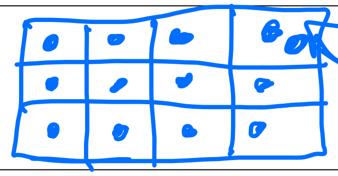
1 The Pigeonhole Principle



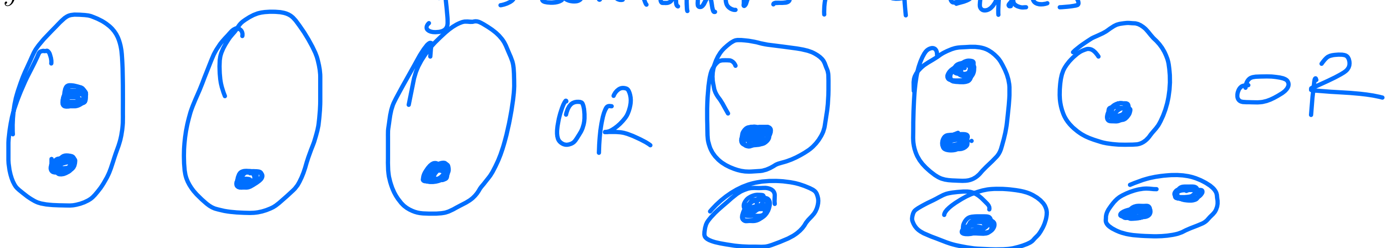
Theorem 8.4:

If pigeons are placed into pigeonholes and there are more pigeons than pigeonholes, then some pigeonhole must contain at least two pigeons. (Not using pigeons: if you have n objects and m containers and $n > m$, at least one container contains more than one object.)

More generally, if the number of pigeons (n) is more than k times the number of pigeonholes (m), then some pigeonhole must contain at least $k+1$ pigeons. (Not using pigeons: if the number of objects is more than k times the number of containers, then some container must contain at least $k+1$ objects.)

$n = mk+1$

Why should this make sense? e.g. 3 containers, 4 balls.



Example: How many English words must be chosen in order to assure at least two begin with the same letter? 26 letters in alphabet

26+1 27

Example: If a committee varies its meeting days, how many meetings must it schedule before we can guarantee that at least two meetings will be held on the same day of the week?

7 days in week

7+1

2 The Multiplication Principle

Theorem 8.5

Consider a procedure that is composed of a sequence of k steps. Suppose that the first step can be performed in n_1 ways; and for each of these, the second step can be performed in n_2 ways; and in general, no matter how the preceding steps are performed, the ith step can be performed in n_i ways ($i = 2, 3, ..., k$). Then the number of different ways in which the entire procedure can be performed is $n_1 * n_2 * ... * n_k$.

Note: the number of options in each step cannot depend on the choice of option at any prior step.

1 – n_1
2 – n_2
3 – n_3
4 – n_4
⋮
k – n_k

Example: Suppose you have two caps (stocking cap and baseball cap) and three coats (raincoat, wintercoat, and windbreaker) and you are willing to wear any cap with any coat.

(a) How many cap/coat combinations could you wear if you must wear 1 cap and 1 coat and are willing to wear any combination of cap with coat?

$2 \cdot 3 = 6$

Cap choice	Coat choice
stocking	raincoat
	wintercoat
	windbreaker
baseball	raincoat
	wintercoat
	windbreaker

(b) How many cap/coat combinations could you wear if you can also wear a cap without a coat, coat without a cap, or neither?

$3 \cdot 4 = 12$

Cap choice	Coat choice
none	none
	raincoat
	wintercoat
	windbreaker
Stocking	none
	raincoat
	wintercoat
	wi–
baseball	none
	n–

Example: How many pizzas can be ordered if a pizza can be selected with any combination of the following ingredients: sausage, pepperoni, tomato, spinach, onion, and feta?

yes
no

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6 = 64$

Example: Suppose we want to use the digits 1-9 to make four-digit positive numbers.

(a) How many different four-digit positive integers can be made if digits can be repeated?

