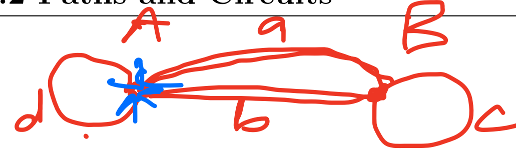
1 Multigraphs

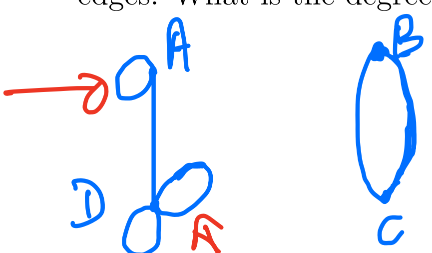


deg A = 4
deg B = 4

Definitions:

A multigraphs is a generalization of a simple graph and consists of a nonempty finite set of vertices and a set of edges, where we allow an edge to join a vertex to itself or a different vertex and where we allow several edges to join the same pair of vertices. An edge from a vertex to itself is called a loop. When there is more than one edge between two vertices, these edges are called parallel edges. The number of edges incident with a vertex V is called the degree of V and denoted deg(V). (A loop is counted twice in deg(V).)

Example: Create a multigraph on 4 vertices with at least one loop and at least one set of parallel edges. What is the degree of each vertex?



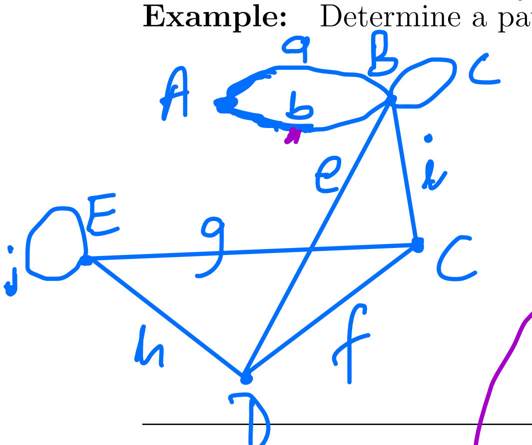
deg(A) = 3
deg(B) = 2
deg(C) = 2
deg(D) = 5

Definitions:

Suppose $\mathcal{G}$ is a multigraph and U and V are vertices, not necessarily distinct. A (u-v) path or a path (from u to v) is an alternating sequence $V_1, e_1, V_2, e_2, ..., V_n, e_n, V_{n+1}$ of vertices and edges in which the first vertex V_1 is U, the last vertex V_{n+1} is V, and edge e_i joins vertices V_i and V_{i+1} for $i = 1, 2, ..., n$. The Length of this path is n, the number of edges listed. U is a path to itself of length 0. If there is no chance of confusion (e.g., in a simple graph), we can choose to represent a path by only its vertices or only its edges. A u-v simple path is a path from U to V in which no vertex and, hence, no edge is repeated.

* A, a, B, i, C, f, D, h, E ← simple path

Example: Determine a path and a simple path from A to E in the multigraph below.



1 A, a, B, c, B, i, C, g, E — path but not simple path

A, b, ~~B~~, e, D, ~~f~~, C, g, E

2 A, a, B, i, C, g, E — path & simple path

* A, b, B, a, A, b, B, i, C, f, D, h, E

3 A, a, B, e, D, f, C, g, E — path & simple path

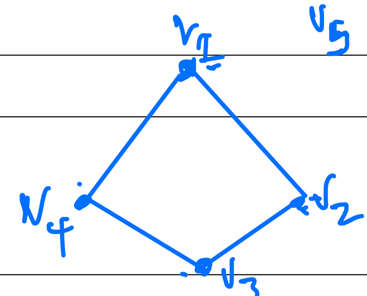
Paths & simple paths between vertices need not be unique.

A, b, B, a, C, F, D, c, E - simple

Theorem 4.4:

Every U-V path contains a U-V simple path.

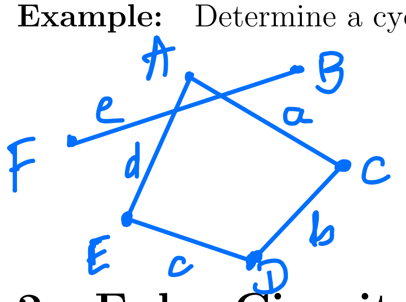
$v_5 = v_1$



Definition:

A multigraph is called connected if there is a path between every pair of vertices. A cycle is a path $V_1, e_1, V_2, e_2, ..., V_n, e_n, V_{n+1}$ where $n > 0, V_1 = V_{n+1}$, and all vertices and edges on the path are distinct.

Example: Determine a cycle in the graph below. Is this graph connected? Why or why not?



A, a, C, b, D, c, E, d, A — cycle

not connected - no path from A to B (for instance)

2 Euler Circuits and Paths

What if we want to model traveling each street once to look at Halloween/Christmas/etc. decorations in a festive part of town? How could we use multigraphs to model this?

Could model road intersections as vertices and streets as edges.

