1 Partial Orderings

Definition:

A relation on a set S is called antisymmetric if, whenever $x\ R\ y$ and $y\ R\ x$ are both true, then $x = y$.

Definition:

A relation R on a set S is called a partial ordering relation (partial order) if it has the following three properties:

(a) R is reflexive; that is, $x\ R\ x$ is true for every $x \in S$.

(b) R is antisymmetric; that is, whenever $y\ R\ x$ and $x\ R\ y$ are true, then $x = y$.

(c) R is transitive; that is, whenever $x\ R\ y$ and $y\ R\ z$ are both true, then $x\ R\ z$ is true.

Example: Is the relation set equality on some particular set S antisymmetric? Is set equality on some particular set S a partial ordering? Why or why not?

yes: $y = x$ and $x = y \Rightarrow x = y$ for equality

yes - previously said equality is an equivalence relation so reflexive & transitive properties are also satisfied

Example: Suppose S is the set of real numbers. Is $\geq$ an antisymmetric relation on S? Is $\geq$ a partial ordering on S? $3 \geq 2$ $2 \ngeq 3$

antisym: If $y \geq x$ and $x \geq y$ does this mean $x = y$? Yes being both $\geq$ and $\leq \Rightarrow =$. antisym. prop.

($4 \geq 2$ but $2 \ngeq 4$ is perfectly fine)

reflexive: Is xRx true for all $x \in S$? Is $x \geq x$ for $x \in$ real numbers? yes

transitive: If xRy & yRz is xRz? If $x \geq y$ and $y \geq z$, is $x \geq z$? yes

$\Rightarrow$ Partial order

Example: Can a relation be both symmetric and antisymmetric? Justify your reasoning.

Yes, equality is both: yRx & xRy

$y = x$ & $x = y \Rightarrow x = y$

(but usually 1 or the other)

Example: Can a partial ordering be an equivalence relation? Must a partial ordering be an equivalence relation? Justify your reasoning. yes - equality is both

$\geq$ is a partial ordering BUT $3 \geq 2$ while $2 \ngeq 3$ - fails the symmetric property

$\Rightarrow$ not an equivalence relation

$\Rightarrow$ but is a partial ordering (passes reflexive, antisymmetric, & transitive properties)

for transitive: (1,1) & (1,2) need (1,2) (1,2) & (2,2) need (1,2) (1,3) & (3,3) need (1,3)
(1,1) & (1,3) need (1,3) (1,2) & (2,3) need (1,3) (2,2) & (2,3) need (2,3)
(2,3) & (3,3) need (2,3)

Example: Determine whether relation $R = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$ is a partial order on set $S = \{1,2,3\}$ and justify.

reflexive: yes - we have (1,1), (2,2), & (3,3)
satisfy xRx for all $x \in S$

antisymmetric: yes - (1,1), (2,2), (3,3) are fine & if have (x,y), $x \neq y$ don't have (y,x)
$1=1$ $2=2$ $3=3$
(don't have (2,1), (3,1), & (3,2) which is good)

transitive: yes - can chain xRy & yRz & have xRz in every case

satisfy all 3 ⇒ partial order

Example: Determine whether relation $R = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$ is a partial order on set $S = \{1,2,3,4\}$ and justify.

Now fail reflexive property - don't have (4,4) & need it because $4 \in S$
(antisymmetric & transitive pass)

Example: Determine whether relation $R = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3),\underline{(3,1)}\}$ is a partial order on set $S = \{1,2,3\}$ and justify.

$\underset{(x,y)}{(3,1)}$ & $\underset{(y,z)}{(1,2)}$ require $\underset{(x,z)}{(3,2)}$ so fails transitive property

(1,3) & (3,1) both true but $1 \neq 3$ so fails antisymmetric property
(reflexive still passes)

Definition:

Suppose R_1 is a partial order on set S_1 and R_2 is a partial order on set S_2. Define relation R by $(a_1, a_2)\ R\ (b_1, b_2)$ if and only if one of the following is true:

(a) $a_1 \neq b_1$ and $a_1\ R_1\ b_1$ OR

(basically do 1 list in order then the other)

(b) $a_1 = b_1$ and $a_2\ R_2\ b_2$.

This relation is called the <u>lexicographic order</u> on $S_1 \times S_2$ or the <u>dictionary</u> order.

