Chapter 2
Complexity of Algorithms

Data Structures and Algorithms

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Outcomes

- **L.O.1.1** - Define concept "computational complexity" and its sepcial cases, best, average, and worst.
- **L.O.1.2** - Analyze algorithms and use Big-O notation to characterize the computational complexity of algorithms composed by using the following control structures: sequence, branching, and iteration (not recursion).
- **L.O.1.3** - List, give examples, and compare complexity classes, for examples, constant, linear, etc.
- **L.O.1.4** - Be aware of the trade-off between space and time in solutions.
- **L.O.1.5** - Describe strategies in algorithm design and problem solving.

Contents

Sources of Materials

1. We would like to thank **Dr. The-Nhan LUONG**, a former instructor of our Department, for the composing of this document.
2. This document also uses figure, sentences and demo source code from the following sources:
   - The old presentation for course *Data Structures and Algorithms* edited by other members in our Department
   - Book entitled **Data Structures - A Pseudocode Approach with C++ (first edition, 2001)** written by Richard F. Gilberg and Behrouz A. Forouzan

Algorithm Efficiency

Algorithm Efficiency

- A problem often has many algorithms.
- Comparing two different algorithms $\Rightarrow$ Computational complexity: measure of the difficulty degree (time and/or space) of an algorithm.
  - How fast an algorithm is?
  - How much memory does it cost?

Algorithm Efficiency

General format

$$\mathbf{efficiency = f(n)}$$

n is the size of a problem (the key number that determines the size of input data)

Linear Loops

```
for (i = 0; i < 1000; i++)
   application code
```

The number of times the body of the loop is replicated is 1000.

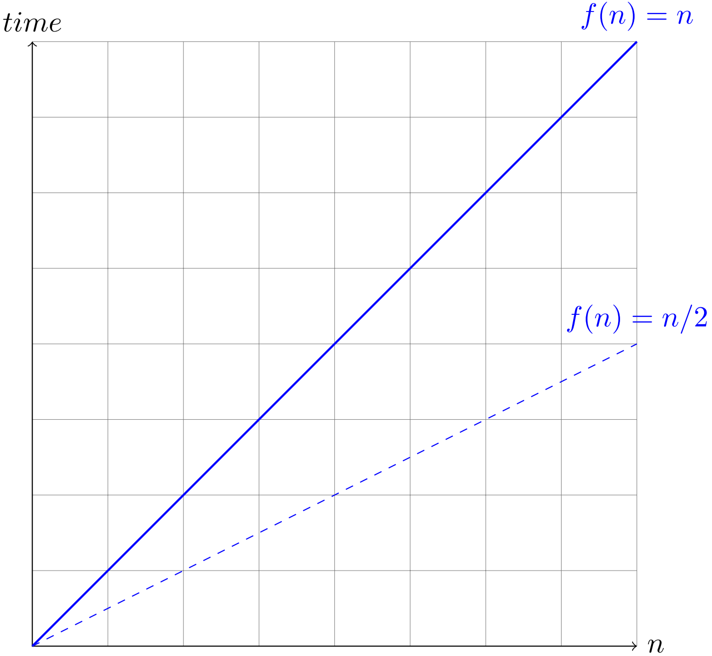
$$f(n) = n$$

```
for (i = 0; i < 1000; i += 2)
   application code
```

The number of times the body of the loop is replicated is 500.

