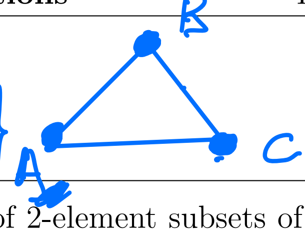
$V = \{A, B, C\}$

$\mathcal{E} = \{\{A,B\}, \{B,C\}, \{A,C\}\}$



$\{A, A\} = \{A\}$

# 1 Simple Graphs

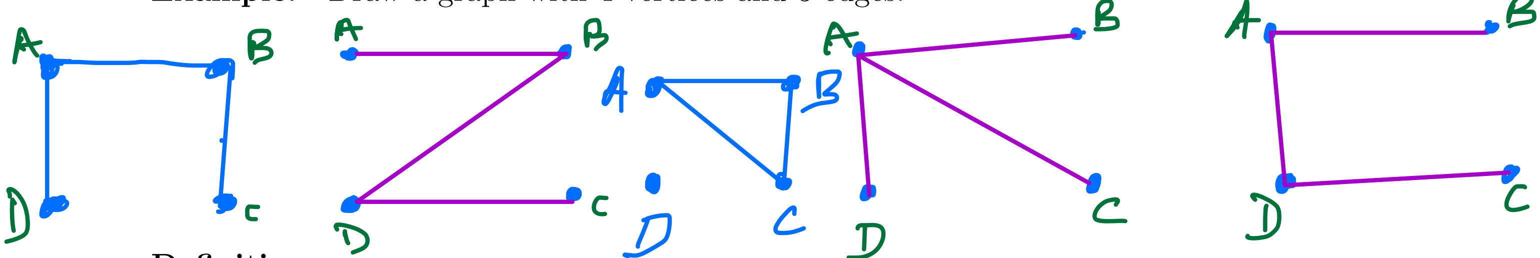
**Definitions:**

A simple graph is a nonempty finite set $\mathcal{V}$ along with a set $\mathcal{E}$ of 2-element subsets of $\mathcal{V}$. The elements of $\mathcal{V}$ are called vertices, and the elements of $\mathcal{E}$ are called edges.

**Example:** Based on this definition, can a vertex have an edge to itself in a (simple) graph? NO!

- Cannot repeat the same element and get a 2-element set.
- Needs two distint elements for a 2-element subset.

**Example:** Draw a graph with 4 vertices and 3 edges.



**Definitions:**

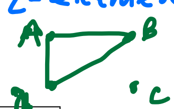
Whenever we have an edge $e = \{U, V\}$, we say that the edge $e$ joins vertices $U$ and $V$ and that $U$ and $V$ are adjacent. We also say edge $e$ is incident with the vertex $U$ and vertex $U$ is incident with edge $e$.

$\{A,B\} = \{B,A\}$

**Example:** Must a vertex be adjacent to at least one other vertex in a simple graph? Can the same two vertices have more than one edge connecting them? Why or why not? NO

NO - Once one subset representing an edge exists, we do not permit repetition of that element in the set of 2-element subsets

- NO requirement for a particular element to be in any 2-element subset describing edges.



**Definitions:**

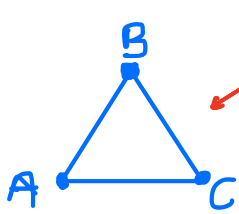
In a graph, the number of edges incident with a vertex $V$ is called the degree of $V$ and is denoted $\deg(V)$.

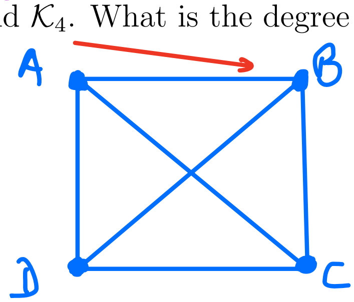
In the complete graph on $n$ vertices, denoted $\mathcal{K}_n$, every vertex is adjacent to all other vertices.

Complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge.

