appendix

Appendix

Deriving Distributions for the Gibbs Sampler

The derivations are based off of @park_bayesian_2008. Notable changes have been made for this specific application. Namely, the model is larger, β and $\sqrt{\theta}$ are not conditioned on σ^2, and the hierarchical structure is redefined to be a *global-local* shrinkage estimator. @park_bayesian_2008 use a hierarchical formulation where the local shrinkage is dependent on the global shrinkage. @park_bayesian_2008 also use an inverse gamma distribution to represent the global shrinkage while this paper opts to use a half Cauchy distribution.

Recall:

$$y_{0,t}(0) = \sum_{j=1}^{J+1} \left(\beta_j + \tilde{\beta}_{j,t}\sqrt{\theta_j}\right) y_{j,t}(0) + \epsilon_t$$

The conditional prior of $\mathbf{y_0}$ is defined as $\mathcal{N}\left(\mathbf{y}\beta_j + (\mathbf{y} * \tilde{\beta}_j)\sqrt{\theta_j}), \sigma^2 I\right)$ where $*$ denotes element wise multiplication. Conditional on α_i^2 and ξ_i^2, the model follows a standard linear regression with normal priors. Textbook tools can be used to derive the distributions for the Gibbs sampler. The joint density is defined as:

$$\begin{aligned}
&f(\mathbf{y_0}|\beta, \sqrt{\theta}, \sigma^2)\pi(\sigma^2)\pi(\lambda^2)\pi(\kappa^2)\prod_{j=1}^{J+1}\pi(\beta_j|\alpha_j^2, \lambda^2)\pi(\alpha_j^2)\pi(\sqrt{\theta_j}|\xi_j^2, \kappa^2)\pi(\xi_j^2) = \\
&\frac{1}{(2\pi\sigma^2)^{\frac{T_0-1}{2}}}\ exp\left\{\frac{-1}{2\sigma^2}\left(\mathbf{y_0} - \mathbf{y}\beta - (\mathbf{y} * \tilde{\beta})\sqrt{\theta}\right)^T\left(\mathbf{y_0} - \mathbf{y}\beta - (\mathbf{y} * \tilde{\beta})\sqrt{\theta}\right)\right\} \\
&\frac{a_2^{a_1}}{\Gamma(a_1)}(\sigma^2)^{-a_1-1} exp\left\{-\frac{a_2}{\sigma^2}\right\}\frac{\frac{1}{\zeta_\beta}^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}(\lambda^2)^{-\frac{1}{2}-1} exp\left\{-\frac{\frac{1}{\zeta_\beta}}{\lambda^2}\right\}\frac{\frac{1}{\zeta_{\sqrt{\theta}}}^{1/2}}{\Gamma\left(\frac{1}{2}\right)}(\kappa^2)^{\frac{-1}{2}-1} exp\left\{-\frac{\frac{1}{\zeta_{\sqrt{\theta}}}}{\kappa^2}\right\} \\
&\frac{1^{1/2}}{\Gamma(1/2)}\zeta_\beta^{-\frac{1}{2}-1} exp\left\{\frac{-1}{\zeta_\beta}\right\}\frac{1^{1/2}}{\Gamma(1/2)}\zeta_{\sqrt{\theta}}^{-\frac{1}{2}-1} exp\left\{\frac{-1}{\zeta_{\sqrt{\theta}}}\right\} \\
&\prod_{j=1}^{J+1}\frac{1}{(2\pi\alpha_j^2\lambda^2)^{\frac{1}{2}}} exp\left\{\frac{-1}{(2\alpha_j^2\lambda^2)}\beta_j^2\right\} exp\left\{-\alpha_j^2\right\}\frac{1}{(2\pi\xi_j^2\kappa^2)^{\frac{1}{2}}} exp\left\{\frac{-1}{(2\xi_j^2\kappa^2)}\sqrt{\theta}_j^2\right\} exp\left\{\xi_j^2\right\}
\end{aligned}$$