$$f(n) = n/2$$

Linear Loops



Logarithmic Loops

Multiply loops

```
i = 1
while (i <= n)
   application code
   i = i x 2
```

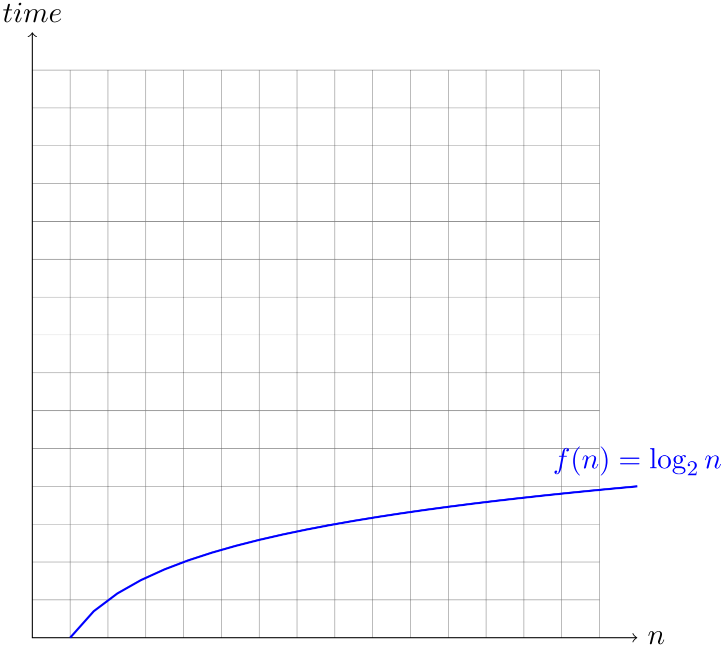
Divide loops

```
i = n
while (i >= 1)
   application code
   i = i / 2
```

The number of times the body of the loop is replicated is

$$\mathbf{f(n)} = \log_2 n$$

Logarithmic Loops



Nested Loops

Iterations $=$ Outer loop iterations $\times$ Inner loop iterations

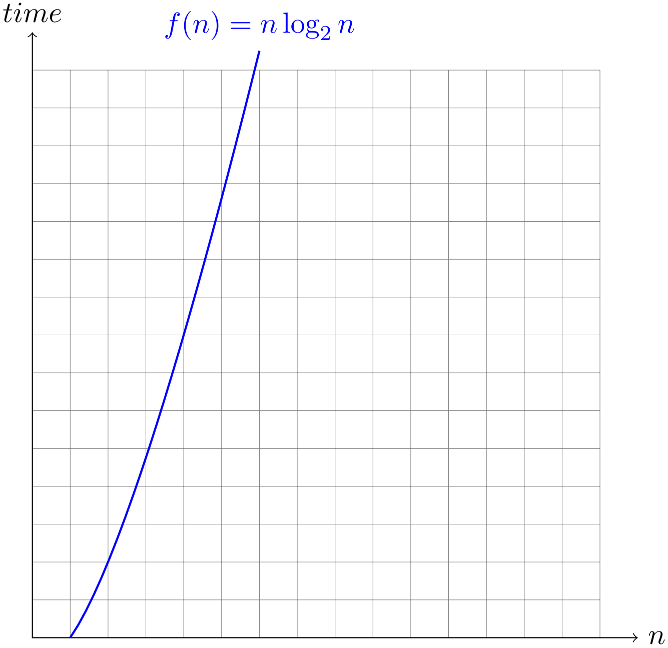
Example

```
i = 1
while (i <= n)
   j = 1
   while (j <= n)
      application code
      j = j * 2
   i = i + 1
```

The number of times the body of the loop is replicated is

$$f(n) = n \log_2 n$$

Nested Loops



Quadratic Loops

Example

```
i = 1
while (i <= n)
   j = 1
   while (j <= n)
      application code
      j = j + 1
   i = i + 1
```

