Math 206 Optional Homework Problems

(1) Section 2.1—Set Operations

(a) For $A = \{1, 3, 5, 9\}$, $B = \{2, 3, 6, 9\}$, $C = \{2, 3, 5, 8\}$ and universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$:

(i) Evaluate $A \cup B$.
(ii) Evaluate $C \cap B$.
(iii) Evaluate $A - C$.
(iv) Evaluate $\overline{B}$.
(v) Evaluate $(A \cup B) \cap (A - C)$.
(vi) Draw a Venn diagram that places each number appropriately.

(b) Suppose $A = \{1, 4, 7, 8, 10\}$, $B = \{2, 5, 8, 10, 11\}$, and $C = \{1, 2, 5, 6, 8, 9, 12\}$ and the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

(i) Evaluate $A \cup C$.
(ii) Evaluate $A \cap B$.
(iii) Evaluate $B - C$.
(iv) Evaluate $\overline{A \cup B}$.
(v) Determine $A \cap (B \cup C)$.
(vi) Draw a Venn diagram that places each number appropriately.

(2) Sections 2.2 & 2.3—Equivalence & Partial Ordering Relations

(a) Suppose $R = \{(5, 5), (5, 4), (5, 3), (4, 5), (4, 4), (4, 3), (3, 5), (3, 4), (3, 3), (2, 2)\}$ and $S = \{2, 3, 4, 5\}$.

(i) Is the reflexive property satisfied by relation R on set S? Justify your conclusion.
(ii) Is the symmetric property satisfied by relation R on set S? Justify your conclusion.
(iii) Is the transitive property satisfied by relation R on set S? Justify your conclusion.
(iv) Is relation R on set S an equivalence relation? Why or why not?
(v) Describe the equivalence class containing 3 in S.

(b) Let $S = \{1, 2, 3, 4\}$, and define $x \; R \; y$ to mean that $x \geq y$.

(i) Is R on S reflexive? Explain.
(ii) Is R on S symmetric? Explain.
(iii) Is R on S antisymmetric? Explain.
(iv) Is R on S transitive? Explain.

(v) Is R on S an equivalence relation, a partial order, both, or neither? Justify your choice.

(3) Section 2.4—Functions

(a) Consider $X = (-\infty, \infty)$, $Y = (-\infty, \infty)$, $Z = [0, \infty)$, $f : X \to Y$ such that $f(x) = x - 1$, and $g : Y \to Z$ such that $g(x) = x^2 - 1$.

(i) Is f one-to-one? Why or why not?

(ii) Is f onto? Why or why not?

(iii) Is f invertible? If so, find its inverse. If not, explain why it is not invertible.

(iv) What is $(g \circ f)(x)$?

(b) Let $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7, 8, 9\}$, and $C = \{2, 4, 6, 8, 10\}$.
Let $f = \{(1, 7), (2, 8), (3, 5)\}$, $g = \{(5, 4), (6, 6), (7, 2), (8, 8), (9, 10)\}$, and
$h = \{(2, 9), (4, 8), (6, 6), (8, 7), (10, 5)\}$. Determine whether each of the following statements are true or false and explain.

(i) f is a one-to-one function from A to B.

(ii) g is an onto function from B to C.

(iii) h is a one-to-one function from C to B.

(iv) $h \circ g(7) = 9$

(v) $g \circ h(8) = 8$

(vi) $h \circ g$ is onto.

(4) Section 9.1 & 9.2—Recurrence Relations & The Method of Iteration

(a) Suppose the recurrence relation is $s_n = 2s_{n-1}$ with initial condition $s_1 = 3$.

(i) Determine s_6.

(ii) Use the method of iteration to find a formula expressing s_n as a function of n.

(b) Suppose $s_0, s_1, s_2, ...$ is a sequence satisfying $s_n = 2s_{n-1} + 1$ for $n \geq 1$ and $s_0 = 4$.

(i) Determine s_5.

(ii) Use the method of iteration to find a formula expressing s_n as a function of n.

