Satellite Formation Flying

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Derivation of Equations of Motion

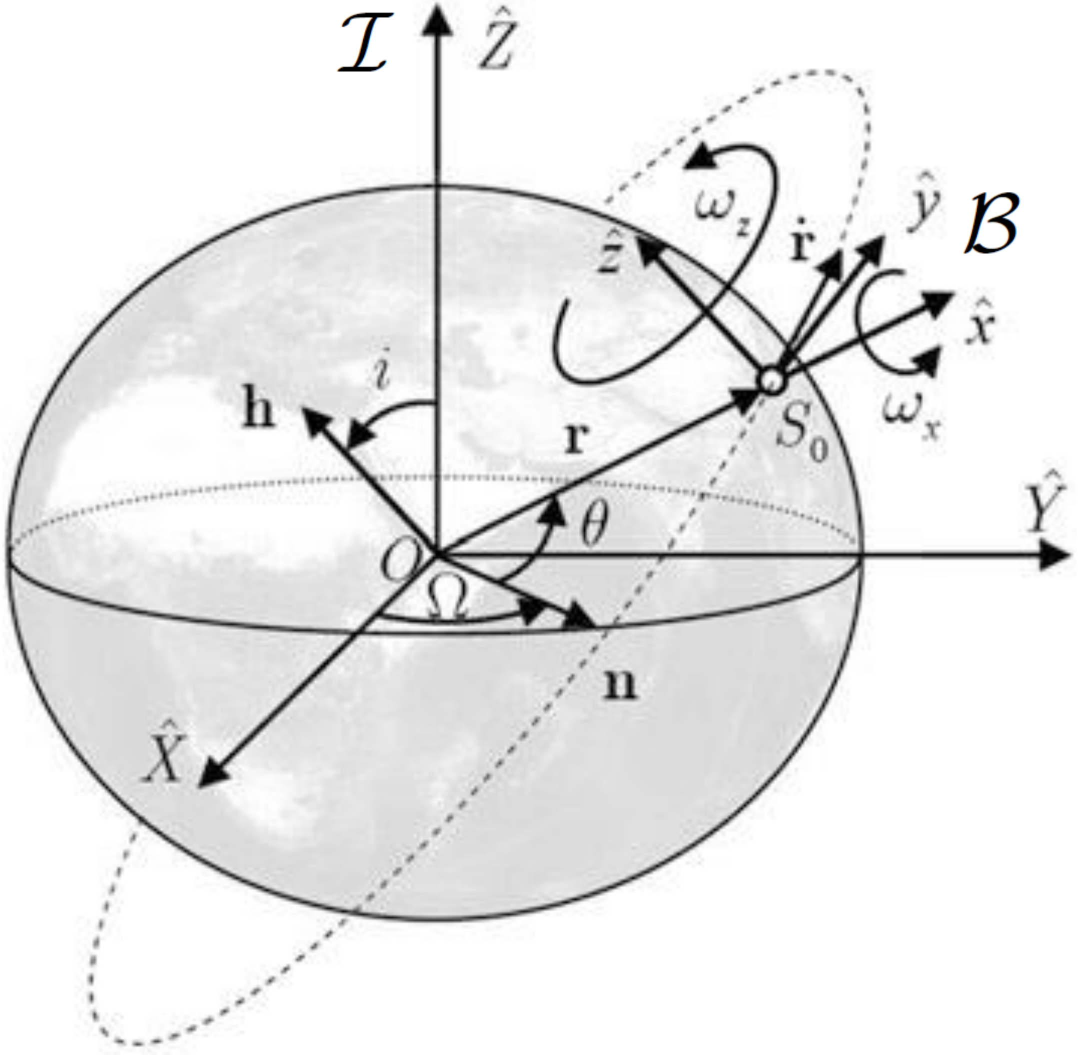
Reference Frames

$$\mathcal{I} = (\mathcal{O}, \boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3})$$

$$\mathcal{B} = (\mathcal{C}, \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$$

$$\boldsymbol{r} = r\hat{\boldsymbol{x}} = X\boldsymbol{e_1} + Y\boldsymbol{e_2} + Z\boldsymbol{e_3}$$

$$\hat{\boldsymbol{x}} = \frac{\boldsymbol{r}}{r} \qquad \hat{\boldsymbol{y}} = \hat{\boldsymbol{z}} \times \hat{\boldsymbol{x}} \qquad \hat{\boldsymbol{z}} = \frac{\boldsymbol{h}}{h}$$



Kinematics in the LVLH and ECI Frames

$$
\begin{aligned}
{}^{\mathcal{I}}R^{\mathcal{B}} &= R_{\Omega,3}R_{I,1}R_{\theta,3} \\
&= \begin{bmatrix} c(\Omega)c(\theta) - c(I)s(\Omega)s(\theta) & -c(\Omega)s(\theta) - c(I)s(\Omega)c(\theta) & s(I)s(\Omega) \\ s(\Omega)c(\theta) + c(I)c(\Omega)s(\theta) & c(I)c(\Omega)c(\theta) - s(\Omega)s(\theta) & -c(\Omega)s(I) \\ s(I)s(\theta) & s(I)c(\theta) & c(I) \end{bmatrix}
\end{aligned}
$$

$$
\boldsymbol{\omega}^{\mathcal{B}/\mathcal{I}} \triangleq \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_{\mathcal{B}} = R_{\theta,3}^T R_{I,1}^T \begin{bmatrix} 0 \\ 0 \\ \dot{\Omega} \end{bmatrix} + R_{\theta,3}^T \begin{bmatrix} \dot{I} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}
$$

$$
\boldsymbol{\omega}^{\mathcal{B}/\mathcal{I}} = \begin{bmatrix} \dot{I}c(\theta) + \dot{\Omega}s(I)s(\theta) \\ \dot{\Omega}s(I)c(\theta) - \dot{I}s(\theta) \\ \dot{\theta} + \dot{\Omega}c(I) \end{bmatrix}_{\mathcal{B}}
$$

