON_refine_batch — refine + certify a list of candidates

Refine a list of candidate left eigenvalues and compute residual certificates. This is the recommended post-processing after calling `leigqNEWTON`.

Setup

```
hasQuat = true;
try
    quaternion(0,0,0,0);
catch
    hasQuat = false;
end
if ~hasQuat
    disp('This toolbox requires MATLAB''s built-in quaternion class
(quaternion(w,x,y,z)).');
    disp('Examples in this page are skipped.');
    return;
end

if exist('leigqNEWTON_refine_batch','file') ~= 2
    thisFile = mfilename('fullpath');
    if ~isempty(thisFile)
        rootGuess = fileparts(fileparts(fileparts(thisFile))); % .../docs/source ->
toolbox root
        if exist(fullfile(rootGuess,'leigqNEWTON_refine_batch.m'),'file')
            addpath(rootGuess);
        end
    end
end

if exist('leigqNEWTON_refine_batch','file') ~= 2
    error('leigqNEWTON_refine_batch not found on the MATLAB path. Add the toolbox
root folder.');
end
```

Syntax

- `[lamR,VR,cert] = leigqNEWTON_refine_batch(A, lam)`
- `... = leigqNEWTON_refine_batch(..., 'DoPolish',true,'TargetResMin',1e-14)`

Notes

- `cert` is a struct; common fields include `resMin` (lambda-only) and `resPair` (pair residual).
- If `DoPolish` is true, each accepted candidate is polished to near machine precision.