Example: Suppose Mr Webster is scheduled to interview three applicants for a summer internship at 9:00, 10:00, and 11:00 and Ms Collins is to interview three other applicants at the same times. Unfortunately both Mr Webster and Ms Collins have become ill so all 6 interviews are to be conducted by Ms Herrera. She has decided to schedule the applicants to be interviewed in time order, alphabetically by interviewer. Thus $S_1 = \{9:00, 10:00, 11:00\}$ and $S_2 = \{\text{Collins, Webster}\}$. What is the order of applicants and how could R_1 and R_2 be characterized?

(9:00, Collins), (9:00, Webster), (10:00, Collins), (10:00, Webster), (11:00, Collins), (11:00, Webster)

R_1 is numerical order (9, 10, 11) & R_2 is alphabetical order (C before W)

Note:

The prior example is an intentional variation on Example 2.22 in text. See that example for another way to do a lexicographic ordering.

Theorem 2.5:

If R_1 is a partial order on set S_1 and R_2 is a partial order on set S_2, then the lexicographic order is a partial order on $S_1 \times S_2$.

Why should this work?

We're maintaining the original orders (which were partial orders) within each set, but just interleaving them (which should not cause problems)

Definition:

A partial order R on a set S is called a total order (or linear order) on S if every pair of elements in S can be compared; that is, for every $x, y \in S$, $x\ R\ y$ or $y\ R\ x$.

Example: What is an example of a relation on a set that has a total order?

$\geq$ on the real numbers – I know how 2 & 1 relate (have (2,1))
I know how 3 & π relate (have (π,3))
I know how 3 & 3 relate (have (3,3))

For $S=\{1,2,3\}$ & $\geq$: $R=\{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3)\}$

Example: What is an example of a relation on a set that only permits partial orders?

= on the real numbers – (1,2) & (2,1) not in relation

For $S=\{1,2,3\}$ & = again 1 & 2 not related either way
$R=\{(1,1),(2,2),(3,3)\}$ don't have (1,2) or (2,1)

Definition:

Let R be a partial order on set S. An element $x \in S$ is called a minimal element of S with respect to R if the only element $s \in S$ satisfying $s\ R\ x$ is x itself (that is, $s\ R\ x$ implies $s = x$).

Let R be a partial order on set S. An element $z \in S$ is called a maximal element of S with respect to R if the only element $s \in S$ satisfying $z\ R\ s$ is z itself (that is, $z\ R\ s$ implies $z = s$).

Note: No minimal or maximal elements need to exist for a given partial order.

Example: Identify the minimal and maximal elements of S with respect to R, if they exist: S is the set of all *integers* x such that $0 \leq x < 2$ and $x\ R\ y$ if $x \leq y$.

$S=\{0,1\}$ $R=\{(0,0),(0,1),(1,1)\}$

0 is minimal

1 is maximal

Example: Identify the minimal and maximal elements of S with respect to R, if they exist: S is the set of all *real numbers* x such that $0 \le x < 1$, and $x \; R \; y$ if $x \le y$.

minimal: 0 (nothing smaller) $[0,1)$

maximal: none (never quite reach 1 & could always make something bigger)

Example: Identify the minimal and maximal elements of S with respect to R, if they exist: S is the set of all real numbers x such that $0 \le x < 1$, and $x \; R \; y$ if $x \ge y$.

minimal: none

maximal: 0

Example: Identify the minimal and maximal elements of S with respect to R, if they exist: S is the set of nonempty subsets of $\{1, 2, 3\}$ and $A \; R \; B$ if $A \subseteq B$. $S = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

minimal: $\{1\}, \{2\}, \{3\}$

maximal: $\{1,2,3\}$

Theorem 2.6:

Let R be a partial order on a finite set S. Then S has both a minimal and a maximal element with respect to R.

Example: Why does this theorem not contradict the example above?

