

Intro to Analysis of Algorithms

Dynamic Programming

Chapter 4

Michael Soltys

CSU Channel Islands

[Ed: 4th, last updated: October 2, 2025]

Longest Monotone Subsequence

Input: $d, a_1, a_2, \dots, a_d \in \mathbb{N}$.

Output: $L =$ length of the longest monotone non-decreasing subsequence.

Note that a subsequence need not be consecutive, that is $a_{i_1}, a_{i_2}, \dots, a_{i_k}$ is a monotone subsequence provided that

$$1 \leq i_1 < i_2 < \dots < i_k \leq d,$$

$$a_{i_1} \leq a_{i_2} \leq \dots \leq a_{i_k}.$$

Dynamic Prog approach

1. Define an array of sub-problems
2. Find the recurrence
3. Write the algorithm

We first define an array of subproblems: $R(j)$ = length of the longest monotone subsequence which ends in a_j . The answer can be extracted from array R by computing $L = \max_{1 \leq j \leq n} R(j)$.

The next step is to find a recurrence. Let $R(1) = 1$, and for $j > 1$,

$$R(j) = \begin{cases} 1 & \text{if } a_i > a_j \text{ for all } 1 \leq i < j \\ 1 + \max_{1 \leq i < j} \{R(i) \mid a_i \leq a_j\} & \text{otherwise} \end{cases}.$$

Algorithm 18

```
1:  $R(1) \leftarrow 1$ 
2: for  $j : 2..d$  do
3:      $\max \leftarrow 0$ 
4:     for  $i : 1..j - 1$  do
5:         if  $R(i) > \max$  and  $a_i \leq a_j$  then
6:              $\max \leftarrow R(i)$ 
7:         end if
8:     end for
9:      $R(j) \leftarrow \max + 1$ 
10: end for
```

Question

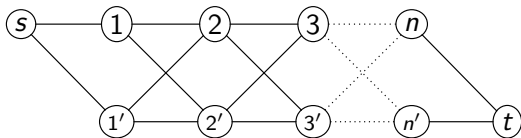
Once R , and L , have been computed how do we build the *actual* monotone subsequence?

All pairs shortest path

Input: Directed graph $G = (V, E)$, $V = \{1, 2, \dots, n\}$, and a cost function $C(i, j) \in \mathbb{N}^+ \cup \{\infty\}$, $1 \leq i, j \leq n$, $C(i, j) = \infty$ if (i, j) is not an edge.

Output: An array D , where $D(i, j)$ the length of the shortest directed path from i to j .

Exponentially many paths Problem: 4.5



Define an array of subproblems: let $A(k, i, j)$ be the length of the shortest path from i to j such that all *intermediate* nodes on the path are in $\{1, 2, \dots, k\}$. Then $A(n, i, j) = D(i, j)$ will be the solution. The convention is that if $k = 0$ then $\{1, 2, \dots, k\} = \emptyset$.

Define a recurrence: we first initialize the array for $k = 0$ as follows: $A(0, i, j) = C(i, j)$.

Now we want to compute $A(k, i, j)$ for $k > 0$.

To design the recurrence, notice that the shortest path between i and j either includes k or does not.

Assume we know $A(k - 1, r, s)$ for all r, s .

Suppose node k is not included. Then, obviously, $A(k, i, j) = A(k - 1, i, j)$.

If, on the other hand, node k occurs on a shortest path, then it occurs exactly once, so $A(k, i, j) = A(k - 1, i, k) + A(k - 1, k, j)$.

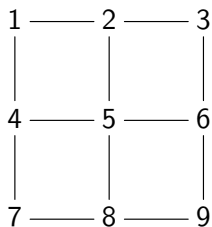
Therefore, the shortest path length is obtained by taking the minimum of these two cases:

$$A(k, i, j) = \min\{A(k - 1, i, j), A(k - 1, i, k) + A(k - 1, k, j)\}.$$

Algorithm 19

```
1: for  $i : 1..n$  do
2:     for  $j : 1..n$  do
3:          $B(i,j) \leftarrow C(i,j)$ 
4:     end for
5: end for
6: for  $k : 1..n$  do
7:     for  $i : 1..n$  do
8:         for  $j : 1..n$  do
9:              $B(i,j) \leftarrow \min\{B(i,j), B(i,k) + B(k,j)\}$ 
10:        end for
11:    end for
12: end for
13: return  $D \leftarrow B$ 
```

Example



$k = 0$ can be read directly from the graph
(assume all edges worth 1).