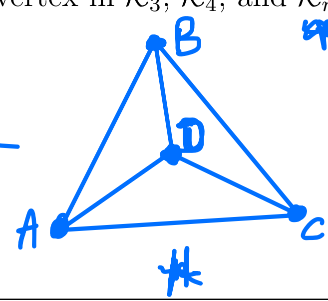
**Example:** Draw $\mathcal{K}_3$ and $\mathcal{K}_4$. What is the degree of each vertex in $\mathcal{K}_3$, $\mathcal{K}_4$, and $\mathcal{K}_n$?

$n - 1$





OR



Each vertex adjacent to all vertices except itself.

**Theorem 4.1:**

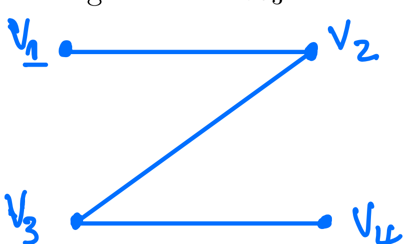
In a simple graph, the sum of the degrees of the vertices equals twice the number of edges.

Each edge is incident with 2 vertices, so adding the degrees of all vertices counts each edge twice.

**Definitions:**

Suppose we have a graph $\mathcal{G}$ with $n$ vertices labeled $V_1$, $V_2$,...,$V_n$. Such a graph is a labeled graph. To represent a labeled graph $\mathcal{G}$ by a matrix, we create an $n \times n$ matrix, where the $i, j$ entry is 1 if there is an edge between $V_i$ and $V_j$ and 0 if not. Such a matrix is the adjacency matrix of $\mathcal{G}$ and denoted $A(\mathcal{G})$. An adjacency list lists each vertex followed by the vertices adjacent to it.

**Example:** Create an adjacency matrix and adjacency list for the graph with 4 vertices and 3 edges and for $\mathcal{K}_3$.




$$\begin{array}{c|c} & V_1 \quad V_2 \quad V_3 \quad V_4 \\ \begin{array}{c} V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} & \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

$V_1$:
$V_2$:
$V_3$:
$V_4$:

*What are some properties adjacency matrices should always have?*

- 0's on main diagonal
- same number of rows and columns (all vertices represent both ways
- symmetry over diagonal (symmetric matrix $A = A^T$)

**Theorem 4.2:**

The sum of the entries in row $i$ of the adjacency matrix of a graph is the degree of the vertex $V_i$ in the graph.

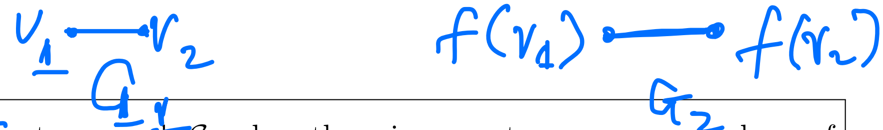
## 2 Isomorphism

Isomorphism, informally, is focused on determining which graphs are essentially the same. Isomorphism comes from the Greek root "isos", meaning 'same,' and "morphos", meaning 'structure'. Because the essential aspects of graphs are the relationships among vertices, what the graph looks like is not really important; rather, we want to know whether the same pattern of adjacency exists. *Why might mathematicians talk about graphs being the same "up to isomorphism"?*

Because any other differences are irrelevant — the picture, the labels/names etc. don't change the underlying relationships between vertices as denoted by edges.