Example 1: refine solver output (2×2)

```
q0 = quaternion(0,0,0,0);
q1 = quaternion(1,0,0,0);
qi = quaternion(0,1,0,0);
qj = quaternion(0,0,1,0);
A  = [ q0, qi;
       qj, q1 ];

lambda1 = leigqNEWTON(A,'Seed',1,'SolveProfile','default');
[lamR1,VR1,cert1] = leigqNEWTON_refine_batch(A, lambda1, 'DoPolish',true);
```

```
=== leigqNEWTON_refine_batch: start (n=2, K=2, mode=auto, target=1.0e-13) ===

[k=1/2] start r_min=2.680e-11
[k=1/2] rand probe: radius=1, NRand=2000
  t=  200/ 2000  best=2.680e-11  dt=0.07s
  t=  400/ 2000  best=2.680e-11  dt=0.12s
  t=  600/ 2000  best=2.680e-11  dt=0.19s
  t=  800/ 2000  best=2.680e-11  dt=0.25s
  t= 1000/ 2000  best=2.680e-11  dt=0.32s
  t= 1200/ 2000  best=2.680e-11  dt=0.38s
  t= 1400/ 2000  best=2.680e-11  dt=0.45s
  t= 1600/ 2000  best=2.680e-11  dt=0.51s
  t= 1800/ 2000  best=2.680e-11  dt=0.58s
  t= 2000/ 2000  best=2.680e-11  dt=0.63s
  done radius=1 -> best=2.680e-11 (dt=0.64s)
[k=1/2] rand probe: radius=0.3, NRand=2000
  t=  200/ 2000  best=2.680e-11  dt=0.06s
  t=  400/ 2000  best=2.680e-11  dt=0.12s
  t=  600/ 2000  best=2.680e-11  dt=0.19s
  t=  800/ 2000  best=2.680e-11  dt=0.25s
  t= 1000/ 2000  best=2.680e-11  dt=0.32s
  t= 1200/ 2000  best=2.680e-11  dt=0.37s
  t= 1400/ 2000  best=2.680e-11  dt=0.44s
  t= 1600/ 2000  best=2.680e-11  dt=0.50s
  t= 1800/ 2000  best=2.680e-11  dt=0.56s
  t= 2000/ 2000  best=2.680e-11  dt=0.62s
  done radius=0.3 -> best=2.680e-11 (dt=0.63s)
[k=1/2] rand probe: radius=0.1, NRand=2000
  t=  200/ 2000  best=2.680e-11  dt=0.06s
  t=  400/ 2000  best=2.680e-11  dt=0.12s
  t=  600/ 2000  best=2.680e-11  dt=0.19s
  t=  800/ 2000  best=2.680e-11  dt=0.25s
  t= 1000/ 2000  best=2.680e-11  dt=0.32s
  t= 1200/ 2000  best=2.680e-11  dt=0.37s
  t= 1400/ 2000  best=2.680e-11  dt=0.44s
  t= 1600/ 2000  best=2.680e-11  dt=0.50s
  t= 1800/ 2000  best=2.680e-11  dt=0.56s
  t= 2000/ 2000  best=2.680e-11  dt=0.62s
  done radius=0.1 -> best=2.680e-11 (dt=0.62s)
[k=1/2] rand probe: radius=0.03, NRand=2000
  t=  200/ 2000  best=2.680e-11  dt=0.06s
  t=  400/ 2000  best=2.680e-11  dt=0.12s
  t=  600/ 2000  best=2.680e-11  dt=0.18s
  t=  800/ 2000  best=2.680e-11  dt=0.24s
  t= 1000/ 2000  best=2.680e-11  dt=0.31s
  t= 1200/ 2000  best=2.680e-11  dt=0.36s
  t= 1400/ 2000  best=2.680e-11  dt=0.43s
  t= 1600/ 2000  best=2.680e-11  dt=0.49s
  t= 1800/ 2000  best=2.680e-11  dt=0.55s
  t= 2000/ 2000  best=2.680e-11  dt=0.62s
  done radius=0.03 -> best=2.680e-11 (dt=0.62s)
```

```
[k=1/2] rand probe: radius=0.01, NRand=2000
  t=  200/ 2000  best=2.680e-11  dt=0.06s
  t=  400/ 2000  best=2.680e-11  dt=0.12s
  t=  600/ 2000  best=2.680e-11  dt=0.19s
  t=  800/ 2000  best=2.680e-11  dt=0.25s
  t= 1000/ 2000  best=2.680e-11  dt=0.31s
  t= 1200/ 2000  best=2.680e-11  dt=0.37s
  t= 1400/ 2000  best=2.680e-11  dt=0.44s
  t= 1600/ 2000  best=2.680e-11  dt=0.50s
  t= 1800/ 2000  best=2.680e-11  dt=0.57s
  t= 2000/ 2000  best=2.680e-11  dt=0.64s
  done radius=0.01 -> best=2.680e-11 (dt=0.65s)
[k=1/2] rand probe: radius=0.003, NRand=2000
  t=  200/ 2000  best=2.680e-11  dt=0.07s
  t=  400/ 2000  best=2.680e-11  dt=0.13s
  t=  600/ 2000  best=2.680e-11  dt=0.19s
  t=  800/ 2000  best=2.680e-11  dt=0.26s
  t= 1000/ 2000  best=2.680e-11  dt=0.33s
  t= 1200/ 2000  best=2.680e-11  dt=0.39s
  t= 1400/ 2000  best=2.680e-11  dt=0.46s
  t= 1600/ 2000  best=2.680e-11  dt=0.52s
  t= 1800/ 2000  best=2.680e-11  dt=0.58s
  t= 2000/ 2000  best=2.680e-11  dt=0.64s
  done radius=0.003 -> best=2.680e-11 (dt=0.64s)
[k=1/2] rand probe: radius=0.001, NRand=2000
  t=  200/ 2000  best=2.680e-11  dt=0.06s
  t=  400/ 2000  best=2.680e-11  dt=0.12s
  t=  600/ 2000  best=2.680e-11  dt=0.18s
  t=  800/ 2000  best=2.680e-11  dt=0.25s
  t= 1000/ 2000  best=2.680e-11  dt=0.31s
  t= 1200/ 2000  best=2.680e-11  dt=0.37s
  t= 1400/ 2000  best=2.680e-11  dt=0.44s
  t= 1600/ 2000  best=2.680e-11  dt=0.50s
  t= 1800/ 2000  best=2.680e-11  dt=0.57s
  t= 2000/ 2000  best=2.680e-11  dt=0.63s
  done radius=0.001 -> best=2.680e-11 (dt=0.63s)
[k=1/2] rand probe: radius=0.0003, NRand=2000
  t=  200/ 2000  best=2.680e-11  dt=0.06s
  t=  400/ 2000  best=2.680e-11  dt=0.12s
  t=  600/ 2000  best=2.680e-11  dt=0.19s
  t=  800/ 2000  best=2.680e-11  dt=0.26s
  t= 1000/ 2000  best=2.680e-11  dt=0.33s
  t= 1200/ 2000  best=2.680e-11  dt=0.39s
  t= 1400/ 2000  best=2.680e-11  dt=0.46s
  t= 1600/ 2000  best=2.680e-11  dt=0.52s
  t= 1800/ 2000  best=2.680e-11  dt=0.59s
  t= 2000/ 2000  best=2.680e-11  dt=0.65s
  done radius=0.0003 -> best=2.680e-11 (dt=0.65s)
[k=1/2] rand probe: radius=0.0001, NRand=2000
  t=  200/ 2000  best=2.680e-11  dt=0.06s
  t=  400/ 2000  best=2.680e-11  dt=0.12s
  t=  600/ 2000  best=2.680e-11  dt=0.19s
  t=  800/ 2000  best=2.680e-11  dt=0.25s
  t= 1000/ 2000  best=2.680e-11  dt=0.31s
  t= 1200/ 2000  best=2.680e-11  dt=0.37s
  t= 1400/ 2000  best=2.680e-11  dt=0.44s
  t= 1600/ 2000  best=2.680e-11  dt=0.50s
  t= 1800/ 2000  best=2.680e-11  dt=0.57s
  t= 2000/ 2000  best=2.680e-11  dt=0.63s
  done radius=0.0001 -> best=2.680e-11 (dt=0.63s)
[k=1] fminsearch starting from r_min=2.680e-11 ...

 Iteration   Func-count         f(x)         Procedure
     0            1      2.68034e-11
```

