sions between a nucleon and a nucleus. Results are presented in Figs. 1 and 2.

We consider first the results for the cross section (3), and $\beta=144$. It can be shown rigorously that as $U$ approaches $U_0$, the mean square angle approaches the value $2/\beta$. This is different from the very small value obtained by the evaluation of (13) and (14) by the method of steepest descents, which, as we have pointed out previously[12] does not give good results when $U$ is comparable with $U_0$. The light portion of the lines in Figs. 1 and 2 represent extrapolations to meet the known values for $U=U_0-0$.

As expected, using the cross section defined by (6), one obtains results differing radically from those already considered only when $U\approx U_0$. With (6), the value of $\langle\theta^2\rangle$ approaches zero, instead of the large value $2/\beta$ resulting from the use of (3). Evidently suitable experiments should be able to decide between the two alternative cross sections.

The results for the mesons are very similar to those for the nucleons, but somewhat smaller, as one might expect on the model adopted. On any other model, the difference between the spread of the mesons and that of the nucleons would be less. This difference could, in principle, enable one to determine which of the various models[7] of meson production, so far considered in cascade theory, is most nearly correct.

On account of its hereditary property, the normalized angular distribution for scattered particles in nuclear collisions represented by (6) is preserved under all conditions throughout the atmosphere and is the same for the mesons as for the nucleons.

[12] H. Messel and H. S. Green, Proc. Phys. Soc. (London) **A65**, 245 (1952); Phys. Rev. **83**, 1279 (1951).

---

PHYSICAL REVIEW VOLUME 88, NUMBER 6 DECEMBER 15, 1952

# The Stopping Power of $K$-Electrons*

M. C. WALSKE
*Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico*
(Received August 27, 1952)

The stopping power of $K$-shell electrons for large incident particle energies is calculated in this paper, adding to and correcting previous work on the subject. A definition is given for the average ionization potential of an atom independent of the energy of the incident particle. An integral representation is derived for Bethe's term $C_K$ which corrects the simple $K$-shell stopping number expression. Finally, asymptotic formulas, applicable at large incident particle energies, are given for use with any element.

## I. INTRODUCTION

THE energy loss of charged particles passing through matter has been calculated by Bethe,[1,2] using the Born approximation. Provided the velocity, $v$, of the incident particle is much larger than the "velocity" of the atomic electrons, the energy loss per cm path length is

$$-\frac{dE}{dx}=\frac{4\pi e^4z^2}{mv^2}NB, \tag{1}$$

where $B=Z\ln(2mv^2/I)$. Here $ez$ is the charge of the incident particle, $m$ the electronic mass, $N$ the number of stopping atoms per cm$^3$, $Z$ the nuclear charge, and $I$ the average excitation potential of the stopping atom. The quantity $B$ is called the stopping number.

If $v$ is not large compared to the "velocity" of some of the atomic electrons, and this will often be true for the inner electrons of heavier elements, (1) must be modified by calculating $B$ for these electrons separately (and without the approximations that impose on (1) a lower limit for $v$)[3]. Bethe[2] has done this for the stopping number, $B_K$, of the $K$-electrons and has given curves of $B_K$ *vs* $\eta_K$. $\eta_K$ is a convenient variable given as the quotient of $mv^2/2$ by the "ideal" ionization potential, $Z_{K\,\mathrm{eff}}^2$ Ry of the $K$-shell. For large $\eta_K$ Bethe gives an asymptotic formula of the form

$$B_K(\theta_K,\eta_K)=S_K(\theta_K)\ln\eta_K+T_K(\theta_K)-C_K(\theta_K,\eta_K), \tag{2}$$

where $\theta_K$ is the observed ionization potential[4] in units $Z_{K\,\mathrm{eff}}^2$ Ry. It is then possible to expand

$$C_K(\theta_K,\eta_K)=U_K(\theta_K)\eta_K^{-1}+V_K(\theta_K)\eta_K^{-2}+\cdots, \tag{3}$$

and Brown[5] has done this, obtaining the constant $U_K(\theta_K)$ for various $\theta_K$. As pointed out by Walske and Bethe,[6] Brown's values need a correction. The details

* Part of this work is included in the author's doctoral thesis at Cornell University, 1951.

[1] H. Bethe, Ann. Physik **5**, 325 (1930).

[2] M. S. Livingston and H. A. Bethe, Revs. Modern Phys. **9**, 263 (1937). A detailed discussion of Bethe's stopping power theory is given in this reference.

[3] These approximations are the ones mentioned at the end of Sec. II.

[4] More accurately, the energy difference between ground state and lowest unoccupied state should be used.

[5] L. M. Brown, Phys. Rev. **79**, 297 (1950).

[6] M. C. Walske and H. A. Bethe, Phys. Rev. **83**, 457 (1951).

of this correction will be presented here, and $V_K(\theta_K)$ will be given. Furthermore, Brown determined the constants $S_K(\theta_K)$ and $T_K(\theta_K)$ in (2). There is a slight numerical error[7] in his values of $T_K(\theta_K)$, which will be corrected in this paper. This error also necessitates new curves for $C_K(\theta_K, \eta_K)$. The $B_K(\theta_K, \eta_K)$ curves as given in Brown's paper remain unchanged.