Kinematics in the LVLH and ECI Frames

$$\dot{\boldsymbol{r}} = \frac{{}^{\mathcal{I}}d}{dt}(r\hat{\boldsymbol{x}}) = \dot{r}\hat{\boldsymbol{x}} + r\boldsymbol{\omega}^{\mathcal{B}/\mathcal{I}} \times \hat{\boldsymbol{x}} = \dot{r}\hat{\boldsymbol{x}} + \frac{h}{r}\hat{\boldsymbol{y}}$$

$$\boldsymbol{h} = \boldsymbol{r} \times \dot{\boldsymbol{r}} = \begin{bmatrix} 0 \\ r^2\omega_y \\ r^2\omega_z \end{bmatrix}_{\mathcal{B}}$$

$$\begin{aligned} \ddot{\boldsymbol{r}} &= \ddot{r}\hat{\boldsymbol{x}} + \dot{r}\frac{h}{r^2}\hat{\boldsymbol{y}} + \frac{r\dot{h} - h\dot{r}}{r^2}\hat{\boldsymbol{y}} - \frac{h^2}{r^3}\hat{\boldsymbol{x}} + \omega_x\frac{h}{r}\hat{\boldsymbol{z}} \\ &= \left(\ddot{r} - \frac{h^2}{r^3}\right)\hat{\boldsymbol{x}} + \frac{\dot{h}}{r}\hat{\boldsymbol{y}} + \frac{\omega_x h}{r}\hat{\boldsymbol{z}} \end{aligned}$$

$$\omega_z = \frac{h}{r^2}$$

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{\omega}^{\mathcal{B}/\mathcal{I}} \times \hat{\boldsymbol{x}} = \omega_z\hat{\boldsymbol{y}} \qquad \dot{\hat{\boldsymbol{y}}} = \boldsymbol{\omega}^{\mathcal{B}/\mathcal{I}} \times \hat{\boldsymbol{y}} = \omega_x\hat{\boldsymbol{z}} - \omega_z\hat{\boldsymbol{x}} \qquad \dot{\hat{\boldsymbol{z}}} = \boldsymbol{\omega}^{\mathcal{B}/\mathcal{I}} \times \hat{\boldsymbol{z}} = -\omega_x\hat{\boldsymbol{y}}$$

Leader Satellite Dynamics

$$\ddot{\boldsymbol{r}} = -\nabla U$$

$$k_{J2} = \frac{3J_2\mu R_e^2}{2}$$

$$U = -\frac{\mu}{r} - \frac{k_{J2}}{r^3}\left(\frac{1}{3} - s^2(\phi)\right)$$

$$\nabla U = \frac{\partial U}{\partial X}\boldsymbol{e_1} + \frac{\partial U}{\partial Y}\boldsymbol{e_2} + \frac{\partial U}{\partial Z}\boldsymbol{e_3}$$

$$\frac{\partial U}{\partial X} = \frac{\mu X}{r^3} + \frac{k_{J2}X}{r^5} - \frac{5k_{J2}Z^2X}{r^7}$$
$$\frac{\partial U}{\partial Y} = \frac{\mu Y}{r^3} + \frac{k_{J2}Y}{r^5} - \frac{5k_{J2}Z^2Y}{r^7}$$
$$\frac{\partial U}{\partial Z} = \frac{\mu Z}{r^3} + \frac{k_{J2}Z}{r^5} + k_{J2}\frac{2Zr^2 - 5Z^3}{r^7}$$

$$s(\phi) = \frac{Z}{r} = s(I)s(\theta)$$

$$\frac{\partial U}{\partial X} = \frac{\mu X}{r^3} + \frac{k_{J2}X}{r^5}\left(1 - 5s^2(\phi)\right)$$
$$\frac{\partial U}{\partial Y} = \frac{\mu Y}{r^3} + \frac{k_{J2}Y}{r^5}\left(1 - 5s^2(\phi)\right)$$
$$\frac{\partial U}{\partial Z} = \frac{\mu Z}{r^3} + \frac{k_{J2}Z}{r^5}\left(1 - 5s^2(\phi)\right) + \frac{2k_{J2}s(\phi)}{r^4}$$

Leader Satellite Dynamics

$$\nabla U = \left(\frac{\mu}{r^2} + \frac{k_{J2}}{r^4}\left(1 - 5s^2(\phi)\right)\right)\hat{\boldsymbol{r}} + \frac{2k_{J2}s(\phi)}{r^4}\hat{\boldsymbol{Z}}$$

$$\hat{\boldsymbol{Z}} = s(\theta)s(I)\hat{\boldsymbol{x}} + c(\theta)s(I)\hat{\boldsymbol{y}} + c(I)\hat{\boldsymbol{z}}$$

$$s(\phi) = s(\theta)s(I)$$

$$\hat{\boldsymbol{r}} = \hat{\boldsymbol{x}}$$

$$\nabla U = \left(\frac{\mu}{r^2} + \frac{k_{J2}}{r^4}\left(1 - 3s^2(I)s^2(\theta)\right)\right)\hat{\boldsymbol{x}} + \frac{k_{J2}s^2(I)s(2\theta)}{r^4}\hat{\boldsymbol{y}} + \frac{k_{J2}s(2I)s(\theta)}{r^4}\hat{\boldsymbol{z}}$$

Leader Satellite Dynamics

$$\nabla U = \left(\frac{\mu}{r^2} + \frac{k_{J2}}{r^4}\left(1 - 3s^2(I)s^2(\theta)\right)\right)\hat{\boldsymbol{x}} + \frac{k_{J2}s^2(I)s(2\theta)}{r^4}\hat{\boldsymbol{y}} + \frac{k_{J2}s(2I)s(\theta)}{r^4}\hat{\boldsymbol{z}}$$

$$\ddot{\boldsymbol{r}} = \left(\ddot{r} - \frac{h^2}{r^3}\right)\hat{\boldsymbol{x}} + \frac{\dot{h}}{r}\hat{\boldsymbol{y}} + \frac{\omega_x h}{r}\hat{\boldsymbol{z}}$$

$$\boldsymbol{\omega}^{\mathcal{B}/\mathcal{I}} = \begin{bmatrix} \dot{I}c(\theta) + \dot{\Omega}s(I)s(\theta) \\ \dot{\Omega}s(I)c(\theta) - \dot{I}s(\theta) \\ \dot{\theta} + \dot{\Omega}c(I) \end{bmatrix}_{\mathcal{B}}$$

$$\ddot{\boldsymbol{r}} = -\nabla U$$

$$\omega_x = -\frac{k_{J2}s(\theta)s(2I)}{hr^3}$$

$$\dot{r} = v_x$$

$$\dot{v}_x = \ddot{r} = \frac{h^2}{r^3} - \frac{\mu}{r^2} - \frac{k_{J2}}{r^4}\left(1 - 3s^2(I)s^2(\theta)\right)$$

