CSC418 Assignment 2

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1. Transformations

1.1 2D Affine Transformation

$$TA = B$$
$$\begin{bmatrix} t_{11} & t_{21} & t_{31} \\ t_{12} & t_{22} & t_{32} \\ t_{13} & t_{23} & t_{33} \end{bmatrix} \begin{bmatrix} a_{1x} & a_{2x} & a_{3x} \\ a_{1y} & a_{2y} & a_{3y} \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} b_{1x} & b_{2x} & b_{3x} \\ b_{1y} & b_{2y} & b_{3y} \\ c_1 & c_2 & c_3 \end{bmatrix}$$
$$TAA^{-1} = T = BA^{-1}$$

$$\det(A) = (a_{2x}a_{3y} - a_{2y}a_{3x}) + (a_{3x}a_{1y} - a_{1x}a_{3y}) + (a_{1x}a_{2y} - a_{1y}a_{2x})$$

The mapping is fully determined when $\det(A) \neq 0$, since the matrix would not be invertible and the transformation could not be specified.

$$\begin{aligned} \det(A) = 0 &= (a_{2x}a_{3y} - a_{2y}a_{3x}) + (a_{3x}a_{1y} - a_{1x}a_{3y}) + (a_{1x}a_{2y} - a_{1y}a_{2x}) \\ 0 &= a_2 \times a_3 + a_3 \times a_1 + a_1 \times a_2 \\ 0 &= \sin\theta_{23} + \sin\theta_{31} + \sin\theta_{12} \end{aligned}$$

Using CFG, this represents the sum of lines formed from $p_1 \rightarrow p_2$, $p_2 \rightarrow p_3$, and $p_3 \rightarrow p_1$.

2. Viewing and Projection

2.1 Pinhole Camera

Since light rays propagate by straight lines, a screen pixel can be determined as a sum of the light sources in a cone through the aperture (assuming a circular aperture). As the aperture size decreases to a "pinhole", the cone of light rays collapses into approximately a straight line through the aperture. Assuming the aperture is $(0, 0)$, screen pixel $(x, y, depth)$ will be determined by the light source in the direction $\langle -x, -y, -depth \rangle$. The resulting image formed is an xy inverted image of the world.

2.2 World to Camera Transformation

$$
\begin{aligned}
\hat{z}' &= \frac{\vec{p} - \vec{c}}{||\vec{p} - \vec{c}||} \\
\hat{y}' &= \hat{u} \\
x' &= \hat{y}' \times \hat{z}' = \hat{u} \times \frac{\vec{p} - \vec{c}}{||\vec{p} - \vec{c}||}
\end{aligned}
$$

$$
\begin{aligned}
&\textbf{from lecture:} \\
&M_{camera} = \begin{bmatrix} \hat{x}'^T & & & 0 \\ \hat{y}'^T & & & 0 \\ \hat{z}'^T & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -c_x \\ 0 & 1 & 0 & -c_y \\ 0 & 0 & 1 & -c_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \left[\hat{u} \times \frac{\vec{p}-\vec{c}}{||\vec{p}-\vec{c}||}\right]^T & & & 0 \\ \hat{u}^T & & & 0 \\ \left[\frac{\vec{p}-\vec{c}}{||\hat{p}-\hat{c}||}\right]^T & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -c_x \\ 0 & 1 & 0 & -c_y \\ 0 & 0 & 1 & -c_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

2.3 Parallel Conditions

Logically, any set of line families that is not perpendicular to z will have a non uniform directional change by the perspective frustrum. That is, any vector with a z component will result in a family of lines with differing z components after the projection, depending on their initial location. Since the frustrum projection will uniformly scale x, y magnitudes along a fixed z, any initial x, y parallel family will be parallel following projection.

Conditions

1. $v \neq \vec{0}$
2. $v \perp \hat{z} \implies v_z = 0$, where all vectors are in the camera space.