(5) Section 2.5—Mathematical Induction

(a) The recurrence relation $s_n = 2s_{n-1} - 3$ for $n \geq 1$ with initial condition $s_0 = 7$ could have its nth term characterized by $s(n) = 4(2)^n + 3$. Describe the steps you need to prove this characterization using mathematical induction. Do not prove!

(b) Describe the steps you need to prove that for all positive integers n, $1+2+...+n = \dfrac{n(n+1)}{2}$ using math induction. Do not prove!

(6) Section 3.1—Congruence

(a) Perform the indicated computations in the given $\mathbb{Z}_m$. Write your answer in the form $[r]$ with $0 \leq r < m$.
   (i) $[29] + [32]$ in $\mathbb{Z}_7$
   (ii) $[784] * [289]$ in $\mathbb{Z}_9$
   (iii) $[37]^{159}$ in $\mathbb{Z}_{36}$

(b) Perform the following calculations in $\mathbb{Z}_{27}$. Write your answer in the form $[r]$ with $0 \leq r < 27$.
   (a) $[32] + [26]$
   (b) $[-45][98]$
   (c) $[26]^{351}$

(7) Section 3.2—The Euclidean Algorithm

(a) (i) Use the Euclidean Algorithm to find the greatest common divisor of 776 and 248.
   (ii) Use the Extended Euclidean Algorithm to find x and y such that $776x+248y = 16$.

(b) (a) Find the greatest common divisor of $m = 315$ and $n = 225$ and integers x and y such that $315x + 225y =$gcd(315, 225) using the Extended Euclidean Algorithm.
   (b) Find, if possible, integers x and y such that $315x + 225y = 990$.
   (c) Find, if possible, integers x and y such that $315x + 225y = 690$.

(8) Section 3.3—The RSA Method

(a) Suppose $n = 187$, $P = 13$, $E = 73$.
   (i) Translate P to ciphertext C.
   (ii) Find b corresponding to n, where b and n are as in the RSA method.
   (iii) Use the extended Euclidean algorithm to find the value of D corresponding to the constants above.

(b) Suppose $P = 19$, $E = 41$, $n = 91$.
   (i) Translate P to ciphertext C.
   (ii) Find b corresponding to n, where b and n are as in the RSA method.
   (iii) Use the extended Euclidean algorithm to find the value of D corresponding to the constants above.

(9) Section 3.4—Error-Detecting and Error-Correcting Codes

(a) (i) Suppose the probability of error in transmission of a single digit is .01. What is the probability of having exactly 1 error in a message of length 9?

(ii) What is the Hamming distance between the codewords 01000111 and 01010101?

(iii) Suppose the minimal Hamming distance between codewords in a certain block code is 4. What is the maximum number of errors that can be detected and what is the maximum number of errors that can be corrected?

(b) (i) Suppose the probability of error in transmission of a single digit is .01. What is the probability of having exactly 0 errors in a message of length 9?

(ii) What is the Hamming distance between the codewords 00000000 and 11111111?

(iii) Suppose the minimal Hamming distance between codewords in a certain block code is 16. What is the maximum number of errors that can be detected and what is the maximum number of errors that can be corrected?

(10) Section 8.1—Pascal's Triangle and the Binomial Theorem

(a) Use the binomial theorem to determine the coefficient of x^6y^5 in the expansion of $(x-3y)^{11}$.

(b) Use the binomial theorem to determine the coefficient of x^8y^5 in the expansion of $(x-2y)^{13}$.

(11) Section 8.2—Three Fundamental Principles

(a) Suppose we want to use the digits 0-9 to make four-digit positive numbers.

(i) How many different four-digit positive integers can be made if digits can be repeated?

(ii) How many different four-digit positive integers can be made if digits cannot be repeated?

(b) Suppose that a license plate must contain a sequence of two letters followed by 5 digits or 3 letters followed by 3 digits.

(i) How many different license plates can be made if repetition is allowed?

