1 Relations

Definition:

A relation from the set A to B is any subset of the Cartesian product $A \times B$. If R is a relation from set A to set B and (x, y) is an element of R, we say x is related to y by R (notation: $x \, R \, y$).

A relation from a set S to itself is called a relation on S

Example: Suppose Math majors need to take Differential Calculus, Ordinary Differential Equations, and Linear Algebra; Computer Science majors need to take Differential Calculus and Discrete Math; and Physics majors need to take Differential Calculus and Ordinary Differential Equations. For $A = \{$Math, Computer Science, Physics$\}$ and
$B = \{$Differential Calculus (DC), Ordinary Differential Equations (ODE), Linear Algebra (LA), Discrete Math (DM)$\}$, what is the relation R representing which majors need to take which classes?

R = {(Math, DC), (Math, ODE) (Math, LA), (Computer Science, DC), (Computer Science, DM), (Physics, DC), (Physics, ODE)}

x's & y's come from here

Example: Let $S = \{1, 2, 3, 4\}$. Define a relation R on S by letting $x \, R \, y$ mean $x > y$. Which elements are related to each other?

R = {(4,3), (2,1), (3,2), (3,1), (4,2), (4,1)}

4R3 (is true) 3R4 is not true so (3,4) is not in R

Definition:

A relation R on a set S may have any of the following special properties:

(a) If for each $x \in S$, $x \, R \, x$ is true, then R is called reflexive.

(b) If $y \, R \, x$ is true whenever $x \, R \, y$ is true, then R is called symmetric

(c) If $x \, R \, z$ is true whenever $x \, R \, y$ and $y \, R \, z$ are both true, then R is called transitive.

Example: Revisit relation R on S where $x \, R \, y$ means $x > y$. Is the relation reflexive, symmetric, and/or transitive?

reflexive: no - counterexample: 4 > 4 is not true

Symmetric: no - counterexample: 4 > 3 but 3 > 4 is not true

transitive: yes: If x > y and y > z that means x > z so the transitive property holds

examples: 3 > 2 and 2 > 1 ⇒ 3 > 1 needed & it is in the set

Example: Suppose S is the set of real numbers and, for $x, y \in S$ (in set S), define $x\ R\ y$ to mean that $x^2 = y^2$. Determine whether this relation is reflexive, symmetric, and/or transitive.

reflexive: For all $x \in S$ (for all x in the real numbers), need $x^2 = x^2$ for xRx.
This is true.
symmetric: Is it true that if xRy then yRx i.e. if $x^2=y^2$ does $y^2 \stackrel{?}{=} x^2$?
Yes - this is true. Note: the fact that $1^2 \neq 2^2$ & $2^2 \neq 4^2$ is fine.
transitive: If xRy & yRz, is xRz? That is, if $x^2=y^2$ & $y^2=z^2$, does $x^2 \stackrel{?}{=} z^2$?
Yes

2 Equivalence Relations

Definition:

A relation on S that is reflexive, symmetric, and transitive is called an equivalence relation.

Example: Have any of the examples up to this point been equivalence relations? If so, which?
Yes (this one)

Definition:

An integer greater than 1, is called prime if its only positive integer divisors are itself and 1.

$\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Example: Determine whether the following are prime or not prime:

(a) 13 just has 13 & 1 as positive integer divisors, so prime

(b) 12 no ($12 \div 2 = 6$ - has other divisors)

(c) -53 53 is prime but we would not -53 is prime (not greater than 1)

(d) 2.5 not an integer so not prime

(e) 57 $57/3 = 19$ so not prime

Example: On the set of integers greater than 1, define $x\ R\ y$ to mean that x has the same number of distinct (not counting repeats) prime divisors as y. Show that R is an equivalence relation on S.

reflexive: x has the same number of prime divisors as x (as itself)

Symmetric: If x has the same number of prime divisors as y, then y has the same number of prime divisors as x.

transitive: If x has the same number of prime divisors as y and y has the same number of prime divisors as z, then x has the same number of prime divisors as z.

Definition:

If R is an equivalence relation on a set S and $x \in S$, the set of elements of S that are related to x is called the equivalence class containing x (notation: $[x]$).

Example: Suppose S is the positive integers. For $x, y \in S$, define $x \; R \; y$ to mean that x and y have the same parity (both odd or both even). Describe the equivalence class containing 7. Describe the equivalence class containing 14. How many distinct equivalence classes of R exist?

$[7] = \{1, 3, 5, 7, 9, \ldots\}$ OR $[7] = \{x : x \text{ is odd}\}$ (such that)

$[14] = \{2, 4, 6, 8, \ldots\}$ OR $[14] = \{x : x \text{ is even}\}$

(2 - evens & odds describe all relevant options

Note: an equivalence class is a set though we may choose 1 element to represent that set.