## II. INTEGRAL REPRESENTATION OF $C_i(\theta_i, \eta_i)$

We shall define for any atomic shell, $i$, $C_i(\theta_i, \eta_i)$ by a formula similar to (2):

$$B_i(\theta_i, \eta_i) = S_i(\theta_i) \ln\eta_i + T_i(\theta_i) - C_i(\theta_i, \eta_i) \qquad (i = K, L, M \cdots), \quad (4)$$

where $\theta_i$ is the observed ionization potential of the $i$th shell divided by the "ideal" ionization potential, $Z_{i\,\text{eff}}^2\,\text{Ry}/n_i^2$ ($n_i$ is the principal quantum number of the $i$th shell), and $\eta_i = mv^2/2Z_{i\,\text{eff}}^2\,\text{Ry}$. It follows then that $B = \sum_i B_i(\theta_i, \eta_i)$ is the total stopping number of the atom. It can be shown[8] that $\sum_i S_i(\theta_i) = Z$, and so one may also write

$$B = Z \ln(2mv^2/I_{\text{Av}}) - \sum_i C_i(\theta_i, \eta_i),$$

where we define $I_{\text{Av}}$ by

$$Z \ln I_{\text{Av}} = \sum_i S_i(\theta_i) \ln(Z_{i\,\text{eff}}^2\,\text{Ry}/\lambda_i(\theta_i) n_i^2),$$

with $\lambda_i(\theta_i)$ given by

$$T_i(\theta_i) = S_i(\theta_i) \ln[4n_i^2 \lambda_i(\theta_i)].$$

As shown in reference 8, $S_i(\theta_i)$ is equal to the number of $i$-electrons times $\frac{1}{2}$ of 1 plus the oscillator strength per $i$-shell electron. This formulation enables one to define a velocity-independent average ionization potential, and also displays clearly the form of the correction needed in (1) at low velocities of the incident particle.

Analogously to Bethe's formula for the $K$-shell,[2] we can write the exact expression for the stopping number for the $i$th shell in an integral form which is not limited by the velocity condition on (1):

$$B_i(\theta_i, \eta_i) = \int_{W_{\min} = \theta_i/n_i^2}^{\infty} W dW \int_{W^2/4\eta_i}^{\infty} \frac{dQ}{Q^2} |F_{W,i}(Q)|^2, \quad (5)$$

where $W$ is the energy transferred to the atomic electron in units $Z_{i\,\text{eff}}^2$ Ry, $Q = q^2 =$ (change in incident particle's momentum)$^2/2mZ_{i\,\text{eff}}^2$ Ry, and $|F_{W,i}(Q)|^2$ is the sum of the squares of the matrix elements of $e^{iqx}$ between the various $i$-shell electron states and the states greater in energy by $W$.

[7] This error was also published in reference 6 where some of Brown's results were quoted. Also in reference 8 the quantity $\lambda$ is slightly changed by the corrected $T_K(\theta_K)$.

Bethe, Brown, and Walske[8] have considered the expression[9]

$$\begin{aligned} x(\theta, \eta) = &\int_0^\infty dW \int_{1/4\eta}^{4\eta} \frac{dQ}{Q} \phi_W(Q) \\ &- \int_{W_{\min}}^\infty dW \int_{1/4\eta}^{W^2/4\eta} \frac{dQ}{Q} \phi_W(0) \\ &- \lim_{Q_1 \to 0} \int_0^{W_{\min}} dW \left[ \int_{1/4\eta}^{Q_1} \frac{dQ}{Q} \phi_W(0) + \int_{Q_1}^\infty \frac{dQ}{Q} \phi_W(Q) \right], \end{aligned} \quad (6)$$

where $\phi_W(Q) = W|F_W(Q)|^2/Q$, and where the integral up to $W_{\min}$ denotes all transitions forbidden by the Pauli principle. They have shown that this expression behaves, for all $\eta$ and for fixed $\theta$, exactly as the sum of a logarithmic term in $\eta$ and a term independent of $\eta$. We shall consider the difference between (5) and (6), and eventually we shall show that this difference goes to zero for large $\eta$, and so it is just the $-C(\theta, \eta)$ of (4). For the purpose of taking the difference of (5) and (6) it is convenient to rewrite (6), using in the first term the sum rule of Bethe (reference 1),[10]

$$\int_0^\infty dW \phi_W(Q) = n^2 = \int_0^\infty dW \phi_W(0). \quad (7)$$

Then we write the third term of (6) as

$$\int_{1/4\eta}^{4\eta} - \int_{Q_1}^{4\eta}.$$

The second of these expressions will be used as it stands; the first is combined with the first term in (6) as transformed by (7), and this result is in turn combined with the second term of (6). One may then take the limit on $Q_1$ and get

$$\begin{aligned} x(\theta, \eta) = &\int_{W_{\min}}^\infty dW \int_{W^2/4\eta}^{4\eta} \frac{dQ}{Q} \phi_W(0) \\ &- \int_0^{W_{\min}} dW \int_0^{4\eta} \frac{dQ}{Q} [\phi_W(Q) - \phi_W(0)] \\ &- \int_0^{W_{\min}} dW \int_{4\eta}^\infty \frac{dQ}{Q} \phi_W(Q). \end{aligned}$$

[8] Bethe, Brown, and Walske, Phys. Rev. **79**, 413 (1950).

