Superlink derivations

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1 Glossary

Variable	Description
Q_{uk}	Discharge at upstream end of superlink k
C_{uk}	Coefficient of discharge at upstream end of superlink k
A_{uk}	Cross-sectional area of flow at upstream end of superlink k
ΔH_{uk}	Head difference at upstream end of superlink k
H_{juk}	Head at junction upstream of superlink k (ground elevation + water depth)
h_{uk}	Water depth at upstream of superlink k
$z_{inv,uk}$	Invert elevation at upstream end of superlink k
Q_{dk}	Discharge at downstream end of superlink k
C_{dk}	Coefficient of discharge at downstream end of superlink k
A_{dk}	Cross-sectional area of flow at downstream end of superlink k
ΔH_{dk}	Head difference at downstream end of superlink k
H_{jdk}	Head at junction downstream of superlink k (ground elevation + water depth)
h_{dk}	Water depth at downstream of superlink k
$z_{inv,dk}$	Invert elevation at downstream end of superlink k
$NBDj$	Number of superlinks with downstream end attached to superjunction j
$NBUj$	Number of superlinks with upstream end attached to superjunction j
H_j	Head at junction j
U_{Ik}, V_{Ik}, W_{Ik}	Coefficients
X_{Ik}, Y_{Ik}, Z_{Ik}	Coefficients

2 Basic equations

The two governing equations for SUPERLINK are continuity and conservation of momentum.

Continuity:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q_0 \tag{1}$$

Conservation of momentum:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}(Qu) + gA\left(\frac{\partial h}{\partial x} - S_0 + S_f + S_L\right) = 0 \tag{2}$$

3 Discretization of momentum

Discretizing the momentum equation:

$$
\begin{aligned}
(Q_{ik}^{t+\Delta t} - Q_{ik}^{t})\frac{\Delta x_{ik}}{\Delta t} + u_{I+1k}Q_{I+1k}^{t+\Delta t} - u_{Ik}Q_{Ik}^{t+\Delta t} \\
+gA(h_{I+1k}^{t+\Delta t} - h_{Ik}^{t+\Delta t}) - gA_{ik}S_{o,ik}\Delta x_{ik} + gA_{ik}(S_{f,ik} + S_{L,ik})\Delta x = 0
\end{aligned} \tag{3}
$$

This equation can be written in terms of the following coefficient equation:

$$
a_{ik}Q_{i-1k}^{t+\Delta t} + b_{ik}Q_{ik}^{t+\Delta t} + c_{ik}Q_{i+1k}^{t+\Delta t} = P_{ik} + gA_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \tag{4}
$$

Where:

$$
a_{ik} = -\max(u_{Ik}, 0) \tag{5}
$$

$$
c_{ik} = -\max(-u_{I+1k}, 0) \tag{6}
$$

$$
b_{ik} = \frac{\Delta x_{ik}}{\Delta t} + \frac{gn_{ik}^2|Q_{ik}^t|\Delta x_{ik}}{A_{ik}R_{ik}^{4/3}} + \frac{A_{ik}|Q_{ik}^t|}{A_{cik}^2C_{ik}^2} - a_{ik} - c_{ik} \tag{7}
$$

$$
P_{ik} = Q_{ik}\frac{\Delta x_{ik}}{\Delta t} + gA_{ik}S_{o,ik}\Delta x_{ik} \tag{8}
$$

Substituting the coefficients:

$$
\begin{aligned}
-\max(u_{Ik}, 0)Q_{i-1k}^{t+\Delta t} + \left(\frac{\Delta x_{ik}}{\Delta t} + \frac{gn_{ik}^2|Q_{ik}^t|\Delta x_{ik}}{A_{ik}R_{ik}^{4/3}} + \frac{A_{ik}|Q_{ik}^t|}{A_{cik}^2C_{ik}^2} + \max(u_{Ik}, 0) + \max(-u_{I+1k}, 0)\right)Q_{ik}^{t+\Delta t} \\
-\max(-u_{I+1k}, 0)Q_{i+1k}^{t+\Delta t} \\
= Q_{ik}\frac{\Delta x_{ik}}{\Delta t} + gA_{ik}S_{o,ik}\Delta x_{ik} + gA_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t})
\end{aligned} \tag{9}
$$

Assuming $u_{ik} > 0$ and $u_{i-1k} \approx u_{ik} \approx u_{i+1k}$:

$$-u_{Ik}Q_{i-1k}^{t+\Delta t} + \left(\frac{\Delta x_{ik}}{\Delta t} + \frac{gn_{ik}^2|Q_{ik}^t|\Delta x_{ik}}{A_{ik}R_{ik}^{4/3}} + \frac{A_{ik}|Q_{ik}^t|}{A_{cik}^2 C_{ik}^2} + u_{Ik}\right)Q_{ik}^{t+\Delta t}$$
$$= Q_{ik}^t\frac{\Delta x_{ik}}{\Delta t} + gA_{ik}S_{o,ik}\Delta x_{ik} + gA_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \quad (10)$$

$$(Q_{ik}^{t+\Delta t} - Q_{ik}^t)\frac{\Delta x_{ik}}{\Delta t} + (Q_{ik}^{t+\Delta t} - Q_{i-1k}^{t+\Delta t})u_{Ik}$$
$$+gA_{ik}\left(\frac{n_{ik}^2|Q_{ik}^t|Q_{ik}^{t+\Delta t}}{A_{ik}^2R_{ik}^{4/3}} + \frac{|Q_{ik}^t|Q_{ik}^{t+\Delta t}}{gC_{ik}^2A_{cik}^2\Delta x_{ik}}\right)\Delta x_{ik}$$
$$= gA_{ik}S_{o,ik}\Delta x_{ik} + gA_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \quad (11)$$

$$(Q_{ik}^{t+\Delta t} - Q_{ik}^t)\frac{\Delta x_{ik}}{\Delta t} + (Q_{ik}^{t+\Delta t} - Q_{i-1k}^{t+\Delta t})u_{Ik} + gA_{ik}(S_{f,ik} + S_{L,ik})\Delta x_{ik}$$
$$= gA_{ik}S_{o,ik}\Delta x_{ik} + gA_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \quad (12)$$

$$(Q_{ik}^{t+\Delta t} - Q_{ik}^t)\frac{\Delta x_{ik}}{\Delta t} + (Q_{ik}^{t+\Delta t} - Q_{i-1k}^{t+\Delta t})u_{Ik}$$
$$+gA_{ik}(h_{I+1k}^{t+\Delta t} - h_{Ik}^{t+\Delta t}) + gA_{ik}(S_{f,ik} - gA_{ik}S_{o,ik}\Delta x_{ik} + S_{L,ik})\Delta x_{ik} \quad (13)$$