iff $x \Longleftrightarrow y$



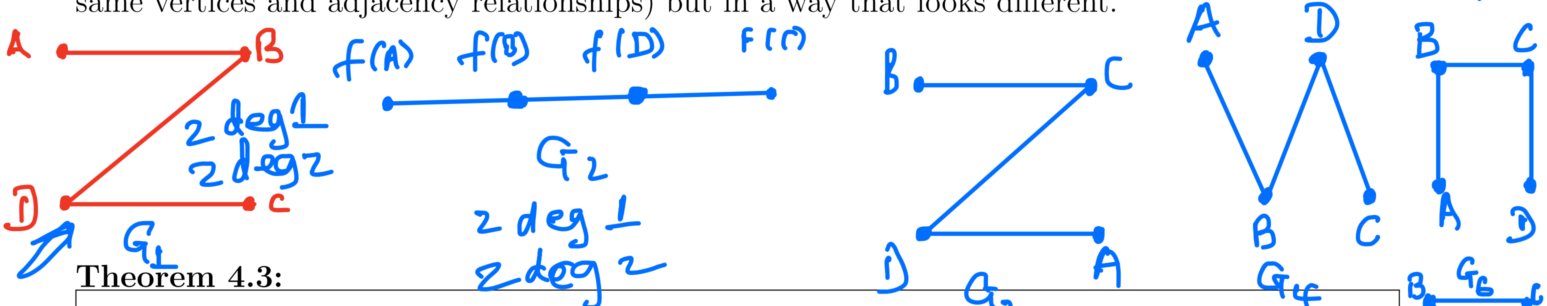

**Definitions:**

A graph $\mathcal{G}_1$ is isomorphic to a graph $\mathcal{G}_2$ when there is a one-to-one correspondence $f$ between the vertices of $\mathcal{G}_1$ and $\mathcal{G}_2$ such that the vertices $U$ and $W$ are adjacent in $\mathcal{G}_1$ if and only if the vertices $f(U)$ and $f(W)$ are adjacent in $\mathcal{G}_2$. The function $f$ is called an isomorphism between $\mathcal{G}_1$ and $\mathcal{G}_2$. "Isomorphic to" is an equivalence relation so we generally say $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic rather than specifying a first and second graph.

$G_1 \cong G_2 \quad G_2 \cong G_3$

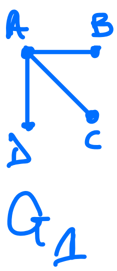
$\Rightarrow G_1 \cong G_3$

**Example:** Draw a graph isomorphic to the graph from before with 4 vertices and 3 edges (i.e., same vertices and adjacency relationships) but in a way that looks different.



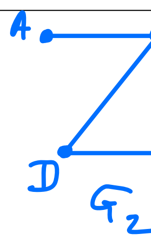

**Theorem 4.3:**

Let $f$ be an isomorphism of graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. For any vertex $V$ in $\mathcal{G}_1$, the degrees of $V$ and $f(V)$ are equal. $G_2$

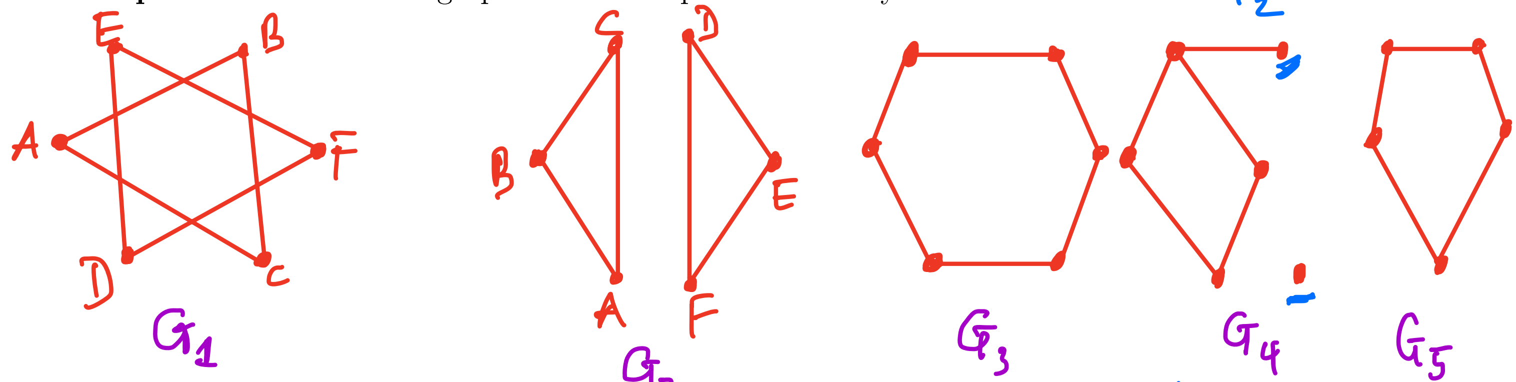



*Why is this useful?*

We can compare the degrees of vertices in two graphs to see if there are same no of vertices of degree 0, 1, ..., n