Conditional Distribution of β and $\sqrt{\theta}$

To solve for the conditional distribution of β and $\sqrt{\theta}$, drop the terms that don't involve β and $\sqrt{\theta}$. This only leaves 3 exponential terms:

$$\begin{aligned}
&exp\left\{\frac{-1}{2\sigma^2}\left(\mathbf{y_0} - \mathbf{y}\beta - (\mathbf{y} * \tilde{\beta})\sqrt{\theta}\right)^T\left(\mathbf{y_0} - \mathbf{y}\beta - (\mathbf{y} * \tilde{\beta})\sqrt{\theta}\right)\right\} \\
&\qquad\prod_{j=1}^{J+1} exp\left\{\frac{-1}{(2\alpha_j^2\lambda^2)}\beta_j^2\right\} exp\left\{\frac{-1}{(2\xi_j^2\kappa^2)}\sqrt{\theta_j}^2\right\}
\end{aligned}$$

Combining exponents yields:

$$exp\left\{\frac{-1}{2\sigma^2}\left(\mathbf{y_0}-\mathbf{y}\beta-(\mathbf{y}*\tilde{\beta})\sqrt{\theta}\right)^T\left(\mathbf{y_0}-\mathbf{y}\beta-(\mathbf{y}*\tilde{\beta})\sqrt{\theta}\right)+\sum_{j=1}^{J+1}\frac{-\sigma^2}{(2\alpha_j^2\lambda^2)}\beta_j^2+\sum_{j=1}^{J+1}\frac{-\sigma^2}{(2\xi_j^2\kappa^2)}\sqrt{\theta_j}^2\right\}$$

Define:

$$\tilde{\mathbf{y}}=\left[\mathbf{y},\mathbf{y}*\tilde{\beta}\right]_{T_0-1,2(J+1)}$$

,

$$\Theta=\left[\beta,\sqrt{\theta}\right]_{2(J+1),2(J+1)}$$

and

$$D=diag\left[\lambda^2\alpha_1^2,...\lambda^2\alpha_{J+1}^2,\kappa^2\xi_1^1,\ldots,\kappa^2\xi_{J+1}^2\right]_{2(J+1),2(J+1)}$$

.

Focusing solely on the exponential term and rearranging yields:

$$\frac{-1}{2\sigma^2}\left[(\mathbf{y_0}-\tilde{\mathbf{y}}\Theta)^T(\mathbf{y_0}-\tilde{\mathbf{y}}\Theta)+\Theta^T\sigma^2V^{-1}\Theta\right]$$

Multiplying out and rearranging yields:

$$\frac{-1}{2\sigma^2}\left[\mathbf{y_0}^T\mathbf{y_0}-2\mathbf{y_0}\tilde{\mathbf{y}}\Theta+\Theta^T(\tilde{\mathbf{y}}^T\mathbf{y_0}+\sigma^2V^{-1})\Theta\right]$$

Focus solely on the terms within the brackets including Θ for a moment. Setting $A=\tilde{\mathbf{y}}^T\tilde{\mathbf{y}}+\sigma^2V^{-1}$ and completing the square yields:

$$\begin{aligned}&\left(\Theta-A^{-1}\tilde{\mathbf{y}}^Ty\right)^TA\left(\Theta-A^{-1}\tilde{\mathbf{y}}^Ty\right)\\&+y^T\left(I-\tilde{\mathbf{y}}A^{-1}\tilde{\mathbf{y}}^T\right)y\end{aligned}$$

Therefore, the part of the conditional distribution that relies on Θ can be written as:

$$\frac{1}{\sqrt{2\pi\sigma^2}}exp\left\{\frac{-1}{2\sigma^2}\left(\Theta-A^{-1}\tilde{\mathbf{y}}^T\mathbf{y_0}\right)^TA\left(\Theta-A^{-1}\tilde{\mathbf{y}}^T\mathbf{y_0}\right)\right\}$$

which can be summarized as Θ conditionally distributed as:

$$\mathcal{N}\left(A^{-1}\tilde{\mathbf{y}}^T\mathbf{y_0},\sigma^2A^{-1}\right)$$

Conditional Distribution of σ^2

Now, I will derive the conditional distribution for σ^2. Returning to the joint probability, drop all terms that do not include σ^2:

$$\begin{aligned}&\frac{1}{(\sigma^2)^{\frac{T_0-1}{2}}}\;exp\left\{\frac{-1}{2\sigma^2}\left(\mathbf{y_0}-\mathbf{y}\beta-(\mathbf{y}*\tilde{\beta})\sqrt{\theta}\right)^T\left(\mathbf{y_0}-\mathbf{y}\beta-(\mathbf{y}*\tilde{\beta})\sqrt{\theta}\right)\right\}\\&(\sigma^2)^{-a_1-1}exp\left\{-\frac{a_2}{\sigma^2}\right\}\end{aligned}$$

