# 1 Characteristics of Functions

**Definition:**

For sets $X$ and $Y$, a ________________________ is a relation from $X$ to $Y$ with the property that, for each element $x$ in $X$, there is *exactly one* element $y$ in $Y$ such that $x\ f\ y$. Note that because a relation from $X$ to $Y$ is a subset of $X \times Y$, a function is a subset $S$ of $X \times Y$ such that for each $x \in X$, there is a unique $y \in Y$ with $(x, y) \in S$ (notation: $f : X \to Y$).

**Example:** Suppose $X = \{1, 3, 5, 7\}$ and $Y = \{2, 4, 6, 8\}$.

(a) Is $f = \{(1, 2), (3, 4), (5, 6), (7, 8)\}$ a function from $X$ to $Y$? Why or why not?

(b) Is $g = \{(1, 2), (3, 2), (5, 2), (7, 2)\}$ a function from $X$ to $Y$? Why or why not?

(c) Is $h = \{(1, 2), (1, 4), (1, 6), (1, 8)\}$ a function from $X$ to $Y$? Why or why not?

(d) Is $j = \{(1, 2), (3, 4), (5, 6)\}$ a function from $X$ to $Y$? Why or why not?

**Definition:**

For function $f : X \to Y$, the ____________ is the set of all possible inputs, here $X$.

The unique element of $Y$ such that $x\ f\ y$ is called the __________________ (notation: $f(x)$).

The ______ is the set of all images under function $f$.

____________: The set $Y$, which contains the range of function $f$.

**Example:** Suppose $k : X \to Y$, where $X = Y = \{x : x \text{ is a real number}\}$ is given by $k(x) = x^2$.

(a) What is the image of $x = 3$?

(b) What are the domain, codomain, and range of $k$?

**Example:** Suppose $m : X \to Y$, where $X = \{x : x \geq 0\}$, $Y = \{y : y \text{ is a real number}\}$ is given by $m(x) = x + 5$.

(a) What is the image of $x = 3$?

(b) What are the domain, codomain, and range of $m$?

**Example:** Suppose $n : X \to Y$, where $X = Y = \{x : x \text{ is a real number}\}$ is given by $n(x) = x^3 - x$.

(a) What is the image of $x = 3$?

(b) What are the domain, codomain, and range of $n$?

**Example:** Suppose $p : X \to Y$, where $X = \{9, 10, 11, 12\}$ and $Y = \{0, 1, 2\}$ is given by $p(x)$ is the remainder when $x$ is divided by 3.

(a) What is the image of $x = 9$?

(b) What are the domain, codomain, and range of $p$?

**Definition:**

A function $f : X \to Y$ is called ____________ if $f(x_1) = f(x_2)$ implies $x_1 = x_2$; that is, for each output of function $f$, there is precisely one input that creates it.

If the range and codomain of a function are equal, then the function is ______.

A function that is both one-to-one and onto is called a ________________________.

For any set $X$, the function $I_X : X \to X$ defined by $I_X(x) = x$ is a one-to-one correspondence called the ________________________.

**Example:** Revisit functions k-p. Which are one-to-one, which are onto, and which are one-to-one correspondences?

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# 2 Composition and Inverses of Functions

**Definition:**

Suppose $f : X \to Y$ and $g : Y \to Z$. The ____________ of $g$ and $f$ is defined as the image of $x$ under $gf = g(f(x))$ for all $x \in X$.

**Example:** Suppose $X$, $Y$, and $Z$ all denote the set of real numbers. Define $f : X \to Y$ by $f(x) = x^2$ and $g : Y \to Z$ by $g(y) = 3y + 2$. Find $gf$ and $fg$.

**Example:** Suppose $X = \{x : x \geq 1\}$, $Y = \{y : y \geq 1\}$, and $Z = \{z : z \text{ is a real number}\}$. Define $f : X \to Y$ by $f(x) = \sqrt{x-1}$ and $g : Y \to Z$ by $g(y) = y^2 + 1$. Find $gf$ and $fg$.

**Definition:**

Suppose $f : X \to Y$ is a one-to-one correspondence. The function with domain $Y$ and codomain $X$ that associates to each $y \in Y$ the unique $x \in X$ such that $y = f(x)$ is the ____________ of function $f$ (notation: $f^{-1}$).

**Theorem 2.7:**

Let $f : X \to Y$ be a one-to-one correspondence. Then

(a) $f^{-1} : Y \to X$ is a one-to-one correspondence.

(b) The inverse function of $f^{-1}$ is $f$.

(c) For all $x \in X$, $f^{-1}f(x) = x$ and for all $y \in Y$, $ff^{-1}(y) = y$. That is, $f^{-1}f = I_X$ and $ff^{-1} = I_Y$.

**Example:** Which example above included inverse functions?

**Example:** Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$. Does $f$ have an inverse? If so, find it. If not,why not?

**Example:** Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^3 - 1$. Does $f$ have an inverse? If so, find it. If not,why not?

**Example:** Suppose $X = \{1, 3, 5, 7\}$ and $f : X \to X$ is given by $f = \{(1, 3), (3, 5), (5, 7), (7, 1)\}$. Does $f$ have an inverse? If so, find it. If not,why not?

*Why is paying attention to the domain and codomain of a function important for computer scientists?*

**Example:** Let $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7, 8, 9\}$, and $C = \{2, 4, 6, 8, 10\}$.
Let $f = \{(1, 7), (2, 8), (3, 5)\}$, $g = \{(5, 4), (6, 6), (7, 2), (8, 8), (9, 10)\}$, and
$h = \{(2, 9), (4, 8), (6, 6), (8, 7), (10, 5)\}$. Determine whether each of the following statements are true or false and explain.

(a) $f$ is a one-to-one function from $A$ to $B$.

(b) $g$ is an onto function from $B$ to $C$.

(c) $h$ is a one-to-one function from $C$ to $B$.

(d) $h \circ g(7) = 9$

(e) $g \circ h(8) = 8$

(f) $h \circ g$ is onto.


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