1	5	2.68034e-11	initial simplex
2	7	2.68034e-11	contract inside
3	8	2.68034e-11	reflect
4	10	2.68034e-11	contract inside
5	12	2.68034e-11	contract inside
6	14	2.68034e-11	contract inside
7	16	2.68034e-11	contract inside
8	17	2.68034e-11	reflect
9	18	2.68034e-11	reflect
10	19	2.68034e-11	reflect
11	21	2.68034e-11	contract inside
12	23	2.68034e-11	contract outside
13	24	2.68034e-11	reflect
14	25	2.68034e-11	reflect
15	27	2.68034e-11	contract outside
16	29	2.68034e-11	contract outside
17	30	2.68034e-11	reflect
18	31	2.68034e-11	reflect
19	33	2.68034e-11	contract inside
20	34	2.68034e-11	reflect
21	36	2.68034e-11	contract outside
22	37	2.68034e-11	reflect
23	39	2.68034e-11	contract outside
24	41	2.68034e-11	contract inside
25	42	2.68034e-11	reflect
26	44	2.68034e-11	contract outside
27	46	2.68034e-11	contract inside
28	48	2.68034e-11	contract inside
29	49	2.68034e-11	reflect
30	50	2.68034e-11	reflect
31	52	2.68034e-11	contract inside
32	53	2.68034e-11	reflect
33	55	2.68034e-11	contract inside
34	57	2.68034e-11	contract outside
35	59	2.68034e-11	contract outside
36	61	2.68034e-11	contract outside
37	62	2.68034e-11	reflect
38	63	2.68034e-11	reflect
39	64	2.68034e-11	reflect
40	66	2.68034e-11	contract inside
41	68	2.68034e-11	contract inside
42	70	2.68034e-11	contract inside
43	72	2.68034e-11	contract inside
44	73	2.68034e-11	reflect
45	75	2.68034e-11	contract inside
46	77	2.68034e-11	contract outside
47	78	2.68034e-11	reflect
48	79	2.68034e-11	reflect
49	80	2.68034e-11	reflect
50	82	2.68034e-11	contract inside
51	83	2.68034e-11	reflect
52	85	2.68034e-11	contract inside
53	86	2.68034e-11	reflect
54	88	2.68034e-11	contract inside
55	90	2.68034e-11	contract inside
56	91	2.68034e-11	reflect
57	93	2.68034e-11	contract inside
58	94	2.68034e-11	reflect
59	95	2.68034e-11	reflect
60	96	2.68034e-11	reflect
61	97	2.68034e-11	reflect
62	99	2.68034e-11	contract inside
63	101	2.68034e-11	contract inside
64	102	2.68034e-11	reflect

65	103	2.68034e-11	reflect
66	105	2.68034e-11	contract inside
67	107	2.68034e-11	contract outside
68	108	2.68034e-11	reflect
69	110	2.68034e-11	contract outside
70	111	2.68034e-11	reflect
71	112	2.68034e-11	reflect
72	113	2.68034e-11	reflect
73	114	2.68034e-11	reflect
74	116	2.68034e-11	contract inside
75	118	2.68034e-11	contract inside
76	120	2.68034e-11	contract inside
77	122	2.68034e-11	contract outside
78	123	2.68034e-11	reflect
79	124	2.68034e-11	reflect
80	126	2.68034e-11	contract outside
81	128	2.68034e-11	contract inside
82	129	2.68034e-11	reflect
83	130	2.68034e-11	reflect
84	132	2.68034e-11	contract outside
85	133	2.68034e-11	reflect
86	135	2.68034e-11	contract inside
87	136	2.68034e-11	reflect
88	138	2.68034e-11	contract inside
89	139	2.68034e-11	reflect
90	141	2.68034e-11	contract outside
91	143	2.68034e-11	contract outside
92	145	2.68034e-11	contract inside
93	147	2.68034e-11	contract outside
94	149	2.68034e-11	contract inside
95	150	2.68034e-11	reflect
96	151	2.68034e-11	reflect
97	152	2.68034e-11	reflect
98	154	2.68034e-11	contract inside
99	156	2.68034e-11	contract outside
100	158	2.68034e-11	contract outside
101	159	2.68034e-11	reflect
102	160	2.68034e-11	reflect
103	161	2.68034e-11	reflect
104	163	2.68034e-11	contract inside
105	165	2.68034e-11	contract inside
106	166	2.68034e-11	reflect
107	167	2.68034e-11	reflect
108	169	2.68034e-11	contract inside
109	171	2.68034e-11	contract inside
110	172	2.68034e-11	reflect
111	173	2.68034e-11	reflect
112	175	2.68034e-11	contract outside
113	177	2.68034e-11	contract inside
114	178	2.68034e-11	reflect
115	180	2.68034e-11	contract outside
116	181	2.68034e-11	reflect
117	183	2.68034e-11	contract inside
118	184	2.68034e-11	reflect
119	185	2.68034e-11	reflect
120	186	2.68034e-11	reflect
121	188	2.68034e-11	contract inside
122	189	2.68034e-11	reflect
123	190	2.68034e-11	reflect
124	192	2.68034e-11	contract outside
125	194	2.68034e-11	contract inside
126	196	2.68034e-11	contract inside
127	198	2.68034e-11	contract outside
128	199	2.68034e-11	reflect

129	200	2.68034e-11	reflect
130	201	2.68034e-11	reflect
131	203	2.68034e-11	contract outside
132	205	2.68034e-11	contract inside
133	207	2.68034e-11	contract inside
134	209	2.68034e-11	contract outside
135	210	2.68034e-11	reflect
136	211	2.68034e-11	reflect
137	212	2.68034e-11	reflect
138	214	2.68034e-11	contract inside
139	215	2.68034e-11	reflect
140	216	2.68034e-11	reflect
141	217	2.68034e-11	reflect
142	219	2.68034e-11	contract inside
143	221	2.68034e-11	contract inside
144	223	2.68034e-11	contract inside
145	225	2.68034e-11	contract inside
146	226	2.68034e-11	reflect
147	228	2.68034e-11	contract outside
148	229	2.68034e-11	reflect
149	231	2.68034e-11	contract outside
150	232	2.68034e-11	reflect
151	233	2.68034e-11	reflect
152	235	2.68034e-11	contract outside
153	236	2.68034e-11	reflect
154	238	2.68034e-11	contract inside
155	240	2.68034e-11	contract inside
156	242	2.68034e-11	contract inside
157	244	2.68034e-11	contract outside
158	246	2.68034e-11	contract outside
159	247	2.68034e-11	reflect
160	248	2.68034e-11	reflect
161	250	2.68034e-11	contract inside
162	251	2.68034e-11	reflect
163	253	2.68034e-11	contract inside
164	255	2.68034e-11	contract outside
165	257	2.68034e-11	contract inside
166	258	2.68034e-11	reflect
167	259	2.68034e-11	reflect
168	260	2.68034e-11	reflect
169	262	2.68034e-11	contract inside
170	264	2.68034e-11	contract outside
171	265	2.68034e-11	reflect
172	266	2.68034e-11	reflect
173	268	2.68034e-11	contract outside
174	269	2.68034e-11	reflect
175	271	2.68034e-11	contract inside
176	273	2.68034e-11	contract outside
177	275	2.68034e-11	contract inside
178	276	2.68034e-11	reflect
179	278	2.68034e-11	contract outside
180	280	2.68034e-11	contract inside
181	281	2.68034e-11	reflect
182	282	2.68034e-11	reflect
183	284	2.68034e-11	contract inside
184	285	2.68034e-11	reflect
185	287	2.68034e-11	contract inside
186	288	2.68034e-11	reflect
187	290	2.68034e-11	contract inside
188	292	2.68034e-11	contract outside
189	294	2.68034e-11	contract inside
190	296	2.68034e-11	contract outside
191	298	2.68034e-11	contract outside
192	299	2.68034e-11	reflect

