Intro to Analysis of Algorithms
Computational Foundations
Section 9.5
Chapter 9

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Part I
Turing machines

Finite control and an infinite tape.

Initially the input is placed on the tape, the head of the tape is reading the first symbol of the input, and the state is q_0.

The other squares contain blanks.

Formally, a *Turing machine* is a tuple $(Q, \Sigma, \Gamma, \delta)$

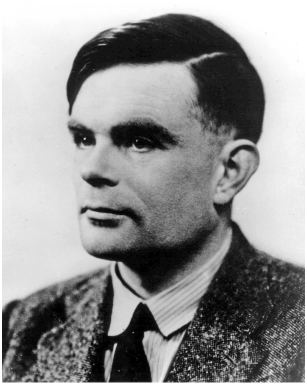
where Q is a finite set of *states* (always including the three special states q_{init}, q_{accept} and q_{reject})

Σ is a finite *input alphabet*

Γ is a finite *tape alphabet*, and it is always the case that $\Sigma \subseteq \Gamma$ (it is convenient to have symbols on the tape which are never part of the input),

$$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{\text{Left}, \text{Right}\}$$

is the *transition function*



Alan Turing

A *configuration* is a tuple (q, w, u) where $q \in Q$ is a state, and where $w, u \in \Gamma^*$, the cursor is on the last symbol of w, and u is the string to the right of w.

A configuration (q, w, u) *yields* (q', w', u') in one step, denoted as $(q, w, u) \stackrel{M}{\to} (q', w', u')$ if one step of M on (q, w, u) results in (q', w', u').

Analogously, we define $\stackrel{M^k}{\to}$, yields in k steps, and $\stackrel{M^*}{\to}$, yields in any number of steps, including zero steps.

The initial configuration, C_{init}, is $(q_{\text{init}}, \triangleright, x)$ where q_{init} is the initial state, x is the input, and $\triangleright$ is the left-most tape symbol, which is always there to indicate the left-end of the tape.

Given a string w as input, we "turn on" the TM in the initial configuration C_{init}, and the machine moves from configuration to configuration.

The computation ends when either the state q_{accept} is entered, in which case we say that the TM *accepts* w, or the state q_{reject} is entered, in which case we say that the TM *rejects* w. It is possible for the TM to never enter q_{accept} or q_{reject}, in which case the computation does not halt.

Given a TM M we define $L(M)$ to be the set of strings accepted by M, i.e., $L(M) = \{x|M \text{ accepts } x\}$, or, put another way, $L(M)$ is the set of precisely those strings x for which $(q_{\text{init}}, \triangleright, x)$ yields an accepting configuration.

Alan Turing showed the existence of a so called *Universal Turing machine* (UTM); a UTM is capable of simulating any TM from its description.

A UTM is what we mean by a *computer*, capable of running any algorithm. The proof is not difficult, but it requires care in defining a consistent way of presenting TMs and inputs.

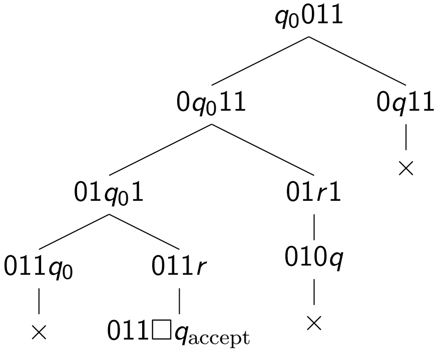
Every Computer Scientist should at some point write a UTM in their favorite programming language ...

This exercise really means: designing your own programming language (how you present descriptions of TMs); designing your own compiler (how your machine interprets those "descriptions"); etc.

NTM

N s.t. $L(N) = \{w \in \{0,1\}^* |$ last symbol of w is 1 $\}$.

$$\begin{aligned}
\delta(q_0, 0) &= \{(q_0, 0, \rightarrow), (q, 0, \rightarrow)\} \\
\delta(q_0, 1) &= \{(q_0, 1, \rightarrow), (r, 1, \rightarrow)\} \\
\delta(r, \square) &= \{(q_{\text{accept}}, \square, \rightarrow)\} \\
\delta(r, 0/1) &= \{(q, 0, \rightarrow)\}
\end{aligned}$$



Different variants of TMs are equivalent (*robustness*): tape infinite in only one direction, or several tapes.

TM = NTM: D maintains a sequence of config's on tape 1:

$\cdots$	config_1	config_2	config_3^*	$\cdots$

and uses a second tape for scratch work.

The marked config (*) is the current config. D copies it to the second tape, and examines it to see if it is accepting. If it is, it accepts.

