ElectroStatic Vlasov-Maxwell (ESVM) code

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1 Equations computed by ESVM

Plasma ions are assumed to be immobile with a homogeneous density n_i and fully ionized with an electrical charge Ze where Z is the plasma ion atomic number and e the elementary charge. The plasma electron distribution function $f_e(x, v_x, t)$ is computed by ESVM according to the plasma electron Vlasov equation

$$\frac{\partial f_e}{\partial t}(x, v_x, t) + \frac{\partial}{\partial x}\left(v_x f_e(x, v_x, t)\right) - \frac{\partial}{\partial v_x}\left(\frac{e}{m_e} E_x(x, t) f_e(x, v_x, t)\right) = 0 \tag{1}$$

that is self-consistently coupled with the Maxwell-Gauss equation

$$\frac{\partial E_x}{\partial x}(x, t) = 4\pi e\left(Z n_i - n_e(x, t)\right) \tag{2}$$

for the electrostatic field $E_x(x, t)$ or, equivalently, self-consistently coupled with the Maxwell-Ampere equation

$$\frac{\partial E_x}{\partial t}(x, t) = -4\pi j_e(x, t) \tag{3}$$

with Maxwell-Gauss equation (2) computed at the simulation start $t = 0$, only. Indeed, by integrating the plasma electron Vlasov equation (1) over the whole plasma electron velocity space $v_x \in [v_{x,\min}, v_{x,\max}]$, one gets the hydrodynamic equation of plasma electron number conservation

$$\frac{\partial n_e}{\partial t}(x, t) + \frac{\partial}{\partial x}\left(n_e v_e(x, t)\right) = 0, \tag{4}$$

which, when injected in the time derivative of Maxwell-Gauss equation (2), provides the Maxwell-Ampere equation (3) if Maxwell-Gauss equation (2) is verified at the simulation start t=0. Here,

$$n_e(x, t) = \int_{v_{x,\min}}^{v_{x,\max}} f_e(x, v_x, t)\, dv_x, \tag{5}$$

$$v_e(x, t) = \frac{1}{n_e(x, t)} \int_{v_{x,\min}}^{v_{x,\max}} f_e(x, v_x, t) v_x\, dv_x \tag{6}$$

and

$$j_e(x, t) = -e n_e(x, t) v_e(x, t) \tag{7}$$

are the plasma electron density, mean velocity and electrical charge current, respectively. ESVM also computes the plasma electron thermal velocity $v_{T_e}(x, t)$ defined according to the plasma electron internal energy density

$$u_{T_e}(x, t) = \frac{m_e}{2} n_e(x, t) v_{T_e}(x, t)^2 = \frac{m_e}{2} \int_{v_{x,\min}}^{v_{x,\max}} f_e(x, v_x, t)\left(v_x - v_e(x, t)\right)^2 dv_x. \tag{8}$$

For example, in 1D plasmas at local Maxwell-Boltzmann equilibrium, $v_{T_e}(x, t) = \sqrt{k_B T_e(x, t)/m_e}$ where k_B is the Boltzmann constant, $T_e(x, t)$ is the local electron temperature and m_e is the electron mass. Maxwell-Gauss equation (2) is computed by using the electrostatic potential definition

$$\frac{\partial \Phi}{\partial x}(x, t) = -E_x(x, t) \tag{9}$$

that gives the Poisson equation

$$\frac{\partial^2 \Phi}{\partial x^2}(x, t) = -4\pi e\left(Z n_i - n_e(x, t)\right) \tag{10}$$

for the electrostatic potential Φ when injected in the Maxwell-Gauss equation (2). When the simulation is running, ESVM stores at every time steps and displays on the terminal at every dumped time steps t_d the total plasma electron internal and kinetic energy (assuming simulations with an area unit perpendicular to the x-axis of ${\lambda_{\text{Debye}}}^2$) and the total electrostatic energy in the simulation box $x \in [x_{\min}, x_{\max}]$

$$U_{T_e}(t_d) = {\lambda_{\text{Debye}}}^2 \int_{x_{\min}}^{x_{\max}} u_{T_e}(x, t_d)\, dx, \tag{11}$$

$$U_{K_e}(t_d) = {\lambda_{\text{Debye}}}^2 \int_{x_{\min}}^{x_{\max}} \frac{m_e}{2} n_e(x,t) v_e(x,t_d)^2 \, dx \tag{12}$$

and

$$U_{E_x}(t_d) = {\lambda_{\text{Debye}}}^2 \int_{x_{\min}}^{x_{\max}} \frac{E_x(x,t_d)^2}{8\pi} \, dx, \tag{13}$$

respectively, as well as the total energy

$$U_{\text{tot}}(t_d) = U_{T_e}(t_d) + U_{K_e}(t_d) + U_{E_x}(t_d) \tag{14}$$

in order to check the energy conservation in the simulation. The user can initialize:

- an initial plasma electron population at Maxwell-Boltzmann equilibrium drifting at the velocty v_d

$$\begin{cases} f_e(x, v_x, t=0) &= \dfrac{Z n_i}{\sqrt{2\pi {v_{T_{e_0}}}^2}} \exp\left[-\dfrac{(v_x - v_d)^2}{2 {v_{T_{e_0}}}^2} \right] \\ E_x(x, t=0) &= 0 \end{cases} \tag{15}$$

  by no imposing any perturbation parameter or

- a provided Plasma Physics academic case; cf. academic_cases.pdf file.

