

Intro to Analysis of Algorithms

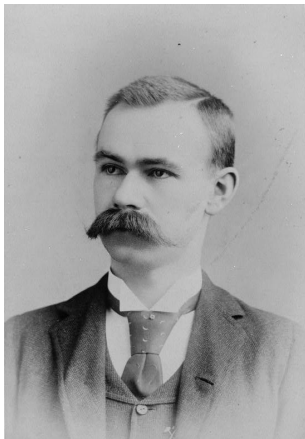
Divide & Conquer

Chapter 3

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Herman Hollerith, 1860–1929

Suppose that we have two lists of numbers that are already sorted.

That is, we have a list $a_1 \leq a_2 \leq \dots \leq a_n$ and $b_1 \leq b_2 \leq \dots \leq b_m$.

We want to combine those two lists into one long sorted list

$c_1 \leq c_2 \leq \dots \leq c_{n+m}$.

The mergesort algorithm sorts a given list of numbers by first dividing them into two lists of length $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$, respectively, then sorting each list recursively, and finally combining the results.

Pre-condition: $a_1 \leq a_2 \leq \dots \leq a_n$ and $b_1 \leq b_2 \leq \dots \leq b_m$

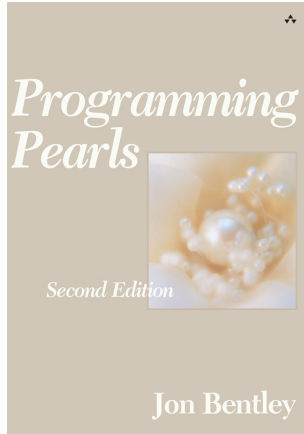
```
1:  $p_1 \leftarrow 1$ ;  $p_2 \leftarrow 1$ ;  $i \leftarrow 1$ 
2: while  $i \leq n + m$  do
3:     if  $a_{p_1} \leq b_{p_2}$  then
4:          $c_i \leftarrow a_{p_1}$ 
5:          $p_1 \leftarrow p_1 + 1$ 
6:     else
7:          $c_i \leftarrow b_{p_2}$ 
8:          $p_2 \leftarrow p_2 + 1$ 
9:     end if
10:     $i \leftarrow i + 1$ 
11: end while
```

Post-condition: $c_1 \leq c_2 \leq \dots \leq c_{n+m}$

Pre-condition: A list of integers a_1, a_2, \dots, a_n

```
1:  $L \leftarrow a_1, a_2, \dots, a_n$ 
2: if  $|L| \leq 1$  then
3:     return  $L$ 
4: else
5:      $L_1 \leftarrow$  first  $\lceil n/2 \rceil$  elements of  $L$ 
6:      $L_2 \leftarrow$  last  $\lfloor n/2 \rfloor$  elements of  $L$ 
7:     return Merge(Mergesort( $L_1$ ), Mergesort( $L_2$ ))
8: end if
```

Post-condition: $a_{i_1} \leq a_{i_2} \leq \dots \leq a_{i_n}$



Multiplication

	1	2	3	4	5	6	7	8
x					1	1	1	0
y					1	1	0	1
s ₁					1	1	1	0
s ₂				0	0	0	0	
s ₃			1	1	1	0		
s ₄		1	1	1	0			
x × y	1	0	1	1	0	1	1	0

Multiply 1110 times 1101, i.e., 14 times 13. Takes $O(n^2)$ steps.

Clever multiplication

Let x and y be two n -bit integers. We break them up into two smaller $n/2$ -bit integers as follows:

$$x = (x_1 \cdot 2^{n/2} + x_0),$$

$$y = (y_1 \cdot 2^{n/2} + y_0).$$

x_1 and y_1 correspond to the high-order bits of x and y , respectively, and x_0 and y_0 to the low-order bits of x and y , respectively.

The product of x and y appears as follows in terms of those parts:

$$\begin{aligned} xy &= (x_1 \cdot 2^{n/2} + x_0)(y_1 \cdot 2^{n/2} + y_0) \\ &= x_1y_1 \cdot 2^n + (x_1y_0 + x_0y_1) \cdot 2^{n/2} + x_0y_0. \end{aligned} \quad (1)$$

A divide and conquer procedure appears surreptitiously. To compute the product of x and y we compute the four products $x_1y_1, x_1y_0, x_0y_1, x_0y_0$, *recursively*, and then we combine them to obtain xy .

