Chapter 8
Heaps

Data Structures and Algorithms

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Outcomes

- **L.O.4.1** - List some applications of Heap.
- **L.O.4.2** - Depict heap structure and relate it to array.
- **L.O.4.3** - List necessary methods supplied for heap structure, and describe them using pseudocode.
- **L.O.4.4** - Depict the working steps of methods that maintain the characteristics of heap structure for the cases of adding/removing elements to/from heap.

Outcomes

- **L.O.4.5** - Implement heap using C/C++.
- **L.O.4.6** - Analyze the complexity and develop experiment (program) to evaluate methods supplied for heap structures.
- **L.O.8.4** - Develop recursive implementations for methods supplied for the following structures: list, tree, heap, searching, and graphs.
- **L.O.1.2** - Analyze algorithms and use Big-O notation to characterize the computational complexity of algorithms composed by using the following control structures: sequence, branching, and iteration (not recursion).

Contents

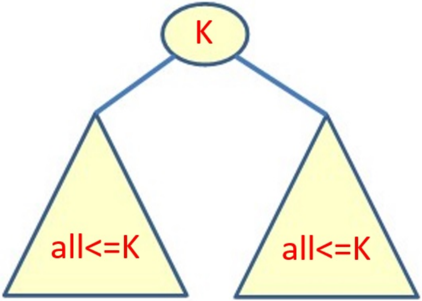
Heap Definition

Heap Definition

Definition

A heap (max-heap) is a binary tree structure with the following properties:

1. The tree is complete or nearly complete.
2. The key value of each node is greater than or equal to the key value in each of its descendents.



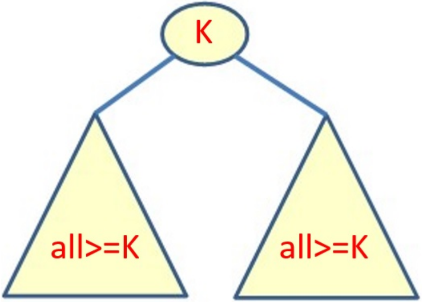
(Source: Data Structures - A Pseudocode Approach with C++)

Heap Definition

Definition

A min-heap is a binary tree structure with the following properties:

1. The tree is complete or nearly complete.
2. The key value of each node is less than or equal to the key value in each of its descendents.



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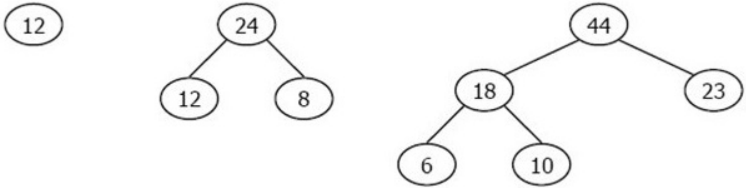
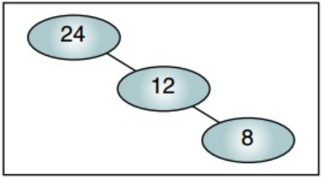
Heap Structure

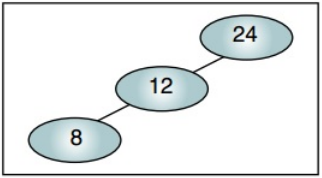
Heap trees



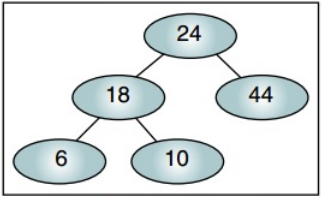
Invalid Heaps



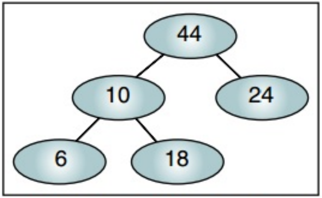
(a) Not nearly complete (rule 1)



(b) Not nearly complete (rule 1)



(c) Root not largest (rule 2)



