Intro to Analysis of Algorithms
Divide & Conquer
Chapter 3

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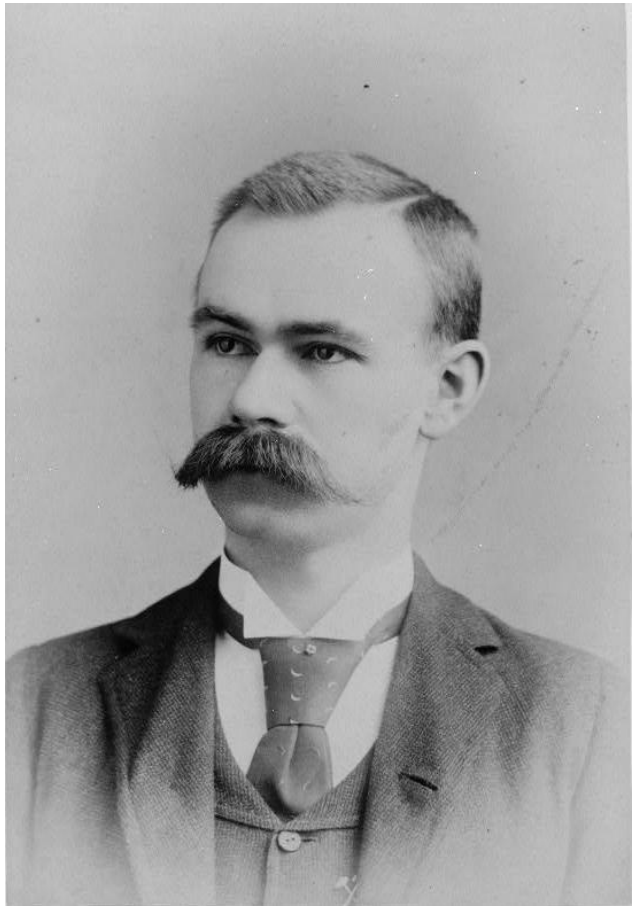


Herman Hollerith, 1860–1929

Suppose that we have two lists of numbers that are already sorted.

That is, we have a list $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_m$.

We want to combine those two lists into one long sorted list $c_1 \leq c_2 \leq \cdots \leq c_{n+m}$.

The mergesort algorithm sorts a given list of numbers by first dividing them into two lists of length $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$, respectively, then sorting each list recursively, and finally combining the results.

Pre-condition: $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_m$

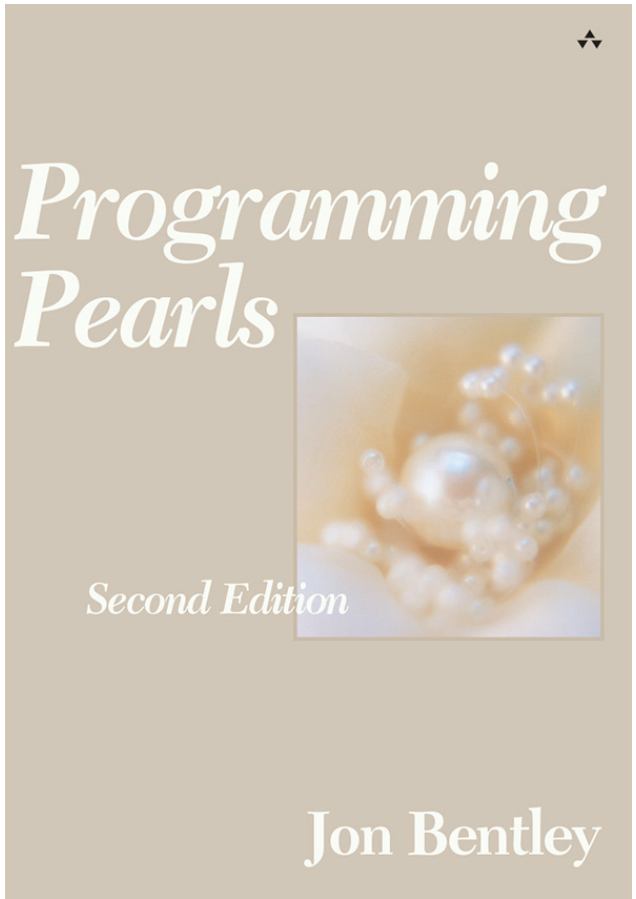
```
 1: p_1 ⟵ 1; p_2 ⟵ 1; i ⟵ 1
 2: while i ≤ n + m do
 3:         if a_{p_1} ≤ b_{p_2} then
 4:                 c_i ⟵ a_{p_1}
 5:                 p_1 ⟵ p_1 + 1
 6:         else
 7:                 c_i ⟵ b_{p_1}
 8:                 p_2 ⟵ p_2 + 1
 9:         end if
10:         i ⟵ i + 1
11: end while
```

Post-condition: $c_1 \leq c_2 \leq \cdots \leq c_{n+m}$

Pre-condition: A list of integers $a_1, a_2, \ldots, a_n$

1: $L \longleftarrow a_1, a_2, \ldots, a_n$
2: **if** $|L| \leq 1$ **then**
3: **return** L
4: **else**
5: $L_1 \longleftarrow$ first $\lceil n/2 \rceil$ elements of L
6: $L_2 \longleftarrow$ last $\lfloor n/2 \rfloor$ elements of L
7: **return** Merge(Mergesort(L_1), Mergesort(L_2))
8: **end if**

Post-condition: $a_{i_1} \leq a_{i_2} \leq \cdots \leq a_{i_n}$


Programming
Pearls
Second Edition
Jon Bentley

Multiplication

	1	2	3	4	5	6	7	8
x					1	1	1	0
y					1	1	0	1
s_1					1	1	1	0
s_2				0	0	0	0	
s_3			1	1	1	0		
s_4		1	1	1	0			
$x \times y$	1	0	1	1	0	1	1	0

Multiply 1110 times 1101, i.e., 14 times 13. Takes $O(n^2)$ steps.

