Chapter 3
Recursion

Data Structures and Algorithms

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Outcomes

- **L.O.8.1** - Describe the basic components of recursive algorithms (functions).
- **L.O.8.2** - Draw trees to illustrate callings and the value of parameters passed to them for recursive algorithms.
- **L.O.8.3** - Give examples for recursive functions written in C/C++.
- **L.O.8.5** - Develop experiment (program) to compare the recursive and the iterative approach.
- **L.O.8.6** - Give examples to relate recursion to backtracking technique.

Contents

Sources of Materials

1. We would like to thank **Dr. The-Nhan LUONG**, a former instructor of our Department, for the composing of this document.
2. This document also uses figure, sentences and demo source code from the following sources:
   - The old presentation for course *Data Structures and Algorithms* edited by other members in our Department
   - Book entitled **Data Structures - A Pseudocode Approach with C++ (first edition, 2001)** written by Richard F. Gilberg and Behrouz A. Forouzan

Recursion and the basic components of recursive algorithms

Recursion

Definition

Recursion is a repetitive process in which an algorithm calls itself.

- Direct : A → A
- Indirect : A → B → A

Example

Factorial

$$Factorial(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \times (n-1) \times (n-2) \times ... \times 3 \times 2 \times 1 & \text{if } n > 0 \end{cases}$$

Using recursion:

$$Factorial(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \times Factorial(n-1) & \text{if } n > 0 \end{cases}$$

Basic components of recursive algorithms

Two main components of a Recursive Algorithm

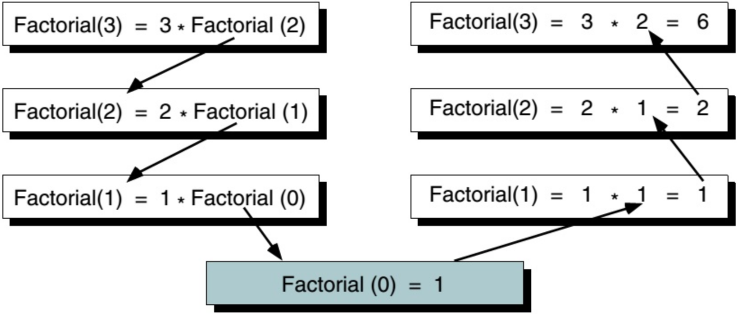
1. Base case (i.e. stopping case)
2. General case (i.e. recursive case)

Example

Factorial

$$Factorial(n) = \begin{cases} 1 & \text{if } n = 0 \quad \text{base case} \\ n \times Factorial(n-1) & \text{if } n > 0 \quad \text{general case} \end{cases}$$

Recursion



Hình: Factorial (3) Recursively (source: Data Structure - A pseudocode Approach with C++)

Recursion

Factorial: Iterative Solution

Algorithm iterativeFactorial(n)

Calculates the factorial of a number using a loop.

Pre: n is the number to be raised factorially

Post: n! is returned - result in factoN

```
i = 1
factoN = 1
while i <= n do
    factoN = factoN * i
    i = i + 1
end
return factoN
End iterativeFactorial
```

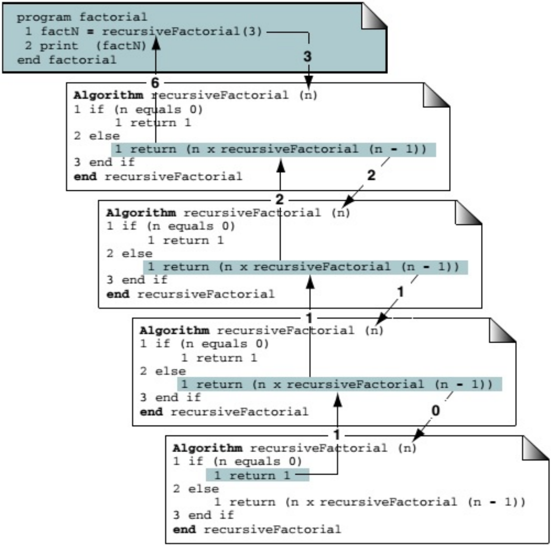
Recursion

Factorial: Recursive Solution

Algorithm recursiveFactorial(n)
Calculates the factorial of a number using a recursion.
Pre: n is the number to be raised factorially
Post: n! is returned

if $n = 0$ **then**
  return 1
else
  return n * recursiveFactorial(n-1)
end
End recursiveFactorial

Recursion



Hình: Calling a Recursive Algorithm (source: Data Structure - A pseudocode Approach with C++)

Properties of recursion

Properties of all recursive algorithms

- A recursive algorithm solves the large problem by using its solution to a simpler sub-problem
- Eventually the sub-problem is simple enough that it can be solved without applying the algorithm to it recursively.
  $\rightarrow$ This is called the base case.

