Intro to Analysis of Algorithms
Computational Foundations
Section 9.4
Chapter 9

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Part I
Context-free languages

A *context-free grammar (CFG)* is $G = (V, T, P, S)$ — Variables, Terminals, Productions, Start variable

Ex. $P \longrightarrow \varepsilon|0|1|0P0|1P1$.

Ex. $G = (\{E, I\}, T, P, E)$ where $T = \{+, *, (,), a, b, 0, 1\}$ and P is the following set of productions:

$$\begin{aligned} E &\longrightarrow I|E+E|E*E|(E) \\ I &\longrightarrow a|b|Ia|Ib|I0|I1 \end{aligned}$$

If $\alpha A\beta \in (V \cup T)^*$, $A \in V$, and $A \longrightarrow \gamma$ is a production, then $\alpha A\beta \Rightarrow \alpha\gamma\beta$. We use $\stackrel{*}{\Rightarrow}$ to denote 0 or more steps.

$L(G) = \{w \in T^* | S \stackrel{*}{\Rightarrow} w\}$

Lemma: $L((\{P\},\{0,1\},\{P\longrightarrow \varepsilon|0|1|0P0|1P1\},P))$ is the set of palindromes over $\{0,1\}$.

Proof: Suppose w is a palindrome; show by induction on $|w|$ that $P\overset{*}{\Rightarrow} w$.

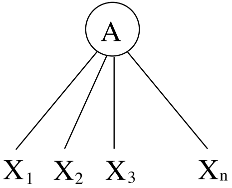
BS: $|w|\leq 1$, so $w=\varepsilon,0,1$, so use $P\longrightarrow\varepsilon,0,1$.
IS: For $|w|\geq 2$, $w=0x0,1x1$, and by IH $P\overset{*}{\Rightarrow} x$.

Suppose that $P\overset{*}{\Rightarrow} w$; show by induction on the number of steps in the derivation that $w=w^R$.

BS: Derivation has 1 step.
IS: $P\Rightarrow 0P0\overset{*}{\Rightarrow} 0x0=w$ (or with 1 instead of 0).

If $S \stackrel{*}{\Rightarrow} \alpha$, then $\alpha \in (V \cup T)^*$, and α is called a *sentential form*. $L(G)$ is the set of those sentential forms which are in T^*.

Given $G = (V, T, P, S)$, the *parse tree* for (G, w) is a tree with S at the root, the symbols of w are the leaves (left to right), and each interior node is of the form:



whenever we have a rule $A \longrightarrow X_1 X_2 X_3 \ldots X_n$

Derivation: head $\longrightarrow$ body
Recursive Inference: body $\longrightarrow$ head

The following five are all equivalent:

1. Recursive Inference
2. Derivation
3. Left-most derivation
4. Right-most derivation
5. Yield of a parse tree.

Ambiguity of Grammars

$E \Rightarrow E + E \Rightarrow E + E * E$
$E \Rightarrow E * E \Rightarrow E + E * E$

Two different parse trees! Different meaning.

A grammar is ambiguous if there exists a string w with two different parse trees.

A *Pushdown Automaton (PDA)* is an ε-NFA with a stack.

Two (equivalent) versions: (i) accept by final state, (ii) accept by empty stack.

PDAs describe CFLs.

The PDA pushes and pops symbols on the stack; the stack is assumed to be as big as necessary.

Ex. What is a simple PDA for $\{ww^R|w \in \{0,1\}^*\}$?

Formal definition of a PDA:

$P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$

Q finite set of states

Σ finite input alphabet

Γ finite stack alphabet, $\Sigma \subseteq \Gamma$

$\delta(q, a, X) = \{(p_1, \gamma_1), \ldots, (p_n, \gamma_n)\}$

if $\gamma = \varepsilon$, then the stack is popped, if $\gamma = X$, then the stack is unchanged, if $\gamma = YZ$ then X is replaced Z, and Y is pushed onto the stack

q_0 initial state

Z_0 start symbol

F accepting states

A *configuration* is a tuple (q, w, γ): state, remaining input, contents of the stack

If $(p, \alpha) \in \delta(q, a, X)$, then $(q, aw, X\beta) \rightarrow (p, w, \alpha\beta)$

Theorem: If $(q, x, \alpha) \rightarrow^* (p, y, \beta)$, then $(q, xw, \alpha\gamma) \rightarrow^* (p, yw, \beta\gamma)$

Acceptance by final state:
$L(P) = \{w | (q_0, w, Z_0) \rightarrow^* (q, \varepsilon, \alpha), q \in F\}$

Acceptance by empty stack: $L(P) = \{w | (q_0, w, Z_0) \rightarrow^* (q, \varepsilon, \varepsilon)\}$

Theorem: L is accepted by PDA by final state iff it is accepted by PDA by empty stack.