**Example:** Which of these graphs are isomorphic? How do you know?




can reach any vertex starting at any vertex

degree 0

5 vertices

$G_1 \cong G_2$

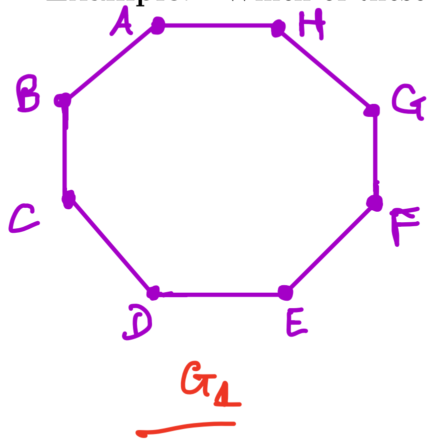
**Definition:**

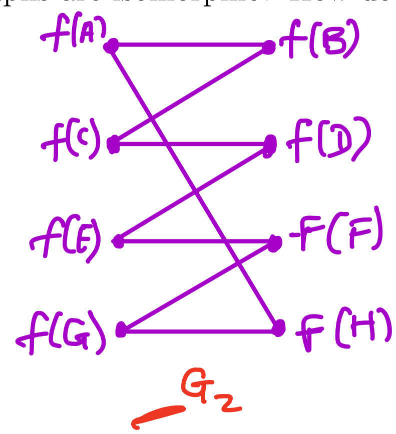
A property is said to be graph isomorphism invariant if, whenever $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic graphs and $\mathcal{G}_1$ has the property, so does $\mathcal{G}_2$.

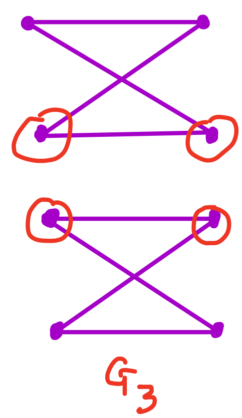
**Example:** What are some properties that should be invariant under an isomorphism?

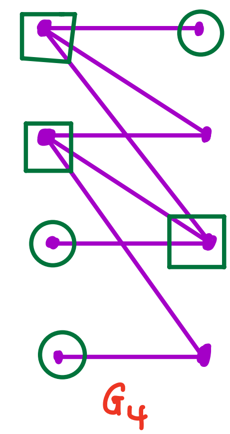
- number of vertices
- number of edges
- number of vertices of the same degree
- connected