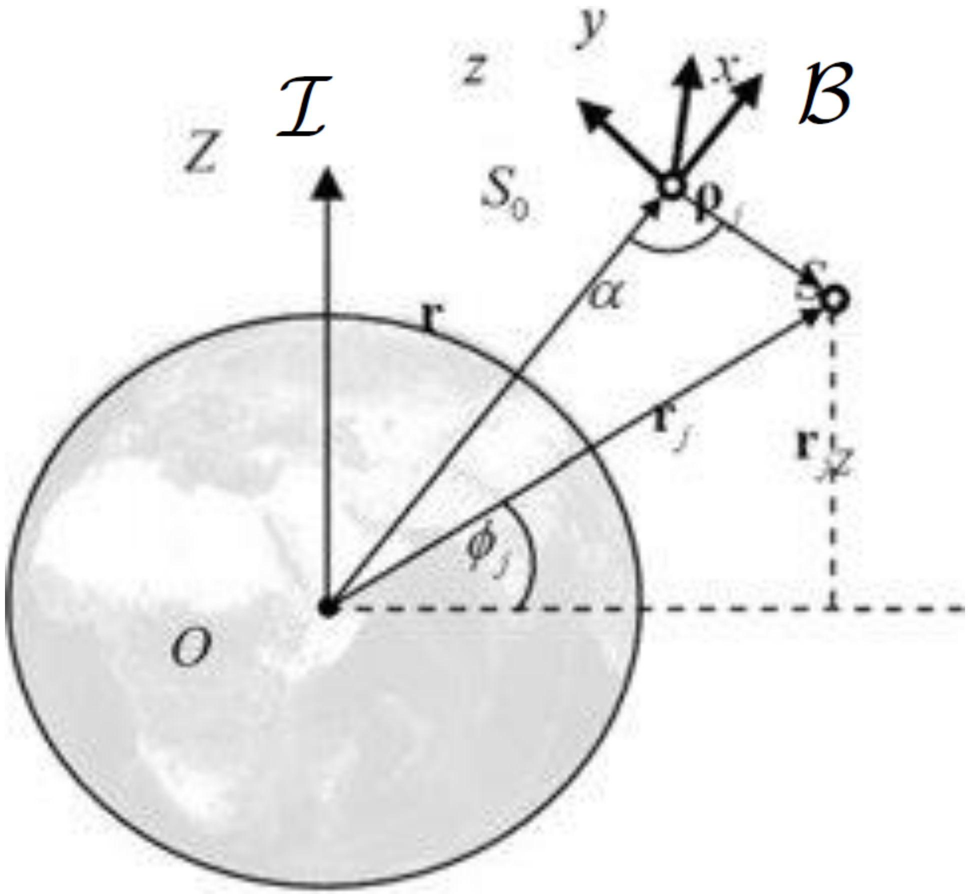
$$\dot{h} = -\frac{k_{J2}s^2(I)s(2\theta)}{r^3}$$

$$\dot{\Omega} = -\frac{2k_{J2}s^2(\theta)c(i)}{hr^3}$$

$$\dot{I} = -\frac{k_{J2}s(2I)s(2\theta)}{2hr^3}$$

$$\dot{\theta} = \frac{h}{r^2} + \frac{2k_{J2}s^2(\theta)c^2(I)}{hr^3}$$

Follower Satellite Kinematics



$$\boldsymbol{r}_j = \boldsymbol{r} + \boldsymbol{\rho}_j = r\hat{\boldsymbol{x}} + (x_j\hat{\boldsymbol{x}} + y_j\hat{\boldsymbol{y}} + z_j\hat{\boldsymbol{z}})$$

$$\boldsymbol{r}_j = (r + x_j)\hat{\boldsymbol{x}} + y_j\hat{\boldsymbol{y}} + z_j\hat{\boldsymbol{z}}$$

$$\dot{\boldsymbol{r}}_{\boldsymbol{j}} = (\dot{r} + \dot{x}_j - y_j\omega_z)\hat{\boldsymbol{x}} + ((r + x_j)\omega_z + \dot{y}_j - z_j\omega_x)\,\hat{\boldsymbol{y}} + (y_j\omega_x + \dot{z}_j)\hat{\boldsymbol{z}}$$

Follower Satellite Dynamics

$$\frac{d}{dt}\left(\frac{\partial L_j}{\partial \dot{\boldsymbol{q}}_j}\right) - \frac{\partial L_j}{\partial \boldsymbol{q}_j} = \boldsymbol{F}_j$$

$$\frac{d}{dt}\left(\frac{\partial K_j}{\partial \dot{\boldsymbol{q}}_j}\right) - \frac{\partial K_j}{\partial \boldsymbol{q}_j} + \frac{\partial U_j}{\partial \boldsymbol{q}_j} = \boldsymbol{0}$$

$$K_j = \tfrac{1}{2}\dot{\boldsymbol{r}}_j \cdot \dot{\boldsymbol{r}}_j$$

$$K_j = \frac{1}{2}(\dot{x}_j + v_x - y_j\omega_z)^2 + \frac{1}{2}\left(\dot{y}_j + (r + x_j)\omega_z - z_j\omega_x\right)^2 + \frac{1}{2}(\dot{z}_j + y_j\omega_x)^2$$

Follower Satellite Dynamics

$$U_j = -\frac{\mu}{r_j} - \frac{k_{J2}}{r_j^3}\left(\frac{1}{3} - s^2(\phi_j)\right)$$

$$s(\phi_j) = \frac{r_{jZ}}{r_j}$$

$$r_{jZ} = \boldsymbol{r_j} \cdot \boldsymbol{\hat{Z}} = (r + x_j)s(I)s(\theta) + y_j s(I)c(\theta) + z_j c(I)$$

$$U_j = -\frac{\mu}{r_j} - \frac{k_{J_2}}{3r_j^3} + \frac{k_{J2} r_{jZ}^2}{r_j^5}$$

Follower Satellite Dynamics

$$\frac{d}{dt}\left(\frac{\partial K_j}{\partial \dot{\boldsymbol{q}}_j}\right) - \frac{\partial K_j}{\partial \boldsymbol{q}_j} + \frac{\partial U_j}{\partial \boldsymbol{q}_j} = \mathbf{0}$$

$$\zeta = \frac{2k_{J2} s(I) s(\theta)}{r^4} \qquad \zeta_j = \frac{2k_{J2} r_{jZ}}{r_j^5}$$

$$\eta^2 = \frac{\mu}{r^3} + \frac{k_{J2}}{r^5} - \frac{5k_{J2} s^2(I) s^2(\theta)}{r^5} \qquad \eta_j^2 = \frac{\mu}{r_j^3} + \frac{k_{J2}}{r_j^5} - \frac{5k_{J2} r_{jZ}^2}{r_j^7}$$