If it is not, and N has k possible moves, D copies the k new config's resulting from these moves at the end of tape 1, and marks the next config as current.

If max nr of choices of N is m, and N makes n moves, D examines $1 + m + m^2 + m^3 + \cdots + m^n \approx nm^n$ many configs.

Undecidability

We can encode every Turing machine with a string over $\{0, 1\}$. For example, if M is a TM:

$$(\{q_1, q_2\}, \{0, 1\}, \delta, \ldots)$$

and $\delta(q_1, 1) = (q_2, 0, \rightarrow)$ is one of the transitions, then it could be encoded as:

$$\underbrace{0}_{q_1}1\underbrace{00}_{1}1\underbrace{00}_{q_2}1\underbrace{0}_{0}1\underbrace{0}_{\rightarrow}11\underbrace{\ldots\ldots\ldots\ldots\ldots\ldots}_{\text{encoding of other transitions}}$$

Not every string is going to be a valid encoding of a TM (for example the string 1 does not encode anything in our convention).

Let all "bad strings" encode a default TM M_{default} which has one state, and halts immediately, so $L(M_{\text{default}}) = \emptyset$.

The intuitive notion of algorithm is captured by the formal definition of a TM.

$$\mathsf{A}_{\mathrm{TM}} = \{\langle M, w\rangle : M \text{ is a TM and } M \text{ accepts } w\},$$

called the *universal language*

Theorem 6.63: A_{TM} is undecidable.

Suppose that it is decidable, and that H decides it. Then, $L(H) = \mathsf{A}_{\mathrm{TM}}$, and H always halts (observe that $L(H) = L(U)$, but U, as we already mentioned, is not guaranteed to be a decider). Define a new machine D (here D stands for "diagonal," since this argument follows Cantor's "diagonal argument"):

$$D(\langle M \rangle) := \begin{cases} \text{accept} & \text{if } H(\langle M, \langle M \rangle \rangle) = \text{reject} \\ \text{reject} & \text{if } H(\langle M, \langle M \rangle \rangle) = \text{accept} \end{cases}$$

that is, D does the "opposite." Then we can see that $D(\langle D \rangle)$ accepts iff it rejects. Contradiction; so A_{TM} cannot be decidable.

It turns out that all nontrivial properties of RE languages are undecidable, in the sense that the language consisting of codes of TMs having this property is not recursive.

E.g., the language consisting of codes of TMs whose languages are empty (i.e., L_e) is not recursive.

A *property* of RE languages is simply a subset of RE. A property is *trivial* if it is empty or if it is everything.

If $\mathcal{P}$ is a property of RE languages, the language $L_{\mathcal{P}}$ is the set of codes for TMs M_i s.t. $L(M_i) \in \mathcal{P}$.

When we talk about the decidability of $\mathcal{P}$, we formally mean the decidability of $L_{\mathcal{P}}$.

Rice's Theorem: Every nontrivial property of RE languages is undecidable.

Proof: Suppose $\mathcal{P}$ is nontrivial. Assume $\emptyset \notin \mathcal{P}$ (if it is, consider $\overline{\mathcal{P}}$ which is also nontrivial).

Since $\mathcal{P}$ is nontrivial, some $L \in \mathcal{P}$, $L \neq \emptyset$.

Let M_L be the TM accepting L.

For a fixed pair (M, w) consider the TM M': on input x, it first simulates $M(w)$, and if it accepts, it simulates $M_L(x)$, and if that accepts, M' accepts.

$\therefore$ $L(M') = \emptyset \notin \mathcal{P}$ if M does not accept w, and $L(M') = L \in \mathcal{P}$ if M accepts w.

Thus, $L(M') \in \mathcal{P} \iff (M, w) \in \mathrm{A}_{\mathrm{TM}}$, $\therefore$ $\mathcal{P}$ is undecidable.

Post's Correspondence Problem (PCP)

An instance of *PCP* consists of two finite lists of strings over some alphabet Σ. The two lists must be of equal length:

$A = w_1, w_2, \ldots, w_k$
$B = x_1, x_2, \ldots, x_k$

For each i, the pair (w_i, x_i) is said to be a *corresponding pair*. We say that this instance of PCP has a solution if there is a sequence of one or more indices:

$$i_1, i_2, \ldots, i_m \qquad m \geq 1$$

such that:

$$w_{i_1} w_{i_2} \ldots w_{i_m} = x_{i_1} x_{i_2} \ldots x_{i_m}$$

The PCP is: given (A, B), tell whether there is a solution.