**Example:** Which of these graphs are isomorphic? How do you know?



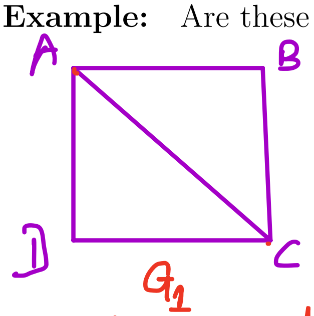



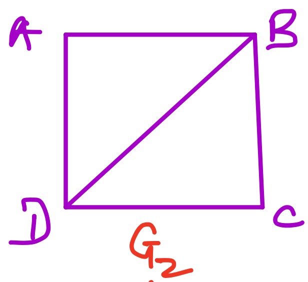




$G_1 \cong G_2$ $G_2 \not\cong G_3$

**Example:** Are these graphs isomorphic? Why or why not?

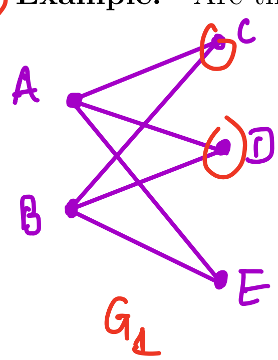


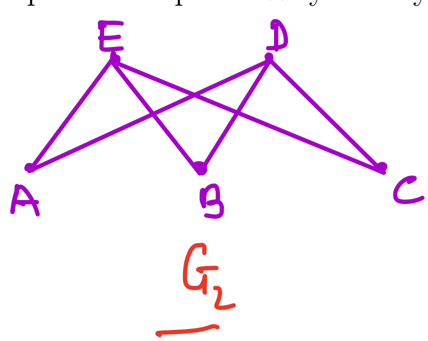



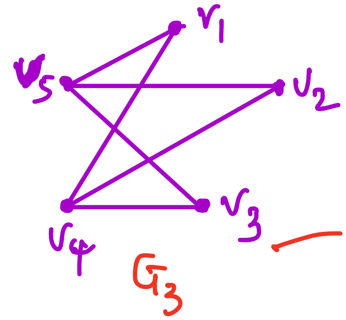

2 deg 2, 2 deg 3 2 deg 2, 2 deg 3

- same adjacency pattern
so isomorphic
* names of vertices are NOT an isomorphism invariant.