193	301	2.68034e-11	contract outside
194	302	2.68034e-11	reflect
195	304	2.68034e-11	contract inside
196	305	2.68034e-11	reflect
197	306	2.68034e-11	reflect
198	308	2.68034e-11	contract inside
199	310	2.68034e-11	contract inside
200	311	2.68034e-11	reflect
201	313	2.68034e-11	contract inside
202	314	2.68034e-11	reflect
203	315	2.68034e-11	reflect
204	316	2.68034e-11	reflect
205	318	2.68034e-11	contract inside
206	320	2.68034e-11	contract inside
207	322	2.68034e-11	contract inside
208	324	2.68034e-11	contract outside
209	326	2.68034e-11	contract outside
210	328	2.68034e-11	contract inside
211	329	2.68034e-11	reflect
212	331	2.68034e-11	contract outside
213	333	2.60885e-11	reflect
214	334	2.60885e-11	reflect
215	336	2.41802e-11	contract inside
216	338	2.41748e-11	contract inside
217	339	2.41748e-11	reflect
218	341	1.15628e-11	contract inside
219	342	1.15628e-11	reflect
220	344	1.15628e-11	contract inside
221	346	1.15628e-11	contract inside
222	348	1.15628e-11	contract inside
223	350	6.49875e-12	reflect
224	351	6.49875e-12	reflect
225	352	6.49875e-12	reflect
226	354	6.49875e-12	contract inside
227	356	6.49875e-12	contract outside
228	358	5.47029e-12	contract inside
229	360	4.66961e-12	contract inside
230	361	4.66961e-12	reflect
231	362	4.66961e-12	reflect
232	364	3.20853e-12	contract inside
233	366	2.81585e-12	contract inside
234	368	2.0131e-12	contract inside
235	370	2.0131e-12	contract inside
236	372	2.0131e-12	contract inside
237	374	1.71612e-12	contract inside
238	376	1.43791e-12	contract inside
239	377	1.43791e-12	reflect
240	379	1.20484e-12	contract inside
241	381	1.11188e-12	contract inside
242	383	6.84523e-13	contract inside
243	384	6.84523e-13	reflect
244	386	6.84523e-13	contract inside
245	388	6.84523e-13	contract inside
246	390	6.64182e-13	reflect
247	391	6.64182e-13	reflect
248	393	3.48049e-13	contract inside
249	395	3.29888e-13	contract inside
250	397	2.93595e-13	contract inside
251	399	2.93595e-13	contract inside
252	401	2.93595e-13	contract outside
253	403	1.69845e-13	contract inside
254	405	1.67972e-13	contract inside
255	407	1.59683e-13	contract inside
256	409	9.77811e-14	contract inside

257	411	9.77811e-14	contract inside
258	413	9.77811e-14	contract inside
259	414	9.77811e-14	reflect
260	416	9.08968e-14	contract outside
261	418	8.36908e-14	contract outside
262	419	8.36908e-14	reflect
263	421	3.74322e-14	contract inside
264	423	3.74322e-14	contract inside
265	425	3.74243e-14	contract inside
266	427	3.74243e-14	contract inside
267	429	3.74243e-14	contract inside
268	431	2.05865e-14	contract inside
269	433	2.05865e-14	contract outside
270	435	1.63141e-14	contract inside
271	437	1.63141e-14	contract inside
272	439	1.63141e-14	contract outside
273	441	1.31348e-14	contract inside
274	443	1.2222e-14	contract outside
275	445	1.152e-14	contract inside
276	447	8.59825e-15	contract inside
277	449	8.59825e-15	contract inside
278	450	8.59825e-15	reflect
279	451	8.59825e-15	reflect
280	453	7.85333e-15	reflect
281	455	7.74141e-15	contract inside
282	456	7.74141e-15	reflect
283	458	6.47942e-15	contract inside
284	460	6.23231e-15	contract inside
285	462	6.23231e-15	contract inside
286	464	6.23231e-15	contract inside
287	466	6.21586e-15	reflect
288	468	5.59735e-15	contract inside
289	470	5.59735e-15	contract inside
290	471	5.59735e-15	reflect
291	473	5.59735e-15	contract inside
292	475	5.59735e-15	contract inside
293	476	5.59735e-15	reflect
294	478	4.53234e-15	expand
295	479	4.53234e-15	reflect
296	480	4.53234e-15	reflect
297	481	4.53234e-15	reflect
298	483	3.62977e-15	reflect
299	485	3.1947e-15	expand
300	487	3.1947e-15	contract inside
301	489	3.02757e-15	reflect
302	491	9.47143e-16	expand
303	492	9.47143e-16	reflect
304	493	9.47143e-16	reflect
305	494	9.47143e-16	reflect
306	495	9.47143e-16	reflect
307	496	9.47143e-16	reflect
308	498	9.47143e-16	contract inside
309	500	8.08755e-16	contract outside
310	502	4.34615e-16	contract outside
311	504	3.7409e-16	contract inside
312	506	3.7409e-16	contract inside
313	508	1.83287e-16	contract inside
314	510	1.83287e-16	contract outside
315	512	5.50481e-17	contract inside
316	514	4.33081e-17	contract inside
317	516	4.33081e-17	contract inside
318	518	4.33081e-17	contract inside
319	520	4.33081e-17	contract inside

```
Optimization terminated:
 the current x satisfies the termination criteria using OPTIONS.TolX of 1.000000e-16
 and F(X) satisfies the convergence criteria using OPTIONS.TolFun of 1.000000e-16

[k=1] after fminsearch: r_min=4.331e-17 (exitflag=1)
[k=1] DONE: r_min 2.680e-11 -> 4.331e-17   (dt=6.17s)

[k=2/2] start r_min=6.309e-17
[k=2] DONE: r_min 6.309e-17 -> 6.309e-17   (dt=0.00s)

=== leigqNEWTON_refine_batch: summary ===
k=1  r_min=3.154e-17  respair=6.210e-16  resInf(polish)=2.371e-16
k=2  r_min=2.113e-17  respair=6.627e-16  resInf(polish)=2.691e-16
min pairwise distance = 1.414e+00 between (2,1)
```

```
median(cert1.resMin), max(cert1.resMin)
```

```
ans =
2.6333e-17
ans =
3.1538e-17
```