Emil Leon Post

Aside: To express **PCP** as a language, we let L_{PCP} be the language:

$$\{\langle A, B\rangle | (A, B) \text{ instance of PCP with solution}\}$$

Example: Consider (A, B) given by:

$A = 1, 10111, 10$
$B = 111, 10, 0$

Then $2, 1, 1, 3$ is a solution as:

$$\underbrace{10111}_{w_2}\underbrace{1}_{w_1}\underbrace{1}_{w_1}\underbrace{10}_{w_3} = \underbrace{10}_{x_2}\underbrace{111}_{x_1}\underbrace{111}_{x_1}\underbrace{0}_{x_3}$$

Note that $2, 1, 1, 3, 2, 1, 1, 3$ is another solution.

On the other hand, you can check that: $A = 10, 011, 101$ & $B = 101, 11, 011$ Does not have a solution.

The **MPCP** has an additional requirement that the first pair in the solution must be the first pair of (A, B).

So $i_1, i_2, \ldots, i_m$, $m \geq 0$, is a solution to the (A, B) instance of **MPCP** if:

$$w_1 w_{i_1} w_{i_2} \ldots w_{i_m} = x_1 x_{i_1} x_{i_2} \ldots x_{i_m}$$

We say that $i_1, i_2, \ldots, i_r$ is a *partial solution* of PCP if one of the following is the prefix of the other:

$$w_{i_1} w_{i_2} \ldots w_{i_r} \qquad x_{i_1} x_{i_2} \ldots x_{i_r}$$

Same def holds for MPCP, but w_1, x_1 must be at the beginning.

We now show:

1. If **PCP** is decidable, then so is **MPCP**.
2. If **MPCP** is decidable, then so is A_{TM}.
3. Since A_{TM} is *not* decidable, neither is **(M)PCP**.

$\boxed{\textbf{PCP decidable} \Longrightarrow \textbf{MPCP decidable}}$

We show that given an instance (A, B) of MPCP, we can construct an instance (A', B') of PCP such that:

$$(A, B) \text{ has solution } \iff (A', B') \text{ has solution}$$

Let (A, B) be an instance of MPCP over the alphabet Σ. Then (A', B') is an instance of PCP over the alphabet $\Sigma' = \Sigma \cup \{*, \$\}$.

If $A = w_1, w_2, w_3, \ldots, w_k$, then
$A' = *\mathbf{w}_1*, \mathbf{w}_1*, \mathbf{w}_2*, \mathbf{w}_3*, \ldots, \mathbf{w}_k*, \$$.

If $B = x_1, x_2, x_3, \ldots, x_k$, then $B' = *\mathbf{x}_1, *\mathbf{x}_1, *\mathbf{x}_2, *\mathbf{x}_3, \ldots, *\mathbf{x}_k, *\$$.

where if $x = a_1 a_2 a_3 \ldots a_n \in \Sigma^*$, then $\mathbf{x} = a_1 * a_2 * a_3 * \ldots * a_n$.

For example: If (A, B) is an instance if MPCP given as:

$A = 1, 10111, 10$
$B = 111, 10, 0$

Then (A', B') is an instance of PCP given as follows:

$A' = *1*, 1*, 1*0*1*1*1*, 1*0*, \$$
$B' = *1*1*1, *1*1*1, *1*0, *0, *\$$

$\boxed{\textbf{MPCP} \text{ decidable} \Longrightarrow \mathsf{A}_{\mathrm{TM}} \text{ decidable}}$

Given a pair (M, w) we construct an instance (A, B) of MPCP such that:

TM M accepts $w \iff (A, B)$ has a solution.

Idea: The MPCP instance (A, B) simulates, in its partial solutions, the computation of M on w.

That is, partial solutions will be of the form:

$$\#\alpha_1\#\alpha_2\#\alpha_3\#\ldots$$

where α_1 is the initial config of M on w, and for all i, $\alpha_i \to \alpha_{i+1}$.

The string from the B list will always be one config ahead of the A list; the A list will be allowed to "catch-up" only when M accepts w.

To simplify things, we may assume that our TM M:

1. Never prints a blank.
2. Never moves left from its initial head position.

The configs of M will always be of the form $\alpha \boldsymbol{q} \beta$, where α, β are non-blank tape symbols and q is a state.