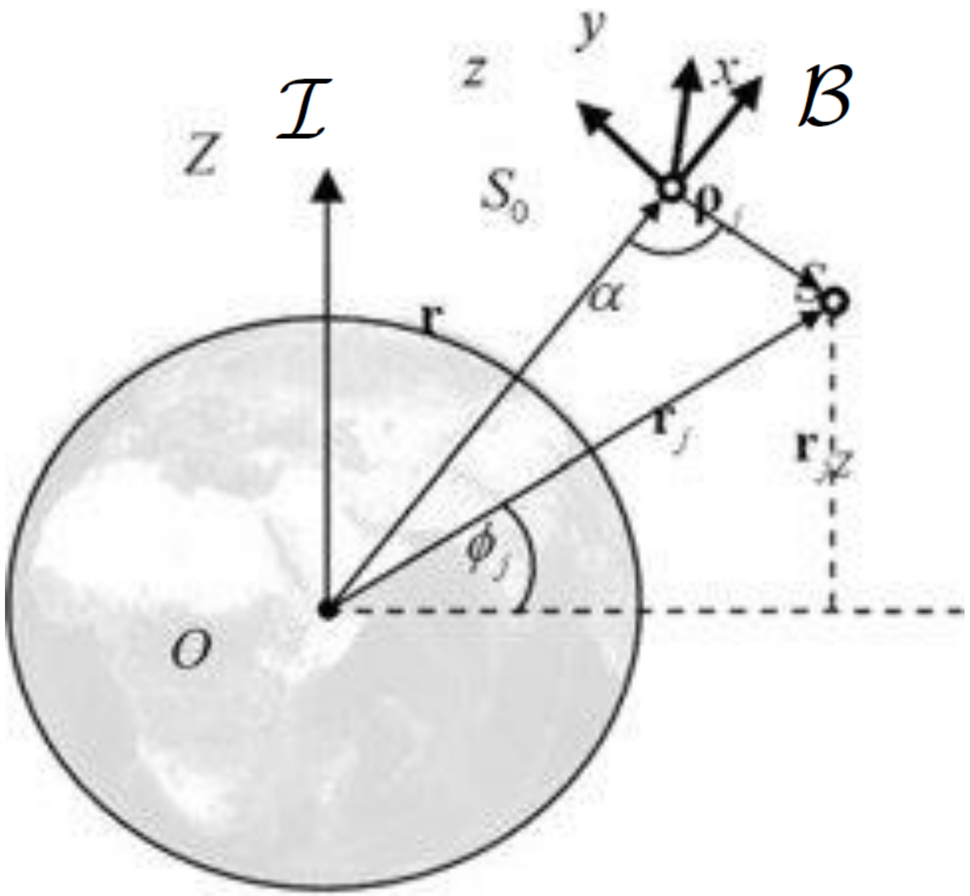
$$\begin{aligned}
\ddot{x}_j &= 2\dot{y}_j\omega_z - x_j(\eta_j^2 - \omega_z^2) + y_j\alpha_z - z_j\omega_x\omega_z - (\zeta_j - \zeta)s(I)s(\theta) - r(\eta_j^2 - \eta^2) \\
\ddot{y}_j &= -2\dot{x}_j\omega_z + 2\dot{z}_j\omega_x - x_j\alpha_z - y_j(\eta_j^2 - \omega_z^2 - \omega_x^2) + z_j\alpha_x - (\zeta_j - \zeta)s(I)c(\theta) \\
\ddot{z}_j &= -2\dot{y}_j\omega_x - x_j\omega_x\omega_z - y_j\alpha_x - z_j(\eta_j^2 - \omega_x^2) - (\zeta_j - \zeta)c(I)
\end{aligned}$$

Linearization of Follower Dynamics

$$r_j^2 = r^2 - 2r\rho_j \cos\alpha + \rho_j^2$$

$$\cos\alpha = \frac{(-\mathbf{r}) \cdot \boldsymbol{\rho}_j}{r\rho_j} = -\frac{x_j}{\rho_j}$$

$$\frac{1}{r_j^{2\lambda}} = \frac{1}{r^{2\lambda}} \left(1 - 2\cos\alpha\left(\frac{\rho_j}{r}\right) + \left(\frac{\rho_j}{r}\right)^2\right)^{-\lambda}$$



Linearization of Follower Dynamics

Gegenbauer Polynomials

$$C_n^{(\lambda)}(u) = \frac{\Gamma(\lambda+1/2)}{\Gamma(2\lambda)} \frac{\Gamma(n+2\lambda)}{\Gamma(n+\lambda+1/2)} \frac{(-1)^n}{2^n n!} (1-u^2)^{-\lambda+1/2} \frac{d^n}{du^n} (1-u^2)^{n+\lambda-1/2}$$

$$\Gamma(n) = (n-1)!$$

$$C_0^{(\lambda)}(u) = 1 \quad C_1^{(\lambda)}(u) = 2\lambda u$$

For $(\lambda+1/2) > 0, |v| < 1$ and $|u| \leq 1$

$$(1-2uv+v^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(u) v^n \approx 1 + (2\lambda u)v$$

$$u = \cos\alpha \quad v = \frac{\rho_j}{r}$$

Linearization of Follower Dynamics

$$\frac{1}{r_j^{2\lambda}} = \frac{1}{r^{2\lambda}}\left(1 - 2\cos\alpha\left(\frac{\rho_j}{r}\right) + \left(\frac{\rho_j}{r}\right)^2\right)^{-\lambda}$$

$$\left(1 - 2uv + v^2\right)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(u)v^n \approx 1 + (2\lambda u)v$$

$$u = \cos\alpha \quad v = \frac{\rho_j}{r}$$

$$\frac{1}{r_j^{2\lambda}} \approx \frac{1}{r^{2\lambda}}\left(1 + 2\lambda\left(-\frac{x_j}{\rho_j}\right)\left(\frac{\rho_j}{r}\right)\right) = \frac{1}{r^{2\lambda}}\left(1 - \frac{2\lambda x_j}{r}\right)$$

Linearization of Follower Dynamics

For $\lambda = (3/2), (5/2), (7/2)$

$$\frac{1}{r_j^3} \approx \frac{1}{r^3} - \frac{3x_j}{r^4}, \quad \frac{1}{r_j^5} \approx \frac{1}{r^5} - \frac{5x_j}{r^6}, \quad \frac{1}{r_j^7} \approx \frac{1}{r^7} - \frac{7x_j}{r^8}$$

$$\zeta_j \approx \zeta - \frac{8k_{J2}x_j s(I)s(\theta)}{r^5} + \frac{2k_{J2}y_j s(I)c(\theta)}{r^5} + \frac{2k_{J2}z_j c(I)}{r^5}$$

$$\eta_j^2 \approx \eta^2 - \frac{3\mu x_j}{r^4} - \frac{5k_{J2}x_j(1 - 5s^2(I)s^2(\theta))}{r^6} - \frac{5k_{J2}y_j s^2(I)s(2\theta)}{r^6} - \frac{5k_{J2}z_j s(2I)s(\theta)}{r^6}$$

Linearization of Follower Dynamics

$$
\begin{aligned}
\ddot{x}_j =&\ 2\dot{y}_j\omega_z + x_j\left(2\eta^2 + \omega_z^2 + \frac{2k_{J2}}{r^5}(1 - s^2(I)s^2(\theta)\right)\\
&+ y_j\left(\alpha_z + \frac{4k_{J2}s^2(I)s(2\theta)}{r^5}\right) - 5z_j\omega_x\omega_z\\
\ddot{y}_j =& -2\dot{x}_j\omega_z + 2\dot{z}_j\omega_x + x_j\left(\frac{4k_{J2}s^2(I)s(2\theta)}{r^5} - \alpha_z\right)\\
&- y_j\left(\frac{2k_{J2}s^2(I)c^2(\theta)}{r^5} + \eta^2 - \omega_z^2 - \omega_x^2\right)\\
&+ z_j\left(\alpha_x - \frac{k_{J2}s(2I)c(\theta)}{r^5}\right)\\
\ddot{z}_j =& -2\dot{y}_j\omega_x - 5x_j\omega_x\omega_z - y_j\left(\frac{k_{J2}s(2I)c(\theta)}{r^5} + \alpha_x\right)\\
&- z_j\left(\eta^2 - \omega_x^2 + \frac{2k_{J2}c^2(I)}{r^5}\right)
\end{aligned}
$$