Proof: When Z_0 is popped, enter an accepting state. For the other direction, when an accepting state is entered, pop all the stack.

Theorem: CFGs and PDAs are equivalent.

Proof: From Grammar to PDA: A left sentential form is $\underbrace{x}_{\in T^*}\overbrace{A\alpha}^{\text{tail}}$

The tail appears on the stack, and x is the prefix of the input that has been consumed so far.

Total input is $w = xy$, and hopefully $A\alpha \stackrel{*}{\Rightarrow} y$.

Suppose PDA is in $(q, y, A\alpha)$. It guesses $A \longrightarrow \beta$, and enters $(q, y, \beta\gamma)$.

The initial segment of β, if it has any terminal symbols, they are compared against the input and removed, until the first variable of β is exposed on top of the stack.

Accept by empty stack.

Ex. Consider $P \longrightarrow \varepsilon|0|1|0P0|1P1$

The PDA has transitions:

$$\delta(q_0, \varepsilon, Z_0) = \{(q, PZ_0)\}$$
$$\delta(q, \varepsilon, P) = \{(q, 0P0), (q, 0), (q, \varepsilon), (q, 1P1), (q, 1)\}$$
$$\delta(q, 0, 0) = \delta(q, 1, 1) = \{(q, \varepsilon)\}$$
$$\delta(q, 0, 1) = \delta(q, 1, 0) = \emptyset$$
$$\delta(q, \varepsilon, Z_0) = (q, \varepsilon)$$

Consider: $P \Rightarrow 1P1 \Rightarrow 10P01 \Rightarrow 100P001 \Rightarrow 100001$

Z	P	1	P	0	P	0	P	0	0	1	Z	
	Z	P	1	P	0	P	0	0	1	Z		
		1	Z	0	1	0	0	1	Z			
		Z		1	Z	0	1	Z				
				Z		1	Z					
						Z						

From PDA to grammar:

Idea: "net popping" of one symbol of the stack, while consuming some input.

Variables: $A_{[pXq]}$, for $p, q \in Q$, $X \in \Gamma$.

$A_{[pXq]} \stackrel{*}{\Rightarrow} w$ iff w takes PDA from state p to state q, and pops X off the stack.

Productions: for all p, $S \longrightarrow A_{[q_0 Z_0 p]}$, and whenever we have:

$$(r, Y_1 Y_2 \ldots Y_k) \in \delta(q, a, X)$$

$A_{[qXr_k]} \longrightarrow a A_{[rY_1 r_1]} A_{[r_1 Y_2 r_2]} \ldots A_{[r_{k-1} Y_k r_k]}$
where $a \in \Sigma \cup \{\varepsilon\}$, $r_1, r_2, \ldots, r_k \in Q$ are all possible lists of states.

If $(r, \varepsilon) \in \delta(q, a, X)$, then we have $A_{[qXr]} \longrightarrow a$.

Claim: $A_{[qXp]} \stackrel{*}{\Rightarrow} w \iff (q, w, X) \rightarrow^* (p, \varepsilon, \varepsilon)$.

A PDA is deterministic if $|\delta(q, a, X)| \leq 1$, and the second condition is that if for some $a \in \Sigma$ $|\delta(q, a, X)| = 1$, then $|\delta(q, \varepsilon, X)| = 0$.

Theorem: If L is regular, then $L = L(P)$ for some deterministic PDA P.

Proof: ignore the stack.

DPDAs that accept by final state are **not** equivalent to DPDAs that accept by empty stack.

L has the *prefix property* if there exists a pair (x, y), $x, y \in L$, such that $y = xz$ for some z.

Ex. $\{0\}^*$ has the prefix property.

Theorem: L is accepted by a DPDA by empty stack $\iff$ L is accepted by a DPDA by final state **and** L does not have the prefix property.

Theorem: If L is accepted by a DPDA, then L is unambiguous.

Eliminating useless symbols from CFG:

$X \in V \cup T$ is *useful* if there exists a derivation such that $S \overset{*}{\Rightarrow} \alpha X \beta \overset{*}{\Rightarrow} w \in T^*$

X is *generating* if $X \overset{*}{\Rightarrow} w \in T^*$

X is *reachable* if there exists a derivation $S \overset{*}{\Rightarrow} \alpha X \beta$

A symbol is useful if it is generating and reachable.