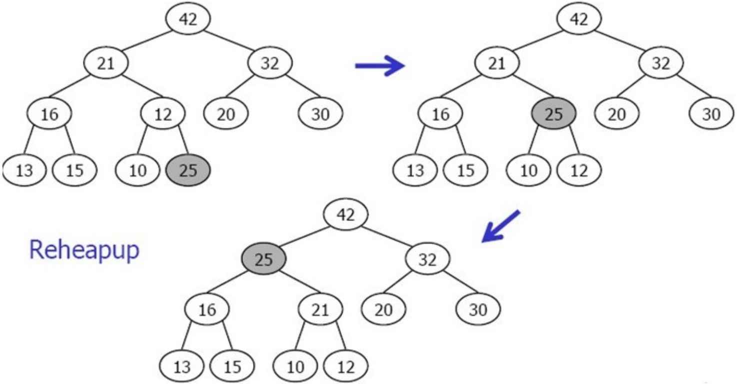
(d) Subtree 10 not a heap (rule 2)

(Source: Data Structures - A Pseudocode Approach with C++)

Basic Heap Algorithms

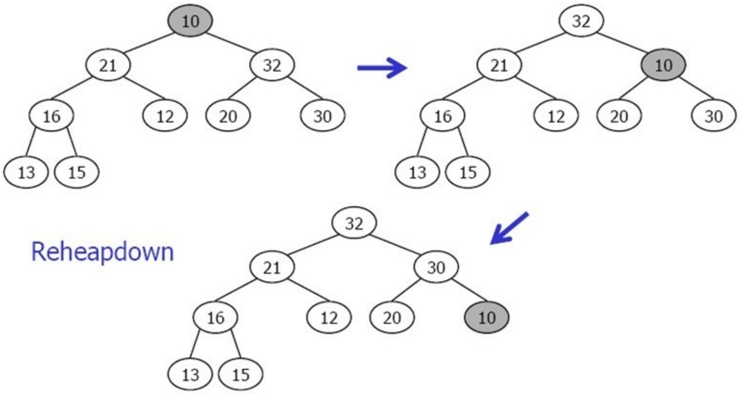
ReheapUp

The reheapUp operation repairs a "broken" heap by floating the last element up the tree until it is in its correct location in the heap.



ReheapDown

The reheapDown operation repairs a "broken" heap by pushing the root down the tree until it is in its correct location in the heap.



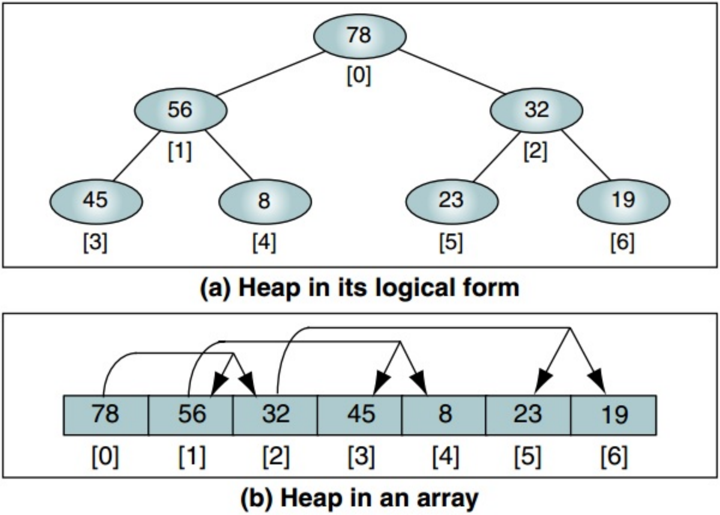
Heap Data Structure

Properties of Heaps

- A complete or nearly complete binary tree.
- If the height is h, the number of nodes N is between 2^{h-1} and $2^h - 1$.
- Complete tree: $N = 2^h - 1$ when last level is full.
- Nearly complete: All nodes in the last level are on the left.

$\rightarrow$ Heap can be represented in an array.

Heap in arrays



(Source: Data Structures - A Pseudocode Approach with C++)

Heap Data Structure

The relationship between a node and its children is fixed and can be calculated:

1. For a node located at index i, its children are found at
   - Left child: $2i + 1$
   - Right child: $2i + 2$
2. The parent of a node located at index i is located at $\lfloor (i-1)/2 \rfloor$.
3. Given the index for a left child, j, its right sibling, if any, is found at $j + 1$. Conversely, given the index for a right child, k, its left sibling, which must exist, is found at $k - 1$.
4. Given the size, N, of a complete heap, the location of the first leaf is $\lfloor N/2 \rfloor$.
5. Given the location of the first leaf element, the location of the last nonleaf element is 1 less.