9
1–9 options

$9 \cdot 9 \cdot 9 \cdot 9 = 9^4$

(b) How many different four-digit positive integers can be made if digits cannot be repeated?

9 8
1–9 all but previous

$9 \cdot 8 \cdot 7 \cdot 6 = \_\_\_\_$

(c) How many different four-digit positive integers can be made that start with 5 (repetition of digits permitted)?

1 9
5 only option 1–9

$1 \cdot 9 \cdot 9 \cdot 9 = 9^3$

3 The Addition Principle

Theorem 8.6:

> Suppose that there are k sets of elements with n_1 elements in the first set, n_2 elements in the second set, etc. If all of the elements are distinct (that is, if all pairs of the k sets are disjoint/the sets are pairwise disjoint) then the number of elements in the union of the sets is $n_1+n_2+...+n_k$.

Example: Suppose 2 Freshmen, 3 Sophomores, 4 Juniors, and 5 Seniors submit an application for one scholarship. How many different winners could the scholarship have?

2 + 3 + 4 + 5

4 Combining Skills

Example: Suppose that a license plate must contain a sequence of two letters followed by 5 digits or 3 letters followed by 3 digits. How many different license plates can be made? (rep. allowed)

26 26 10 10 10 10 10 + 26 26 26 10 10 10

$26^2 \cdot 10^5 + 26^3 \cdot 10^3 =$ 85,176,000

$26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 + 26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 =$ 30,888,000

Example: In a departmental awards ceremony, the math department will present awards to 5 seniors and 4 juniors. In how many different orders can the awards be presented under the following conditions:

(a) The awards can be presented in any order.

real pool: 9 people

$9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 =$ 362880

(b) Awards are presented to juniors before awards are presented to seniors.

$4 \cdot 3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 =$ 2880

(c) The first and last awards are presented to juniors.

$4 \cdot \underbrace{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}_{\text{anyone}} \cdot 3 = 60480$

juniors

(d) The first and last awards are presented to seniors.

$5 \cdot \underbrace{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}_{\text{anyone}} \cdot 4 = 100{,}800$

seniors

0 2 2 2

~~2 2 2~~

Example: Suppose we want to use the digits 0-9 to make four-digit positive numbers.

(a) How many different four-digit positive integers can be made if digits can be repeated?

$9 \cdot 10 \cdot 10 \cdot 10 = 9 \cdot 10^3$

N N N N

9 9 8 7

(b) How many different four-digit positive integers can be made if digits cannot be repeated?

$9 \cdot 9 \cdot 8 \cdot 7 = 9^2 \cdot 8 \cdot 7$

× × × ×

2 3 × ×
2 × 3 ×
2 × × 3
× × 2 3
× 2 × 3
× 2 3 ×

2 N 3 N

(c) How many different four-digit positive integers (no digit repetition) can be made that contain 2 and 3?

$\underbrace{2}_{\text{2 or 3}} \cdot \underbrace{1}_{\text{other}} \cdot \underbrace{8}_{\text{0-9 except 2\&3}} \cdot \overset{\text{0-9 except prev 2\&3}}{7} + \underbrace{7}_{\text{1,4-9}} \cdot \underbrace{2}_{\text{2 or 3}} \cdot \overset{\text{other}}{1} \cdot \underbrace{7}_{\text{0,1,4-9 except first digit}} + 7 \cdot 7 \cdot 2 \cdot 1 + 2 \cdot 8 \cdot 1 \cdot 7$

$+ 2 \cdot 8 \cdot 7 \cdot 1 + 7 \cdot 2 \cdot 7 \cdot 1 =$ ________

(d) How many different four-digit positive integers can be made without any digits being repeated (across those numbers)?

2

_ _ _ _ _ _ _ _ _ _ _ _ _ _

at best 0-9 used   repeat

2 3 x x
2 x 3 x
2 x x 3
x x 2 3
x 2 x 3
x 2 3 x

2 2 x x
2 · 1 · 8 · 7

2 x x 3
2 8 7 1

2 x 3 x
2 · 8 · 1 7

x 2 3 x
7 2 1 7

x 2 x 3
7 2 7 1

x x 2 3
7 · 7 2 1