**Example:** Are these graphs isomorphic? Why or why not?



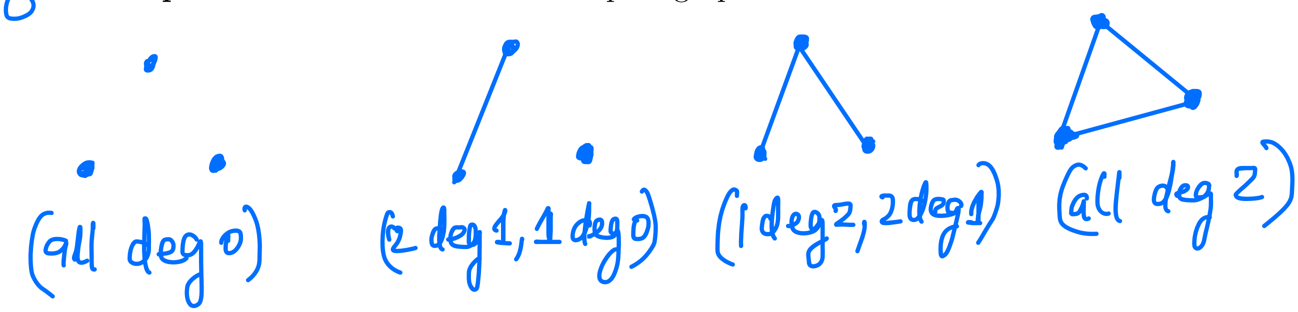

$A \to E$ $G_1 \cong G_2$
$B \to D$
$D \to B$ 2 deg 3
$E \to C$ 3 deg 2

$A \to v_1$ $G_2 \cong G_3$
$B \to v_2$
$C \to v_3$
$D \to v_4$
$E \to v_5$

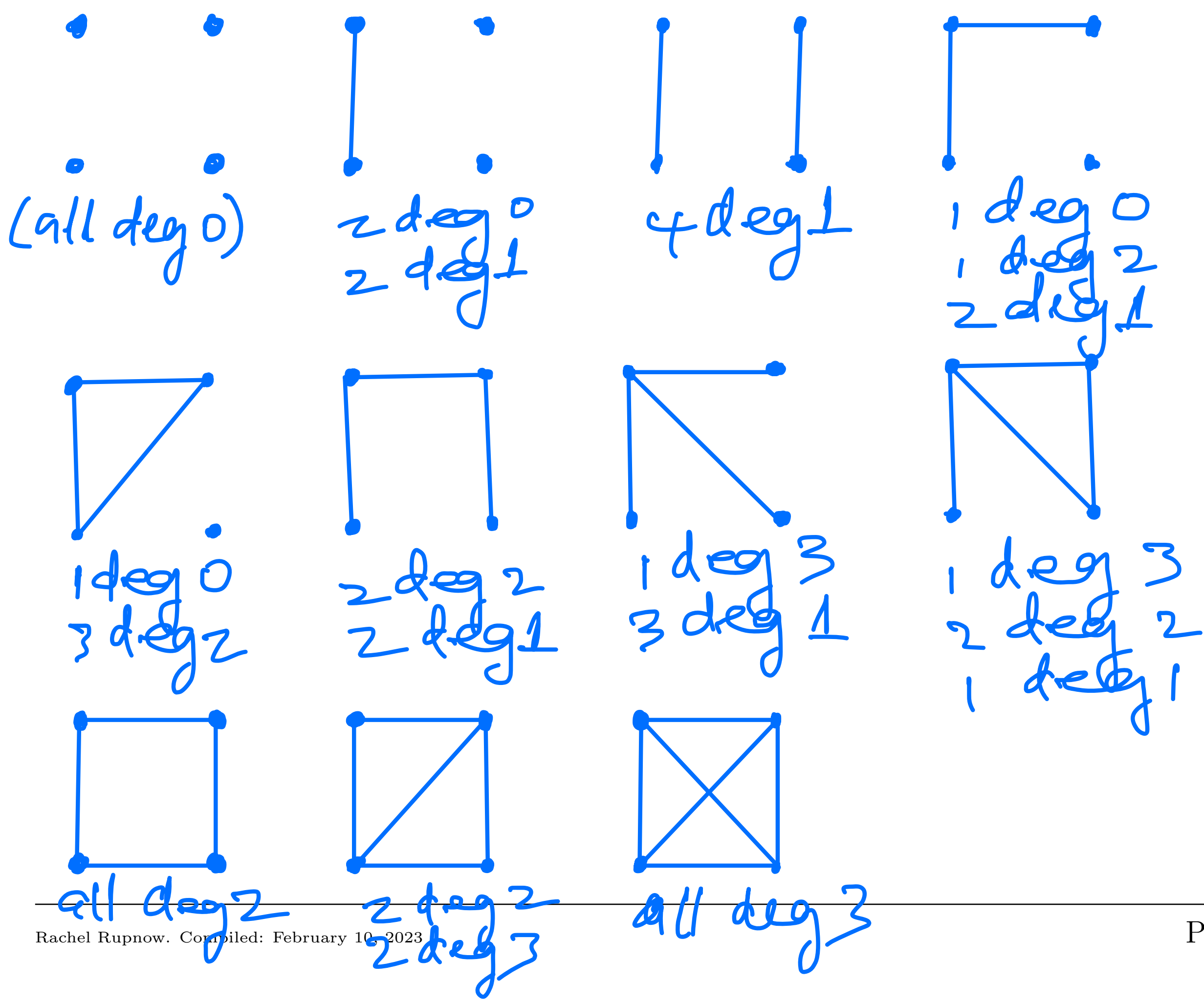
$G_1 \cong G_2$ & $G_2 \cong G_3$
$\Rightarrow G_1 \cong G_3$
(transitive property)