[9] We now drop the $i$-shell subscripts.

[10] $\int_0^\infty dW \phi_W(Q)$ is to be interpreted as $\sum_{\text{all } n} \phi_n(Q)$, where $\phi_n(Q) = (E_n - E_i)|F_n(Q)|^2/Q$. The $n^2$ occurs in (7) from our taking the effect of the $n_i^2$ distinct electrons in a given shell into $|F_n(Q)|^2$ and $|F_W(Q)|^2$.

The difference of (5) and (6), $-C(\theta,\eta)$, is then

$$-C(\theta,\eta)=\int_{W_{\min}}^{\infty}dW\int_{4\eta}^{\infty}\frac{dQ}{Q}\phi_W(Q)+\int_0^{\infty}dW\int_{W^2/4\eta}^{4\eta}\frac{dQ}{Q}[\phi_W(Q)-\phi_W(0)]+\int_0^{W_{\min}}dW\int_0^{W^2/4\eta}\frac{dQ}{Q}[\phi_W(Q)-\phi_W(0)]+\int_0^{W_{\min}}dW\int_{4\eta}^{\infty}\frac{dQ}{Q}\phi_W(Q).$$

We may, to order $\eta^{-4}$, neglect the last integral above because for large $Q$ and *fixed* $W$, $\phi_W(Q)\sim Q^{-5}$ [see Eq. (9) for the $K$-shell case]; therefore, the last integral goes as $\eta^{-5}$ for large $\eta$. We shall now rewrite the second term above. We denote it by $G$ so that we may write

$$G=\int_0^{4\eta}dW\int_{W^2/4\eta}^{4\eta}\frac{dQ}{Q}[\phi_W(Q)-\phi_W(0)]-\int_{4\eta}^{\infty}dW\int_{4\eta}^{W^2/4\eta}\frac{dQ}{Q}[\phi_W(Q)-\phi_W(0)].$$

(Note our reversal of the limits in the inner integral of the second term of $G$.) Reversing the order of integration, the first term of $G$ is just

$$G_1=\int_0^{4\eta}\frac{dQ}{Q}\int_0^{(4\eta Q)^{\frac{1}{2}}}dW[\phi_W(Q)-\phi_W(0)]-\int_0^{4\eta}\frac{dQ}{Q}\int_0^{\infty}dW[\phi_W(Q)-\phi_W(0)],$$

since the inner integral of the second term vanishes by the sum rule (7). Combining,

$$G_1=-\int_0^{4\eta}\frac{dQ}{Q}\int_{(4\eta Q)^{\frac{1}{2}}}^{\infty}dW[\phi_W(Q)-\phi_W(0)],$$

and reversing the order of integrations once again,

$$G_1=-\int_0^{4\eta}dW\int_0^{W^2/4\eta}\frac{dQ}{Q}[\phi_W(Q)-\phi_W(0)]-\int_{4\eta}^{\infty}dW\int_0^{4\eta}\frac{dQ}{Q}[\phi_W(Q)-\phi_W(0)].$$

Inserting this result in $G$ above, we have

$$G=-\int_0^{\infty}dW\int_0^{W^2/4\eta}\frac{dQ}{Q}[\phi_W(Q)-\phi_W(0)].$$

Thus, to order $\eta^{-4}$ we may write

$$-C(\theta,\eta)=\int_{W_{\min}}^{\infty}dW\int_{4\eta}^{\infty}\frac{dQ}{Q}\phi_W(Q)-\int_{W_{\min}}^{\infty}dW\int_0^{W^2/4\eta}\frac{dQ}{Q}[\phi_W(Q)-\phi_W(0)]. \quad (8)$$

We now write out $\phi_{W,K}(Q)$ for the $K$-shell[11] explicitly (from reference 1) and study the properties of (8):

$$\phi_{W,K}(Q)=\frac{2^7W}{1-e^{-2\pi/k}}\times\frac{(Q+\frac{1}{3}k^2+\frac{1}{3})\exp\left[-\frac{2}{k}\tan^{-1}\left(\frac{2k}{Q-k^2+1}\right)\right]}{[(Q-k^2+1)^2+4k^2]^3}, \quad (9)$$

with $W=k^2+1$. If we consider fixed $W$, then in the first integral of (8) the integrand goes as $Q^{-6}$, and so the integral converges well. In the second integral a logarithmic divergence is avoided at small $Q$ by the fact that $\phi_W(Q)-\phi_W(0)$ goes to zero for small $Q$. For large $W$ and $Q$ we have convergence through a cancellation of the first integral with the first term of the second integral.

Anticipating our results in expanding (8) in a power series in $\eta^{-1}$ with no constant term, we may conclude that (8) is indeed the $-C(\theta,\eta)$ of (4). In addition, since $x(\theta,\eta)$ was obtained in reference 5 by rewriting $B(\theta,\eta)$ with a $Q_{\max}$ of $4\eta$ instead of the true $Q_{\max}$ of infinity and with the assumption $Q_{\min}\ll 1$ for all $W$, it follows that the error introduced by these two approximations is of order $\eta^{-1}$. Bethe has used this fact frequently in his stopping power theory where large $\eta$ were being considered.