(ii) How many different license plates can be made if repetition is not allowed?

(12) Section 8.3—Permutations and Combinations

(a) Eight people are running for three at-large seats on a school board. In how many different ways can the three seats be filled?

(b) Eight people are running for all three positions of President, Vice-President, and Treasurer seats on a school board. In how many different ways can the three seats be filled?

(13) Section 8.4—Arrangements and Selections with Repetitions

(a) How many ways can 15 distinct books be distributed so that Alice receives 7, Bob receives 5, and Charlie receives 3?

(b) How many different three digit numbers can be formed using the digits 1,2,3,4,5 with repetition?

(c) How many different 8-digit numbers can be formed using the digits in the number 42,644,266?

(14) Section 4.1—Graphs and their Representations

(a) (i) Draw a multigraph with exactly 5 vertices, exactly 8 edges, at least one loop, and at least one pair of parallel edges. Identify the loop(s) and pair(s) of parallel edges.

(ii) Create an adjacency list for the graph in (i)

(iii) Draw a multigraph with exactly 5 vertices and exactly 8 edges that is *not* isomorphic to the graph you made in (i). Explain why it is not isomorphic.

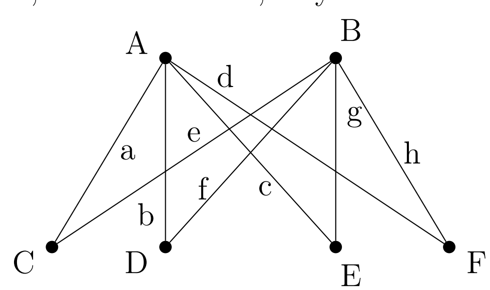
(b) (i) Draw a multigraph with exactly 4 vertices, exactly 7 edges, at least one loop, and at least one pair of parallel edges. Identify the loop(s) and pair(s) of parallel edges.

(ii) Create an adjacency list for the graph in (i)

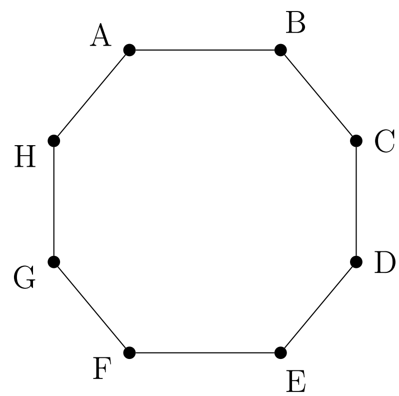
(iii) Draw a multigraph with exactly 4 vertices and exactly 7 edges that is *not* isomorphic to the graph you made in (i). Explain why it is not isomorphic.

(15) Section 4.2—Paths and Circuits

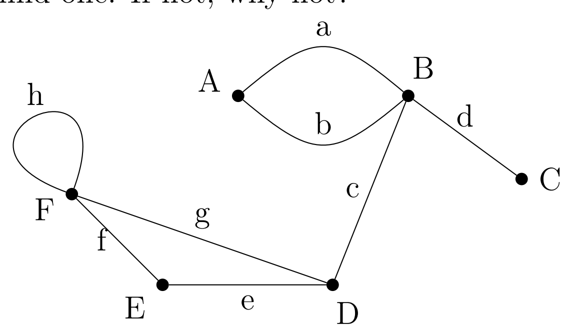
(a) (i) Consider the graph shown here. Does it permit an Euler path or circuit? If so, find one. If not, why not?



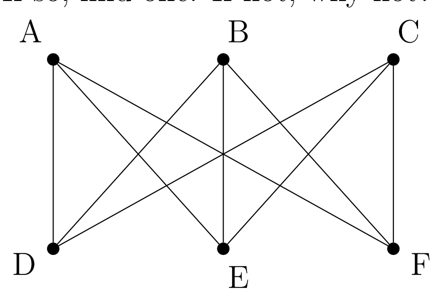
(ii) Consider the graph shown here. Does it permit a Hamiltonian path or cycle? If so, find one. If not, why not?



