1 Characteristics of Functions

Definition:

For sets X and Y, a function from X to Y is a relation from X to Y with the property that, for each element x in X, there is *exactly one* element y in Y such that $x\ f\ y$. Note that because a relation from X to Y is a subset of $X \times Y$, a function is a subset S of $X \times Y$ such that for each $x \in X$, there is a unique $y \in Y$ with $(x, y) \in S$ (notation: $f : X \to Y$).

(each input from X produces exactly 1 output in Y)

Example: Suppose $X = \{1, 3, 5, 7\}$ and $Y = \{2, 4, 6, 8\}$.

(a) Is $f = \{(1, 2), (3, 4), (5, 6), (7, 8)\}$ a function from X to Y? Why or why not?

all elements of the domain (input set X) are used as inputs once (everywhere defined & well-defined) so function

(b) Is $g = \{(1, 2), (3, 2), (5, 2), (7, 2)\}$ a function from X to Y? Why or why not?

all elements of domain used (everywhere defined) & used once (well-defined) so function

(c) Is $h = \{(1, 2), (1, 4), (1, 6), (1, 8)\}$ a function from X to Y? Why or why not?

input 1 is associated with different outputs (2,4,6,8) so not well-defined & 3, 5, 7 are not used so not everywhere defined & not a function

(d) Is $j = \{(1, 2), (3, 4), (5, 6)\}$ a function from X to Y? Why or why not?

x=7 is not used ⇒ not everywhere defined ⇒ not a function (each used input associated w/ exactly 1 output so is well-defined)

Definition:

For function $f : X \to Y$, the domain is the set of all possible inputs, here X.

The unique element of Y such that $x\ f\ y$ is called the image of x under f (notation: $f(x)$).

The range is the set of all images under function f. (everything hit by function)

Codomain: The set Y, which contains the range of function f.

Example: Suppose $k : X \to Y$ ($\mathbb{R} \to \mathbb{R}$) where $X = Y = \{x : x \text{ is a real number}\}$ (: such that) is given by $k(x) = x^2$.

(a) What is the image of $x = 3$? $k(3) = 3^2 = 9$

(b) What are the domain, codomain, and range of k?

$\mathbb{R}$ (real numbers)

$\mathbb{R}$ (real numbers)

$k(x) \geq 0$ positive & 0 real numbers

Example: Suppose $m : X \to Y$, where $X = \{x \overset{\text{such that}}{:} x \geq 0\}$, $Y = \{y : y \text{ is a real number}\}$ is given by $m(x) = x + 5$.

(a) What is the image of $x = 3$? $m(3) = 3+5 = 8$

(b) What are the domain, codomain, and range of m?

$x \geq 0$ $[0, \infty)$ all real numbers $y \geq 5$

Example: Suppose $n : X \to Y$, where $X = Y = \{x : x \text{ is a real number}\}$ is given by $n(x) = x^3 - x$.

(a) What is the image of $x = 3$? $n(3) = 3^3 - 3 = 27 - 3 = 24$

(b) What are the domain, codomain, and range of n?

reals reals reals

Example: Suppose $p : X \to Y$, where $X = \{9, 10, 11, 12\}$ and $Y = \{0, 1, 2\}$ is given by $p(x)$ is the remainder when x is divided by 3.

(a) What is the image of $x = 9$? 0 $(9 \div 3 = 3 \text{ r.} 0)$

(b) What are the domain, codomain, and range of p?

$\{9, 10, 11, 12\}$ (allowed inputs) $\{0, 1, 2\}$ $\{0, 1, 2\}$

$9 \div 3 = 3 \text{ r.} 0$
$10 \div 3 = 3 \text{ r.} 1$
$11 \div 3 = 3 \text{ r.} 2$
$12 \div 3 = 4 \text{ r.} 0$

Definition:

A function $f : X \to Y$ is called one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$; that is, for each output of function f, there is precisely one input that creates it.

If the range and codomain of a function are equal, then the function is onto.

A function that is both one-to-one and onto is called a one-to-one correspondence

For any set X, the function $I_X : X \to X$ defined by $I_X(x) = x$ is a one-to-one correspondence called the identity function on X (do-nothing function)

Example: Revisit functions $\overset{k(x) = x^2 \text{ on } \mathbb{R}}{k}$-p. Which are one-to-one, which are onto, and which are one-to-one correspondences? none

$k(2) = 2^2 = 4 \sim f(x_1) = 4$ $x_1 = 2$
$k(-2) = (-2)^2 = 4 \sim f(x_2) = 4$, $x_2 = -2$
$f(x_1) = f(x_2)$ but $x_1 \neq x_2$ so not 1-1
not onto – no way to produce negatives (e.g. $k(x) \neq -1$ for any $x \in \mathbb{R}$)

m (if adjusted so domain was all reals, m would pass)

for $n(1) = 1^3 - 1 = 0$
$n(0) = 0^3 - 0 = 0$
(not 1-1)
will be onto (hit all $\mathbb{R}$)

$p(9) = p(12) = 0$
so not 1-1
but is onto

2 Composition and Inverses of Functions

Definition:

Suppose $f : X \to Y$ and $g : Y \to Z$. The Composition of g and f is defined as the image of x under $gf = g(f(x))$ for all $x \in X$.

