Intro to Analysis of Algorithms
Dynamic Programming
Chapter 4

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Longest Monotone Subsequence

Input: $d, a_1, a_2, \ldots, a_d \in \mathbb{N}$.
Output: $L =$ length of the longest monotone non-decreasing subsequence.

Note that a subsequence need not be consecutive, that is $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ is a monotone subsequence provided that

$$1 \leq i_1 < i_2 < \ldots < i_k \leq d,$$
$$a_{i_1} \leq a_{i_2} \leq \ldots \leq a_{i_k}.$$

Dynamic Prog approach

1. Define an array of sub-problems
2. Find the recurrence
3. Write the algorithm

We first define an array of subproblems: $R(j) =$ length of the longest monotone subsequence which ends in a_j. The answer can be extracted from array R by computing $L = \max_{1 \leq j \leq n} R(j)$.

The next step is to find a recurrence. Let $R(1) = 1$, and for $j > 1$,

$$R(j) = \begin{cases} 1 & \text{if } a_i > a_j \text{ for all } 1 \leq i < j \\ 1 + \max_{1 \leq i < j}\{R(i) | a_i \leq a_j\} & \text{otherwise} \end{cases}.$$

Algorithm 21

```
R(1) ← 1
for j : 2..d do
    max ← 0
    for i : 1..j − 1 do
        if R(i) > max and a_i ≤ a_j then
            max ← R(i)
        end if
    end for
    R(j) ← max +1
end for
```

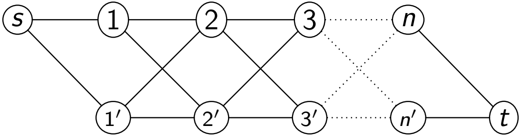
Question

Once R, and L, have been computed how do we build the *actual* monotone subsequence?

All pairs shortest path

Input: Directed graph $G = (V, E)$, $V = \{1, 2, \ldots, n\}$, and a cost function $C(i, j) \in \mathbb{N}^+ \cup \{\infty\}$, $1 \leq i, j \leq n$, $C(i, j) = \infty$ if (i, j) is not an edge.
Output: An array D, where $D(i, j)$ the length of the shortest directed path from i to j.

Exponentially many paths Problem: 4.5



Define an array of subproblems: let $A(k,i,j)$ be the length of the shortest path from i to j such that all *intermediate* nodes on the path are in $\{1,2,\ldots,k\}$. Then $A(n,i,j) = D(i,j)$ will be the solution. The convention is that if $k=0$ then $\{1,2,\ldots,k\} = \emptyset$.

Define a recurrence: we first initialize the array for $k = 0$ as follows: $A(0, i, j) = C(i, j)$.

Now we want to compute $A(k, i, j)$ for $k > 0$.

To design the recurrence, notice that the shortest path between i and j either includes k or does not.

Assume we know $A(k-1, r, s)$ for all r, s.

Suppose node k is not included. Then, obviously, $A(k, i, j) = A(k-1, i, j)$.

If, on the other hand, node k occurs on a shortest path, then it occurs exactly once, so $A(k, i, j) = A(k-1, i, k) + A(k-1, k, j)$.

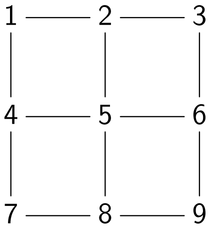
Therefore, the shortest path length is obtained by taking the minimum of these two cases:

$$A(k,i,j) = \min\{A(k-1,i,j), A(k-1,i,k) + A(k-1,k,j)\}.$$

Algorithm 22

```
for i : 1..n do
    for j : 1..n do
        B(i,j) ⟵ C(i,j)
    end for
end for
for k : 1..n do
    for i : 1..n do
        for j : 1..n do
            B(i,j) ⟵ min{B(i,j), B(i,k) + B(k,j)}
        end for
    end for
end for
return D ⟵ B
```

Example



$k = 0$ can be read directly from the graph
(assume all edges worth 1).

