1 Relations

Definition:

A ________________________ is any subset of the Cartesian product $A \times B$. If R is a relation from set A to set B and (x, y) is an element of R, we say _______________ (notation: $x\ R\ y$).

A relation from a set S to itself is called a ____________.

Example: Suppose Math majors need to take Differential Calculus, Ordinary Differential Equations, and Linear Algebra; Computer Science majors need to take Differential Calculus and Discrete Math; and Physics majors need to take Differential Calculus and Ordinary Differential Equations. For $A = \{$Math, Computer Science, Physics$\}$ and
$B = \{$Differential Calculus, Ordinary Differential Equations, Linear Algebra, Discrete Math$\}$, what is the relation R representing which majors need to take which classes?

Example: Let $S = \{1, 2, 3, 4\}$. Define a relation R on S by letting $x\ R\ y$ mean $x > y$. Which elements are related to each other?

Definition:

A relation R on a set S may have any of the following special properties:

(a) If for each $x \in S$, $x\ R\ x$ is true, then R is called ____________.

(b) If $y\ R\ x$ is true whenever $x\ R\ y$ is true, then R is called ____________.

(c) If $x\ R\ z$ is true whenever $x\ R\ y$ and $y\ R\ z$ are both true, then R is called ____________.

Example: Revisit relation R on S where $x\ R\ y$ means $x > y$. Is the relation reflexive, symmetric, and/or transitive?

Example: Suppose S is the set of real numbers and, for $x, y \in S$, define $x \ R \ y$ to mean that $x^2 = y^2$. Determine whether this relation is reflexive, symmetric, and/or transitive.

2 Equivalence Relations

Definition:

A relation on S that is reflexive, symmetric, and transitive is called an ________________________.

Example: Have any of the examples up to this point been equivalence relations? If so, which?

Definition:

An integer greater than 1 is called ______ if its only positive integer divisors are itself and 1.

Example: Determine whether the following are prime or not prime:

(a) 13

(b) 12

(c) -53

(d) 2.5

(e) 57

Example: On the set of integers greater than 1, define $x \ R \ y$ to mean that x has the same number of distinct (not counting repeats) prime divisors as y. Show that R is an equivalence relation on S.

Definition:

If R is an equivalence relation on a set S and $x \in S$, the set of elements of S that are related to x is called the ________________ containing x (notation: $[x]$).

Example: Suppose S is the positive integers. For $x, y \in S$, define $x\ R\ y$ to mean that x and y have the same parity (both odd or both even). Describe the equivalence class containing 7. Describe the equivalence class containing 14. How many distinct equivalence classes of R exist?

Example: Suppose S is the set of real numbers and for $x, y \in S$, $x\ R\ y$ means $x^2 = y^2$. Describe the equivalence class containing 7. Describe the equivalence class containing 14. How many distinct equivalence classes of R exist?

Example: Let S be the set of ordered pairs of positive integers. Define R on S so that (x_1, x_2) R (y_1, y_2) means that $x_1 + y_2 = y_1 + x_2$. Describe the equivalence class containing $z = (5, 8)$. How many distinct equivalence classes of R exist?

Theorem 2.3:

Let R be an equivalence relation on set S.

(a) If x and y are elements of S, then x is related to y by R if and only if $[x] = [y]$.

(b) Two equivalence classes of R are either equal or disjoint.

Proof summary:

Note:

One standard example of an equivalence relation is equality.

Definition:

The equivalence classes of an equivalence relation R on set S divide S into disjoint subsets. This family of subsets of S is called a ____________ of S and has the following properties:

(a) No subset is empty.

(b) Each element of S belongs to some subset.

(c) Two distinct subsets are disjoint.

Example: Suppose $A = \{1, 3, 5, 7, 9\}$, $B = \{8\}$, and $C = \{2, 4, 6\}$.

(a) Is $\mathcal{P} = \{A, B, C\}$ a partition of $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$?

(b) Is $\mathcal{P} = \{A, B, C\}$ a partition of $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$?

Theorem 2.4:

(a) An equivalence relation R gives rise to a partition $\mathcal{P}$ in which the members of $\mathcal{P}$ are the equivalence classes of R.

(b) Conversely, a partition $\mathcal{P}$ induces an equivalence relation R in which two elements are related by R whenever they lie in the same member of $\mathcal{P}$. Moreover, the equivalence classes of this relation are the members of $\mathcal{P}$.

What does this mean?

Example: Write the equivalence relation on $\{1, 2, 3, 4, 5\}$ that is induced by the partition with $\{2, 3\}, \{1, 4\}, \{5\}$ as its partitioning subsets.

Example: Let $A = \{1, 2, 3, 4, 5\}$ and suppose that R is an equivalence relation on A. Suppose we also know $1R3$, $4R5$, and $2R5$ are true and $2R3$ is not true. Determine the following:

(a) Is $4R2$ true? Why or why not?

(b) Is $1R5$ true? Why or why not?

(c) What is the equivalence class $[4]$?