Designing recursive algorithms

The Design Methodology

Every recursive call must either solve a part of the problem or reduce the size of the problem.

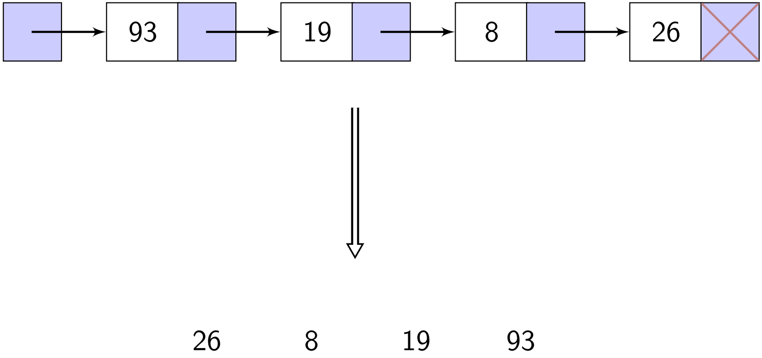
Rules for designing a recursive algorithm

1. Determine the base case (stopping case).
2. Then determine the general case (recursive case).
3. Combine the base case and the general cases into an algorithm.

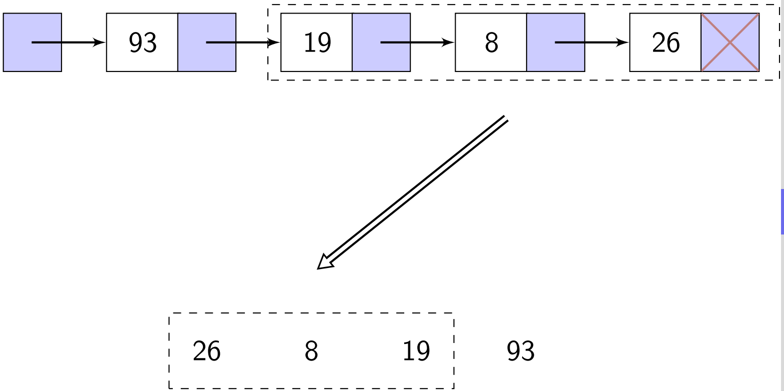
Limitations of Recursion

- A recursive algorithm generally runs more slowly than its nonrecursive implementation.
- BUT, the recursive solution shorter and more understandable.

Print List in Reverse



Print List in Reverse



Print List in Reverse

Algorithm printReverse(list)
Prints a linked list in reverse.
Pre: list has been built
Post: list printed in reverse

if *list is null* **then**
    return
end
printReverse (list -> next)
print (list -> data)
End printReverse

Greatest Common Divisor

Definition

$$\gcd(a,b) = \left[\begin{array}{ll} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ \gcd(b, a \mod b) & \text{otherwise} \end{array}\right.$$

Example

$\gcd(12, 18) = 6$
$\gcd(5, 20) = 5$

Greatest Common Divisor

Algorithm gcd(a, b)
Calculates greatest common divisor using the Euclidean algorithm.
Pre: a and b are integers
Post: greatest common divisor returned

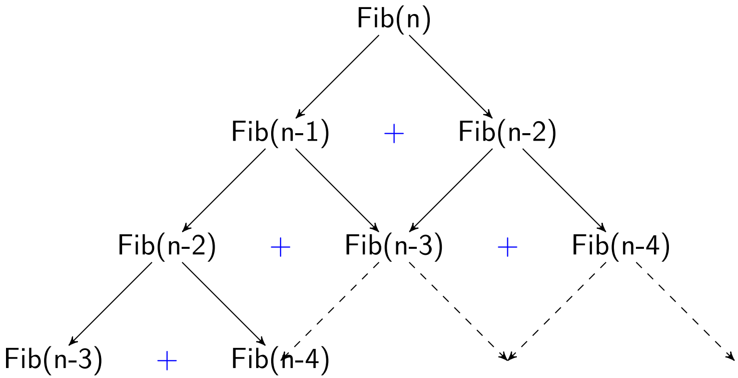
if $b = 0$ **then**
    return a
end
if $a = 0$ **then**
    return b
end
return gcd(b, a mod b)
End gcd