Linearization of Follower Dynamics

$$\frac{d}{dt}\begin{bmatrix}\dot{x}_j\\ \dot{y}_j\\ \dot{z}_j\end{bmatrix} = A_1(t)\begin{bmatrix}\dot{x}_j\\ \dot{y}_j\\ \dot{z}_j\end{bmatrix} + A_2(t)\begin{bmatrix}x_j\\ y_j\\ z_j\end{bmatrix}$$

$$A_1(t) = \begin{bmatrix} 0 & 2\omega_z & 0\\ -2\omega_z & 0 & 2\omega_x\\ 0 & -2\omega_x & 0\end{bmatrix}$$

$$A_2(t) = \begin{bmatrix} 2\eta^2 + \omega_z^2 + \frac{2k_{J2}}{r^5}(1 - s^2(I)s^2(\theta)) & \alpha_z + \frac{4k_{J2}s^2(I)s(2\theta)}{r^5} & -5\omega_x\omega_z\\ \frac{4k_{J2}s^2(I)s(2\theta)}{r^5} - \alpha_z & \frac{2k_{J2}s^2(I)c^2(\theta)}{r^5} + \eta^2 - \omega_z^2 - \omega_x^2 & \alpha_x - \frac{k_{J2}s(2I)c(\theta)}{r^5}\\ -5\omega_x\omega_z & \frac{k_{J2}s(2I)c(\theta)}{r^5} + \alpha_x & \eta^2 - \omega_x^2 + \frac{2k_{J2}c^2(I)}{r^5}\end{bmatrix}$$

Simulation and Analysis of Satellite Formations

Initial Conditions

$$\begin{bmatrix} x_0 & y_0 & z_0 & v_{x0} & v_{y0} & v_{z0} \end{bmatrix} \longleftrightarrow \begin{bmatrix} a_0 & e_0 & \nu_0 & I_0 & \Omega_0 & \omega_0 \end{bmatrix}$$

$$r_0 = \left\| \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \right\|_2$$

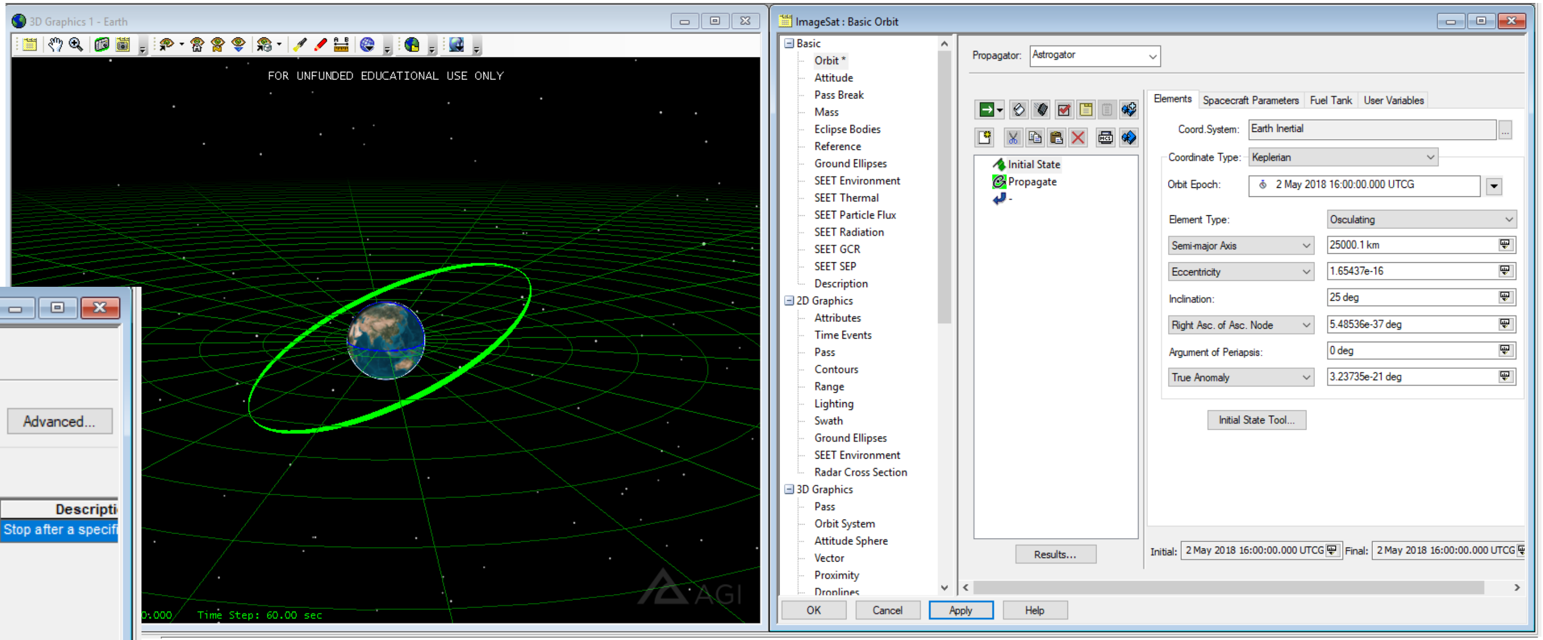
$$\dot{r}_0 = \begin{bmatrix} v_{x0} \\ v_{y0} \\ v_{z0} \end{bmatrix} \cdot \hat{\mathbf{r}}$$

