Equivalence relations and classes, partial order

Definition:
A relation on S that is reflexive, symmetric, and transitive is called an **equivalence relations**.

Definition:
A relation on S that is reflexive, antisymmetric, and transitive is called a **partial order**.

Exercises :

1. In Examples 1 through 8, determine if the relations are equivalence or partial order.
2. Can a partial ordering be an equivalence relation? Must a partial ordering be an equivalence relation? Justify your reasoning.

Definition:
If R is an equivalence relation on a set S and $x \in S$, the set of elements of S that are related to x is called the **equivalence class** containing x (notation: $[x]$) ($[x] = \{a \in S : a\ R\ x\}$).

Examples :

1. Suppose S is the positive integers. For $x, y \in S$, define $x\ R\ y$ to mean that x and y have the same parity (both odd or both even). Describe the equivalence class containing 7. Describe the equivalence class containing 14. How many distinct equivalence classes of R exist?

2. Suppose S is the set of real numbers and for $x, y \in S$, $x \; R \; y$ means $x^2 = y^2$. Describe the equivalence class containing 7. Describe the equivalence class containing 14. How many distinct equivalence classes of R exist?

3. Let S be the set of ordered pairs of positive integers. Define R on S so that (x_1, x_2) R (y_1, y_2) means that $x_1 + y_2 = y_1 + x_2$. Describe the equivalence class containing $z = (5, 8)$. How many distinct equivalence classes of R exist?

Theorem 2.3:
Let R be an equivalence relation on set S.

(a) If x and y are elements of S, then x is related to y by R if and only if $[x] = [y]$.

(b) Two equivalence classes of R are either equal or disjoint.

Example :
Suppose $R = \{(1,1),(1,3),(3,1),(3,3),(3,5),(5,1),(1,5),(5,3),(5,5),(2,2),(2,6),(6,2),(6,6),(4,4)\}$ is an equivalence relation on set $S = \{1,2,3,4,5,6\}$. Describe the following equivalence classes: [1], [2], [3], [4], [5], and [6]. How many distinct classes of R exist?

Definition:

The equivalence classes of an equivalence relation R on set S divide S into disjoint subsets.

This family of subsets of S is called a **partition** of S and has the following properties:

(a) No subset is empty.

(b) Each element of S belongs to some subset.

(c) Two distinct subsets are disjoint.

Example :

Suppose $A = \{1, 3, 5, 7, 9\}$, $B = \{8\}$, and $C = \{2, 4, 6\}$.

(a) Is $\mathcal{P} = \{A, B, C\}$ a partition of $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$?

(b) Is $\mathcal{P} = \{A, B, C\}$ a partition of $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$?

Theorem 2.4:

(a) An equivalence relation R gives rise to a partition $\mathcal{P}$ in which the members of $\mathcal{P}$ are the equivalence classes of R.

(b) Conversely, a partition $\mathcal{P}$ induces an equivalence relation R in which two elements are related by R whenever they lie in the same member of $\mathcal{P}$. Moreover, the equivalence classes of this relation are the members of $\mathcal{P}$.

Example :

Write the equivalence relation on $S = \{1, 2, 3, 4, 5\}$ that is induced by the partition with $\{2, 3\}, \{1, 4\}, \{5\}$ as its partitioning subsets.

Definition: Suppose R_1 is a partial order on set S_1 and R_2 is a partial order on set S_2. Define relation R by $(a_1, a_2)\ R\ (b_1, b_2)$ if and only if one of the following is true:

(a) $a_1 \neq b_1$ and $a_1\ R_1\ b_1$ OR

(b) $a_1 = b_1$ and $a_2\ R_2\ b_2$.

This relation is called the **lexicographic order** on $S_1 \times S_2$ or the dictionary order.

Example :

Suppose Mr Webster is scheduled to interview three applicants for a summer internship at 9:00, 10:00, and 11:00 and Ms Collins is to interview three other applicants at the same times. Unfortunately both Mr Webster and Ms Collins have become ill so all 6 interviews are to be conducted by Ms Herrera. She has decided to schedule the applicants to be interviewed in time order, alphabetically by interviewer. Thus $S_1 = \{9:00, 10:00, 11:00\}$ and $S_2 = \{\text{Collins, Webster}\}$. What is the order of applicants and how could R_1 and R_2 be characterized?

Theorem 2.5: If R_1 is a partial order on set S_1 and R_2 is a partial order on set S_2, then the lexicographic order is a partial order on $S_1 \times S_2$.

Definition: A partial order R on a set S is called a **total order** on S if every pair of elements in S can be compared; that is, for every $x, y \in S$, $x \ R \ y$ or $y \ R \ x$.

Examples :

1. What is an example of a relation on a set that has a total order?

2. What is an example of a relation on a set that only permits partial orders?