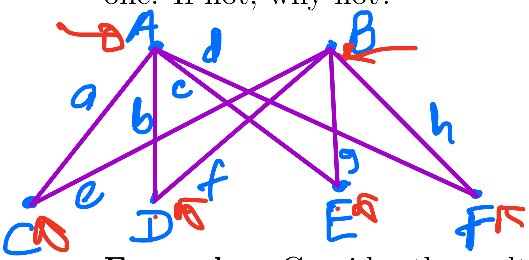
Definitions:

A path in a multigraph $\mathcal{G}$ that includes exactly once all the edges of $\mathcal{G}$ and has different first and last vertices is called an Euler path. A path that includes exactly once all of the edges of $\mathcal{G}$ and has the same first and last vertex is called an Euler circuit.

Theorem 4.5:

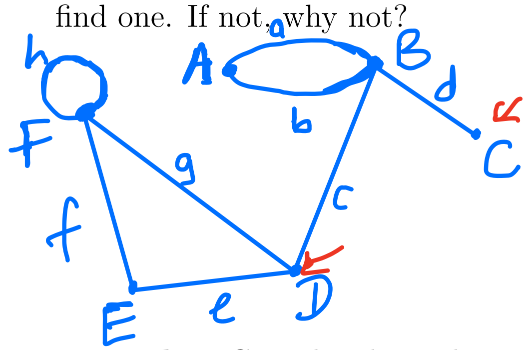
Suppose a multigraph $\mathcal{G}$ is connected. Then $\mathcal{G}$ has an Euler circuit if and only if every vertex has even degree. Furthermore, $\mathcal{G}$ has an Euler path if and only if every vertex has even degree except for two distinct vertices, which have odd degree. When this is the case, the Euler path starts at one and ends at the other of these two vertices of odd degree.

Example: Consider the graph shown here. Does it permit an Euler path or circuit? If so, find one. If not, why not?



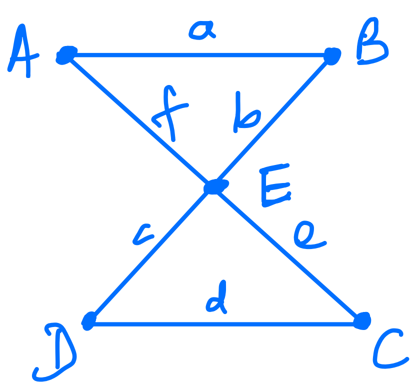
A, a, C, e, B, f, D, b, A, c, E, g, B, h, F, d, A

Example: Consider the multigraph shown here. Does it permit an Euler path or circuit? If so, find one. If not, why not?



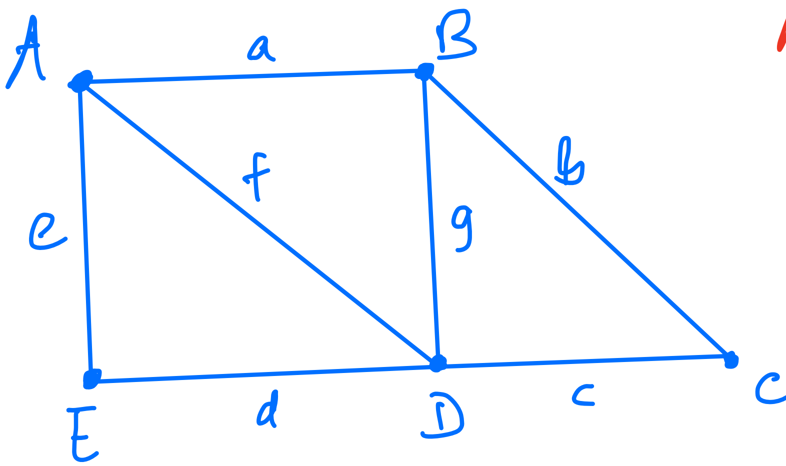
D, e, E, f, F, h, F, g, D, c, B, b, A, a, B, d, C

Example: Consider the multigraph shown here. Does it permit an Euler path or circuit? If so, find one. If not, why not?



A, a, B, b, E, c, D, d, C, e, E, f, A

Example: Consider the multigraph shown here. Does it permit an Euler path or circuit? If so, find one. If not, why not?



A, e, E, d, D, f, A, a, B, g, D, c, C, b, B

B - - — —

3 Hamiltonian Cycles and Paths

Suppose a truck driver is delivering goods around town. To save time, they don't want to visit the same location multiple times. How could we use graphs to model this? Make a graph representing each location with a vertex and roads with edges. Try to avoid revisiting vertices

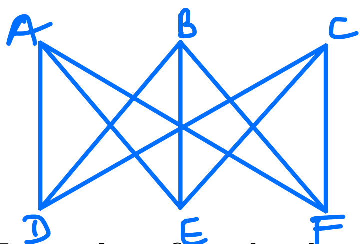
Definitions:

In a graph, a Hamiltonian path is a path that contains each vertex of the graph exactly once. A Hamiltonian cycle is a cycle that includes each vertex of the graph once (except the start/end vertex which is used twice). Note: the existence of a Hamiltonian cycle implies the existence of a Hamiltonian path, but the existence of a Hamiltonian path does not imply the existence of a Hamiltonian cycle.