- Finally, specific Plasma Physics simulations can easily be added in ESVM by implementing their initialization in the subroutines INIT_SIMU and/or DRIVE in the library.f90 source file.

2 ESVM units

The code units consist in the commonly used electrostatic units : the electron mass m_e for masses, the elementary charge e for electrical charges, the inverse of the Langmuir plasma electron angular frequency $\omega_p = \sqrt{4\pi Z n_i e^2 / m_e}$ for times, the Debye electron screening length $\lambda_{\text{Debye}} = v_{T_{e_0}} / \omega_p$ and the average plasma electron density $n_0 = Z n_i$ for spatial densities. $v_{T_{e_0}}$ is therefore an important unit parameter of normalization since it fixes indirectly the space unit. It can be defined more generally as the initial plasma electron velocity distribution standard deviation if the plasma is not initialized at Maxwell-Boltzmann thermodynamic equilibrium (15); cf. (8). Injecting these units in the equations computed by the code, detailed in the previous section, one deduces the resulting normalized energies, electrostatic field, electrostatic potential, plasma electron electrical current and distribution function that consequently reads $\underline{U_X} = U_X / \left({n_0 \lambda_{\text{Debye}}}^3 m_e {v_{T_{e_0}}}^2 \right)$ where $X = T_e$, K_e or E_x, $\underline{E_x} = e E_x / \left(m_e \omega_p v_{T_{e_0}} \right)$, $\underline{\Phi} = e\Phi / \left(m_e {v_{T_{e_0}}}^2 \right)$, $\underline{j_e} = j_e / \left(n_0 e v_{T_{e_0}} \right)$ and $\underline{f_e} = f_e v_{T_{e_0}} / n_0$, respectively.

3 ESVM numerical stability

The spatial grid cells should be chosen lower than the Debye length $\Delta x < \lambda_{\text{Debye}}$ for the simulation to be Physical. $v_{x,\min}$ and $v_{x,\max}$ should be chosen sufficiently large $|v_{x,\min/\max}| \gg v_{T_{e_0}}$ in such a way that there is no plasma electrons outside the simulation velocity space during the whole simulation; cf. the continuity equation (4). The simulation velocity bin size should be chosen lower than the thermal electron velocity $\Delta v_x < v_{T_{e_0}}$ and also sufficiently small to capture the desired Physics. The CFL stability condition, from the name of its finder R. Courant, K. Friedrichs and H. Lewy [@Courant:1928], is implemented inside the code in such a way that the user just needs to specify in the input deck the scalar parameter cfl < 1 that fixes the normalized simulation time step according to

$$\underline{\Delta t}_n = \text{cfl} \times F^n(\underline{\Delta x}, \underline{\Delta v}_x) < F^n(\underline{\Delta x}, \underline{\Delta v}_x) \tag{16}$$

at the time step $\underline{t}_n = \sum_{m=1}^{n} \underline{\Delta t}_m$ at time iteration n where $F^n(\underline{\Delta x}, \underline{\Delta v}_x)$ depends on the chosen numerical scheme. For example, if one notes

$${\underline{f_e}}^{n,i}(\underline{v_x}) = \frac{1}{\underline{\Delta x}} \int_{\underline{x}_{i-1/2}}^{\underline{x}_{i+1/2}} \underline{f_e}\left(\underline{x}, \underline{v_x}, \underline{t}_n \right) \, d\underline{x} \tag{17}$$

the finite volume plasma electron distribution function at the phase-space bin located in between $\underline{x}_{i-1/2} = \underline{x}_i - \underline{\Delta x}/2$ and $\underline{x}_{i+1/2} = \underline{x}_i + \underline{\Delta x}/2$ and one considers the Lax-Wendroff method to compute the advection

$$\frac{\partial \underline{f_e}}{\partial \underline{t}} + \underline{v_x} \frac{\partial \underline{f_e}}{\partial \underline{x}} = 0 \tag{18}$$

of plasma electrons along the spatial $\underline{x}$-axis in the phase-space, the numerical scheme reads

$$\frac{\underline{f_e}^{n+1,i} - \underline{f_e}^{n,i}}{\underline{\Delta t}_n} + \underline{v_x}\frac{\underline{F_x}^{n,i+1/2} - \underline{F_x}^{n,i-1/2}}{\underline{\Delta x}} = 0 \tag{19}$$

where the plasma electron fluxes across the bin volume interfaces located at $\underline{x}_{i\pm1/2}$ are given by

$$\underline{F_x}^{n,i+1/2} = \frac{\underline{f_e}^{n,i+1} + \underline{f_e}^{n,i}}{2} - \frac{\underline{v_x}\underline{\Delta t}_n}{\underline{\Delta x}}\frac{\underline{f_e}^{n,i+1} - \underline{f_e}^{n,i}}{2} \tag{20}$$

and

$$\underline{F_x}^{n,i-1/2} = \frac{\underline{f_e}^{n,i} + \underline{f_e}^{n,i-1}}{2} - \frac{\underline{v_x}\underline{\Delta t}_n}{\underline{\Delta x}}\frac{\underline{f_e}^{n,i} - \underline{f_e}^{n,i-1}}{2}. \tag{21}$$