Alternatively, assuming $u_{ik} < 0$ and $u_{i-1k} \approx u_{ik} \approx u_{i+1k}$:

$$u_{I+1k}Q_{i+1k}^{t+\Delta t} + \left(\frac{\Delta x_{ik}}{\Delta t} + \frac{gn_{ik}^2|Q_{ik}^t|\Delta x_{ik}}{A_{ik}R_{ik}^{4/3}} + \frac{A_{ik}|Q_{ik}^t|}{A_{cik}^2 C_{ik}^2} - u_{I+1k}\right)Q_{ik}^{t+\Delta t}$$
$$= Q_{ik}^t\frac{\Delta x_{ik}}{\Delta t} + gA_{ik}S_{o,ik}\Delta x_{ik} + gA_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \quad (14)$$

$$(Q_{ik}^{t+\Delta t} - Q_{ik}^t)\frac{\Delta x_{ik}}{\Delta t} + (Q_{i+1k}^{t+\Delta t} - Q_{ik}^{t+\Delta t})u_{I+1k}$$
$$+gA_{ik}\left(\frac{n_{ik}^2|Q_{ik}^t|Q_{ik}^{t+\Delta t}}{A_{ik}^2R_{ik}^{4/3}} + \frac{|Q_{ik}^t|Q_{ik}^{t+\Delta t}}{gC_{ik}^2A_{cik}^2\Delta x_{ik}}\right)\Delta x_{ik}$$
$$= gA_{ik}S_{o,ik}\Delta x_{ik} + gA_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \quad (15)$$

$$(Q_{ik}^{t+\Delta t} - Q_{ik}^t)\frac{\Delta x_{ik}}{\Delta t} + (Q_{i+1k}^{t+\Delta t} - Q_{ik}^{t+\Delta t})u_{I+1k} + gA_{ik}(S_{f,ik} + S_{L,ik})\Delta x_{ik}$$
$$= gA_{ik}S_{o,ik}\Delta x_{ik} + gA_{ik}(h_{Ik}^{t+\Delta t} - h_{I+1k}^{t+\Delta t}) \quad (16)$$

$$(Q_{ik}^{t+\Delta t} - Q_{ik}^t)\frac{\Delta x_{ik}}{\Delta t} + (Q_{i+1k}^{t+\Delta t} - Q_{ik}^{t+\Delta t})u_{I+1k}$$
$$+gA_{ik}(h_{I+1k}^{t+\Delta t} - h_{Ik}^{t+\Delta t}) + gA_{ik}(S_{f,ik} - gA_{ik}S_{o,ik}\Delta x_{ik} + S_{L,ik})\Delta x_{ik} \quad (17)$$

4 Recurrence relationships

4.1 Forward recurrence

Starting at the upstream end of superlink k:

$$Q_{2k}^{t+\Delta t} - Q_{1k}^{t+\Delta t} + E_{2k}h_{2k}^{t+\Delta t} = D_{2k} \tag{18}$$

$$a_{1k}Q_{0k}^{t+\Delta t} + b_{1k}Q_{1k}^{t+\Delta t} + c_{1k}Q_{2k}^{t+\Delta t} = P_{1k} + gA_{1k}(h_{1k}^{t+\Delta t} - h_{2k}^{t+\Delta t}) \tag{19}$$

Assuming $Q_{0k}^{t+\Delta t} = Q_{1k}^{t+\Delta t}$:

$$a_{1k}Q_{1k}^{t+\Delta t} + b_{1k}Q_{1k}^{t+\Delta t} + c_{1k}(Q_{1k}^{t+\Delta t} - E_{2k}h_{2k}^{t+\Delta t} + D_{2k}) = P_{1k} + gA_{1k}(h_{1k}^{t+\Delta t} - h_{2k}^{t+\Delta t}) \tag{20}$$

$$(a_{1k} + b_{1k} + c_{1k})Q_{1k}^{t+\Delta t} = E_{2k}c_{2k}h_{2k}^{t+\Delta t} + (P_{1k} + c_{1k}D_{2k}) + gA_{1k}(h_{1k}^{t+\Delta t} - h_{2k}^{t+\Delta t}) \tag{21}$$

$$(a_{1k} + b_{1k} + c_{1k})Q_{1k}^{t+\Delta t} = (E_{2k}c_{1k} - gA_{1k})h_{2k}^{t+\Delta t} + (P_{1k} - D_{2k}c_{1k}) + gA_{1k}h_{1k}^{t+\Delta t} \tag{22}$$

$$Q_{1k}^{t+\Delta t} = \frac{(E_{2k}c_{1k} - gA_{1k})h_{2k}^{t+\Delta t} + (P_{1k} - D_{2k}c_{1k}) + gA_{1k}h_{1k}^{t+\Delta t}}{a_{1k} + b_{1k} + c_{1k}} \tag{23}$$

Thus for the upstream end of superlink k:

$$\boxed{Q_{1k}^{t+\Delta t} = U_{1k}h_{2k}^{t+\Delta t} + Y_{1k} + Z_{1k}h_{1k}^{t+\Delta t}} \tag{24}$$

Where:

$$\boxed{T_{1k} = a_{1k} + b_{1k} + c_{1k}} \tag{25}$$

$$\boxed{U_{1k} = \frac{E_{2k}c_{1k} - gA_{1k}}{T_{1k}}} \tag{26}$$

$$\boxed{V_{1k} = \frac{P_{1k} - D_{2k}c_{1k}}{T_{1k}}} \tag{27}$$

$$\boxed{W_{1k} = \frac{gA_{1k}}{T_{1k}}} \tag{28}$$

For the next element downstream:

$$Q_{3k}^{t+\Delta t} - Q_{2k}^{t+\Delta t} + E_{3k}h_{3k}^{t+\Delta t} = D_{3k} \tag{29}$$

$$a_{2k}Q_{1k}^{t+\Delta t} + b_{2k}Q_{2k}^{t+\Delta t} + c_{2k}Q_{3k}^{t+\Delta t} = P_{2k} + gA_{2k}(h_{2k}^{t+\Delta t} - h_{3k}^{t+\Delta t}) \tag{30}$$