→ notice S is not finite

2 Hasse Diagrams

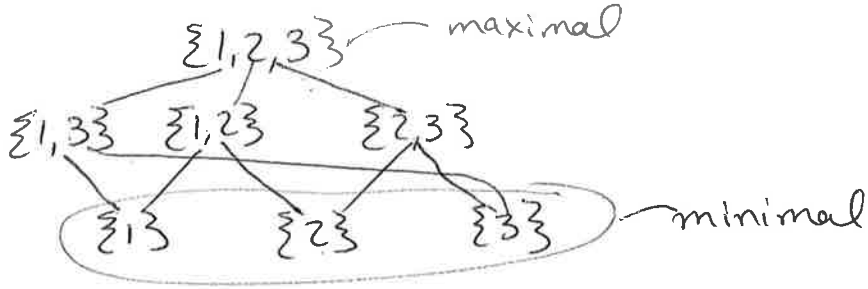
Description:

A Hasse diagram represents a partial order R on a finite set S by representing each element of S by a point and each pair of distinct points related by R has a line connecting them. Each line is arranged so that the initial point is below its terminal point; that is, an arrow is drawn from the point representing x to the point representing y when $x \; R \; y$ and there is no $s \in S$ other than x and y such that $x \; R \; s$ and $s \; R \; y$. Hasse diagrams are read from bottom to top so all line segments between points are regarded as pointing upward.

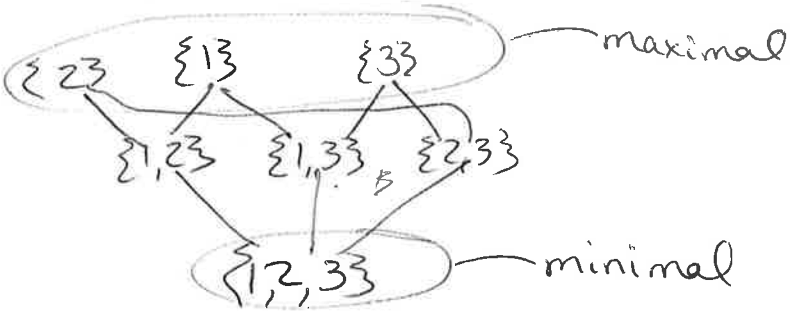
Note: segments need not be drawn with arrows. The direction is implicit in how the Hasse diagram is drawn.

Example: Where will minimal and maximal element(s) of partial order R on set S appear in a Hasse diagram? at the bottom at the top

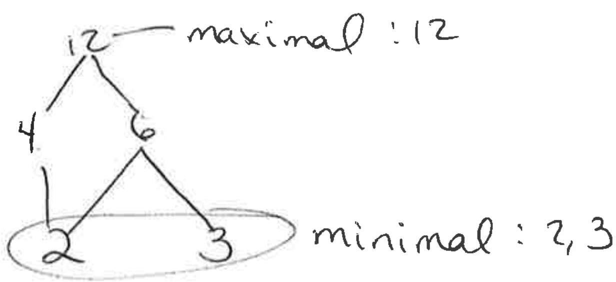
Example: Create a Hasse diagram for relation R on set S: S is the set of nonempty subsets of $\{1,2,3\}$ and $A\ R\ B$ if $A \subseteq B$. $S = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$



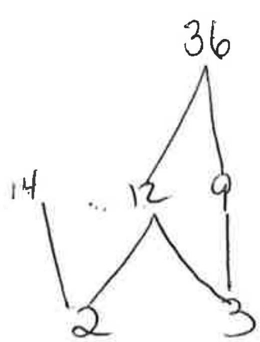
Example: Create a Hasse diagram for relation R on set S: S is the set of nonempty subsets of $\{1,2,3\}$ and $A\ R\ B$ if $B \subseteq A$.



Example: Create a Hasse diagram for relation R on set S: let R be the partial order "divides" on the set $S = \{2,3,4,6,12\}$. What does this Hasse diagram indicate about the minimal and maximal element(s) of S? i.e. x divides y



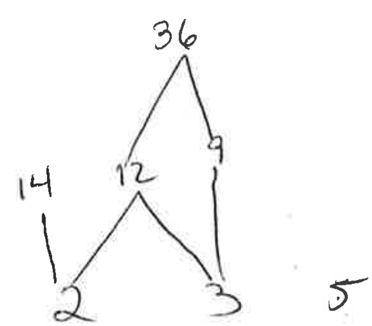
Example: Create a Hasse diagram for relation R on set S: let R be the partial order "divides" on the set $S = \{2,3,9,12,14,36\}$. What does this Hasse diagram indicate about the minimal and maximal element(s) of S?



maximal: 14, 36

minimal: 2, 3

if changed $S = \{2,3,5,9,12,14,36\}$



max: 5, 14, 36

min: 2, 3, 5