2.4 Line Convergence

$$
\begin{aligned}
\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} &= \lim_{t\to\infty} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1+\frac{f}{n} & -f \\ 0 & 0 & \frac{1}{n} & 0 \end{bmatrix} \begin{bmatrix} xt+x_0 \\ yt+y_0 \\ zt+z_0 \\ 1 \end{bmatrix} \\
&= \lim_{t\to\infty} \begin{bmatrix} xt+x_0 \\ yt+y_0 \\ (zt+z_0)(1+\frac{f}{n})-f \\ \frac{zt+z_0}{n} \end{bmatrix} \\
&= \lim_{t\to\infty} \begin{bmatrix} \frac{n(xt+x_0)}{zt+z_0} \\ yt+y_0 \\ (zt+z_0)(1+\frac{f}{n})-f \\ 1 \end{bmatrix}
\end{aligned}
$$

3. Surfaces

3.1 Surface Normal at p

$$
\begin{aligned}
f(x,y,z) &= (R - \sqrt{x^2+y^2})^2 + z^2 - r^2 = 0 \\
\nabla f(x,y,z) &= \begin{bmatrix} 2(R-\sqrt{x^2+y^2})(-\frac{1}{2}(x^2+y^2)^{-0.5})(2x) \\ 2(R-\sqrt{x^2+y^2})(-\frac{1}{2}(x^2+y^2)^{-0.5})(2y) \\ 2z \end{bmatrix} \\
&= \begin{bmatrix} 2x\left[1-\frac{R}{\sqrt{x^2+y^2}}\right] \\ 2y\left[1-\frac{R}{\sqrt{x^2+y^2}}\right] \\ 2z \end{bmatrix} = \text{surface normal at } (x,y,z)
\end{aligned}
$$

3.2 Tangent Plane at p

$$
p = \begin{bmatrix} a \\ b \\ c \end{bmatrix}
$$

$$
\nabla f(x,y,z)\Big|_{(x,y,z)=a,b,c} \cdot \begin{bmatrix} x-a \\ y-b \\ z-c \end{bmatrix} = 0
$$

$$
2a\left[1-\frac{R}{\sqrt{a^2+b^2}}\right](x-a) + 2b\left[1-\frac{R}{\sqrt{a^2+b^2}}\right](y-c) + 2c(z-c) = 0
$$

3.3 Parametric Curve

$$
\begin{aligned}
f(R\cos\lambda, R\sin\lambda, r) &= \left[R - \sqrt{(R\cos\lambda)^2 + (R\sin\lambda)^2}\right]^2 + r^2 - r^2 \\
&= \left[R - R\sqrt{\cos^2\lambda + \sin^2\lambda}\right]^2 + 0 \\
&= 0 \\
&\therefore \text{the curve lies on the surface}
\end{aligned}
$$

3.4 Tangent of q

$$
\begin{aligned}
\text{tangent}(q(\lambda)) &= \frac{d}{d\lambda} q(\lambda) \\
&= \begin{bmatrix} \frac{d}{d\lambda} R\cos\lambda \\ \frac{d}{d\lambda} R\sin\lambda \\ \frac{d}{d\lambda} r \end{bmatrix} \\
&= \begin{bmatrix} -R\sin\lambda \\ R\cos\lambda \\ 0 \end{bmatrix}
\end{aligned}
$$

3.5 Tangent of q on Tangent Plane

$$
\begin{gathered}
\text{tangent}(q(\lambda)) \in \text{tangent plane} \iff \text{tangent}(q(\lambda)) \cdot (\text{surface plane normal}) = 0 \;\forall\; \lambda \in [0, 2\pi] \\
\text{since the tangent and surface normal originate from the same point}
\end{gathered}
$$

$$
\begin{aligned}
\begin{bmatrix} -R\sin\lambda \\ R\cos\lambda \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2x\left[1 - \frac{R}{\sqrt{x^2+y^2}}\right] \\ 2y\left[1 - \frac{R}{\sqrt{x^2+y^2}}\right] \\ 2z \end{bmatrix}_{(x,y,z)=p} &= \begin{bmatrix} -R\sin\lambda \\ R\cos\lambda \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2R\cos\lambda\left[1 - \frac{R}{R}\right] \\ 2R\sin\lambda\left[1 - \frac{R}{R}\right] \\ 2r \end{bmatrix} \\
&= \begin{bmatrix} -R\sin\lambda \\ R\cos\lambda \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 2r \end{bmatrix} \\
&= 0 \\
&\therefore \text{ the tangent vector of } q \text{ lies on the tangent plane}
\end{aligned}
$$