**Example:** Draw all of the nonisomorphic graphs with 2 vertices.

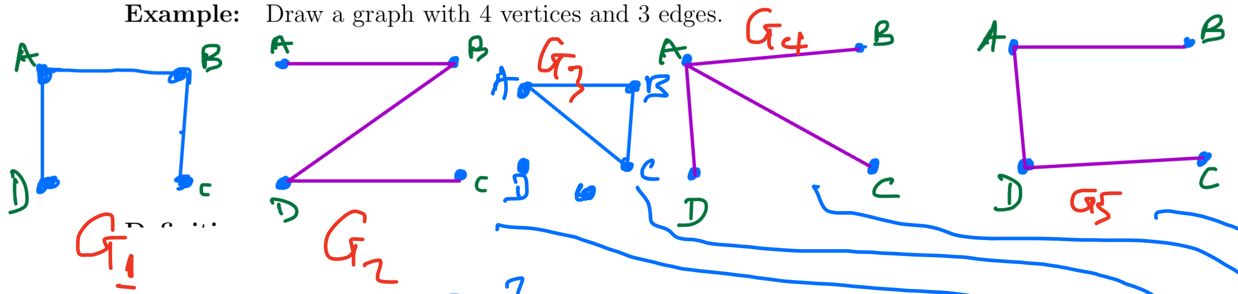

**Example:** Draw all of the nonisomorphic graphs with 3 vertices.




**Example:** Draw all of the nonisomorphic graphs with 4 vertices.



**Example:** Draw a graph with 4 vertices and 3 edges.

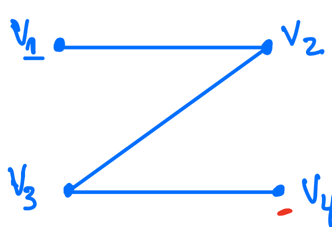



$$V_{G_1} = \{A, B, C, D\}$$

$$E_{G_1} = \{\{A, D\}, \{A, B\}, \{B, C\}\}$$

$$V = \{A, B, C, D\}$$

$$E_{G_3} = \{\{A, B\}, \{A, C\}, \{B, C\}\}$$




adjacency matrix

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

upper triangular matrix

adjacency list

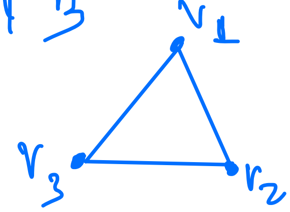
$v_1 : v_2$

$v_2 : v_1, v_3$

$v_3 : v_2, v_4$

$v_4 : v_3$

$C_3$




adjacency matrix

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

diagonal elements

adjacency list

$v_1 : v_2, v_3$

$v_2 : v_1, v_3$

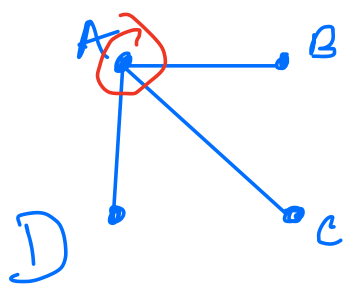
$v_3 : v_1, v_2$

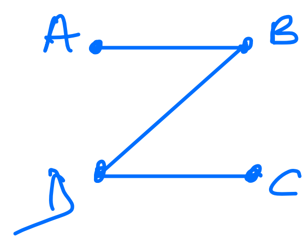
$f$ — 1-1 correspondence

$$G_1 \xrightarrow{f} G_2$$

$v_1 — v_2$ $\qquad$ $f(v_1) — f(v_2)$

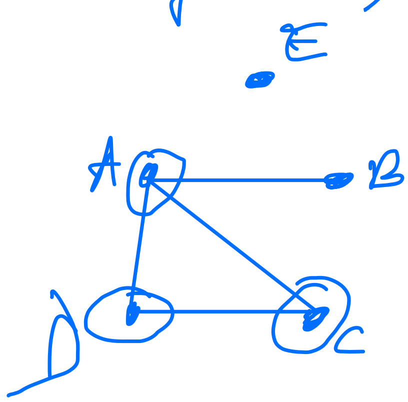
$G_1$ $\qquad\qquad$ $G_2$




no of vertices same $\qquad$ $G_1 \ncong G_2$

no of edges same



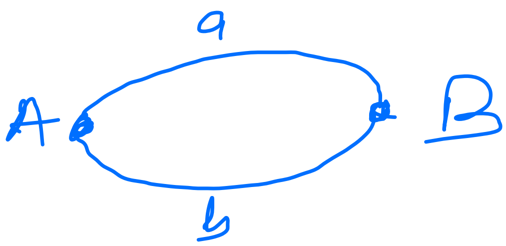

deg A = 3

deg D = 2

deg C = 2

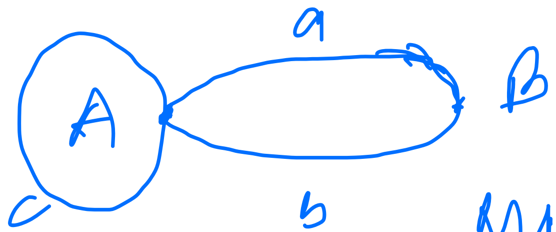
deg B = 1

deg E = 0




parallel

95%




Multigraph