$k = 1$								
	1	∞	1	∞	∞	∞	∞	∞
		1	2	1	∞	∞	∞	∞
			∞	∞	1	∞	∞	∞
				1	∞	1	∞	∞
					1	∞	1	∞
						∞	∞	1
							1	∞
								1

$k = 2$								
	1	2	1	2	∞	∞	∞	∞
		1	2	1	∞	∞	∞	∞
			3	2	1	∞	∞	∞
				1	∞	1	∞	∞
					1	∞	1	∞
						∞	∞	1
							1	∞
								1

The “overwriting” trick

“Overwriting” not a problem on line 9 of algorithm.

Bellman-Ford algorithm: §4.2.1

$$\text{OPT}(i, v) = \min\{\text{OPT}(i-1, v), \min_{w \in V}\{c(v, w) + \text{OPT}(i-1, w)\}\}$$

where $\text{OPT}(i, v)$ is the shortest i -path from v to t (we want the shortest path from s to t).

Knapsack Problem

Input: $w_1, w_2, \dots, w_d, C \in \mathbb{N}$, where C is the knapsack's capacity.

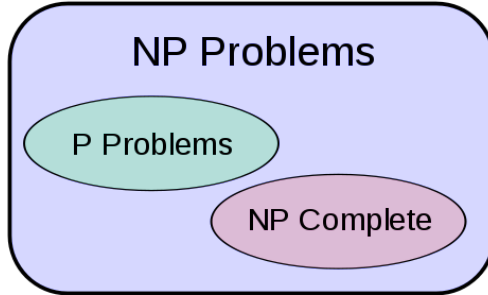
Output: $\max_S \{K(S) | K(S) \leq C\}$, where $S \subseteq [d]$ and

$$K(S) = \sum_{i \in S} w_i.$$

First example of an NP-hard problem.

CHOTCHKIES RESTAURANT	
~ APPETIZERS ~	
MIXED FRUIT	2.15
FRENCH FRIES	2.75
SIDE SALAD	3.35
HOT WINGS	3.55
MOZZARELLA STICKS	4.20
SAMPLER PLATE	5.80
~ SANDWICHES ~	
BARBECUE	6.55







Define an array of subproblems: we consider the first i weights (i.e., $[i]$) summing up to an *intermediate* weight limit j .

We define a Boolean array R as follows:

$$R(i, j) = \begin{cases} \text{T} & \text{if } \exists S \subseteq [i] \text{ such that } K(S) = j \\ \text{F} & \text{otherwise} \end{cases},$$

for $0 \leq i \leq d$ and $0 \leq j \leq C$.

Once we have computed all the values of R we can obtain the solution M as follows: $M = \max_{j \leq C} \{j \mid R(d, j) = \text{T}\}$.

Define a recurrence: we initialize $R(0, j) = F$ for $j = 1, 2, \dots, C$, and $R(i, 0) = T$ for $i = 0, 1, \dots, d$.

We now define the recurrence for computing R , for $i, j > 0$, in a way that hinges on whether we include object i in the knapsack.

Suppose that we do *not* include object i . Then, obviously, $R(i, j) = T$ iff $R(i - 1, j) = T$.

Suppose, on the other hand, that object i is included. Then it must be the case that $R(i, j) = \text{T}$ iff $R(i - 1, j - w_i) = \text{T}$ and $j - w_i \geq 0$, i.e., there is a subset $S \subseteq [i - 1]$ such that $K(S)$ is exactly $j - w_i$ (in which case $j \geq w_i$).

R	0	\dots	$j-w_i$	\dots	j	\dots	C
0	T	F...F	F	F...F	F	F...F	F
	T						
	\vdots						
	T						
$i-1$	T		c		b		
i	T				a		
	T						
	\vdots						
	T						
d	T						

Putting it all together we obtain the following recurrence for $i, j > 0$:

$$R(i, j) = T \iff R(i-1, j) = T \vee (j \geq w_i \wedge R(i-1, j-w_i) = T).$$

```

1:  $S(0) \leftarrow T$ 
2: for  $j : 1..C$  do
3:      $S(j) \leftarrow F$ 
4: end for
5: for  $i : 1..d$  do
6:     for decreasing  $j : C..1$  do
7:         if  $(j \geq w_i \text{ and } S(j - w_i) = T)$  then
8:              $S(j) \leftarrow T$ 
9:         end if
10:    end for
11: end for

```

General Knapsack Problem

Input: $w_1, w_2, \dots, w_d, v_1, \dots, v_d, C \in \mathbb{N}$

Output: $\max_{S \subseteq [d]} \{V(S) | K(S) \leq C\}$, $K(S) = \sum_{i \in S} w_i$,
 $V(S) = \sum_{i \in S} v_i$.

$$V(i, j) = \max\{V(S) | S \subseteq [i] \text{ and } K(S) = j\},$$

for $0 \leq i \leq d$ and $0 \leq j \leq C$.

Problem: what is the recurrence for this problem?

Approximating SKS

Greedy “solution” to SKS:

order the weights from heaviest to lightest, keep adding for as long as possible.

