1 Simple Graphs

Definitions:

A simple graph is a nonempty finite set $\mathcal{V}$ along with a set $\mathcal{E}$ of 2-element subsets of $\mathcal{V}$. The elements of $\mathcal{V}$ are called vertices, and the elements of $\mathcal{E}$ are called edges.

Example: Based on this definition, can a vertex have an edge to itself in a (simple) graph?

Example: Draw a graph with 4 vertices and 3 edges.

Definitions:

Whenever we have an edge $e = \{U, V\}$, we say that the edge e joins vertices U and V and that U and V are adjacent. We also say edge e is incident with the vertex U and vertex U is incident with edge e.

Example: Must a vertex be adjacent to at least one other vertex in a simple graph? Can the same two vertices have more than one edge connecting them? Why or why not?

Definitions:

In a graph, the number of edges incident with a vertex V is called the degree of V and is denoted $\deg(V)$.

In the complete graph on n vertices, denoted $\mathcal{K}_n$, every vertex is adjacent to all other vertices.

Comple graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge.

Example: Draw $\mathcal{K}_3$ and $\mathcal{K}_4$. What is the degree of each vertex in $\mathcal{K}_3$, $\mathcal{K}_4$, and $\mathcal{K}_n$?

Theorem 4.1:

In a simple graph, the sum of the degrees of the vertices equals twice the number of edges.

Definitions:

Suppose we have a graph $\mathcal{G}$ with n vertices labeled V_1, V_2,...,V_n. Such a graph is a labeled graph. To represent a labeled graph $\mathcal{G}$ by a matrix, we create an $n \times n$ matrix, where the i, j entry is 1 if there is an edge between V_i and V_j and 0 if not. Such a matrix is the adjacency matrix of $\mathcal{G}$ and denoted $A(\mathcal{G})$. An adjacency list lists each vertex followed by the vertices adjacent to it.

Example: Create an adjacency matrix and adjacency list for the graph with 4 vertices and 3 edges and for $\mathcal{K}_3$.

What are some properties adjacency matrices should always have?

Theorem 4.2:

The sum of the entries in row i of the adjacency matrix of a graph is the degree of the vertex V_i in the graph.

2 Isomorphism

Isomorphism, informally, is focused on determining which graphs are essentially the same. Isomorphism comes from the Greek root "isos", meaning 'same,' and "morphos", meaning 'structure'. Because the essential aspects of graphs are the relationships among vertices, what the graph looks like is not really important; rather, we want to know whether the same pattern of adjacency exists. *Why might mathematicians talk about graphs being the same "up to isomorphism"?*

Definitions:

A graph $\mathcal{G}_1$ is isomorphic to a graph $\mathcal{G}_2$ when there is a one-to-one correspondence f between the vertices of $\mathcal{G}_1$ and $\mathcal{G}_2$ such that the vertices U and W are adjacent in $\mathcal{G}_1$ if and only if the vertices $f(U)$ and $f(W)$ are adjacent in $\mathcal{G}_2$. The function f is called an isomorphism between $\mathcal{G}_1$ and $\mathcal{G}_2$. "Isomorphic to" is an equivalence relation so we generally say $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic rather than specifying a first and second graph.

Example: Draw a graph isomorphic to the graph from before with 4 vertices and 3 edges (i.e., same vertices and adjacency relationships) but in a way that looks different.

Theorem 4.3:

Let f be an isomorphism of graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. For any vertex V in $\mathcal{G}_1$, the degrees of V and $f(V)$ are equal.

Why is this useful?

Example: Which of these graphs are isomorphic? How do you know?

Definition:

A property is said to be graph isomorphism invariant if, whenever $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic graphs and $\mathcal{G}_1$ has the property, so does $\mathcal{G}_2$.

Example: What are some properties that should be invariant under an isomorphism?

Example: Which of these graphs are isomorphic? How do you know?

Example: Are these graphs isomorphic? Why or why not?

Example: Are these graphs isomorphic? Why or why not?

Rachel Rupnow. Compiled: February 10, 2023

Example: Draw all of the nonisomorphic graphs with 2 vertices.

Example: Draw all of the nonisomorphic graphs with 3 vertices.

Example: Draw all of the nonisomorphic graphs with 4 vertices.