Example: Suppose X, Y, and Z all denote the set of real numbers. Define $f : X \to Y$ by $f(x) = x^2$ and $g : Y \to Z$ by $g(y) = 3y + 2$. Find gf and fg.

$g(f(x)) = 3x^2 + 2$

$f(g(y)) = (3y+2)^2 = 9y^2 + 12y + 4$

Example: Suppose $X = \{x : x \geq 1\}$, $Y = \{y : y \geq 1\}$, and $Z = \{z : z \text{ is a real number}\}$. Define $f : X \to Y$ by $f(x) = \sqrt{x-1}$ and $g : Y \to Z$ by $g(y) = y^2 + 1$. Find gf and fg.

$g(f(x)) = (\sqrt{x-1})^2 + 1 = x - 1 + 1 = x$

$f(g(x)) = \sqrt{(\cancel{x}^2+1) - 1} = \sqrt{x^2} = x$ $\quad f(g(y)) = \sqrt{(y^2+1)-1} = \sqrt{y^2} = y$

Definition: (x, y)

Suppose $f : X \to Y$ is a one-to-one correspondence. The function with domain Y and codomain X that associates to each $y \in Y$ the unique $x \in X$ such that $y = f(x)$ is the inverse of function f (notation: f^{-1}).

1-1 & onto

(y, x)

Theorem 2.7:

Let $f : X \to Y$ be a one-to-one correspondence. Then

1-1 & onto

(a) $f^{-1} : Y \to X$ is a one-to-one correspondence.

1-1 & onto

(b) The inverse function of f^{-1} is f.

(c) For all $x \in X$, $f^{-1}f(x) = x$ and for all $y \in Y$, $ff^{-1}(y) = y$. That is, $f^{-1}f = I_X$ and $ff^{-1} = I_Y$.

x

y

when inverse functions exist, they undo each other to create the identity: a do-nothing function

Example: Which example above included inverse functions?

inverses

Example: Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$. Does f have an inverse? If so, find it. If not,why not?

no $(-1)^2 = 1$, $(1)^2 = 1$ $\Rightarrow f(x)$ is not 1-1 with $\mathbb{R}$ as domain

If we restrict f: pos. reals $+0$ → pos. reals $+0$ then $f(x) = x^2$ would be invertible with $f^{-1}(x) = \sqrt{x}$ $+0$

Example: Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^3 - 1$. Does f have an inverse? If so, find it. If not,why not?

$y = x^3 - 1$

$\boxed{\text{Swap roles of } x \,\&\, y}$

$x = y^3 - 1$

$x + 1 = y^3$

$\sqrt[3]{x+1} = y = f^{-1}$

Suppose $x_1^3 - 1 = x_2^3 - 1$

$\Rightarrow x_1^3 = x_2^3$

$\Rightarrow x_1 = x_2$

$\Rightarrow f(x)$ is 1-1

outputs all reals (range is all reals) $\Rightarrow$ onto

Example: Suppose $X = \{1, 3, 5, 7\}$ and $f : X \to X$ is given by $f = \{(1, 3), (3, 5), (5, 7), (7, 1)\}$. Does f have an inverse? If so, find it. If not,why not?

yes - is both 1-1 & onto

$f^{-1} = \{(3,1), (5,3), (7,5), (1,7)\}$

Why is paying attention to the domain and codomain of a function important for computer scientists? Computer security issues from lazy specification of inputs (e.g. Software bugs like division by 0) or outputs (allowing people who should not have access to get access to inputs/add malicious entries to databases (SQL injection attack)

gets passed along when codomain too broad & permits a chance to do bad things once in system

Example: Let $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7, 8, 9\}$, and $C = \{2, 4, 6, 8, 10\}$. Let $f = \{(1, 7), (2, 8), (3, 5)\}$, $g = \{(5, 4), (6, 6), (7, 2), (8, 8), (9, 10)\}$, and $h = \{(2, 9), (4, 8), (6, 6), (8, 7), (10, 5)\}$. Determine whether each of the following statements are true or false and explain.

(a) f is a one-to-one function from A to B.

passes 1-1 but we didn't use all inputs so not a function

(b) g is an onto function from B to C.

uses all of B (function) once (function) & hits all of C (onto)

(c) h is a one-to-one function from C to B.

uses all of C once (function) & each output hit once (1-1)

(d) $h \circ g(7) = 9$

$g(7) = 2$

$h(2) = 9$ yes

(e) $g \circ h(8) = 8$

$h(8) = 7$

$g(7) = 2 \neq 8$ no

(f) $h \circ g$ is onto.

yes