Example 2: Pan–Ng 4×4 (start from solver hits)

```
q = @(w,x,y,z) quaternion(w,x,y,z);
a = q(-2, 1, 1, 4);
b = q( 2, 4, 1, 1);
c = q( 1, 3, 2, 2);
d = q(-1, 2, 2, 3);
A = [a b c d;
     d a b c;
     c d a b;
     b c d a];

lambda2 = leigqNEWTON(A,'Seed',1,'SolveProfile','reliable','K',2*size(A,1));
[lamR2,VR2,cert2] = leigqNEWTON_refine_batch(A, lambda2,
'DoPolish',true,'TargetResMin',1e-14);
```

```
=== leigqNEWTON_refine_batch: start (n=4, K=4, mode=auto, target=1.0e-14) ===

[k=1/4] start r_min=2.978e-16
[k=1] DONE: r_min 2.978e-16 -> 2.978e-16   (dt=0.00s)

[k=2/4] start r_min=1.405e-15
[k=2] DONE: r_min 1.405e-15 -> 1.405e-15   (dt=0.00s)

[k=3/4] start r_min=5.199e-16
[k=3] DONE: r_min 5.199e-16 -> 5.199e-16   (dt=0.00s)

[k=4/4] start r_min=3.767e-12
[k=4/4] rand probe: radius=1, NRand=2000
  t=  200/ 2000  best=3.767e-12  dt=0.09s
  t=  400/ 2000  best=3.767e-12  dt=0.17s
  t=  600/ 2000  best=3.767e-12  dt=0.25s
  t=  800/ 2000  best=3.767e-12  dt=0.33s
  t= 1000/ 2000  best=3.767e-12  dt=0.40s
  t= 1200/ 2000  best=3.767e-12  dt=0.49s
  t= 1400/ 2000  best=3.767e-12  dt=0.57s
```

```
  t= 1600/ 2000  best=3.767e-12  dt=0.64s
  t= 1800/ 2000  best=3.767e-12  dt=0.72s
  t= 2000/ 2000  best=3.767e-12  dt=0.80s
  done radius=1 -> best=3.767e-12 (dt=0.80s)
[k=4/4] rand probe: radius=0.3, NRand=2000
  t=  200/ 2000  best=3.767e-12  dt=0.07s
  t=  400/ 2000  best=3.767e-12  dt=0.16s
  t=  600/ 2000  best=3.767e-12  dt=0.23s
  t=  800/ 2000  best=3.767e-12  dt=0.31s
  t= 1000/ 2000  best=3.767e-12  dt=0.39s
  t= 1200/ 2000  best=3.767e-12  dt=0.47s
  t= 1400/ 2000  best=3.767e-12  dt=0.55s
  t= 1600/ 2000  best=3.767e-12  dt=0.64s
  t= 1800/ 2000  best=3.767e-12  dt=0.72s
  t= 2000/ 2000  best=3.767e-12  dt=0.80s
  done radius=0.3 -> best=3.767e-12 (dt=0.80s)
[k=4/4] rand probe: radius=0.1, NRand=2000
  t=  200/ 2000  best=3.767e-12  dt=0.08s
  t=  400/ 2000  best=3.767e-12  dt=0.16s
  t=  600/ 2000  best=3.767e-12  dt=0.24s
  t=  800/ 2000  best=3.767e-12  dt=0.33s
  t= 1000/ 2000  best=3.767e-12  dt=0.41s
  t= 1200/ 2000  best=3.767e-12  dt=0.50s
  t= 1400/ 2000  best=3.767e-12  dt=0.58s
  t= 1600/ 2000  best=3.767e-12  dt=0.66s
  t= 1800/ 2000  best=3.767e-12  dt=0.74s
  t= 2000/ 2000  best=3.767e-12  dt=0.82s
  done radius=0.1 -> best=3.767e-12 (dt=0.82s)
[k=4/4] rand probe: radius=0.03, NRand=2000
  t=  200/ 2000  best=3.767e-12  dt=0.08s
  t=  400/ 2000  best=3.767e-12  dt=0.16s
  t=  600/ 2000  best=3.767e-12  dt=0.24s
  t=  800/ 2000  best=3.767e-12  dt=0.33s
  t= 1000/ 2000  best=3.767e-12  dt=0.41s
  t= 1200/ 2000  best=3.767e-12  dt=0.50s
  t= 1400/ 2000  best=3.767e-12  dt=0.57s
  t= 1600/ 2000  best=3.767e-12  dt=0.65s
  t= 1800/ 2000  best=3.767e-12  dt=0.73s
  t= 2000/ 2000  best=3.767e-12  dt=0.82s
  done radius=0.03 -> best=3.767e-12 (dt=0.82s)
[k=4/4] rand probe: radius=0.01, NRand=2000
  t=  200/ 2000  best=3.767e-12  dt=0.08s
  t=  400/ 2000  best=3.767e-12  dt=0.15s
  t=  600/ 2000  best=3.767e-12  dt=0.23s
  t=  800/ 2000  best=3.767e-12  dt=0.32s
  t= 1000/ 2000  best=3.767e-12  dt=0.40s
  t= 1200/ 2000  best=3.767e-12  dt=0.48s
  t= 1400/ 2000  best=3.767e-12  dt=0.56s
  t= 1600/ 2000  best=3.767e-12  dt=0.65s
  t= 1800/ 2000  best=3.767e-12  dt=0.72s
  t= 2000/ 2000  best=3.767e-12  dt=0.81s
  done radius=0.01 -> best=3.767e-12 (dt=0.81s)
[k=4/4] rand probe: radius=0.003, NRand=2000
  t=  200/ 2000  best=3.767e-12  dt=0.08s
  t=  400/ 2000  best=3.767e-12  dt=0.16s
  t=  600/ 2000  best=3.767e-12  dt=0.24s
  t=  800/ 2000  best=3.767e-12  dt=0.33s
  t= 1000/ 2000  best=3.767e-12  dt=0.40s
  t= 1200/ 2000  best=3.767e-12  dt=0.49s
  t= 1400/ 2000  best=3.767e-12  dt=0.56s
  t= 1600/ 2000  best=3.767e-12  dt=0.65s
  t= 1800/ 2000  best=3.767e-12  dt=0.73s
  t= 2000/ 2000  best=3.767e-12  dt=0.81s
  done radius=0.003 -> best=3.767e-12 (dt=0.81s)
```

```
[k=4/4] rand probe: radius=0.001, NRand=2000
  t=  200/ 2000  best=3.767e-12  dt=0.08s
  t=  400/ 2000  best=3.767e-12  dt=0.16s
  t=  600/ 2000  best=3.767e-12  dt=0.24s
  t=  800/ 2000  best=3.767e-12  dt=0.32s
  t= 1000/ 2000  best=3.767e-12  dt=0.40s
  t= 1200/ 2000  best=3.767e-12  dt=0.49s
  t= 1400/ 2000  best=3.767e-12  dt=0.56s
  t= 1600/ 2000  best=3.767e-12  dt=0.65s
  t= 1800/ 2000  best=3.767e-12  dt=0.73s
  t= 2000/ 2000  best=3.767e-12  dt=0.81s
  done radius=0.001 -> best=3.767e-12 (dt=0.81s)
[k=4/4] rand probe: radius=0.0003, NRand=2000
  t=  200/ 2000  best=3.767e-12  dt=0.08s
  t=  400/ 2000  best=3.767e-12  dt=0.16s
  t=  600/ 2000  best=3.767e-12  dt=0.24s
  t=  800/ 2000  best=3.767e-12  dt=0.33s
  t= 1000/ 2000  best=3.767e-12  dt=0.41s
  t= 1200/ 2000  best=3.767e-12  dt=0.49s
  t= 1400/ 2000  best=3.767e-12  dt=0.57s
  t= 1600/ 2000  best=3.767e-12  dt=0.66s
  t= 1800/ 2000  best=3.767e-12  dt=0.74s
  t= 2000/ 2000  best=3.767e-12  dt=0.82s
  done radius=0.0003 -> best=3.767e-12 (dt=0.82s)
[k=4/4] rand probe: radius=0.0001, NRand=2000
  t=  200/ 2000  best=3.767e-12  dt=0.08s
  t=  400/ 2000  best=3.767e-12  dt=0.17s
  t=  600/ 2000  best=3.767e-12  dt=0.25s
  t=  800/ 2000  best=3.767e-12  dt=0.33s
  t= 1000/ 2000  best=3.767e-12  dt=0.41s
  t= 1200/ 2000  best=3.767e-12  dt=0.49s
  t= 1400/ 2000  best=3.767e-12  dt=0.57s
  t= 1600/ 2000  best=3.767e-12  dt=0.66s
  t= 1800/ 2000  best=3.767e-12  dt=0.73s
  t= 2000/ 2000  best=3.767e-12  dt=0.82s
  done radius=0.0001 -> best=3.767e-12 (dt=0.82s)
[k=4] fminsearch starting from r_min=3.767e-12 ...

 Iteration   Func-count         f(x)         Procedure
     0            1      3.76692e-12
     1            5      3.76692e-12         initial simplex
     2            7      3.76692e-12         contract outside
     3            9      3.76692e-12         contract inside
     4           10      3.76692e-12         reflect
     5           11      3.76692e-12         reflect
     6           12      3.76692e-12         reflect
     7           14      3.76692e-12         contract inside
     8           16      3.76692e-12         contract inside
     9           18      3.76692e-12         contract outside
    10           20      3.76692e-12         contract inside
    11           22      3.76692e-12         contract outside
    12           24      3.76692e-12         contract outside
    13           26      3.76692e-12         contract inside
    14           27      3.76692e-12         reflect
    15           29      3.76692e-12         contract inside
    16           31      3.76692e-12         contract outside
    17           33      3.76692e-12         contract outside
    18           35      3.76692e-12         contract inside
    19           37      3.76692e-12         contract inside
    20           39      3.76692e-12         contract inside
    21           41      3.76692e-12         contract inside
    22           43      3.76692e-12         contract inside
    23           45      3.76692e-12         contract inside
    24           47      3.76692e-12         contract inside
```