Substituting:

$$\begin{aligned} a_{2k}(Q_{2k}^{t+\Delta t} + E_{2k}h_{2k}^{t+\Delta t} - D_{2k}) + (b_{2k})Q_{2k}^{t+\Delta t} + c_{2k}(Q_{2k} - E_{3k}h_{3k}^{t+\Delta t} + D_{3k}) \\ -P_{2k} - gA_{2k}(h_{2k}^{t+\Delta t} - h_{3k}^{t+\Delta t}) = 0 \end{aligned} \tag{31}$$

$$\begin{aligned} (a_{2k} + b_{2k} + c_{2k})Q_{2k}^{t+\Delta t} + (E_{2k}a_{2k} - gA_{2k})h_{2k}^{t+\Delta t} \\ +(gA_{2k} - c_{2k}E_{3k})h_{3k}^{t+\Delta t} + (-D_{2k}a_{2k} + D_{3k}c_{2k} - P_{2k}) = 0 \end{aligned} \tag{32}$$

Multiplying $h_{2k}^{t+\Delta t}$ by $(U_{1k} - E_{2k})/(U_{1k} - E_{2k})$ and rearranging:

$$\begin{aligned} (a_{2k} + b_{2k} + c_{2k})Q_{2k}^{t+\Delta t} + \frac{(E_{2k}a_{2k} - gA_{2k})(U_{1k} - E_{2k})}{(U_{1k} - E_{2k})}h_{2k}^{t+\Delta t} \\ +(gA_{2k} - c_{2k}E_{3k})h_{3k}^{t+\Delta t} + (-P_{2k} - D_{2k}a_{2k} + D_{3k}c_{2k}) = 0 \end{aligned} \tag{33}$$

Note that:

$$U_{1k}h_{2k}^{t+\Delta t} = (Q_{1k}^{t+\Delta t} - V_{1k} - W_{1k}h_{1k}^{t+\Delta t}) \tag{34}$$

$$E_{2k}h_{2k}^{t+\Delta t} = (D_{2k} - Q_{2k}^{t+\Delta t} + Q_{1k}^{t+\Delta t}) \tag{35}$$

Thus:

$$
\begin{aligned}
&(a_{2k}+b_{2k}+c_{2k})Q_{2k}^{t+\Delta t} \\
&+\frac{(E_{2k}a_{2k}-gA_{2k})}{(U_{1k}-E_{2k})}[(Q_{1k}^{t+\Delta t}-V_{1k}-W_{1k}h_{1k}^{t+\Delta t})-(D_{2k}-Q_{2k}^{t+\Delta t}+Q_{1k}^{t+\Delta t})] \\
&+(gA_{2k}-c_{2k}E_{3k})h_{3k}^{t+\Delta t}+(-P_{2k}-D_{2k}a_{2k}+D_{3k}c_{2k})=0
\end{aligned}
\tag{36}
$$

Allowing $Q_{1k}^{t+\Delta t}$ to be eliminated:

$$
\begin{aligned}
&(a_{2k}+b_{2k}+c_{2k})Q_{2k}^{t+\Delta t}+\frac{(E_{2k}a_{2k}-gA_{2k})}{U_{1k}-E_{2k}}Q_{2k}^{t+\Delta t} \\
&+\frac{(E_{2k}a_{2k}-gA_{2k})(-W_{1k})}{U_{1k}-E_{2k}}h_{1k}^{t+\Delta}+(gA_{2k}-c_{2k}E_{3k})h_{3k}^{t+\Delta t} \\
&+(-P_{2k}-D_{2k}a_{2k}+D_{3k}c_{2k}+(E_{2k}a_{2k}-gA_{2k})\frac{(-V_{1k}-D_{2k})}{(U_{1k}-E_{2k})})=0
\end{aligned}
\tag{37}
$$

Rearranging:

$$
\begin{aligned}
&\left(a_{2k}+b_{2k}+c_{2k}-\frac{gA_{2k}-E_{2k}a_{2k}}{U_{1k}-E_{2k}}\right)Q_{2k}^{t+\Delta t} \\
&+\frac{(gA_{2k}-E_{2k}a_{2k})W_{1k}}{U_{1k}-E_{2k}}h_{1k}^{t+\Delta}+(gA_{2k}-c_{2k}E_{3k})h_{3k}^{t+\Delta t} \\
&+\left(-P_{2k}-D_{2k}a_{2k}+D_{3k}c_{2k}+(gA_{2k}-E_{2k}a_{2k})\frac{V_{1k}+D_{2k}}{U_{1k}-E_{2k}}\right)=0
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
&\left(a_{2k}+b_{2k}+c_{2k}-\frac{gA_{2k}-E_{2k}a_{2k}}{U_{1k}-E_{2k}}\right)Q_{2k}^{t+\Delta t} \\
&=(E_{3k}c_{2k}-gA_{2k})h_{3k}^{t+\Delta t} \\
&+\left(P_{2k}+D_{2k}a_{2k}-D_{3k}c_{2k}-(gA_{2k}-E_{2k}a_{2k})\frac{V_{1k}+D_{2k}}{(U_{1k}-E_{2k})}\right) \\
&-\frac{(gA_{2k}-E_{2k}a_{2k})W_{1k}}{U_{1k}-E_{2k}}h_{1k}^{t+\Delta}
\end{aligned}
\tag{39}
$$

Generalizing for $i=2$, $I=2$:

$$\left(a_{ik} + b_{ik} + c_{ik} - \frac{gA_{ik} - E_{Ik}a_{ik}}{U_{I-1k} - E_{Ik}}\right)Q_{ik}^{t+\Delta t}$$
$$= (E_{I+1k}c_{ik} - gA_{ik})h_{I+1k}^{t+\Delta t}$$
$$+\left(P_{ik} + D_{Ik}a_{ik} - D_{I+1k}c_{ik} - (gA_{ik} - E_{ik}a_{ik})\frac{V_{I-1k} + D_{Ik}}{U_{I-1k} - E_{Ik}}\right) \tag{40}$$
$$-\frac{(gA_{ik} - E_{Ik}a_{ik})W_{I-1k}}{U_{I-1k} - E_{Ik}}h_{1k}^{t+\Delta}$$

Condensing in terms of coefficients:

$$\boxed{Q_{ik}^{t+\Delta t} = U_{Ik}h_{I+1k}^{t+\Delta t} + V_{Ik} + W_{Ik}h_{1k}^{t+\Delta t}} \tag{41}$$

Where:

$$\boxed{U_{Ik} = \frac{E_{I+1k}c_{ik} - gA_{ik}}{T_{ik}}} \tag{42}$$

$$\boxed{V_{Ik} = \frac{P_{ik} + D_{Ik}a_{ik} - D_{I+1k}c_{ik} - (gA_{ik} - E_{Ik}a_{ik})\frac{V_{I-1k}+D_{Ik}}{U_{I-1k}-E_{Ik}}}{T_{ik}}} \tag{43}$$

$$\boxed{W_{Ik} = -\frac{(gA_{ik} - E_{Ik}a_{ik})W_{I-1k}}{U_{I-1k} - E_{Ik}}} \tag{44}$$

$$\boxed{T_{ik} = \left(a_{ik} + b_{ik} + c_{ik} - \frac{gA_{ik} - E_{Ik}a_{ik}}{U_{I-1k} - E_{Ik}}\right)} \tag{45}$$

