$C(n,n) = 1$

$\frac{2!}{1!(2-1)!} = \frac{2}{1\cdot 1}$

1 Arrangements with Repetition

The permutation formula we learned in Section 8.3 applies when there is no repetition of elements. We need to adjust our approach slightly when repetition is permitted. However, arrangements still emphasize that order matters (i.e., in the context of the problem, when changing the order makes for a distinguishable outcome).

Example: Determine the number of indistinguishable arrangements of the letters in the word "ball".

- the l's are indistinguishable
- C(4,2)
- The remaining positions for a & b can be chosen in C(2,1) & C(1,1)

$C(4,2)\cdot C(2,1)\cdot C(1,1) = 12$

$6 \cdot 2 \cdot 1$

$\frac{4!}{2!(4-2)!} = \frac{4!}{2\cdot 2!} = \frac{4\cdot 3\cdot 2}{2\cdot 2}$

ball abll
blla allb
llba llab
lbal labl
lbla lalb
blal albl

Theorem 8.7:

Let S be a collection containing n objects of k different types. (Objects of the same type are indistinguishable, and objects of different types are distinguishable.) Suppose that each object is of exactly one type and that there are n_1 objects of type 1, n_2 objects of type 2, and, in general, n_i objects of type i. Then the number of different arrangements of the objects in S is

$$C(n, n_1) * C(n - n_1, n_2) * C(n - n_1 - n_2, n_3) \cdots C(n - n_1 - n_2 - ... - n_{k-1}, n_k) = \frac{n!}{n_1! n_2! \cdots n_k!}.$$

Note: $n = n_1 + n_2 + ... + n_k$.

Why should this make sense? In each successive choice, we lost $n_1, n_2, \dots$ more options. We frame this as a multiplication because each succesive phase has the same number of options. Each phase is framed as a combination because objects within a type are indistinguishable.

Example: How many arrangements are there of the letters in the word "Mississippi"?

$C(11,4)\cdot C(11-4,4)\cdot C(11-4-4,2)\cdot C(11-4-4-2,1)$

$= C(11,4)\cdot C(7,4)\cdot C(3,2)\cdot C(1,1)$

$\boxed{\frac{11!}{4!\,4!\,2!\,1!}}$

Example: How many different 8-digit numbers can be formed using the digits in the number 42,644,266?

$\boxed{C(8,3)\cdot(8-3,3)\cdot C(8-3-3,2)} = \boxed{\frac{8!}{3!\,3!\,2!}}$

No 4 – 3
No 6 – 3
no 2 – 2

Example: How many different three digit numbers can be formed using the digits 1,2,3,4,5 with repetition?

$\underbrace{1,2,3,4,5}_{n}$

n n n r

5 5 5 $\Rightarrow 5\cdot 5\cdot 5 = 5^3$

n^r

2 Collections with Repetition

The combination formula we learned in Section 8.3 applies when there is no repetition of elements. We need to adjust our approach slightly when repetition is permitted. Collection-thinking is relevant when changing the order does not make for a disguishable outcome.

Theorem 8.8:

> If repetition is allowed, the number of selections of s elements that can be made from a set containing t distinct elements is $C(s+t-1, s)$.

Example: Suppose six people in a van are ordering food from the McDonald's drive-thru. If their sandwich choices are limited to Quarter Pounder, Big Mac, Crispy Chicken, and Spicy Chicken, how many different selections of six sandwiches are possible?

$$C(6+4-1, 6) = C(9,6) = \frac{9!}{6!\,3!}$$

Example: How many different boxes with 12 wedges of cheese can be made using wedges of Cheddar, Gouda, Parmesan, Swiss, and Colby?

$$C(12+5-1, 12) = C(16, 12) = \frac{16!}{12!\,4!}$$

Example: In how many different ways can 11 identical quarters be distributed to five people?

$$C(11+5-1, 11) = C(15, 11) = \frac{15!}{11!\,4!}$$

$$\frac{15 \cdot \cancel{14}^{7} \cdot 13 \cdot \cancel{12} \cdot \cancel{11!}}{\cancel{11!}\ \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}$$

3 Summary and Combining Skills

Summary Chart:

Repetition of items	# of arrangement (ordered list)	# of collections (unordered lists)
no repetition	$P(n,r)$	$C(n,r)$
with repetition	n^r	$C(n+r-1,r)$
When we have multiple types specified	$\frac{n!}{n_1!\,n_2!\cdots n_k!}$	if k distinguishable types.

Example: How many ways can 15 distinct books be distributed so that Alice receives 7, Bob receives 5, and Charlie receives 3?

Thm 8.7

$$\frac{15!}{7!5!3!}$$

$$C(15,7)\cdot C(15-7,5)\cdot C(15-7-5,3)$$

Example: How many numbers greater than 50,000,000 can be formed by rearranging the digits of the number 12,562,651?

5 7 digits

6 7 digits

1 – 2x
2 – 2x
5 – 2x
6 – 2x

$$\frac{7!}{2!2!2!1!}+\frac{7!}{2!2!2!1!}$$

Example: In how many sequences can we list 4 novels followed by 6 biographies if there are 8 novels and 10 biographies from which to choose?

$$P(8,4)\cdot P(10,6)$$

N N N N B B B B B B
8 7 6 5 10 9 8 7 6 5

Example: Suppose that 5 freshmen, 3 sophomores, 6 juniors, and 4 seniors have been nominated to serve on a student advisory committee. How many different committees can be formed if the committee is to consist of three persons from different classes?

$$\underbrace{3\cdot 6\cdot 4}_{\text{no Fr}}+\underbrace{5\cdot 6\cdot 4}_{\text{no So}}+\underbrace{5\cdot 3\cdot 4}_{\text{no Jr}}+\underbrace{5\cdot 3\cdot 6}_{\text{no Sr}}$$

Example: A domino contains two indistinguishable squares, each of which is marked with 0,1,2,3,4,5, or 6 dots. How many distinct dominoes are possible?

$C(2+7-1,2) = C(8,2) = \frac{8!}{2!6!}$

Example: A bag with sugar and sugar-substitute packets contains 13 packets of sugar, 16 packets of Splenda, 9 packets of Sweet 'n Low, 25 packets of Equal, and 5 packets of Truvia. If you can't see into the bag, how many packets must you grab from the bag to guarantee grabbing at least 3 of one type? Pigeonhole principle —

at worst grab 2 of each (total of 10), then the 11th must be the third of something

$2 \cdot 5 + 1 = 11$

Example: A bag with sugar and sugar-substitute packets contains 13 packets of sugar, 16 packets of Splenda, 9 packets of Sweet 'n Low, 25 packets of Equal, and 5 packets of Truvia. If you can't see into the bag, how many packets must you grab from the bag to guarantee grabbing at least 3 sugars?

At worst grab everything except sugar + 3 sugars

$16 + 9 + 25 + 5 + 3$

Example: A restaurant offers extra cheese, pepperoni, sausage, canadian bacon, onion, green pepper, and tomatoes as possible toppings. (You can't order multiple copies of the same topping.) How many different 3-topping pizzas are possible?

$C(7,3) = \frac{7!}{3!(7-3)!}$