Theorem 4.6:

Suppose a graph $\mathcal{G}$ is a graph with n vertices, where $n > 2$. If for each pair of nonadjacent vertices U and V we have $\deg(U) + \deg(V) \geq n$, then $\mathcal{G}$ has a Hamiltonian cycle.

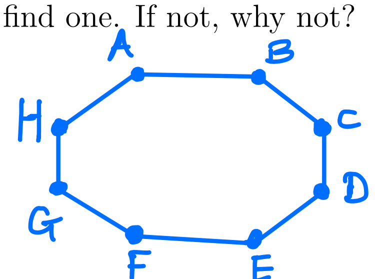
Example: Consider the graph shown here. Does it permit a Hamiltonian path or cycle? If so, find one. If not, why not?



A, D, B, E, C, F – Hamiltonian path

A, D, B, E, C, F, A – Hamiltonian cycle

Example: Consider the graph shown here. Does it permit a Hamiltonian path or cycle? If so, find one. If not, why not?



Can't use theorem 4.6

BUT we have both a Hamiltonian cycle and path

Path: A, B, C, D, E, F, G, H

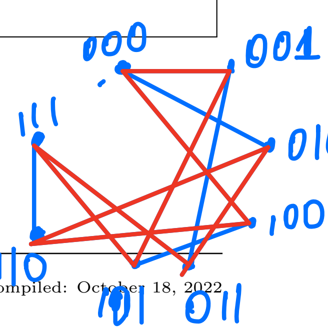
Cycle: A, B, C, D, E, F, G, H, A

Definition:

A Gray code is a listing of n-bit strings in which each n-bit string differs from the preceding string in exactly 1 position and the last n-bit string differs from the first string in exactly 1 position.

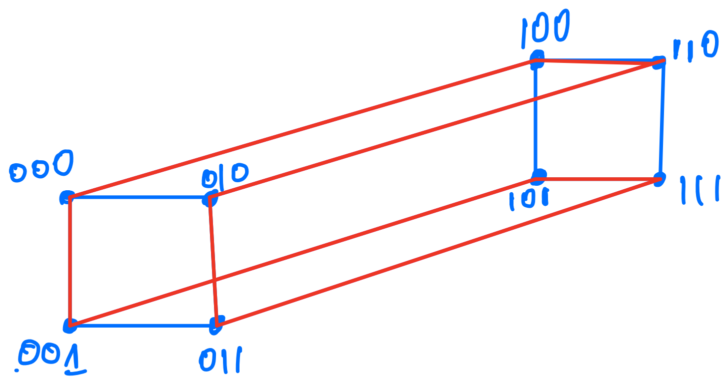
Example: Use Hamiltonian cycles to find a Gray code for $n = 3$.

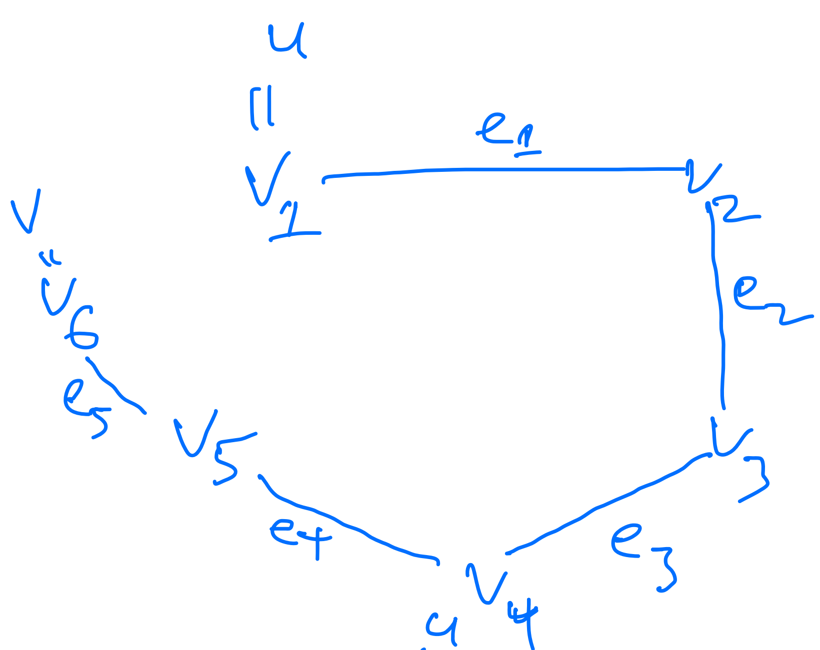
construct a graph with 2^n vertices representing all possible n-bit strings and use edges to represent strings that differ in 1 position



Rachel Rupnow. Compiled: October 18, 2022

Gray code / Hamiltonian cycle is
000, 001, 101, 111, 011, 010, 110, 100, 000





u - v path

u - v path $\quad$ $u = v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5, e_5, v_6 = v$