Let M be the optimal solution, and let \bar{M} be the solution obtained from the greedy approach.

Performance: $1/2$.

Let S_0 be the set of weights we got from greedy, so $K(S_0) = \bar{M}$.

If $S_0 = \emptyset$, then $\bar{M} = M$.

If $S_0 = S$ (all weights in), then $\bar{M} = M$.

OTHERWISE:

Assume we throw out weights greater than C (they won't be added anyway). Let w_j be the first weight that has been rejected, after some weights have been added

Activity Selection

Input: A list of activities $(s_1, f_1, p_1), \dots, (s_n, f_n, p_n)$, where $p_i > 0$, $s_i < f_i$ and s_i, f_i, p_i are non-negative real numbers.

Output: A set $S \subseteq [n]$ of selected activities such that no two selected activities overlap, and the profit $P(S) = \sum_{i \in S} p_i$ is as large as possible.

An *activity* i has a fixed start time s_i , finish time f_i and profit p_i . Given a set of activities, we want to select a subset of non-overlapping activities with maximum total profit.

Define an array of subproblems: sort the activities by their finish times, $f_1 \leq f_2 \leq \dots \leq f_n$.

As it is possible that activities finish at the same time, we select the *distinct* finish times, and denote them $u_1 < u_2 < \dots < u_k$, where, clearly, $k \leq n$.

For instance, if we have activities finishing at times 1.24, 4, 3.77, 1.24, 5 and 3.77, then we partition them into four groups: activities finishing at times $u_1 = 1.24$, $u_2 = 3.77$, $u_3 = 4$, $u_4 = 5$.

Let u_0 be $\min_{1 \leq i \leq n} s_i$, i.e., the earliest start time. Thus,

$$u_0 < u_1 < u_2 < \dots < u_k,$$

as it is understood that $s_i < f_i$. Define an array $A(0..k)$ as follows:

$$A(j) = \max_{S \subseteq [n]} \{P(S) \mid S \text{ is feasible and } f_i \leq u_j \text{ for each } i \in S\},$$

where S is *feasible* if no two activities in S overlap. Note that $A(k)$ is the maximum possible profit for all feasible schedules S .

Define a recurrence for $A(0..k)$.

In order to give such a recurrence we first define an auxiliary array $H(1..n)$ such that $H(i)$ is the index of the largest distinct finish time no greater than the start time of activity i .

Formally, $H(i) = \ell$ if ℓ is the largest number such that $u_\ell \leq s_i$. To compute $H(i)$, we need to search the list of distinct finish times.

To do it efficiently, for each i , apply the binary search procedure that runs in logarithmic time in the length of the list of distinct finish times (try $\ell = \lfloor \frac{k}{2} \rfloor$ first).

Since the length k of the list of distinct finish times is at most n , and we need to apply binary search for each element of the array $H(1..n)$, the time required to compute all entries of the array is $O(n \log n)$.

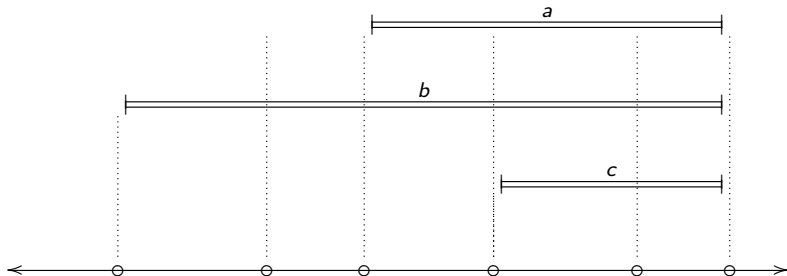
We initialize $A(0) = 0$, and we want to compute $A(j)$ given that we already have $A(0), \dots, A(j-1)$.

Consider $u_0 < u_1 < u_2 < \dots < u_{j-1} < u_j$.

Can we beat profit $A(j-1)$ by scheduling some activity that finishes at time u_j ? Try all activities that finish at this time and compute maximum profit in each case. We obtain the following recurrence:

$$A(j) = \max\{A(j-1), \max_{1 \leq i \leq n} \{p_i + A(H(i)) \mid f_i = u_j\}\},$$

where $H(i)$ is the greatest ℓ such that $u_\ell \leq s_i$.



$$s_b = u_H(b)$$

$$u_H(a)$$

$$s_b$$

$$s_c = u_H(c)$$

$$u_{j-1}$$

$$u_j$$

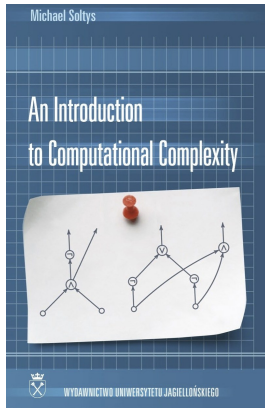
```

 $A(0) \leftarrow 0$ 
for  $j : 1..k$  do
     $\max \leftarrow 0$ 
    for  $i = 1..n$  do
        if  $f_i = u_j$  then
            if  $p_i + A(H(i)) > \max$  then
                 $\max \leftarrow p_i + A(H(i))$ 
            end if
        end if
    end for
    if  $A(j-1) > \max$  then
         $\max \leftarrow A(j-1)$ 
    end if
     $A(j) \leftarrow \max$ 
end for