$k = 1$								
	1	∞	1	∞	∞	∞	∞	∞
		1	2	1	∞	∞	∞	∞
			∞	∞	1	∞	∞	∞
				1	∞	1	∞	∞
					1	∞	1	∞
						∞	∞	1
							1	∞
								1

$k=2$								
	1	2	1	2	∞	∞	∞	∞
		1	2	1	∞	∞	∞	∞
			3	2	1	∞	∞	∞
				1	∞	1	∞	∞
					1	∞	1	∞
						∞	∞	1
							1	∞
								1

The “overwriting” trick

“Overwriting” not a problem on line 9 of algorithm.

Bellman-Ford algorithm: §4.2.1

$$\textsc{Opt}(i, v) = \min\{\textsc{Opt}(i-1, v), \min_{w \in V}\{c(v, w) + \textsc{Opt}(i-1, w)\}\}$$

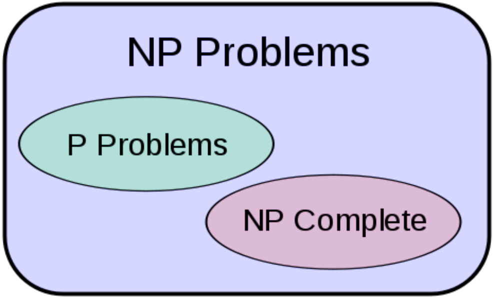
where $\textsc{Opt}(i, v)$ is the shortest i-path from v to t (we want the shortest path from s to t).

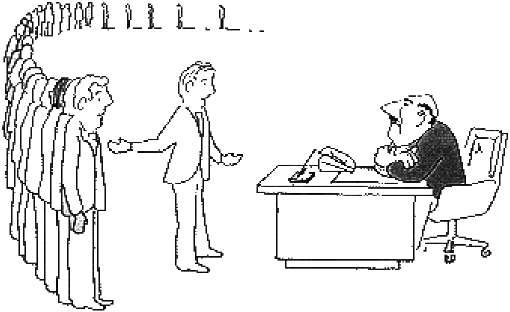
Knapsack Problem

Input: $w_1, w_2, \ldots, w_d, C \in \mathbb{N}$, where C is the knapsack's capacity.
Output: $\max_S\{K(S)|K(S) \leq C\}$, where $S \subseteq [d]$ and
$K(S) = \sum_{i \in S} w_i$.

First example of an NP-hard problem.


CHOTCHKIES RESTAURANT
APPETIZERS
MIXED FRUIT 2.15
FRENCH FRIES 2.75
SIDE SALAD 3.35
HOT WINGS 3.55
MOZZARELLA STICKS 4.20
SAMPLER PLATE 5.80
SANDWICHES
WE'D LIKE EXACTLY $15.05 WORTH OF APPETIZERS, PLEASE.
...EXACTLY? UHH...
HERE, THESE PAPERS ON THE KNAPSACK PROBLEM MIGHT HELP YOU OUT.
LISTEN, I HAVE SIX OTHER TABLES TO GET TO –
– AS FAST AS POSSIBLE, OF COURSE. WANT SOMETHING ON TRAVELING SALESMAN?


NP Problems
P Problems
NP Complete



Define an array of subproblems: we consider the first i weights (i.e., $[i]$) summing up to an *intermediate* weight limit j.

We define a Boolean array R as follows:

$$R(i,j) = \begin{cases} \mathrm{T} & \text{if } \exists S \subseteq [i] \text{ such that } K(S) = j \\ \mathrm{F} & \text{otherwise} \end{cases},$$

for $0 \leq i \leq d$ and $0 \leq j \leq C$.

Once we have computed all the values of R we can obtain the solution M as follows: $M = \max_{j \leq C}\{j | R(d,j) = \mathrm{T}\}$.

Define a recurrence: we initialize $R(0, j) = \mathrm{F}$ for $j = 1, 2, \ldots, C$, and $R(i, 0) = \mathrm{T}$ for $i = 0, 1, \ldots, d$.