Generating symbols: Every symbol in T is generating, and if $A \longrightarrow \alpha$ is a production, and every symbol in α is generating (or $\alpha = \varepsilon$) then A is also generating.

Reachable symbols: S is reachable, and if A is reachable, and $A \longrightarrow \alpha$ is a production, then every symbol in α is reachable.

If L has a CFG, then $L - \{\varepsilon\}$ has a CFG without productions of the form $A \longrightarrow \varepsilon$

A variable is *nullable* if $A \overset{*}{\Rightarrow} \varepsilon$

To compute nullable variables: if $A \longrightarrow \varepsilon$ is a production, then A is nullable, if $B \longrightarrow C_1 C_2 \dots C_k$ is a production and all the C_i's are nullable, then so is B.

Once we have all the nullable variables, we eliminate ε-productions as follows: eliminate all $A \longrightarrow \varepsilon$.

If $A \longrightarrow X_1 X_2 \dots X_k$ is a production, and $m \leq k$ of the X_i's are nullable, then add the 2^m versions of the rule the the nullable variables present/absent (if $m = k$, do not add the case where they are *all* absent).

Eliminating unit productions: $A \longrightarrow B$
If $A \overset{*}{\Rightarrow} B$, then (A, B) is a unit pair.

Find all unit pairs: (A, A) is a unit pair, and if (A, B) is a unit pair, and $B \longrightarrow C$ is a production, then (A, C) is a unit pair.

To eliminate unit productions: compute all unit pairs, and if (A, B) is a unit pair and $B \longrightarrow \alpha$ is a non-unit production, add the production $A \longrightarrow \alpha$. Throw out all the unit productions.

A CFG is in *Chomsky Normal Form* if all the rules are of the form $A \longrightarrow BC$ and $A \longrightarrow a$.

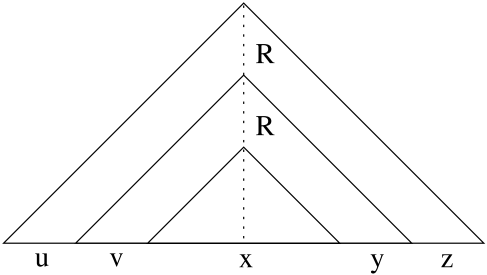
Theorem: Every CFL without ε has a CFG in CNF.

Proof: Eliminate ε-productions, unit productions, useless symbols. Arrange all bodies of length ≥ 2 to consist of only variables (by introducing new variables), and finally break bodies of length ≥ 3 into a cascade of productions, each with a body of length exactly 2.

Pumping Lemma for CFLs: There exists a p so that any s, $|s| \geq p$, can be written as $s = uvxyz$, and:

1. $uv^i xy^i z$ is in the language, for all $i \geq 0$,
2. $|vy| > 0$,
3. $|vxy| \leq p$

Proof:



Ex. The lang $\{0^n1^n2^n | n \geq 1\}$ is not CF.

So CFL are not closed under intersection: $L_1 = \{0^n1^n2^i | n, i \geq 1\}$ and $L_2 = \{0^i1^n2^n | n, i \geq 1\}$ are CF, but $L_1 \cap L_2 = \{0^n1^n2^n | n \geq 1\}$ is not.

Theorem: If L is a CFL, and R is a regular language, then $L \cap R$ is a CFL.

$L = \{ww : w \in \{0,1\}^*\}$ is not CF, but L^c is CF. So CFLs are not close under complementation either.

We design a CFG for L^c. First note that no odd strings are of the form ww, so the first rule should be:

$$S \longrightarrow O|E$$
$$O \longrightarrow a|b|aaO|abO|baO|bbO$$

here O generates all the odd strings.

E generates even length strings not of the form ww, i.e., all strings of the form:

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X=|_____0__|_____1__|   or   Y=|_____1__|_____0__|
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We need the rule:

$$E \longrightarrow X|Y$$

and now

$$\begin{array}{ll} X \longrightarrow PQ & Y \longrightarrow VW \\ P \longrightarrow RPR & V \longrightarrow SVS \\ P \longrightarrow a & V \longrightarrow b \\ Q \longrightarrow RQR & W \longrightarrow SWS \\ Q \longrightarrow b & W \longrightarrow a \\ R \longrightarrow a|b & S \longrightarrow a|b \end{array}$$

Ex.

