What if we want to use a minimal number of edges to be able to reach everywhere?

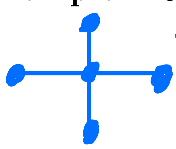
want a graph no cycles  connected

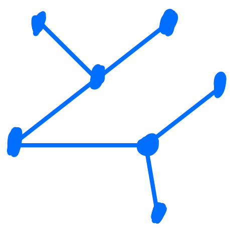
cycles cause redundancy

Definition:

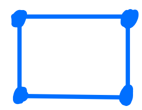
Any graph that is connected and has no cycles is called a tree.

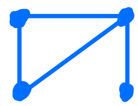
Example: Create 2 trees and 2 graphs that are not trees.

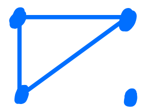
1  ← 5 vertices 4 edges

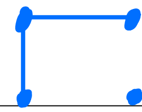
2  7 vertices 6 edge

non trees

1  connected cycle

2  connected cycle

3  disconnected cycle

4  disconnected not cycle

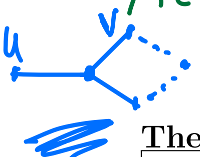
Theorem 5.1:

Let U and V be vertices in a tree. Then there is exactly one simple path from U to V.

Why should this make sense?

At least 1 way: tree is connected graph (by definition). This means we can get from each vertex to another vertex.

At most 1 way: suppose there are 2 simple paths from U to V. Then there must be a vertex where they differ and they must come back together at some point (at least at V). But this forms a cycle.



Theorem 5.2:

In a tree $\mathcal{T}$ with more than one vertex, there are at least two vertices of degree 1.

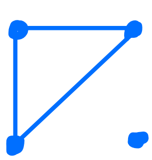
Why should this make sense?

At the beginning and ending of the longest path the degree has to be 1. Otherwise we could extend the path (or there would be cycles).

Theorem 5.3:

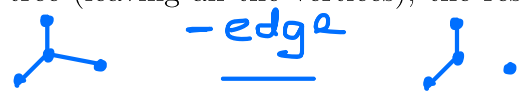
A tree with n vertices has exactly $n-1$ edges.

Example: If a graph has $n-1$ edges and n vertices, must it be a tree? Why or why not?

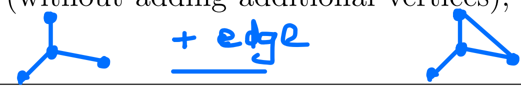
 3 edges 4 vertices

Theorem 5.4:

(a) When an edge is removed from a tree (leaving all the vertices), the resulting graph is not connected and hence is not a tree.



(b) When an edge is added to a tree (without adding additional vertices), the resulting graph has a cycle and hence is not a tree.



Why should this make sense?

By Thm 5.3, if we start with a tree, adding or subtracting an edge makes for the wrong number of edges. For connection, we need to be able to reach everything and trees are minimal (in an edge sense)

Theorem 5.5:

The following statements are equivalent for a graph $\mathcal{T}$:

$A \Longleftrightarrow B$ $\qquad A \Rightarrow B$

A iff B $\qquad B \Rightarrow A$

(a) $\mathcal{T}$ is a tree.

(b) $\mathcal{T}$ is connected, and the number of vertices is one more than the number of edges.

(c) $\mathcal{T}$ has no cycles, and the number of vertices is one more than the number of edges.

(d) There is exactly one simple path between each pair of vertices in $\mathcal{T}$.

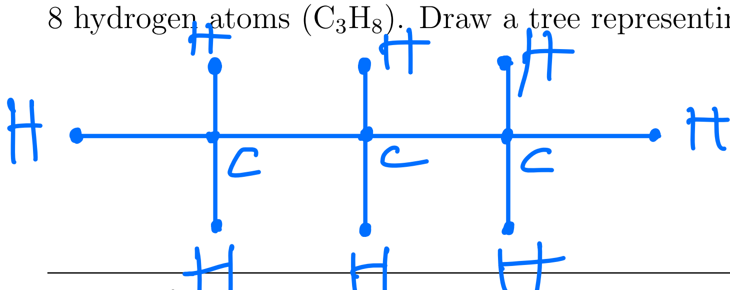
(e) $\mathcal{T}$ is connected, and the removal of any edge of $\mathcal{T}$ results in a graph that is not connected.

(f) $\mathcal{T}$ has no cycles, and the addition of any edge between two nonadjacent vertices results in a graph with a cycle.

How should I read this theorem?

All statements are iff and only if. For example, if (a) then (b); if (b) then (a); if (d) then (f) etc.

Example: Hydrogen has one free electron (so it can form 1 bond) and carbon has 4 free electrons (so it can form 4 bonds). Propane is a saturated hydrocarbon (meaning it has single bonds between atoms and has the maximal number of hydrogens for each carbon atom) with 3 carbon atoms and 8 hydrogen atoms (C_3H_8). Draw a tree representing the chemical structure of propane.



11 vertices

10 edges

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