## III. SEPARATION OF $C(\theta,\eta)$ INTO CONTRIBUTIONS FROM HIGH AND LOW MOMENTUM TRANSFERS

If in (8) we break off the $W$ integration at $4\eta\alpha$, with $\alpha$ a constant such that $\frac{1}{2}<\alpha<1$, then to order $\eta^{-4}$:

$$-C(\theta,\eta)=C_1(\theta,\eta)+C_2(\eta), \quad (10)$$

where

$$C_1=\int_{W_{\min}}^{4\eta\alpha}dW\int_{4\eta}^{\infty}\frac{dQ}{Q}\phi_W(Q)-\int_{W_{\min}}^{4\eta\alpha}dW\int_0^{W^2/4\eta}\frac{dQ}{Q}[\phi_W(Q)-\phi_W(0)], \quad (10a)$$

and

$$C_2=\int_{4\eta\alpha}^{\infty}dW\int_{4\eta}^{\infty}\frac{dQ}{Q}\phi_W(Q)-\int_{4\eta\alpha}^{\infty}dW\int_0^{W^2/4\eta}\frac{dQ}{Q}[\phi_W(Q)-\phi_W(0)]. \quad (10b)$$

[11] $W$ for the $K$-shell has been called $\epsilon$ elsewhere in the literature. The $\tan^{-1}$ in (9) is to be taken in the first and second quadrants.

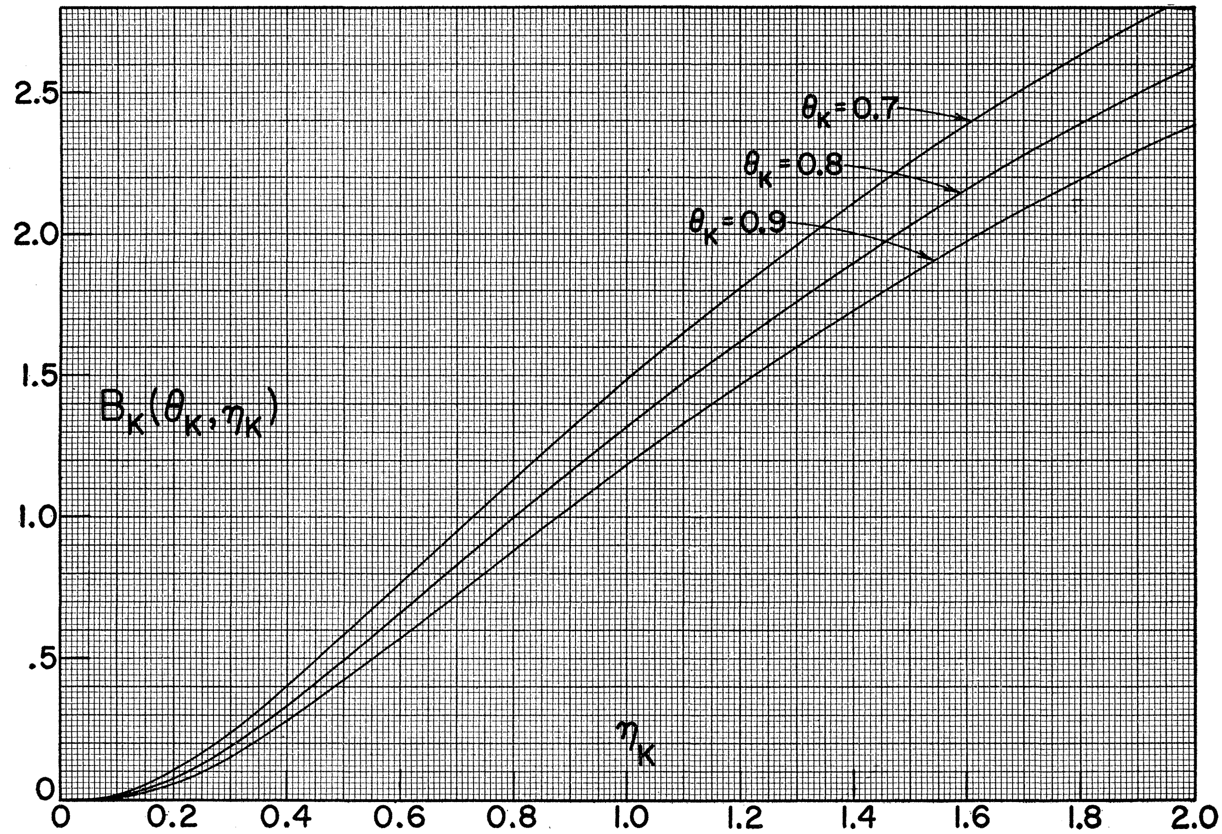


FIG. 1. $B_K(\theta_K, \eta_K)$, stopping number contribution of $K$-electrons.