25	49	3.76692e-12	contract inside
26	51	3.76692e-12	contract outside
27	52	3.76692e-12	reflect
28	54	3.76692e-12	contract inside
29	56	3.76692e-12	contract inside
30	58	3.76692e-12	contract inside
31	60	3.76692e-12	contract inside
32	62	3.76692e-12	contract inside
33	64	3.76692e-12	contract inside
34	66	3.76692e-12	contract outside
35	68	3.76692e-12	contract inside
36	70	3.76692e-12	contract inside
37	72	3.76692e-12	contract outside
38	74	3.76692e-12	contract inside
39	76	3.76692e-12	contract inside
40	77	3.76692e-12	reflect
41	79	3.76692e-12	contract inside
42	81	3.76692e-12	contract inside
43	83	3.76692e-12	contract inside
44	85	3.76692e-12	contract inside
45	87	3.76692e-12	contract inside
46	89	3.76692e-12	contract inside
47	91	3.76692e-12	contract outside
48	93	3.76692e-12	contract inside
49	95	3.76692e-12	contract outside
50	96	3.76692e-12	reflect
51	98	3.76692e-12	contract inside
52	100	3.76692e-12	contract inside
53	102	3.76692e-12	contract inside
54	104	3.76692e-12	contract outside
55	106	3.76692e-12	contract inside
56	108	3.76692e-12	contract inside
57	109	3.76692e-12	reflect
58	111	3.76692e-12	contract inside
59	113	3.76692e-12	contract inside
60	114	3.76692e-12	reflect
61	116	3.76692e-12	contract inside
62	118	3.76692e-12	contract outside
63	120	3.76692e-12	contract outside
64	122	3.76692e-12	contract inside
65	124	3.76692e-12	contract inside
66	126	3.76692e-12	contract inside
67	128	3.76692e-12	contract outside
68	130	3.76692e-12	contract inside
69	132	3.76692e-12	contract inside
70	134	3.76692e-12	contract inside
71	136	3.76692e-12	contract outside
72	138	3.76692e-12	contract inside
73	140	3.76692e-12	contract inside
74	142	3.76692e-12	contract inside
75	144	3.76692e-12	contract outside
76	146	3.76692e-12	contract inside
77	148	3.76692e-12	contract inside
78	150	3.76692e-12	contract inside
79	152	3.76692e-12	contract outside
80	154	3.76692e-12	contract inside
81	156	3.76692e-12	contract inside
82	158	3.76692e-12	contract inside
83	160	3.76692e-12	contract outside
84	162	3.76692e-12	contract outside
85	164	3.76692e-12	contract inside
86	166	3.76692e-12	contract inside
87	168	3.76692e-12	contract inside
88	170	3.76692e-12	contract inside