Let $T(n)$ be the number of operations that are required to compute the product of two n -bit integers using the divide and conquer procedure:

$$T(n) \leq 4T(n/2) + cn, \quad (2)$$

since we have to compute the four products $x_1y_1, x_1y_0, x_0y_1, x_0y_0$ (this is where the $4T(n/2)$ factor comes from), and then we have to perform three additions of n -bit integers (that is where the factor cn , where c is some constant, comes from).

Notice that we do not take into account the product by 2^n and $2^{n/2}$ as they simply consist in shifting the binary string by an appropriate number of bits to the left (n for 2^n and $n/2$ for $2^{n/2}$). These shift operations are inexpensive, and can be ignored in the complexity analysis.

It appears that we have to make four recursive calls; that is, we need to compute the four multiplications $x_1y_1, x_1y_0, x_0y_1, x_0y_0$.

But we can get away with only three multiplications, and hence three recursive calls: x_1y_1, x_0y_0 and $(x_1 + x_0)(y_1 + y_0)$; the reason being that

$$(x_1y_0 + x_0y_1) = (x_1 + x_0)(y_1 + y_0) - (x_1y_1 + x_0y_0). \quad (3)$$

	multiplications	additions	shifts
Method 1	4	3	2
Method 2	3	4	2

Algorithm takes $T(n) \leq 3T(n/2) + dn$ operations.

Thus, the running time is $O(n^{\log 3}) \approx O(n^{1.59})$.

Recursive Binary Mult A18

Pre-condition: Two n -bit integers x and y

```
1: if  $n = 1$  then  
2:     if  $x = 1 \wedge y = 1$  then  
3:         return 1  
4:     else  
5:         return 0  
6:     end if  
7: end if  
8:  $(x_1, x_0) \leftarrow$  (first  $\lfloor n/2 \rfloor$  bits, last  $\lceil n/2 \rceil$  bits) of  $x$   
9:  $(y_1, y_0) \leftarrow$  (first  $\lfloor n/2 \rfloor$  bits, last  $\lceil n/2 \rceil$  bits) of  $y$   
10:  $z_1 \leftarrow \text{Multiply}(x_1 + x_0, y_1 + y_0)$   
11:  $z_2 \leftarrow \text{Multiply}(x_1, y_1)$   
12:  $z_3 \leftarrow \text{Multiply}(x_0, y_0)$   
13: return  $z_2 \cdot 2^n + (z_1 - z_2 - z_3) \cdot 2^{\lceil n/2 \rceil} + z_3$ 
```

Savitch's Algorithm

We have a directed graph, and we want to establish whether we have a path from s to t .

Savitch's algorithm solves the problem in *space* $O(\log^2 m)$.

$$R(G, u, v, i) \iff (\exists w)[R(G, u, w, i-1) \wedge R(G, w, v, i-1)]. \quad (4)$$

```

1: if  $i = 0$  then
2:     if  $u = v$  then
3:         return T
4:     else if  $(u, v)$  is an edge then
5:         return T
6:     end if
7: else
8:     for every vertex  $w$  do
9:         if  $R(G, u, w, i - 1)$  and  $R(G, w, v, i - 1)$  then
10:            return T
11:        end if
12:    end for
13: end if
14: return F

```

Example run

●¹ ——— ●² ——— ●³ ——— ●⁴

Then the recursion stack would look as follows for the first 6 steps:

		$R(1, 4, 0)$	F	$R(2, 4, 0)$	F
		$R(1, 1, 0)$	T	$R(1, 2, 0)$	T
	$R(1, 4, 1)$	$R(1, 4, 1)$	$R(1, 4, 1)$	$R(1, 4, 1)$	$R(1, 4, 1)$
	$R(1, 1, 1)$	$R(1, 1, 1)$	$R(1, 1, 1)$	$R(1, 1, 1)$	$R(1, 1, 1)$
$R(1, 4, 2)$	$R(1, 4, 2)$	$R(1, 4, 2)$	$R(1, 4, 2)$	$R(1, 4, 2)$	$R(1, 4, 2)$
Step 1	Step 2	Step 3	Step 4	Step 5	Step 6

Quicksort & git bisect

```
qsort [] = []  
qsort (x:xs) = qsort smaller ++ [x] ++ qsort larger  
  where  
    smaller = [a | a <- xs, a <= x]  
    larger  = [b | b <- xs, b > x]
```