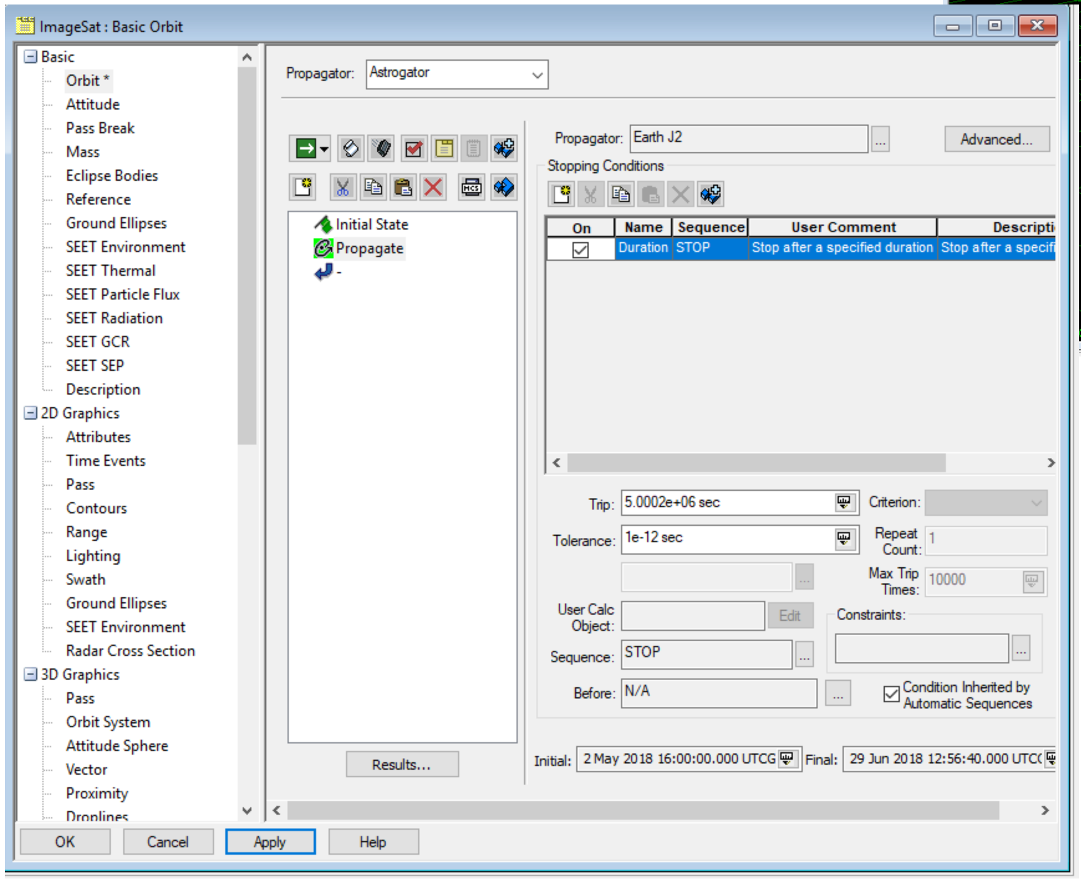
$$h_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \times \begin{bmatrix} v_{x0} \\ v_{y0} \\ v_{z0} \end{bmatrix}$$

$$\theta_0 = \omega_0 + \nu_0$$

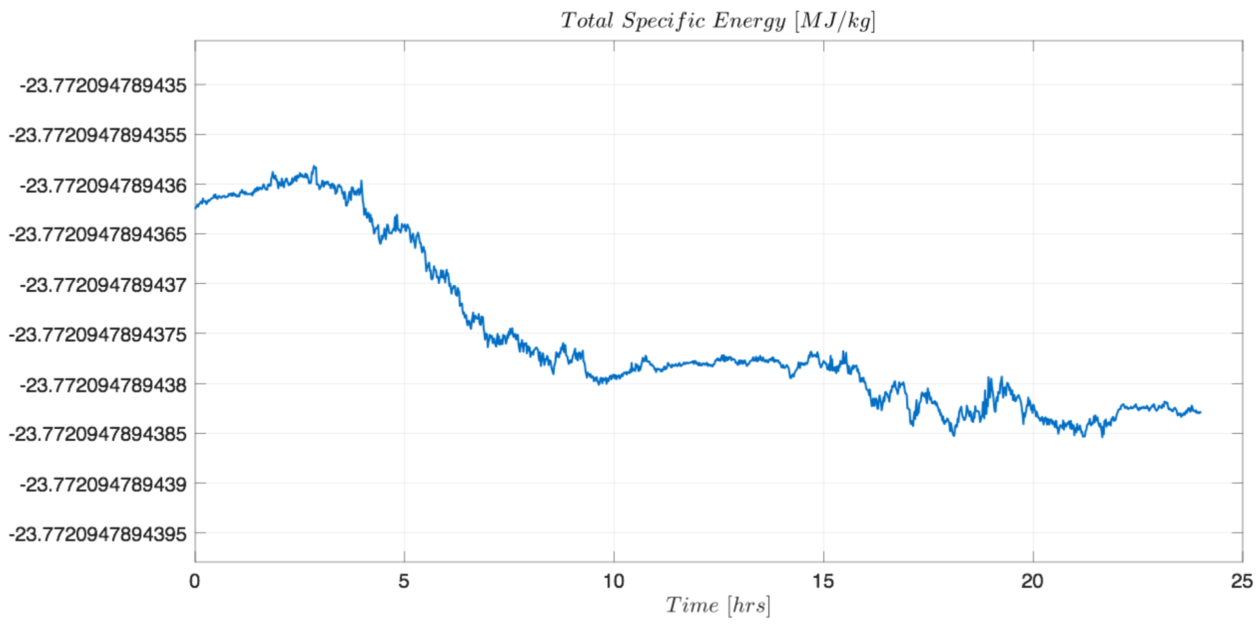
$$z_0 = \begin{bmatrix} r_0 & \dot{r}_0 & h_0 & \Omega_0 & I_0 & \theta_0 \end{bmatrix}$$