Clever multiplication

Let x and y be two n-bit integers. We break them up into two smaller $n/2$-bit integers as follows:

$$x = (x_1 \cdot 2^{n/2} + x_0),$$
$$y = (y_1 \cdot 2^{n/2} + y_0).$$

x_1 and y_1 correspond to the high-order bits of x and y, respectively, and x_0 and y_0 to the low-order bits of x and y, respectively.

The product of x and y appears as follows in terms of those parts:

$$\begin{aligned} xy &= (x_1 \cdot 2^{n/2} + x_0)(y_1 \cdot 2^{n/2} + y_0) \\ &= x_1 y_1 \cdot 2^n + (x_1 y_0 + x_0 y_1) \cdot 2^{n/2} + x_0 y_0. \end{aligned} \tag{1}$$

A divide and conquer procedure appears surreptitiously. To compute the product of x and y we compute the four products $x_1y_1, x_1y_0, x_0y_1, x_0y_0$, *recursively*, and then we combine them to obtain xy.

Let $T(n)$ be the number of operations that are required to compute the product of two n-bit integers using the divide and conquer procedure:

$$T(n) \leq 4T(n/2) + cn, \tag{2}$$

since we have to compute the four products $x_1y_1, x_1y_0, x_0y_1, x_0y_0$ (this is where the $4T(n/2)$ factor comes from), and then we have to perform three additions of n-bit integers (that is where the factor cn, where c is some constant, comes from).

Notice that we do not take into account the product by 2^n and $2^{n/2}$ as they simply consist in shifting the binary string by an appropriate number of bits to the left (n for 2^n and $n/2$ for $2^{n/2}$). These shift operations are inexpensive, and can be ignored in the complexity analysis.

It appears that we have to make four recursive calls; that is, we need to compute the four multiplications $x_1y_1, x_1y_0, x_0y_1, x_0y_0$.

But we can get away with only three multiplications, and hence three recursive calls: x_1y_1, x_0y_0 and $(x_1 + x_0)(y_1 + y_0)$; the reason being that

$$(x_1y_0 + x_0y_1) = (x_1 + x_0)(y_1 + y_0) - (x_1y_1 + x_0y_0). \qquad (3)$$

	multiplications	additions	shifts
Method 1	4	3	2
Method 2	3	4	2

Algorithm takes $T(n) \leq 3T(n/2) + dn$ operations.

Thus, the running time is $O(n^{\log 3}) \approx O(n^{1.59})$.

Recursive Binary Mult A18

Pre-condition: Two n-bit integers x and y

1: **if** $n = 1$ **then**
2: **if** $x = 1 \wedge y = 1$ **then**
3: **return** 1
4: **else**
5: **return** 0
6: **end if**
7: **end if**
8: $(x_1, x_0) \longleftarrow$ (first $\lfloor n/2 \rfloor$ bits, last $\lceil n/2 \rceil$ bits) of x
9: $(y_1, y_0) \longleftarrow$ (first $\lfloor n/2 \rfloor$ bits, last $\lceil n/2 \rceil$ bits) of y
10: $z_1 \longleftarrow \text{Multiply}(x_1 + x_0, y_1 + y_0)$
11: $z_2 \longleftarrow \text{Multiply}(x_1, y_1)$
12: $z_3 \longleftarrow \text{Multiply}(x_0, y_0)$
13: **return** $z_2 \cdot 2^n + (z_1 - z_2 - z_3) \cdot 2^{\lceil n/2 \rceil} + z_3$

Savitch's Algorithm

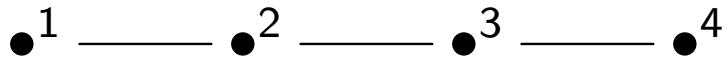
We have a directed graph, and we want to establish whether we have a path from s to t.

Savitch's algorithm solves the problem in *space* $O(\log^2 m)$.

$$\mathrm{R}(G,u,v,i) \iff (\exists w)[\mathrm{R}(G,u,w,i-1) \wedge \mathrm{R}(G,w,v,i-1)]. \quad (4)$$

```
 1: if i = 0 then
 2:         if u = v then
 3:                 return T
 4:         else if (u, v) is an edge then
 5:                 return T
 6:         end if
 7: else
 8:         for every vertex w do
 9:                 if R(G, u, w, i − 1) and R(G, w, v, i − 1) then
10:                         return T
11:                 end if
12:         end for
13: end if
14: return F
```

Example run



Then the recursion stack would look as follows for the first 6 steps:

		$R(1,4,0)$	F	$R(2,4,0)$	F
		$R(1,1,0)$	T	$R(1,2,0)$	T
	$R(1,4,1)$	$R(1,4,1)$	$R(1,4,1)$	$R(1,4,1)$	$R(1,4,1)$
	$R(1,1,1)$	$R(1,1,1)$	$R(1,1,1)$	$R(1,1,1)$	$R(1,1,1)$
$R(1,4,2)$	$R(1,4,2)$	$R(1,4,2)$	$R(1,4,2)$	$R(1,4,2)$	$R(1,4,2)$
Step 1	Step 2	Step 3	Step 4	Step 5	Step 6

Quicksort & git bisect

```
qsort [] = []
qsort (x:xs) = qsort smaller ++ [x] ++ qsort larger
  where
    smaller = [a | a <- xs, a <= x]
    larger = [b | b <- xs, b > x]
```