4.2 Backward recurrence

Starting at the downstream end of superlink k:

$$Q_{nk}^{t+\Delta t} - Q_{nk-1}^{t+\Delta t} + E_{Nk} h_{Nk}^{t+\Delta t} = D_{Nk} \tag{46}$$

$$a_{nk} Q_{nk-1}^{t+\Delta t} + b_{nk} Q_{nk}^{t+\Delta t} + c_{nk} Q_{nk+1}^{t+\Delta t} = P_{nk} + g A_{nk} (h_{Nk}^{t+\Delta t} - h_{Nk+1}^{t+\Delta t}) \tag{47}$$

Assuming $Q_{nk}^{t+\Delta t} = Q_{nk+1}^{t+\Delta t}$:

$$a_{nk}(Q_{nk}^{t+\Delta t} + E_{Nk} h_{Nk}^{t+\Delta t} - D_{Nk}) + b_{nk} Q_{nk}^{t+\Delta t} + c_{nk} Q_{nk}^{t+\Delta t} = P_{nk} + g A_{nk} (h_{Nk}^{t+\Delta t} - h_{Nk+1}^{t+\Delta t}) \tag{48}$$

$$(a_{nk} + b_{nk} + c_{nk}) Q_{nk}^{t+\Delta t} = -E_{Nk} a_{nk} h_{Nk}^{t+\Delta t} + (P_{nk} + a_{nk} D_{Nk}) + g A_{nk} (h_{Nk}^{t+\Delta t} - h_{Nk+1}^{t+\Delta t}) \tag{49}$$

$$(a_{nk} + b_{nk} + c_{nk}) Q_{nk}^{t+\Delta t} = (g A_{nk} - E_{Nk} a_{nk}) h_{Nk}^{t+\Delta t} + (P_{nk} + D_{Nk} a_{nk}) - g A_{nk} h_{Nk+1}^{t+\Delta t} \tag{50}$$

$$Q_{nk}^{t+\Delta t} = \frac{(g A_{nk} - E_{Nk} a_{nk}) h_{Nk}^{t+\Delta t} + (P_{nk} + D_{Nk} a_{nk}) - g A_{nk} h_{Nk+1}^{t+\Delta t}}{(a_{nk} + b_{nk} + c_{nk})} \tag{51}$$

Thus for the downstream end of superlink k:

$$\boxed{Q_{nk}^{t+\Delta t} = X_{Nk} h_{Nk}^{t+\Delta t} + Y_{Nk} + Z_{Nk} h_{Nk+1}^{t+\Delta t}} \tag{52}$$

Where:

$$\boxed{O_{nk} = a_{nk} + b_{nk} + c_{nk}} \tag{53}$$

$$\boxed{X_{Nk} = \frac{(g A_{nk} - E_{Nk} a_{nk})}{O_{nk}}} \tag{54}$$

$$\boxed{Y_{Nk} = \frac{P_{nk} + D_{Nk} a_{nk}}{O_{nk}}} \tag{55}$$

$$\boxed{Z_{Nk} = -\frac{gA_{nk}}{O_{nk}}} \tag{56}$$

For the next element upstream:

$$Q_{nk-1}^{t+\Delta t} - Q_{nk-2}^{t+\Delta t} + E_{Nk-1} h_{Nk-1}^{t+\Delta t} = D_{Nk-1} \tag{57}$$

$$a_{nk-1} Q_{nk-2}^{t+\Delta t} + b_{nk-1} Q_{nk-1}^{t+\Delta t} + c_{nk-1} Q_{nk}^{t+\Delta t} = P_{nk-1} + gA_{nk-1}(h_{Nk-1}^{t+\Delta t} - h_{Nk}^{t+\Delta t}) \tag{58}$$

$$\begin{split} a_{nk-1}(Q_{nk-1}^{t+\Delta t} + E_{Nk-1} h_{Nk-1}^{t+\Delta t} - D_{Nk-1}) + (b_{nk-1}) Q_{nk-1}^{t+\Delta t} + c_{nk-1}(Q_{nk-1} - E_{Nk} h_{Nk}^{t+\Delta t} + D_{Nk}) \\ -P_{nk-1} - gA_{nk-1}(h_{Nk-1}^{t+\Delta t} - h_{Nk}^{t+\Delta t}) = 0 \end{split} \tag{59}$$

$$\begin{split} (a_{nk-1} + b_{nk-1} + c_{nk-1}) Q_{nk-1}^{t+\Delta t} + (-E_{Nk} c_{nk-1} + gA_{nk-1}) h_{Nk}^{t+\Delta t} \\ +(E_{Nk-1} a_{nk-1} - gA_{nk-1}) h_{Nk-1}^{t+\Delta t} + (-D_{Nk-1} a_{nk-1} + D_{Nk} c_{nk-1} - P_{nk-1}) = 0 \end{split} \tag{60}$$

Multiplying $h_{Nk}^{t+\Delta t}$ by $(X_{Nk} + E_{Nk})/(X_{Nk} + E_{Nk})$ and rearranging:

$$\begin{split} (a_{nk-1} + b_{nk-1} + c_{nk-1}) Q_{nk-1}^{t+\Delta t} + \frac{(gA_{nk-1} - E_{Nk} c_{nk-1})(X_{Nk} + E_{Nk})}{(X_{Nk} + E_{Nk})} h_{Nk}^{t+\Delta t} \\ +(E_{Nk-1} a_{nk-1} - gA_{nk-1}) h_{Nk-1}^{t+\Delta t} + (-P_{nk-1} - D_{Nk-1} a_{nk-1} + D_{Nk} c_{nk-1}) = 0 \end{split} \tag{61}$$

Note that:

$$X_{Nk} h_{Nk}^{t+\Delta t} = (Q_{nk}^{t+\Delta t} - Y_{Nk} - Z_{Nk} h_{Nk+1}^{t+\Delta t}) \tag{62}$$

$$E_{Nk} h_{Nk}^{t+\Delta t} = (D_{Nk} - Q_{nk}^{t+\Delta t} + Q_{nk-1}^{t+\Delta t}) \tag{63}$$