```

Introduction to Complexity

This material is not in the IAA textbook but here:



A TM M is of *time complexity* $T(n)$ if whenever M is given an input w , $|w| = n$, then M *halts* after making at most $T(n)$ many moves.

$L \in \text{TIME}(f(n))$ if there exists a deterministic TM M of time complexity $O(f(n))$ that decides L .

$L \in \text{NTIME}(f(n))$ if there exists a nondeterministic TM M of time complexity $O(f(n))$ that decides L .

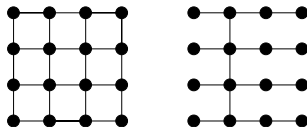
L is in the class P if $L \in \text{TIME}(n^k)$ for some fixed k .

L is in the class NP if $L \in \text{NTIME}(n^k)$ for some fixed k .

Observation: $P \subseteq NP$; Question: $NP \subseteq P$?

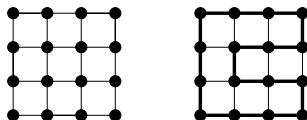
Ex. of a language in P:

$\{\langle G, k \rangle \mid G \text{ has a spanning tree of weight } \leq k\}$. ($k = 15$)

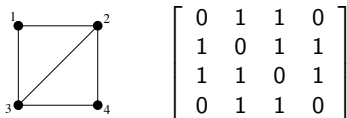


Ex. of a language in NP believed not to be in P:

$\{\langle G, k \rangle \mid G \text{ has a complete cycle of weight } \leq k\}$. ($k = 16$)



A graph G can be encoded as an adjacency matrix. For example, the graph given below would have the adjacency matrix given by:



If P is a *decision problem*, the related language L_P consists of the encodings (under some fixed convention) of all the “yes” instances of P .

Feasibility Thesis:

Polynomial time algorithm \equiv polynomial time TM.

A problem P_1 is *reducible in polynomial time* to a problem P_2 if there exists a polynomial time function f such that:

$$\langle I \rangle \in L_{P_1} \iff \langle f(I) \rangle \in L_{P_2}$$

L is *NP-complete* if:

1. $L \in \text{NP}$
2. Every language $L' \in \text{NP}$ is polynomial time reducible to L .

Ex. Traveling Salesman Problem

L is NP-complete is *evidence* of L not being in P

(see *Computers and Intractability* by Michael Garey and David Johnson.)

Theorem: If P_1 is NP-complete, P_2 is in NP, and there is a polynomial time reduction of P_1 to P_2 , then P_2 is also NP-complete.

Proof: Every language L in NP is reducible to L_{P_1} , by completeness, and P_1 is reducible to P_2 . Enough to show transitivity of reductions.

Theorem: If some NP-complete problem P is in P, then $P=NP$.

Proof: Follows from the fact that all languages in NP are polynomial time reducible to P .

Satisfiability

Boolean Expressions are built from: Boolean variables x, y, z, \dots , Boolean values 0, 1, and Boolean connectives: \vee, \wedge, \neg , and parenthesis.

Ex. $\neg x \vee (y \wedge z)$

If ϕ is a Boolean expression, then a *truth assignment* T is an assignment of truth values to the variables of ϕ .

Ex. $T(x) = 0, T(y) = 1, T(z) = 1$, then
 $T(\neg x \vee (y \wedge z)) = \neg 0 \vee (1 \wedge 1) = 1 \vee 1 = 1$.

T *satisfies* ϕ if $T(\phi) = 1$, and ϕ is *satisfiable* if $\exists T$ s.t. $T(\phi) = 1$.

The *satisfiability problem* is: given a Boolean expression, is it satisfiable?

$\text{SAT} = \{\langle \phi \rangle \mid \phi \text{ is satisfiable}\}$

(i.e., SAT is the language corresponding to the satisfiability problem).

Cook's Theorem: SAT is NP-complete.

PROOF: SAT is in NP.

Let L be any language in NP.

We show there exists a polynomial time function f s.t.:

$$w \in L \iff f(w) = \phi \in \text{SAT}$$

\exists non-det TM M s.t. $L = L(M)$ and M always halts within n^k many steps on inputs w , $|w| = n$, for fixed k .

Given w , f outputs a Boolean formula ϕ which encodes a computation of M on w and is satisfiable $\iff M$ accepts w .