First we consider $C_1$. Since the maximum of $\phi_W(Q)$ occurs at about $W=Q$ (see (9)), and since with our choice of $\alpha$, $Q-W$ is at least of order $\eta$, the first term of $C_1$ contributes only to order $\eta^{-5}$. We may neglect it since we are ultimately interested in an expansion of $C(\theta, \eta)$ in the form (3) only to the term in $\eta^{-2}$. In the second term the main contribution arises from small $Q$, and for this reason we call $C_1$ the low momentum transfer contribution to $C(\theta, \eta)$. The integrand in the second term may be expanded in a power series in $Q$ which converges inside the circle of convergence in the complex $Q$ plane. The radius of this circle is the distance from the origin to the nearest singularity of the integrand which for the $K$-shell[12] [see (9)] occurs at $(Q-k^2+1)^2+4k^2=0$. Hence, the radius of the circle of convergence is $k^2+1=W$. Thus, the power series is convergent for real $Q<W$, but from the upper limit on the $Q$ integration we see that the maximum $Q$ is $W^2/4\eta$, which is at most at the maximum $W$ just $(4\eta\alpha/4\eta)W = \alpha W < W$ since we have chosen $\frac{1}{2}<\alpha<1$. Thus, the expansion is quite valid here, and using it $C_1$ is correctly obtained to order $\eta^{-4}$ from the second term of (10a).

Brown [reference 5, Eqs. (25) to (31)] has done this for $W_{\min}=0$[13] in the case of the $K$-shell obtaining the term in $\eta^{-1}$. However, in Brown's case the upper limit of the $W$ integration is infinity rather than $4\eta\alpha$. Still his method gives $C_1$ (with $W_{\min}=0$) to order $\eta^{-2}$ since one may easily show using (9) that the different $W_{\max}$ makes a difference only in order $\eta^{-5/2}$. If one extends Brown's work to the $\eta^{-2}$ term, the net result for the

TABLE I. Comparison of computed and asymptotic $C_K(\theta_K, \eta_K)$.

| | Computed values of $C_K(\theta_K, \eta_K)$ $\theta_K=0.7$ | 0.8 | 0.9 | $C_K(\theta_K, \eta_K)$ from (19) $\theta_K=0.7$ | 0.8 | 0.9 |
|---|---|---|---|---|---|---|
| $\eta_K=10$ | 0.2407 | 0.2460 | 0.2472 | 0.2799 | 0.2852 | 0.2864 |
| $\eta_K=20$ | 0.1153 | 0.1180 | 0.1186 | 0.1216 | 0.1243 | 0.1249 |

TABLE II. Stopping number contribution of $K$-electrons, $B_K(\theta_K, \eta_K)$.

| $\eta_K$ | $\theta_K=0.7$ | $\theta_K=0.8$ | $\theta_K=0.9$ |
|---|---|---|---|
| 1.5 | 2.249 | 2.031 | 1.857 |
| 1.75 | 2.573 | 2.337 | 2.142 |
| 2.0 | 2.851 | 2.595 | 2.385 |
| 2.5 | 3.366 | 3.077 | 2.841 |
| 3.5 | 4.122 | 3.782 | 3.508 |
| 5.0 | 4.931 | 4.537 | 4.221 |
| 10.0 | 6.406 | 5.900 | 5.496 |

[12] The same general conclusions to be drawn for the $K$-shell can be shown to hold for the $L$-shell, and undoubtedly hold for all shells.

[13] This is just the case of a hydrogen atom except that it is for 2 $K$-electrons.

$K$-shell is

$$C_{1,K}(W_{\min}=0,\eta_K) = -\eta_K^{-1}-(19/6)\eta_K^{-2} \text{ (to order } \eta_K^{-2}). \quad (11)$$

## IV. EVALUATION OF $C_{2,K}(\theta_K, \eta_K)$, HIGH MOMENTUM TRANSFER CONTRIBUTION TO $C_K(\theta_K, \eta_K)$, TO ORDER $\eta_K^{-2}$

For the $K$-shell $C_2$ is explicitly [see (10b) and (9)]

$$C_{2,K}=2^7\int_{4\eta\alpha}^{\infty}\frac{WdW}{1-e^{-2\pi/k}}\times\Bigg\{\int_{4\eta}^{\infty}\frac{dQ}{Q}\frac{(Q+\frac{1}{3}k^2+\frac{1}{3})\exp\left[-\frac{2}{k}\tan^{-1}\left(\frac{2k}{Q-k^2+1}\right)\right]}{[(Q-k^2+1)^2+4k^2]^3}$$
$$-\int_0^{W^2/4\eta}\frac{dQ}{Q}\left[\frac{(Q+\frac{1}{3}k^2+\frac{1}{3})\exp\left[-\frac{2}{k}\tan^{-1}\left(\frac{2k}{Q-k^2+1}\right)\right]}{[(Q-k^2+1)^2+4k^2]^3}-\frac{1}{3}\frac{\exp\left(-\frac{4}{k}\tan^{-1}k\right)}{(k^2+1)^5}\right]\Bigg\}.$$