89	172	3.76692e-12	contract inside
90	174	3.76692e-12	contract inside
91	176	3.76692e-12	contract inside
92	178	3.76692e-12	contract inside
93	180	3.76692e-12	contract inside
94	182	3.76692e-12	contract inside
95	184	3.76692e-12	contract inside
96	186	3.76692e-12	contract inside
97	188	3.76692e-12	contract outside
98	189	3.76692e-12	reflect
99	191	3.76692e-12	contract inside
100	193	3.76692e-12	contract inside
101	199	3.76692e-12	shrink
102	201	3.76692e-12	contract outside
103	203	3.76692e-12	contract outside
104	205	3.76692e-12	contract inside
105	207	3.76692e-12	contract inside
106	209	3.76692e-12	contract inside
107	211	3.76692e-12	contract outside
108	212	3.76692e-12	reflect
109	214	3.76692e-12	contract inside
110	216	3.76692e-12	contract outside
111	218	3.76692e-12	contract inside
112	220	3.76692e-12	contract outside
113	222	3.76692e-12	contract inside
114	224	3.76692e-12	contract outside
115	225	3.76692e-12	reflect
116	227	3.76692e-12	contract inside
117	229	3.76692e-12	contract inside
118	230	3.76692e-12	reflect
119	232	3.76692e-12	contract inside
120	234	3.76692e-12	contract outside
121	236	2.52777e-12	contract inside
122	238	2.52777e-12	contract inside
123	240	2.52777e-12	contract outside
124	242	2.39765e-12	contract inside
125	244	1.33557e-12	contract inside
126	246	1.33557e-12	contract outside
127	248	1.1612e-12	contract inside
128	250	1.09099e-12	contract inside
129	252	6.92566e-13	contract outside
130	254	6.49651e-13	contract inside
131	256	3.64665e-13	contract inside
132	257	3.64665e-13	reflect
133	259	3.64665e-13	contract inside
134	261	1.77102e-13	contract inside
135	263	1.77102e-13	contract inside
136	264	1.77102e-13	reflect
137	266	1.77102e-13	contract inside
138	268	1.72467e-13	contract inside
139	270	1.72467e-13	contract inside
140	272	1.5983e-13	contract inside
141	274	6.54049e-14	contract inside
142	276	6.54049e-14	contract inside
143	278	6.54049e-14	contract inside
144	280	6.54049e-14	contract inside
145	282	6.54049e-14	contract inside
146	283	6.54049e-14	reflect
147	285	2.64575e-14	contract inside
148	287	2.64575e-14	contract inside
149	289	2.64575e-14	contract outside
150	290	2.64575e-14	reflect
151	292	2.32462e-14	contract inside
152	294	2.10865e-14	contract inside