We now define the recurrence for computing R, for $i, j > 0$, in a way that hinges on whether we include object i in the knapsack.

Suppose that we do *not* include object i. Then, obviously, $R(i, j) = \mathrm{T}$ iff $R(i - 1, j) = \mathrm{T}$.

Suppose, on the other hand, that object i *is* included. Then it must be the case that $R(i,j) = \mathrm{T}$ iff $R(i-1, j-w_i) = \mathrm{T}$ and $j - w_i \geq 0$, i.e., there is a subset $S \subseteq [i-1]$ such that $K(S)$ is exactly $j - w_i$ (in which case $j \geq w_i$).

R	0	$\cdots$	$j-w_i$	$\cdots$	j	$\cdots$	C
0	T	F⋯F	F	F⋯F	F	F⋯F	F
	T ⋮ T						
$i-1$	T		**c**		**b**		
i	T				**a**		
	T ⋮ T						
d	T						

Putting it all together we obtain the following recurrence for $i, j > 0$:

$$R(i, j) = \mathsf{T} \iff R(i-1, j) = \mathsf{T} \vee (j \geq w_i \wedge R(i-1, j-w_i) = \mathsf{T}).$$

```
S(0) ⟵ T
for j : 1..C do
    S(j) ⟵ F
end for
for i : 1..d do
    for decreasing j : C..1 do
        if (j ≥ w_i and S(j − w_i) = T) then
            S(j) ⟵ T
        end if
    end for
end for
```

General Knapsack Problem

Input: $w_1, w_2, \ldots, w_d, v_1, \ldots, v_d, C \in \mathbb{N}$
Output: $\max_{S \subseteq [d]}\{V(S) | K(S) \leq C\}$, $K(S) = \sum_{i \in S} w_i$,
$V(S) = \sum_{i \in S} v_i$.

$$V(i,j) = \max\{V(S) | S \subseteq [i] \text{ and } K(S) = j\},$$

for $0 \leq i \leq d$ and $0 \leq j \leq C$.

Problem: what is the recurrence for this problem?

Approximating SKS

Greedy "solution" to SKS:

order the weights from heaviest to lightest, keep adding for as long as possible.

Let M be the optimal solution, and let $\bar{M}$ be the solution obtained from the greedy approach.

Performance: $1/2$.

Let S_0 be the set of weights we got from greedy, so $K(S_0) = \bar{M}$.

If $S_0 = \emptyset$, then $\bar{M} = M$.
If $S_0 = S$ (all weights in), then $\bar{M} = M$.

OTHERWISE:

Assume we throw out weights greater than C (they won't be added anyway). Let w_j be the first weight that has been rejected, after some weights have been added ....

Activity Selection

Input: A list of activities $(s_1, f_1, p_1), \ldots, (s_n, f_n, p_n)$, where $p_i > 0$, $s_i < f_i$ and s_i, f_i, p_i are non-negative real numbers.
Output: A set $S \subseteq [n]$ of selected activities such that no two selected activities overlap, and the profit $P(S) = \sum_{i \in S} p_i$ is as large as possible.

An *activity* i has a fixed start time s_i, finish time f_i and profit p_i. Given a set of activities, we want to select a subset of non-overlapping activities with maximum total profit.

Define an array of subproblems: sort the activities by their finish times, $f_1 \leq f_2 \leq \ldots \leq f_n$.

As it is possible that activities finish at the same time, we select the *distinct* finish times, and denote them $u_1 < u_2 < \ldots < u_k$, where, clearly, $k \leq n$.

For instance, if we have activities finishing at times 1.24, 4, 3.77, 1.24, 5 and 3.77, then we partition them into four groups: activities finishing at times $u_1 = 1.24$, $u_2 = 3.77$, $u_3 = 4$, $u_4 = 5$.