Since the main contribution to $C_{2,K}$ arises from $Q\sim 4\eta$ which is large, we call $C_{2,K}$ the high momentum contribution to $C_K(\theta,\eta)$. In order to rewrite the two inside integrals with the same limits of integration we let $Q=y$ in the first, and $Q=W^2/y$ in the second. We make the further transformation $z=y-k^2+1$, and remembering that $W=k^2+1$, we obtain

$$C_{2,K}=\frac{2^7}{3}\int_{k0}^{\infty}dk\frac{2k(k^2+1)}{1-e^{-2\pi/k}}\int_{a^2-k^2-1}^{\infty}\frac{dz}{z+k^2-1}\times\left[\frac{(3z+4k^2-2)\exp\left(-\frac{2}{k}\tan^{-1}\frac{2k}{z}\right)}{(z^2+4k^2)^3}-\frac{(z+4k^2+2)(z+k^2-1)^5\exp\left(\frac{2}{k}\tan^{-1}\frac{2k}{z}\right)}{(z^2+4k^2)^3}\cdot\frac{\exp\left(-\frac{4}{k}\tan^{-1}k\right)}{(k^2+1)^5}+\frac{\exp\left(-\frac{4}{k}\tan^{-1}k\right)}{(k^2+1)^5}\right], \quad (12)$$

where $k_0=(\alpha a^2-1)^{\frac{1}{2}}$, $a^2=4\eta$, and the $\tan^{-1}$ are in the first and second quadrants.

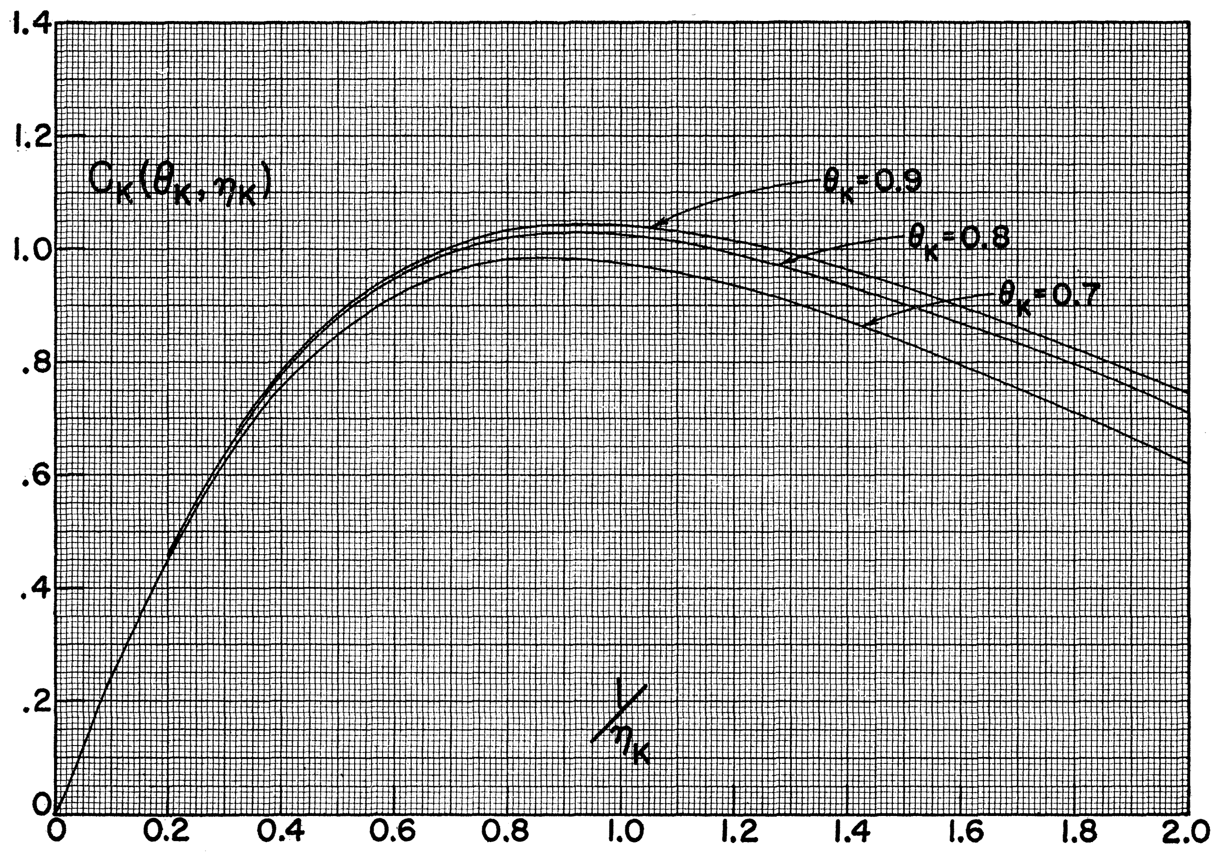


FIG. 2. $C_K(\theta_K, \eta_K)$, correction to $K$-electrons' stopping number contribution.

So far we have chosen $\alpha$ so that $\frac{1}{2}<\alpha<1$. It is now convenient to further restrict $\alpha$ so that the lower limit on the $z$ integration, $a^2-k^2+1$, is never greater than $k^2$.[14] That is, we want $a^2-k^2+1\leq k^2$, or $k\geq[(a^2+1)/2]^{\frac{1}{2}}$. This will be true if $k_{\min}=k_0=[(a^2+1)/2]^{\frac{1}{2}}$. In terms of $\alpha$ this means $\alpha a^2-1=(a^2+1)/2$, or $\alpha=1/2+(3/2)a^{-2}=1/2+(3/8)\eta^{-1}$, which for reasonably large $\eta$ is consistent with $\frac{1}{2}<\alpha<1$.