Thus:

$$\begin{split} (a_{nk-1} + b_{nk-1} + c_{nk-1}) Q_{nk-1}^{t+\Delta t} \\ + \frac{(gA_{nk-1} - E_{Nk} c_{nk-1})}{(X_{Nk} + E_{Nk})} [(Q_{nk}^{t+\Delta t} - Y_{Nk} - Z_{Nk} h_{Nk+1}^{t+\Delta t}) + (D_{Nk} - Q_{nk}^{t+\Delta t} + Q_{nk-1}^{t+\Delta t})] \\ +(E_{Nk-1} a_{nk-1} - gA_{nk-1}) h_{Nk-1}^{t+\Delta t} + (-P_{nk-1} - D_{Nk-1} a_{nk-1} + D_{Nk} c_{nk-1}) = 0 \end{split} \tag{64}$$

Allowing $Q_{nk}^{t+\Delta t}$ to be eliminated:

$$
\begin{aligned}
&(a_{nk-1} + b_{nk-1} + c_{nk-1})Q_{nk-1}^{t+\Delta t} \\
&+ \frac{(gA_{nk-1} - E_{Nk}c_{nk-1})}{(X_{Nk} + E_{Nk})}Q_{nk-1}^{t+\Delta t} + \frac{(gA_{nk-1} - E_{Nk}c_{nk-1})(-Z_{Nk})}{(X_{Nk} + E_{Nk})}h_{Nk+1}^{t+\Delta t} \\
&+(E_{Nk-1}a_{nk-1} - gA_{nk-1})h_{Nk-1}^{t+\Delta t} \\
&+\left(-P_{nk-1} - D_{Nk-1}a_{nk-1} + D_{Nk}c_{nk-1} + \frac{(gA_{nk-1} - E_{Nk}c_{nk-1})(-Y_{Nk} + D_{Nk})}{(X_{Nk} + E_{Nk})}\right) = 0
\end{aligned}
\tag{65}
$$

Rearranging:

$$
\begin{aligned}
&\left(a_{nk-1} + b_{nk-1} + c_{nk-1} + \frac{(gA_{nk-1} - E_{Nk}c_{nk-1})}{(X_{Nk} + E_{Nk})}\right)Q_{nk-1}^{t+\Delta t} \\
&+(E_{Nk-1}a_{nk-1} - gA_{nk-1})h_{Nk-1}^{t+\Delta t} - \frac{(gA_{nk-1} - E_{Nk}c_{nk-1})Z_{Nk}}{(X_{Nk} + E_{Nk})}h_{Nk+1}^{t+\Delta t} \\
&+\left(-P_{nk-1} - D_{Nk-1}a_{nk-1} + D_{Nk}c_{nk-1} + (gA_{nk-1} - E_{Nk}c_{nk-1})\frac{(D_{Nk} - Y_{Nk})}{(X_{Nk} + E_{Nk})}\right) = 0
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
&\left(a_{nk-1} + b_{nk-1} + c_{nk-1} + \frac{(gA_{nk-1} - E_{Nk}c_{nk-1})}{(X_{Nk} + E_{Nk})}\right)Q_{nk-1}^{t+\Delta t} \\
&= (gA_{nk-1} - E_{Nk-1}a_{nk-1})h_{Nk-1}^{t+\Delta t} \\
&+\left(P_{nk-1} + D_{Nk-1}a_{nk-1} - D_{Nk}c_{nk-1} - (gA_{nk-1} - E_{Nk}c_{nk-1})\frac{(D_{Nk} - Y_{Nk})}{(X_{Nk} + E_{Nk})}\right) \\
&+\frac{(gA_{nk-1} - E_{Nk}c_{nk-1})Z_{Nk}}{(X_{Nk} + E_{Nk})}h_{Nk+1}^{t+\Delta t}
\end{aligned}
\tag{67}
$$

Generalizing for $i = nk - 1$, $I = Nk - 1$:

$$
\begin{aligned}
&\left(a_{ik} + b_{ik} + c_{ik} + \frac{(gA_{ik} - E_{I+1k}c_{ik})}{(X_{I+1k} + E_{I+1k})}\right)Q_{ik}^{t+\Delta t} \\
&= (gA_{ik} - E_{Ik}a_{ik})h_{Ik}^{t+\Delta t} \\
&+\left(P_{ik} + D_{Ik}a_{ik} - D_{I+1k}c_{ik} - (gA_{ik} - E_{I+1k}c_{ik})\frac{(D_{I+1k} - Y_{I+1k})}{(X_{I+1k} + E_{I+1k})}\right) \\
&+\frac{(gA_{ik} - E_{I+1k}c_{ik})Z_{I+1k}}{(X_{I+1k} + E_{I+1k})}h_{Nk+1}^{t+\Delta t}
\end{aligned}
\tag{68}
$$

Condensing in terms of coefficients:

$$\boxed{Q_{ik}^{t+\Delta t} = X_{ik}h_{Ik}^{t+\Delta t} + Y_{Ik} + Z_{Ik}h_{Nk+1}^{t+\Delta t}} \tag{69}$$

Where:

$$\boxed{X_{Ik} = \frac{gA_{ik} - E_{Ik}a_{ik}}{O_{ik}}} \tag{70}$$

$$\boxed{Y_{Ik} = \frac{P_{ik} + D_{Ik}a_{ik} - D_{I+1k}c_{ik} - (gA_{ik} - E_{I+1k}c_{ik})\frac{(D_{I+1k} - Y_{I+1k})}{(X_{I+1k} + E_{I+1k})}}{O_{ik}}} \tag{71}$$

$$\boxed{Z_{Ik} = \frac{(gA_{ik} - E_{I+1k}c_{ik})Z_{I+1k}}{(X_{I+1k} + E_{I+1k})O_{ik}}} \tag{72}$$

$$\boxed{O_{ik} = \left(a_{ik} + b_{ik} + c_{ik} + \frac{gA_{ik} - E_{I+1k}c_{ik}}{X_{I+1k} + E_{I+1k}}\right)} \tag{73}$$

5 Inlet hydraulics

5.1 Depth at upstream end of superlink

The discharge at the upstream end of a superlink is given by:

$$Q_{uk} = C_{uk} A_{uk} \sqrt{2g \Delta H_{uk}} \tag{74}$$

Where:

$$\Delta H_{uk} = H_{juk} - h_{uk} - z_{inv,uk} \tag{75}$$

Squaring and rearranging provides the depth boundary condition at the upstream end:

$$Q_{uk}^2 = 2C_{uk}^2 A_{uk}^2 g (H_{juk} - h_{uk} - z_{inv,uk}) \tag{76}$$

$$|Q_{uk}^t| Q_{uk}^{t+\Delta t} = 2C_{uk}^2 A_{uk}^2 g (H_{juk} - h_{uk} - z_{inv,uk}) \tag{77}$$

$$h_{uk} = -\frac{|Q_{uk}^t| Q_{uk}^{t+\Delta t}}{2C_{uk}^2 A_{uk}^2 g} + H_{juk} - z_{inv,uk} \tag{78}$$