Let u_0 be $\min_{1\le i\le n} s_i$, i.e., the earliest start time. Thus,

$$u_0 < u_1 < u_2 < \ldots < u_k,$$

as it is understood that $s_i < f_i$. Define an array $A(0..k)$ as follows:

$$A(j) = \max_{S\subseteq[n]}\{P(S)|S \text{ is feasible and } f_i \le u_j \text{ for each } i \in S\},$$

where S is *feasible* if no two activities in S overlap. Note that $A(k)$ is the maximum possible profit for all feasible schedules S.

Define a recurrence for $A(0..k)$.

In order to give such a recurrence we first define an auxiliary array $H(1..n)$ such that $H(i)$ is the index of the largest distinct finish time no greater than the start time of activity i.

Formally, $H(i) = \ell$ if ℓ is the largest number such that $u_\ell \leq s_i$. To compute $H(i)$, we need to search the list of distinct finish times.

To do it efficiently, for each i, apply the binary search procedure that runs in logarithmic time in the length of the list of distinct finish times (try $\ell = \lfloor \frac{k}{2} \rfloor$ first).

Since the length k of the list of distinct finish times is at most n, and we need to apply binary search for each element of the array $H(1..n)$, the time required to compute all entries of the array is $O(n \log n)$.

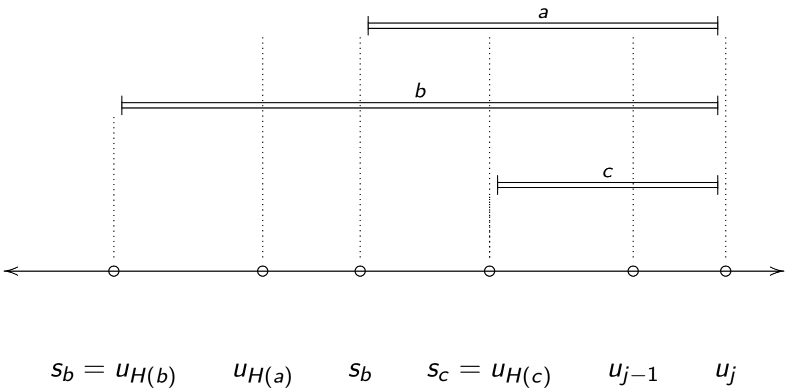
We initialize $A(0) = 0$, and we want to compute $A(j)$ given that we already have $A(0), \ldots, A(j-1)$.

Consider $u_0 < u_1 < u_2 < \ldots < u_{j-1} < u_j$.

Can we beat profit $A(j-1)$ by scheduling some activity that finishes at time u_j? Try all activities that finish at this time and compute maximum profit in each case. We obtain the following recurrence:

$$A(j) = \max\{A(j-1), \max_{1 \le i \le n}\{p_i + A(H(i)) \mid f_i = u_j\}\},$$

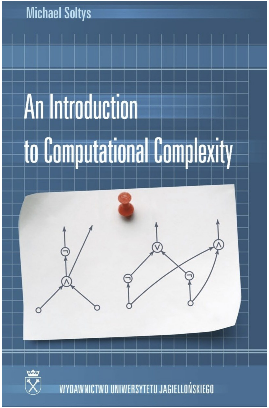
where $H(i)$ is the greatest ℓ such that $u_\ell \le s_i$.


a
b
c
$s_b = u_{H(b)}$
$u_{H(a)}$
s_b
$s_c = u_{H(c)}$
u_{j-1}
u_j

$A(0) \longleftarrow 0$
for $j : 1..k$ **do**
    $\max \longleftarrow 0$
    for $i = 1..n$ **do**
        if $f_i = u_j$ **then**
            if $p_i + A(H(i)) > \max$ **then**
                $\max \longleftarrow p_i + A(H(i))$
            end if
        end if
    end for
    if $A(j-1) > \max$ **then**
        $\max \longleftarrow A(j-1)$
    end if
    $A(j) \longleftarrow \max$
end for

Introduction to Complexity

This material is not in the IAA textbook but here:



A TM M is of *time complexity* $T(n)$ if whenever M is given an input w, $|w| = n$, then M *halts* after making at most $T(n)$ many moves.