We shall now show that the contribution, say $H$, to $C_{2,K}$ from $k^2<z\leq\infty$ may be neglected without error to order $\eta^{-2}$. $H$ is written exactly as (12) except that the $z$-limits are $k^2$ to infinity. Expanding the square bracket of (12) in powers of $k^2/z$, one sees that the last term in the bracket serves its purpose of canceling the logarithmic divergence arising in the second term at large $z$. The result is that the square bracket is $O(k^{-8}z^{-1})$, making the inside integral $O(k^{-10})$, and so to highest order

$$H=O\int_{k_0}^{\infty}k^4dk\int_{k^2}^{\infty}dzk^{-8}z^{-2}=O\int_{k_0}^{\infty}dkk^{-6}$$
$$=O(k_0^{-5})=O(\eta^{-5/2}).$$

Thus, since we are calculating to order $\eta^{-2}$, we may neglect $H$. Also, since the third term in the square bracket of (12) is only important as noted above in $H$, we may neglect it henceforth in $C_{2,K}$ to order $\eta^{-2}$. Thus, to order $\eta^{-2}$,

$$C_{2,K}=\frac{2^7}{3}\int_{k_0}^{\infty}dkk(k^2+1)\operatorname{csch}\pi/k$$
$$\times\int_{a^2-k^2+1}^{k^2}dz\frac{1}{z+k^2-1}\cdot\frac{1}{(z^2+4k^2)^3}$$
$$\times\left\{(4k^2+3z-2)\exp\left(\frac{2}{k}\tan^{-1}\frac{z}{2k}\right)\right.$$
$$-(4k^2+z+2)(k^2+z-1)^5\exp\left(-\frac{2}{k}\tan^{-1}\frac{z}{2k}\right)$$
$$\left.\times\exp\left(\frac{4}{k}\tan^{-1}\frac{1}{k}\right)\Big/(k^2+1)^5\right\},$$

where $\tan^{-1}$ is now to be taken in the first and fourth quadrants.

One is now in a position to expand the integrand to the degree necessary for obtaining the terms which give $C_{2,K}$ to order $\eta^{-2}$. This is an extremely long and tedious job so we shall only present here the results and a few remarks. One can show that the factor $(z+k^2-1)^{-1}$ need be expanded in powers of $(z-1)/k^2$ to four terms for the desired accuracy. The factor $(z^2+4k^2)^{-3}$ is not expanded. Using the fact that $k\gg1$, $(k^2+1)^{-5}$ is expanded as are the exponentials and $\tan^{-1}1/k$, but $\tan^{-1}z/2k$ is not expanded. The result is that all the $z$ integrals are elementary. The $k$ integration can then not be done explicitly, but to order $\eta^{-2}$ the $\eta$-dependence can be determined by breaking the $k$ integration into appropriate ranges and separating leading terms. It is particularly noteworthy that the parameter $k_0$ does not affect the result directly. The terms involving $k_0$ cancel out, independent of the fact that $k_0=[(a^2+1)/2]^{\frac{1}{2}}$. This is as it should be since the precise choice of $k_0$ (or $\alpha$) is only felt in order $\eta^{-5/2}$. Our result to order $\eta^{-2}$ is

$$C_{2,K}=-\eta_K^{-1}-(25/6)\eta_K^{-2}. \tag{13}$$

Adding this to $C_{1,K}(W_{\min}=0,\eta_K)$ [see (11)] we get, to order $\eta_K^{-2}$,

$$C(W_{\min}=0,\eta_K)=2\eta_K^{-1}+(22/3)\eta_K^{-2}. \tag{14}$$

## V. ASYMPTOTIC FORMULAS FOR $B_K(\theta_K, n_K)$ TO ORDER $n_K^{-2}$

In order to calculate $C_K(\theta_K,\eta_K)$ for elements other than hydrogen we must subtract from (14) the contributions arising from transitions to hydrogen-like states which are forbidden in the heavier elements by the Pauli principle, i.e., occupied states. Brown (reference 5) has worked out in detail the method of doing this, and has applied it to obtain $C_K(\theta_K,\eta_K)$ to order $\eta_K^{-1}$. Except for his omission of the high momentum transfer contribution, $C_{2,K}$, and a numerical error in his determination of $T_K(\theta_K)$ [see (2)], his results are correct to order $\eta_K^{-1}$. We have extended this work in a straightforward way to obtain the term in $\eta_K^{-2}$. The procedure is to find the contribution of transitions from the $K$-shell to all discrete states for hydrogen, subtract this from $B_K(W_{\min}=0,\eta_K)$ corresponding to the $C_K(W_{\min}=0,\eta_K)$ given by (14) to obtain the continuum contribution in a hydrogen-like atom with no outer screening, and then add to the latter the contribution from transitions to continuum states with $W<1$ (and unoccupied discrete states) in the actual atom. It is in this last contribution, called $D_K(\theta)$ by Brown, that a numerical error was made in the $\eta$-independent term. In calculating the $\eta$-independent term of his Eq. (22) from his (20), he mistakenly used $1+n^{-2}$ instead of $1-n^{-2}$ in (20). With this term corrected, and with the $\eta_K^{-2}$ term added, $D_K(\theta)$ is