(b) (i) Consider the multigraph shown here. Does it permit an Euler path or circuit? If so, find one. If not, why not?

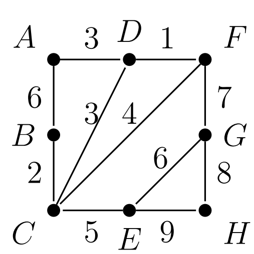


(ii) Consider the graph shown here. Does it permit a Hamiltonian path or cycle? If so, find one. If not, why not?



(16) Section 4.3—Shortest Paths and Distance

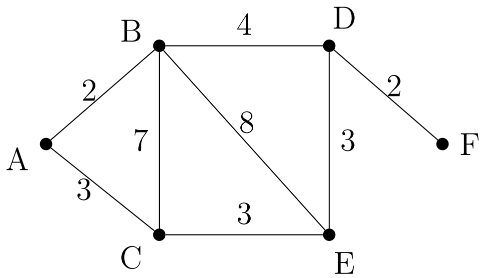
(a) Consider this graph:



i. Use Dijkstra's algorithm to determine the distance from A to all other vertices in the weighted graph.

ii. Find a shortest path from A to H.

(b) Consider this graph:

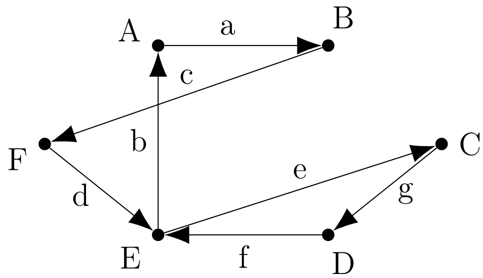


i. Use Dijkstra's algorithm to determine the distance from B to all other vertices in the weighted graph.

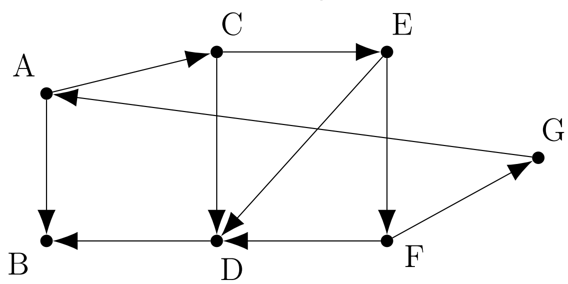
ii. Find a shortest path from B to C.

(17) Section 4.5—Directed Graphs and Multigraphs

(a) (i) Consider the directed graph shown here. Does it permit a directed Euler path or circuit? If so, find one. If not, why not?

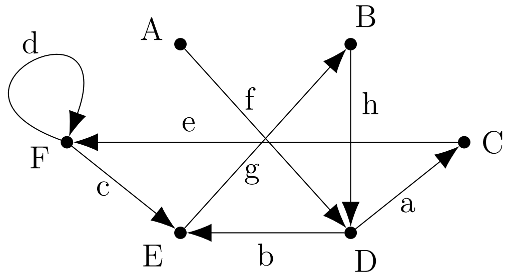


(ii) Use the Breadth-First Search Algorithm to determine the shortest path from A to G in the directed graph.

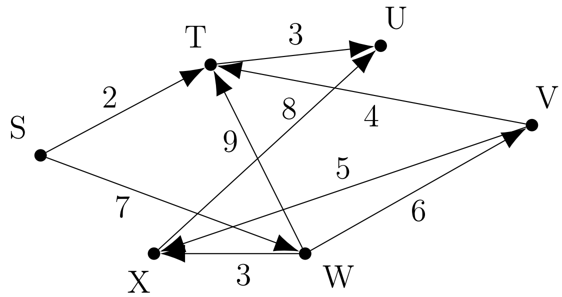


(iii) Find the distance from A to G in the directed graph given in (ii).

(b) (i) Consider the directed graph shown here. Does it permit a directed Euler path or circuit? If so, find one. If not, why not?