Let M be a TM and $w \in \Sigma^*$. We construct an instance (A, B) of MPCP as follows:

1. A: #
   B: $\#q_0 w\#$
2. A: $X_1, X_2, \ldots, X_n, \#$
   B: $X_1, X_2, \ldots, X_n, \#$
   where the X_i are all the tape symbols.
3. To simulate a move of M, for all non-accepting $q \in Q$:

list A	list B	
qX	Yp	if $\delta(q, X) = (p, Y, \rightarrow)$
ZqX	pZY	if $\delta(q, X) = (p, Y, \leftarrow)$
$q\#$	$Yp\#$	if $\delta(q, B) = (p, Y, \rightarrow)$
$Zq\#$	$pZY\#$	if $\delta(q, B) = (p, Y, \leftarrow)$

4. If the config at the end of B has an accepting state, then we need to allow A to catch up with B. So we need for all accepting states q, and all symbols X, Y:

list A	list B
XqY	q
Xq	q
qY	q

5. Finally, after using 4 and 3 above, we end up with $x\#$ and $x\#q\#$, where x is a long string. Thus we need $q\#\#$ in A and $\#$ in B to complete the catching up.

Ex. $\delta(q_1, 0) = (q_2, 1, \rightarrow), \delta(q_1, 1) = (q_2, 0, \leftarrow), \delta(q_1, B) = (q_2, 1, \leftarrow)$
$\delta(q_2, 0) = (q_3, 0, \leftarrow), \delta(q_2, 1) = (q_1, 0, \rightarrow), \delta(q_2, B) = (q_2, 0, \rightarrow)$

Rule	list A	list B	Source
1	#	# q_1 01#	
2	0	0	
	1	1	
	#	#	
3	$q_1 0$	$1 q_2$	$\delta(q_1, 0) = (q_2, 1, \rightarrow)$
	$0 q_1 1$	$q_2 00$	$\delta(q_1, 1) = (q_2, 0, \leftarrow)$
	$1 q_1 1$	$q_2 10$	$\delta(q_1, 1) = (q_2, 0, \leftarrow)$
	$0 q_1$#	$q_2$01#	$\delta(q_1, B) = (q_2, 1, \leftarrow)$
	$1 q_1$#	$q_2$11#	$\delta(q_1, B) = (q_2, 1, \leftarrow)$
	$0 q_2 0$	$q_3$00#	$\delta(q_2, 0) = (q_3, 0, \leftarrow)$
	$1 q_2 0$	$q_3$10#	$\delta(q_2, 0) = (q_3, 0, \leftarrow)$
	$q_2 1$	$0 q_1$	$\delta(q_2, 1) = (q_1, 0, \rightarrow)$
	q_2#	$0 q_2$#	$\delta(q_2, B) = (q_2, 0, \rightarrow)$

4	$0 q_3 0$	q_3	
	$0 q_3 1$	q_3	
	$1 q_3 0$	q_3	
	$1 q_3 1$	q_3	
	$0 q_3$	q_3	
	$1 q_3$	q_3	
	$q_3 0$	q_3	
	$q_3 1$	q_3	
5	q_3##	#	

The TM M accepts the input 01 by the sequence of moves:

$$q_1 01 \rightarrow 1 q_2 1 \rightarrow 10 q_1 \rightarrow 1 q_2 01 \rightarrow q_3 101$$

We examine the sequence of partial solutions that mimics this computation of M and eventually leads to a solution.

We must start with the first pair (MPCP):

A: #
B: #$q_1$01#

The only way to extend this partial solution is with the corresponding pair $(q_1 0, 1 q_2)$, so we obtain:

A: #$q_1$0
B: #$q_1$01#1q_2

Now using copying pairs we obtain:

A: $\#q_1 01\#1$
B: $\#q_1 01\#1q_2 1\#1$

Next corresponding pair is $(q_2 1, 0q_1)$:

A: $\#q_1 01\#1q_2 1$
B: $\#q_1 01\#1q_2 1\#10q_1$

Now careful! We only copy the next two symbols to obtain:

A: $\#q_1 01\#1q_2 1\#1$
B: $\#q_1 01\#1q_2 1\#10q_1\#1$

because we need the $0q_1$ as the head now moves left, and use the next appropriate corresponding pair which is $(0q_1\#, q_2 01\#)$ and obtain:

A: $\#q_1 01\#1q_2 1\#10q_1\#$
B: $\#q_1 01\#1q_2 1\#10q_1\#1q_2 01\#$

We can now use another corresponding pair $(1q_2 0, q_3 10)$ right away to obtain:

A: $\#q_1 01\#1q_2 1\#10q_1\#1q_2 0$
B: $\#q_1 01\#1q_2 1\#10q_1\#1q_2 01\#q_3 10$

and note that we have an accepting state! We use two copying pairs to get:

A: $\#q_1 01\#1q_2 1\#10q_1\#1q_2 01\#$
B: $\#q_1 01\#1q_2 1\#10q_1\#1q_2 01\#q_3 101\#$

and we can now start using the rules in 4. to make A catch up with B:

A: $\ldots\#q_3 1$
B: $\ldots\#q_3 101\#q_3$

and we copy three symbols:

A: $\ldots\#q_3 101\#$
B: $\ldots\#q_3 101\#q_3 01\#$

And again catch up a little:

A: $\ldots\#q_3101\#q_30$
B: $\ldots\#q_3101\#q_301\#q_3$

Copy two symbols:

A: $\ldots\#q_3101\#q_301\#$
B: $\ldots\#q_3101\#q_301\#q_31\#$

and catch up:

A: $\ldots\#q_3101\#q_301\#q_31$
B: $\ldots\#q_3101\#q_301\#q_31\#q_3$

and copy:

A: $\ldots\#q_3101\#q_301\#q_31\#$
B: $\ldots\#q_3101\#q_301\#q_31\#q_3\#$

And now end it all with the corresponding pair $(q_3\#\#, \#)$ given by rule 5. to get matching strings:

A: $\dots\#q_3101\#q_301\#q_31\#q_3\#\#$
B: $\dots\#q_3101\#q_301\#q_31\#q_3\#\#$

THEREFORE: we reduced A_{TM} to the MPCP. Now, we can solve A_{TM} by producing a carefully crafted instance of MPCP (A, B), and asking if it has a solution. If yes, then we know that M accepts w.

Since we have already shown that A_{TM} is undecidable, MPCP must also be undecidable. Thus, PCP is undecidable.

NEXT: We can now use the fact that PCP is undecidable to show that a number of questions about CFLs are undecidable.

Let $A = w_1, w_2, \ldots, w_k$, let G_A be the related CFG given by:

$$A \longrightarrow w_1 A a_1 | w_2 A a_2 | \cdots | w_k A a_k | w_1 a_1 | w_2 a_2 | \cdots | w_k a_k$$

Let $L_A = L(G_A)$, the language of the list A, and $a_1, a_2, \ldots, a_k$ are distinct index symbols not in alphabet of A.

The terminal strings of G_A are of the form:

$$w_{i_1} w_{i_2} \ldots w_{i_m} a_{i_m} \ldots a_{i_2} a_{i_1}$$

Let G_{AB} be a CFG consisting of G_A, G_B, with $S \longrightarrow A|B$.

$\therefore$ G_{AB} is ambiguous $\iff$ the PCP (A, B) has a solution.

Theorem: It is undecidable whether a CFG is ambiguous.

$\overline{L_A}$ is also a CFL; we show this by giving a PDA P.

$\Gamma_P = \Sigma_A \cup \{a_1, a_2, \ldots, a_k\}$.

As long as P sees a symbol in Σ_A it stores it on the stack.

As soon as P sees a_i, it pops the stack to see if top of string is w_i^R. (i) if not, then accept no matter what comes next. (ii) if yes, there are two subcases:

(iia) if stack is not yet empty, continue.

(iib) if stack is empty, and the input is finished, reject.

If after an a_i, P sees a symbol in Σ_A, it accepts.

Theorem: G_1, G_2 are CFGs, and R is a reg. exp., then the following are undecidable problems:

1. $L(G_1) \cap L(G_2) \stackrel{?}{=} \emptyset$
2. $L(G_1) \stackrel{?}{=} L(G_2)$
3. $L(G_1) \stackrel{?}{=} L(R)$
4. $L(G_1) \stackrel{?}{=} T^*$
5. $L(G_1) \stackrel{?}{\subseteq} L(G_2)$
6. $L(R) \stackrel{?}{\subseteq} L(G_2)$

Proofs: 1. Let $L(G_1) = L_A$ and $L(G_2) = L_B$, then $L(G_1) \cap L(G_2) \neq \emptyset$ iff PCP (A, B) has a solution.

2. Let G_1 be the CFG for $\overline{L_A} \cup \overline{L_B}$ (CFGs are closed under union). Let G_2 be the CFG for the reg. lang. $(\Sigma \cup \{a_1, a_2, \ldots, a_k\})^*$.

Note $L(G_1) = \overline{L_A} \cup \overline{L_B} = \overline{L_A \cap L_B} =$ everything but solutions to PCP (A, B).

$\therefore L(G_1) = L(G_2)$ iff (A, B) has no solution.

3. Shown in 2.

4. Again, shown in 2.

5. Note that $A = B$ iff $A \subseteq B$ and $B \subseteq A$, so it follows from 2.

6. By 3. and 5.