Fibonacci Numbers

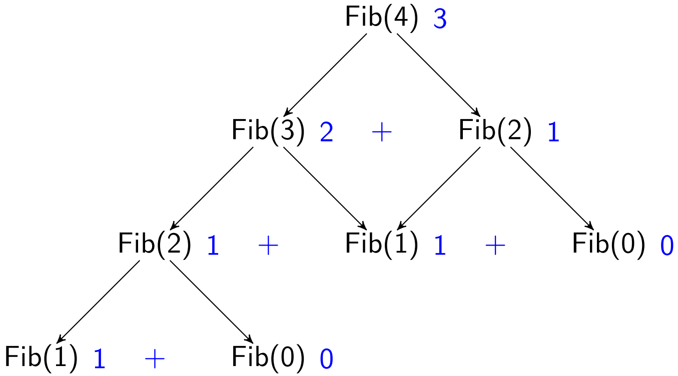
Definition

$$Fibonacci(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ Fibonacci(n-1) + Fibonacci(n-2) & \text{otherwise} \end{cases}$$

Fibonacci Numbers



Fibonacci Numbers



Result

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

Fibonacci Numbers

Algorithm fib(n)
Calculates the n^{th} Fibonacci number.
Pre: n is postive integer
Post: the n^{th} Fibonnacci number returned

if *n = 0 or n = 1* **then**
    return n
end
return fib(n-1) + fib(n-2)
End fib

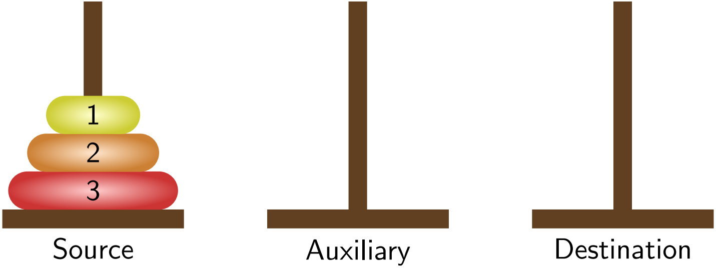
Fibonacci Numbers

No	Calls	Time	No	Calls	Time
1	1	< 1 sec.	11	287	< 1 sec.
2	3	< 1 sec.	12	465	< 1 sec.
3	5	< 1 sec.	13	753	< 1 sec.
4	9	< 1 sec.	14	1,219	< 1 sec.
5	15	< 1 sec.	15	1,973	< 1 sec.
6	25	< 1 sec.	20	21,891	< 1 sec.
7	41	< 1 sec.	25	242,785	1 sec.
8	67	< 1 sec.	30	2,692,573	7 sec.
9	109	< 1 sec.	35	29,860,703	1 min.
10	177	< 1 sec.	40	331,160,281	13 min.

The Towers of Hanoi

Move disks from Source to Destination using Auxiliary:

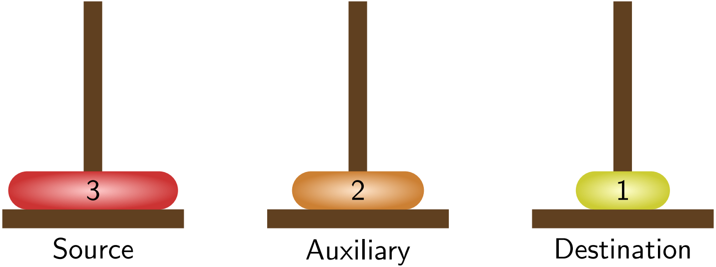
1. Only one disk could be moved at a time.
2. A larger disk must never be stacked above a smaller one.
3. Only one auxiliary needle could be used for the intermediate storage of disks.



The Towers of Hanoi

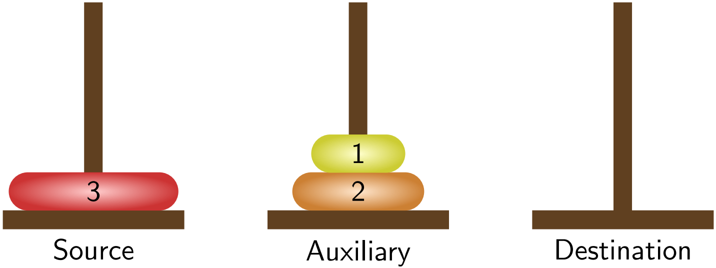
Moved disc from pole 1 to pole 3.

