1 Directed Graphs

What if the relationship between two objects only goes one way? Then, rather than indicate a symmetric relationship, as we have with (undirected) edges, we want to assign directions to our edges.

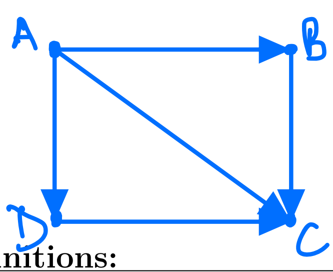
Definitions:

A directed graph is a finite nonempty set $\mathcal{V}$ and a set $\mathcal{E}$ of ordered pairs of distinct elements of $\mathcal{V}$. The elements of $\mathcal{V}$ are called vertices and the elements of $\mathcal{E}$ are called directed edges. If there is a directed edge $e = (A, B)$, it is said that e is a directed edge from A to B.

Note: whereas (undirected) graphs' edges are represented by sets (because order does not matter), we represent directed graphs' edges with ordered pairs (because order matters).

The number of directed edges from vertex A is called the outdegree of A and is denoted outdeg(A). The number of directed edges to vertex A is called the indegree of A and is denoted indeg(A).

Example: Create a directed graph on 4 vertices with 5 directed edges. What are the indegrees and outdegrees of each vertex?



indegrees
indeg(A) =
indeg(B) =
indeg(C) =
indeg(D) =

outdegrees
outdeg(A) =
outdeg(B) =
outdeg(C) =
outdeg(D) =

Definitions:

A directed graph $\mathcal{D}$ with n vertices labeled $V_1, V_2, ..., V_n$ is called a labeled graph. The adjacency matrix with respect to the labeling of $\mathcal{D}$ is an $n \times n$ matrix in which the i, j entry is 1 if there is a directed edge from the vertex V_i to the vertex V_j and 0 if it is not, and is denoted $A(\mathcal{D})$. The adjacency List of a directed graph lists vertices to which there is a directed edge from the given vertex.

Based on this definition, must the adjacency matrix of a directed graph be symmetric? Why or why not?

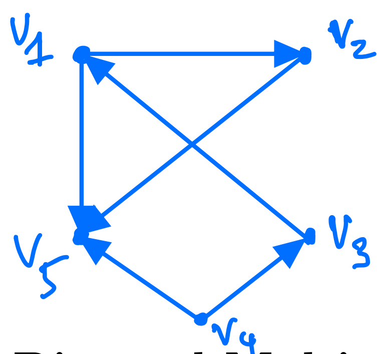
NO — just because a vertex has a directed edge from V_1 to V_2 does not mean it has a directed edge from V_2 to V_1.

represent outdegree

Theorem 4.11

> The sum of the entries in row i of the adjacency matrix of a directed graph equals the outdegree of the vertex V_i, and the sum of the entries in column j equals the indegree of the vertex V_j.

Example: Create an adjacency matrix and adjacency list for the directed graph below.



Adjacency list

Adjacency matrix

2 Directed Multigraphs

The ideas in directed graphs can be extended to multigraphs analogously. (See p.205 in text for descriptions of directed multigraph, directed loop, and parallel directed edges.)

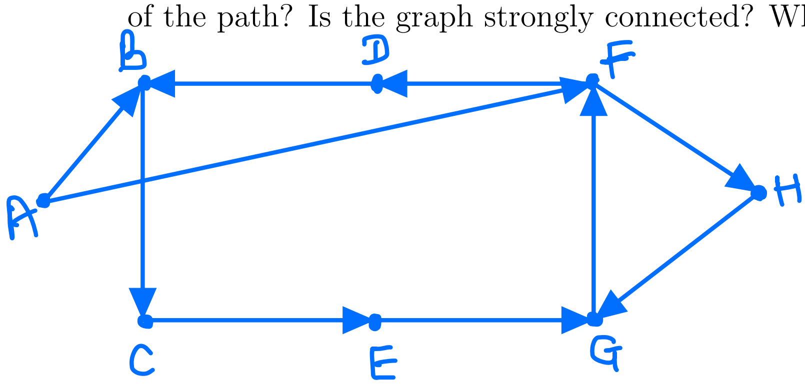
Definitions:

> An alternating sequence of vertices and directed edges $V_1, e_1, V_2, e_2, ..., V_n, e_n, V_{n+1}$ is called a directed path from V_1 to V_{n+1} if $e_i = (V_i, V_{i+1})$ for each $i = 1, 2, ..., n$. The length of this directed path is n, the number of directed edges. A simple directed path is a directed path with no vertex repeated. A directed cycle is a directed path of positive length from V_i to V_i in which no other vertex is visited twice. A directed multigraph $\mathcal{D}$ is called strongly connected if, for every pair V_i and V_j of vertices in $\mathcal{D}$, there is a directed path from V_i to V_j.

Theorem 4.12:

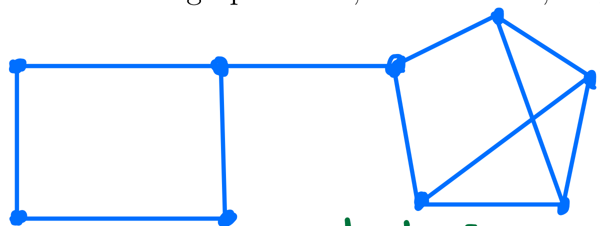
> Every U-V directed path contains a U-V simple directed path.

Example: Using the graph below, find a simple directed path from B to D. What is the length of the path? Is the graph strongly connected? Why or why not?



 Rachel Rupnow. Compiled: October 21, 2022

Example: Can a direction be assigned to each edge of the graph below which results in a strongly connected directed graph? If so, do so. If not, why not?



if a graph has an edge whose removal makes the graph not connected, the graph is not strongly connected.

