Hasse diagrams

Definition:

Let R be a partial order on set S. An element $x \in S$ is called a **minimal element** of S with respect to R if the only element $s \in S$ satisfying $s\ R\ x$ is x itself (that is, $s\ R\ x$ implies $s = x$).

Let R be a partial order on set S. An element $z \in S$ is called a **maximal element** of S with respect to R if the only element $s \in S$ satisfying $z\ R\ s$ is z itself (that is, $z\ R\ s$ implies $z = s$).

Note: No minimal or maximal elements need to exist for a given partial order.

Examples :

1. Identify the minimal and maximal elements of S with respect to R, if they exist: S is the set of all *integers* x such that $0 \leq x < 2$ and $x\ R\ y$ if $x \leq y$.

2. Identify the minimal and maximal elements of S with respect to R, if they exist: S is the set of all *real numbers* x such that $0 \leq x < 1$, and $x\ R\ y$ if $x \leq y$.

3. Identify the minimal and maximal elements of S with respect to R, if they exist: S is the set of all real numbers x such that $0 \leq x < 1$, and $x \; R \; y$ if $x \geq y$.

4. Identify the minimal and maximal elements of S with respect to R, if they exist: S is the set of nonempty subsets of $\{1, 2, 3\}$ and $A \; R \; B$ if $A \subseteq B$.

Theorem 2.6:
Let R be a partial order on a finite set S. Then S has both a minimal and a maximal element with respect to R.

Hasse Diagrams

Description:
A **Hasse diagram** represents a partial order R on a **finite** set S by representing each element of S by a point and each pair of distinct points related by R has a line connecting them.

Each line is arranged so that the initial point is below its terminal point; that is, an arrow is drawn from the point representing x to the point representing y when $x \; R \; y$ and there is no $s \in S$ other than x and y such that $x \; R \; s$ and $s \; R \; y$.

Hasse diagrams are read from bottom to top so all line segments between points are regarded as pointing upward.

Note: segments need not be drawn with arrows. The direction is implicit in how the Hasse diagram is drawn.

$\boxed{Examples:}$

1. Where will minimal and maximal element(s) of partial order R on set S appear in a Hasse diagram?

2. Create a Hasse diagram for relation R on set S: S is the set of nonempty subsets of $\{1,2,3\}$ and $A\ R\ B$ if $A \subseteq B$.

3. Create a Hasse diagram for relation R on set S: S is the set of nonempty subsets of $\{1, 2, 3\}$ and $A\ R\ B$ if $B \subseteq A$.

4. Create a Hasse diagram for relation R on set S: let R be the partial order "divides" on the set $S = \{2, 3, 4, 6, 12\}$. What does this Hasse diagram indicate about the minimal and maximal element(s) of S?

5. Create a Hasse diagram for relation R on set S: let R be the partial order "divides" on the set $S = \{2, 3, 9, 12, 14, 36\}$. What does this Hasse diagram indicate about the minimal and maximal element(s) of S?