In section 8.3, we learned about combinations denoted $C(n, r)$, which give the number of (unordered) collections of r objects from n choices without repetition. These combinations form a pattern, which can be seen from a triangular array.

1 Pascal's Triangle

Theorem 8.1:

If r and n are integers such that $1 \le r < n$, then $C(n, r) = C(n-1, r-1) + C(n-1, r)$.

Example: Find $C(5, 2)$ and $C(5, 3)$ and use them to find $C(6, 3)$.

$C(6,3) = C(5,2) + C(5,3)$

Example: This suggests we can make a triangular array that shows combinations. This array is called Pascal triangle even though it was known in the Middle East, India, and China well before Pascal wrote about it. Create the first six rows (a) in terms of combination notation (e.g., C(0,0)) and (b) in terms of the number of combinations (e.g., 1).

n r

n=0 C(0,0)

n=1 C(1,0) C(1,1)

n=2 C(2,0) C(2,1) C(2,2)

n=3 C(3,0) C(3,1) C(3,2) C(3,3)

n=4 C(4,0) C(4,1) C(4,2) C(4,3) C(4,4)

n=5 C(5,0) C(5,1) C(5,2) C(5,3) C(5,4) C(5,5)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Theorem 8.2:

If r and n are integers such that $0 \le r \le n$, then $C(n, r) = C(n, n-r)$.

n=6 C(6,0)

$C(n,r) = C(n,n-r)$

1 n=0
1 1 n=1
1 2 1 n=2
1 3 3 1 n=3
1 4 6 4 1 n=4
1 5 10 10 5 1 n=5

1 6 15 20 15 6 1 $n=6$

1 7 21 35 35 21 7 1 $n=7$

2 Binomial Theorem

We can also use combinations to tell us the value of particular coefficients when expanding $(x+y)^n$ and other binomials of a similar pattern.

Theorem 8.3 (Binomial Theorem):

For every positive integer n,

$(x+y)^n = C(n,0)x^n + C(n,1)x^{n-1}y + \ldots + C(n,n-1)xy^{n-1} + C(n,n)y^n$.

$$(x+y)^3 = C(3,0)x^3y^0 + C(3,1)x^2y^1 + C(3,2)x^1y^2 + C(3,3)x^0y^3$$
$$= 1x^3 + 3x^2y + 3xy^2 + y^3$$

Example: Use the binomial theorem to determine the coefficient of x^4y^2 in the expansion of $(x+y)^6$.

$$C(6,2) = \frac{6!}{2!\,4!} = \frac{6\cdot 5\cdot \cancel{4!}}{2\cdot \cancel{4!}} = 15$$

Example: Use the binomial theorem to determine the expansion of $(x+y)^7$.

1 7 21 35 35 21 7 1 $n=7$

$$1x^7y^0 + 7x^6y^1 + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7x^1y^6 + 1x^0y^7 = (x+y)^7$$

Example: Use the binomial theorem to determine the coefficient of x^3y^7 in the expansion of $(x+2y)^{10}$.

y

$(x-3y)^{10}$

$C(10,7)\cdot 2^7$

$C(10,7)\cdot 3^7$

$(x+y)^{10}$.

$x^3(2y)^7 = x^3y^72^7$

Example: How many four-element subsets of $\{a,b,c,d,e,f,g,h,i,j,k,l,m\}$ contain no vowels?

$$C(10,4) = \frac{10!}{4!\,6!} = \frac{10\cdot \overset{3}{\cancel{9}}\cdot \cancel{8}\cdot 7\cdot \cancel{6!}}{\cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1\cdot \cancel{6!}}$$

Use the binomial theorem to determine the coefficient of $x^6 y^5$ in the expansion of

$(x-2y)^{11}$