$L \in \mathit{TIME}(f(n))$ if there exists a deterministic TM M of time complexity $O(f(n))$ that decides L.

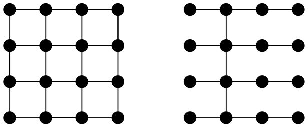
$L \in \mathit{NTIME}(f(n))$ if there exists a nondeterministic TM M of time complexity $O(f(n))$ that decides L.

L is in the class P if $L \in \mathrm{TIME}(n^k)$ for some fixed k.

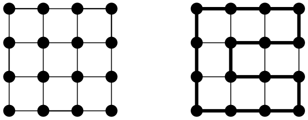
L is in the class NP if $L \in \mathrm{NTIME}(n^k)$ for some fixed k.

Observation: P⊆NP; Question: NP⊆P ?

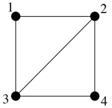
Ex. of a language in P:
$\{\langle G, k\rangle | G$ has a spanning tree of weight $\leq k\}$. $(k = 15)$



Ex. of a language in NP believed not to be in P:
$\{\langle G, k\rangle | G$ has a complete cycle of weight $\leq k\}$. $(k = 16)$



A graph G can be encoded as an adjacency matrix. For example, the graph given below would have the adjacency matrix given by:



$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

If P is a *decision problem*, the related language L_P consists of the encodings (under some fixed convention) of all the "yes" instances of P.

Feasibility Thesis:

$$\text{Polynomial time algorithm} \equiv \text{polynomial time TM.}$$

A problem P_1 is *reducible in polynomial time* to a problem P_2 if there exists a polynomial time function f such that:

$$\langle I \rangle \in L_{P_1} \iff \langle f(I) \rangle \in L_{P_2}$$

L is *NP-complete* if:

1. $L \in \mathrm{NP}$
2. Every language $L' \in \mathrm{NP}$ is polynomial time reducible to L.

Ex. Traveling Salesman Problem

L is NP-complete is *evidence* of L not being in P

(see *Computers and Intractability* by Michael Garey and David Johnson.)

Theorem: If P_1 is NP-complete, P_2 is in NP, and there is a polynomial time reduction of P_1 to P_2, then P_2 is also NP-complete.

Proof: Every language L in NP is reducible to L_{P_1}, by completeness, and P_1 is reducible to P_2. Enough to show transitivity of reductions.

Theorem: If some NP-complete problem P is in P, then P=NP.

Proof: Follows from the fact that all languages in NP are polynomial time reducible to P.

Satisfiability

Boolean Expressions are built from: Boolean variables $x, y, z, \ldots$, Boolean values $0, 1$, and Boolean connectives: $\vee, \wedge, \neg$, and parenthesis.

Ex. $\neg x \vee (y \wedge z)$

If ϕ is a Boolean expression, then a *truth assignment* T is an assignment of truth values to the variables of ϕ.

Ex. $T(x) = 0, T(y) = 1, T(z) = 1$, then
$T(\neg x \vee (y \wedge z)) = \neg 0 \vee (1 \wedge 1) = 1 \vee 1 = 1$.

T *satisfies* ϕ if $T(\phi) = 1$, and ϕ is *satisfiable* if $\exists T$ s.t. $T(\phi) = 1$.

The *satisfiability problem* is: given a Boolean expression, is it satisfiable?

SAT $= \{\langle\phi\rangle | \phi$ is satisfiable$\}$
(i.e., SAT is the language corresponding to the satisfiability problem).

Cook's Theorem: SAT is NP-complete.

PROOF: SAT is in NP.
Let L be any language in NP.
We show there exists a polynomial time function f s.t.:

$$w \in L \iff f(w) = \phi \in \text{SAT}$$

$\exists$ non-det TM M s.t. $L = L(M)$ and M always halts within n^k many steps on inputs w, $|w| = n$, for fixed k.
Given w, f outputs a Boolean formula ϕ which encodes a computation of M on w and is satisfiable $\iff$ M accepts w.