According to the Taylor expansion of $\underline{f_e}^{n,i+i}$, $\underline{f_e}^{n,i-i}$ and $\underline{f_e}^{n+1,i}$ close to $(\underline{x}_i, \underline{t}_n)$ up to the third order in space and time, one can check the Lax-Wendroff numerical consistency error is indeed of second order :

$$\begin{aligned}\underline{\epsilon}^{n,i} &= \frac{\underline{f_e}^{n+1,i} - \underline{f_e}^{n,i}}{\underline{\Delta t}_n} + \underline{v_x}\frac{\underline{F_x}^{n,i+1/2} - \underline{F_x}^{n,i-1/2}}{\underline{\Delta x}} - \left(\frac{\partial \underline{f_e}}{\partial \underline{t}}\bigg|^{n,i} + \underline{v_x}\frac{\partial \underline{f_e}}{\partial \underline{x}}\bigg|^{n,i}\right) \\ &= \frac{\underline{\Delta t}_n{}^2}{6}\frac{\partial^3 \underline{f_e}}{\partial \underline{t}^3}\bigg|^{n,i} + \underline{v_x}\frac{\underline{\Delta x}^2}{6}\frac{\partial^3 \underline{f_e}}{\partial \underline{x}^3}\bigg|^{n,i} + O\left(\underline{\Delta t}_n{}^3 + \underline{\Delta x}^3 + \underline{\Delta t}_n\underline{\Delta x}^2\right).\end{aligned} \tag{22}$$

By using the Von Neumann stability analysis, assuming periodic boundary conditions for simplicity and noting

$$\widehat{\underline{f_e}}^{\,n}(\underline{k}^p) = \frac{1}{N_x}\sum_{i=1}^{N_x}\underline{f_e}^{n,i}\exp\left(-\iota\underline{k}^p\underline{x}_i\right) \Leftrightarrow \underline{f_e}^{n,i} = \sum_{p=1}^{N_x}\widehat{\underline{f_e}}^{\,n}(\underline{k}^p)\exp\left(\iota\underline{k}^p\underline{x}_i\right) \tag{23}$$

with $\iota^2 = -1$, $N_x = 1 + (\underline{x}_{\max} - \underline{x}_{\min})/\underline{\Delta x}$ the number of spatial grid points and $\underline{k}^p = 2\pi(p-1)/(\underline{x}_{\max} - \underline{x}_{\min})$ the discrete Fourier mode, one gets by injecting (23) in (19)

$$\frac{\widehat{\underline{f_e}}^{\,n+1}(\underline{k}^p)}{\widehat{\underline{f_e}}^{\,n}(\underline{k}^p)} = 1 - \frac{\underline{v_x}\underline{\Delta t}_n}{\underline{\Delta x}}\iota\sin\left(\underline{k}^p\underline{\Delta x}\right) + \left(\frac{\underline{v_x}\underline{\Delta t}_n}{\underline{\Delta x}}\right)^2\left[\cos\left(\underline{k}^p\underline{\Delta x}\right) - 1\right] \tag{24}$$

for each term p of the series. It implies the numerical scheme is stable,

$$\text{meaning } \left|\frac{\widehat{\underline{f_e}}^{\,n+1}(\underline{k}^p)}{\widehat{\underline{f_e}}^{\,n}(\underline{k}^p)}\right| < 1, \text{ if } \underline{\Delta t}_n < \frac{\underline{\Delta x}}{\underline{v_x}}. \tag{25}$$

Performing the same reasoning when also considering the advection of plasma electrons along the $\underline{v_x}$-axis in the phase-space due to the action of the electrostatic field, with bins centered at $\underline{v_x}^{\ell} = \underline{v_{x,\min}} + (\ell - 1)\underline{\Delta v}_x$ where $\ell \in [1, N_{v_x}]$ and $N_{v_x} = 1 + (\underline{v_{x,\max}} - \underline{v_{x,\min}})/\underline{\Delta v}_x$, in order to compute the Vlasov equation (1) and update all $\underline{f_e}^{n,i,\ell}$ with each numerical scheme implemented in ESVM, one finds (sometimes empirically when it is too difficult analytically) that

$$F^n(\underline{\Delta x}, \underline{\Delta v}_x) = \frac{1/2}{\dfrac{\max\limits_{\ell\in[1,N_{v_x}]}\{\underline{v_x}^{\ell}\}}{\underline{\Delta x}} + \dfrac{\max\limits_{i\in[1,N_x]}\{\underline{E_x}^{n,i}\}}{\underline{\Delta v}_x}}. \tag{26}$$

is a sufficient CFL stability condition for all numerical schemes implemented in ESVM to be stable.