Heap Algorithms

ReheapUp Algorithm

Algorithm reheapUp(ref heap <array>, val position <integer>)
Reestablishes heap by moving data in `position` up to its correct location.

Pre: All data in the heap above this `position` satisfy key value order of a heap, except the data in `position`

Post: Data in `position` has been moved up to its correct location.

ReheapUp Algorithm

```
if position > 0 then
    parent = (position-1)/2
    if heap[position].key > heap[parent].key
    then
        swap(position, parent)
        reheapUp(heap, parent)
    end
end
return
End reheapUp
```

ReheapDown Algorithm

Algorithm reheapDown(ref heap <array>, val position <integer>, val lastPosition <integer>)

Reestablishes heap by moving data in `position` down to its correct location.

Pre: All data in the subtree of `position` satisfy key value order of a heap, except the data in `position`

`lastPosition` is an index to the last element in heap

Post: Data in `position` has been moved down to its correct location.

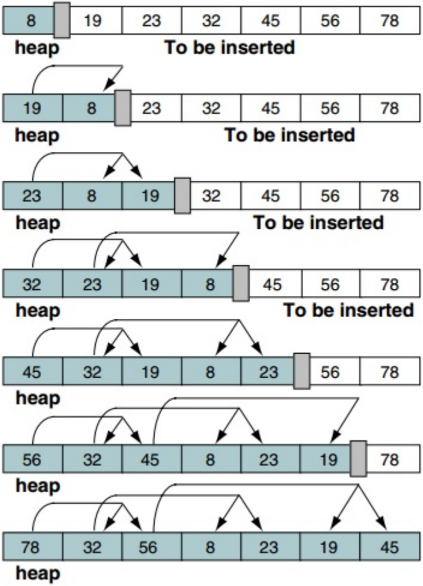
ReheapDown Algorithm

leftChild = position * 2 + 1
rightChild = position * 2 + 2
if *leftChild <= lastPosition* **then**
    if *(rightChild <= lastPosition) AND (heap[rightChild].key > heap[leftChild].key* **then**
        largeChild = rightChild
    else
        largeChild = leftChild
    end
    if *heap[largeChild].key > heap[position].key* **then**
        swap(largeChild, position)
        reheapDown(heap, largeChild, lastPosition)
    end
end
return
End reheapDown

Build a Heap

- Given a filled array of elements in random order, to build the heap we need to rearrange the data so that each node in the heap is greater than its children.
- We begin by dividing the array into two parts, the left being a heap and the right being data to be inserted into the heap. Note the "wall" between the first and second parts.
- At the beginning the root (the first node) is the only node in the heap and the rest of the array are data to be inserted.
- Each iteration of the insertion algorithm uses reheap up to insert the next element into the heap and moves the wall separating the elements one position to the right.

Build a Heap



(Source: Data Structures - A Pseudocode Approach with C++)

Build a Heap

Algorithm buildHeap(ref heap <array>, val size <integer>)
Given an array, rearrange data so that they form a heap.

Pre: `heap` is array containing data in nonheap order
`size` is number of elements in array

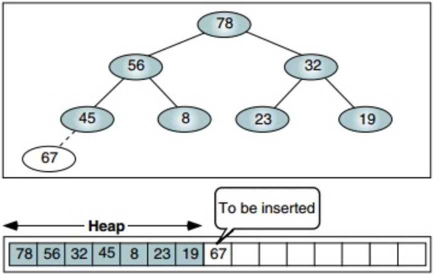
Post: array is now a heap.

walker = 1
while *walker < size* **do**
    reheapUp(heap, walker)
    walker = walker + 1
end
End buildHeap

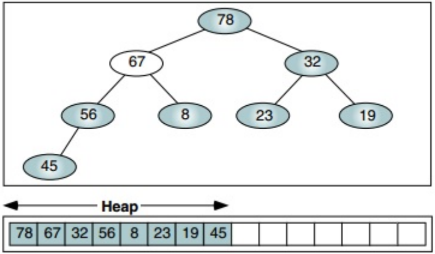
Insert a Node into a Heap

- To insert a node, we need to locate the first empty leaf in the array.
- We find it immediately after the last node in the tree, which is given as a parameter.
- To insert a node, we move the new data to the first empty leaf and reheap up.