5.2 Depth at downstream end of superlink

The discharge at the downstream end of a superlink is given by:

$$Q_{dk} = C_{dk} A_{dk} \sqrt{2g \Delta H_{dk}} \tag{79}$$

Where:

$$\Delta H_{dk} = h_{dk} + z_{inv,dk} - H_{jdk} \tag{80}$$

Squaring and rearranging provides the depth boundary condition at the downstream end:

$$Q_{dk}^2 = 2C_{dk}^2 A_{dk}^2 g (h_{dk} + z_{inv,dk} - H_{jdk}) \tag{81}$$

$$|Q_{dk}^t| Q_{dk}^{t+\Delta t} = 2C_{dk}^2 A_{dk}^2 g(h_{dk} + z_{inv,dk} - H_{jdk}) \tag{82}$$

$$h_{dk} = \frac{|Q_{dk}^t| Q_{dk}^{t+\Delta t}}{2C_{dk}^2 A_{dk}^2 g} + H_{jdk} - z_{inv,dk} \tag{83}$$

5.3 Superlink boundary conditions

From the recurrence relations:

$$Q_{uk}^{t+\Delta t} = X_{1k} h_{uk}^{t+\Delta t} + Y_{1k} + Z_{1k} h_{dk}^{t+\Delta t} \tag{84}$$

$$Q_{dk}^{t+\Delta t} = U_{Nk} h_{dk}^{t+\Delta t} + V_{Nk} + W_{Nk} h_{uk}^{t+\Delta t} \tag{85}$$

From the depth boundary conditions at the ends of each superlink:

$$h_{uk} = \gamma_{uk} Q_{uk}^{t+\Delta t} + H_{juk} - z_{inv,uk} \tag{86}$$

$$h_{dk} = \gamma_{dk} Q_{dk}^{t+\Delta t} + H_{jdk} - z_{inv,dk} \tag{87}$$

Where:

$$\gamma_{uk} = -\frac{|Q_{uk}^t|}{2C_{uk}^2 A_{uk}^2 g} \tag{88}$$

$$\gamma_{dk} = \frac{|Q_{dk}^t|}{2C_{dk}^2 A_{dk}^2 g} \tag{89}$$

Substituting into the recurrence relations:

$$Q_{uk}^{t+\Delta t} = X_{1k}(\gamma_{uk} Q_{uk}^{t+\Delta t} + H_{juk} - z_{inv,uk}) + Y_{1k} + Z_{1k}(\gamma_{dk} Q_{dk}^{t+\Delta t} + H_{jdk} - z_{inv,dk}) \tag{90}$$

$$Q_{dk}^{t+\Delta t} = U_{Nk}(\gamma_{dk} Q_{dk}^{t+\Delta t} + H_{jdk} - z_{inv,dk}) + V_{Nk} + W_{Nk}(\gamma_{uk} Q_{uk}^{t+\Delta t} + H_{juk} - z_{inv,uk}) \tag{91}$$

Expanding:

$$Q_{uk}^{t+\Delta t} = X_{1k}\gamma_{uk}Q_{uk}^{t+\Delta t} + X_{1k}H_{juk}^{t+\Delta t} - X_{1k}z_{inv,uk} + Y_{1k} + Z_{1k}\gamma_{dk}Q_{dk}^{t+\Delta t} + Z_{1k}H_{jdk}^{t+\Delta t} - Z_{1k}z_{inv,dk} \tag{92}$$

$$Q_{dk}^{t+\Delta t} = U_{Nk}\gamma_{dk}Q_{dk}^{t+\Delta t} + U_{Nk}H_{jdk} - U_{Nk}z_{inv,dk} + V_{Nk} + W_{Nk}\gamma_{uk}Q_{uk}^{t+\Delta t} + W_{Nk}H_{juk} - W_{Nk}z_{inv,uk} \tag{93}$$

Rearranging:

$$0 = (X_{1k}\gamma_{uk} - 1)Q_{uk}^{t+\Delta t} + Z_{1k}\gamma_{dk}Q_{dk}^{t+\Delta t} + X_{1k}H_{juk}^{t+\Delta t} + Z_{1k}H_{jdk}^{t+\Delta t} + \pi_1 \tag{94}$$

$$0 = W_{Nk}\gamma_{uk}Q_{uk}^{t+\Delta t} + (U_{Nk}\gamma_{dk} - 1)Q_{dk}^{t+\Delta t} + W_{Nk}H_{juk} + U_{Nk}H_{jdk} + \pi_2 \tag{95}$$

Where:

$$\pi_1 = Y_{1k} - X_{1k}z_{inv,uk} - Z_{1k}z_{inv,dk} \tag{96}$$

$$\pi_2 = V_{Nk} - W_{Nk}z_{inv,uk} - U_{Nk}z_{inv,dk} \tag{97}$$

$$\tag{98}$$

Writing as a matrix equation:

$$\begin{bmatrix} (X_{1k}\gamma_{uk} - 1) & Z_{1k}\gamma_{dk} \\ W_{Nk}\gamma_{uk} & (U_{Nk}\gamma_{dk} - 1) \end{bmatrix} \begin{bmatrix} Q_{uk}^{t+\Delta t} \\ Q_{dk}^{t+\Delta t} \end{bmatrix} = \begin{bmatrix} -X_{1k}H_{juk}^{t+\Delta t} - Z_{1k}H_{jdk}^{t+\Delta t} - \pi_1 \\ -W_{Nk}H_{juk}^{t+\Delta t} - U_{Nk}H_{jdk}^{t+\Delta t} - \pi_2 \end{bmatrix} \tag{99}$$

Taking the matrix inverse:

$$\begin{bmatrix} Q_{uk}^{t+\Delta t} \\ Q_{dk}^{t+\Delta t} \end{bmatrix} = \frac{1}{D_k^*} \begin{bmatrix} (U_{Nk}\gamma_{dk} - 1) & -Z_{1k}\gamma_{dk} \\ -W_{Nk}\gamma_{uk} & (X_{1k}\gamma_{uk} - 1) \end{bmatrix} \begin{bmatrix} -X_{1k}H_{juk}^{t+\Delta t} - Z_{1k}H_{jdk}^{t+\Delta t} - \pi_1 \\ -W_{Nk}H_{juk}^{t+\Delta t} - U_{Nk}H_{jdk}^{t+\Delta t} - \pi_2 \end{bmatrix} \tag{100}$$