(ii) Use Dijkstra's algorithm to determine the distance from S to all other vertices in the directed weighted graph below.



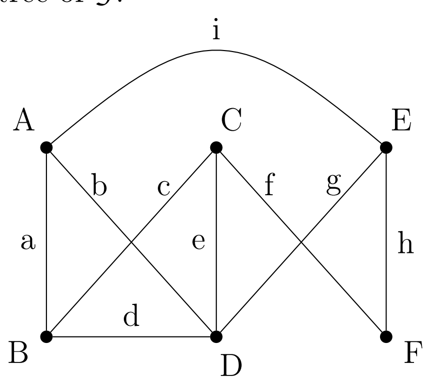
(18) Section 5.1—Properties of Trees

(a) Create 2 trees and 2 graphs that are not trees.

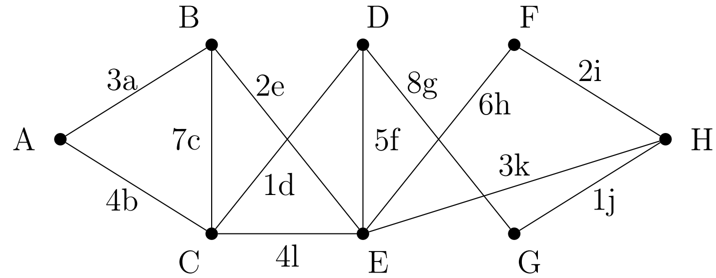
(b) Hydrogen has one free electron (so it can form 1 bond) and carbon has 4 free electrons (so it can form 4 bonds). Propane is a saturated hydrocarbon (meaning it has single bonds between atoms and has the maximal number of hydrogens for each carbon atom) with 3 carbon atoms and 8 hydrogen atoms (C_3H_8). Draw a tree representing the chemical structure of propane.

(19) Section 5.2—Spanning Trees

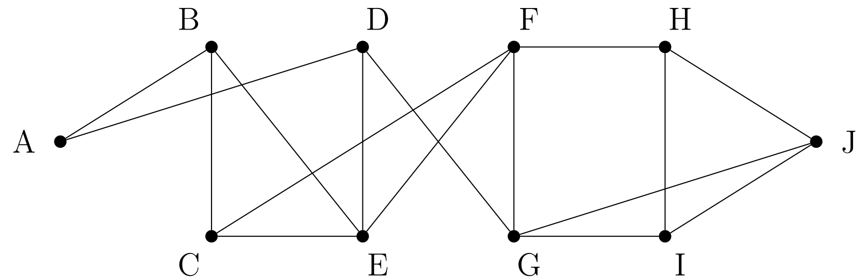
(a) (i) Consider the graph below. Remove edges as appropriate to make a spanning tree of $\mathcal{G}$.



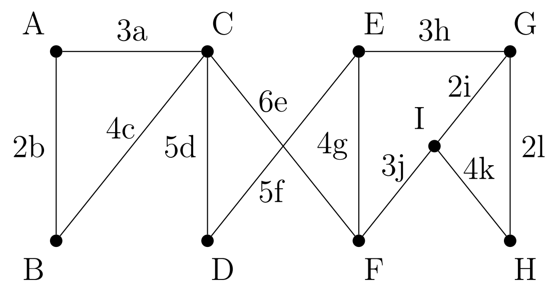
(ii) Use Prim's algorithm to find a minimal spanning tree and maximal spanning tree for the graph below. Give the weight of each.



(b) (i) Use the Breadth-First Search Algorithm to help you create a spanning tree for the graph below.

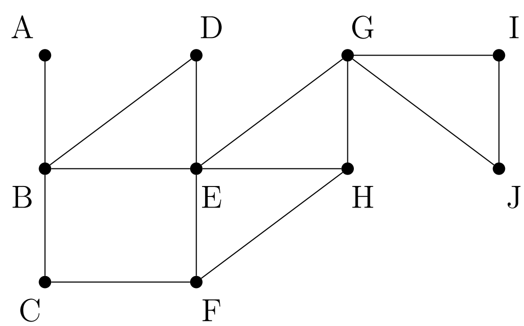


(ii) Use Prim's algorithm to find a minimal spanning tree and maximal spanning tree for the graph below. Give the weight of each.