Insert a Node into a Heap



(a) Before reheap up



(b) After reheap up

(Source: Data Structures - A Pseudocode Approach with C++)

Insert a Node into a Heap

Algorithm insertHeap(ref heap <array>, ref last <integer>, val data <dataType>)
Inserts `data` into heap.

Pre: `heap` is a valid heap structure
`last` is reference parameter to last node in heap
`data` contains data to be inserted

Post: `data` have been inserted into heap.

Return `true` if successful; `false` if array full

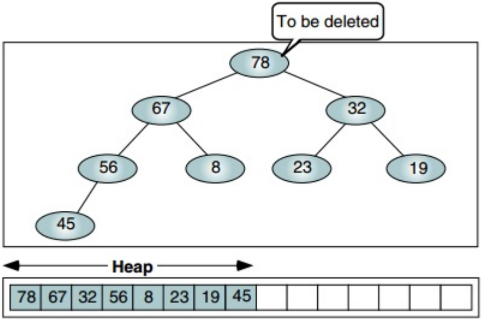
Insert a Node into a Heap

```
if heap full then
    return false
end
last = last + 1
heap[last] = data
reheapUp(heap, last)
return true
End insertHeap
```

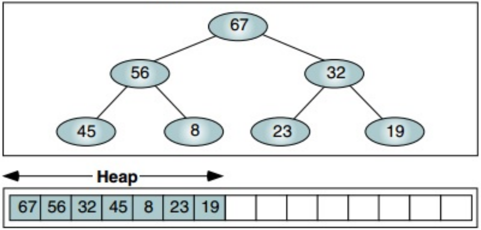
Delete a Node from a Heap

- When deleting a node from a heap, the most common and meaningful logic is to delete the root.
- After it has been deleted, the heap is thus left without a root.
- To reestablish the heap, we move the data in the last heap node to the root and reheap down.

Delete a Node from a Heap



(a) Before delete



(b) After delete

(Source: Data Structures - A Pseudocode Approach with C++)

Delete a Node from a Heap

Algorithm deleteHeap(ref heap <array>, ref last <integer>, ref dataOut <dataType>)
Deletes root of heap and passes data back to caller.

Pre: `heap` is a valid heap structure
`last` is reference parameter to last node
`dataOut` is reference parameter for output data

Post: root deleted and heap rebuilt
root data placed in `dataOut`

Return `true` if successful; `false` if array empty

Delete a Node from a Heap

```
if heap empty then
    return false
end
dataOut = heap[0]
heap[0] = heap[last]
last = last - 1
reheapDown(heap, 0, last)
return true
End deleteHeap
```

Complexity of Binary Heap Operations

- ReheapUp: $O(\log_2 n)$
- ReheapDown: $O(\log_2 n)$
- Build a Heap: $O(n \log_2 n)$
- Insert a Node into a Heap: $O(\log_2 n)$
- Delete a Node from a Heap: $O(\log_2 n)$

Heap Applications

Heap Applications

Three common applications of heaps are:

1. selection algorithms,
2. priority queues,
3. and sorting.

We discuss heap sorting in Chapter 10 and selection algorithms and priority queues here.

Selection Algorithms

Problem

Determining the k^{th} element in an unsorted list.

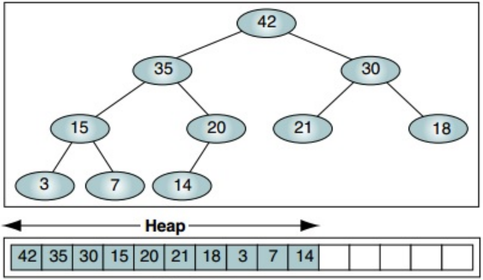
Two solutions:

1. Sort the list and select the element at location k. The complexity of a simple sorting algorithm is $O(n^2)$.
2. Create a heap and delete $k-1$ elements from the heap, leaving the desired element at the top. The complexity is $O(n \log_2 n)$.