Where:

$$D_k^* = (X_{1k}\gamma_{uk} - 1)(U_{Nk}\gamma_{dk} - 1) - (Z_{1k}\gamma_{dk})(W_{Nk}\gamma_{uk}) \tag{101}$$

Expanding:

$$\begin{bmatrix} Q_{uk}^{t+\Delta t} \\ Q_{dk}^{t+\Delta t} \end{bmatrix} = \frac{1}{D_k^*} \begin{bmatrix} (U_{Nk}\gamma_{dk}-1)(-X_{1k}H_{juk}^{t+\Delta t} - Z_{1k}H_{jdk}^{t+\Delta t} - \pi_1) + (-Z_{1k}\gamma_{dk})(-W_{Nk}H_{juk}^{t+\Delta t} - U_{Nk}H_{jdk}^{t+\Delta t} - \pi_2) \\ (-W_{Nk}\gamma_{uk})(-X_{1k}H_{juk}^{t+\Delta t} - Z_{1k}H_{jdk}^{t+\Delta t} - \pi_1) + (X_{1k}\gamma_{uk}-1)(-W_{Nk}H_{juk}^{t+\Delta t} - U_{Nk}H_{jdk}^{t+\Delta t} - \pi_2) \end{bmatrix} \tag{102}$$

Arranging in terms of the unknown heads:

$$\begin{aligned} Q_{uk}^{t+\Delta t} = {} & [(U_{Nk}\gamma_{dk}-1)(-X_{1k}) + (-Z_{1k}\gamma_{dk})(-W_{Nk})]H_{juk}^{t+\Delta t} + \\ & [(U_{Nk}\gamma_{dk}-1)(-Z_{1k}) + (-Z_{1k}\gamma_{dk})(-U_{Nk})]H_{jdk}^{t+\Delta t} + \\ & [(U_{Nk}\gamma_{dk}-1)(-\pi_1) + (-Z_{1k}\gamma_{dk})(-\pi_2)] \end{aligned} \tag{103}$$

$$\begin{aligned} Q_{dk}^{t+\Delta t} = {} & [(-W_{Nk}\gamma_{uk})(-X_{1k}) + (X_{1k}\gamma_{uk}-1)(-W_{Nk})]H_{juk}^{t+\Delta t} + \\ & [(-W_{Nk}\gamma_{uk})(-Z_{1k}) + (X_{1k}\gamma_{uk}-1)(-U_{Nk})]H_{jdk}^{t+\Delta t} + \\ & [(-W_{Nk}\gamma_{uk})(-\pi_1) + (X_{1k}\gamma_{uk}-1)(-\pi_2)] \end{aligned} \tag{104}$$

Finally, the upstream and downstream flows can be expressed as:

$$Q_{uk}^{t+\Delta t} = \alpha_{uk}H_{juk}^{t+\Delta t} + \beta_{uk}H_{jdk}^{t+\Delta t} + \chi_{uk} \tag{105}$$

$$Q_{dk}^{t+\Delta t} = \alpha_{dk}H_{juk}^{t+\Delta t} + \beta_{dk}H_{jdk}^{t+\Delta t} + \chi_{dk} \tag{106}$$

Where:

$$\alpha_{uk} = \frac{(1-U_{Nk}\gamma_{dk})X_{1k} + Z_{1k}\gamma_{dk}W_{Nk}}{D_k^*} \tag{107}$$

$$\beta_{uk} = \frac{(1-U_{Nk}\gamma_{dk})Z_{1k} + Z_{1k}\gamma_{dk}U_{Nk}}{D_k^*} \tag{108}$$

$$\chi_{uk} = \frac{(1-U_{Nk}\gamma_{dk})(Y_{1k} - X_{1k}z_{inv,uk} - Z_{1k}z_{inv,dk}) + (Z_{1k}\gamma_{dk})(V_{Nk} - W_{Nk}z_{inv,uk} - U_{Nk}z_{inv,dk})}{D_k^*} \tag{109}$$

$$\alpha_{dk} = \frac{(1-X_{1k}\gamma_{uk})W_{Nk} + W_{Nk}\gamma_{uk}X_{1k}}{D_k^*} \tag{110}$$

$$\beta_{dk} = \frac{(1 - X_{1k}\gamma_{uk})U_{Nk} + W_{Nk}\gamma_{uk}Z_{1k}}{D_k^*} \tag{111}$$

$$\chi_{dk} = \frac{(1 - X_{1k}\gamma_{uk})(V_{Nk} - W_{Nk}z_{inv,uk} - U_{Nk}z_{inv,dk}) + (W_{Nk}\gamma_{uk})(Y_{1k} - X_{1k}z_{inv,uk} - Z_{1k}z_{inv,dk})}{D_k^*} \tag{112}$$

$$D_k^* = (X_{1k}\gamma_{uk} - 1)(U_{Nk}\gamma_{dk} - 1) - (Z_{1k}\gamma_{dk})(W_{Nk}\gamma_{uk}) \tag{113}$$

$$\gamma_{uk} = -\frac{|Q_{uk}^t|}{2C_{uk}^2 A_{uk}^2 g} \tag{114}$$

$$\gamma_{dk} = \frac{|Q_{dk}^t|}{2C_{dk}^2 A_{dk}^2 g} \tag{115}$$

5.4 Boundary conditions using Ji's method

In a similar manner, one can define the superlink coefficients using the linearized boundary conditions from Ji (1998):

$$h_{uk}^{t+\Delta t} = \kappa_{uk} Q_{uk}^{t+\Delta t} + \lambda_{uk} H_{juk}^{t+\Delta t} + \mu_{uk} \tag{116}$$

$$h_{dk}^{t+\Delta t} = \kappa_{dk} Q_{dk}^{t+\Delta t} + \lambda_{dk} H_{jdk}^{t+\Delta t} + \mu_{dk} \tag{117}$$

$$\kappa_{uk} = \frac{2A_{uk}\Delta H_{uk}}{Q_{uk}(2\Delta H_{uk}B_{uk} - A_{uk})} \tag{118}$$

$$\lambda_{uk} = -\frac{A_{uk}}{2\Delta H_{uk}B_{uk} - A_{uk}} \tag{119}$$

$$\mu_{uk} = \frac{A_{uk}(H_{juk} - h_{uk})}{2\Delta H_{uk}B_{uk} - A_{uk}} \tag{120}$$

$$\kappa_{dk} = \frac{2A_{dk}\Delta H_{dk}}{Q_{dk}(2\Delta H_{dk}B_{dk} + A_{dk})} \tag{121}$$

