Mathematical Induction

> **The Principle of Mathematical Induction:**
> Let $S(n)$ be a statement involving the integer n. Suppose that for some fixed integer n_0,
>
> (1) $S(n_0)$ is true (that is, the statement is true if $n = n_0$) AND
>
> (2) whenever k is an integer such that $k \geq n_0$, and $S(k)$ is true, then $S(k+1)$ is true.
>
> Then $S(n)$ is true for all integers $n \geq n_0$.

> **The Strong Principle of Mathematical Induction:**
> Let $S(n)$ be a statement involving the integer n. Suppose that for some fixed integer n_0,
>
> (1) $S(n_0)$ is true (that is, the statement is true if $n = n_0$) AND
>
> (2) whenever k is an integer such that $k \geq n_0$, and $S(n_0), S(n_0+1), ..., S(k)$ are all true, then $S(k+1)$ is true.
>
> Then $S(n)$ is true for all integers $n \geq n_0$.

> In strong induction, we assume all cases 1 through k are true (rather than just case k). Strong induction is needed when dealing with recursions:

Examples :

1. Recall in section 9.2, we used the method of iteration to compute $1+2+3+...+n = \dfrac{n(n+1)}{2}$. Use the principle of mathematical induction to prove this equality.

$n \geq 1$

$S(n)$: $\quad 1+2+3+\cdots+n = \dfrac{n(n+1)}{2}$

base case : $n = 1$

$$1 \overset{?}{=} \frac{1(1+1)}{2} = \frac{2}{2} = 1 \checkmark$$

Assume that $S(k)$ is true

$$\boxed{1+2+3+\cdots+k = \frac{k(k+1)}{2}} \leftarrow$$

Our task : Show that $S(k+1)$ is true

$$1+2+3+\cdots+k+k+1 = \frac{(k+1)(k+2)}{2} \; ?$$

$$\underbrace{1+2+3+\cdots+k}+k+1 = k(k+1) + k+1$$

Conclusion that $S(n)$ is true for all $n \geq 1$

$$= \frac{k(k+1)^2 + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

2. Prove for all positive integers n, $1^2 + 3^2 + ... + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$ using math induction.

$$S(n) = 1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Base case: $n = 1$

$$1^2 = \frac{1(2-1)(2+1)}{3} = \frac{1 \cdot 1 \cdot 3}{3} = 1$$

Assume that $S(k)$ is true

$$\Rightarrow 1^2 + 3^2 + \cdots + (2k-1)^2 = \boxed{\frac{k(2k-1)(2k+1)}{3}}$$

Our task: show that $S(k+1)$ is true

$$\boxed{1^2 + 3^2 + \cdots + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}}$$

$$\underbrace{1^2 + 3^2 + \cdots + (2k-1)^2}_{} + (2k+1)^2 = \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

$$\frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

$$= \frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3}$$

$$= \frac{(2k+1)[k(2k-1) + 3(2k+1)]}{3}$$

$$= \frac{(2k+1)[2k^2 - k + 6k + 3]}{3}$$

$$= \boxed{\frac{(2k+1)(2k+3)(k+2)}{3}}$$

$2k^2 + 7k + 6 = 2k^2 + 2k + 3k + 6$
$= 2k(k+2) + 3(k+2) = (k+2)(2k+3)$

3. Prove $n! > 3^n$ for every integer $n \geq 7$ using math induction.

Base case: $S(7)$ $\quad 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$k+1 > k \quad 7! > 3^7 \quad 3^{k+1}$

Assume $k! > 3^k$ $\quad \boxed{k \geq 7}$ $\quad (3k! > 3^k \cdot 3$

$S(k)$ is true

Our task is show that $S(k+1)$ is true

$\boxed{(k+1)! > 3^{k+1}}$

$k \geq 7$
$\Rightarrow (k+1) > 7 > 3$

$(k+1)! = (k+1) \cdot k! > 3 \cdot k! > 3^{k+1}$ $\quad n! > 3^n \quad n \geq 7$

4. Recall in sections 9.1 and 9.2, we saw the recurrence relation $p_n = p_{n-1} + 2$ for $n \geq 1$ with initial condition $p_0 = 92$ could have its nth term characterized by $p(n) = 2n + 92$. Prove this characterization using mathematical induction.

Base case: $n = 0$
$p(0) = 2 \cdot 0 + 92 = 92 = P_0$

Assume $P_0 = 92,\ P_k = P_{k-1} + 2$ for all case 0 through k; $\boxed{P_k = 2k + 92}$

Our task: $n = k+1$

$\boxed{P_{k+1} = 2(k+1) + 92 = 2k + 94}$

$P_{k+1} = P_k + 2 = (2k + 92) + 2 = \boxed{2(k+1) + 92}$

5. Recall in sections 9.1 and 9.2, we saw the recurrence relation $t_n = 2t_{n-1}$ for $n \geq 2$ with initial condition $t_1 = 3$ could have its nth term characterized by $t(n) = 3(2)^{n-1}$. Prove this characterization using mathematical induction.

Base case: $t(1) = 3 \cdot 2^{1-1} = 3 = t_1$ ✓

Assume that $t_1 = 3$, $t_k = 2t_{k-1}$ for all cases 1 through k; $t_k = 3 \cdot 2^{k-1}$

Our task: $t_{k+1} = 3 \cdot 2^k$

$t_{k+1} = 2t_k = 2 \cdot (3 \cdot 2^{k-1}) = 3 \cdot 2^{k-1+1}$
$= 3 \cdot 2^k$

6. Recall in section 9.2, we saw the recurrence relation $s_n = 2s_{n-1} - 3$ for $n \geq 1$ with initial condition $s_0 = 7$ could have its nth term characterized by $s(n) = 4(2)^n + 3$. Prove this characterization using mathematical induction.

Base case: $n = 0$
$s(0) = 4 \cdot 2^0 + 3 = 7 = s_0$ ✓

Assume that $s_0 = 7$, & $s_k = 2s_{k-1} - 3$ for all cases 1 through k;
$\boxed{s_k = 4 \cdot 2^k + 3}$.

Our task: $S(k+1)$ is true

$s_{k+1} = 2 \cdot s_k - 3 = 2 \cdot (4 \cdot 2^k + 3) - 3$
$= 4 \cdot 2^{k+1} + 6 - 3 = 4 \cdot 2^{k+1} + 3$

$S(k+1) = 4 \cdot 2^{k+1} + 3$

PMI:

$S(n_0)$

n_0

$k \geq n_0$

Step 1 : We check the base case to see if is true.

Step 2 : We assume that the the statement is true for some $k \geq n_0$

Step 3 : Our task of showing that $S(k+1)$ is true.

Strong PMI

Step 1 : Base case

Step 2

Step 3

$P(x) = x^2 + 5x + 6$
$d(x) = x + 1$

$\frac{P(x)}{d(x)}$ $P(x) = (x+1)\cdot(x+4) + 2$

$$x + 4$$

$$x+1 \;\big)\; x^2 + 5x + 6$$

$$-\;x^2 + x$$

$$0 + 4x + 6$$

$$4x + 4$$

$$0 + 2$$

LONG

DIVISION