1 Partial Orderings

Definition:

A relation on a set S is called ____________________ if, whenever $x\ R\ y$ and $y\ R\ x$ are both true, then $x = y$.

Definition:

A relation R on a set S is called a ___________________________ if it has the following three properties:

(a) R is reflexive; that is, $x\ R\ x$ is true for every $x \in S$.

(b) R is antisymmetric; that is, whenever $y\ R\ x$ and $x\ R\ y$ are true, then $x = y$.

(c) R is transitive; that is, whenever $x\ R\ y$ and $y\ R\ z$ are both true, then $x\ R\ z$ is true.

Example: Is the relation set equality on some particular set S antisymmetric? Is set equality on some particular set S a partial ordering? Why or why not?

Example: Suppose S is the set of real numbers. Is $\geq$ an antisymmetric relation on S? Is $\geq$ a partial ordering on S?

Example: Can a relation be both symmetric and antisymmetric? Justify your reasoning.

Example: Can a partial ordering be an equivalence relation? Must a partial ordering be an equivalence relation? Justify your reasoning.

Example: Determine whether relation $R = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$ is a partial order on set $S = \{1,2,3\}$ and justify.

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Example: Determine whether relation $R = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3),(3,1)\}$ is a partial order on set $S = \{1,2,3\}$ and justify.

Definition:

Suppose R_1 is a partial order on set S_1 and R_2 is a partial order on set S_2. Define relation R by $(a_1, a_2)\ R\ (b_1, b_2)$ if and only if one of the following is true:

(a) $a_1 \neq b_1$ and $a_1\ R_1\ b_1$ OR

(b) $a_1 = b_1$ and $a_2\ R_2\ b_2$.

This relation is called the ________________________ on $S_1 \times S_2$ or the dictionary order.

Example: Suppose Mr Webster is scheduled to interview three applicants for a summer internship at 9:00, 10:00, and 11:00 and Ms Collins is to interview three other applicants at the same times. Unfortunately both Mr Webster and Ms Collins have become ill so all 6 interviews are to be conducted by Ms Herrera. She has decided to schedule the applicants to be interviewed in time order, alphabetically by interviewer. Thus $S_1 = \{9:00, 10:00, 11:00\}$ and $S_2 = \{\text{Collins, Webster}\}$. What is the order of applicants and how could R_1 and R_2 be characterized?

Note:

The prior example is an intentional variation on Example 2.22 in text. See that example for another way to do a lexicographic ordering.

Theorem 2.5:

If R_1 is a partial order on set S_1 and R_2 is a partial order on set S_2, then the lexicographic order is a partial order on $S_1 \times S_2$.

Why should this work?

Definition:

A partial order R on a set S is called a ________________________________ on S if every pair of elements in S can be compared; that is, for every $x, y \in S$, $x\ R\ y$ or $y\ R\ x$.

Example: What is an example of a relation on a set that has a total order?

Example: What is an example of a relation on a set that only permits partial orders?

Definition:

Let R be a partial order on set S. An element $x \in S$ is called a ___________________ of S with respect to R if the only element $s \in S$ satisfying $s\ R\ x$ is x itself (that is, $s\ R\ x$ implies $s = x$).

Let R be a partial order on set S. An element $z \in S$ is called a ___________________ of S with respect to R if the only element $s \in S$ satisfying $z\ R\ s$ is z itself (that is, $z\ R\ s$ implies $z = s$).

Note: No minimal or maximal elements need to exist for a given partial order.

Example: Identify the minimal and maximal elements of S with respect to R, if they exist: S is the set of all *integers* x such that $0 \leq x < 2$ and $x\ R\ y$ if $x \leq y$.

Example: Identify the minimal and maximal elements of S with respect to R, if they exist: S is the set of all *real numbers* x such that $0 \le x < 1$, and $x\ R\ y$ if $x \le y$.

Example: Identify the minimal and maximal elements of S with respect to R, if they exist: S is the set of all real numbers x such that $0 \le x < 1$, and $x\ R\ y$ if $x \ge y$.

Example: Identify the minimal and maximal elements of S with respect to R, if they exist: S is the set of nonempty subsets of $\{1, 2, 3\}$ and $A\ R\ B$ if $A \subseteq B$.

Theorem 2.6:

Let R be a partial order on a finite set S. Then S has both a minimal and a maximal element with respect to R.

Example: Why does this theorem not contradict the example above?

2 Hasse Diagrams

Description:

A ____________________ represents a partial order R on a finite set S by representing each element of S by a point and each pair of distinct points related by R has a line connecting them. Each line is arranged so that the initial point is below its terminal point; that is, an arrow is drawn from the point representing x to the point representing y when $x\ R\ y$ and there is no $s \in S$ other than x and y such that $x\ R\ s$ and $s\ R\ y$. Hasse diagrams are read from bottom to top so all line segments between points are regarded as pointing upward.

Note: segments need not be drawn with arrows. The direction is implicit in how the Hasse diagram is drawn.

Example: Where will minimal and maximal element(s) of partial order R on set S appear in a Hasse diagram?

Example: Create a Hasse diagram for relation R on set S: S is the set of nonempty subsets of $\{1, 2, 3\}$ and $A\ R\ B$ if $A \subseteq B$.

Example: Create a Hasse diagram for relation R on set S: S is the set of nonempty subsets of $\{1, 2, 3\}$ and $A\ R\ B$ if $B \subseteq A$.

Example: Create a Hasse diagram for relation R on set S: let R be the partial order "divides" on the set $S = \{2, 3, 4, 6, 12\}$. What does this Hasse diagram indicate about the minimal and maximal element(s) of S?

Example: Create a Hasse diagram for relation R on set S: let R be the partial order "divides" on the set $S = \{2, 3, 9, 12, 14, 36\}$. What does this Hasse diagram indicate about the minimal and maximal element(s) of S?