4. Curves

4.1 Tangents at P_4

$$\begin{aligned} B_1(t) &= \sum_{i=0}^{3} \binom{3}{i} (1-t)^{3-i} t^i P_{i+1} \\ B_1{}'(t) &= \sum_{i=0}^{3} \binom{3}{i} P_{i+1} \Big[-(3-i)(1-t)^{2-i} t^i + (1-t)^{3-i}(i)t^{i-1}) \Big] \\ B_1{}'(1) &= \Big[\binom{3}{2} P_3 \cdot (-1) \Big] + \Big[\binom{3}{3} P_4 \cdot 3 \Big] \\ &= 3[P_4 - P_3] \\ &= \text{ray formed from } P_3 \rightarrow P_4 \end{aligned}$$

Replacing $P_{1,4}$ with $P_{4,7}$ will yield the same result with the substituted points and time ($t = t - 1$)

$$\begin{aligned} B_2{}'(t) &= \sum_{i=0}^{3} \binom{3}{i} P_{i+4} \Big[-(3-i)(1-t)^{2-i} t^i + (1-t)^{3-i}(i)t^{i-1} \Big] \\ B_2{}'(0) &= 3[P_5 - P_4] \\ &= \text{ray formed from } P_4 \rightarrow P_5 \end{aligned}$$

4.2 Second Derivatives at P_4

From 4.1,

$$\begin{aligned} B_1{}'(t) &= \sum_{i=0}^{3} \binom{3}{i} P_{i+1} \Big[-(3-i)(1-t)^{2-i} t^i + (1-t)^{3-i}(i)t^{i-1}) \Big] \\ B_1{}''(t) &= \sum_{i=0}^{3} \binom{3}{i} P_{i+1} \begin{bmatrix} (i-3)\big[(i-2)(1-t)^{1-i} t^i + i(1-t)^{2-i} t^{i-1}\big] + \\ i\big[(i-3)(1-t)^{2-i} t^{i-1} + (1-t)^{3-i}(i-1)t^{i-2}\big] \end{bmatrix} \\ B_1{}''(1) &= 6P_2 - 12P_3 + 6P_4 \\ &= 6[(P_2 - P_3) + (P_4 - P_3)] \end{aligned}$$

Substituting points $P_{4,7}$

$$B_2{}''(0) = 6[(P_5 - P_6) + (P_7 - P_6)], \text{where t begins at 0 for } B_2$$

4.3 Continuous

In order for the curve to be C^2 continuous given a fixed P_1, P_2, P_3, P_4

$$\begin{aligned} B_1{}''(1) &= B_2{}''(0) \\ 6[(P_2 - P_3) + (P_4 - P_3)] &= 6[(P_5 - P_6) + (P_7 - P_6)] \\ P_2 - P_3 + P_4 - P_3 &= P_5 - P_6 + P_7 - P_6 \\ \Big[P_2 - 2P_3 + P_4\Big]_{\text{constant}} &= P_5 - 2P_6 + P_7 \end{aligned}$$

$P_{5,7}$ are constrained by the above equality.

4.4 Bezier Popularity

1. Bezier curves are serializable: They can be modeled continuously by a designer/modeller, and can be serialized into a small set of discrete-valued points. A 3D program can then "deserialize" the points and recreate the exact curve that was modelled.
2. Bezier curves can be interpolated to any level of detail. Since the curve models a continuous function, the curve can be interpolated by a graphics program to any level of detail/number of linear segments.
3. Bezier curves are intuitive to design: Humans can easily model with bezier curves, since the control points alone give a rough idea of what the curve should look like. The first two and last two control points control the endpoint tangents, which is also useful in design.
4. Bezier curves have easily computable derivatives.