$$\begin{aligned}
D_K(0.7)\ &=0.3783_1\ln\eta_K+0.3503_9\\
&\quad+0.0528_0\eta_K^{-1}+0.0133_0\eta_K^{-2},\\
D_K(0.75)&=0.28729\ln\eta_K+0.2945_2\\
&\quad+0.01910\eta_K^{-1}+0.01850\eta_K^{-2},\\
D_K(0.8)\ &=0.21072\ln\eta_K+0.2362_6\\
&\quad-0.00063\eta_K^{-1}+0.01887\eta_K^{-2},\\
D_K(0.85)&=0.14569\ln\eta_K+0.1769_0\\
&\quad-0.01008\eta_K^{-1}+0.01616\eta_K^{-2},\\
D_K(0.9)\ &=0.08996\ln\eta_K+0.1173_8\\
&\quad-0.01190\phi_K^{-1}+0.01161\eta_K^{-2}.
\end{aligned} \tag{15}$$

[14] Note that $z_{\min}=a^2-k^2+1$ is automatically greater than $-k^2$.

Brown's result for the contribution of the discrete states to $B_K$ for hydrogen, extended to the $\eta_K^{-2}$ term, is

$$0.56500 \ln\eta_K + 0.46869 + 0.11896\eta_K^{-1} + 0.00462\eta_K^{-2}, \quad (16)$$

and his result for $B_K(W_{\min}=0, \eta_K)$[15] as corrected by (14) is

$$B_K(W_{\min}=0, \eta_K) = 2\ln\eta_K + 2.57861 - 2\eta_K^{-1} - (22/3)\eta_K^{-2}. \quad (17)$$

Subtracting (16) from (17) we obtain the continuum contribution, $E_K$:

$$E_K = 1.43500 \ln\eta_K + 2.10992 - 2.11896\eta_K^{-1} - 7.33795\eta_K^{-2}. \quad (18)$$

Finally, adding (18) to (15) we get $B_K(\theta_K, \eta_K)$, the asymptotic formulas for the stopping number of the $K$-shell:

$$\begin{aligned}
B_K(0.7, \eta_K) &= 1.8133 \ln\eta_K + 2.4603 - 2.0662\eta_K^{-1} - 7.3246\eta_K^{-2},\\
B_K(0.75, \eta_K) &= 1.7223 \ln\eta_K + 2.4044 - 2.0999\eta_K^{-1} - 7.3194\eta_K^{-2},\\
B_K(0.8, \eta_K) &= 1.6457 \ln\eta_K + 2.3462 - 2.1196\eta_K^{-1} - 7.3191\eta_K^{-2}, \quad (19)\\
B_K(0.85, \eta_K) &= 1.5807 \ln\eta_K + 2.2868 - 2.1290\eta_K^{-1} - 7.3218\eta_K^{-2},\\
B_K(0.9, \eta_K) &= 1.5250 \ln\eta_K + 2.2273 - 2.1309\eta_K^{-1} - 7.3263\eta_K^{-2}.
\end{aligned}$$

Of course, $C_K(\theta_K, \eta_K)$ to order $\eta_K^{-2}$ is just the negative of the last two terms of (19).

As a check on the last two terms of (19), $C_K(\theta_K, \eta_K)$ has been evaluated numerically for $\eta_K = 10$ and 20 using the expression (8). A comparison with $C_K(\theta_K, \eta_K)$ as obtained from the last two terms of (19) is given in Table I. From this table one can see that the asymptotic terms for $C_K$ are high by about 16 percent at $\eta_K = 10$ and by about 5.5 percent at $\eta_K = 20$ for all $\theta_K$. If one adds to the asymptotic formula for $B_K$ (or subtracts from $C_K$) a term $45\eta_K^{-3}$ these errors are reduced to about $-2.4$ percent and 0.6 percent, respectively, for all $\theta_K$. This should therefore be done. That the $\theta_K$-variation of $C_K$ is given to greater accuracy by (19) than the full value of $C_K$ can be seen by differencing the $C_K$'s of different $\theta_K$ for the two sets of values in Table I.

For convenience we reprint in Fig. 1 curves of $B_K(\theta_K, \eta_K)$ for $\theta_K = 0.7$, 0.8, 0.9, and for $0 \leq \eta_K \leq 1.5$ which were calculated by the author from expression (5) and previously published by L. M. Brown in reference 5. In Table II we give a continuation of these calculations up to $\eta_K = 10$. These values are accurate to at least one percent. At $\eta_K = 10$ we have two completely independent calculations of $B_K$; one is to take the first two terms of (19) and to subtract the exact numerically calculated value of $C_K$ from Table I, the other is the direct calculation of $B_K$ reported in Table II. The values for the successive $\theta_K$ calculated via $C_K$ are 6.401, 5.895, and 5.497, in excellent agreement with Table II. In Fig. 2 we have combined the values from Table II and the first two terms of (19) in order to plot $C_K(\theta_K, \eta_K)$ *vs* $1/\eta_K$. Near the origin on this graph we have relied upon the asymptotic formula and the exact calculations of $C_K$ at $\eta_K = 10$ and 20. This curve is changed from the one in reference 5 by the correction to the constant term in (19).

It is a pleasure for the author to acknowledge his indebtedness to Professor H. A. Bethe for his friendly guidance and encouragement of this work.

[15] This is just twice the stopping number of a hydrogen atom since it is written for two $K$-electrons.