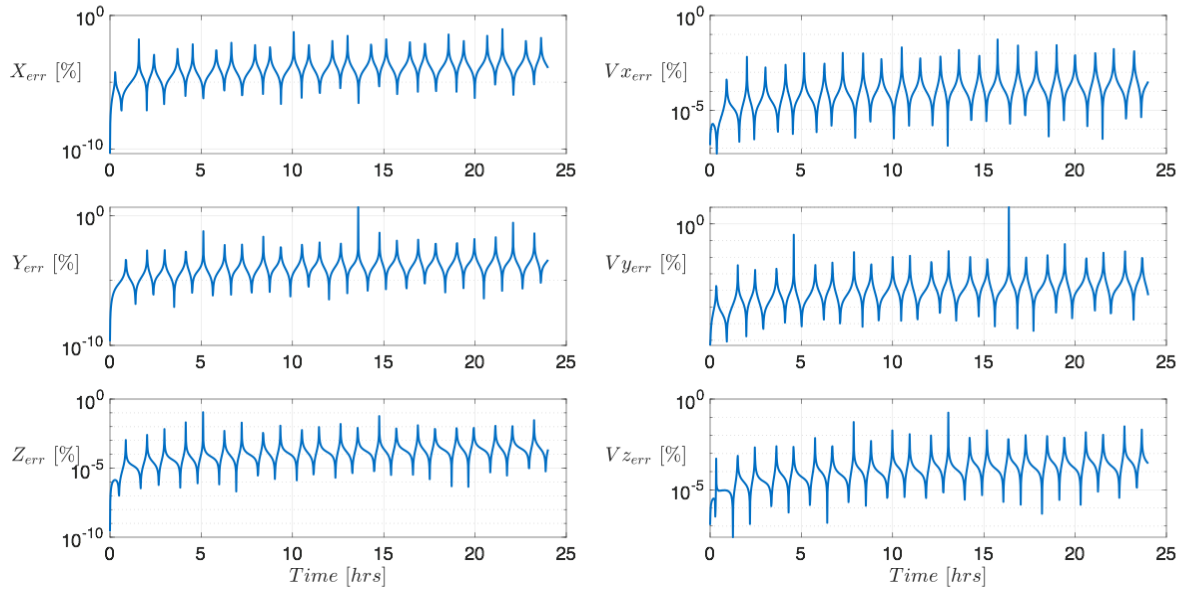
Nonlinear Dynamics Validation with STK

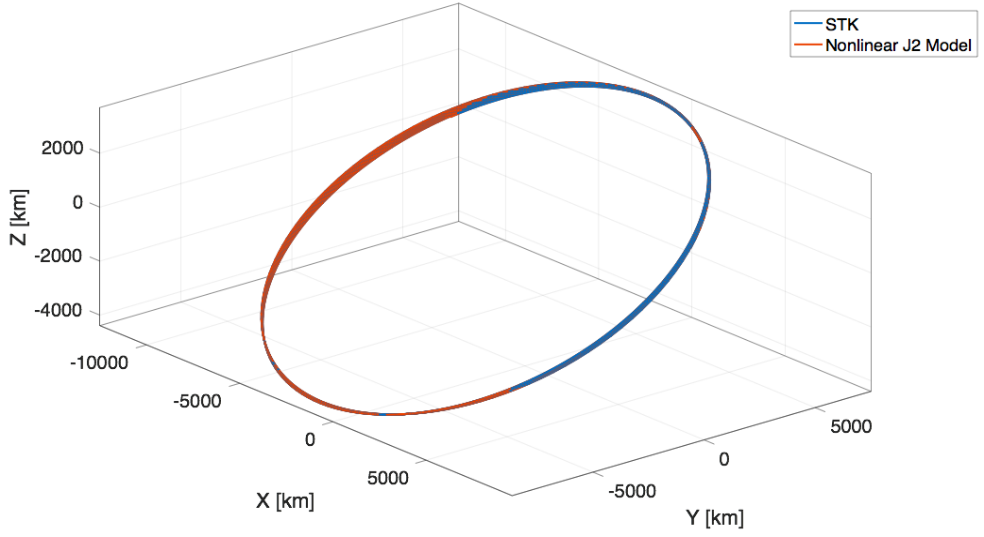




Nonlinear Dynamics Validation with STK

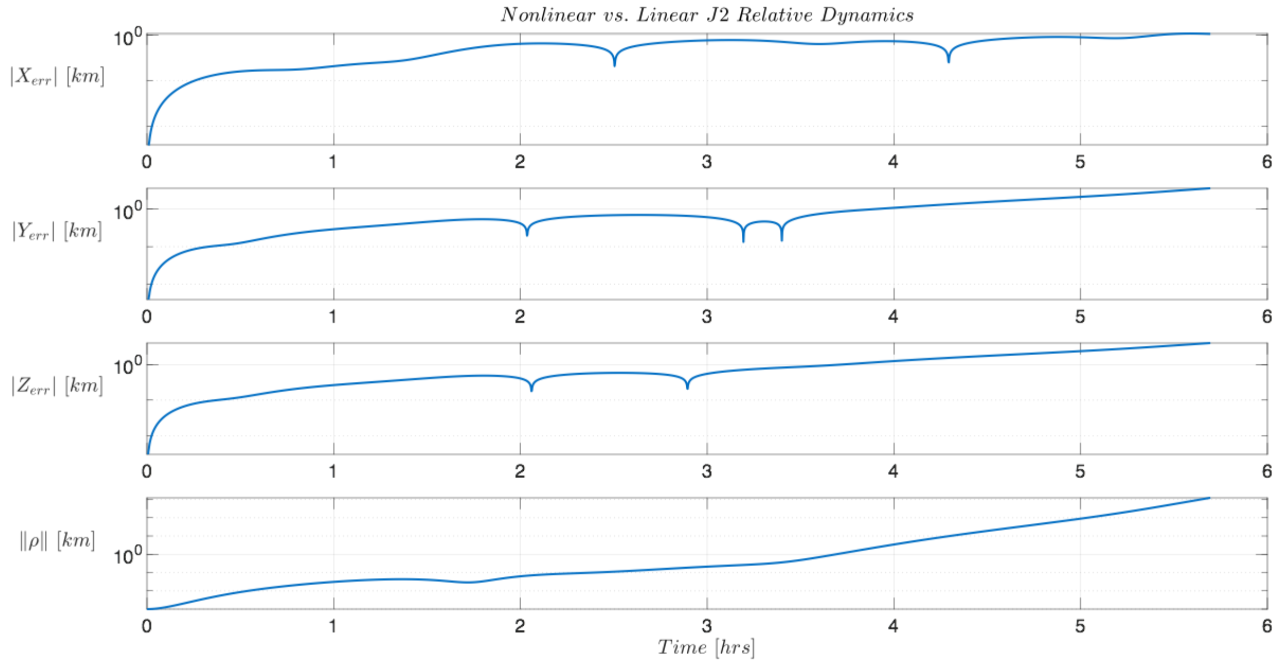


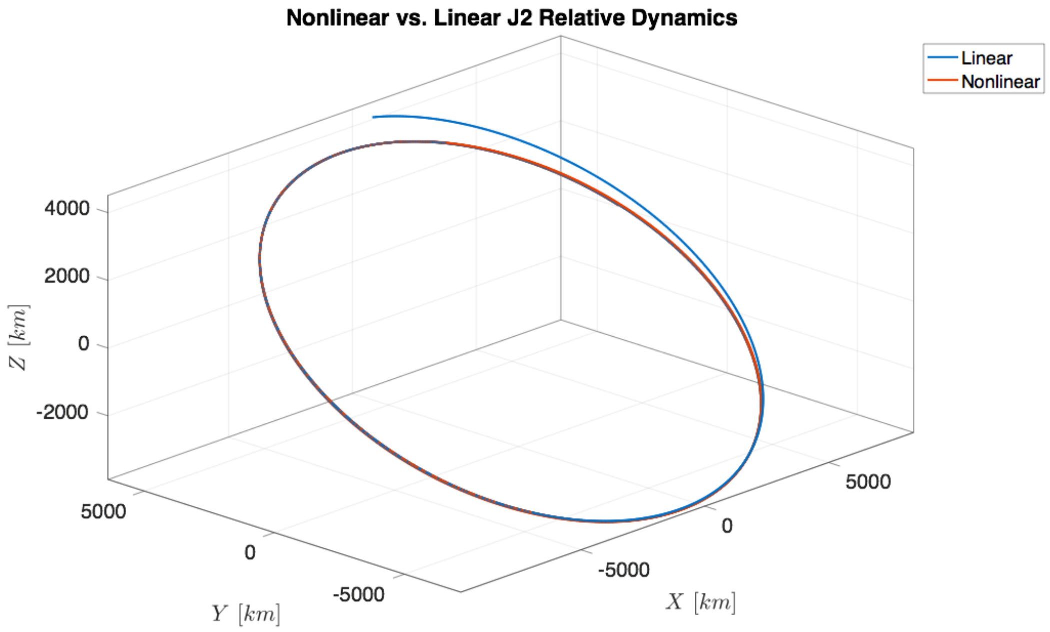




$$\epsilon_{tot} = K_j + U_j = \frac{1}{2}{}^{\mathcal{I}}\mathbf{v}_j^{T}\,{}^{\mathcal{I}}\mathbf{v}_j + \left(-\frac{\mu}{r_j} - \frac{k_{J_2}}{3r_j^3} + \frac{k_{J_2} r_j^2 Z}{r_j^5}\right)$$