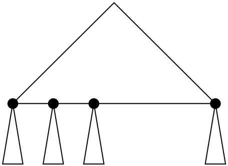
$$\begin{aligned} X &\Rightarrow PQ \Rightarrow RPRQ \Rightarrow RRPRRQ \Rightarrow RRRPRRRQ \Rightarrow RRRRPRRRRQ \\ &\Rightarrow RRRRRPRRRRRQ \Rightarrow RRRRRaRRRRRQ \Rightarrow RRRRRaRRRRRRQR \\ &\Rightarrow RRRRRaRRRRRRRQRR \Rightarrow RRRRRaRRRRRRRbRR \end{aligned}$$

and now the R's can be replaced at will by a's and b's.

CFL are closed under substitution: for every $a \in \Sigma$ we choose L_a, which we call $s(a)$. For any $w \in \Sigma^*$, $s(w)$ is the language of $x_1 x_2 \dots x_n$, $x_i \in s(a_i)$.

Theorem: If L is a CFL, and $s(a)$ is a CFL $\forall a \in \Sigma$, then $s(L) = \cup_{w \in L} s(w)$ is also CF.

Proof:



CFL are closed under union, concatenation, $*$ and $+$, homomorphism (just define $s(a) = \{h(a)\}$, so $h(L) = s(L)$), and reversal (just replace each $A \longrightarrow \alpha$ by $A \longrightarrow \alpha^R$).

We can test for emptiness: just check whether S is generating. Test for membership: use CNF of the CYK algorithm (more efficient).

However, there are many **undecidable** properties of CFL:

1. Is a given CFG G ambiguous?
2. Is a given CFL inherently ambiguous?
3. Is the intersection of two CFL empty?
4. Given G_1, G_2, is $L(G_1) = L(G_2)$?
5. Is a given CFL everything?

CYK[1] alg: Given G in CNF, and $w = a_1 a_2 \dots a_n$, build an $n \times n$ table. $w \in L(G)$ if $S \in (1, n)$. $(X \in (i,j) \iff X \overset{*}{\Rightarrow} a_i a_{i+1} \dots a_j.)$

Let $V = \{X_1, X_2, \dots, X_m\}$. Initialize T as follows:

for $(i = 1; i \leq n; i++)$

  for $(j = 1; j \leq m; j++)$ Put X_j in (i,i) iff $\exists X_j \longrightarrow a_i$

Then, for $i < j$:

for $(k = i; k < j; k++)$

  if $(\exists \quad X_p \in (i,k)$ & $X_q \in (k+1, j)$ & $X_r \longrightarrow X_p X_q)$

  Put X_r in (i,j)

x	(2,2)	(2,3)	(2,4)	**(2,5)**
x	x			(3,5)
x	x	x		(4,5)
x	x	x	x	(5,5)

[1] Cocke-Kasami-Younger

Context-sensitive grammars (CSG) have rules of the form:

$$\alpha \to \beta$$

where $\alpha, \beta \in (T \cup V)^*$ and $|\alpha| \leq |\beta|$. A language is *context sensitive* if it has a CSG.

Fact: It turns out that $\text{CSL} = \text{NTIME}(n)$

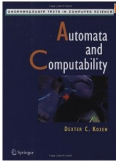
A *rewriting system* (also called a *Semi-Thue system*) is a grammar where there are no restrictions; $\alpha \to \beta$ for arbitrary $\alpha, \beta \in (V \cup T)^*$.

Fact: It turns out that a rewriting system corresponds to the most general model of computation; i.e., a language has a rewriting system iff it is "computable."

Enter Turing machines ...

Chomsky-Schutzenberger Theorem: If L is a CFL, then there exists a regular language R, an n, and a homomorphism h, such that $L = h(\text{PAREN}_n \cap R)$.

Parikh's Theorem: If $\Sigma = \{a_1, a_2, \ldots, a_n\}$, the signature of a string $x \in \Sigma^*$ is $(\#a_1(x), \#a_2(x), \ldots, \#a_n(x))$, i.e., the number of ocurrences of each symbol, in a fixed order. The signature of a language is defined by extension; regular and CFLs have the same signatures.



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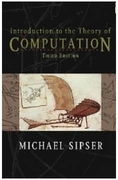

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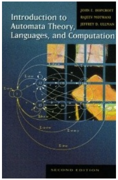

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