(20) Section 5.3—Depth-First Search

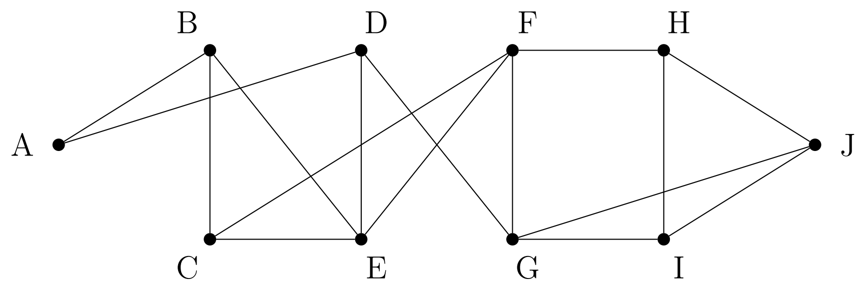
(a) Consider the graph:



(i) Strating from A, apply the depth-first search algorithm to obtain a depth-first search numbering of the vertices and use that numbering to create a spanning tree.

(ii) List the back edges of the graph.

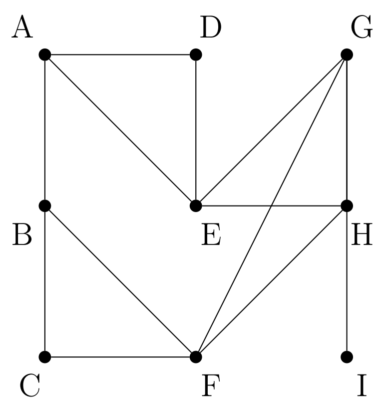
(b) Consider the graph:



(i) Strating from A, apply the depth-first search algorithm to obtain a depth-first search numbering of the vertices and use that numbering to create a spanning tree.

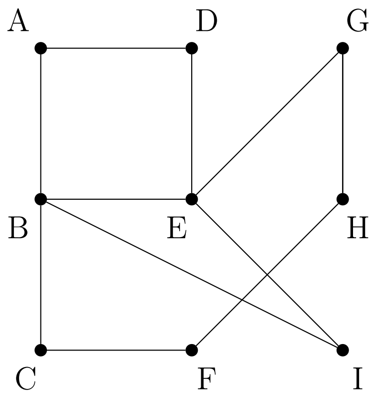
(ii) List the back edges of the graph.

(c) Consider the graph:

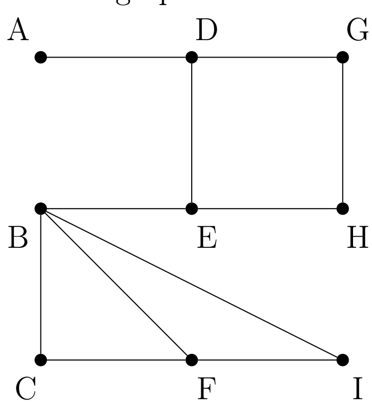


Strating from A, apply the (i) breadth-first search algorithm and (ii) depth-first search algorithm to create a spanning tree.

(d) i. Are there any bridges in the graph below? If so, where? If not, apply the depth-first search algorithm to obtain a depth first search numbering of the vertices to assign directions to edges that will make the graph a strongly connected directed graph.



ii. Are there any bridges in the graph below? If so, where? If not, apply the depth-first search algorithm to obtain a depth first search numbering of the vertices to assign directions to edges that will make the graph a strongly connected directed graph.



Submission of solutions to these homework problems is entirely voluntary. However, submission by December 10, 2023, will attract bonus points that may be added to the worst test score.