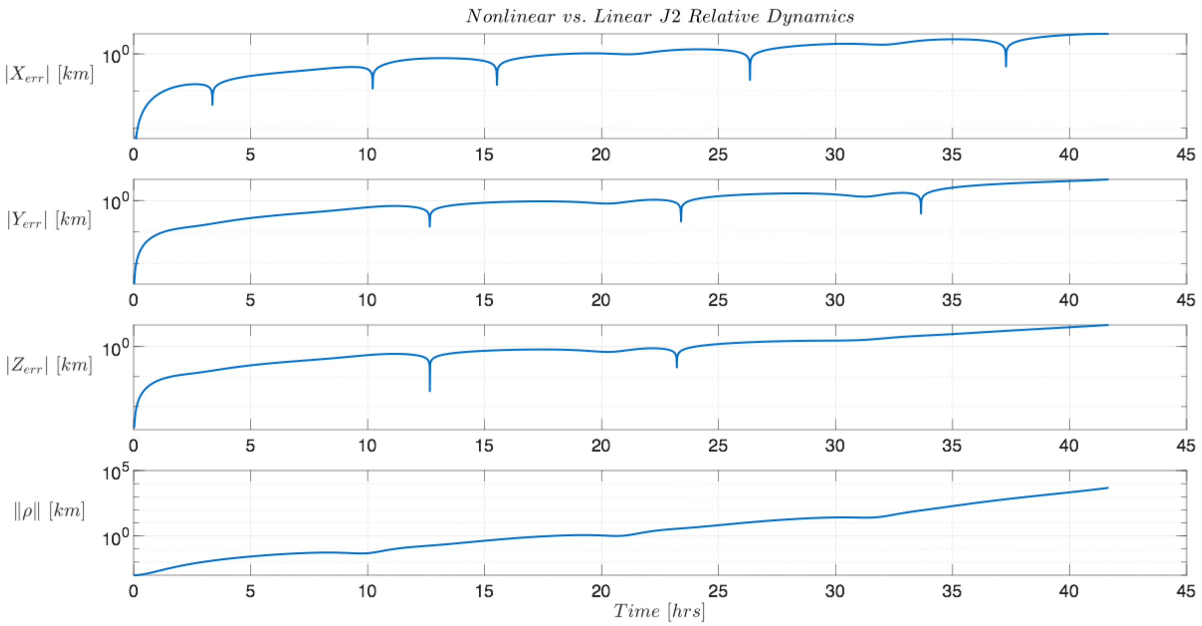
Nonlinear vs. Linear (close approach)

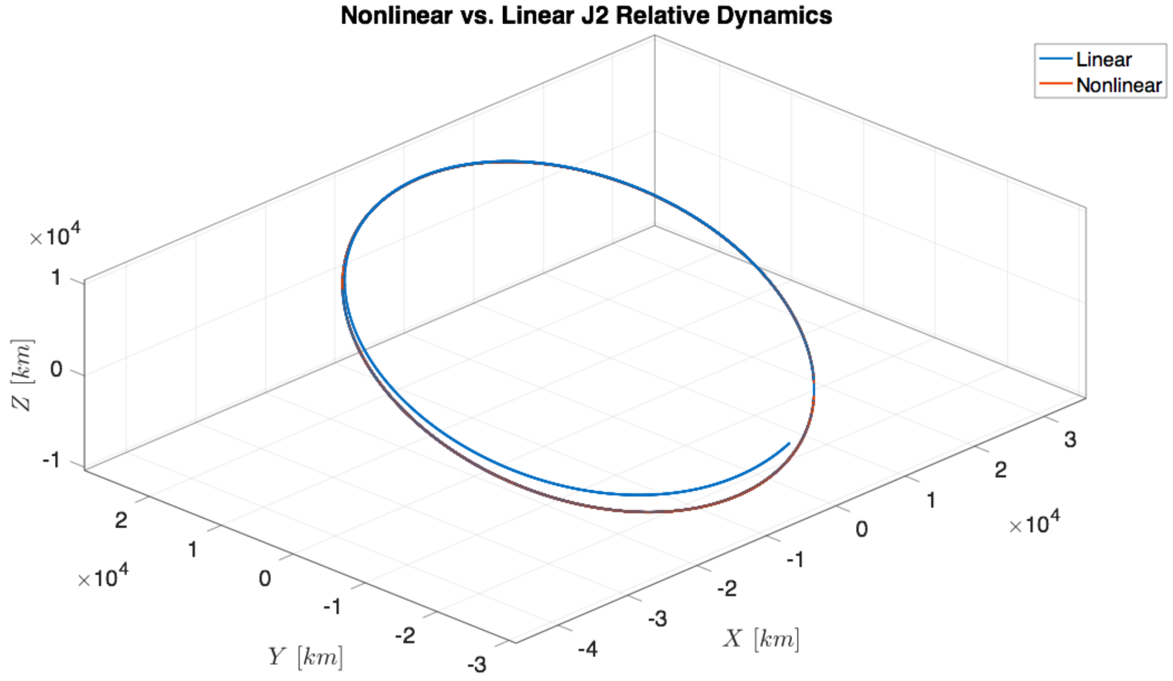




$$r_p = 6687\ km$$

Nonlinear vs. Linear (High orbit)





$$r_p = 20000.1\ km$$

Relative Dynamics for General Perturbations

$$\ddot{\mathbf{r}} = -\nabla U + \mathbf{a}_{dist}$$

$$\frac{d}{dt}\left(\frac{\partial K_j}{\partial \dot{\mathbf{q}}_j}\right) - \frac{\partial K_j}{\partial \mathbf{q}_j} + \frac{\partial U_j}{\partial \mathbf{q}_j} = Q_{dist}$$

Conclusions

- Incorporate J2 effects into relative dynamics, derived in LVLH frame
- Linear J2 model can be developed from exact nonlinear model
- STK used to validate nonlinear model
- Linear model works much better in high orbits (large perigee) than close approaches (small perigee)

References

Wang, D., Wu, B. and Poh, E. (2017). *Satellite Formation Flying*. Singapore: Springer Singapore.

Xu, G. and Wang, D. (2008). Nonlinear Dynamic Equations of Satellite Relative Motion Around an Oblate Earth. *Journal of Guidance, Control, and Dynamics*, 31(5), pp. 1521-1524.

Questions?