Definitions:

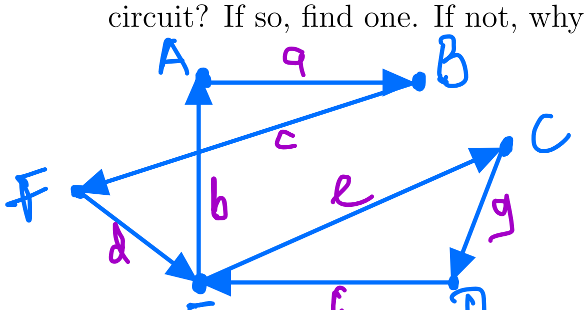
A directed path in a directed multigraph $\mathcal{D}$ that includes exactly once all the directed edges of $\mathcal{D}$ and has different initial and terminal vertices is called a directed Euler path. A directed cycle that includes exactly once all the directed edges of $\mathcal{D}$ is called a directed Euler circuit.

Theorem 4.13:

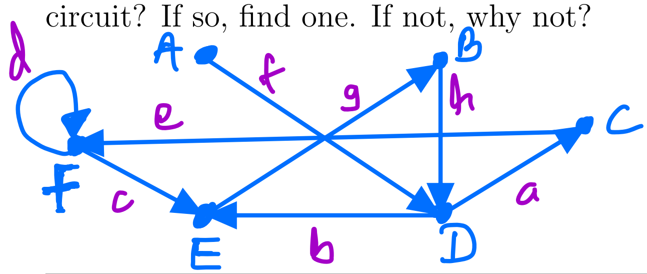
Suppose the directed multigraph $\mathcal{D}$ has the property that, whenever the directions are ignored on the directed edges, the resulting multigraph is connected. Then $\mathcal{D}$ has a directed Euler circuit if and only if, for each vertex of $\mathcal{D}$, the indegree is the same as the outdegree. Furthermore, $\mathcal{D}$ has a directed Euler path if and only if every vertex of $\mathcal{D}$ has its indegree equal to its outdegree except for two distinct vertices, B and C, where the outdegree of B exceeds its indegree by 1 and where the indegree of C exceeds its outdegree by 1. When this is the case, the directed Euler path begins at B and ends at C.

Note: the approach to finding a directed Euler circuit is basically the same as finding an Euler circuit, just using directed edges.

Example: Consider the directed graph shown here. Does it permit a directed Euler path or circuit? If so, find one. If not, why not?

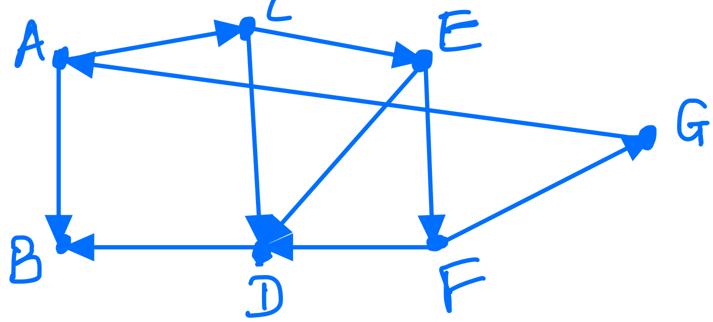


Example: Consider the directed graph shown here. Does it permit a directed Euler path or circuit? If so, find one. If not, why not?



Much like the Euler Circuit approach, the Breadth-First Search Algorithm applies in a similar way to directed graphs.

Example: Use the Breadth-First Search Algorithm to determine the shortest path from A to G in the directed graph.



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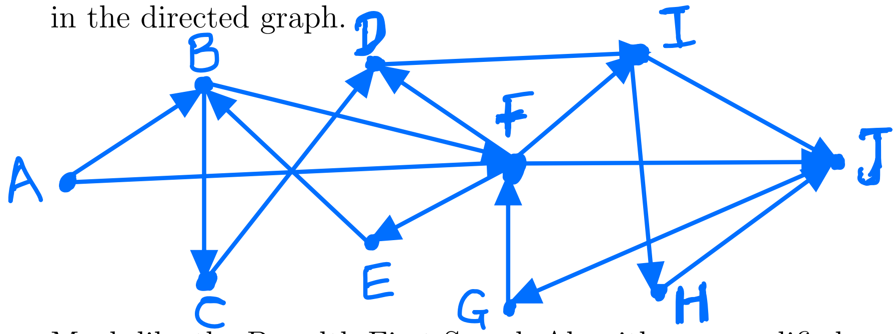
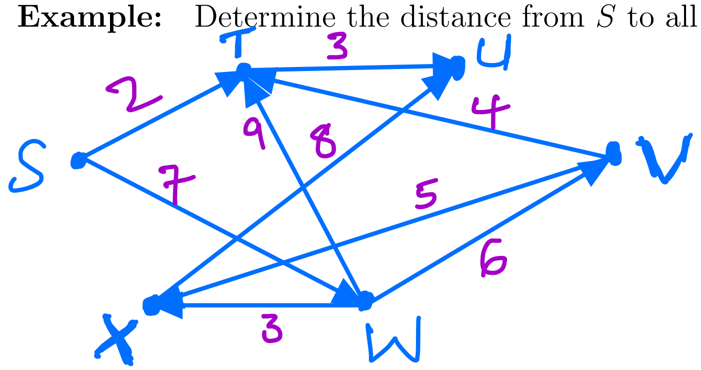


Much like the Breadth-First Search Algorithm, a modified version of Dijkstra's Algorithm applies in a similar way to directed graphs.

Example: Determine the distance from S to all other vertices in the directed weighted graph.



Example: Determine the distance from S to all other vertices in the directed weighted graph.