The Towers of Hanoi



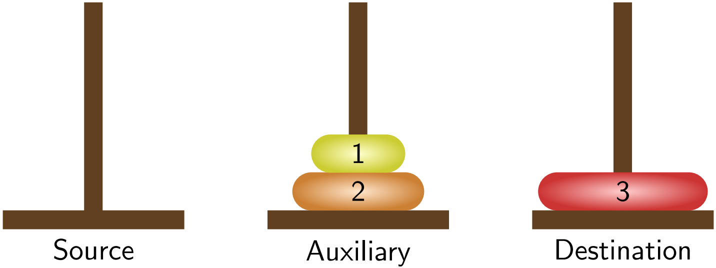
Moved disc from pole 1 to pole 2.

The Towers of Hanoi



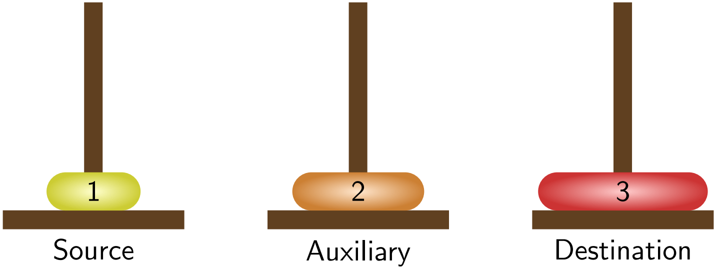
Moved disc from pole 3 to pole 2.

The Towers of Hanoi



Moved disc from pole 1 to pole 3.

The Towers of Hanoi



Moved disc from pole 2 to pole 1.

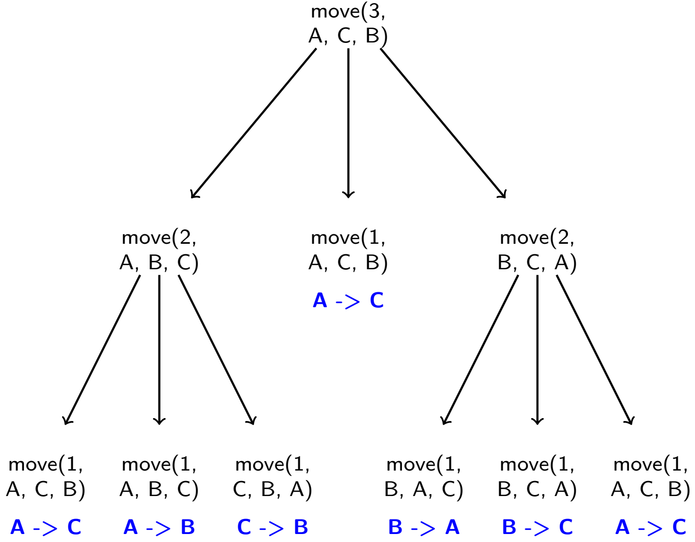
The Towers of Hanoi

Moved disc from pole 2 to pole 3.

The Towers of Hanoi

Moved disc from pole 1 to pole 3.

The Towers of Hanoi



The Towers of Hanoi : General

move(n, A, C, B)

move(n-1, A, B, C)   move(1, A, C, B)   move(n-1, B, C, A)

Complexity

$$T(n) = 1 + 2T(n-1)$$

The Towers of Hanoi

Complexity

$$T(n) = 1 + 2T(n-1)$$
$$=> T(n) = 1 + 2 + 2^2 + ... + 2^{n-1}$$
$$=> T(n) = 2^n - 1$$
$$=> T(n) = O(2^n)$$

- With 64 disks, total number of moves: $2^{64} - 1 \approx 2^4 \times 2^{60} \approx 2^4 \times 10^{18} = 1.6 \times 10^{19}$
- If one move takes 1s, 2^{64} moves take about 5×10^{11} years (500 billions years).

The Towers of Hanoi

Algorithm move(val disks <integer>, val source <character>, val destination <character>, val auxiliary <character>)

Move disks from source to destination.