$$\lambda_{dk} = \frac{A_{dk}}{Q_{dk}(2\Delta H_{dk} B_{dk} + A_{dk}} \tag{122}$$

$$\mu_{dk} = \frac{A_{dk}(h_{dk} - H_{jdk})}{2\Delta H_{dk} B_{dk} + A_{uk}} \tag{123}$$

$$\alpha_{uk} = \frac{(1 - U_{Nk}\kappa_{dk})X_{1k}\lambda_{uk} + \kappa_{dk}\lambda_{uk}Z_{1k}W_{Nk}}{D_k^*} \tag{124}$$

$$\beta_{uk} = \frac{(1 - U_{Nk}\kappa_{dk})X_{1k}\lambda_{uk} + \kappa_{dk}\lambda_{uk}Z_{1k}W_{Nk}}{D_k^*} \tag{125}$$

$$\chi_{uk} = \frac{(1 - U_{Nk}\kappa_{dk})(X_{1k}\mu_{uk} + Z_{1k}\mu_{dk} + Y_{1k}) + (Z_{1k}\kappa_{dk})(W_{Nk}\mu_{uk} + U_{Nk}\mu_{dk} + V_{Nk})}{D_k^*} \tag{126}$$

$$\alpha_{dk} = \frac{(1 - X_{1k}\kappa_{uk})W_{Nk}\lambda_{uk} + \kappa_{uk}\lambda_{uk}W_{Nk}X_{1k}}{D_k^*} \tag{127}$$

$$\beta_{dk} = \frac{(1 - X_{1k}\kappa_{uk})U_{Nk}\lambda_{dk} + \kappa_{uk}\lambda_{dk}W_{Nk}Z_{1k}}{D_k^*} \tag{128}$$

$$\chi_{dk} = \frac{(1 - X_{1k}\kappa_{uk})(W_{Nk}\mu_{uk} + U_{Nk}\mu_{dk} + V_{Nk}) + (W_{Nk}\kappa_{uk})(X_{1k}\mu_{uk} + Z_{1k}\mu_{dk} + Y_{1k})}{D_k^*} \tag{129}$$

$$D_k^* = (X_{1k}\kappa_{uk} - 1)(U_{Nk}\kappa_{dk} - 1) - (Z_{1k}\kappa_{dk})(W_{Nk}\kappa_{uk}) \tag{130}$$

6 Forming the solution matrix

The equations for the flows at the ends of each superlink are given by:

$$\sum_{l=1}^{NBDj} Q_{dk_l}^{t+\Delta t} - \sum_{m=1}^{NBUj} Q_{uk_m}^{t+\Delta t} + Q_{o,j} = \frac{A_{sj}(H_j^{t+\Delta t} - H_j)}{\Delta t} \tag{131}$$

Substituting the linear expressions for the upstream and downstream flows:

$$
\begin{aligned}
\frac{A_{sj}(H_j^{t+\Delta t} - H_j^t)}{\Delta t} = & \sum_{l=1}^{NBDj} (\alpha_{dk_l} H_{juk_l}^{t+\Delta t} + \beta_{dk_l} H_{jdk_l}^{t+\Delta t} + \chi_{dk_l}) \\
& - \sum_{m=1}^{NBUj} (\alpha_{uk_m} H_{juk_m}^{t+\Delta t} + \beta_{uk_m} H_{jdk_m}^{t+\Delta t} + \chi_{uk_m}) + Q_{o,j}
\end{aligned} \tag{132}
$$

Because $H_{jdk_l} = H_j$ and $H_{juk_m} = H_j$:

$$
\begin{aligned}
\frac{A_{sj}(H_j^{t+\Delta t} - H_j^t)}{\Delta t} = & \sum_{l=1}^{NBDj} (\alpha_{dk_l} H_{juk_l}^{t+\Delta t} + \beta_{dk_l} H_j^{t+\Delta t} + \chi_{dk_l}) \\
& - \sum_{m=1}^{NBUj} (\alpha_{uk_m} H_j^{t+\Delta t} + \beta_{uk_m} H_{jdk_m}^{t+\Delta t} + \chi_{uk_m}) + Q_{o,j}
\end{aligned} \tag{133}
$$

Rearranging:

$$
\begin{aligned}
& \frac{A_{sj}(H_j^t)}{\Delta t} + \sum_{l=1}^{NBDj} \chi_{dk_l} - \sum_{m=1}^{NBUj} \chi_{uk_m} + Q_{o,j} \\
& = \left(\frac{A_{sj}}{\Delta t} + \sum_{m=1}^{NBUj} \alpha_{uk_m} - \sum_{l=1}^{NBDj} \beta_{dk_l} \right) H_j^{t+\Delta t} \\
& \quad - \sum_{l=1}^{NBDj} \alpha_{dk_l} H_{juk_l}^{t+\Delta t} + \sum_{m=1}^{NBUj} \beta_{uk_m} H_{jdk_m}^{t+\Delta t}
\end{aligned} \tag{134}
$$

For the example network in Ji (1998):

$$
Ax = b \tag{135}
$$

$$
A = \begin{bmatrix}
(\frac{A_{s0}}{\Delta t} + \alpha_{u0}) & \beta_{u0} & 0 & 0 & 0 & 0 \\
-\alpha_{d0} & (\frac{A_{s1}}{\Delta t} + \alpha_{u1} + \alpha_{u3} - \beta_{d0}) & \beta_{u1} & 0 & \beta_{u3} & 0 \\
0 & -\alpha_{d1} & (\frac{A_{s2}}{\Delta t} + \alpha_{u2} - \beta_{d1} - \beta_{d5}) & \beta_{u2} & -\alpha_{d5} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & -\alpha_{d3} & \beta_{u5} & 0 & (\frac{A_{s4}}{\Delta t} + \alpha_{u4} + \alpha_{u5} - \beta_{d3}) & \beta_{u4} \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \tag{136}
$$

$$
b = \begin{bmatrix} \frac{A_{s0}H_0^t}{\Delta t} - \chi_{u0} + Q_{o0} \\ \frac{A_{s1}H_1^t}{\Delta t} + \chi_{d0} - (\chi_{u1} + \chi_{u3}) + Q_{o1} \\ \frac{A_{s2}H_2^t}{\Delta t} + (\chi_{d1} + \chi_{d5}) - \chi_{u2} + Q_{02} \\ H_{3,bc} \\ \frac{A_{s4}H_4^t}{\Delta t} + \chi_{d3} - (\chi_{u4} + \chi_{u5}) + Q_{04} \\ H_{5,bc} \end{bmatrix} \tag{137}
$$