153	296	1.44621e-14	contract inside
154	298	1.44621e-14	contract inside
155	300	1.15599e-14	contract inside
156	302	7.22256e-15	contract inside
157	304	7.22256e-15	contract outside
158	306	4.37682e-15	contract inside
159	308	4.37682e-15	contract outside
160	310	3.89828e-15	contract inside
161	312	2.60486e-15	contract inside
162	314	2.10951e-15	contract inside
163	315	2.10951e-15	reflect
164	317	5.65456e-16	contract inside
165	319	5.65456e-16	contract inside
166	320	5.65456e-16	reflect
167	322	2.61128e-16	contract inside
168	324	2.61128e-16	contract inside
169	326	2.61128e-16	contract inside
170	327	2.61128e-16	reflect
171	329	1.40004e-16	contract inside
172	331	1.40004e-16	contract inside
173	332	1.40004e-16	reflect
174	334	9.9982e-17	contract inside
175	340	9.4032e-17	shrink
176	342	9.4032e-17	contract inside
177	344	9.4032e-17	contract outside
178	350	9.4032e-17	shrink
179	352	9.28756e-17	contract inside
180	353	9.28756e-17	reflect
181	355	9.28756e-17	contract inside
182	357	9.28756e-17	contract inside
183	359	7.87333e-17	reflect
184	365	4.34872e-17	shrink
185	366	4.34872e-17	reflect
186	372	9.24455e-18	shrink
187	373	9.24455e-18	reflect
188	374	9.24455e-18	reflect
189	375	9.24455e-18	reflect
190	376	9.24455e-18	reflect
191	382	9.24455e-18	shrink
192	383	9.24455e-18	reflect
193	384	9.24455e-18	reflect
194	385	9.24455e-18	reflect
195	391	9.24455e-18	shrink
196	393	9.24455e-18	contract outside
197	394	9.24455e-18	reflect
198	395	9.24455e-18	reflect
199	397	9.24455e-18	contract inside
200	399	9.24455e-18	contract inside
201	400	9.24455e-18	reflect
202	401	9.24455e-18	reflect
203	402	9.24455e-18	reflect
204	404	9.24455e-18	contract outside
205	410	9.24455e-18	shrink
206	411	9.24455e-18	reflect
207	413	9.24455e-18	contract inside
208	419	9.24455e-18	shrink
209	420	9.24455e-18	reflect
210	421	9.24455e-18	reflect
211	427	9.24455e-18	shrink
212	428	9.24455e-18	reflect
213	430	9.24455e-18	contract inside
214	436	9.24455e-18	shrink
215	438	9.24455e-18	contract inside
216	444	9.24455e-18	shrink

```
   217          450      9.24455e-18         shrink
   218          451      9.24455e-18         reflect
   219          457      9.24455e-18         shrink
   220          459      9.24455e-18         contract outside
   221          460      9.24455e-18         reflect
   222          466      9.24455e-18         shrink
   223          467      9.24455e-18         reflect
   224          468      9.24455e-18         reflect
   225          474      9.24455e-18         shrink
   226          475      9.24455e-18         reflect
   227          476      9.24455e-18         reflect
   228          477      9.24455e-18         reflect
   229          478      9.24455e-18         reflect
   230          484      9.24455e-18         shrink
   231          485      9.24455e-18         reflect
   232          487      4.54138e-18         reflect
   233          488      4.54138e-18         reflect
   234          494      4.54138e-18         shrink
   235          495      4.54138e-18         reflect
   236          501      4.54138e-18         shrink
   237          503      4.54138e-18         contract outside
   238          505      4.54138e-18         contract inside
   239          507      4.54138e-18         contract inside
   240          509      4.54138e-18         contract inside
   241          515      4.54138e-18         shrink
   242          517      4.54138e-18         contract outside
   243          519      4.54138e-18         contract inside
   244          521      4.54138e-18         contract inside

Optimization terminated:
 the current x satisfies the termination criteria using OPTIONS.TolX of 1.000000e-16
 and F(X) satisfies the convergence criteria using OPTIONS.TolFun of 1.000000e-16

[k=4] after fminsearch: r_min=4.541e-18 (exitflag=1)
[k=4] DONE: r_min 3.767e-12 -> 4.541e-18   (dt=7.76s)

=== leigqNEWTON_refine_batch: summary ===
k=1  r_min=6.736e-18  respair=3.326e-15  resInf(polish)=8.204e-17
k=2  r_min=1.188e-17  respair=2.794e-15  resInf(polish)=5.586e-16
k=3  r_min=7.289e-17  respair=3.001e-15  resInf(polish)=6.668e-16
k=4  r_min=7.960e-16  respair=2.895e-15  resInf(polish)=9.366e-16
min pairwise distance = 9.272e-01 between (3,2)
```

```
median(cert2.resMin), max(cert2.resMin)
```

```
ans =
4.2385e-17
ans =
7.9596e-16
```

See also

leigqNEWTON, leigqNEWTON_refine_polish, leigqNEWTON_cert_resMin, checkNEWTON