$[7] = [1] = [3]$

Example: Suppose S is the set of real numbers and for $x, y \in S$, $x \; R \; y$ means $x^2 = y^2$. Describe the equivalence class containing 7. Describe the equivalence class containing 14. How many distinct equivalence classes of R exist?

$[7] = \{7, -7\}$ $\{x : x^2 = 7^2 = 49\}$

$[14] = \{14, -14\}$

Infinitely many equivalence classes (e.g. $[1]$, $[\pi]$, $[7]$, $[14]$, $[3\text{billion}]$)

Example: Let S be the set of ordered pairs of positive integers. Define R on S so that (x_1, x_2) R (y_1, y_2) means that $x_1 + y_2 = y_1 + x_2$. Describe the equivalence class containing $z = (5, 8)$. How many distinct equivalence classes of R exist?

$5 + y_2 = y_1 + 8$

$y_2 = y_1 + 3$ OR $y_2 - y_1 = 3$ $[(5,8)] = \{(y_1, y_2) : y_2 - y_1 = 3\}$

(examples include $(6,9)$, $(7,10)$ infinitely many ordered pairs in this set

$[(5,9)]$ belongs to a different equivalence class: $\{(y_1, y_2) : y_2 - y_1 = 4\}$

infinitely many equivalence classes because there are infinitely many possible differences between the coordinates of the ordered pair

Theorem 2.3:

Let R be an equivalence relation on set S.

(a) If x and y are elements of S, then x is related to y by R if and only if $[x] = [y]$. $\Leftrightarrow$
in same equivalence class $\Leftrightarrow$ related

(b) Two equivalence classes of R are either equal or disjoint. – whole set is same OR no shared elements

Proof summary:

a) Show equality by showing mutual inclusion $[x] \subseteq [y]$ and $[y] \subseteq [x]$. We do this by using the equivalence relation def (use reflexive, symmetric, & transitive properties)

b) If not disjoint, the 2 equivalence classes share an element. By part a, that means the whole set is the same (equal sets). Otherwise, disjoint.

Note:

One standard example of an equivalence relation is equality.

Definition:

The equivalence classes of an equivalence relation R on set S divide S into disjoint subsets. This family of subsets of S is called a partition of S and has the following properties:

(a) No subset is empty.

(b) Each element of S belongs to some subset.

(c) Two distinct subsets are disjoint.

Example: Suppose $A = \{1, 3, 5, 7, 9\}$, $B = \{8\}$, and $C = \{2, 4, 6\}$.

(a) Is $\mathcal{P} = \{A, B, C\}$ a partition of $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$? yes – no empty sets (A, B, C) every element accounted for, all sets (A, B, C) disjoint

(b) Is $\mathcal{P} = \{A, B, C\}$ a partition of $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$?

no – 10 not accounted for (fails b)

Theorem 2.4:

(a) An equivalence relation R gives rise to a partition $\mathcal{P}$ in which the members of $\mathcal{P}$ are the equivalence classes of R.

(b) Conversely, a partition $\mathcal{P}$ induces an equivalence relation R in which two elements are related by R whenever they lie in the same member of $\mathcal{P}$. Moreover, the equivalence classes of this relation are the members of $\mathcal{P}$.

What does this mean?

"Equivalence relations" & "partitions" describe the same thing.

Example: Write the equivalence relation on $\{1, 2, 3, 4, 5\}$ that is induced by the partition with $\{2, 3\}, \{1, 4\}, \{5\}$ as its partitioning subsets.

$R = \{(2,2), (2,3), (3,2), (3,3), (4,1), (1,4), (1,1), (4,4), (5,5)\}$

Example: Let $A = \{1, 2, 3, 4, 5\}$ and suppose that R is an equivalence relation on A. Suppose we also know $1R3$, $4R5$, and $2R5$ are true and $2R3$ is not true. Determine the following:

(1,3) (4,5) (2,5)

(a) Is $4R2$ true? Why or why not?

$2R5 \Rightarrow 5R2$ (symmetric property)

$4R5$ and $5R2$ true $\Rightarrow 4R2$ true (transitive property)

$R = \{(1,3), (4,5), (2,5), (3,1), (5,4), (5,2), (5,5), (1,1), (2,2), (4,4), (3,3), (4,2), (2,4)\}$

(b) Is $1R5$ true? Why or why not?

$2R3$ not true $\Rightarrow$ can't have $3R2$ & so our chain from 1 to 5 is in trouble

(c) What is the equivalence class [4]?

$\{4,5,2\}$ $\{1,3\}$ serve as partition

$[2] = [5] = [4] = \{2,4,5\}$

$[1] = [3] = \{1,3\}$