Rearranging yields:

$$(\sigma^2)^{-\frac{T_0-1}{2}-a_1-1}\ exp\left\{\frac{-1}{2\sigma^2}\left(\mathbf{y_0}-\mathbf{y}\beta-(\mathbf{y}*\tilde{\beta})\sqrt{\theta}\right)^T\left(\mathbf{y_0}-\mathbf{y}\beta-(\mathbf{y}*\tilde{\beta})\sqrt{\theta}\right)-\frac{a_2}{\sigma^2}\right\}$$

which is an inverse gamma distribution without the scaling term. Therefore, σ^2 is conditionally inverse gamma with *shape* parameter $\frac{T_0-1}{2}+a_1$ and *scale* parameter $\frac{1}{2}\left(\mathbf{y_0}-\mathbf{y}\beta-(\mathbf{y}*\tilde{\beta})\sqrt{\theta}\right)^T\left(\mathbf{y_0}-\mathbf{y}\beta-(\mathbf{y}*\tilde{\beta})\sqrt{\theta}\right)+a_2$.

Conditional Distribution of α_j^2 and ξ_j^2

Focusing only on terms involving α_j^2, the conditional distribution is:

$$\frac{1}{(\alpha_j^2)^{1/2}}exp\left\{\frac{-1}{(2\alpha_j^2\lambda^2)}\beta_j^2-\alpha_j^2\right\}$$

@park_bayesian_2008 note that by setting $\frac{1}{\alpha_j^2}=\zeta^2$, the density can be rewritten proportionally as an inverse Gaussian:

$$\begin{aligned}(\zeta^2)^{-3/2}exp\left\{-\left(\frac{\beta_j^2\zeta_j^2}{2\lambda^2}+\alpha_j^2\right)\right\}&\propto(\zeta^2)^{-3/2}exp\left\{\frac{-\beta_j^2}{2\zeta^2\lambda^2}\left[\zeta^2-\sqrt{\frac{2\lambda^2}{\beta_j^2}}\right]^2\right\}\\&=(\zeta^2)^{-3/2}exp\left\{\frac{-2}{2\zeta^2\frac{2\lambda^2}{\beta_j^2}}\left[\zeta^2-\sqrt{\frac{2\lambda^2}{\beta_j^2}}\right]^2\right\}\end{aligned}$$

This is one of many parameterizations of the Inverse Gaussian distribution. The Inverse Gaussian distribution can be written as:

$$f(x)=\sqrt{\frac{\lambda'}{2\pi}}\ x^{-3/2}exp\left\{-\frac{\lambda'(x-\mu')^2}{2(\mu')^2x}\right\}$$

with mean parameter μ' and scale parameter λ.

Therefore, $\frac{1}{\alpha_j^2}$ is conditionally distributed Inverse Gaussian with mean parameters $\frac{2\lambda^2}{\beta_j^2}$ and scale parameter $\lambda'=2$. ξ_j^2 is derived following the same steps.

Conditional Distribution of λ^2 and κ^2

Focusing solely on λ^2 in the joint distribution yields:

$$\left(\lambda^2\right)^{-\frac{J+2}{2}-1}exp\left\{\left(-\frac{\sum_{j=1}^{J+1}\alpha_j^2}{2}-\frac{1}{\zeta_\beta}\right)\frac{1}{\lambda^2}\right\}$$

which is proportional to an inverse gamma distribution with *shape* parameter $\frac{J+1}{2}$ and *rate* parameter $\frac{1}{\zeta_\beta}+\frac{1}{2}\sum_{j=1}^{J+1}\frac{\beta_j^2}{\alpha_j^2}$.

Similarly, κ^2 will follow an inverse gamma distribution with *shape* parameter $\frac{J+1}{2}$ and *rate* parameter $\frac{1}{\zeta_{\sqrt{\theta}}}+\frac{1}{2}\sum_{j=1}^{J+1}\frac{\sqrt{\theta_j}^2}{\xi_j^2}$.

Sample $\zeta_{\sqrt{\theta}}$ and ζ_β

Finally, ζ_β will follow an inverse gamma with shape 1 and rate $1 + \frac{1}{\lambda^2}$. Similarly, $\zeta_{\sqrt{\theta}}$ will follow an inverse gamma with shape 1 and rate $1 + \frac{1}{\kappa^2}$.