The number of times the body of the loop is replicated is

$$f(n) = n^2$$

Dependent Quadratic Loops

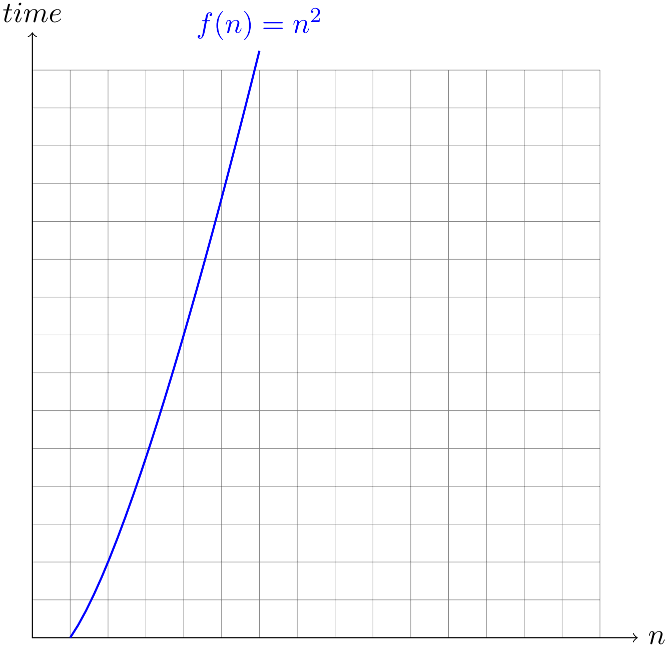
Example

```
i = 1
while (i <= n)
   j = 1
   while (j <= i)
      application code
      j = j + 1
   i = i + 1
```

The number of times the body of the loop is replicated is

$$1 + 2 + \ldots + n = n(n+1)/2$$

Quadratic Loops



Asymptotic Complexity

- Algorithm efficiency is considered with only big problem sizes.
- We are not concerned with an exact measurement of an algorithm's efficiency.
- Terms that do not substantially change the function's magnitude are eliminated.

Big-O notation

Big-O notation

Example

$f(n) = c.n \Rightarrow f(n) = O(n)$
$f(n) = n(n+1)/2 = n^2/2 + n/2 \Rightarrow f(n) = O(n^2)$

- Set the coefficient of the term to one.
- Keep the largest term and discard the others.

Some example of Big-O:
$\log_2 n \quad n \quad n\log_2 n \quad n^2 \ldots n^k \ldots 2^n \quad n!$

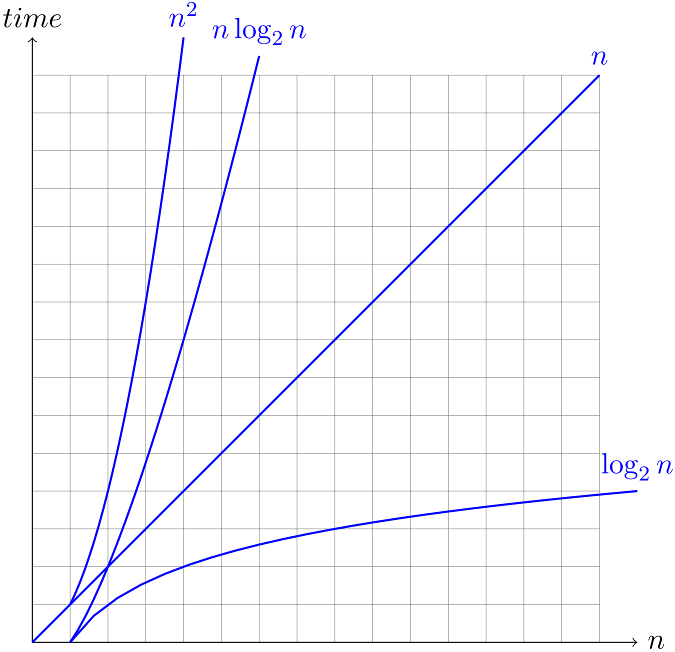
Standard Measures of Efficiency

Efficiency	Big-O	Iterations	Est. Time
logarithmic	$O(\log_2 n)$	14	microseconds
linear	$O(n)$	10 000	0.1 seconds
linear log	$O(n \log_2 n)$	140 000	2 seconds
quadratic	$O(n^2)$	10000^2	15-20 min.
polynomial	$O(n^k)$	10000^k	hours
exponential	$O(2^n)$	2^{10000}	intractable
factorial	$O(n!)$	$10000!$	intractable

Assume instruction speed of 1 microsecond and 10 instructions in loop.