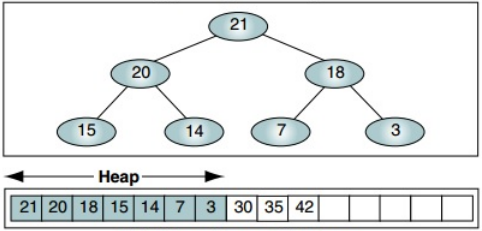
Rather than simply discarding the elements at the top of the heap, a better solution would be to place the deleted element at the end of the heap and reduce the heap size by 1.

After the k^{th} element has been processed, the temporarily removed elements can then be inserted into the heap.

Selection Algorithms



(a) Original heap



(b) After three deletions

(Source: Data Structures - A Pseudocode Approach with C++)

Selection Algorithms

Algorithm selectK(ref heap <array>, ref k <integer>, ref last <integer>)
Select the k-th largest element from a list.

Pre: `heap` is an array implementation of a heap
`k` is the ordinal of the element desired
`last` is reference parameter to last element

Post: k-th largest value returned

Selection Algorithms

```
if k > last + 1 then
    return 0
end
i = 1
originalSize = last + 1
while i < k do
    temp = heap[0]
    deleteHeap(heap, last, dataOut)
    heap[last + 1] = temp
    i = i + 1
end
```

// Desired element is now at top of heap
holdOut = heap[0]

// Reconstruct heap
while *last < originalSize* **do**
    last = last + 1
    reheapUp(heap, last)
end
return holdOut
End selectK

Priority Queues

The heap is an excellent structure to use for a priority queue.

Example

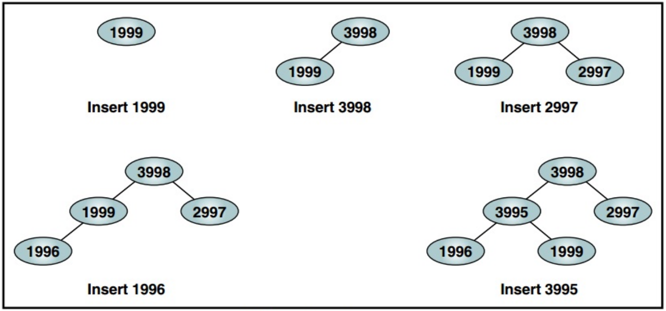
Assume that we have a priority queue with three priorities: high (3), medium (2), and low (1).
Of the first five customers who arrive, the second and the fifth are high-priority customers, the third is medium priority, and the first and the fourth are low priority.

Arrival	Priority	Priority
1	low	1999 (1 & (1000 – 1)
2	high	3998 (3 & (1000 – 2)
3	medium	2997 (2 & (1000 – 3)
4	low	1996 (1 & (1000 – 4)
5	high	3995 (3 & (1000 – 5)

(Source: Data Structures - A Pseudocode Approach with C++)

Priority Queues

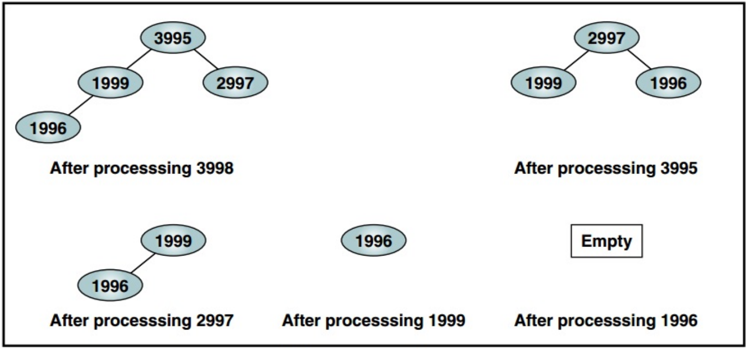
The customers are served according to their priority and within equal priorities, according to their arrival. Thus we see that customer 2 (3998) is served first, followed by customer 5 (3995), customer 3 (2997), customer 1 (1999), and customer 4 (1996).



(a) Insert customers

(Source: Data Structures - A Pseudocode Approach with C++)

Priority Queues



(b) Process customers

(Source: Data Structures - A Pseudocode Approach with C++)