Pre: `disks` is the number of disks to be moved

Post: steps for moves printed

print("Towers: ", disks, source, destination, auxiliary)

if *disks* = *1* **then**

  print ("Move from", source, "to", destination)

else

  move(disks - 1, source, auxiliary, destination)

  move(1, source, destination, auxiliary)

  move(disks - 1, auxiliary, destination, source)

end

return

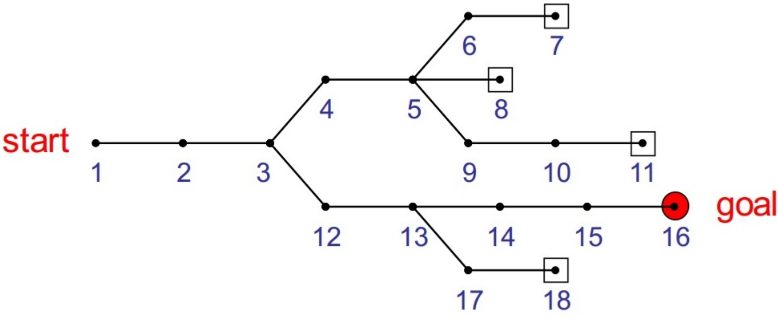
End move

Recursion and backtracking

Backtracking

Definition

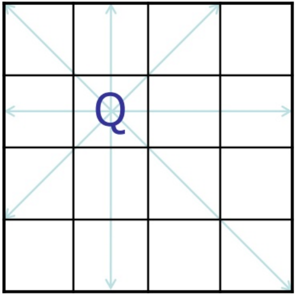
A process to go back to previous steps to try unexplored alternatives.

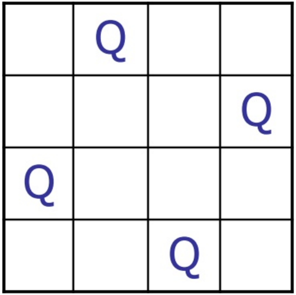


Hình: Goal seeking

Eight Queens Problem

Place eight queens on the chess board in such a way that no queen can capture another.





Eight Queens Problem

Algorithm putQueen(ref board <array>, val r <integer>)
Place remaining queens safely from a row of a chess board.

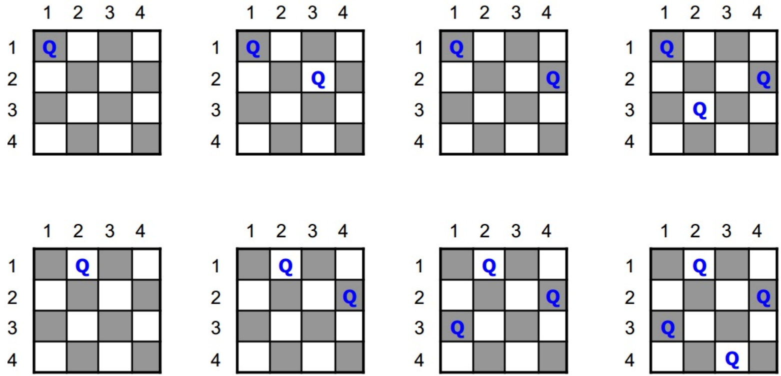
Pre: `board` is nxn array representing a chess board
`r` is the row to place queens onwards

Post: all the remaining queens are safely placed on the board; or backtracking to the previous rows is required

Eight Queens Problem

```
for every column c on the same row r do
    if cell r,c is safe then
        place the next queen in cell r,c
        if r < n-1 then
            putQueen (board, r + 1)
        else
            output successful placement
        end
        remove the queen from cell r,c
    end
end
return
End putQueen
```

Eight Queens Problem



Recursion implementation in C/C++

Fibonacci Numbers

```
#include <iostream>
using namespace std;

long fib(long num);

int main () {
  int num;
  cout << "What Fibonacci number
          do you want to calculate? ";
  cin >> num;
  cout << "The " << num << "th Fibonacci number
          is: " << fib(num) << endl;
  return 0;
}

long fib(long num) {
  if (num == 0 || num == 1)
    return num;
  return fib(num-1) + fib(num-2);
}
```

The Towers of Hanoi

```
#include <iostream>
using namespace std;

void move(int n, char source,
          char destination, char auxiliary);

int main () {
  int numDisks;
  cout << "Please␣enter␣number␣of␣disks:␣";
  cin >> numDisks;
  cout << "Start␣Towers␣of␣Hanoi" << endl;
  move(numDisks, 'A', 'C', 'B');
  return 0;
}
```

The Towers of Hanoi

```
void move(int n, char source,
          char destination, char auxiliary){
  static int step = 0;

  if (n == 1)
    cout << "Step " << ++step << ": Move from "
      << source << " to " << destination << endl;
  else {
    move(n-1, source, auxiliary, destination);
    move(1, source, destination, auxiliary);
    move(n - 1, auxiliary, destination, source);
  }
  return;
}
```