$n = 10000$

Standard Measures of Efficiency



Big-O Analysis Examples

Algorithm addMatrix(val matrix1<matrix>, val matrix2<matrix>, val size<integer>, ref matrix3<matrix>)
Add matrix1 to matrix2 and place results in matrix3
Pre: matrix1 and matrix2 have data
size is number of columns and rows in matrix
Post: matrices added - result in matrix3

```
r = 1
while r <= size do
    c = 1
    while c <= size do
        matrix3[r, c] = matrix1[r, c] + matrix2[r, c]
        c = c + 1
    end
    r = r + 1
end
return matrix3
End addMatrix
```

Big-O Analysis Examples

Nested linear loop:
$f(size) = O(size^2)$

Time Costing Operations

- The most time consuming: data movement to/from memory/storage.
- Operations under consideration:
  - Comparisons
  - Arithmetic operations
  - Assignments

Problems and common complexities

Binary search

Recurrence Equation (Phương trình hồi quy)

An equation or inequality that describes a function in terms of its value on smaller input.

1	2	3	5	8	13	21	34	55	89

$$T(n) = 1 + T(n/2) \Rightarrow T(n) = O(\log_2 n)$$

Binary search

- Best case: when the number of steps is smallest. $T(n) = O(1)$
- Worst case: when the number of steps is largest. $T(n) = O(\log_2 n)$
- Average case: in between. $T(n) = O(\log_2 n)$

Sequential search

8	5	21	2	1	13	4	34	7	18

- Best case: $T(n) = O(1)$
- Worst case: $T(n) = O(n)$
- Average case: $T(n) = \sum_{i=1}^{n} i.p_i$
  p_i : probability for the target being at a[i]
  $p_i = 1/n \Rightarrow T(n) = (\sum_{i=1}^{n} i)/n = O(n(n+1)/2n) = O(n)$

Quick sort

19	8	3	15	28	10	22	4	12	83

Recurrence Equation

$$T(n) = O(n) + 2T(n/2)$$

- Best case: $T(n) = O(n \log_2 n)$
- Worst case: $T(n) = O(n^2)$
- Average case: $T(n) = O(n \log_2 n)$

P and NP Problems

P and NP Problems

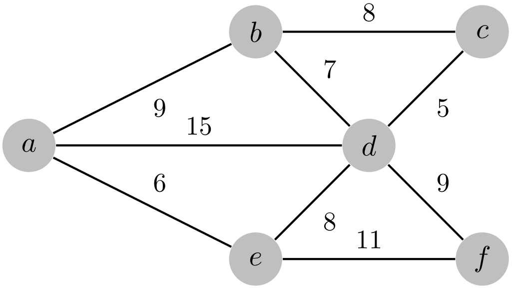
- P: Polynomial (can be solved in polynomial time on a deterministic machine).
- NP: Nondeterministic Polynomial (can be solved in polynomial time on a nondeterministic machine).

P and NP Problems

Travelling Salesman Problem:

A salesman has a list of cities, each of which he must visit exactly once. There are direct roads between each pair of cities on the list.

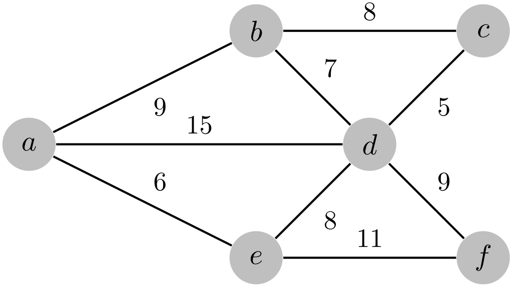
Find the route the salesman should follow for the shortest possible round trip that both starts and finishes at any one of the cities.



P and NP Problems

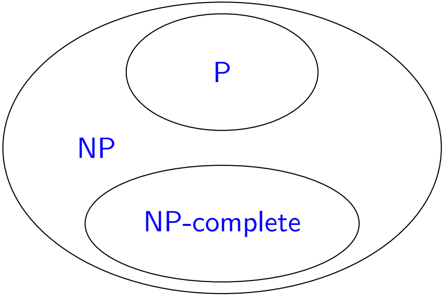
Travelling Salesman Problem:

Deterministic machine: $f(n) = n(n-1)(n-2)\ldots 1 = O(n!)$
$\Rightarrow$ NP problem



P and NP Problems

NP-complete: NP and every other problem in NP is polynomially reducible to it.



$$P = NP?$$